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Letter

Particle Acceleration by Electrostatic Waves Propagating in an Inhomogeneous Plasma

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Swift [1968] has suggested that an efficient accelerating process for electrons could be provided by electrostatic waves that propagate downward in the inhomogeneous plasma of the upper ionosphere. These waves would be excited by anisotropies in the distribution function of the ring current. Numerical computation by *Gary et al.* [1968] showed that this conjecture is correct: if the phase velocity of the waves increases, a few particles may remain trapped in the potential wells of the waves, and their velocities would then be multiplied by a factor of 4 in the chosen example.

We shall demonstrate that these results can be obtained by analytically studying the motion of particles in the wave field. Conditions for acceleration will be given in terms of rate of increment of the phase velocity, the magnetic field gradient, and the amplitude of the wave.

We assume that an electrostatic wave is emitted in a region of space with a fixed frequency ω and propagates along field lines downward to the earth. The wavelength perpendicular to the magnetic field is supposed to be much longer than the electron gyroradius and the frequency much smaller than the electron gyrofrequency. In the Wentzel-Kramer-Brillouin approximation, the component E of the electric field parallel to a magnetic field line can be written:

$$E = E(Z) \sin \left[\omega t - \int_0^{z} k(Z')\, dZ' \right] \qquad (1)$$

where E and k are slowly varying functions of the parallel coordinate Z. The motion of an electron along the magnetic field line is then given by [*Northrop*, 1963, p. 6]

$$\frac{d^2 Z}{dt^2} = -\frac{\mu}{m}\frac{dB}{dZ} + \frac{e}{m} E(Z) \cdot \sin\left[\omega t - \int_0^Z k(Z')\, dZ' \right] \qquad (2)$$

where e, m, and μ are the charge, the mass, and the magnetic moment of the electron ($\mu = mv^2/2B$), and B is the intensity of the magnetic field. As we are looking for particles that are in resonance with the wave, we put:

$$\frac{d}{dt} Z = Y + \bar{Z}$$

where $\bar{Z}$ is defined by

$$\bar{Z} = \omega/k(\bar{Z})$$

Equation (2) becomes:

$$\frac{d^2 Y}{dt^2} = \frac{\omega^2}{k^3}\frac{dk}{d\bar{Z}} - \frac{\mu}{m}\frac{dB}{d\bar{Z}} - \frac{e}{m} E(\bar{Z} + Y) \cdot \sin \int_{\bar{Z}}^{\bar{Z}+Y} k(Z')\, dZ' \qquad (3)$$

k, B, and E are slowly varying functions, so that in the lowest-order approximation we can consider them as constants if y remains small enough.

To fulfill this last requirement, equation 3 must describe an oscillatory motion in the lowest order, which is possible only if

$$\left| \frac{m\omega^2}{k^3}\frac{dk}{d\bar{Z}} - \mu \frac{dB}{d\bar{Z}} \right| < eE \qquad (4)$$

The physical meaning of this condition is that the displacement of the averaged position of the particle relative to $\bar{Z}$ remains small compared with the wavelength. This condition, necessary for a particle to remain trapped in the wave and consequently accelerated, is also sufficient in our example as we now demonstrate.

If we set

$$\frac{1}{2}\left(\frac{dY}{dt}\right)^2 = \epsilon + \frac{e}{m}\frac{E}{k}\cos kY \qquad (5)$$

ϵ is, in the lowest order, a constant of the motion; if $\epsilon < eE/mk$, the motion is oscillatory in the wave frame of reference with a period τ given by

$$\tau = \oint dY\left[2\left(\epsilon + \frac{eE}{mk}\cos kY\right)\right]^{-1/2} \qquad (6)$$

In the next order of approximation, the time derivative of ϵ averaged over the rapidly varying phase is obtained from (3), (5), and (6):

$$\oint \frac{d\epsilon}{dt}\,dt = -\frac{e}{m}\oint \frac{d}{d\bar{Z}}\left(\frac{E}{k}\cos kY\right)\frac{d\bar{Z}}{dt}\,dt \qquad (7)$$

This equation proves the conversation of the adiabatic invariant J defined by

$$J = \oint \left[\epsilon + \frac{eE}{mk}\cos kY\right]^{1/2} dY \qquad (8)$$

as one can verify by taking dJ/dt and using the definition of dY/dt in (5), if we set

$$\lambda = \left(\frac{1}{2} + \frac{mk\epsilon}{2eE}\right)^{1/2}$$

we get

$$J = 8(2^{1/2})\left(\frac{eE}{mk^3}\right)^{1/2}\lambda^2\{K(\lambda) - D(\lambda)\} \qquad (9)$$

where K and D are complete elliptic integrals.

The particle will remain trapped if λ remains smaller than unity when Z varies, J being kept constant.

We can distinguish two cases:

1. If

$$\frac{E(Z)}{E(0)} > \left(\frac{k(Z)}{k(0)}\right)^3 \quad \text{for} \quad k(Z) < k(0)$$

λ decreases when Z increases and the particle remains trapped forever in the wave (in the frame of the WKB approximation).

2. If

$$\frac{E(Z)}{E(0)} < \left(\frac{k(Z)}{k(0)}\right)^3$$

λ increases with Z and the particle is trapped in the wave as long as $\lambda(Z)$ remains less than 1.

The conservation of J gives a bound to the acceleration process only in the second case.

For the electrostatic waves we are thinking about, if we assume that the wavelength perpendicular to the magnetic field does not depend on Z and that amplification or damping are negligible, $E(Z)$ is proportional to $k^{3/2}$.

To verify this result, we remark that the scalar potential ϕ of the wave satisfies the equation:

$$\left(-k_\perp^2 + \frac{d^2}{dZ^2}\right)\phi = \frac{d}{dZ}\left(\frac{\omega_p^2}{\omega^2}\frac{d\phi}{dZ}\right) \qquad (10)$$

ω_p being the plasma frequency ($\omega_p^2 = ne^2/m\epsilon_0$), and $k_\perp$ the wave number perpendicular to the magnetic field.

Equation 10 is simply the equation of propagation of plasma waves for a cold dilute plasma if the plasma frequency is much smaller than the electronic cyclotron frequency.

Looking for a WKB solution, $\phi(Z)$ exp $i\int k(Z')\,dZ'$, we find that $\phi \sim k^{1/2}$, and consequently $E = d\phi/dZ \sim k^{3/2}$.

In this case, the particle remains trapped in the wave and is accelerated as long as condition 4 is satisfied. We can verify, in the quoted numerical example of *Gary et al.* [1968], that detrapping occurs when condition 4 has been violated. Then particle trapping in an accelerated electrostatic wave appears as a possible accelerating process and above all as a precipitation mechanism for low-energy electrons (since the mirror effect is easily suppressed for a very small electrostatic field amplitude).

Finally, let us notice that time variation of the frequency or of the amplitude could be easily taken into account if it remains small enough. The adiabatic invariant is not changed but new conditions of validity would appear. They have not been mentioned here because of our lack of knowledge about the parameters involved.

References

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