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# Motion of a single charged particle in electromagnetic fields with cyclotron resonances

By BORIS WEYSSOW

Physique Statistique, Plasmas et Optique Nonlinéaire, Faculté des Sciences, Université Libre de Bruxelles, CP 231 Campus Plaine, 1050 Bruxelles, Belgium

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The general Hamiltonian-averaging transformation developed to study the motion of a charged particle in a strong magnetic field and various electromagnetic perturbations permits a clear definition of the dynamics of the guiding centre. In the case of a high-frequency electromagnetic perturbation, the equations of evolution of the phase-space co-ordinates are sums of guiding-centre terms, resonant terms at any of the cyclotron resonance frequencies $n(v_{\parallel} k_{\parallel} - \omega) + l\Omega \approx 0$ ($l$ and $n$ are integers) and ponderomotive terms. In this paper we consider the $n = 1$ resonances giving a contribution one order stronger than the ponderomotive terms to the equations of motion. The guiding-centre transformation therefore suffices to derive the leading terms of the averaged dynamics. A simple case of isolated resonance is then considered for which, depending on the value of the external parameters (initial particle energy, amplitude of the perturbation), the phase space may possess one or two trapping regions. Even in the latter situation, the particle trajectories can be described analytically by the set of 12 Jacobian elliptic functions.

## 1. Introduction

In this paper we consider the motion of a charged particle in a quasi-static magnetic field in resonance with a high-frequency electromagnetic field. This study is the natural continuation of our previous work on Hamiltonian-averaging methods applied to the study of a single charged particle in an electromagnetic field. Indeed, in our previous paper (Weyssow & Balescu 1988, hereinafter referred to as WB3) we saw that the averaging transformation, used to suppress the dependence of the dynamics on two fast variables (the eikonal function of the wave and the gyrophase), breaks down because of the presence of a small denominator i.e. owing to a resonance: the two fast variables combine to give one slow variable, which is therefore not removed from the dynamics.

As a result, the equations of motion taking account of cyclotron resonances can also be obtained by a Hamiltonian generalized oscillation-centre transformation.

Our objective here is to provide a set of averaged equations of motion for a single particle that permits us to write down a drift kinetic equation including the effects of a cyclotron resonance. (A complete derivation of the drift kinetic equation using the formalism presented here was given by Balescu (1988).) A study of the gyrokinetic equation with perpendicular ion-cyclotron resonance was made by Lashmore-Davies & Dendy (1989). Since both equations must be in Hamiltonian form, we require an averaging transformation that obviously respects the Hamiltonian structure of the particle dynamics. As has been well known since the work of Kruskal (1962), the guiding-centre transformation, which we intend to extend here, cannot be developed in a suitable form by using only canonical transformations. As first shown by Littlejohn (1979, 1981, 1983), one needs to develop a new *pseudo-canonical* averaging transformation: i.e. a non-canonical transformation that respects the Hamiltonian structure of the dynamics. This can be accomplished, as Littlejohn did, by using Lie transformations and Darboux theorem. We, however, adopt a different approach.

First, as we showed in Weyssow & Balescu (1986, hereinafter referred to as WB1) (see also Balescu 1988), the fundamental structure underlying the Hamiltonian formalism is a Lie algebra. Therefore the operations included in this algebra should be sufficient for the development of an averaging transformation. There is no fundamental reason to use Darboux theorem.

Secondly, since there is a vast choice of non-canonical co-ordinates suitable for the description of the particle dynamics, it is intuitively justifiable to use a set of *Non-canonical but physical co-ordinates* such as the true particle velocity or the total energy of the system. Again, we stress that our point of view is different from Littlejohn's since he prefers a set of co-ordinates that are as close as possible to canonical ones. Apart from the relative simplicity of our averaging Hamiltonian transformation compared with previous ones, we also gain the possibility of simple comparisons with results of non-canonical averaging transformations. This is due to our use of the particle velocity and total energy as physical variables. The coordinates used here are those required in the application of the well-known non-Hamiltonian averaging method of Bogoliubov & Mitropolsky (1962).

The derivation of the drift kinetic equation including cyclotron resonance will not be given here. Rather, we shall continue our analysis of single-particle motion in this new context of cyclotron resonance.

The cyclotron resonances are defined by the condition $n(v_\parallel k_\parallel - \omega) + l\Omega \approx 0$, where $n$ is an integer associated with the order $\lambda^n$ of the averaging transformation and $l$ is associated with an expansion of the function exp $[iS(\mathbf{z}(\mathbf{Z})]$ in a series of Bessel functions of integer indices $l$. Hence, $n = 1$ corresponds to a series of resonances that contribute linearly in the wave amplitude to the equations of motion; $n = 2$ corresponds to a series of resonances that contribute quadratically in the wave amplitude to the equations of motion, and which are therefore of the same order of magnitude as the ponderomotive terms.

In practice, only the values $n = 1$ or $n = 2$ are accessible because the derivation of the averaged dynamics rapidly becomes tedious with increasing power of the expansion parameter $\eta$ (the usual guiding-centre parameter). The case $n = 2$, considered for example by Kitsenko, Pankratov & Stepanov (1975), is a series of resonances provided by terms of order $\lambda^2\eta$ in the equations of motion, i.e. by terms of the order of the ponderomotive terms discussed in WB3 (the small parameter $\lambda$ is the ratio of the amplitude of the strong magnetic field to the amplitude of the wave). The $n = 1$ resonances considered here (see also

Kitsenko, Pankratov & Stepanov 1974; Dendy 1987; and references therein) comes from terms of order $\lambda\eta$, linear in the amplitude of the perturbation and accordingly one order higher in $\lambda$ than the ponderomotive terms. The average dynamics to first order in $\lambda$ can be derived via the guiding-centre transformation (see WB1), which is a simpler version of the generalized averaging transformation in that it does not introduce an expansion in $\lambda$.

Previous authors who studied the derivation of the equations of motion with cyclotron resonance were not particularly interested in the structure in phase space of the isolated resonance (a single resonance $l$ that is not perturbed by the nearest resonances $l-1$ and $l+1$) or in a search for an analytical description of trajectories in phase space. Resonances were always reduced to the simple case for which the phase space presents only one trapping region. However, cases of resonances with many trapping regions certainly do exist (see e.g. Kitsenko and Pankratov 1984).

Landau damping, and, at the single-particle level, detrapping of particles, are processes that can hardly be understood without an analytic description of the particle motion. These important processes depend also on the structure of phase space. One reason for this is that in multiply connected trapping regions in phase space a particle that is untrapped from one trapping region may well be trapped by a neighbouring one. This could increase the average trapping time of the particle in comparison with the usual case of phase space with only one trapping region.

In the second part of this paper we study a simple case of isolated resonance with two trapping regions, for which we can still give an analytical description of phase-space trajectories. The condition for isolated resonance is important since it excludes any possibility of stochastic heating, as studied, for example, for the case of an obliquely propagating electrostatic field by Smith & Kaufman (1981) or for the case of a perpendicularly propagating electrostatic wave by Karney & Bers (1977) and Fukuyama, Mamota & Itatani (1977). The waves that could be used to heat the plasma stochastically are in the range of hybrid waves, as shown by Gell & Nakach (1980). Other waves, in the ion-cyclotron or electron-cyclotron frequencies, have resonance zones that are indeed isolated owing mainly to problems of wave propagation and mode conversion.

The equations of motion with the isolated $l = 2$ cyclotron resonance are simplified because of the following conditions imposed on the electromagnetic fields; the confining magnetic field is constant in space and time (equivalently, the width of the resonance is small compared with the length scale of variation of the magnetic field). A second condition is that the perpendicular wave vector is sufficiently small in order that $\mathscr{W} = v_\perp k_\perp/\Omega \leqslant 1$. This condition is usually valid for electrons in the Tokamaks since they have a very high Larmor frequency. Finally, we choose the electric component of the perturbing wave parallel to the wave vector to be one order weaker than the other two components. The wave vector, frequency and amplitude are also constants to the order of the calculations.

These conditions are preparatory to a perturbation scheme where a slow time dependence is added to the wave amplitude in order to simulate a slow Landau-damping effect. This procedure is already known for the case of a single trapping region as considered by Pocobelli (1981), who described analytically separatrix crossing by a particle.

The conditions summarized here are sufficient for reducing the system of equations of motion to only two dependent equations of motion with two parameters $a$ and $\mathscr{K}$. This system can be cast in Hamiltonian form, where a new nonlinearity in the action variable generates the multiple-trapping regions in phase space. This Hamiltonian is reduced to the usual pendulum Hamiltonian studied by Chirikov (1979) only in the case of small parameters.

The solutions for this system can be written in terms of the 12 Jacobian elliptic functions instead of only 3 as in the case of the pendulum Hamiltonian.

The condition of isolated resonance requires that the parameter $a$ must be close to zero. Various plots of the trajectories in phase space, using the analytical solutions, are presented and reveal the deformation of the resonance as the parameter $\mathscr{K}$ varies. Section 2 presents the derivation of the resonance equations using the guiding-centre transformation. Comparisons are made with previous derivations using the averaging method of Bogoliubov & Mitropolsky (Kitsenko & Pankratov 1984; Palmadesso 1973) and the Lagrangian method due to Littlejohn (Dendy 1987). Section 3 describes the reduction of the set of equations of motion to two equations under the conditions specified above; in addition the structure of phase space is analysed. Section 4 presents a derivation of the analytical solutions and some pictures of phase space. Section 5 describes our conclusions.

## 2. Derivation of the resonance equations

A charged particle in a strong magnetic field may enter into resonance with a low-amplitude high-frequency electromagnetic field at any one of the normal $(l > 0)$ or, anomalous $(l < 0)$ Doppler cyclotron resonances (Sagdeev & Zaslavskii 1986):

$$v_{\parallel} k_{\parallel} - \omega + l\Omega \approx 0, \tag{2.1}$$

where $v_{\parallel}$ and $k_{\parallel}$ are the velocity of the particle and the wave vector along the quasi-static magnetic field $\mathbf{B}_{(0)}$, $\Omega = eB_{(0)}/mc$ is the Larmor frequency, $\omega$ is the wave frequency and $l$ is an integer. The motion of such a particle is best described by a set of averaged equations on the fast gyrophase. We begin this section with a description of the electromagnetic field leading to the resonances (2.1) and then give a summary of the Hamiltonian pseudo-canonical averaging transformation that we developed when dealing with the guiding- and oscillation-centre problems (WB1; Weyssow & Balescu 1987, hereinafter referred to as WB2; WB3). This averaging procedure is then applied to the derivation of the equations of motion for a resonant particle.

We consider a charged particle in the presence of an electromagnetic field of the form

$$\left.\begin{aligned} \mathbf{B}(\mathbf{q},t) &= \mathbf{B}_{(0)}(\mathbf{q},t) + \lambda \mathbf{B}_{(1)}(\mathbf{q},t), \\ \mathbf{E}(\mathbf{q},t) &= \mathbf{E}_{(0)}(\mathbf{q},t) + \lambda \mathbf{E}_{(1)}(\mathbf{q},t), \end{aligned}\right\} \tag{2.2}$$

where $\lambda$ is a small parameter characterizing the strength of the high-frequency electromagnetic field $\mathbf{B}_{(1)}(\mathbf{q},t)$, $\mathbf{E}_{(1)}(\mathbf{q},t)$ compared with the strength of the quasi-static magnetic field $\mathbf{B}_{(0)}(\mathbf{q},t)$. A second small parameter $\eta$ is introduced to characterize the spatial and temporal scales of variation of the dynamical quantities. The following definitions of the multiple-scales formalism are used:

$$\left.\begin{aligned} \frac{dt^{-1}}{dt} &= \eta^{-1}, & \frac{dt^{0}}{dt} &= \eta^{0} = 1, \quad \ldots, \\ \frac{dq^{-1r}}{dq^{s}} &= \eta^{-1}\delta^{rs}, & \frac{dq^{0r}}{dq^{s}} &= \delta^{rs}, \quad \ldots \quad . \end{aligned}\right\} \tag{2.3}$$

In terms of these variables we assume the following form for the electromagnetic field:

$$\left.\begin{aligned} \mathbf{B}_{(1)}(\mathbf{q},t) &= \tilde{\mathbf{B}}_{(1)}(\mathbf{q}^{0},t^{0})\exp\left[iS(\mathbf{q}^{-1},t^{-1},\mathbf{q}^{2},t^{2})\right]+\text{c.c.}, \\ \mathbf{E}_{(1)}(\mathbf{q},t) &= \tilde{\mathbf{E}}_{(1)}(\mathbf{q}^{0},t^{0})\exp\left[iS(\mathbf{q}^{-1},t^{-1},\mathbf{q}^{2},t^{2})\right]+\text{c.c.}, \end{aligned}\right\} \tag{2.4}$$

where

$$\left.\begin{aligned} \nabla S &= \frac{1}{\eta}\nabla^{-1}S+O(\eta^{0}) \equiv \frac{1}{\eta}\mathbf{k}+O(\eta^{0}), \\ \partial_{t} S &= \frac{1}{\eta}\partial_{t^{-1}}S+O(\eta^{0}) \equiv \frac{1}{\eta}\omega+O(\eta^{0}). \end{aligned}\right\} \tag{2.5}$$

The eikonal approximation to the electromagnetic wave made here is used in the description of high-frequency waves (Bernstein 1975; Brambilla & Cardinali 1982). Even if the resonant equations are solved for a constant-$\mathbf{B}_{(0)}$ magnetic field (i.e. in a shear-less slab geometry), for later purposes we retain the possibility of introducing the effects of the curvature of the field lines in the equations of motion. We therefore choose $\mathbf{B}_{(0)}(\mathbf{q},t) = \mathbf{B}_{(0)}(\mathbf{q}^{1},t^{2})$ in accordance with the guiding-centre ordering (see WB1). We also introduce the usual guiding-centre parameter $\epsilon$, defined as the ratio of the Larmor radius to the characteristic length of variation of the fields. In the present problem this parameter is small because the quasi-static magnetic field is chosen to be strong enough so that the motion of the particle can be approximated by the motion of its guiding centre. Going back to (2.1) and comparing the wave frequency in (2.5) with the Larmor frequency $\Omega$ of order $\epsilon^{-1}$, we see that for a resonant particle we must have $\eta$ of order $\epsilon:\eta = O(\epsilon)$.

The final form of the electro-magnetic field that will be used is therefore

$$\left.\begin{aligned} \tilde{\mathbf{B}}(\mathbf{q},t) &= \frac{1}{\eta}\mathbf{B}_{(0)(-1)}(\mathbf{q}^{1},t^{2})+\lambda\sum_{n\geqslant 0}\eta^{n}\,\tilde{\mathbf{B}}_{(1)(n)}(\mathbf{q}^{0},t^{0})\exp(iS)+\text{c.c.} \\ \mathbf{E}(\mathbf{q},t) &= \sum_{n\geqslant 0}\eta^{n}\tilde{\mathbf{E}}_{(1)(n)}(\mathbf{q}^{0},t^{0})\exp(iS)+\text{c.c.}, \end{aligned}\right\} \tag{2.6}$$

where for simplicity we have neglected fields of order $\lambda^{0}\eta^{n}$, $n > -1$. In the following we shall use the simpler notation $\mathbf{B}_{(0)}$ for the quasi-static magnetic field.

The motion of a charged particle in an electromagnetic field may be described using a Hamiltonian formalism on an extended phase space with canonically conjugate co-ordinates $\mathbf{z}' = \{\mathbf{q},\mathbf{p},t,h\}$. The dynamics can be represented by

$$\mathscr{D}(\mathbf{z}') = \{H(\mathbf{z}'),[\mathbf{z}'^{i},\mathbf{z}'^{j}] = l^{ij}\}, \tag{2.7}$$

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where $H$ is the extended Hamiltonian,

$$H(\mathbf{z}') = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi + h,$$

and $l^{ij}$ is a set of Poisson brackets between the phase-space co-ordinates,

$$[q^i, p^j] = \delta^{ij}, \quad [t, h] = 1.$$

The only formula involving the Lie brackets that will be used in the averaging transformation is

$$[A(\mathbf{z}'), B(\mathbf{z}')] = \frac{\partial A(\mathbf{z}')}{\partial z^i}[z^i, z^j]\frac{\partial B(\mathbf{z}')}{\partial z^j},$$

where $A(\mathbf{z}')$ and $B(\mathbf{z}')$ are two dynamical functions. This formula also serves to derive the equation of evolution of a dynamical function, say $A(\mathbf{z}')$. In this case $B(\mathbf{z}')$ is simply the Hamiltonian and one has

$$\dot{A}(\mathbf{z}') \equiv [A(\mathbf{z}'), H(\mathbf{z}')].$$

It is easily verified that this equation reduces to the usual Hamilton equation when $A$ is one of the canonical phase-space co-ordinates.

A set of pseudo-canonical transformations, summarized below but more fully explained in our previous works (WB1–3), is necessary to obtain a description of the motion of the particle in terms of the physical quantities $\mathbf{z} = \{\mathbf{q}, v_\parallel, v_\perp, \varphi, t, \mathscr{E}\}$.

(*a*) First a pseudo-canonical transformation

$$\mathbf{v} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right), \quad k = h + e\Phi \tag{2.8}$$

is necessary to replace the dependence of the dynamics on the potentials $\mathbf{A}$ and $\Phi$ by a dependence on the electromagnetic field $\mathbf{E}$, $\mathbf{B}$:

$$\mathbf{E} = -\nabla\Phi + \frac{1}{c}\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.9}$$

(*b*) Since the variable $k$ has no obvious physical meaning, we prefer to use the Hamiltonian as the variable conjugate to the time $t$:

$$\mathscr{E} \equiv H = \tfrac{1}{2}mv^2 + k. \tag{2.10}$$

(*c*) The gyration of the particle is best described by the use of the cylindrical components of the velocity:

$$\mathbf{v} = v_\parallel \mathbf{b} + v_\perp \mathbf{n}_1. \tag{2.11}$$

The reference frames defining (2.11) are as follows. First we have a moving local reference frame comprising three unit vectors;

- $\mathbf{b}$ along the quasi-static magnetic field;
- $\mathbf{n}_1$ along the projection of the velocity in the plane orthogonal to the quasi-static magnetic field; and
- $\mathbf{n}_2$ orthogonal to $\mathbf{b}$ and $\mathbf{n}_1$.

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Secondly, we have a fixed reference frame comprising

- $\mathbf{b}$ along the quasi-static magnetic field;
- $\mathbf{e}_1$ along the principal normal to the magnetic field line; and
- $\mathbf{e}_2$ orthogonal to $\mathbf{b}$ and $\mathbf{e}_1$.

The relation between the two frames is

$$\mathbf{n}_1 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2, \tag{2.12}$$

where $\varphi$ is the gyrophase of the particle.

The transformation defined in (*a*)–(*c*) applied to the original dynamics gives a new representation that is still Hamiltonian but in non-canonical form ($l^{ij}$ now depends on the phase-space co-ordinates):

$$\mathscr{D}(\mathbf{z}) = \{H(\mathbf{z}), [z^i, z^j] = l^{ij}(\mathbf{z})\}, \tag{2.13}$$

where
$$H(\mathbf{z}) = \mathscr{E}$$

and
$$l^{ij} \equiv [z^i, z^j] = \sum_{n=-1}^{+\infty} \eta^n l^{ij}_{(0)(n)} + \sum_{n=0}^{+\infty} \lambda\eta^n l^{ij}_{(1)(n)}.$$

The explicit form of the equations of motion is

$$\left.\begin{aligned}
&[\mathbf{q}, \mathscr{E}] = v_\parallel \mathbf{b} + v\,\mathbf{n}_1,\\
&[t, \mathscr{E}] = 1,\\
&[v_\parallel, \mathscr{E}] = \frac{e}{m}\mathbf{b}.\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - v_\perp[\mathbf{v}.\nabla\mathbf{n}_1.\mathbf{b} + \partial_t\,\mathbf{n}_1.\mathbf{b}],\\
&[v_\perp, \mathscr{E}] = \frac{e}{m}\mathbf{n}_1.\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) - v_\parallel[\mathbf{v}.\nabla\mathbf{b}.\mathbf{n}_1 + \partial_t\,\mathbf{b}.\mathbf{n}_1],\\
&[\varphi, \mathscr{E}] = -\frac{e}{mv_\perp}\mathbf{n}_2.\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\\
&\qquad + \frac{1}{v_\perp}\{v_\parallel[\mathbf{v}.\nabla\mathbf{b}.\mathbf{n}_2 + \partial_t\,\mathbf{b}.\mathbf{n}_2] + v_\perp[\mathbf{v}.\nabla\mathbf{n}_1.\mathbf{n}_2 + \partial_t\,\mathbf{n}_1.\mathbf{n}_2]\},
\end{aligned}\right\} \tag{2.14}$$

where we still have to introduce the ordering of the guiding centre (see WB1) and the electromagnetic field as given by (2.6). As a result, the dynamical functions are divided into two groups. First, we find the fast dynamical functions depending on the fast variables $\varphi$ and $S$:

$$[\varphi, H] = \frac{1}{\eta}\Omega + O(\eta^0), \quad [\mathrm{S}, \mathrm{H}] = \frac{1}{\eta}(v_\parallel k_\parallel + v_\perp\,\mathbf{n}_1.\mathbf{k} - \omega) + O(\eta^0).$$

Secondly, we have the slow dynamical functions $f$ independent of $\varphi$ and $S$:

$$[f, H] = O(\eta^0).$$

This classification shows that the equations of motion (2.14) are in a form suitable for averaging. Such a result can *only* be reached by using non-canonical co-ordinates. This is already apparent in Kruskal's (1962) paper, which discusses the guiding-centre transformation. The first Hamiltonian averaging transformation for the guiding-centre problem was given by Littlejohn (1979).

In contrast with Littlejohn (1983), because of our use of *pseudo-canonical* but *physical* co-ordinates such as the velocity and the total energy, we get a Hamiltonian that depends on only one of the phase-space variables. We can therefore construct an averaging transformation that keeps the Hamiltonian *form-invariant*, and the total energy is therefore conserved through the averaging processes.

We recall that the transformation to oscillation-centre co-ordinates is introduced in order to delete all the fast terms from the dynamics. In the resonant case the two fast variables, the gyrophase $\varphi$ and the eikonal function $S$, combine to give a slow co-ordinate (later referred to as the phase coordinate $\xi_1$); to lowest order in $\lambda$ and $\eta$, its total time derivative is of the form $(v_\parallel k_\parallel - \omega) + l\Omega$, where $l$ is an integer.

To obtain the resonant dynamics at resonance (2.1), we need only consider the guiding-centre transformation, which is completely defined by the terms of zeroth order in $\lambda$ of the dynamics.

(i) A new set of phase-space co-ordinates $\mathbf{Z}$ is constructed as a power series in the small guiding-centre parameter about the old co-ordinates $\mathbf{z}$:

$$\mathbf{Z} = \mathbf{z}_{(0)}(\mathbf{z}) + \eta \mathbf{z}_{(1)}(\mathbf{z}) + O(\eta^2), \tag{2.15}$$

where
$$\mathbf{z}_{(0)} \equiv \mathbf{z} = \{\mathbf{q}, v_\parallel, v_\perp, \varphi, t, \mathscr{E}\}$$

and accordingly
$$\mathbf{Z} = \{\mathscr{Y}, \mathscr{U}, \mathscr{W}, \theta, t, \mathscr{E}\}.$$

The correspondence $\mathscr{E} \mapsto \mathscr{E}$, $t \mapsto t$ from the old to the new co-ordinates simply shows that the two variables have no expansion (2.15).

(ii) The dynamics $\mathscr{D}(\mathbf{z})$ is rewritten as a function of the new co-ordinates:

$$\mathscr{D}(\mathbf{z}(\mathbf{Z})) \equiv \mathscr{D}(\mathbf{Z}). \tag{2.16}$$

(iii) The new dynamics $\mathscr{D}(\mathbf{Z})$ is said to be averaged if it is independent of the fast variable $\theta$ corresponding to the gyrophase $\varphi$ in the original phase-space co-ordinates:

$$\partial_\theta \mathscr{D}(\mathbf{Z}) = 0. \tag{2.17}$$

(iv) Applying this condition, we get separately a set of partial differential equations for the $\mathbf{z}_{(n)}(\mathbf{z})$ that are easily integrated and the average dynamics.

The derivation of the guiding centre by this method can be found in WB1. We now turn to terms of first order in $\lambda$, which we rewrite in guiding-centre co-ordinates. First we recall from WB3 that the fast eikonal function $S$ transforms according to

$$S(\mathbf{z}) \mapsto S(\mathbf{Z}) - \mathbf{z}_{(1)} . \nabla^{-1} S(\mathbf{Z}) \equiv \chi(\mathbf{Z}) \tag{2.18}$$

to lowest order in $\eta$ through the guiding-centre transformation. In (2.18) $\mathbf{z}_{(1)} = \mathbf{q}_{(1)}$ is the first correction to the guiding-centre position:

$$\mathbf{q}_{(1)} = -v_\perp \mathbf{n}_2/\Omega.$$

Two other relations are also useful since they help in representing the oscillatory terms by their Fourier series:

$$\left.\begin{aligned} \left(\frac{iv_\perp k_\perp}{\Omega}\mathbf{n}_1.\hat{\mathbf{k}}\right)_{\mathbf{z}\equiv\mathbf{Z}} \exp(i\chi) &= \partial_\theta \exp(i\chi), \\ \left(\frac{iB_0 k_\perp}{mv_\perp \Omega}\mathbf{n}_2.\hat{\mathbf{k}}\right)_{\mathbf{z}=\mathbf{Z}} \exp(i\chi) &= \partial_{\mathscr{U}} \exp(i\chi), \end{aligned}\right\} \tag{2.19}$$

where $\mathscr{M} = m\mathscr{W}^2/2B_{(0)}$. The Fourier series are defined by

$$A(\mathbf{Z})\exp(i\chi) = \sum_{l=-\infty}^{+\infty} A^{(1)}(\mathbf{Z}')\,J_l\left(\frac{\mathscr{W}k_\perp}{\Omega}\right)\exp(i\zeta_1), \tag{2.20}$$

where $J_1$ is a Bessel function of order $l$, $\mathbf{k} = k_\parallel \mathbf{b} + k_\perp \hat{\mathbf{k}}$, $\zeta_1 = S(\mathbf{Z}) + l\Phi + \frac{1}{2}l\pi$ and $\mathbf{Z}'$ is the set of all co-ordinates except for $\theta$ that are contained in $\mathbf{Z}$. The Lie brackets between the new co-ordinates are all of the form (2.20). Far from resonance, they vary rapidly because of the exponential, which depends on $S$ and $\theta$. In the case of cyclotron resonance, the combination $\zeta_1 = S(\mathbf{Z}) + l\Phi + \frac{1}{2}l\pi$ is a slowly varying variable since we find

$$\dot{\zeta}_1 = Uk_\parallel - \omega + l\Omega,$$

which is $O(1)$ instead of $O(\eta^{-1})$.

Using the formula presented above, we write the $O(\eta^1)$ contribution of $O(\lambda^0)$ and $O(\lambda^1)$ terms in the equations of motion:

$$[\mathscr{Y}, \mathscr{E}] = \mathscr{U}\mathbf{b}, \tag{2.21a}$$

$$[\mathscr{U}, \mathscr{E}] = -\mathscr{M}\mathbf{b}.\nabla B_{(0)} + \sum_l \frac{e}{m\omega}\left\{\left[(\omega - l\Omega)\,\tilde{\mathbf{E}}.\mathbf{b} + l\Omega\frac{k_\parallel}{k_\perp}\tilde{\mathbf{E}}.\hat{\mathbf{k}}\right]J_1 + ik_\parallel \mathscr{W}\tilde{\mathbf{E}}.\mathbf{e}_2 J_1'\right\}\exp(i\zeta_1) + \text{c.c.}, \tag{2.21b}$$

$$[\mathscr{W}, \mathscr{E}] = \frac{\mathscr{U}\mathscr{W}}{2B_{(0)}}\mathbf{b}.\nabla B_{(0} + \sum_l \frac{e}{m\omega}\left\{\frac{l\Omega}{\mathscr{W}}\left[\mathscr{U}\tilde{\mathbf{E}}.\mathbf{b} + \frac{\omega - \mathscr{U}k_\parallel}{k_\perp}\tilde{\mathbf{E}}.\hat{\mathbf{k}}\right]J_1 + i(\omega - \mathscr{U}k_\parallel)\,\tilde{\mathbf{E}}.\mathbf{e}_2 J_1'\right\}\exp(i\zeta_1) + \text{c.c.}, \tag{2.21c}$$

$$[\theta, \mathscr{E}] = \frac{e}{mc}B_0(\mathscr{Y}, t) + \sum_l \frac{e}{m\omega}\mathscr{W}^{-2}\left\{i\mathscr{W}[\mathscr{U}k_\perp\tilde{\mathbf{E}}.\mathbf{b} + (\omega - \mathscr{U}k_\parallel)\tilde{\mathbf{E}}.\hat{\mathbf{k}}]J_1' + \frac{1}{k}[(\mathscr{W}k_\perp)^2 - l\Omega(\omega - \mathscr{U}k_\parallel)]\,\tilde{\mathbf{E}}.\mathbf{e}_2 J_1\right\}\exp(i\zeta_1) + \text{c.c.} \tag{2.21d}$$

Because of the Hamiltonian form of the Lie brackets (2.21 *a–d*), we can consider using them to write down a drift kinetic equation that includes the effects of a cyclotron resonant wave. This will not be done here, but it does represent a possible extension of the derivation of the drift kinetic equation as done by Balescu (1988).

The set of equations (2.21 *a–d*) can be compared with the equations of motion obtained by other averaging methods: the Bogoliubov–Mitropolskii method (Kitsenko & Pankratov 1984, Palmadesso 1973) or the Lagrangian approach developed by Littlejohn (Dendy 1987). The reduction of the equations to the electrostatic case is obtained by setting

$$\tilde{\mathbf{E}}.\mathbf{e}_2 = 0, \quad \tilde{\mathbf{E}}.\hat{\mathbf{k}} = \frac{k_\perp}{\mathrm{k}_\parallel}\tilde{\mathbf{E}}.\mathbf{b} \tag{2.22}$$

in (2.21 *a–d*). The result is the same as that given by Dendy (1987), except for a forgotten $n$ in his equation (17). We should also add that the oscillation of the co-ordinate of the guiding centre present in Dendy (1987) and in Kitsenko & Pankratov (1984) is one order higher in $\eta$ than the oscillatory terms in the other equations of motion. The more general case of an electromagnetic wave given by (2.21$a$–$d$) can be compared with the work of Palmadesso (1973). The equations of motion for the parallel and perpendicular velocities are almost the same (a misprint in his equation (19) is corrected by replacing the perpendicular velocity by a parallel velocity in front of the square-bracketed term). The equations of evolution of the phase variable are, however, slightly different.

In the following sections we shall study a very simple case of resonance contained in (2.21$a$–$d$) for which we can obtain an analytical expression for the trajectories in phase space.

## 3. The second-harmonic resonant equations of motion

In this section we consider the motion of a charged particle in a single resonance $l$. This supposes that the nearest resonances $l+1$ and $l-1$ are sufficiently far from the chosen resonance that they do not perturb each other. Also, the two fast variables, the gyrophase and the eikonal function, combine to give a new slow variable, which we use instead of the gyrophase:

$$S' = \xi_1 = S + l\theta + \tfrac{1}{2}l\pi. \tag{3.1}$$

To prepare for the study of the equations of motion (2.21$a$–$d$) in the case of the resonance $l$, we introduce dimensionless time ($\tau'$), position ($\mathscr{Y}'$), parallel-velocity ($\mathscr{U}'$) and perpendicular-velocity ($\mathscr{W}'$) variables by

$$\tau' = l\Omega\tau, \quad \mathscr{Y}' = k_\parallel \mathscr{Y}, \quad \mathscr{U}' = k_\parallel \frac{\mathscr{U}}{\omega}, \quad \mathscr{W}' = \frac{k_\perp \mathscr{W}}{\Omega};$$

the gyrophase $\theta' = \theta$ is unchanged.

We also define $\mathscr{M}' = m\mathscr{W}'^2/2B_{(0)}$, corresponding to the variable $\mathscr{M}$ used in (2.19). We also redefine the parameters of the problem:

$$\left.\begin{aligned} e_b &= \frac{\tilde{\mathbf{E}}\cdot\mathbf{b}}{B_{(0)}}, \quad e_k = \frac{\tilde{\mathbf{E}}\cdot\hat{\mathbf{k}}}{B_{(0)}}, \quad e_\perp = \frac{\tilde{\mathbf{E}}\cdot\mathbf{e}_2}{B_{(0)}}, \\ a_{k\Omega} &= \frac{ck_\perp}{l\Omega}, \quad a_\Omega = \frac{l\Omega}{\omega}, \quad \alpha_\kappa = \frac{\kappa_\parallel}{k}. \end{aligned}\right\} \tag{3.2}$$

With these changes (and dropping the primes), (2.21$a$–$d$) now become

$$\dot{\mathscr{Y}} = \frac{1}{a_\Omega}\mathscr{U}\mathbf{b}, \tag{3.3$a$}$$

$$\begin{aligned}\dot{\mathscr{U}} = -\frac{\mathscr{M}}{a_\Omega}\mathbf{b}\cdot\nabla B_{(0)}\left(\frac{a_k a_\Omega}{l}\right)^2 + \frac{1}{l}a_{k\Omega}a_\Omega a_k \Big\{&[(1-a_\Omega)e_b + a_\Omega a_k e_k]J_1 \\ &+ \frac{i}{l}a_k a_\Omega e_\perp \mathscr{W} J_1'\Big\}e^{iS} + \text{c.c.}\end{aligned} \tag{3.3$b$}$$

$$\begin{aligned}\dot{\mathscr{W}} = \frac{\mathscr{U}\mathscr{W}}{a_\Omega}\frac{\mathbf{b}\cdot\nabla\Omega}{2\Omega} + \frac{a_{k\Omega}}{\mathscr{W}}\Big\{&\left[\frac{e_b\mathscr{U}}{a_k} + (1-\mathscr{U})e_k\right]lJ_1 \\ &+ i(1-\mathscr{U})e_\perp J_1'\Big\}e^{iS} + \text{c.c.},\end{aligned} \tag{3.3$c$}$$

$$\begin{aligned}\dot{S} = \frac{1}{a_\Omega}(a_\Omega - 1 + \mathscr{U}) + \frac{la_{k\Omega}}{\mathscr{W}}\Big\{&i\mathscr{W}\left[\frac{\mathscr{U}e_b}{a_k} + (1-\mathscr{U})e_k\right]J_1' \\ &+ \left[\frac{a_\Omega\mathscr{W}^2}{l} - l(l-\mathscr{U})\right]e_\perp J_1\Big\}e^{iS} + \text{c.c.}\end{aligned} \tag{3.3$d$}$$

Following Palmadesso (1973), it is easily shown that the equations of motion ($3.3a$–$d$) can be reduced to just two equations of motion. However, this reduction is only possible if the quasi-static magnetic field is spatially constant in the resonant region. This approximation is valid for an isolated resonance whose width is smaller than the lengthscale of variation of the quasi-static magnetic field.

A first approximate integral of the motion is the kinetic energy of the particle in a frame moving at the wave's parallel velocity. Indeed, comparison of the equations of evolution for the parallel and perpendicular velocities (the strong magnetic field is constant) leads to

$$\mathscr{W}\dot{\mathscr{W}}-\left(\frac{1}{a_k\,a_\Omega}\right)^2(\dot{\mathscr{U}}-\mathscr{U}\dot{\mathscr{U}})=\frac{a_{k\Omega}}{a_k}J_1\,e_b\frac{a_\Omega+\mathscr{U}-1}{a_\Omega}e^{iS}+\text{c.c.} \tag{3.4}$$

This equation is easily integrated to give to order $\lambda^2$ (we have used the equation of evolution of the phase variable $S$ to approximate the integration of the last term in (3.4))

$$\tfrac{1}{2}\mathscr{W}^2-\left(\frac{1}{a_k\,a_\Omega}\right)^2(\mathscr{U}-\tfrac{1}{2}\mathscr{U}^2)=\frac{a_{k\Omega}}{a_k}J_1(e_b\,e^{iS}+\text{c.c.})+\tfrac{1}{2}\mathscr{K}'. \tag{3.5}$$

In the same way, it is easily shown that

$$\frac{(a_\Omega+\mathscr{U}-1)^2}{2a_\Omega}=a_{k\Omega}\,a_\Omega\,a_k\{[(1-la_\Omega)\,e_b+la_k\,a_\Omega\,a_k]\,J_1 \\ +i\mathscr{W}a_k\,a_\Omega\,e_\perp\,J_1'\}\,e^{iS}+\text{c.c.}+\mathscr{L} \tag{3.6}$$

is a second integral of the motion to order $\lambda^2$. If the high frequency electric field has no parallel components then the integral (3.5) is exact (there are no $\lambda$ terms involved).

Before using the results (3.5) and (3.6), we reduce ($3.3a$–$d$) to a simpler system, which we solve exactly in §4. In the following we shall set $l = 2$ in order to work on a specific example. Other values of $l$ are also of interest, but in the case of a single-component plasma $l = 2$ is the first cyclotron resonance used to heat the particles (Adam 1987).

A first simplification of the equations of motion is obtained by choosing the amplitudes of the fields to belong to the set of real functions:

$$e_b,\quad e_k,\quad e_\perp\quad \text{are real functions}$$

(this is not the choice made in Palmadesso (1973), where $e_b$ and $e_k$ are purely imaginary functions). A second simplification is needed in order to expand the Bessel functions in power series of their argument (in the following we shall keep only the first contribution of the expansions):

$$\mathscr{W}\ll 1.$$

This condition is usually satisfied for electrons in a tokamak magnetic field. We therefore have the inequality

$$k_\perp \ll \frac{\Omega}{v_\perp}. \tag{3.7}$$

Inspecting the equations of motion (3.3$a$–$d$) reveals (using 3.8) with the definitions of $a_{k\Omega}$ and $a_k$ from (3.2) that the coefficients of the parallel component of the electric field are one order less than the coefficients of the perpendicular components of the electric field. Therefore the parallel component of the electric field does not appear in the leading terms of the equations of motion. For convenience, we also choose

$$e_k = 0. \tag{3.8}$$

With the above simplifications and the use of (3.4), we can write the equations of motion for the parallel velocity and the phase in a very simple form:

$$\left.\begin{aligned} \dot{\mathscr{U}} &= -\tfrac{1}{2}a_{k\Omega}(\tfrac{1}{2}a_k a_\Omega)^2\left[\mathscr{K}' - \frac{4}{(a_k a_\Omega)^2}(\mathscr{U}^2 - 2\mathscr{U})\right]e_\perp \sin S, \\ \dot{S} &= \frac{1}{a_\Omega}(\mathscr{U} - 1 + a_\Omega) - (1-\mathscr{U})\, a_{k\Omega}\, e_\perp \cos S. \end{aligned}\right\} \tag{3.9}$$

A more convenient parallel-velocity variable is obtained by a shift of the origin and division by $a_\Omega$:

$$u = \frac{1-\mathscr{U}}{a_\Omega}. \tag{3.10}$$

Introducing also

$$a = a_\Omega a_{k\Omega} e_\perp, \quad \mathscr{K} = a_\Omega^{-2}[1 + (\tfrac{1}{2}a_\Omega a_k)^2 \mathscr{K}'], \tag{3.11}$$

we obtain

$$\left.\begin{aligned} \dot{u} &= \tfrac{1}{2}a(\mathscr{K} - u^2)\sin S, \\ \dot{S} &= (1-u) - u\, a \cos S. \end{aligned}\right\} \tag{3.12}$$

We now proceed to a discussion of the type of solutions expected when varying the two parameters $a$ and $\mathscr{K}$. The system of equations (3.12) can be written in Hamiltonian form by considering the parallel velocity and the phase as two canonically conjugate phase-space co-ordinates. The Hamiltonian is constructed as the solution of

$$\dot{\mathscr{H}} = \frac{\partial \mathscr{H}}{\partial u}\dot{u} + \frac{\partial \mathscr{H}}{\partial S}\dot{S}. \tag{3.13}$$

Trying a function $\mathscr{H}$ of the form

$$\mathscr{H} = A(u)\cos S + B(u) \tag{3.14}$$

(we expect (3.13) to be the second integral of the motion (3.4)) and using (3.12) and (3.14) in (3.13), we get

$$\mathscr{H} = \frac{a}{2}\left[(u^2 - \mathscr{K})\cos S + \frac{u^2}{a} - 2\frac{u}{a}\right]. \tag{3.15}$$

We note that, compared with the usual pendulum Hamiltonian (Chirikov 1979), this Hamiltonian contains a new non-linearity through the term $u^2 \cos S$. This term is responsible for a new set of fixed points and therefore for the resonance with multiple-trapping regions in phase space.

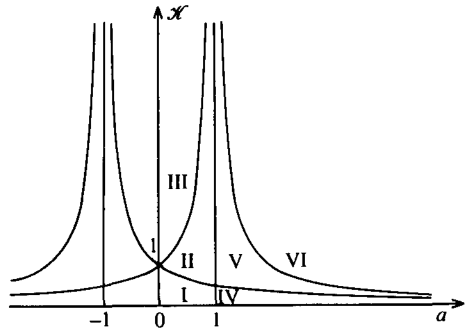


FIGURE 1. Picture of parameter space $\{a, \mathscr{K}\}$. The various regions I–VI are defined according to the number of fixed points present in phase space: in I we have $P_1$ hyperbolic and $P_2$ elliptic; in II we have $P_1$, $P_2$ elliptic and $P_3$, $P_4$ hyperbolic; in III we have $P_1$ elliptic and $P_2$ hyperbolic; in IV we have $P_1$, $P_2$ hyperbolic; in V we have $P_1$ elliptic and $P_2$, $P_3$, $P_4$ hyperbolic; in VI we have $P_1$, $P_2$ elliptic and $P_3$, $P_4$, $P_5$, $P_6$ hyperbolic.

With this Hamiltonian, we are ready to find the analytic solutions of (3.12). The method followed for the integration of the equations of motion is a classical one and can be found in Abramowitz & Stegun (1970). Before introducing the analysis of phase space as a function of the position in parameter space $\{a, \mathscr{K}\}$, we note that (3.12) is invariant under the transformation

$$a \rightarrow -a, \quad S \rightarrow S+\pi, \tag{3.16}$$

and we therefore consider only the positive values of the parameter $a$. In order to give a complete description of phase space using the analytic solutions, we first determine the number of fixed points and localize the stable (unstable) branches of the separatrix passing through the hyperbolic fixed points.

Solving (3.12) with $\dot{u} = 0$ and $\dot{S} = 0$ and substituting the result into (3.15), we get the following six possible fixed points:

$$P_1 = \left\{S_1 = 0, \quad u_1 = \frac{1}{1+a}, \quad \mathscr{H}(P_1) = \frac{1}{2}\left(\frac{-1}{1+a} - a\mathscr{K}\right)\right\}, \tag{3.17a}$$

$$P_2 = \left\{S_2 = \pi, \quad u_2 = \frac{1}{1-a}, \quad \mathscr{H}(P_2) = \frac{1}{2}\left(\frac{-1}{1-a} + a\mathscr{K}\right)\right\}, \tag{3.17b}$$

$$P_{3,4} = \left\{S_{3,4} = \mp \operatorname{arc\,cos}\left(\frac{1-\mathscr{K}^{\frac{1}{2}}}{a\mathscr{K}^{\frac{1}{2}}}\right), \quad u_{3,4} = \mathscr{K}^{\frac{1}{2}}, \quad \mathscr{H}(P_{3,4}) = \tfrac{1}{2}\mathscr{K} - \mathscr{K}^{\frac{1}{2}}\right\}, \tag{3.17c}$$

$$P_{5,6} = \left\{S_{5,6} = \mp \operatorname{arc\,cos}\left(\frac{-1-\mathscr{K}^{\frac{1}{2}}}{a\mathscr{K}^{\frac{1}{2}}}\right), \quad u_{5,6} = -\mathscr{K}^{\frac{1}{2}}, \quad \mathscr{H}(P_{5,6}) = \tfrac{1}{2}\mathscr{K} + \mathscr{K}^{\frac{1}{2}}\right\}. \tag{3.17d}$$

As is obvious, the existence of the fixed points depends on the values of the two parameters $a$ and $\mathscr{K}$: the various situations are shown in figure 1. Introducing the two quantities

$$\alpha = \epsilon\frac{a}{2}\left(\mathscr{K}^{\frac{1}{2}} - \frac{1}{1+\epsilon a}\right)\left(\mathscr{K}^{\frac{1}{2}} + \frac{1}{1+\epsilon a}\right),$$

$$\beta = -\tfrac{1}{2}a(1+\epsilon a),$$

where $\epsilon = 1$ for $P_1$ and $\epsilon = -1$ for $P_2$.

If $\alpha$ and $\beta$ have the same sign then the fixed points $P_1$ and $P_2$ are hyperbolic; otherwise they are elliptic. The fixed points $P_3$, $P_4$ and $P_5$, $P_6$ are always hyperbolic. Therefore, by varying the parameters $a$ and $\mathscr{K}$, one can have from two to six fixed points in phase space. This situation is summarized in figure 1. The motion of the particle in such a resonance structure has not received much attention until now. Chirikov (1979) considered the simpler case of only two fixed points and therefore only one trapping region; because of their interest in the stochasticity due to the overlapping of the separatrix of two 'isolated' resonances, Lichtenberg & Lieberman (1983) always reduced their Hamiltonian to the case of the forced pendulum Hamiltonian, which is not the Hamiltonian we are considering here.

In §4 we show that the full set of elliptic functions is necessary to give an analytical description of particle motion in a resonance with multiple-trapping regions.

## 4. Solutions of the equations of motion with an isolated resonance

The set of two resonant equations (3.12) of §3 is integrable because we already know one integral of the motion. The derivation of an explicit expression for the solution of the equations will be presented in this section. The method we use is a classical one and is extracted from Abramovitz & Stegun (1970):

(i) we solve for the time as a function of the phase variable $S$ by inverting the evolution equation for $S$ and replacing $u$ as a function of $S$ through the Hamiltonian $H$;

(ii) a second-degree polynomial in $\cos S$ is replaced by a fourth-order polynomial in a new variable;

(iii) a simple manipulation allows us to write this polynomial as the product of two second-degree polynomials;

(iv) a new change of variables is necessary to rewrite the two polynomials in the form of two perfect squares;

(v) The final form is identified with the canonical integral form of the elliptic functions.

We now carry out this procedure.

(i) We first write $u$ as a function of $S$ through the Hamiltonian $H$ given in (3.15);

$$u = \frac{1 \mp [1+(a\cos S+1)(2\mathscr{H}+\mathscr{K}a\cos S)]^{\frac{1}{2}}}{a\cos S+1}. \tag{4.1}$$

We now invert the evolution equation of the phase variable $S$ (see (3.12)):

$$dt = \frac{-dS}{\mp(A+B\cos S+C\cos^2 S)^{\frac{1}{2}}}, \tag{4.2}$$

where $\quad A = 1+2\mathscr{H}, \quad B = a(\mathscr{K}+2\mathscr{H}), \quad C = \mathscr{K}a^2.$

(ii) The second-order polynomial in $\cos S$ in (4.2) is written as a fourth-order polynomial in $X$ (cf. Gradshteyn & Ryzhik 1965), where

$$X = \tan\tfrac{1}{2}S. \tag{4.3}$$

We then get a new expression for (4.2):

$$dt = -\frac{2\,dX}{\mp|\alpha|^{\frac{1}{2}}(\epsilon X^4+2bX^2+c)^{\frac{1}{2}}}, \tag{4.4}$$

where

$$\alpha = A-B+C, \quad b=\frac{A-C}{\alpha}, \quad c=\frac{A+B+C}{\alpha}$$

and $\epsilon = \operatorname{sgn}\alpha$.

(iii) The argument of the square root is written as the product of two second-order polynomials in $X$:

$$X^4+2bX^2+c = (X^2+\rho_1 X+\xi_1)(X^2+\rho_2 X+\xi_2) \tag{4.5}$$

Two cases must be studied separately according to the sign of $b^2-c$.

*First case*: if $b^2-c > 0$, we have

$$\left.\begin{aligned} X^4+2bX^2+c &= (X^2+\xi_1)(X^2+\xi_2),\\ \text{with}\quad \xi_1 = b+(b^2-c)^{\frac{1}{2}}, &\quad \xi_2 = b-(b^2-c)^{\frac{1}{2}},\end{aligned}\right\} \tag{4.6}$$

and we directly have the canonical form for the integral (4.2);

$$dt = -\frac{2dX}{\mp a\alpha^{\frac{1}{2}}[(X^2+\xi_1)(X^2+\xi_2)]^{\frac{1}{2}}}. \tag{4.7}$$

This representation is valid for all values of $b$ if $c$ is negative, and for $b > c^{\frac{1}{2}}$ and $b > -c^{\frac{1}{2}}$ if $c$ is positive.

*Second case*: the representation of the integral for positive $c$ and $-c^{\frac{1}{2}} < b < c^{\frac{1}{2}}$, is obtained by using another decomposition of the fourth-order polynomial:

$$X^4+2bX^2+c = (X^2+\rho X+\xi)(X^2+\rho X+\xi) \equiv \mathscr{P}\mathscr{Q}, \tag{4.8}$$

with $$\xi = c^{\frac{1}{2}}, \quad \rho = [2(c^{\frac{1}{2}}-b)]^{\frac{1}{2}}.$$

The product on the right-hand side of (4.8) is still not in the form to be used in (4.2). Following Abramowitz & Stegun (1970), we construct a linear combination of $\mathscr{P}$ and $\mathscr{Q}$ giving a perfect square in $X$:

$$\mathscr{P}-\lambda_{1,2}\mathscr{Q} = (1-\lambda_{1,2})X^2+\rho(1+\lambda_{1,2})X+\delta(1-\lambda_{1,2}) \equiv (1-\lambda_{1,2})(X+\delta_{1,2})^2. \tag{4.9}$$

Solving this equation for $\lambda_{1,2}$ and $\delta_{1,2}$, we find

$$\left.\begin{aligned} \delta_1 = \xi^{\frac{1}{2}}, \quad \lambda_1 &= \frac{2\xi^{\frac{1}{2}}-\rho}{2\xi^{\frac{1}{2}}+\rho},\\ \delta_2 = -\xi^{\frac{1}{2}}, \quad \lambda_2 &= \frac{2\xi^{\frac{1}{2}}+\rho}{2\xi^{\frac{1}{2}}-\rho}.\end{aligned}\right\} \tag{4.10}$$

The two solutions are necessary to obtain the values of $\mathscr{P}$ and $\mathscr{Q}$ separately:

$$\left.\begin{aligned} \mathscr{P} &= \lambda_2\frac{1-\lambda_1}{\lambda_2-\lambda_1}(X+\delta_1)^2-\lambda_1\frac{1-\lambda_2}{\lambda_2-\lambda_1}(X+\delta_2)^2,\\ \mathscr{Q} &= \frac{1-\lambda_1}{\lambda_2-\lambda_1}(X+\delta_1)^2-\frac{1-\lambda_2}{\lambda_2-\lambda_1}(X+\delta_2)^2.\end{aligned}\right\} \tag{4.11}$$

A change of variable

$$Y = \frac{X+\delta_1}{X+\delta_2} \tag{4.12}$$

and the use of (4.11) in (4.4) give the canonical form

$$dt = -\frac{2(\lambda_2-\lambda_1)\,dY}{\mp a(\lambda_2\,\alpha)^{\frac{1}{2}}(\delta_1+\delta_2)\,(\lambda_1-1)\,[(Y^2+a_2^2)\,(Y^2+a_2^2)]^{\frac{1}{2}}}, \tag{4.13}$$

where

$$a_1 = \left[\frac{\lambda_1(\lambda_2-1)}{\lambda_2(1-\lambda_1)}\right]^{\frac{1}{2}}, \quad a_2 = \left(\frac{\lambda_2-1}{1-\lambda_1}\right)^{\frac{1}{2}}.$$

(iv) The integrals (4.7) or (4.13) are known to be solvable in terms of the 12 Jacobian elliptic functions. Before giving the explicit solution, we discuss some properties of the solutions in relation to the two signs appearing in the integrals. If $f(t)$ represents one of the 12 Jacobian elliptic functions, then the solution of (4.7) or (4.13) will be given by

$$X_{\mp} = f(t_{\mp}), \tag{4.14}$$

according to the sign of the argument in the integral. Going back to the original set of co-ordinates $u$ and $S$, we also have

$$\left.\begin{aligned} S_{\mp} &= 2\tan^{-1}X_{\mp}, \\ u_{\mp} &= \frac{1\mp[1+(1+a\cos S_{\mp})\,(2\mathscr{H}+\mathscr{K}a\cos S_{\mp})]^{\frac{1}{2}}}{1+a\cos S_{\mp}}. \end{aligned}\right\} \tag{4.15}$$

It can be shown (Weyssow 1990) that the two solutions + and − are sometimes necessary to completely define one trajectory in phase space. This is mainly due to the periodicity and the evenness or oddness of the elliptic functions (Abramowitz & Stegun 1970).

We now list the solutions of the integrals (4.7) and (4.13).

*The solutions of* (4.7)

($a$) $\alpha > 0$

(i) $\delta_1 > \delta_2 > 0$ (for $\delta_2 > d_1 > 0$ we simply permute $\delta_1 \leftrightarrow \delta_2$):

$$X_{\mp} = \delta_2^{\frac{1}{2}}\,\mathrm{sc}\left(t\,\middle|\,\frac{\delta_1-\delta_2}{\delta_1}\right).$$

One sign gives a trajectory, the other sign gives the same trajectory but travelled backwards. The periodicity is $2K$ where $K$ is the complete elliptical integral of the first kind.

(ii) $\delta_1 > 0 > \delta_2$ (for $\delta_2 > 0 > \delta_1$ we simply permute $\delta_1 \leftrightarrow \delta_2$):

$$X_{\mp} = (-\delta_2)^{\frac{1}{2}}\,\mathrm{nc}\left(t\,\middle|\,\frac{\delta_1}{\delta_1-\delta_2}\right).$$

The trajectory is composed of solution '+' for $-2K \leqslant t \leqslant 0$ followed by solution '−' for $0 \leqslant t \leqslant 2K$ (or the contrary).

(iii) $\delta_1 < \delta_2 < 0$ (for $\delta_2 < \delta_1 < 0$ simply permute $\delta_1 \leftrightarrow \delta_2$). Two solutions of the integral are acceptable:

$$X_{1\mp} = (-\delta_1)^{\frac{1}{2}} \operatorname{sc}\left(t \left| \frac{\delta_2}{\delta_1}\right.\right),$$

$$X_{1\mp} = (-\delta_2)^{\frac{1}{2}} \operatorname{sc}\left(t \left| \frac{\delta_2}{\delta_1}\right.\right).$$

Here we have two trajectories in phase space. The first is given by $X_1$ with $-$ sign for $-2K \leqslant t \leqslant 0$ then $+$ sign for $0 \leqslant t \leqslant 2K$ (or the contrary). The second trajectory is given by $X_2$ with $-$ sign for $-K \leqslant t \leqslant K$ (or $+$ sign for reversed motion)

(*b*) $\alpha < 0$

(i) $\delta_1 < 0 < \delta_2$ (for $\delta_2 < 0 < \delta_1$ we permute $\delta_1 \leftrightarrow \delta_2$):

$$X_{\mp} = \operatorname{sd}\left(t \left| \frac{\delta_1}{\delta_1 - \delta_2}\right.\right).$$

In this case we have one trajectory defined by one sign for $-2K \leqslant t \leqslant 0$ followed by the other sign for $0 \leqslant t \leqslant 2K$.

(ii) $\delta_1 < \delta_2 < 0$ (for $\delta_2 < \delta_1 < 0$ we permute $\delta_1 \leftrightarrow \delta_2$):

$$X_{\mp} = (-\delta_2)^{\frac{1}{2}} \operatorname{sc}\left(t \left| \frac{\delta_1 - \delta_2}{\delta_1}\right.\right).$$

Again, the two signs are needed to define one trajectory. The periodicity is $2K$.

*The solution of* (4.13)

For $\alpha\lambda_2 > 0$:

$$Y_{\mp} = a_1 \operatorname{sc}\left(t \left| \frac{a_1^2 - a_2^2}{a_1^2}\right.\right).$$

One sign of the solution gives a trajectory, while the other sign gives the reversed motion. The periodicity is $2K$.

The solutions presented above will now be used to obtain a series of plots of phase space in which the locations of the fixed points will be marked by a cross. In §3 it was shown that the parameter space $\{a, \mathscr{K}\}$ can be divided into six regions (see also figure 1) according to the number of fixed points generated in phase space.

We first plot in figures 2–4 phase space for regions IV–VI of parameter space. The three figures clearly show a resonance extending over the whole of phase space. This contradicts the notion of isolated resonances. Looking more carefully at this question, we see that the condition for isolated resonances is simply that the parameter $a$ must be close to zero, leaving us with the three remaining regions of figure 1. Phase spaces for these three regions and for small $a$ are plotted in figures 5–7. Two remarks can be made about these last three plots. First, since the plots are obtained from explicit solutions of the equations of motion, the Jacobian elliptic functions are sufficient for representing a simple case of resonance with a small number of trapping regions. In the present case

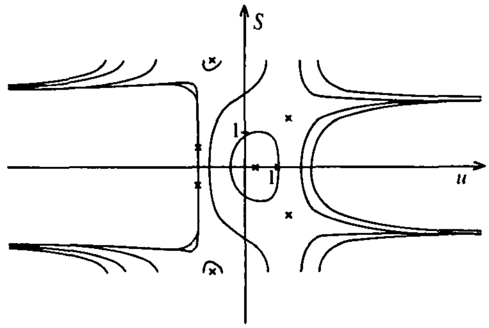


FIGURE 2. Plot of trajectories in phase space for the parameters $a = 2$ and $\mathscr{K} = 2$ from region VI of figure 1.

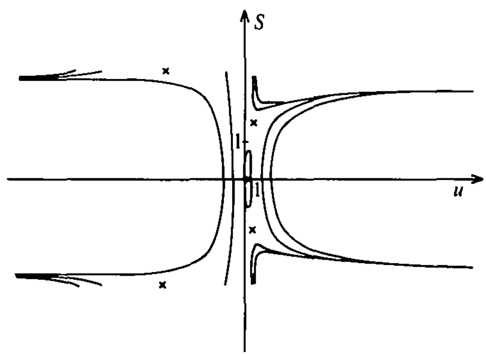


FIGURE 3. Plot of trajectories in phase space for the parameters $a = 1{\cdot}1$ and $\mathscr{K} = 1$ from region V of figure 1,

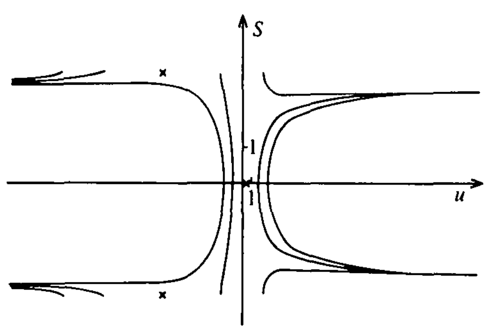


FIGURE 4. Plot of trajectories in phase space for the parameters $a = 1{\cdot}1$ and $\mathscr{K} = 0{\cdot}1$ from region IV of figure 1.

figure 6 represents a phase space with two trapping regions. The second remark concerns the 'dynamics' of phase space as the parameters change. From figures 5–7 we can follow the process of interchange of stability of two fixed points: an elliptic point becomes hyperbolic, and *vice versa*. This process is possible owing to the appearance of new fixed points represented in figure 6. If the parameters of the problem of resonance are time-dependent, as in the case of wave damping, then such an effect must arise and modify some properties of the plasma.

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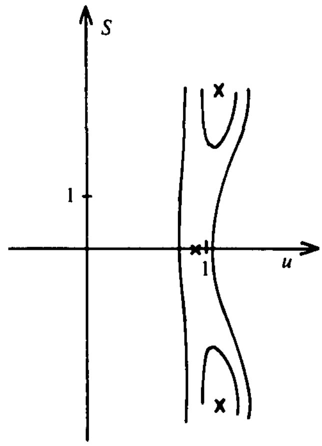


FIGURE 5. Plot of trajectories in phase space for the parameters $a = 0{\cdot}1$ and $\mathscr{K} = 0{\cdot}8$ from region I of figure 1.

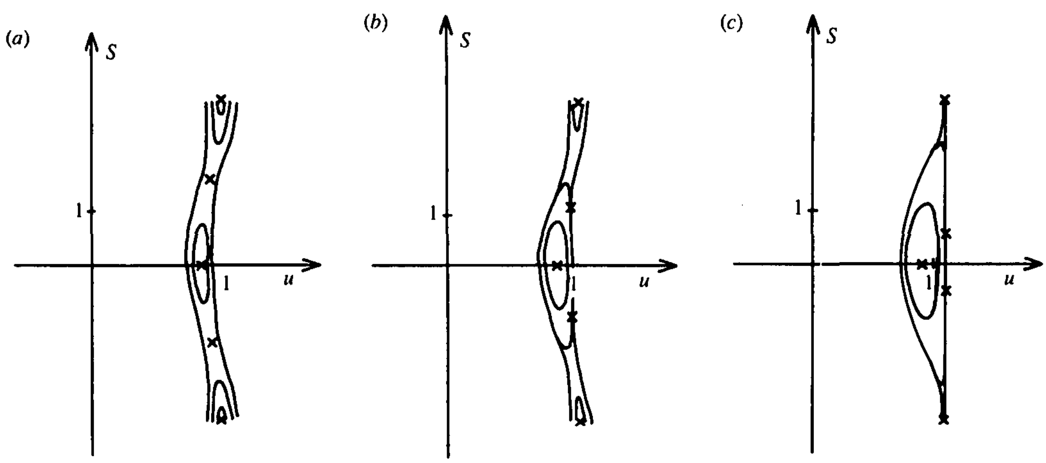


FIGURE 6. Plot of trajectories in phase space for the parameters $a = 0{\cdot}1$ and $\mathscr{K} = 1$: ($a$), $1{\cdot}1$; ($b$), $1{\cdot}2$; ($c$) from region II of figure 1.

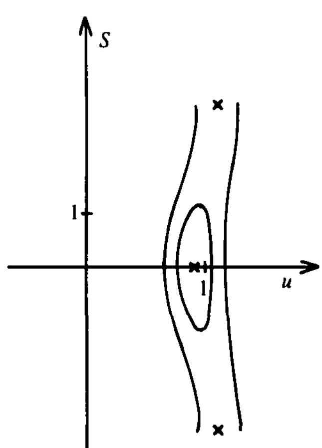


FIGURE 7. Plot of trajectories in phase space for the parameters $a = 0{\cdot}1$ and $\mathscr{K} = 1{\cdot}25$ from region III of figure 1.

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## 5. Conclusions

The Hamiltonian averaging transformation applied to the motion of a charged particle in a strong magnetic field perturbed by an electromagnetic wave yields an averaged dynamics in which we can easily recognize the guiding-centre terms, the ponderomotive terms and a series of resonant terms.

The Hamiltonian guiding-centre transformation used in this paper yields only the guiding-centre drift terms and resonant terms linear in the wave amplitude. Higher-order resonances can only be obtained by considering the more general oscillation-centre transformation. We have also studied a particularly simple situation of an isolated resonance, choosing a constant confining magnetic field and a perpendicular wave vector sufficiently small in order to satisfy the condition $v_\perp/\Omega \ll 1$. In this case two integrals of motion are known and the problem reduces to the study of a one-dimensional Hamiltonian system. The solutions of the equations of motion can be written in terms of the 12 Jacobian elliptic functions. Phase space presents a complex resonant structure with a few trapping regions. Applying the condition of isolated resonances, we find the usual resonance with one trapping region but also, for particular values of the parameters, a resonance with two trapping regions. In view of the plots of phase space, this last situation is to be understood as a necessary step in a process of exchange of stability between two fixed points.

As we have an analytical description of the trajectories, nonlinear wave damping and related processes can be analysed in more complicated situations than is usually done.

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