RESEARCH ARTICLE | FEBRUARY 01 2000

Relativistic motion of a charged particle in a superposition of circularly polarized plane electromagnetic waves and a uniform magnetic field

Bao-Liang Qian

Phys. Plasmas 7, 537–543 (2000)
https://doi.org/10.1063/1.873839

Relativistic motion of a charged particle in a superposition of circularly polarized plane electromagnetic waves and a uniform magnetic field

Bao-Liang Qian
Department of Applied Physics, National University of Defense Technology, Changsha 410073, Hunan, People's Republic of China

(Received 22 July 1999; accepted 20 October 1999)

The relativistic motion of a charged particle in a superposition of circularly polarized plane electromagnetic waves and a uniform magnetic field is studied by deriving an exact solution to the Lorentz force equation of the charged particle. All of the circularly polarized plane electromagnetic waves propagate parallel to the uniform magnetic field. The explicit expressions of the charged particle position, velocity, and energy are obtained for arbitrary initial conditions, and the behavior of the particle motion is studied both analytically and numerically. It is found that for certain initial conditions, the particle gains energy from the waves and the energy gain could reach a maximum value during the time evolution of the charged particle motion in the nonresonance case. In addition, the particle could be accelerated to a much higher energy at the cyclotron resonance, and, in comparison with the situation of a single wave, the use of the superposition wave that consists of a group of circularly polarized plane electromagnetic waves with different frequencies increases the chance of the occurrence of cyclotron resonance for a charged particle, making it easier for the particle to be accelerated. It has also been observed that the interaction of the charged particle with the superposition of electromagnetic waves can be improved significantly at cyclotron resonance when the frequency and phase differences of the waves remain so small that the phases of the waves are in the same quadrant. The results of the present paper are of interest to particle acceleration and heating applications, as well as to basic plasma processes.
[S1070-664X(00)01702-X]

I. INTRODUCTION

Electromagnetic waves have been employed for accelerating or heating particles in many physical systems, such as fusion devices, particle accelerators, and space plasmas. Therefore, the interaction of the charged particles with the electromagnetic waves is of great importance. In order to understand the fundamental physics of the interaction, Roberts and Buchsbaum[1] have studied the relativistic motion of a charged particle in the presence of a uniform magnetic field and a circularly polarized electromagnetic wave by deriving an exact solution for the particle energy as a function of time. Later, Villalon and Burke[2] studied this problem in a different situation using a numerical method, and obtained some valuable results. Several other important works focusing on the solutions of the Lorentz force equation of the charged particle in a variety of situations can be found in Refs. 3–6.

In the area of electron–cyclotron resonance heating (ECRH), theoretical and experimental results[7–10] reveal that multifrequency ECRH causes an increase in energy density over the single-frequency level for the same total input power. This increase is attributed to the overlapping of cyclotron resonances of different waves. These results encourage the consideration of a charged particle interacting with the electromagnetic waves that possess different frequencies.[11] The purpose of this paper is to study the interaction of a charged particle with a superposition of circularly polarized plane electromagnetic waves and a uniform magnetic field by deriving an exact solution to the relativistic Lorentz force equation of the charged particle. The superposition wave, consisting of a number of electromagnetic waves with different frequencies, propagates parallel to the uniform magnetic field and increases the chance of the occurrence of cyclotron resonance for the charged particle. The explicit expressions of particle position, velocity, and energy are obtained for arbitrary initial conditions. The behavior of the particle motion is studied both analytically and numerically.

The organization of this paper is as follows. In Sec. II the exact solution of the relativistic Lorentz force equation of a charged particle driven by a superposition of circularly polarized plane electromagnetic waves and a uniform magnetic field is derived. In Sec. III the behavior of the charged particle at cyclotron resonance is analyzed. In Sec. IV the numerical results are present, and conclusions are given in Sec. V.

II. SOLUTION OF THE RELATIVISTIC LORENTZ FORCE EQUATION

Consider a particle with charge q, rest mass m, position $\vec{r}=(x,y,z)$, and velocity $\vec{\nu}=(\nu_x,\nu_y,\nu_z)$, moving in the presence of a uniform magnetic field

$$\vec{B}_0=B_0\hat{z} \tag{1}$$

and a superposition wave consisting of a group of circularly polarized plane electromagnetic waves with different frequencies. The electric and magnetic fields of the superposition wave are expressed as

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t} \tag{2}$$

and

$$\vec{B}=\nabla\times\vec{A}, \tag{3}$$

where

$$\vec{A}=\sum_{i=1}^{N}\hat{A}_i(\hat{x}\cos\phi_i+\hat{y}\sin\phi_i), \tag{4}$$

with N being the number of waves in the group and

$$\phi_i=\omega_i\left(\frac{z}{c}-t\right)+\theta_{i0}. \tag{5}$$

Here, ϕ_i are the phases of the waves, θ_{i0} are the constant phases, ω_i indicate the angular frequencies, $\hat{A}_i$ represent the wave amplitudes, and c is the speed of light in vacuum.

The relativistic Lorentz force equation of the charged particle is given by

$$\frac{d\vec{p}}{dt}=q[\vec{E}+\vec{v}\times(\vec{B}_0+\vec{B})], \tag{6}$$

where $\vec{p}=m\gamma\vec{v}$ is the momentum of the particle, $\gamma=(1-\beta_\perp^2-\beta_z^2)^{-1/2}$ denotes the relativistic factor, $\beta_\perp=(v_x^2+v_y^2)^{1/2}/c$, and $\beta_z=v_z/c$. The components of the Lorentz force equation can be expressed as

$$\frac{d}{dt}(p_x+qA_x)=qv_yB_0, \tag{7}$$

$$\frac{d}{dt}(p_y+qA_y)=-qv_xB_0, \tag{8}$$

and

$$\frac{dp_z}{dt}=qv_x\frac{\partial A_x}{\partial z}+qv_y\frac{\partial A_y}{\partial z}, \tag{9}$$

where A_x and A_y are the $\hat{x}$ and $\hat{y}$ components of $\vec{A}$. The expression for the rate of change of energy of the charged particle is given by

$$\frac{d}{dt}(mc^2\gamma)=-qv_x\frac{\partial A_x}{\partial t}-qv_y\frac{\partial A_y}{\partial t}. \tag{10}$$

Equations (9) and (10) can be combined to yield

$$\frac{d}{dt}(mc\gamma-p_z)=0, \tag{11}$$

and the solution of (11) is written as

$$mc\gamma-p_z=mc\gamma_0-p_{z0}, \tag{12}$$

where γ_0 is the initial relativistic factor and $p_{z0}=m\gamma_0v_{z0}$, with v_{z0} being the initial axial velocity of the particle. In addition, integrating (7) and (8), and using the initial conditions of $p_x=p_{\perp 0}\cos\theta_0=m\gamma_0c\beta_{\perp 0}\cos\theta_0$, $p_y=p_{\perp 0}\sin\theta_0=m\gamma_0c\beta_{\perp 0}\sin\theta_0$, and $\vec{r}=(x_0,y_0,z_0)$ at $t=0$, one can get

$$p_x-p_{\perp 0}\cos\theta_0+qA_x-qA_{x0}=q(y-y_0)B_0 \tag{13}$$

and

$$p_y-p_{\perp 0}\sin\theta_0+qA_y-qA_{y0}=-q(x-x_0)B_0, \tag{14}$$

where

$$A_{x0}=\sum_{i=1}^{N}\hat{A}_i\cos\phi_{i0}, \tag{15}$$

and

$$A_{y0}=\sum_{i=1}^{N}\hat{A}_i\sin\phi_{i0}. \tag{16}$$

Here, $\phi_{i0}=\omega_iz_0/c+\theta_{i0}$ are the initial phases of the waves.

Defining an important variable of

$$\tau=\frac{z}{c}-t, \tag{17}$$

and using (12), one can obtain

$$\frac{dx}{d\tau}=\frac{v_x}{d\tau/dt}=\frac{cp_x}{p_z-m\gamma c}=-\frac{p_x}{m\gamma_0(1-\beta_{z0})}, \tag{18}$$

$$\frac{dy}{d\tau}=\frac{v_y}{d\tau/dt}=\frac{cp_y}{p_z-m\gamma c}=-\frac{p_y}{m\gamma_0(1-\beta_{z0})}, \tag{19}$$

and

$$\frac{dz}{d\tau}=\frac{v_z}{d\tau/dt}=\frac{cp_z}{p_z-m\gamma c}=\frac{\gamma_0(1-\beta_{z0})-\gamma}{\gamma_0(1-\beta_{z0})}c, \tag{20}$$

where $\beta_{z0}=v_{z0}/c$. In addition, substituting (12) into $\gamma^2=1+p^2/m^2c^2$ leads to

$$\gamma=\frac{1+\gamma_0^2(1-\beta_{z0})^2+p_\perp^2/(m^2c^2)}{2\gamma_0(1-\beta_{z0})}. \tag{21}$$

Employing (13) and (14), and noting the initial conditions mentioned above, one can get the solution of (18), (19), and (20), which is written as

$$\begin{aligned}x-x_0=&\sum_{i=1}^{N}\frac{q\hat{A}_i[\omega_i\sin(\omega_0\Delta\tau+\phi_{i0})-\omega_0\sin(\omega_i\Delta\tau+\phi_{i0})]}{m\gamma_0\Omega_0(\omega_0-\omega_i)}\\&+\sum_{i=1}^{N}\frac{q\hat{A}_i}{m\gamma_0\Omega_0}\sin\phi_{i0}-\frac{p_{\perp 0}}{m\gamma_0\Omega_0}\\&\times[\sin(\omega_0\Delta\tau+\theta_0)-\sin\theta_0],\end{aligned} \tag{22}$$

$$\begin{aligned}y-y_0=&\sum_{i=1}^{N}\frac{q\hat{A}_i[\omega_0\cos(\omega_i\Delta\tau+\phi_{i0})-\omega_i\cos(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma_0\Omega_0(\omega_0-\omega_i)}\\&-\sum_{i=1}^{N}\frac{q\hat{A}_i}{m\gamma_0\Omega_0}\cos\phi_{i0}+\frac{p_{\perp 0}}{m\gamma_0\Omega_0}\\&\times[\cos(\omega_0\Delta\tau+\theta_0)-\cos\theta_0],\end{aligned} \tag{23}$$

and

$$z-z_0=-\frac{z_1+z_2}{2\gamma_0^2(1-\beta_{z0})^2}, \tag{24}$$

where

$$z_1=[1-\gamma_0^2(1-\beta_{z0})^2]\Delta\tau c+\frac{p_{\perp 0}^2\Delta\tau}{m^2c}+\sum_{i=1}^{N}\frac{2q^2\hat{A}_i^2\omega_i^2}{m^2c}\frac{\Delta\tau(\omega_0-\omega_i)-\sin[(\omega_0-\omega_i)\Delta\tau]}{(\omega_0-\omega_i)^3}$$
$$+\sum_{i=1}^{N}\frac{2q\hat{A}_i\omega_i p_{\perp 0}}{m^2c(\omega_0-\omega_i)}\left(\frac{\sin[(\omega_i-\omega_0)\Delta\tau+\phi_{i0}-\theta_0]-\sin(\phi_{i0}-\theta_0)}{\omega_i-\omega_0}-\Delta\tau\cos(\phi_{i0}-\theta_0)\right) \tag{25}$$

and

$$z_2=\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{q^2\hat{A}_i\hat{A}_j\omega_i\omega_j}{m^2c(\omega_0-\omega_i)(\omega_0-\omega_j)}$$
$$\times\left(\frac{\sin[(\omega_i-\omega_j)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_i-\omega_j}-\frac{\sin[(\omega_0-\omega_j)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_0-\omega_j}\right.$$
$$\left.+\Delta\tau\cos(\phi_{i0}-\phi_{j0})-\frac{\sin[(\omega_i-\omega_0)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_i-\omega_0}\right), \tag{26}$$

with $\Omega_0=qB_0/(m\gamma_0)$, $\omega_0=\Omega_0/(1-\beta_{z0})$, and $\Delta\tau=\tau-z_0/c=\tau-\tau_0$.

The components of the particle velocity are given by

$$\nu_x=\sum_{i=1}^{N}\frac{q\hat{A}_i\omega_i[\cos(\omega_i\Delta\tau+\phi_{i0})-\cos(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma(\omega_0-\omega_i)}+\frac{p_{\perp 0}\cos(\omega_0\Delta\tau+\theta_0)}{m\gamma}, \tag{27}$$

$$\nu_y=\sum_{i=1}^{N}\frac{q\hat{A}_i\omega_i[\sin(\omega_i\Delta\tau+\phi_{i0})-\sin(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma(\omega_0-\omega_i)}+\frac{p_{\perp 0}\sin(\omega_0\Delta\tau+\theta_0)}{m\gamma}, \tag{28}$$

and

$$\nu_z=\frac{p_{z0}+mc(\gamma-\gamma_0)}{m\gamma}, \tag{29}$$

where the relativistic factor γ is given by (21) with

$$p_\perp^2=\left(\sum_{i=1}^{N}\frac{q\hat{A}_i\omega_i}{\omega_0-\omega_i}[\cos(\omega_i\Delta\tau+\phi_{i0})-\cos(\omega_0\Delta\tau+\phi_{i0})]+p_{\perp 0}\cos(\omega_0\Delta\tau+\phi_0)\right)^2$$
$$+\left(\sum_{i=1}^{N}\frac{q\hat{A}_i\omega_i}{\omega_0-\omega_i}[\sin(\omega_i\Delta\tau+\phi_{i0})-\sin(\omega_0\Delta\tau+\phi_{i0})]+p_{\perp 0}\sin(\omega_0\Delta\tau+\theta_0)\right)^2. \tag{30}$$

Using (21) and (29), one can find that

$$\beta_\perp^2=\frac{\nu_x^2+\nu_y^2}{c^2}\leqslant\frac{\gamma_0^2(1-\beta_{z0})^2}{1+\gamma_0^2(1-\beta_{z0})^2} \tag{31}$$

and

$$\nu_z-\nu_{z0}=\frac{\gamma-\gamma_0}{\gamma}(1-\beta_{z0})c. \tag{32}$$

It can be seen from (32) that the axial acceleration may occur in the case of $\gamma-\gamma_0>0$.

In addition, the normalized particle energy is defined by

$$\hat{E}=\frac{\gamma-\gamma_0}{\gamma_0}=\frac{p_\perp^2-p_{\perp 0}^2}{2m^2\gamma_0^2c^2(1-\beta_{z0})}, \tag{33}$$

and it is easy to show that, for $p_{\perp 0}=0$, $\hat{E}$ satisfies

$$0\leqslant\hat{E}<4(1+\beta_{z0})\left(\sum_{i=1}^{N}\frac{q\hat{A}_i\omega_i}{mc(\omega_0-\omega_i)}\right)^2. \tag{34}$$

The trajectory and velocity expressions of the charged particle, (22)–(24) and (27)–(29), imply that the particle oscillates in the $\hat{x}$ and $\hat{y}$ directions. It can be seen from (30), (32), (33), and (34) that, in the case of $p_{\perp 0}=0$, the particle gains energy from the electromagnetic waves at $t>0$ and can be considerably accelerated. Additionally, the inequality (34) indicates that there exists a maximum energy gain for $p_{\perp 0}=0$ when the charged particle travels in the waves and the uniform magnetic field.

It should be noticed that the exact solution given in this paper continues to hold as $N\to\infty$. In addition, the solution can also be generalized to the situation of $\vec{A}=A_x(\tau)\hat{x}+A_y(\tau)\hat{y}$, *where* $A_x(\tau)$ *and* $A_y(\tau)$ are general functions of $\tau=z/c-t$, and in this case one should physically choose appropriate expressions of $A_x(\tau)$ *and* $A_y(\tau)$ in order to solve the Lorentz force equation by using the method of Laplace transform.

III. BEHAVIOR OF THE CHARGED PARTICLE AT CYCLOTRON RESONANCE

The results are quite interesting if the charged particle is near the cyclotron resonance. Assume that the charged particle is in cyclotron resonance with one of the circularly polarized plane electromagnetic waves belonging to the superposition wave, i.e., $\omega_k=\omega_0$, where ω_k is the angular frequency of the specified wave. The solution for the cyclotron resonance can be obtained directly by taking the limit of $\omega_k \rightarrow \omega_0$ in the expressions given in Sec. II. Thus, the expressions of (22)–(26) take the form

$$x-x_0=\sum_{\substack{i=1\\ i\neq k}}^{N} \frac{q\hat{A}_i[\omega_i \sin(\omega_0\Delta\tau+\phi_{i0})-\omega_0 \sin(\omega_i\Delta\tau+\phi_{i0})]}{m\gamma_0\Omega_0(\omega_0-\omega_i)}+\frac{q\hat{A}_k}{m\gamma_0\Omega_0}[\omega_0\Delta\tau\cos(\omega_0\Delta\tau+\phi_{k0})-\sin(\omega_0\Delta\tau+\phi_{k0})]$$
$$-\frac{p_{\perp 0}}{m\gamma_0\Omega_0}[\sin(\omega_0\Delta\tau+\theta_0)-\sin\theta_0]+\sum_{i=1}^{N}\frac{q\hat{A}_i}{m\gamma_0\Omega_0}\sin\phi_{i0}, \tag{35}$$

$$y-y_0=\sum_{\substack{i=1\\ i\neq k}}^{N} \frac{q\hat{A}_i[\omega_0 \cos(\omega_i\Delta\tau+\phi_{i0})-\omega_i \cos(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma_0\Omega_0(\omega_0-\omega_i)}+\frac{q\hat{A}_k}{m\gamma_0\Omega_0}[\omega_0\Delta\tau\sin(\omega_0\Delta\tau+\phi_{k0})+\cos(\omega_0\Delta\tau+\phi_{k0})]$$
$$+\frac{p_{\perp 0}}{m\gamma_0\Omega_0}[\cos(\omega_0\Delta\tau+\theta_0)-\cos\theta_0]-\sum_{i=1}^{N}\frac{q\hat{A}_i}{m\gamma_0\Omega_0}\cos\phi_{i0}, \tag{36}$$

and

$$z-z_0=-\frac{z_1+z_2+z_3}{2\gamma_0^2(1-\beta_{z0})^2}, \tag{37}$$

with

$$z_1=[1-\gamma_0^2(1-\beta_{z0})^2]\Delta\tau c+\frac{p_{\perp 0}^2\Delta\tau}{m^2c}+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{2q^2\hat{A}_i^2\omega_i^2}{m^2c}\frac{\Delta\tau(\omega_0-\omega_i)-\sin[(\omega_0-\omega_i)\Delta\tau]}{(\omega_0-\omega_i)^3}$$
$$+\sum_{\substack{i=l\\ i\neq k}}^{N}\frac{2q\hat{A}_i\omega_i p_{\perp 0}}{m^2c(\omega_0-\omega_i)}\left(\frac{\sin[(\omega_i-\omega_0)\Delta\tau+\phi_{i0}-\theta_0]-\sin(\phi_{i0}-\theta_0)}{\omega_i-\omega_0}-\Delta\tau\cos(\phi_{i0}-\theta_0)\right), \tag{38}$$

$$z_2=\sum_{\substack{i,j=l\\ i\neq j\neq k}}^{N}\frac{q^2\hat{A}_i\hat{A}_j\omega_i\omega_j}{m^2c(\omega_0-\omega_i)(\omega_0-\omega_j)}\left(\frac{\sin[(\omega_i-\omega_j)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_i-\omega_j}\right.$$
$$-\frac{\sin[(\omega_0-\omega_j)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_0-\omega_j}+\Delta\tau\cos(\phi_{i0}-\phi_{j0})$$
$$\left.-\frac{\sin[(\omega_i-\omega_0)\Delta\tau+\phi_{i0}-\phi_{j0}]-\sin(\phi_{i0}-\phi_{j0})}{\omega_i-\omega_0}\right), \tag{39}$$

and

$$z_3=\frac{q^2\hat{A}_k^2\omega_0^2(\Delta\tau)^3}{3m^2c}+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{q^2\hat{A}_i\hat{A}_k\omega_i\omega_k(\Delta\tau)^2}{m^2c(\omega_0-\omega_i)}$$
$$\times\sin(\phi_{i0}-\phi_{k0})+\frac{q\hat{A}_k\omega_0 p_{\perp 0}(\Delta\tau)^2}{m^2c}\sin(\phi_{k0}-\theta_0). \tag{40}$$

In addition, the expressions of particle velocity, (27)–(30), become

$$\nu_x=\frac{q\hat{A}_k\omega_0\Delta\tau}{m\gamma}\sin(\omega_0\Delta\tau+\phi_{k0})$$
$$+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{q\hat{A}_i\omega_i[\cos(\omega_i\Delta\tau+\phi_{i0})-\cos(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma(\omega_0-\omega_i)}$$
$$+\frac{p_{\perp 0}}{m\gamma}\cos(\omega_0\Delta\tau+\theta_0), \tag{41}$$

$$\nu_y=-\frac{q\hat{A}_k\omega_0\Delta\tau}{m\gamma}\cos(\omega_0\Delta\tau+\phi_{k0})$$
$$+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{q\hat{A}_i\omega_i[\sin(\omega_i\Delta\tau+\phi_{i0})-\sin(\omega_0\Delta\tau+\phi_{i0})]}{m\gamma(\omega_0-\omega_i)}$$
$$+\frac{p_{\perp 0}}{m\gamma}\sin(\omega_0\Delta\tau+\theta_0), \tag{42}$$

and

$$\nu_z=\frac{p_{z0}+mc(\gamma-\gamma_0)}{m\gamma}, \tag{43}$$

where the relativistic factor is still given by (21) with

$$p_{\perp}^{2}=\Bigg(q\hat{A}_{k}\omega_{0}\Delta\tau\sin(\omega_{0}\Delta\tau+\phi_{k0})+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{q\hat{A}_{i}\omega_{i}}{\omega_{0}-\omega_{i}}\times[\cos(\omega_{i}\Delta\tau+\phi_{i0})-\cos(\omega_{0}\Delta\tau+\phi_{i0})]+p_{\perp 0}\times\cos(\omega_{0}\Delta\tau+\theta_{0})\Bigg)^{2}+\Bigg(-q\hat{A}_{k}\omega_{0}\Delta\tau\times\cos(\omega_{0}\Delta\tau+\phi_{k0})+\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{q\hat{A}_{i}\omega_{i}}{\omega_{0}-\omega_{i}}[\sin(\omega_{i}\Delta\tau+\phi_{i0})-\sin(\omega_{0}\Delta\tau+\phi_{i0})]+p_{\perp 0}\sin(\omega_{0}\Delta\tau+\theta_{0})\Bigg)^{2}. \quad (44)$$

It can be seen from (35)–(44) that the oscillation amplitudes of the particle trajectory and velocity increase with the time evolution of the particle motion, indicating the occurrence of the cyclotron resonance. In addition, (33) and (44) illustrate that the energy gain of the particle could be increased rapidly due to the particle cyclotron resonance.

IV. NUMERICAL RESULTS

In this section the numerical evaluation of the formulas derived in Secs. II and III is employed to further explore the behavior of motion of the charged particle. For the sake of convenience and simplicity, it is necessary to use dimensionless variables in the numerical calculations. In addition, the formulas given in Secs. II and III indicate that the expressions of particle position, velocity, and energy are explicit functions of $\Delta\tau$; however, one can easily obtain the relation between $\Delta\tau$ and t by using $t=(z-z_0)/c-\Delta\tau$ in the numerical calculations.

Although the number N of the circularly polarized plane electromagnetic waves contained in the superposition wave may be infinite, in practice, we must choose a finite value of N for the calculations. Typically, we use a superposition wave consisting of three circularly polarized plane electromagnetic waves with amplitudes of $\hat{A}_1$, $\hat{A}_2$, and $\hat{A}_3$, angular frequencies of ω_1, ω_2, and ω_3, and initial phases of ϕ_{10}, ϕ_{20}, and ϕ_{30}.

The results of the nonresonance case are shown in Fig. 1, where the normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ is plotted as a function of $\omega_0 t$ for $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\omega_1/\omega_0=2.0$, $\omega_2/\omega_0=3.0$, $\omega_3/\omega_0=4.0$, $\phi_{10}=0.0$, $\phi_{20}=0.5\pi$, $\phi_{30}=\pi$, $\beta_{z0}=0.8$, and $\theta_0=0.25\pi$ in the cases of $\beta_{\perp 0}=0.0$ (solid line), and $\beta_{\perp 0}=0.56$ (dashed line). One can see from Fig. 1 that the value of the normalized particle energy oscillates periodically with $\omega_0 t$ and the energy gain reaches a maximum value at certain values of $\omega_0 t$. Figure 1 also implies that in the case of $\beta_{\perp 0}=0$ the normalized particle energy $\hat{E}$ satisfies $\hat{E}\geqslant 0$ for any value of $\omega_0 t$, which is identical with (34).

The results are quite different in the case of cyclotron resonance. For simplicity, we use a variable of $\Delta r=[(x-x_0)^2+(y-y_0)^2]^{1/2}$ to describe the transverse motion of the charged particle. The numerical results of cyclotron resonance are shown in Figs. 2, 3, 4, and 5, where the angular frequency ω_3 is equal to ω_0 for each curve. The parameters employed in these figures are $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\omega_3/\omega_0=1.0$, $\phi_{10}=0.2\pi$, $\phi_{20}=0.5\pi$, $\phi_{30}=0.3\pi$, $\beta_{z0}=0.8$, $\beta_{\perp 0}=0.1$, *and* $\theta_0=0.25\pi$. The three curves in each figure correspond to the cases of $\omega_1/\omega_0=2.0$ *and* $\omega_2/\omega_0=3.0$, $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=3.0$, and $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=1.2$, respectively. Figure 2 illustrates the normalized transverse position $\Delta r\Omega_0/c$ of the charged particle versus $\omega_0 t$ for three different situations. One can see from Fig. 2 that near the cyclotron resonance the values of $\Delta r\Omega_0/c$ swingingly increase as $\omega_0 t$ becomes larger. In addition, whether ω_1 and ω_2 are near ω_0 or not considerably affect the motion of the charged particle when $\omega_3=\omega_0$. Figure 3 shows the normalized transverse velocity $\beta_\perp$ of the charged particle as a function of $\omega_0 t$, and

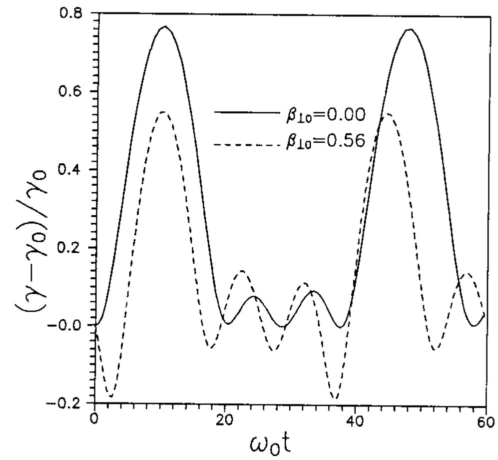

FIG. 1. The normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ as a function of $\omega_0 t$ for $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\omega_1/\omega_0=2.0$, $\omega_2/\omega_0=3.0$, $\omega_3/\omega_0=4.0$, $\phi_{10}=0.0$, $\phi_{20}=0.5\pi$, $\phi_{30}=\pi$, $\beta_{z0}=0.8$, *and* $\theta_0=0.25\pi$ in the cases of $\beta_{\perp 0}=0.0$ (solid line) and $\beta_{\perp 0}=0.56$ (dashed line).

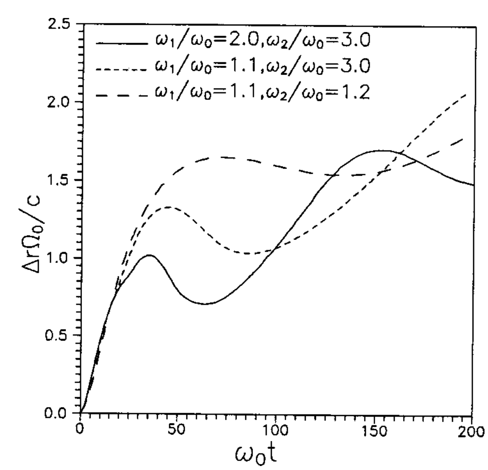

FIG. 2. The normalized transverse position $\Delta r\Omega_0/c$ of the charged particle as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\phi_{10}=0.2\pi$, $\phi_{20}=0.5\pi$, $\phi_{30}=0.3\pi$, $\beta_{z0}=0.8$, $\beta_{\perp 0}=0.1$, *and* $\theta_0=0.25\pi$. The three curves correspond to the situations of $\omega_1/\omega_0=2.0$ *and* $\omega_2/\omega_0=3.0$, $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=3.0$, *and* $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=1.2$.

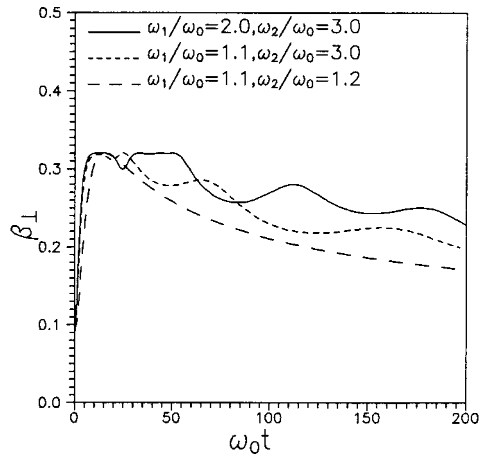

FIG. 3. The normalized transverse velocity $\beta_\perp$ of the charged particle as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for the same parameters as Fig. 2.

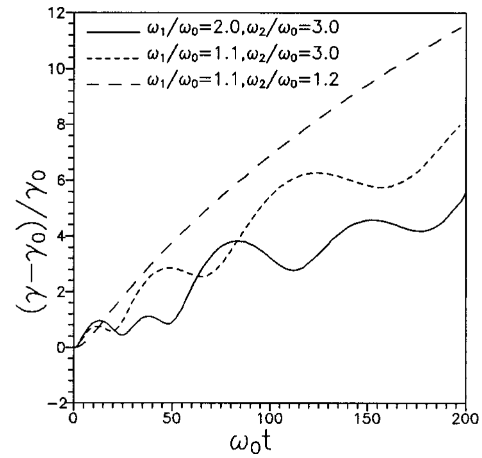

FIG. 5. The normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for the same parameters as Fig. 2.

Fig. 4 depicts the normalized axial velocity β_z of the charged particle versus $\omega_0 t$. As can be seen from Fig. 3, around the cyclotron resonance the transverse velocity of the charged particle first increases sharply and then tends to decrease with $\omega_0 t$, which is identical with (31), though it swings in the cases of $\omega_1/\omega_0=2.0$ *and* $\omega_2/\omega_0=3.0$, and $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=3.0$. Figure 4 states that the axial velocity of the charged particle can be increased when $\omega_0 t$ increases. One can also observe from Figs. 3 and 4 that, in the case of $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=1.2$, $\beta_\perp$ and β_z vary monotonically without any obvious swings when $\omega_3=\omega_0$ is satisfied. Figure 5 gives the normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ as a function of $\omega_0 t$ for three different cases. Note from Fig. 5 that the particle kinetic energy can be increased greatly when $\omega_3=\omega_0$. Additionally, Fig. 5 also indicates that, in the case of $\omega_1/\omega_0=1.1$ and $\omega_2/\omega_0=1.2$, the interaction between the particle and the waves is improved significantly in such a way that the particle kinetic energy can be much higher than those of the other cases ($\omega_1/\omega_0=2.0$ *and* $\omega_2/\omega_0=3.0$, *and* $\omega_1/\omega_0=1.1$ *and* $\omega_2/\omega_0=3.0$) for the same value of $\omega_0 t$.

It is worth noting that, at cyclotron resonance, the initial phases of the waves can also have an effect on the interaction of the charged particle with the waves. This is shown in Fig. 6, where the normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ is plotted as a function of $\omega_0 t$ for $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\omega_1/\omega_0=1.1$, $\omega_2/\omega_0=1.2$, $\omega_3/\omega_0=1.0$, $\phi_{10}=0.2\pi$, $\phi_{20}=0.5\pi$, $\beta_{z0}=0.8$, $\beta_{\perp 0}=0.1$, *and* $\theta_0=0.25\pi$. The solid and dashed lines in Fig. 6 represent the situations of $\phi_{30}=0.3\pi$ *and* $\phi_{30}=1.2\pi$, respectively. One can determine from Fig. 6 that when the angular frequencies ω_1, ω_2, *and* ω_3 are all around ω_0 the normalized particle energy is much higher for $\phi_{30}=0.3\pi(\phi_{30}-\phi_{10}=0.1\pi)$ than for $\phi_{30}=1.2\pi(\phi_{30}-\phi_{10}=\pi)$ at certain values of $\omega_0 t$. This result implies that the interaction between the charged particle and the waves can be much stronger at cyclotron reso-

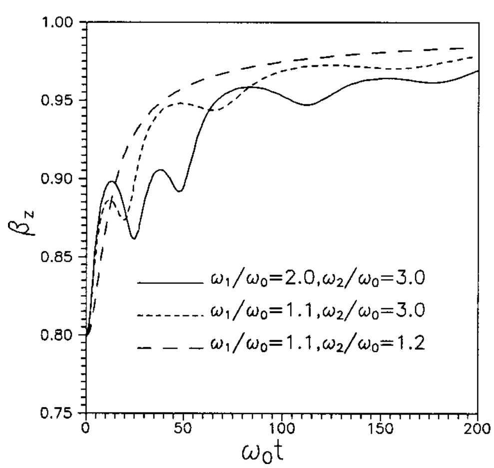

FIG. 4. The normalized axial velocity β_z of the charged particle as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for the same parameters as Fig. 2.

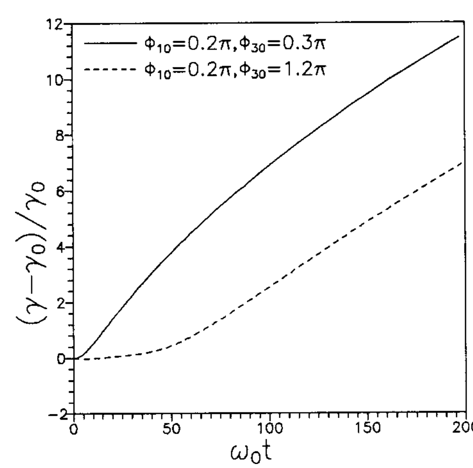

FIG. 6. The normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for $q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, $q\hat{A}_3/mc=0.20$, $\omega_1/\omega_0=1.1$, $\omega_2/\omega_0=1.2$, $\phi_{10}=0.2\pi$, $\phi_{20}=0.5\pi$, $\beta_{z0}=0.8$, $\beta_{\perp 0}=0.1$, and $\theta_0=0.25\pi$. The solid and dashed lines represent the cases of $\phi_{30}=0.3\pi$ and $\phi_{30}=1.2\pi$, respectively.

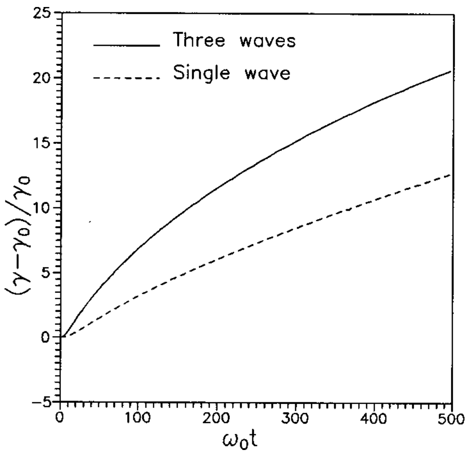

FIG. 7. The normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ as a function of $\omega_0 t$ at the cyclotron resonance of $\omega_3=\omega_0$ for a superposition of three waves ($q\hat{A}_1/mc=0.10$, $q\hat{A}_2/mc=0.15$, and $q\hat{A}_3/mc=0.20$) and a single wave ($q\hat{A}_1/mc=0.0$, $q\hat{A}_2/mc=0.0$, and $q\hat{A}_3/mc=0.20$). The other parameters are the same as those in Fig. 6, except $\phi_{30}=0.3\pi$.

nance when the phase differences of the waves remain so small that the phases of the waves are in the same quadrant.

Figure 7 shows the normalized particle energy $\hat{E}=(\gamma-\gamma_0)/\gamma_0$ vs $\omega_0 t$ for the cases of a superposition of three waves ($q\hat{A}/mc=0.10$, $q\hat{A}_2/mc=0.15$, and $q\hat{A}_3/mc=0.20$) and a single wave ($q\hat{A}_1/mc=0.0$, $q\hat{A}_2/mc=0.0$, and $q\hat{A}_3/mc=0.20$). The other parameters employed in Fig. 7 are $\omega_1/\omega_0=1.1$, $\omega_2/\omega_0=1.2$, $\omega_3/\omega_0=1.0$, $\phi_{10}=0.2\pi$, $\phi_{20}=0.5\pi$, $\phi_{30}=0.3\pi$, $\beta_{z0}=0.8$, $\beta_{\perp 0}=0.1$, *and* $\theta_0=0.25\pi$. Figure 7 indicates that, around the cyclotron resonance, the energy gain of the charged particle is much larger for the superposition of three waves than for a single wave at certain values of $\omega_0 t$. This result means that the use of a superposition of electromagnetic waves with different frequencies has enhanced the chance of the occurrence of cyclotron resonance.

V. CONCLUSIONS

In this paper the relativistic motion of a charged particle driven by a uniform magnetic field and a superposition wave consisting of a number of circularly polarized plane electromagnetic waves with different frequencies is studied by deriving an exact solution to the Lorentz force equation of the charged particle for arbitrary initial conditions. The expressions of the particle position and velocity are found to be explicit functions of the phases of the waves. It is found that, for certain initial conditions, the particle gains energy from the waves and the energy gain could reach a maximum value during the time evolution of the charged particle motion in the nonresonance case. The results also show that the particle motion is constrained by $\beta_\perp^2 \leqslant \gamma_0^2(1-\beta_{z0})^2/[1+\gamma_0^2(1-\beta_{z0})^2]$, and that the axial velocity of the particle can be enhanced considerably. Additionally, the particle energy gain could be increased rapidly at the particle cyclotron resonance. It is noticed that, in comparison with the situation of a single wave, the use of the superposition wave, including a group of electromagnetic waves that possess different frequencies, increases the chance of the occurrence of cyclotron resonance for a charged particle, making it easier for the particle to be accelerated. Numerical results also show that the interaction of the charged particle with the superposition of electromagnetic waves can be improved significantly at cyclotron resonance when the frequency and phase differences of the waves remain so small that the phases of the waves are in the same quadrant. The results of the present paper may find applications in particle acceleration and heating, as well as in basic plasma processes.

[1]C. S. Roberts and S. J. Buchsbaum, Phys. Rev. A **135**, 381 (1964).
[2]E. Villalon and W. J. Burke, Phys. Fluids **30**, 3695 (1987).
[3]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), p. 118.
[4]J. V. Shebalin, IEEE Trans. Plasma Sci. **PS-16**, 390 (1988).
[5]S. Acharya and A. C. Saxena, IEEE Trans. Plasma Sci. **PS-21**, 257 (1993).
[6]H. R. Jory and A. W. Trivelpiece, J. Appl. Phys. **39**, 3053 (1968).
[7]B. H. Quan, R. A. Dandl, N. H. Lazar, and R. F. Wuerker, Bull. Am. Phys. Soc. **26**, 893 (1981).
[8]J. E. Howard, A. J. Lichtenberg, M. A. Lieberman, and R. H. Cohen, Physica D **20**, 259 (1986).
[9]A. J. Lichtenberg, M. A. Lieberman, J. E. Howard, and R. H. Cohen, Phys. Fluids **29**, 1061 (1986).
[10]T. D. Rognlien, Nucl. Fusion **23**, 163 (1983).
[11]K. Hizanidis, Phys. Fluids B **1**, 675 (1989).