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ORIGINAL PAPER

Oblique Whistler-Mode Waves in the Inhomogeneous Magnetospheric Plasma: Resonant Interactions with Energetic Charged Particles

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Abstract A consistent account is given of the theory of resonant interactions between energetic charged particles and a whistler-mode wave propagating obliquely to the non-uniform geomagnetic field in the inhomogeneous magnetospheric plasma. The basic equations for the wave field and charged particle dynamics are presented, with the emphasis being placed on the parameters governing the problem. A Hamiltonian approach is consistently used in the analysis of the particle equations of motion which are discussed in detail and solved analytically in various cases. Two applications of the theory are considered. First, we calculate the growth (or damping) rate for a whistler-mode wave propagating obliquely to geomagnetic field in the magnetosphere. Secondly, we estimate the proton precipitation into the upper atmosphere induced by a VLF transmitter signal.

Keywords Resonant interaction · Inhomogeneous magnetospheric plasma · Whistler-mode wave · Oblique propagation

1 Introduction

Resonant wave–particle interactions are a basic problem in space plasma physics, particularly in magnetospheric physics. Beyond distances from the Earth as little as a few hundred kilometers, the matter present is significantly ionized. Therefore the plasma, charged particles and related electromagnetic fields are the main features encountered in near-Earth space. In the absence of external sources, self-consistent electromagnetic fields

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exist in the space plasma in the form of electromagnetic waves. The generation, propagation, and interaction of these waves with the plasma affect the dynamics of the near-Earth space environment in a crucial manner. These processes as such constitute a "self-consistent" element of the dynamics: the waves excited as a result of the development of plasma instabilities depend on the geomagnetic activity conditions and on the state of the medium, while the interactions of these waves with charged particles affect the state of the environmental plasma.

An important feature of the near-Earth space plasma is that it represents an open system, the dynamics of which is strongly influenced by solar and geomagnetic activity. Another important point is that plasma processes in the magnetosphere are essentially determined by the ambient geomagnetic field. Thus, from the viewpoint of plasma physics, the dynamics of charged particles and fields in the magnetosphere may be regarded as a problem of wave–particle interactions in a magnetized plasma. The result of these interactions shows itself as an evolution of the wave fields and charged particle distributions. Depending on the physical conditions, the wave–particle interactions may have essentially different characters. It is convenient to represent the physical parameters of the problem with the help of corresponding frequencies. Thus, in a homogeneous plasma there are five characteristic frequencies: the wave frequency ω which is a kinematic feature of the wave, the nonlinear frequency ω_{NL} related to the wave amplitude, the cyclotron frequency Ω describing the ambient magnetic field, the damping (or growth) rate γ which depends on the particle distribution function, and plasma frequency ω_{p}.

In an inhomogeneous plasma the problem becomes more complicated. Firstly, there appears an inhomogeneity parameter, denoted by α later on, which determines the destruction rate of a cyclotron resonance due to the inhomogeneity

$$|\alpha| \propto \Omega_d^2 \equiv \left|\frac{d}{dt}(\omega - k_{\|}v_{\|} - n\Omega)\right|, \tag{1.1}$$

where $k_{\|}$ and $v_{\|}$ are parallel (i.e. directed along the ambient magnetic field $\mathbf{B}_0$) components of the wave normal vector and charged particle velocity, respectively, and n is the order of cyclotron resonance, $n = 0$ corresponding to Cerenkov resonance. As the derivative on the right-hand side (r.h.s.) of (1.1) has the dimension of frequency squared, there appears a new characteristic frequency of the problem, namely, $\Omega_{\mathrm{d}} \propto |\alpha|^{1/2}$. The ratio of this frequency to ω_{NL}, which determines the relative importance of inhomogeneity and nonlinearity, becomes a fundamental parameter of the problem. While the frequency Ω_{d} is defined by the equality (1.1), a strict definition of the inhomogeneity parameter α will be given later.

The frequency of particle transitions between cyclotron resonances is another distinctive frequency in the problem of wave–particle interactions in an inhomogeneous plasma. This quantity, however, may be expressed through those already introduced. Indeed, for a particle in the n-th cyclotron resonance with a wave, the quantity $u_n \equiv k_{\|}v_{\|} - \omega + n\Omega = 0$, while in the $(n \pm 1)$-th resonance the same quantity is equal to $\mp\Omega$. Accounting for (1.1), we find that the transition frequency between resonances is equal to $|\Omega_d^2/\Omega|$.

In closed plasma configurations like the magnetosphere at low L-shells, there is one more inhomogeneity related characteristic frequency, namely, the frequency Ω_b of particle oscillations between mirror points, called the bounce frequency. Obviously, with a given set of parameters, a universal solution to the problem of wave–particle interactions in an inhomogeneous plasma is impossible; rather, every particular case requires a special consideration. This is already clear from the fact that, depending on the parameters of the problem, the charged particle motion may be either regular or stochastic, and a unified approach describing both of these situations is hardly possible.

To give an overall framework for the subsequent considerations, we briefly describe the general features of the problem depending on the relations between the parameters. First of all, we should mention that in most cases related to whistler-mode waves in a space plasma, the following relationships hold:

$$\max(\gamma, \Omega_b, \omega_{NL}, \Omega_d) \ll \max(\omega, \Omega). \tag{1.2}$$

If the growth rate γ exceeds all frequencies in parentheses on the left-hand side (l.h.s.) of (1.2), the variation of the wave amplitude should be taken into account when investigating resonant wave–particle interactions. Apart from numerical methods, there is only one efficient approach to the problem in this case, i.e. to use the linear Vlasov–Landau approximation, which, of course, is appropriate only for a stable plasma. Here, we will mainly deal with the opposite situation, namely, when

$$\gamma \ll \max(\Omega_b, \omega_{NL}, \Omega_d). \tag{1.3}$$

The inequality (1.3) corresponds to the so-called approximation of the given field developed in the papers written by Mazitov (1965), Al'tshul and Karpman (1965), and O'Neil (1965). In the frame of this approximation, the charged particle distribution function is calculated in a given wave field, while the effect of the particles back on the wave is evaluated using the successive approximation method in small parameters proportional to γ. Thus, for instance, for a homogeneous plasma when $\Omega_d = \Omega_b = 0$, the inequality (1.3) leads to the condition of applicability for the approximation of the given field which reads $\gamma \ll \omega_{NL}$ (O'Neil 1965). The papers mentioned above dealt with Langmuir waves in an isotropic homogeneous plasma, while we shall later discuss resonant interactions for whistler-mode waves in a magnetized inhomogeneous plasma.

The investigation of whistlers in the magnetosphere began with the classical research by Eckersley (1935) and Storey (1953a, b) among others, and it has continued up to the present time. The first and the most profound summary of research in this field was given in the book by Helliwell (1965), which made a superlative contribution to whistler studies. In spite of a great number of papers dealing with various aspects of whistler phenomena, the number of monographs especially devoted to whistlers is still not large. Apart from already mentioned classical book by Helliwell (1965), we here refer to the book by Sazhin (1993) which concentrates on peculiarities of whistler-mode propagation in a hot plasma, and the book "*Whistler Phenomena*" (2001) edited by Ferencz et al., mainly devoted to full wave solutions for short impulses. And only recently, an excellent book by Trakhtengerts and Rycroft (2008) that covers a large number of problems in magnetospheric whistler studies has been published.

Theoretical studies of the resonant interactions between whistler-mode waves and energetic electrons also has a history of more than half a century. An account of this history is beyond the scope of present review which aims just to present the results of this theory. We only list several—perforce far from all—publications where additional references may be found. To restrict ourselves even further, we mainly refer to those papers in which the inhomogeneities of the plasma and of the ambient magnetic field were taken into account. This pared-down list of publications includes: Sudan and Ott (1971), Nunn (1971, 1974), Matsumoto and Kimura (1971), Karpman et al. (1975a), Roux and Pellat (1978), Inan et al. (1978), Matsumoto (1979), Vomvoridis and Denavit (1980), Omura and Matsumoto (1982), Bell (1984), Shklyar (1986), Omura et al. (1991), Albert (2000, 2002), Trakhtengerts and Rycroft (2000), Trakhtengerts et al. (2003), Omura et al. (2008), and Trakhtengerts and Rycroft (2008). We should mention that the largest part of theoretical work on whistler-mode wave–particle interactions in an inhomogeneous plasma deals with parallel propagation, i.e. with ducted waves, while here we focus on oblique wave propagation. Although whistler wave propagation is a special part of whistler studies and by no means falls within the scope of the present review, this point needs at least a short comment.

As has been shown by Storey (1953a), the group velocity of low-frequency ($\omega \ll \Omega_e$) whistler waves does not deviate from the direction of the ambient magnetic field by angles larger than 19.47 degrees. This natural guiding, however, is insufficient to explain multiple whistlers between magnetospherically conjugate points and the observations of echoing whistlers on the ground. As ray tracing studies in a smooth magnetosphere show, wave packets leave their initial geomagnetic field lines and move to field lines where the condition $\omega \ll \Omega_e$ is not satisfied all along the ray path. These waves then enter the so-called quasi-resonance regime of propagation and suffer magnetospheric reflection from the region where their frequency is close to the lower hybrid resonance frequency (Kimura 1966). The observation of multi-hop whistlers, or whistler trains, as well as the smaller than would be expected attenuation of signal spectral amplitudes led Storey (1953b) to the idea of "ducting or focussing" which should occur in the magnetospheric propagation of whistlers. The major importance of ducting in the propagation of whistlers at middle latitudes was first recognized by Helliwell and his colleagues (see Helliwell 1965). Along with whistler wave events which cannot be explained without making use of the idea of ducts, there are other VLF phenomena which are definitely related to nonducted propagation. The most remarkable of these are magnetospherically reflected (MR) whistlers and their particular type, the so-called Nu-whistlers, virtually predicted by Kimura (1966) and first observed and explained by Smith and Angerami (1968) (see also Edgar 1976). Thus, both ducted and nonducted whistler wave propagation through the magnetosphere can occur, and this viewpoint is generally accepted nowadays. As was already mentioned, the main focus of the present review is on resonant wave–particle interactions in the case of oblique nonducted propagation. A schematic picture of magnetospheric particle interaction with monochromatic wave field from ground-based VLF transmitter is shown in Fig.1, while Fig.2 presents the wave characteristics along a particular ray trajectory. We will be returning to these pictures later on.

2 Governing Equations for the Wave Field

2.1 Presentation of the Wave Field in an Inhomogeneous Plasma

In the case of a homogeneous plasma, and in the absence of resonant wave–particle interactions, a linear solution of the basic equations for the wave field can be found in the form of a plane wave, with a constant wave frequency ω and a wave normal vector $\mathbf{k}$ satisfying a dispersion relation. A more general form of the wave field in that case is a wave packet having a slowly varying complex amplitude. Geometrical optics (see, for instance, Landau and Lifshitz 1959, 1975) suggests the form in which we may look for a solution of the field equations in an inhomogeneous plasma in presence of wave–particle interactions. Namely, we write the expression for the wave electric field in the form

$$\mathcal{E}_j = Re\{E_j\} \equiv Re\{E_{0j}(t, \mathbf{r})e^{i\psi(\mathbf{r})-i\omega t}\}, \tag{2.1}$$

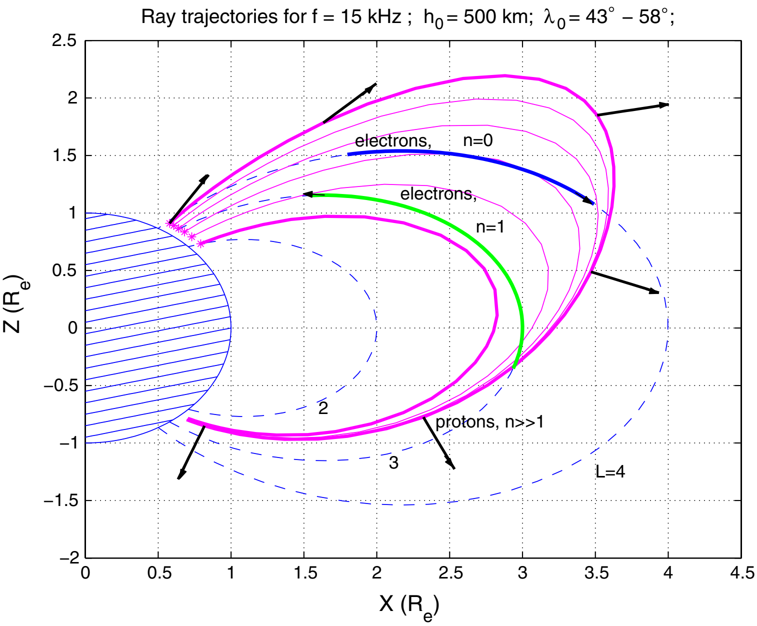

Fig. 1 Electromagnetic field emitted by a ground-based VLF transmitter initially extends into the Earth-ionosphere waveguide. While spreading in the ducts, the field partly leaks into the magnetosphere and starts propagating as nonducted whistler-mode waves. Due to refractive properties of the ionosphere, all waves have almost vertical direction of the wave normal vector at the top of the ionosphere

where the complex amplitudes $E_{0j}(t, \mathbf{r})$ are slowly varying functions of time and the spatial coordinates as compared to the wave frequency ω and the wave number $\mathbf{k}$ introduced below. Presentation of the wave field in the form (2.1), with the restrictions formulated above, is a strong assumption but, in this study, we limit ourselves to electromagnetic fields that may be represented in this form. In particular, presentation (2.1) is assumed to be valid for the wave field excited by a ground-based VLF transmitter in the area between inner-most and outermost ray trajectories shown in Fig. 1.

In the case of a homogeneous plasma, $\psi(\mathbf{r}) = \mathbf{k}{\cdot}\mathbf{r} \equiv k_x x + k_y y + k_z z$, so that $\nabla\psi = \mathbf{k}$, where $\mathbf{k}$ is a constant vector. In an inhomogeneous case, the quantity $\nabla\psi$, which is now a function of $\mathbf{r}$, is still denoted by $\mathbf{k}$ and is called a local wave normal vector:

$$\mathbf{k}(\mathbf{r}) = \nabla\psi(\mathbf{r}). \tag{2.2}$$

The direction of the local wave normal vector along the outermost trajectory is indicated by vectors of equal length in Fig.1.

An essential assumption concerning this presentation (2.1) is that ω, which is a constant central frequency of the wave packet, and the local wave normal vector $\mathbf{k(r)}$ satisfy a local dispersion relation

$$\omega = H[\mathbf{k}(\mathbf{r}), \mathbf{r}], \tag{2.3}$$

where $H(\mathbf{k}, \mathbf{r})$ as a function of $\mathbf{k}$ is determined by the same expression which defines the dispersion relation in the homogeneous case, but with the plasma parameters taken at the point $\mathbf{r}$, which gives rise to the dependence of H on $\mathbf{r}$ as the second argument. As long as

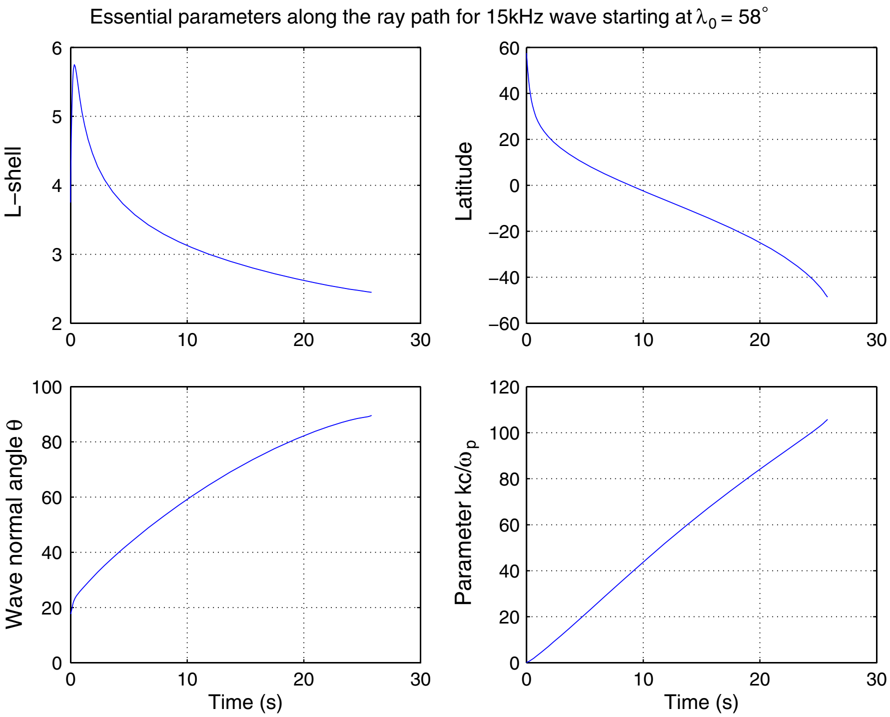

Fig. 2 Wave characteristics along the outermost ray shown in Fig.1, for which the directions of the wave normal vector is indicated. Note the propagation almost along a field line ($L \simeq$ const.) in the Southern hemisphere far enough from the equator

all the plasma parameters vary slowly on the scale of a wavelength, which we will assume, the quantity $\mathbf{k}(\mathbf{r})$ should be a slowly varying vector in the sense that

$$\max\left|\frac{\partial k_i}{\partial x_j}\right|\frac{1}{k(\mathbf{r})} \ll k(\mathbf{r}); \quad (k(\mathbf{r}) \equiv |\mathbf{k}(\mathbf{r})|).$$

Taking account of the definition (2.2), the relation (2.3) may be rewritten in the form:

$$\omega = H\left(\frac{\partial\psi}{\partial\mathbf{r}}, \mathbf{r}\right) \tag{2.4}$$

giving a nonlinear partial differential equation of the first order for the function $\psi(\mathbf{r})$, which is often called *eikonal*. We should mention that, in order that the complex amplitudes E_{0j} are slowly varying functions of t and $\mathbf{r}$, not only should the phase of the field satisfy the eikonal equation (2.4), but the amplitudes E_{0j} should also be related by the local polarization coefficients (see below).

2.2 Energy Conservation and Equation for the Wave Amplitude

The wave electromagnetic field is described by Maxwell's equations which, in the standard way, are reduced to the set of equations for the wave electric field

$$\text{grad div } \boldsymbol{\mathcal{E}} - \Delta\boldsymbol{\mathcal{E}} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{\mathcal{E}}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial\mathbf{j}}{\partial t} = 0, \tag{2.5}$$

where Δ is Laplace's operator and $\mathbf{j}$ is the total current density of plasma particles (further on called simply the "current"). The main simplification which we shall adopt in dealing with this equation is the following. We assume that the plasma may be divided into two components (not to be confused with different plasma species!) The first is a dense "cold" component which determines plasma dispersion and the propagation properties, and which we may describe using a linear approximation. The second is the much less dense energetic component involved in the resonant interactions with the wave which, therefore, is responsible for the wave growth or damping, but which has little influence on the wave dispersion and propagation characteristics. Further insight into the meaning of such a separation of components will be obtained in the course of the following developments. For the present, we simply divide the current in (2.5) into two parts

$$\mathbf{j} = \mathbf{j}_{\mathrm{LIN}} + \mathbf{j}_{\mathrm{RES}},$$

where $\mathbf{j}_{\mathrm{LIN}}$ and $\mathbf{j}_{\mathrm{RES}}$ correspond to the cold dense plasma component and to the energetic component, respectively. Since in the case of a cold plasma the current at any point $\mathbf{r}$ is determined only by the local electric field at the same point (in other words, when spatial dispersion is absent), the linear current in an inhomogeneous plasma is expressed in exactly the same way as in the case of a homogeneous plasma, i.e. with the help of the local dielectric tensor $\varepsilon_{ij}(\omega', \mathbf{r})$. The resonance current in (2.5), however, should be retained as it is. Thus, instead of (2.5) we will have

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\mathcal{E}_j}{\partial x_j}\right) - \frac{\partial^2\mathcal{E}_i}{\partial x_j\partial x_j} - \int\limits_{-\infty}^{\infty}\frac{\omega'^2}{c^2}\varepsilon_{ij}(\omega',\mathbf{r})\mathcal{E}_j(\omega',\mathbf{r})e^{-i\omega' t}\frac{d\omega'}{2\pi} = -\frac{4\pi}{c^2}\left(\frac{\partial\mathbf{j}_{\mathrm{RES}}}{\partial t}\right)_i, \tag{2.6}$$

where $i, j = (x, y, z)$. Here and further on, we use subscripts "i" and "j" as vector indices, with the usual convention about summation over repeated indices. The same letters are used to denote the imaginary unit ("i") and the current density ("j") which should not lead to any confusion since these two quantities never appear as subscripts. We now substitute (2.1) into (2.6). Using the relation

$$\varepsilon_{ij}(-\omega',\mathbf{r}) = \varepsilon^*_{ij}(\omega',\mathbf{r}). \tag{2.7}$$

which is the direct consequence of the fact that the dielectric tensor $\varepsilon_{ij}(t - t', \mathbf{r})$ is a real quantity, one can easily prove that

$$\int\limits_{-\infty}^{\infty}\frac{\omega'^2}{c^2}\varepsilon_{ij}(\omega',\mathbf{r})\mathcal{E}_j(\omega',\mathbf{r})e^{-i\omega' t}\frac{d\omega'}{2\pi} = Re\left\{\int\limits_{-\infty}^{\infty}\frac{\omega'^2}{c^2}\varepsilon_{ij}(\omega',\mathbf{r})E_j(\omega',\mathbf{r})e^{-i\omega' t}\frac{d\omega'}{2\pi}\right\}, \tag{2.8}$$

where $E_j(\omega', \mathbf{r})$ is Fourier transform over time of $E_j(t, \mathbf{r}) = E_{0j}(t, \mathbf{r}) \exp[-i\omega t + i\psi(\mathbf{r})]$ [cf (2.1)]. Since, according to our assumption of slow time variations of the amplitude $\mathbf{E}_0$, the function $E_j(\omega', \mathbf{r})$ is peaked at $\omega' = \omega$, and since $\omega'^2\varepsilon_{ij}(\omega')$ is a smooth function of ω', we may expand it around ω to the linear term, so that the integral term in (2.6) takes the form

$$\begin{aligned}&\frac{1}{c^2}\int\limits_{-\infty}^{\infty}\left\{\omega^2\varepsilon_{ij}(\omega)+\frac{\partial}{\partial\omega}\left[\omega^2\varepsilon_{ij}(\omega)\right](\omega'-\omega)\right\}E_j(\omega',\mathbf{r})e^{-i\omega' t}\frac{d\omega'}{2\pi}\\&=\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega)E_j(t,\mathbf{r})+\frac{1}{c^2}\frac{\partial}{\partial\omega}\left[\omega^2\varepsilon_{ij}(\omega)\right]\left[i\frac{\partial E_j(t,\mathbf{r})}{\partial t}-\omega E_j(t,\mathbf{r})\right].\end{aligned} \tag{2.9}$$

Substituting (2.1) into (2.6) and (2.9), and neglecting the second derivatives of the amplitude $\mathbf{E}_0$ over time and space, as well as the second derivatives of $\mathbf{k} = \nabla\,\psi(\mathbf{r})$ over spatial coordinates, we obtain:

$$\begin{aligned}Re\left\{\Lambda_{ij}E_{0j}e^{i\psi(\mathbf{r})-i\omega t}\right\}=&-\frac{4\pi}{c^2}\left(\frac{\partial \mathbf{j}_{\mathrm{RES}}}{\partial t}\right)_i+Re\left\{\left[i\frac{\partial}{\partial\omega}\left(\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega)\right)\frac{\partial E_{0j}}{\partial t}\right.\right.\\&\left.\left.+\,i\left(2k_j\frac{\partial E_{0i}}{\partial x_j}-k_i\frac{\partial E_{0j}}{\partial x_j}-k_j\frac{\partial E_{0j}}{\partial x_i}+\frac{\partial k_j}{\partial x_j}E_{0i}-\frac{\partial k_j}{\partial x_i}E_{0j}\right)\right]e^{i\psi(\mathbf{r})-i\omega t}\right\},\end{aligned} \tag{2.10}$$

where

$$\Lambda_{ij}=k^2\delta_{ij}-k_ik_j-\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega), \tag{2.11}$$

with $k^2 = k_jk_j \equiv k_x^2 + k_y^2 + k_z^2$. As follows from (2.10), in order that the quantities E_{0j} were finite, but slowly varying functions, the left hand sides of Eqs. (2.10) should be small for finite E_{0j}. More precisely, if we solve Eq. (2.10) by means of successive approximations, as we do, then, in the lowest approximation, the l.h.s. of the equation should be equal to zero, which requires

$$\|\Lambda_{ij}\| = 0. \tag{2.12}$$

While in the homogeneous case Eq. (2.12) gives the dispersion relation, in the inhomogeneous case under discussion equation (2.12) is nothing but the eikonal equation, which we assume to be satisfied for the quantities $\psi(\mathbf{r})$ and ω that enter the phase of the wave packet (2.1). Thus, the local wave normal vector $\mathbf{k} = \nabla\psi(\mathbf{r})$ is also assumed to be known. When (2.12) is fulfilled, the set of equations $\Lambda_{ij}E_{0j} = 0$ has a non-trivial solution which is determined to an arbitrary complex scalar function E and which can be written in the form:

$$E_{0j} = Ea_j,$$

a_j being the components of the wave polarization vector. The relation above is nothing but the zero-order solution to Eq. (2.10). By definition, vector $\boldsymbol{a}$ is a non-trivial solution of the equation

$$\Lambda_{ij}a_j = 0 \tag{2.13}$$

for $\|\Lambda_{ij}\| = 0$, i.e. for ω and $\mathbf{k}$ connected by (2.12). Thus, $\boldsymbol{a}$ may be considered as a function of the local wave normal vector $\mathbf{k}$. While the quantities E_{0j} are completely determined by Eq. (2.10) and the corresponding initial and boundary conditions, the components of the polarization vector are not defined uniquely: they can be renormalized simultaneously with the quantity E without changing the values of E_{0j}. At the same time, the ratios between the components of the polarization vector are fixed. Thus, without any loss of generality, we may arbitrarily set one component of the polarization vector (for example, put $a_x = 1$.) We should stress that the exact choice of polarization coefficients will affect not only the magnitude of E, but also its phase (see below).

The matrix Λ_{ij} is a hermitian one, i.e. $\Lambda_{ij} = \Lambda_{ji}^*$, provided that $\varepsilon_{ij}(\omega)$ is hermitian (see the definition (2.11)). Since $\Lambda_{ij} a_j = 0$ and Λ_{ij} is a hermitian matrix,

$$a_i^* \Lambda_{ij} = 0 \tag{2.14}$$

also holds. Multiplying (2.13) by k_i and a_i^* and summing up over the index "i", we find two other identities which are satisfied if ω and $\mathbf{k}$ obey the local dispersion relation (i.e. the eikonal equation), and $\boldsymbol{a}$ is the vector function of $\mathbf{k}$ specified above. Those identities read:

$$\begin{aligned} k_i \Lambda_{ij} a_j &= k_i \varepsilon_{ij}(\omega) a_j = 0; \\ G \equiv a_i^* \Lambda_{ij} a_j &= a_i^* \left[k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{c^2} \varepsilon_{ij}(\omega) \right] a_j = 0. \end{aligned} \tag{2.15}$$

We have introduced the notation G for the quantity $a_i^* \Lambda_{ij} a_j$ that will be used in further relations. We also need the expressions for the partial derivatives of the quantity G with respect to ω and k_j for ω and $\mathbf{k}$ satisfying the local dispersion relation:

$$\frac{\partial G}{\partial \omega} = -\frac{\partial}{\partial \omega} \left[\frac{\omega^2}{c^2} \varepsilon_{ij}(\omega) \right] a_i^* a_j. \tag{2.16}$$

$$\frac{\partial G}{\partial k_j} = 2 k_j a_i^* a_i - k_i a_i^* a_j - k_i a_j^* a_i. \tag{2.17}$$

We should mention that, although $\boldsymbol{a}$ is the function of $\mathbf{k}$, all terms in $\partial G / \partial k_j$ containing the derivatives of $\boldsymbol{a}$ vanish due to the relations (2.13) and (2.14).

Let us denote by Ψ the total phase of the field

$$\Psi(t, \mathbf{r}) = \psi(\mathbf{r}) - \omega t. \tag{2.18}$$

We now write expressions of the form $Re\{A_i e^{i\Psi}\}$ in (2.10) as $(A_i e^{i\Psi} + A_i^* \, e^{-i\Psi})/2$. Then we multiply both parts of equation (2.10) by $-2\, i a_i^* \, e^{-i\Psi}$, sum up over the index "i" and average over the wave period. Under this procedure, the term proportional to $\Lambda_{ij} a_i^*$ becomes zero due to the relation (2.14), while all terms proportional to $e^{-i\Psi}$ vanish due to averaging. Thus, all the terms on the l.h.s. disappear, and as the result we obtain

$$\begin{aligned} &\frac{\partial}{\partial \omega} \left(\frac{\omega^2}{c^2} \varepsilon_{ij}(\omega) \right) \frac{\partial E_{0j}}{\partial t} a_i^* + \left(2 k_j \frac{\partial E_{0i}}{\partial x_j} - k_i \frac{\partial E_{0j}}{\partial x_j} - k_j \frac{\partial E_{0j}}{\partial x_i} \right. \\ &\qquad\qquad \left. + \frac{\partial k_j}{\partial x_j} E_{0i} - \frac{\partial k_j}{\partial x_i} E_{0j} \right) a_i^* = -\frac{8\pi i}{c^2} \left\langle \left(\frac{\partial \mathbf{j}_{\mathrm{RES}}}{\partial t} \right)_i a_i^* e^{-i\Psi} \right\rangle \end{aligned} \tag{2.19}$$

where $\langle \cdots \rangle$ stands for averaging over the wave period.

Let us first consider the r.h.s. of Eq. (2.19). Writing the equality

$$\frac{\partial \mathbf{j}_{\mathrm{RES}}}{\partial t} \boldsymbol{a}^* e^{-i\Psi} = \frac{\partial}{\partial t} \left(\mathbf{j}_{\mathrm{RES}} \boldsymbol{a}^* e^{-i\Psi} \right) - i\omega \mathbf{j}_{\mathrm{RES}} \boldsymbol{a}^* e^{-i\Psi}$$

and taking into account that the average value of the time derivative is equal to zero we find

$$\left\langle \frac{\partial \mathbf{j}_{\mathrm{RES}}}{\partial t} \boldsymbol{a}^* e^{-i\Psi} \right\rangle = -i\omega \left\langle \mathbf{j}_{\mathrm{RES}} \boldsymbol{a}^* e^{-i\Psi} \right\rangle; \quad \left\langle \frac{\partial \mathbf{j}_{\mathrm{RES}}}{\partial t} \boldsymbol{a} e^{i\Psi} \right\rangle = i\omega \left\langle \mathbf{j}_{\mathrm{RES}} \boldsymbol{a} e^{i\Psi} \right\rangle, \tag{2.20}$$

the second relation being the complex conjugate of the first one. Since Eq. (2.19) contains only small quantities, we may use the relation between electric field components in the zero-order approximation (see above):

$$\mathbf{E}_0(t,\mathbf{r}) = \boldsymbol{a}(\mathbf{k},\mathbf{r})E(t,\mathbf{r}). \tag{2.21}$$

Here we have indicated in an explicit way the dependence of the polarization coefficients on coordinates which is related to a spatial dependence of the plasma parameters in the inhomogeneous plasma. Apart from this, they depend on **k**, of course, which in turn is a function of the coordinates according to (2.2). We multiply both parts of Eq. (2.19) by E, write the complex conjugate equation, and then sum them up. Making use of the equalities

$$\varepsilon_{ij}^* = \varepsilon_{ji}; \quad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i} \tag{2.22}$$

[the last one follows directly from (2.2)] we find

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega}\left(\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega)\right)E_{0i}^*E_{0j}\right] + \frac{\partial}{\partial x_j}(2k_jE_{0i}^*E_{0i} - k_iE_{0i}E_{0j}^* - k_iE_{0i}^*E_{0j}) = -\frac{16\pi\omega}{c^2}\langle\mathbf{j}_{\mathrm{RES}}\cdot\boldsymbol{\mathcal{E}}\rangle, \tag{2.23}$$

where (2.20) and definition (2.1) have been used to evaluate the r.h.s. Using (2.21) and relations (2.16), (2.17) we can write (2.23) in the form

$$\frac{\partial}{\partial t}\left(-\frac{\partial G}{\partial\omega}|E|^2\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial G}{\partial k_j}|E|^2\right) = -\frac{16\pi\omega}{c^2}\langle\mathbf{j}_{\mathrm{RES}}\cdot\boldsymbol{\mathcal{E}}\rangle. \tag{2.24}$$

Introducing the wave energy density U and the group velocity v_{gj} according to the relations

$$U = -\frac{c^2}{16\pi\omega}\frac{\partial G}{\partial\omega}|E|^2 \equiv \frac{1}{16\pi\omega}\frac{\partial}{\partial\omega}(\omega^2\varepsilon_{ij})E_{0i}^*E_{0j} \tag{2.25}$$

$$v_{gj} = -\frac{\partial G/\partial k_j}{\partial G/\partial\omega} \equiv \frac{\partial\omega}{\partial k_j}, \tag{2.26}$$

we rewrite (2.24) in the form of the energy conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x_j}(v_{gj}U) = -\langle\mathbf{j}_{\mathrm{RES}}\cdot\boldsymbol{\mathcal{E}}\rangle, \tag{2.27}$$

which shows that the rate of wave energy density variation in time is equal, with the minus sign, to the sum of divergence of the wave energy flux density $\mathbf{v}_gU$ and the rate of resonant particle kinetic energy variation in a unit volume.

2.3 Equation for Nonlinear Phase Evolution

The quantity $E(t, \mathbf{r})$ which determines the wave field according to (2.1) and (2.21) is complex:

$$E(t,\mathbf{r}) = |E(t,\mathbf{r})|e^{i\phi(t,\mathbf{r})}. \tag{2.28}$$

Equation (2.27) describes the variation of $|E|$ and, thus, the variation of the wave field energy. The quantity ϕ is the nonlinear phase. To find the equation which governs the variation of ϕ, we follow the same steps which lead us from (2.19) to (2.23), but instead of summing up the complex conjugate equations we now subtract one from another. Again using relations (2.22) we come to equation for the nonlinear phase ϕ

$$
\begin{aligned}
&2i\frac{\partial\phi}{\partial t}\frac{\partial}{\partial\omega}\left(\frac{\omega^2}{c^2}\varepsilon_{ij}(\omega)\right)a_i^*a_j|E|^2+2i\frac{\partial\phi}{\partial x_j}(2k_ja_i^*a_i-k_ia_ia_j^*-k_ia_i^*a_j)|E|^2\\
&\quad+2i\widetilde{S}|E|^2=-\frac{8\pi\omega}{c^2}\left\langle\mathbf{j}_{\mathrm{RES}}(\mathbf{E}_0^*e^{-i\Psi}-\mathbf{E}_0e^{i\Psi})\right\rangle,
\end{aligned}
\tag{2.29}
$$

where the notation

$$
2i\widetilde{S}=2k_j\left(\frac{\partial a_i}{\partial x_j}a_i^*-\frac{\partial a_i^*}{\partial x_j}a_i\right)-k_i\left(\frac{\partial a_j}{\partial x_j}a_i^*-\frac{\partial a_j^*}{\partial x_j}a_i\right)-k_i\left(\frac{\partial a_i}{\partial x_j}a_j^*-\frac{\partial a_i^*}{\partial x_j}a_j\right)
\tag{2.30}
$$

has been introduced. According to its definition (2.1), the partial derivative of the wave electric field over time is equal to

$$
\dot{\boldsymbol{\mathcal{E}}}\simeq-\frac{i\omega}{2}(\mathbf{E}_0e^{i\Psi}-\mathbf{E}_0^*e^{-i\Psi}),
$$

thus, the r.h.s. of Eq. (2.29) is equal to

$$
\frac{16\pi i}{c^2}\left\langle\mathbf{j}_{\mathrm{RES}}\cdot\dot{\boldsymbol{\mathcal{E}}}\right\rangle.
$$

Using this expression, and the definitions of the wave energy density U and the wave group velocity $\mathbf{v}_g$, we rewrite (2.29) in the form

$$
\frac{\partial\phi}{\partial t}+v_{gj}\frac{\partial\phi}{\partial x_j}-\left(\frac{\partial G}{\partial\omega}\right)^{-1}\widetilde{S}=\frac{1}{2\omega U}\left\langle\mathbf{j}_{\mathrm{RES}}\cdot\dot{\boldsymbol{\mathcal{E}}}\right\rangle.
\tag{2.31}
$$

We see that, in an inhomogeneous plasma, the phase ϕ may vary even in the absence of wave–particle interactions, as the quantity $\widetilde{S}$, in general, is not equal to zero. The reason for this is readily apparent. As was mentioned above, the polarization coefficients a_i are not defined uniquely, as only the quantities $E_{0i}=a_iE$ have real physical sense. Thus, multiplication of all a_i by a complex scalar $C=|C|e^{i\arg(C)}$ only leads to renormalization of the complex amplitude E such that both its magnitude and phase are changed. This renormalization does not affect the equation for the wave energy U as the expression for U contains only the quantities E_{0i}. However, the phase of the complex amplitude E certainly depends on the phase of the normalization factor.

Further on, we will limit ourselves to the consideration of $2D$ geometry, i.e. we will assume that the varying wave normal vector and the ambient non-uniform magnetic field always remains in one, say, the (x, z)-plane, and that plasma parameters do not depend on the y-coordinate. In this case the polarization vector $\boldsymbol{a}$ of a whistler-mode wave may be chosen in the form

$$
\boldsymbol{a}=\boldsymbol{a}_1+i\boldsymbol{a}_2;\quad \boldsymbol{a}_1=(a_{1x},0,a_{1z});\quad \boldsymbol{a}_2=(0,a_{2y},0).
$$

One can easily check that with this choice of the polarization vector and under the assumptions formulated above, the quantity $\widetilde{S}$ is identically equal to zero, so that the equation for the nonlinear phase takes the form

$$
\frac{\partial\phi}{\partial t}+\mathbf{v}_g\frac{\partial\phi}{\partial\mathbf{r}}=\frac{1}{2\omega U}\left\langle\mathbf{j}_{\mathrm{RES}}\cdot\dot{\boldsymbol{\mathcal{E}}}\right\rangle.
\tag{2.32}
$$

3 Particle Dynamics in the Field of a Quasi-Monochromatic Wave Packet

3.1 General Hamiltonian form for Charged Particle Equations of Motion in an Electromagnetic Field

In this Section, we study the motion of resonant charged particles in the field of a whistler-mode wave packet propagating at an arbitrary angle with respect to a non-uniform ambient magnetic field, in an inhomogeneous plasma.

The resonant particle current, which enters the equations for the wave amplitude and phase, is determined by the distribution function governed by Boltzmann-Vlasov equation

$$\frac{\partial F_s}{\partial t} + \mathbf{v}\frac{\partial F_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}\left\{\boldsymbol{\mathcal{E}} + \frac{1}{c}[\mathbf{v} \times (\mathbf{B}_0 + \boldsymbol{\mathcal{B}})]\right\}\frac{\partial F_s}{\partial \mathbf{v}} = 0. \tag{3.1}$$

For the sake of convenience, the ambient magnetic field $\mathbf{B}_0$ is extracted from the total magnetic field. In this Subsection, we will omit the subscript "s" wherever the consideration is appropriate for all plasma species. According to Liouville's theorem, if we solve the characteristic set of Eq. (3.1)

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}; \quad \frac{d\mathbf{v}}{dt} = \frac{q}{m}\left\{\boldsymbol{\mathcal{E}} + \frac{1}{c}[\mathbf{v} \times (\mathbf{B}_0 + \boldsymbol{\mathcal{B}})]\right\} \tag{3.2}$$

which coincides with the particle equations of motion, then the distribution function $F(t, \mathbf{r}, \mathbf{v})$ will be given by

$$F(t, \mathbf{r}, \mathbf{v}) = F_0[\mathbf{r}_0(t, \mathbf{r}, \mathbf{v}), \;\; \mathbf{v}_0(t, \mathbf{r}, \mathbf{v})], \tag{3.3}$$

where $F_0(\mathbf{r}, \mathbf{v})$ is particle initial distribution function

$$F_0(\mathbf{r}, \mathbf{v}) = F(t = 0, \mathbf{r}, \mathbf{v}), \tag{3.4}$$

and $\mathbf{r}_0(t, \mathbf{r}, \mathbf{v})$, $\mathbf{v}_0(t, \mathbf{r}, \mathbf{v})$ are equal to the particle initial coordinate and velocity, respectively, expressed through the current values of t, $\mathbf{r}$, $\mathbf{v}$ from the solution of (3.2). Thus, solving the particle equations of motion is the first step in finding the particle distribution function.

As is well known (see, for instance, Landau and Lifshitz 1959), the particle equations of motion in an electromagnetic field can be written in the Hamiltonian form, with the Hamiltonian function

$$H = \frac{1}{2m}\left[\mathbf{P} - \frac{q}{c}\mathbf{A}(t, \mathbf{r})\right]^2, \tag{3.5}$$

canonically conjugated variables being $\mathbf{P}$ and $\mathbf{r}$. The quantity $\mathbf{A}(t, \mathbf{r})$ which enters (3.5) is the vector potential of electromagnetic field, such that

$$\mathbf{E} = -\frac{1\partial \mathbf{A}}{c\,\partial t}; \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{3.6}$$

and the canonical momentum $\mathbf{P}$ is equal to

$$\mathbf{P} = m\mathbf{v} + \frac{q}{c}\mathbf{A}(t, \mathbf{r}). \tag{3.7}$$

Here, the Coulomb gauge of the wave field potentials that sets the scalar potential to zero is assumed. Similarly to the convention on the notation adopted for the magnetic field, we will denote by $\mathbf{A}_0$ and $\boldsymbol{\mathcal{A}}$ vector potentials of the ambient magnetic field and of the wave magnetic field, respectively.

We should stress that, while canonical coordinates in (3.5) coincide with the usual space coordinates, canonical momenta $\mathbf{P}$ are expressed through $\mathbf{v}$, $\mathbf{r}$, and t. From (3.5) and (3.7) it follows that the Hamiltonian function is equal to the particle kinetic energy $W = mv^2/2$ expressed through canonical variables $\mathbf{P}$, $\mathbf{r}$ and time t. The reason for dealing with Hamiltonian equations is that they allow us to use powerful methods of canonical transformations.

3.2 Canonical Transformation of the Hamiltonian to Curvilinear Dipole Coordinates

When the ambient magnetic field is non-uniform, using Cartesian coordinates in the equations of motion becomes inconvenient. Much more opportune is a coordinate system in which one of the coordinate axes coincides with the line of force of the ambient magnetic field $\mathbf{B}_0$. In such a coordinate system $\mathbf{B}_0$ has only one component. For the sake of definiteness, we will assume the ambient magnetic field to be dipole, and consider the particle equation of motion in dipole coordinates L, Φ, M. Definitions of the dipole coordinates and of related formulas are given in Appendix A. Let us now perform a canonical transformation of the particle equations of motion, choosing L, Φ, and M as new canonical coordinates. (For general formulas related to canonical transformations see, for instance, Landau and Lifshitz 1976). This transformation is provided by the generating function S expressed through old coordinates $\mathbf{r}$ and new momenta $p_{\mathrm{L}}, p_\Phi, p_{\mathrm{M}}$ in the following form:

$$S(\mathbf{r}, p_L, p_\Phi, p_M) = L(\mathbf{r})p_L + \Phi(\mathbf{r})p_\Phi + M(\mathbf{r})p_M, \tag{3.8}$$

where $L(\mathbf{r})$, $\Phi(\mathbf{r})$, and $M(\mathbf{r})$ are determined in (A.1). According to general formulas, the relation between the new and old variables is determined by

$$L \equiv \frac{\partial S}{\partial p_{\mathrm{L}}} = L(\mathbf{r}); \quad \Phi \equiv \frac{\partial S}{\partial p_\Phi} = \Phi(\mathbf{r}); \quad M \equiv \frac{\partial S}{\partial p_{\mathrm{M}}} = M(\mathbf{r}), \tag{3.9}$$

so that the new coordinates coincide indeed with the dipole coordinates, and

$$P_i \equiv \frac{\partial S}{\partial r_i} = p_{\mathrm{L}}\frac{\partial L(\mathbf{r})}{\partial r_i} + p_\Phi \frac{\partial \Phi(\mathbf{r})}{\partial r_i} + p_{\mathrm{M}}\frac{\partial M(\mathbf{r})}{\partial r_i}, \quad (r_i = x, y, z). \tag{3.10}$$

We see that the new momenta are expressed through P_x, P_y, and P_z and coordinates x, y, and z in quite a complicated way. However, they are expressed much more simply through the orthogonal projections of the vector $\mathbf{P}$ on the dipole coordinate system, namely

$$p_{\mathrm{L}} = h_{\mathrm{L}}P_{\mathrm{L}}; \quad p_\Phi = h_\Phi P_\Phi; \quad p_{\mathrm{M}} = h_{\mathrm{M}}P_{\mathrm{M}}. \tag{3.11}$$

Appendix B contains the proofs of these relations, as well as some other formulas used below.

Returning to the canonical transformation of the equations of motion, we notice that the generating function (3.8) is independent of time; thus the new Hamiltonian function is equal to the old one, i.e. to that determined by (3.5), but expressed in terms of the new variables:

$$H = \frac{1}{2mh_{\mathrm{L}}^2}\left(p_L - \frac{q}{c}h_{\mathrm{L}}A_{\mathrm{L}}\right)^2 + \frac{1}{2mh_\Phi^2}\left(p_\Phi - \frac{q}{c}h_\Phi A_\Phi\right)^2 + \frac{1}{2mh_{\mathrm{M}}^2}\left(p_{\mathrm{M}} - \frac{q}{c}h_{\mathrm{M}}A_{\mathrm{M}}\right)^2. \tag{3.12}$$

The vector potential $\mathbf{A}$ consists of two terms in a sum: one is related to the ambient magnetic field $\mathbf{B}_0$, and the other is related to the wave electromagnetic field. Since in dipole coordinates the ambient magnetic field $\mathbf{B}_0$ has only a M-component, the vector potential of the ambient magnetic field can be chosen in the form:

$$A_{0\mathrm{L}} = A_{0\mathrm{M}} = 0; \quad A_{0\Phi} h_\Phi = a_0(L), \tag{3.13}$$

which gives, according to (B.7)

$$\mathbf{B}_0 \equiv B_0 \mathbf{e}_\mathrm{M} = \frac{\mathbf{e}_\mathrm{M}}{h_\mathrm{L} h_\Phi} \frac{\partial a_0}{\partial L}. \tag{3.14}$$

Using the expressions for $\mathbf{B}_0$ (A.4) and Lamé coefficients (A.2) one can find that in the case of a dipole magnetic field

$$a_0 = -\frac{B_e R_e^2}{L}. \tag{3.15}$$

Further on, we will extract $\mathbf{A}_0$ from the vector potential, thus writing the total potential in the form $\mathbf{A}_0 + \boldsymbol{\mathcal{A}}$.

Once performing a canonical transformation, we may immediately write the equations of motion in new variables, since they again have the standard Hamiltonian form:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{3.16}$$

where $q_i = L, \Phi, M$, $p_i = p_\mathrm{L}, p_\Phi, p_\mathrm{M}$, and the Hamiltonian function $H(t, L, \Phi, M, p_\mathrm{L}, p_\Phi, p_\mathrm{M})$ is determined by (3.12).

Another advantage of using the Hamiltonian approach is that we can simplify the Hamiltonian function, rather than the equations of motion, which is much easier, and, in fact, more consistent (Naife 1973). We will assume that the vector potential $\mathbf{A}$ does not depend on the azimuth angle Φ. In terms of the wave properties, this means that the wave normal vector does not have an azimuthal component. In the case of the magnetosphere, this corresponds to wave propagation in the meridian plane. Since the Hamiltonian function does not depend on Φ, the canonical momentum p_Φ is a constant of the motion

$$p_\Phi = \mathrm{const}. \tag{3.17}$$

Thus, for particles with a given value of p_Φ, the problem becomes two-dimensional.

Let us consider the second term (3.12), the only one which contains $\mathbf{A}_0$

$$\frac{1}{2mh_\Phi^2}\left[p_\Phi - \frac{q}{c}a_0(L) - \frac{q}{c}h_\Phi \mathcal{A}_\Phi\right]^2 \tag{3.18}$$

We denote by L_{p_Φ} the solution of the equation

$$p_\Phi - \frac{q}{c}a_0(L) = 0, \tag{3.19}$$

and expand $p_\Phi - qa_0(L)/c$ around L_{p_Φ}

$$p_\Phi - \frac{q}{c}a_0(L) = -\frac{q}{c}(L - L_{p_\Phi})\left(\frac{\partial a_0}{\partial L}\right)_{L=L_{p_\Phi}} \equiv -\frac{q}{c}l\left(\frac{\partial a_0}{\partial L}\right)_{L=L_{p_\Phi}}; \quad l = L - L_{p_\Phi}. \tag{3.20}$$

As we will see below, the quantity $h_\mathrm{L} l$ is of the order of the Larmor radius; thus, the expansion can be limited to the first order terms, provided that the inhomogeneity scale is much larger than Larmor radius. Under the same assumption that we may put $L = L_{p_\Phi}$ in Lamé coefficients (but not in $\mathbf{A}$, of course), they become functions of M only. Obviously, we can use the variable l defined in (3.20) instead of L as the coordinate canonically conjugated to p_{L}. The Hamiltonian (3.12) then takes the form

$$H = \frac{1}{2mh_{\mathrm{L}}^2(M)}\left(p_{\mathrm{L}} - \frac{q}{c}h_{\mathrm{L}}\mathcal{A}_{\mathrm{L}}\right)^2 + \frac{q^2}{2mh_{\Phi^2}(M)c^2}\left(\frac{\partial a_0}{\partial L}l + h_{\Phi}\mathcal{A}_{\Phi}\right)^2 + \frac{1}{2mh_{\mathrm{M}}^2(M)}\left(p_{\mathrm{M}} - \frac{q}{c}h_{\mathrm{M}}\mathcal{A}_{\mathrm{M}}\right)^2, \tag{3.21}$$

where the canonically conjugated variables now are (p_{L}, l), and (p_{M}, M) as before.

By definition (3.14), the particle gyrofrequency $\mathbf{\Omega}$ is expressed through $\partial a_0/\partial L$ as

$$\mathbf{\Omega} = \Omega \cdot \mathbf{e}_M; \quad \Omega \equiv \frac{q}{mch_{\mathrm{L}}h_{\Phi}}\left(\frac{\partial a_0}{\partial L}\right)_{L=L_{p_\Phi}}. \tag{3.22}$$

We should mention that the scalar quantity Ω defined above, which is further on referred to as the particle gyrofrequency (or cyclotron frequency) has the sign of q. It is important to note that in the framework of the equations of motion assigned by the Hamiltonian (3.21), for a given p_Φ and, thus, $L_{p\Phi}$, the particle gyrofrequency is a function of the coordinate M along the ambient magnetic field and does not depend on l.

Using (3.22) and the definition of canonical momenta, we obtain from (3.20) the relation which will prove useful to further analysis

$$v_\Phi + \frac{q}{mc}\mathcal{A}_\Phi = -\Omega h_{\mathrm{L}} l. \tag{3.23}$$

Although in this work we study nonlinear effects in wave–particle interactions, the wave fields that we consider cannot be arbitrarily strong. In particular, we will assume that the variation of the resonant particle velocity during the wave period is much smaller than the velocity itself. For this to be true, it is necessary that the second order terms, proportional to the vector potential of the wave field, in parentheses in (3.21), were much smaller than the first order terms. These requirements, which we will assume to be fulfilled in what follows, permit us to neglect the second order terms in $\mathcal{A}$ in (3.21), and to write the Hamiltonian as

$$\begin{aligned} H(t; p_{\mathrm{L}}, l; p_{\mathrm{M}}, M) = &\frac{p_{\mathrm{L}}^2}{2mh_{\mathrm{L}}^2(M)} + \frac{m}{2}\Omega^2 h_{\mathrm{L}}^2 l^2 + \frac{p_{\mathrm{M}}^2}{2mh_{\mathrm{M}}^2(M)} \\ &- \frac{q}{mc}\left(\frac{p_{\mathrm{L}}\mathcal{A}_{\mathrm{L}}}{h_{\mathrm{L}}} - m\Omega\mathcal{A}_\Phi h_{\mathrm{L}} l + \frac{p_{\mathrm{M}}\mathcal{A}_{\mathrm{M}}}{h_{\mathrm{M}}}\right). \end{aligned} \tag{3.24}$$

3.3 Further Transformation of the Hamiltonian

We now perform one more canonical transformation, from the variables (l, p_{L}) and (M, p_{M}) to new variables (φ, μ) and (s, $p_\parallel$), defined by the generating function

$$\mathcal{F}(l, \varphi, M, p_\parallel) = \frac{m|\Omega|h_{\mathrm{L}}^2}{2} l^2 \operatorname{ctan}\varphi + p_\parallel \int_0^M h_{\mathrm{M}}(M')dM'. \tag{3.25}$$

The relations between the old and new variables are determined by general formulas and have the form:

$$p_{\mathrm{L}} \equiv \frac{\partial \mathcal{F}}{\partial l} = m|\Omega| h_{\mathrm{L}}^2 l \operatorname{ctan} \varphi; \quad p_{\mathrm{M}} \equiv \frac{\partial \mathcal{F}}{\partial M} = p_{\|} h_{\mathrm{M}} + \frac{ml^2}{2} \operatorname{ctan} \varphi \frac{d}{dM}(|\Omega| h_{\mathrm{L}}^2);$$
$$\mu \equiv -\frac{\partial \mathcal{F}}{\partial \varphi} = \frac{m|\Omega| h_{\mathrm{L}}^2 l^2}{2\sin^2 \varphi}; \quad s \equiv \frac{\partial \mathcal{F}}{\partial p_{\|}} = \int_0^M h_{\mathrm{M}}(M')dM'. \tag{3.26}$$

Notice that the quantity μ is always positive. The first and the third relations in (3.26) can be resolved in a form which expresses the old variables as functions of the new ones

$$h_{\mathrm{L}} l = \left(\frac{2\mu}{m|\Omega|}\right)^{1/2} \sin \varphi; \quad \frac{p_{\mathrm{L}}}{h_{\mathrm{L}}} = (2m\mu|\Omega|)^{1/2} \cos \varphi, \tag{3.27}$$

while the inverse relations have the form:

$$\mu = \frac{m}{2|\Omega|}\left(\frac{p_{\mathrm{L}}^2}{m^2 h_{\mathrm{L}}^2} + h_{\mathrm{L}}^2 l^2 \Omega^2\right); \quad \tan \varphi = \frac{m|\Omega| h_{\mathrm{L}}^2 l}{p_{\mathrm{L}}}; \tag{3.28}$$

We should mention that the signs in front of the square roots in (3.27) are a matter of choice and, in fact, we can choose plus for both the electrons and for the ions. However, as we will see below, in the Hamiltonian of the interaction, in particular, the term proportional to $\mathcal{A}_{\Phi}$ in (3.24), depends explicitly on the sign of the charge of the particle.

Let us now discuss the physical meaning of the new variables. According to (3.26), the new coordinate s is uniquely related to the old coordinate M, and represents the length along the field line $L = L_{p\Phi}$. Before turning to the physical meaning of the other variables, the following remarks are in order. As mentioned above, from the very beginning, the canonical momenta which we deal with depend, apart from the particle velocity, on coordinates and time, which enter through the wave vector potential $\boldsymbol{\mathcal{A}}(t, \mathbf{r})$. This is an important point which is connected with the fact that the Lorentz force depends on velocity and, due to this fact, particle coordinates and velocities are not canonically conjugate variables. However, that is no problem, provided that the particle velocity can be expressed through canonical variables. In this case, we may solve the canonical equation and then find the particle velocity; moreover, in many cases the solution in terms of canonical variables is sufficient.

Another important point that should be mentioned is the following. In general, the Hamiltonian function H, or simply the Hamiltonian, depends on canonical coordinates, canonical momenta, and time, all of which should be treated as independent variables. However, as is well known, the variation of the Hamiltonian along a charged particle trajectory obeys the equation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{3.29}$$

and, if the Hamiltonian does not depend explicitly on time, it is a constant of the motion. As has been shown above, the initial Hamiltonian is equal to the particle kinetic energy

$$W = \frac{mv^2}{2}.$$

Since up to now all the generating functions that we used for canonical transformations were time-independent and, thus, led only to expressions of the Hamiltonian through new variables, but did not change its value, we can write an exact equation

$$\frac{dW}{dt} = \frac{\partial H}{\partial t}. \tag{3.30}$$

Returning to the physical meaning of the variables which we have introduced, we should say that these variables, as they stand, are determined by Eqs. (3.27) and (3.28). Their relations to the particle velocity can be traced back following successive definitions of the variables determined by (3.11) and (3.7). Accounting for (3.18) and (3.20) we find

$$\begin{aligned} p_{\|} &= mv_{\|} + \frac{q}{c}\mathcal{A}_{\|} - \frac{ml^2}{2h_{\mathrm{M}}}\mathrm{ctan}\,\varphi\frac{d(|\Omega|h_{\mathrm{L}}^2)}{dM}; \\ \mu &= \frac{mv_{\perp}^2}{2|\Omega|} + \frac{qv_{\mathrm{L}}\mathcal{A}_{\mathrm{L}}}{|\Omega|c} - \frac{|q|h_{\mathrm{L}}l\mathcal{A}_{\Phi}}{c} \\ \tan\varphi &= \frac{|\Omega|h_{\mathrm{L}}l}{v_{\mathrm{L}} + (q/mc)\mathcal{A}_{\mathrm{L}}}. \end{aligned} \tag{3.31}$$

In the expressions above, we have used the subscript "$\|$" instead of "M" in $v_{\|}$ and $\mathcal{A}_{\|}$ to emphasize that the M-components of the vectors coincide with the parallel ones, i.e. with the component along the ambient magnetic field, and that

$$v_{\perp} = \sqrt{v_L^2 + v_{\Phi}^2}$$

is the magnitude of the particle transverse velocity, i.e. of the component perpendicular to the ambient magnetic field. The quantity $p_{\|}$ differs from $mv_{\|}$ by two terms: one is related to the wave field, and the second is related to the plasma inhomogeneity. If the inhomogeneity scale is $\mathcal{L}$, the second term is of the order of $mv_{\perp}(\lambda_r/\mathcal{L})$, where λ_r is the Larmor radius. According to our assumption, this term is small compared with $mv_{\perp}$. In contrast to $p_{\|}$, the quantity μ differs from $mv_{\perp}^2/2|\Omega|$ only by terms related to the wave field. We see that in a homogeneous plasma and in the absence of a wave field, the quantities $p_{\|}$, μ and φ are equal to the particle parallel momentum, transverse (first) adiabatic invariant, and gyrophase, respectively. In an inhomogeneous plasma in the presence of a wave field $p_{\|}$, μ and φ oscillate around the corresponding values of parallel momentum, transverse adiabatic invariant, and gyrophase. However, we will still call the quantities $p_{\|}$, μ and φ determined by relations (3.31) in the same way, on the understanding that this will not lead to any confusion.

As we will see later, the variations of the quantities $p_{\|}$, μ and φ due to wave–particle interactions for resonant particles is, in fact, much larger than the corresponding additional terms in (3.31). Thus these terms are not essential in the solutions. This by no means implies that we can neglect these terms in the Hamiltonian (which would lead to absurd results) since any incorrect omission in the Hamiltonian leads to accumulative errors.

Having established the physical meaning of the new canonical variables (s, $p_{\|}$) and (φ, μ) determined by relations (3.26)–(3.28), we turn to the Hamiltonian (3.24) itself, which in the new variables takes the form:

$$\begin{aligned} H(t;p_{\|},s;\mu,\varphi) = {} & \frac{p_{\|}^2}{2m} + \mu|\Omega| + \frac{p_{\|}\mu}{m}\frac{1}{|\Omega|h_{\mathrm{L}}^2}\frac{d(|\Omega|h_{\mathrm{L}}^2)}{ds}\sin\varphi\cos\varphi \\ & - q\left[\left(\frac{2\mu|\Omega|}{m}\right)^{1/2}\cos\varphi\frac{\mathcal{A}_{\mathrm{L}}}{c} - \mathrm{sign}(q)\left(\frac{2\mu|\Omega|}{m}\right)^{1/2}\sin\varphi\frac{\mathcal{A}_{\Phi}}{c} + \frac{p_{\|}}{m}\frac{\mathcal{A}_{\mathrm{M}}}{c}\right], \end{aligned} \tag{3.32}$$

where, as before, we have neglected terms of second order in the wave vector potential, as well as those of second order in the inverse inhomogeneity scale, i.e. the terms $\propto \mathcal{L}^{-2}$. Quantitative criteria for these omissions to be valid are given by the following inequalities:

$$\frac{l^2}{2}\frac{d(|\Omega|h_{\mathrm{L}}^2)}{ds} \ll \frac{p_{\|}}{m}; \quad \frac{q\mathcal{A}}{c} \ll \min[p_{\|}, (2m\mu|\Omega|)^{1/2}], \tag{3.33}$$

which, in terms of physical parameters, read:

$$v_{\perp}\frac{\lambda_r}{\mathcal{L}} \ll v_{\|}; \quad \frac{q\mathcal{E}}{m\omega} \ll \min[v_{\|}, v_{\perp}]. \tag{3.34}$$

We should notice one feature of the Hamiltonian (3.32), namely that the Hamiltonian of the interaction depends on the sign of the charge in an explicit way. However, the unperturbed Hamiltonian, i.e. that in the absence of the wave field, does not depend on the sign of the particle charge. Also, charged particles of different sign rotate in the ambient magnetic field in opposite directions; this reveals itself as different relative phases of the variation between L and Φ, or v_L and v_Φ, for electrons and ions.

To define the Hamiltonian (3.32) completely, we must specify the orthogonal components $\mathcal{A}_{\mathrm{L}}, \mathcal{A}_{\Phi}, \mathcal{A}_{\mathrm{M}}$ of the wave vector potential. Using the relation (3.6) which expresses the wave electric field through the wave vector potential, we can easily show that the vector potential $\boldsymbol{\mathcal{A}}$ corresponding to the electric field (2.1), (2.21) may be chosen in the form

$$\boldsymbol{\mathcal{A}} = Re\{\boldsymbol{a}A(t, L, M)e^{i\psi(L,M)-i\omega t}\} \tag{3.35}$$

with the same polarization vector $\boldsymbol{a}$ as for the wave electric field. As we assume that the field does not depend on the azimuth angle Φ, the amplitude A and the phase ψ of the vector potential are functions of only two coordinates L and M.

We have seen above that the quantity $\boldsymbol{a}$ has the sense of a local polarization vector, and is determined in the reference frame where the ambient magnetic field is directed along the z-axis, and the wave normal vector lies in the (x, z)-plane. Thus, we have the following correspondence between the curvilinear coordinates L, Φ, M and the local Cartesian coordinate frame x, y, z in which the polarization vector is determined: $x \Leftrightarrow L$; $y \Leftrightarrow \Phi$; $z \Leftrightarrow M$ and, thus, $a_{\mathrm{L}} = a_x$; $a_\Phi = a_y$; $a_{\mathrm{M}} = a_z$. Since, as we know, only the relative phases between complex polarization coefficients are important, without any loss of generality we may choose a_{L} to be real. From the general expressions for wave polarization coefficients in a cold magnetized plasma (see, for instance, Shafranov 1967), it then follows that a_{M} is also real, while a_Φ is purely imaginary. We have already mentioned and used these properties of the polarization coefficients in Subsection 2.3 when deriving the equation for the nonlinear phase.

We now expand the phase $\psi(L, M)$ around $L_{p\Phi}$ determined in (3.19)

$$\psi(L, M) = \psi(L_{p_\Phi}, M) + \frac{\partial\psi}{\partial L}(L - L_{p_\Phi}). \tag{3.36}$$

By definition, the local wave normal vector $\mathbf{k} = \nabla\,\psi$. Using the expression (B.6) for the gradient in curvilinear coordinates we may write

$$k_{\perp} = \frac{1}{h_{\mathrm{L}}}\frac{\partial\psi}{\partial L}; \quad k_{\|} = \frac{1}{h_{\mathrm{M}}}\frac{\partial\psi}{\partial M}. \tag{3.37}$$

Taking into account (3.37) and the equalities $L - L_{p\Phi} = l$, $h_{\mathrm{M}}dM = ds$, we can rewrite the expression (3.36) for the phase ψ in the form:

$$\psi(L, M) = \int^{s} k_{\|}(s')ds' + k_{\perp}h_{\mathrm{L}}l + \frac{\pi}{2}. \tag{3.38}$$

Since the phase of the field is determined to an arbitrary constant, the lower limit of the integral in (3.38) need not be specified. The additional item $\pi/2$ in (3.38) is chosen for convenience of writing exact solutions later on.

Substituting (3.35), (3.38), and (3.27) in (3.32) we obtain

$$H(t;p_\parallel,s;\mu,\varphi)=\frac{p_\parallel^2}{2m}+\mu|\Omega|+\frac{p_\parallel\mu}{m}\frac{1}{|\Omega|h_\mathrm{L}^2}\frac{d(|\Omega|h_\mathrm{L}^2)}{ds}\sin\varphi\cos\varphi+q|A|\times$$
$$\times\left[\left(\frac{2\mu|\Omega|}{mc^2}\right)^{1/2}a_\mathrm{L}\cos\varphi\sin\zeta+\mathrm{sign}(q)\left(\frac{2\mu|\Omega|}{mc^2}\right)^{1/2}ia_\Phi\sin\varphi\cos\zeta+\frac{p_\parallel}{mc}a_\mathrm{M}\sin\zeta\right] \tag{3.39}$$

where

$$\zeta=\int^{s}k_\parallel(s')ds'+k_\perp\left(\frac{2\mu}{m|\Omega|}\right)^{1/2}\sin\varphi-\omega t+\phi, \tag{3.40}$$

and ϕ is the phase of the complex amplitude of the wave vector potential

$$A=|A|e^{i\phi}. \tag{3.41}$$

As was mentioned above, we have omitted the second order terms in $\boldsymbol{\mathcal{A}}$ and the inverse scale of the inhomogeneity in the Hamiltonian. For clarity, we will write in an explicit form the expressions for the real components of the wave vector potential (3.35) expressed through the variable ζ which we used above (remembering that a_Φ is purely imaginary):

$$\mathcal{A}_\mathrm{L}=-a_\mathrm{L}|A|\sin\zeta;\quad \mathcal{A}_\Phi=ia_\Phi|A|\cos\zeta;\quad \mathcal{A}_\mathrm{M}=-a_\mathrm{M}|A|\sin\zeta. \tag{3.42}$$

The corresponding components of the wave electric field $\mathcal{E}=c^{-1}\partial\mathcal{A}/\partial t$ are

$$\mathcal{E}_\mathrm{L}=-\frac{\omega}{c}a_\mathrm{L}|A|\cos\zeta;\quad \mathcal{E}_\Phi=-i\frac{\omega}{c}a_\Phi|A|\sin\zeta;\quad \mathcal{E}_\mathrm{M}=-\frac{\omega}{c}a_\mathrm{M}|A|\cos\zeta. \tag{3.43}$$

We see that small terms in the Hamiltonian are of a different nature: one is explicitly related to the plasma inhomogeneity and is proportional to the derivative of $|\Omega|\,h_\mathrm{L}^2$ along the field line, but does not depend on the wave field, and others, which are proportional to the wave vector potential, do not contain derivatives of the plasma parameters. As we will see below, these terms are of different character from the viewpoint of resonant wave–particle interactions.

The next transformation of the Hamiltonian which reveals resonance effects in wave–particle interactions is based on the well known expansion and functional relations between Bessel functions and their derivatives:

$$\begin{gathered}\exp(i\rho\sin\varphi)=\sum_{n=-\infty}^{\infty}J_n(\rho)\exp(in\varphi);\\ J_{n-1}(\rho)+J_{n+1}(\rho)=(2n/\rho)J_n(\rho);\quad J_{n-1}(\rho)-J_{n+1}(\rho)=2J_n'(\rho),\end{gathered} \tag{3.44}$$

where $J_n(\rho)$ is Bessel function of integer order n, ρ is a real parameter, and $J_n{}'(\rho)$ is the derivative of the Bessel function with respect to argument ρ. Using (3.44), the following presentation of the Hamiltonian (3.39) can be derived

$$H(t;p_\parallel,s;\mu,\varphi)=\frac{p_\parallel^2}{2m}+\mu|\Omega|+\frac{p_\parallel\mu}{m}\frac{1}{|\Omega|h_\mathrm{L}^2}\frac{d(|\Omega|h_\mathrm{L}^2)}{ds}\sin\varphi\cos\varphi+q|A|\sum_{n=-\infty}^{\infty}V_n\sin\xi_n, \tag{3.45}$$

where

$$
\begin{aligned}
V_n &= \left(\frac{n|\Omega|}{k_\perp c}a_{\mathrm{L}} + \frac{p_\parallel}{mc}a_{\mathrm{M}}\right)J_n(\rho) + \frac{i\rho\Omega}{k_\perp c}a_\Phi J_n'(\rho) \\
\rho &= k_\perp\left(\frac{2\mu}{m|\Omega|}\right)^{1/2}; \quad \xi_n = \int^s k_\parallel ds' + n\varphi - \omega t + \phi.
\end{aligned}
\tag{3.46}
$$

As we have seen above, the quantity μ differs from the transverse adiabatic invariant $mv_\perp^2/2\ |\Omega|$ by small terms related to the wave vector potential, thus

$$\rho \simeq \frac{k_\perp v_\perp}{|\Omega|}. \tag{3.47}$$

As follows from the derivation, the expressions (3.45) for the Hamiltonian and (3.46) for the effective amplitude of interaction V_n are valid for any electromagnetic (not necessarily whistler-mode) wave, the only difference being in the expressions for the polarization coefficients. We will give the expressions for V_n in two particular cases, namely, for wave propagation along the ambient magnetic field (parallel propagation), and for wave propagation in the quasi-resonance regime for whistler-mode waves. In the first case, $\rho \propto k_\perp \to 0$, and we may use asymptotic relations for the Bessel functions

$$J_n(\rho) \simeq \frac{1}{n!}\left(\frac{\rho}{2}\right)^n, \quad (\rho \to 0,\ n \geq 0), \tag{3.48}$$

as well as

$$J_{-n}(\rho) = (-1)^n J_n(\rho),$$

along with the functional relations (3.44), of course. Then, choosing $a_{\mathrm{L}} > 0$ (which we may always do) and taking into account that for a parallel propagating whistler-mode wave $a_{\mathrm{M}} = 0$, $a_\Phi \equiv ia_{\mathrm{L}}$, from (3.46) we find that for electrons ($\Omega < 0$) V_n is equal to zero for all n except $n = 1$, while

$$V_1 = a_{\mathrm{L}}\left(\frac{2\mu|\Omega|}{mc^2}\right)^{1/2} \simeq a_{\mathrm{L}}\frac{v_\perp}{c}. \tag{3.49}$$

For protons, or any positive ions with $\Omega > 0$, we have $V_n = 0$ for $n \neq -1$, and

$$V_{-1} = -a_{\mathrm{L}}\left(\frac{2\mu|\Omega|}{mc^2}\right)^{1/2} \simeq -a_{\mathrm{L}}\frac{v_\perp}{c}. \tag{3.50}$$

For quasi-resonance waves when $\boldsymbol{a} \parallel \mathbf{k}$ we have $a_\Phi = 0$; $a_{\mathrm{M}} = a_{\mathrm{L}}(k_\parallel/k_\perp)$, and thus

$$V_n = a_{\mathrm{L}}\frac{n|\Omega| + k_\parallel p_\parallel/m}{k_\perp c}J_n(\rho) \simeq a_{\mathrm{L}}\frac{n|\Omega| + k_\parallel v_\parallel}{k_\perp c}J_n(\rho), \tag{3.51}$$

the most natural choice for a_{L}, a_{M} being

$$a_{\mathrm{L}} = \frac{k_\perp}{k}; \quad a_{\mathrm{M}} = \frac{k_\parallel}{k}. \tag{3.52}$$

We have given the expressions for V_n through the particle transverse velocity $v_\perp$ and parallel velocity $v_\parallel$ since these quantities are sometimes more convenient to use. Moreover, as we neglected the second order terms in $\boldsymbol{\mathcal{A}}$ and $\mathcal{L}^{-1}$ in the Hamiltonian, in the expressions for V_n we should not distinguish between $p_\parallel$ and $mv_\parallel$, and between μ and

$mv_\perp^2/2|\Omega|$. However, we should bear in mind that neither $v_\parallel$ nor $v_\perp$ are canonical variables; thus, when deriving equations from the Hamiltonian, or in any canonical transformations, only $p_\parallel$ and μ should be used.

3.4 Resonant Effects in Wave–Particle Interactions

As we will see below, the third term in the Hamiltonian (3.45) proportional to $\sin\varphi\cos\varphi$ does not contribute to resonant wave–particle interactions and, thus, may be omitted if we are interested in resonance effects. Then, the equations of motion derived from the Hamiltonian (3.45) take the form:

$$\begin{aligned}
&\frac{ds}{dt}=\frac{p_\parallel}{m}+q|A|\sum_{n=-\infty}^{\infty}\frac{\partial V_n}{\partial p_\parallel}\sin\xi_n; \quad \frac{dp_\parallel}{dt}=-\mu\frac{d|\Omega|}{ds}-q|A|k_\parallel\sum_{n=-\infty}^{\infty}V_n\cos\xi_n;\\
&\frac{d\varphi}{dt}=|\Omega|+q|A|\sum_{n=-\infty}^{\infty}\frac{\partial V_n}{\partial\mu}\sin\xi_n; \quad \frac{d\mu}{dt}=-q|A|\sum_{n=-\infty}^{\infty}nV_n\cos\xi_n.
\end{aligned} \tag{3.53}$$

Using the definitions of the quantities $p_\parallel$ and V_n one can easily check that $ds/dt=v_\parallel$ which, however, is not a canonical variable. We will also write the equation for the variation of the Hamiltonian itself, which, as we know, coincides with the equation for the particle kinetic energy W

$$\frac{dW}{dt}=-q|A|\omega\sum_{n=-\infty}^{\infty}V_n\cos\xi_n. \tag{3.54}$$

The variation of charged particle energy strongly depends on the behavior of the quantities ξ_n, in particular, their full derivatives over time along the particle trajectory, $\dot{\xi}_n$, which are determined by Eqs. (3.53) and in the lowest approximation are equal to

$$\dot{\xi}_n(t)\equiv\frac{d}{dt}\left(\int^s k_\parallel(s')ds'+n\varphi-\omega t\right)\simeq k_\parallel v_\parallel+n|\Omega|-\omega. \tag{3.55}$$

For a given particle, the quantity $\dot{\xi}_n$ may be close to zero for one particular n, which corresponds to the n-th cyclotron resonance between the wave and the particle:

$$k_\parallel v_\parallel+n|\Omega|-\omega=0. \tag{3.56}$$

In this case, the corresponding term in (3.54), being a slowly varying one, brings the main contribution to the particle energy variation. For non-resonant particles, for which all $\dot{\xi}_n$ are far from zero, the variation of kinetic energy has the character of fast oscillations and is much smaller in magnitude than the energy variation of the resonant particles.

Let us take a more careful look at the resonance conditions. They may be written in another form

$$v_\parallel=v_{\mathrm{R}n}\equiv\frac{\omega-n|\Omega|}{k_\parallel}, \tag{3.57}$$

which expresses the particle parallel velocity through the wave and plasma parameters, as well as the order n of the cyclotron resonance, of course. The quantity $v_{\mathrm{R}n}$ is the resonance velocity corresponding to the n-th cyclotron resonance. The special nature of the resonance interactions in an inhomogeneous plasma is that the quantities $v_\parallel$, $|\Omega|$, and $k_\parallel$ vary along a particle trajectory, even in the absence of a wave field. Thus, a non-resonant particle may become a resonant one, and vice versa, and, in principle, a particle may cross multiple cyclotron resonances. However, at any instant of time, a particle can only be in one exact resonance with a wave. If it is the n-th cyclotron resonance, then the n-th term in the Hamiltonian (3.45) is slowly varying, which in fact is the definition of the n-th resonance [see (3.55), (3.56)], while the closest terms in the Hamiltonian (3.45) oscillate with the frequency $|\Omega|$.

The question which arises is how should we deal with the equations of motion for resonant particles. We can again direct our efforts to a correct calculation of the particle energy variation, and use it as a guide in solving the whole set of equations. First of all, from the remark made above it is clear that, in an inhomogeneous plasma, it might be impossible to treat the motion of a given particle uniquely on an infinite time scale, at least because its attribute as a resonant or non-resonant particle may be changed. Thus, let us look for a solution of the equations of motion on a finite time scale t_{NL} which, however, is much larger than the interval of validity of the linear solution. This definition explains the reason why t_{NL} is called nonlinear time. Obviously, t_{NL} should be much larger than the wave period $2\pi/\omega$ which is the smallest time scale of the problem. The exact definition of the nonlinear time depends on the wave and plasma parameters, and should be specified in each particular case.

Imagine now that we integrate equation (3.54) over a time of the order of t_{NL}, for a particle in the n-th cyclotron resonance with the wave. The main contribution to the energy variation comes from slowly varying terms, first of all from the n-th term. If during the time t_{NL}, all terms in the Hamiltonian, except the n-th one, are rapidly varying, we may expect that the contribution from these terms to the energy variation is small on the average and, thus, may be neglected. Let us find the conditions for this requirement to be fulfilled. Evidently, the terms closest to the n-th one, i.e. the $n \pm 1$-th terms, are the most "dangerous", since particles satisfying $\dot{\xi}_n = 0$ oscillate most slowly as compared to all but the n-th term. From the definition (3.55) of the quantity $\dot{\xi}_n$ we can write

$$\dot{\xi}_{n\pm1} = \dot{\xi}_n \pm |\Omega|.$$

Thus, in order for the contribution to the energy variation from the n-th term to be much larger than from the $n \pm 1$-th ones, it is necessary that during the time t_{NL}

$$|\dot{\xi}_n| \ll |\Omega|. \tag{3.58}$$

Since at exact resonance $\dot{\xi}_n = 0$, the condition above is equivalent to

$$|\Delta\dot{\xi}_n| \ll |\Omega|, \tag{3.59}$$

where $\Delta\dot{\xi}_n$ is the variation of the quantity $\dot{\xi}_n$ in the resonance region. If the requirement (3.58) [or (3.59)] is fulfilled, the resonances are in a sense isolated, and, for particles in the n-th resonance with the wave, we may retain in the Hamiltonian only the n-th slowly varying term. Since such a procedure is equivalent to averaging the rapidly varying terms in the Hamiltonian over the nonlinear time, this method of solving the equations of motion is called the averaging method. Its strict justification can be found, for example, in Hinch (1991).

If condition (3.58) is violated, then several terms in the Hamiltonian contribute simultaneously to the variation of particle energy and other quantities, i.e. the coordinates and momenta, during the time t_{NL}. As has been shown by Chirikov and other authors (see, for example, Chirikov 1979), in this case the particle motion becomes stochastic in the nonlinear time. Accordingly, the condition opposite to (3.59), i.e.

$$|\Delta\dot{\xi}_n| \gtrsim |\Omega| \tag{3.60}$$

is called Chirikov's criterion of stochasticity.

The conditions (3.59), (3.60) as presented above are formulated in a general qualitative way. They will be specified and defined more precisely later on. We should emphasize that, although Chirikov's criterion of stochasticity has initially been derived for a homogeneous plasma, in the terms formulated above it is valid for both homogeneous and inhomogeneous cases. However, this criterion is related to the particle motion only on the nonlinear time scale. As we will see further, in an inhomogeneous plasma, the particle motion in phase space on a larger time scale may have a diffusive character, even when the condition (3.59) is fulfilled.

Along with the resonance velocity, an important characteristic of wave–particle interactions is the distance between resonances in velocity space $v_{\|}$, i.e. the quantity

$$v_{\mathrm{Rn}\pm 1} - v_{\mathrm{Rn}} = \mp\frac{|\Omega|}{k_{\|}}. \tag{3.61}$$

We should stress that, in an inhomogeneous plasma, both resonance velocity v_{Rn} and the distance between resonances $\mp |\Omega|/k_{\|}$ are local quantities, and, for a given wave and the order of resonance n, these quantities depend on the coordinate s, i.e. they vary along the ambient magnetic field line. From (3.57) and (3.61) we see that, in order for the resonances to be separated, it is necessary that the variation of the particle parallel velocity in the resonance region is much less than the distance between resonances (3.61):

$$|\Delta v_{\|}| \ll \frac{|\Omega|}{k_{\|}}. \tag{3.62}$$

The inequality (3.62) is similar, but not identical, to conditions of isolated resonances (3.58), (3.59), since it deals only with the variation of $v_{\|}$, while (3.58) and (3.59) take into account the variation of other parameters which determine the resonance conditions, and which also vary in an inhomogeneous plasma. This difference is especially pronounced in the case of quasi-perpendicular propagation $(k_{\|} \to 0)$ when the condition (3.62) becomes meaningless, while (3.58) and (3.59) still make sense. This consideration already shows that a treatment of the case $k_{\|} \to 0$ requires special attention. The peculiarity of the case $k_{\|} \to 0$ is just related to the fact that, for $k_{\|} \to 0$, the resonance conditions determine the particle position on the field line of the ambient magnetic field, but not its parallel velocity. In this case, the variation of the effective amplitude of interaction V_n in the resonance region, as well as amplitude dependent terms in the equation for the phase φ which are not taken into account in (3.55), become important. Later on, we will always specify exactly when the consideration is restricted to the case $k_{\|} \not\to 0$, and when it is pertinent to arbitrary $k_{\|}$. We should notice that, for whistler-mode waves with frequencies not too close to the lower hybrid frequency, the wave normal angle θ cannot approach $\pi/2$ and, thus, $k_{\|}$ is always finite.

The previous consideration has been relevant for both electrons and ions. Further in this Section, we will study whistler-mode wave resonant interactions with electrons. Two important factors facilitate the consideration in this case. Firstly, for typical plasma parameters and wave amplitudes, the distance between resonances in units of parallel velocities $v_{\|}$, which is equal to $|\Omega|/k_{\|}$, exceeds the nonlinear resonance width $\Delta v_{\|}$ essentially, and, as a rule, exceeds the "thermal" velocity of the energetic component of the electron distribution participating in the resonance interaction with the wave (see quantitative estimates below). Therefore, for electrons, the approximation of isolated resonances is always valid and, moreover, only two resonances, corresponding to the smallest magnitudes of the resonant velocities, namely, the first cyclotron resonance

$$v_{\mathrm{R1}} = \frac{\omega - \omega_{\mathrm{c}}}{k_{\|}} \tag{3.63}$$

and the Cerenkov (sometimes also called the Landau) resonance

$$v_{\mathrm{R0}} = \frac{\omega}{k_{\|}} \tag{3.64}$$

play the main role. Here and further on, we denote by ω_{c} the magnitude of the electron cyclotron frequency in the relations which are specific for electrons. Since the distribution function, as a rule, drops rapidly with increasing particle parallel velocity, and, thus, with increasing particle energy, other resonances, which correspond to larger values of the resonant energy, do not contribute essentially to particle interactions with the wave. Thus, we can limit ourselves to considerations of two resonances: $n = 1$ and $n = 0$ for electrons. We should stress that, in the case of protons, when the distance between resonances $\Omega_{ci}/k_{\|}$ is small, the situation is significantly changed. Namely, higher order resonances for which $\omega - n\Omega_{ci} \simeq 0$ become the most important. Proton interactions with whistler-mode waves are studied in Sect. 5.

3.5 Wave–Particle Interactions in the Approximation of Isolated Resonances

Let us proceed with the investigation of electron interactions with a VLF radio wave. In the case of the n-th isolated cyclotron resonance this interaction is described by the Hamiltonian [see (3.45)]

$$H_n(t; p_{\|}, s; \mu, \varphi) = \frac{p_{\|}^2}{2m} + \mu|\Omega| + q|A|V_n \sin\left(\int^{s} k_{\|} ds' + n\varphi - \omega t + \phi\right). \tag{3.65}$$

Obviously, the third term in the Hamiltonian (3.45), which oscillates with the frequency $2|\Omega|$, does not contribute to resonance wave–particle interactions, and may be omitted along with other non-resonance terms. The corresponding equations of motion take the form

$$\begin{aligned} \frac{ds}{dt} &= \frac{p_{\|}}{m} + q|A|\frac{\partial V_n}{\partial p_{\|}} \sin\xi_n; \quad & \frac{dp_{\|}}{dt} &= -\mu\frac{d|\Omega|}{ds} - q|A|k_{\|}V_n \cos\xi_n; \\ \frac{d\varphi}{dt} &= |\Omega| + q|A|\frac{\partial V_n}{\partial \mu} \sin\xi_n; \quad & \frac{d\mu}{dt} &= -q|A|nV_n \cos\xi_n \end{aligned} \tag{3.66}$$

In the approximation of isolated resonances, the total derivative of the Hamiltonian over time, which coincides with the equation for the particle kinetic energy, is equal to

$$\frac{dH_n}{dt} \equiv \frac{dW}{dt} = \frac{\partial H_n}{\partial t} = -q|A|\omega V_n \cos\xi_n. \tag{3.67}$$

From (3.66) and (3.67) it follows that the quantity

$$C_n \equiv nW - \omega\mu = \mathrm{const.} \tag{3.68}$$

is the constant of the motion. This integral of motion is the consequence of the fact that the variables φ and t enter the Hamiltonian (3.65) only in the combination $n\varphi - \omega t$ and, in fact, can be obtained from the general equations $dH_n/dt = \partial H_n/\partial t$ and $d\mu/dt = -\partial H_n/\partial\varphi$, without writing the equations in an explicit form.

The problem described by the Hamiltonian (3.65), and the corresponding Eqs. (3.66), is two dimensional. The existence of the additional integral of motion (3.68) permits us to reduce the problem to a one-dimensional set of equations. For that, it is sufficient to choose (ξ_n, W) as new variables, the coordinate s as a new independent variable, and exclude $p_\|$ from the equations of motion with the help of the integral (3.68). Straightforward but somewhat cumbersome calculations given in Appendix C lead to the following one dimensional Hamiltonian

$$\widetilde{H}_n(s, \xi_n, W) = [k_\|(s)W - \omega p_0(s, W; C_n)] + \frac{m\omega q|A|}{p_0(s, W; C_n)} V_n \sin\xi_n, \tag{3.69}$$

where ξ_n and W play the roles of canonical coordinate and momentum, respectively, s is an independent variable similar to time, C_n is a constant parameter of the problem, and

$$p_0 = \sqrt{2m\left[\left(1 - \frac{n\omega_c}{\omega}\right)W + \frac{\omega_c}{\omega}C_n\right]}\,\mathrm{sign} p_\|. \tag{3.70}$$

For $k_\| \nrightarrow 0$, and under some additional conditions which will be specified below, the Hamiltonian (3.69) can be reduced to a standard form, i.e. to a sum of virtual kinetic and potential energies. For this aim, it is convenient to change to new independent variable $\tilde{t}$ which is a simple function of s:

$$\tilde{t} = \int^s \frac{k_\|^2(s')ds'}{m\omega v_{\mathrm{R}n}(s')}, \tag{3.71}$$

that results in the Hamilton transformation to

$$\mathcal{H} = \mathcal{H}_0 + \frac{m^2\omega^2 v_{\mathrm{R}n} q|A|}{k_\|^2 p_0(s, W; C_n)} V_n \sin\xi_n; \quad \mathcal{H}_0 = \frac{m\omega v_{\mathrm{R}n}}{k_\|^2(s)}[k_\|(s)W - \omega p_0(s, W; C_n)]. \tag{3.72}$$

The equality $\partial\mathcal{H}_0/\partial W = 0$ determines the resonance value of the energy W_R as a function of $\tilde{t}$ and the parameter C_n:

$$k_\| - \frac{m(\omega - n\omega_c)}{p_0} = 0 \Rightarrow W = W_\mathrm{R} \equiv \frac{m\omega v_{\mathrm{R}n}^2 - 2\omega_c C_n}{2(\omega - n\omega_c)}. \tag{3.73}$$

For $W = W_\mathrm{R}$, $p_0 = mv_{\mathrm{R}n}$ [see (3.73)]. Expanding $\mathcal{H}_0$ around W_R to quadratic terms, and putting $W = W_\mathrm{R}$ into the second term in (3.72), which is already proportional to the wave amplitude, we obtain

$$\mathcal{H} = \frac{1}{2}(W - W_\mathrm{R})^2 + \frac{m\omega^2 q|A|}{k_\|^2} V_n(W_\mathrm{R}) \sin\xi_n. \tag{3.74}$$

The domain of applicability of (3.74) is determined by inequalities

$$\frac{k_\|\Delta(W - W_\mathrm{R})}{m\omega v_{\mathrm{R}n}} \ll 1; \quad \left(\frac{\partial V_n}{\partial W}\right)_{C_n} \cdot \Delta(W - W_\mathrm{R}) \ll V_n. \tag{3.75}$$

Explicit expressions for the inequalities (3.75), as well as for (3.59), depend on the relation between inhomogeneity and nonlinearity, and will be given later.

The Hamiltonian (3.74) has a standard form in the sense that it consists of a sum of "kinetic energy" $(W - W_{\mathrm{R}})/2$ and "potential energy" $(m\omega^2 q|A|/k_{\|}^2)V_n(W_{\mathrm{R}})\sin\xi_n$. An essential feature, however, is that the effective kinetic energy is centered on the varying value W_{R}, while the effective potential energy depends on "time" $\tilde{t}$, which comes in through the quantity W_{R}.

The Hamiltonian (3.74) sets the following equations of motion

$$\begin{aligned} \frac{d\xi_n}{d\tilde{t}} &= W - W_{\mathrm{R}}, \\ \frac{dW}{d\tilde{t}} &= -\frac{m\omega^2 q|A|}{k_{\|}^2} V_n(W_{\mathrm{R}}) \cos\xi_n. \end{aligned} \tag{3.76}$$

The quantity W_{R} varies due to the inhomogeneity (in a homogeneous plasma, $W_{\mathrm{R}} = \mathrm{const.}$) and is characterized by its derivative, which we denote by α_n

$$\begin{aligned} \alpha_n \equiv \frac{dW_{\mathrm{R}}}{d\tilde{t}} &= \frac{m\omega^2}{k_{\|}^3}\left(\frac{nW_{\mathrm{R}} - C_n}{\omega}\frac{d\omega_{\mathrm{c}}}{ds} + \frac{m}{2}\frac{dv_{\mathrm{Rn}}^2}{ds}\right) \\ &\equiv \frac{m\omega^2}{k_{\|}^3}\left(\mu_{\mathrm{R}}\frac{d\omega_{\mathrm{c}}}{ds} + \frac{m}{2}\frac{dv_{\mathrm{Rn}}^2}{ds}\right), \end{aligned} \tag{3.77}$$

where μ_{R} is the resonance value of μ related to W_{R} by (3.68). The quantity α_n is an inhomogeneity parameter for the n-th cyclotron resonance which plays an important role in the theory of wave–particle interactions in an inhomogeneous plasma.

The solution of the equations of motion (3.76) strongly depends on the relation between inhomogeneity and nonlinearity. It can be seen most clearly if we transform to the new momentum w

$$w = (W - W_{\mathrm{R}}), \tag{3.78}$$

so that the basic set of equations of motion take the form

$$\begin{aligned} \frac{d\xi_n}{d\tilde{t}} &= w, \\ \frac{dw}{d\tilde{t}} &= \beta_n \cos\xi_n - \alpha_n, \end{aligned} \tag{3.79}$$

where

$$\beta_n = -\frac{m\omega^2 q|A|}{k_{\|}^2} V_n(W_{\mathrm{R}}). \tag{3.80}$$

For brevity, we will further omit the subscript "n" in ξ_n, α_n, and β_n when the analysis concerns one particular resonance. The Hamiltonian h from which Eqs. (3.79) can be derived

$$h = \frac{w^2}{2} + \alpha\xi - \beta\sin\xi \tag{3.81}$$

is already a sum of virtual kinetic energy $w^2/2$ and potential energy $\alpha\xi - \beta\sin\xi$. This potential has a qualitatively different character for $|\alpha| < |\beta|$ and $|\alpha| > |\beta|$ (see Fig. 3). In particular, when $|\alpha| < |\beta|$, the potential has potential troughs, while for $|\alpha| > |\beta|$ the

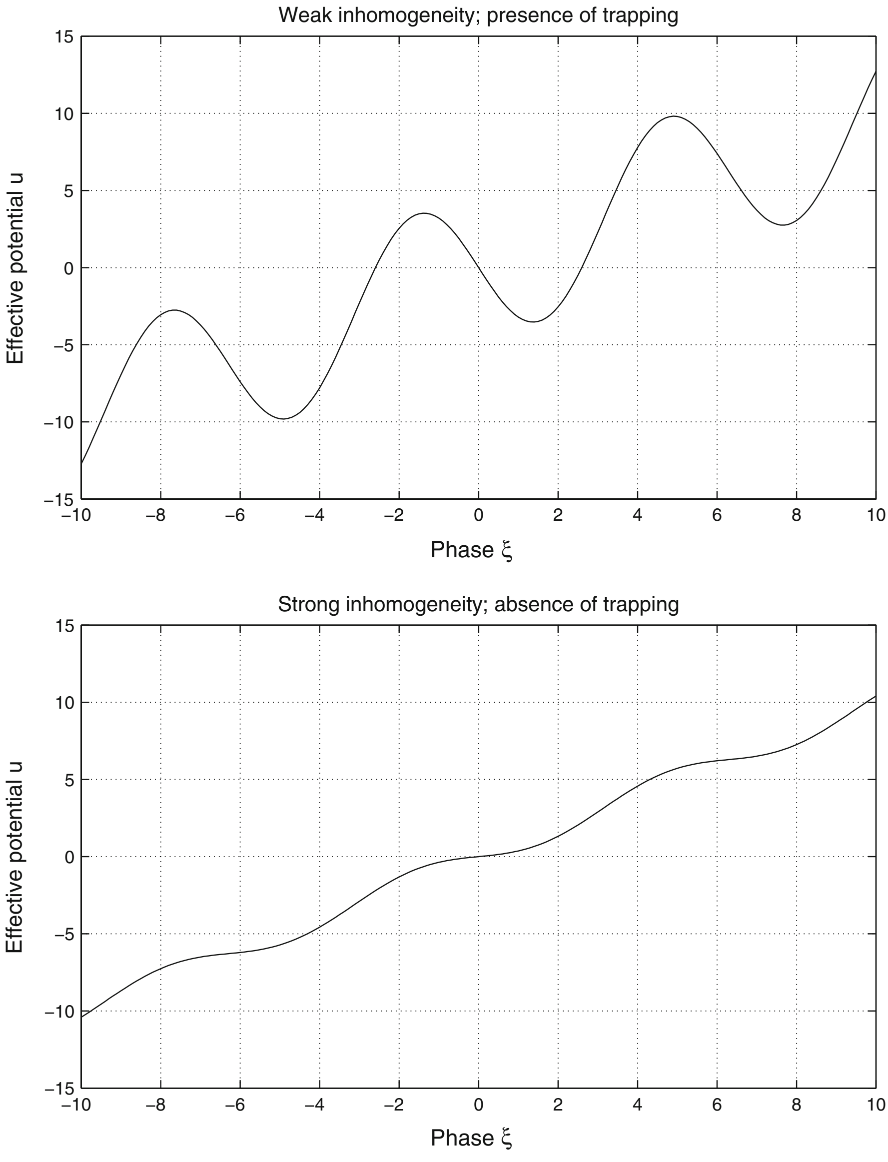

Fig. 3 Effective potential of the basic equation

potential is a monotonic function of the coordinate ξ. We will refer to these cases as to the cases of weak and strong inhomogeneity, respectively.

We see that exact resonance corresponds to $w = 0$. On the other hand, as we know, resonance conditions are determined by the equality $v_{\|} = v_{\mathrm{Rn}}$. Thus, close to resonance, the quantity w should be proportional to $v_{\|} - v_{\mathrm{Rn}}$. Writing w as $W - W_{\mathrm{R}}$ and using the definitions (3.73) and (C.6) of the quantities W_{R} and C_n, respectively, we find that under the conditions (3.75)

$$w = \frac{m\omega(v_{\|} - v_{\mathrm{Rn}})}{k_{\|}}. \tag{3.82}$$

3.6 Basic Equations of Motion in the Field of a Quasi-Monochromatic Wave in an Inhomogeneous Plasma

Equations (3.79) may be written as an ordinary second order differential equation

$$\frac{d^2\xi}{d\tilde{t}^2} = \beta\cos\xi - \alpha, \tag{3.83}$$

which describes the particle motion in the potential

$$u = \alpha\xi - \beta\sin\xi. \tag{3.84}$$

As was mentioned above, this potential has a different character depending on the ratio $|\beta/\alpha|$ (see Fig.3), and we may expect that the wave–particle interaction is also quite different in the cases of strong ($|\alpha| > |\beta|$) and weak ($|\alpha| < |\beta|$) inhomogeneity. Since Eq. (3.83) [or Eqs. (3.79)] are the final equations to which the problem of particle motion in the field of a monochromatic wave is reduced, it plays an important role in the theory of wave–particle interactions in an inhomogeneous plasma. In this subsection, we will study this equation in detail.

We should notice that Eq. (3.83) up to notation coincides with the equation which describes particle dynamics in the field of a Langmuir wave in an isotropic, inhomogeneous plasma. Thus, the approximation when Eq. (3.83) is applicable could be called the approximation of Langmuir similarity. An essential difference, however, is that for whistler-mode waves, the coefficients α and β depend not only on wave and plasma parameters, but also on particle transverse velocity. Thus, speaking about strong and weak inhomogeneity, we should think not only of a typical wave amplitude and inhomogeneity scale, but also about a typical value of particle transverse velocity. Moreover, even for the validity of whistler-Langmuir similarity, the quantity β may have monotonic or oscillating character for $\rho \equiv k_\perp v_\perp/|\Omega| < 1$ and $\rho \equiv k_\perp v_\perp/|\Omega| \gg 1$, respectively.

3.6.1 Strong Inhomogeneity Case

We will start the investigation of the Eq. (3.83) from the case of a strong inhomogeneity, i.e. $|\alpha| > |\beta|$. As we will see below, in this case the time of resonance interaction is much shorter than the typical time of α and β variations, so that we may consider them constant. This can already be anticipated from a qualitative analysis of the particle motion based on the shape of the potential in the case $|\alpha| > |\beta|$. As the potential is a monotonic function and does not have potential wells, all particles are untrapped. Exact resonance $w = 0$ corresponds to particle reflection from the potential, after which $|w|$ increases rapidly, and a particle moves out of resonance.

For $\alpha = \text{const.}$, $\beta = \text{const.}$ equation (3.83) has the integral of the motion, which we denote as ε

$$\epsilon = \frac{w^2}{2} + \alpha\xi - \beta\sin\xi, \tag{3.85}$$

and which coincides with the Hamiltonian (3.81), of course. The factor which permits us to find an exact solution of Eq. (3.83) in the case of strong inhomogeneity is that the potential

u is a monotonoic function of the coordinate ξ, and thus the coordinate can be expressed as a function of potential. This solution was first found by Karpman and Shklyar (1975) (see, also, Karpman et al. 1975a) and has the form

$$w - w_0 + \alpha\tilde{t} = \sum_{l=1}^{\infty}\left(\frac{4\pi|\alpha|}{l}\right)^{1/2} J_l\left(\frac{l\beta}{|\alpha|}\right)\left\{\cos\left(\frac{l\epsilon}{|\alpha|}\right)\left[C\left(\sqrt{\frac{l}{2|\alpha|}}w_0\right) - C\left(\sqrt{\frac{l}{2|\alpha|}}w\right)\right]\right. \\ \left. + \sin\left(\frac{l\epsilon}{|\alpha|}\right)\left[S\left(\sqrt{\frac{l}{2|\alpha|}}w_0\right) - S\left(\sqrt{\frac{l}{2|\alpha|}}w\right)\right]\right\}, \tag{3.86}$$

where $J_l(z)$ is a Bessel function of integer order l, $C(\zeta)$, $S(\zeta)$ are Fresnel's integrals, and ε is determined in (3.85). The exact resonance $w = 0$ corresponds to particle reflection from the potential (3.84) and the change of sign of the velocity w. As follows from the solution of the equations of motion, in the case of a strong inhomogeneity, the time of the resonant interaction between particle and wave (in units of $\tilde{t}$) is $\Delta\tilde{t}_R \sim |\alpha|^{-1/2}$. The corresponding variation of the quantity w in the resonance region is $\Delta w \sim |\alpha|^{1/2}$.

3.6.2 *Weak Inhomogeneity Case* ($|\beta| \gg |\alpha|$)

For arbitrary $|\beta|/|\alpha|$, the solution of Eq. (3.83) has not yet been found, although a general expression for the wave growth (damping) rate has been obtained (see Karpman et al. 1975a for details). Here we will consider the case $|\beta| \gg |\alpha|$, which is less general, but much more illustrative, and gives a better understanding of wave–particle interactions in the case of a weak inhomogeneity.

As was mentioned above, the Hamiltonian (3.74) and the corresponding Eqs. (3.76) are equivalent to the Hamiltonian (3.81) and Eqs. (3.79); thus any of them may be used, the choice being a matter of convenience. The latter presentation clearly shows the existence of phase trapped particles in the case of a weak inhomogeneity. Moreover, for varying α and β, the phase volume (on the (ξ, w)-plane) of phase trapped particles varies. Therefore, there is an exchange between trapped and untrapped particles. This leads to quite complicated particle dynamics in this case. For the sake of simplicity, we will limit ourselves to a consideration of the simpler case α = const, β = const. The case of varying α and β has been discussed by Solov'ev and Shklyar (1986).

For constant α and β, the Hamiltonian (3.81) is independent of time and, thus, is a constant of the motion. This constant is given by the same expression (3.85) as in the case of a strong inhomogeneity, and has the meaning of particle total effective energy. However, the shape of the potential $\alpha\xi - \beta\sin\xi$ is qualitatively different. In particular, it now has potential wells. Accordingly, there are phase trapped particles, for which the phase ξ varies over a limited interval, as well as phase untrapped particles undergoing infinite motion. For brevity, we will omit the word "phase" when referring to these types of particles. Phase trajectories on the (ξ, w)-plane corresponding to the Hamiltonian (3.81) are shown in Fig. 4. Each trajectory on the plot corresponds to a particular value of effective energy ε.

For trapped particles, the average variation of the phase ξ is equal to zero. So, from (3.79), (3.76) we obtain

$$\overline{w} = \overline{W} - W_R = 0. \tag{3.87}$$

This relation may also be written in the form [see (3.77)]:

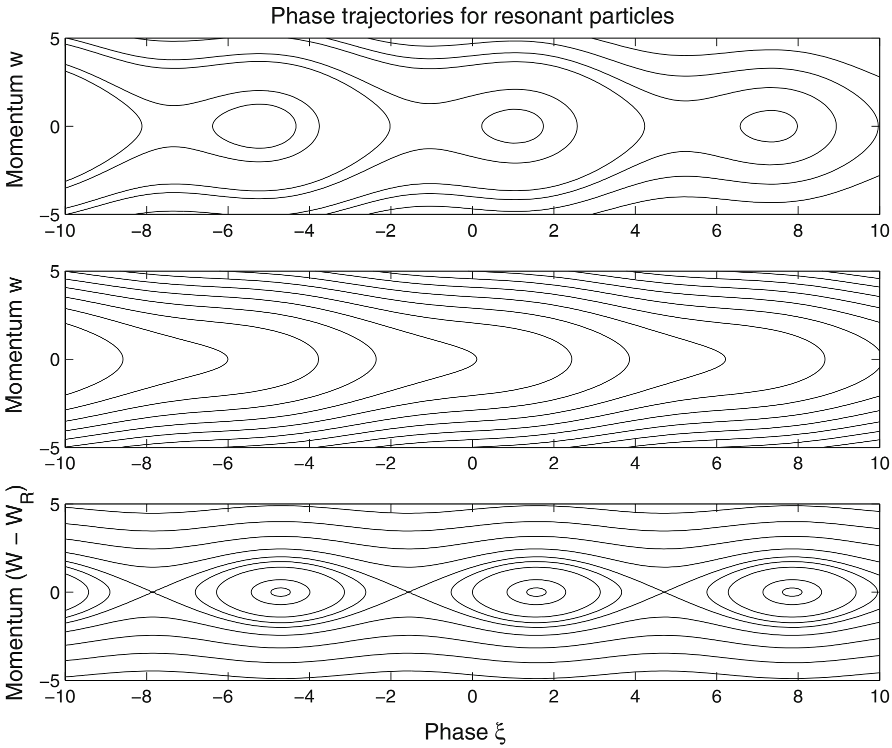

Fig. 4 Phase trajectories in various cases: weak inhomogeneity, the presence of trapping (*upper panel*); strong inhomogeneity, the absence of trapping (*middle panel*); phase trajectories corresponding to a homogeneous case (W_R = const.) that should be used for calculating the adiabatic invariants (*bottom panel*)

$$\frac{d\overline{W}}{d\tilde{t}} = \alpha. \tag{3.88}$$

Since $w \equiv W - W_R = 0$ is nothing but the resonance conditions, Eq. (3.87) means that trapped particles are always in resonance with the wave, i.e. for these particles $\overline{v_\parallel} = v_{Rn}$, which agrees with (3.82), of course. Taking into account the definition (3.71) of the quantity $\tilde{t}$ and the equality $ds/dt = v_{Rn}$ we find from (3.88) that the kinetic energy of trapped particles varies with time according to the equation

$$\frac{d\overline{W}}{dt} = \frac{k_\parallel^2}{m\omega}\alpha. \tag{3.89}$$

Notice that the sign of the energy variation for trapped particles coincides with the sign of α, and that the total variation of kinetic energy increases in magnitude with increasing time.

As one can see from the phase plot in Fig.4, for untrapped particles, the time of resonant interaction and, thus, the total variation of particle kinetic energy W is limited and in general is smaller than for trapped particles. This fact by no means implies that trapped particles play a more important role in wave–particle interactions than untrapped particles. The net variation of particle energy determines the deviation of the distribution function from an unperturbed value, while the interaction itself is characterized by the rate of energy variation, in which the phase trapped particles have no dominance. An important difference between trapped and untrapped particles is that the phase space of resonant untrapped particles is permanently renewed by "fresh", formerly nonresonant particles, in contrast to phase trapped particles.

This consideration was based on the analysis of the Hamiltonian (3.81) and the corresponding Eqs. (3.79). For quantitative estimations we will use the Hamiltonian (3.74) which for $|\alpha| \ll |\beta|$ appears to be a slowly varying function of $\tilde{t}$ entering the Hamiltonian through W_R. We should stress that the Hamiltonians $\mathcal{H}$ (3.74) and h (3.81) are not equal; they differ by the term $\alpha\ \xi$ as the generating function of the corresponding canonical transformation depends explicitly on "time" $\tilde{t}$.

For constant W_R, which corresponds to $\alpha = 0$, the particle motion is periodic (see Fig.4 for the phase space in this case) and, thus, the motion described by the Hamiltonian (3.74) and Eqs. (3.76) show where the theory of adiabatic invariants is applicable. As is well known (see, for instance, Landau and Lifshitz 1976), the adiabatic invariant I is determined by the integral

$$I = \frac{1}{2\pi} \oint W(\varepsilon, \xi; W_R) d\xi. \tag{3.90}$$

where $W(\varepsilon,\ \xi;\ W_R)$ is the particle momentum expressed as a function of the coordinate ξ, energy ε, and the parameter W_R from the relation $\varepsilon = \mathcal{H}(W, \xi; W_R)$, and the integral with respect to the coordinate ξ is taken over the period of particle motion at constant ε and W_R. In our case we obtain

$$W = W_R \pm \sqrt{2(\varepsilon + \beta \sin \xi)}, \tag{3.91}$$

the sign of the square root corresponding to the sign of $d\xi$. When calculating adiabatic invariants, it is useful to be guided by the plot of phase space (see Fig.4). From (3.74) we can see that the minimum energy ε is equal to $-|\beta|$, and the energy range $(-|\beta|,\ |\beta|)$ corresponds to trapped particles, which undergo an oscillatory motion with a limited variation of the canonical coordinate (or phase) ξ. Particles with energy $\varepsilon > |\beta|$ are untrapped, and their phase ξ varies monotonically. By substituting (3.91) into (3.90), we notice that the calculation is reduced to standard integrals, the adiabatic invariant being expressed through the quantity κ determined as

$$\kappa = \sqrt{\frac{2|\beta|}{\varepsilon + |\beta|}} \mathrm{sign}(W - W_R). \tag{3.92}$$

For untrapped particles $|\kappa| < 1$, and (3.90) gives

$$\mathrm{sign}(W - W_R) \cdot I = W_R + \frac{4}{\pi} \sqrt{|\beta|} \frac{E(\kappa)}{\kappa} = \mathrm{const}, \tag{3.93}$$

while for trapped particles $|\kappa| > 1$, and the adiabatic invariant takes the form

$$I = \frac{8}{\pi} \sqrt{|\beta|} \left[E\left(\frac{1}{\kappa}\right) - \frac{\kappa^2 - 1}{\kappa^2} K\left(\frac{1}{\kappa}\right) \right] = \mathrm{const}, \tag{3.94}$$

where $K(x)$ and $E(x)$ are complete elliptical integrals of the first and second kind, respectively.

We see that resonance particles are trapped or untrapped depending on the value of parameter κ (3.92), the value $|\kappa| = 1$ defining the separatrix. From the definition (3.90) of the adiabatic invariant it can be inferred that the phase volume of trapped particles on the (ξ, W)-plane is equal to $\Gamma_T = 2\pi I(\kappa = 1)$, i.e.

$$\Gamma_T = 16\sqrt{|\beta|}, \tag{3.95}$$

where (3.94) and $E(1) = 1$ have been used.

From (3.91)–(3.94) it follows that, in the case of weak inhomogeneity, the variation of the quantity ($W - W_R$) in the resonance region for trapped and untrapped particles has the following order of magnitude

$$\Delta(W - W_R) \sim |\beta|^{1/2}. \tag{3.96}$$

The relation (3.96) permits us to write conditions (3.59) and (3.75) in an explicit form. The criterion of isolated resonances (3.59) in the case of weak inhomogeneity turns into

$$\frac{k_\parallel^2}{m\omega}|\beta|^{1/2} \ll \Omega,$$

while the condition (3.75) takes the form

$$\frac{nk_\perp^2}{\rho m\omega}|\beta|^{1/2} \ll \Omega.$$

3.7 Electron Acceleration by a Whistler-Mode Wave

The following consideration takes into account the effects of oblique propagation, however, as the entire review, the consideration is non-relativistic. Simultaneous account for the both effects is a tempting vista for future work.

As we have seen above, the averaged (over the nonlinear period) kinetic energy W of phase trapped particles varies monotonically along the particle trajectory [see (3.88), (3.89)] and, depending on the sign of the inhomogeneity parameter α, the energy increases (at $\alpha > 0$) or decreases (at $\alpha < 0$). We will now discuss this question in a little more detail, aiming at an estimation of the total energy gain (or loss) by a trapped particle. Further discussion of this problem may be found in the papers listed at the end of this subsection.

The integral of motion (3.68) is valid for both trapped and untrapped particles. Apart from this, the essence of phase trapping is that the particle parallel velocity $v_\parallel$ oscillates around the local value of the resonance velocity v_{Rn} which varies along the particle trajectory due to the inhomogeneity of the plasma and the ambient magnetic field. Thus, for phase trapped particles $v_\parallel^2 \simeq v_{Rn}^2$. Combining this with (3.68) we find the relation

$$\left(n - \frac{\omega}{\omega_c}\right)W + \frac{mv_{Rn}^2}{2}\frac{\omega}{\omega_c} = \text{const}, \tag{3.97}$$

which permits us to express the variation of particle kinetic energy through the variations of quantities ω_c and v_{Rn}^2:

$$\Delta W = \frac{\omega}{\omega - n\omega_c}\left[\mu\Delta\omega_c + \frac{m}{2}\Delta v_{Rn}^2\right]. \tag{3.98}$$

For $n = 0$ we get $\Delta W = \mu\,\Delta\omega_c + (m/2)\,\Delta v_{Rn}^2$, which is a consequence of the fact that, according to (3.68), for particles interacting with the wave at Cerenkov resonance, $\mu = \text{const}$. For $n = 1$, which corresponds to the first cyclotron resonance—the only one that exists in the case of parallel propagation—the variation of the kinetic energy has the opposite sign to variations of the quantities ω_c and v_{Rn}^2. If the wave packet is situated on both sides of the magnetospheric equator ($s = 0$) and has $k_{\|} > 0$, then $\alpha > 0$ at $s > 0$ and $\alpha < 0$ at $s < 0$ (provided that the term $d\omega_c/ds$ dominates in the expression (3.77) for α, or both terms have the same sign). Then at $s > 0$, the electron kinetic energy increases for both $n = 0$ and $n = 1$, while at $s < 0$ the particle energy decreases. This entirely agrees with the general equation (3.89). It does not contradict the relation (3.98), of course, since for $n = 0$ and $n = 1$ resonant particles move in different directions: for $n = 0$ they move in the direction of $k_{\|}$, while for $n = 1$ the resonant particles move in the opposite direction. Thus, in the same region of space with respect to the equator, the quantities $\Delta\omega_c$, Δv_{Rn}^2 have different signs depending on the resonance number. For known geometry of the trapping region with respect to the equator, which is determined by particle dynamics in the field of the wave packet and, thus, depends on the wave and plasma parameters, relation (3.98) defines the variation of the particle kinetic energy. When estimating the quantity ΔW it is necessary to bear in mind that particle trapping may take place only in the case of weak inhomogeneity (or strong nonlinearity), when the effective amplitude β (3.80) exceeds the inhomogeneity factor α. It is important to note that, in the case of oblique wave propagation in an inhomogeneous plasma, the effective amplitude β is a varying, non-monotonic function [see the definitions (3.80), 3.46), (3.73)].

Electron acceleration by a Langmuir wave in an inhomogeneous plasma has been studied in a series of papers by Istomin et al. (1975, 1976a, b) and Karpman et al. (1975b, c). This effect is quite similar to particle acceleration by a whistler-mode wave in view of the above mentioned similarity in the description of resonant interactions for Langmuir and whistler-mode waves (Karpman et al. 1975a). The peculiarities of particle heating by a whistler-mode wave in an inhomogeneous plasma have been considered by Solov'ev and Shklyar (1986). Recently, electron acceleration up to relativistic energies by whistler-mode waves has excited substantial interest in connection with the problem of the formation and loss of the Earth's radiation belts, and the evaluation of the risks to the performance and lifetime of Earth-orbiting satellites. The number of publications on this subject is rapidly increasing. We will cite only a few of them, including the latest ones, where the references to the most important publications on this subject could be found (Hobara et al. 2000; Albert, 2002; Meredith et al. 2002; Trakhtengerts et al. 2003; Horne and Thorne 2003; Omura et al., 2007; Katoh and Omura 2007; Summers and Omura 2007; Trakhtengerts and Rycroft 2008).

4 Effects of Resonant Electrons Upon the Wave: Growth (or damping) Rate due to Resonant Interactions

To find the evolution of a wave caused by interactions with resonant electrons, we need to calculate the r.h.s. of Eq. (2.27). It is conventional to introduce the wave growth (or damping) rate γ uniquely related to the r.h.s. of Eq. (2.27) according to the relation

$$\gamma \equiv -\frac{1}{2U}\left\langle (\mathbf{j}_{\mathrm{RES}})_i \mathcal{E}_i \right\rangle = \frac{e}{2U}\left\langle \int (f_{\mathrm{RES}} - f_0) v_i d\mathbf{v} \cdot \mathcal{E}_i \right\rangle, \tag{4.1}$$

where $-e$ is electron charge, f_{RES} is the resonant electron distribution function in the wave field, f_0 is the corresponding unperturbed distribution, and U is the wave energy density determined by (2.25). The unperturbed distribution f_0, which does not contribute to the electron current, of course, is introduced for convenience (see below). As is known, in a stationary case, the charged particle distribution is a function of the integrals of motion. In the absence of a wave, those constants of the motion are the particle kinetic energy W and the transverse adiabatic invariant μ. In compliance with this, we will assume that the unperturbed distribution function $f_0 = f(W, \mu)$. According to Liouville's theorem [see (3.3)]

$$f_{\mathrm{RES}}(\mathbf{v}, \mathbf{r}, t) = f_0(W_0, \mu_0), \tag{4.2}$$

where W_0 and μ_0 are initial values of the particle kinetic energy and adiabatic invariant expressed through $\mathbf{v}$, $\mathbf{r}$, and t (or any other variables) from the equations of motion. In fact, we can express the distribution function in any variables which are convenient. We should only keep in mind that the normalization of the distribution function which is set by the requirement that the number of particles in the phase volume $d\mathbf{v}d\mathbf{r}$ centered on $\mathbf{v}$, $\mathbf{r}$ is $dN = fd\mathbf{v}d\mathbf{r}$, and whatever variables are used, the differential $d\mathbf{v}d\mathbf{r}$ should be transformed and expressed through those variables.

As the growth (damping) rate γ is determined by resonant particles (the contribution from non-resonant particles being included into the wave energy through the dielectric tensor), the contribution to the integral in (4.1) comes only from resonance regions, which, in fact, we should always be sure about. Thus, we can expand the difference $f_{\mathrm{RES}} - f_0$ around each resonance. For the n-th cyclotron resonance we get

$$\delta f_n \equiv f_{\mathrm{RESn}} - f_0 = f_0(W_0, \mu_0) - f_0(W, \mu) = \tag{4.3}$$

$$= (W_0 - W) \cdot \left(\frac{\partial f_0}{\partial W}\right)_{W = mv_{\mathrm{Rn}}^2/2 + \mu\omega_{\mathrm{c}}} + (\mu_0 - \mu) \cdot \frac{\partial f_0}{\partial \mu}. \tag{4.4}$$

The subscript at the derivative $\partial f_0/\partial W$ indicates that the derivative is taken at $v_{\|} = v_{\mathrm{Rn}}$. Taking into account the integral of motion (3.68) which relates the variations of W and μ for particles at the n-th cyclotron resonance with the wave and gives

$$\mu_0 - \mu = \frac{n}{\omega}(W_0 - W),$$

we may combine two terms in (4.3) to obtain

$$\delta f_n = f'_{0n} \cdot (W_0 - W); \quad f'_{0n} = \left(\frac{\partial f_0}{\partial W} + \frac{n \partial f_0}{\omega \partial \mu}\right)_{W = mv_{\mathrm{Rn}}^2/2 + \mu\omega_{\mathrm{c}}}. \tag{4.5}$$

Substituting (3.43) into (4.1) and transforming to cylindrical coordinates in velocity space, i.e.

$$v_{\mathrm{L}} = v_{\perp} \cos\varphi; \quad v_{\Phi} = -\mathrm{sign}(q) v_{\perp} \sin\varphi, \tag{4.6}$$

[see (3.23), (3.27)] we obtain

$$\begin{aligned} \gamma = -\frac{e|A|\omega}{2U} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} d\zeta \int_{-\pi}^{\pi} d\varphi \int_{-\infty}^{\infty} dv_{\|} \int_0^{\infty} dv_{\perp} v_{\perp} \delta f_n \\ \times \left\{ \frac{v_{\perp}}{c} a_{\mathrm{L}} \cos\varphi \cos\zeta + i \frac{v_{\perp}}{c} a_{\Phi} \sin\varphi \sin\zeta + \frac{v_{\|}}{c} a_{\mathrm{M}} \cos\zeta \right\}, \end{aligned} \tag{4.7}$$

where the sum is over all cyclotron resonances, and the averaging over the wave phase ζ is written in an explicit way. We now change the angle variables for each resonance [see (3.40), (3.46)]

$$\varphi, \zeta \to \varphi, \xi_n = n\varphi - \rho \sin \varphi + \zeta. \tag{4.8}$$

The Jacobian of this transformation is equal to 1. Substituting into (4.7) $\zeta = \xi_n - n\varphi + \rho \sin \varphi$ [see (4.8)] and performing the integration over φ using the following tabulated integrals:

$$\begin{aligned} \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - \rho \sin \varphi) d\varphi &= J_n(\rho) \\ \frac{1}{\pi}\int_0^{\pi} \cos \varphi \cos(n\varphi - \rho \sin \varphi) d\varphi &= \frac{n}{\rho} J_n(\rho) \\ \frac{1}{\pi}\int_0^{\pi} \sin \varphi \sin(n\varphi - \rho \sin \varphi) d\varphi &= J'_n(\rho) \end{aligned} \tag{4.9}$$

(the last two being, in fact, simple consequences of the integral representation of Bessel functions of integer order given by the first relation) we obtain from (4.7)

$$\gamma = -\frac{e|A|\omega}{2U} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{\infty} dv_\| \int_0^{2\pi} \cos \xi_n d\xi_n V_n(v_\perp, v_{\mathrm{R}n}) \delta f_n, \tag{4.10}$$

where V_n is determined in (3.46). In (4.10), as in the solutions of the equations of motion, the parallel velocity $v_\|$ has been replaced by the resonance value $v_{\mathrm{R}n}$ in the expression for the effective amplitude of the interaction V_n.

As has already been pointed out, smaller magnitudes of resonant velocities correspond to the cyclotron resonances $n = 0, 1$. Since in any real situation the number of particles, as a rule, decreases with increasing velocity, the main contribution to wave damping (or growth) comes from those two resonances, namely, the Cerenkov ($n = 0$) and the first cyclotron ($n = 1$) resonances.

We begin calculations of the damping (growth) rate (4.10) for the case of a strong inhomogeneity. Then $W_0 - W$ in the expression $\delta f_n = f_{0n}{}' \cdot (W_0 - W)$ is determined by the relation [see (3.77)–(3.78)]

$$W_0 - W = w_0 - w - \int \alpha d\tilde{t} \equiv w_0 - w - (W_{\mathrm{R}} - W_{R0}) \tag{4.11}$$

together with the solution (3.86). As can be seen from the phase plots corresponding to the case of a strong inhomogeneity, for large enough $\tilde{t}$, the quantity w_0 for resonant particles satisfies the relations $|w_0|/\sqrt{|\alpha|} \gg 1, \mathrm{sign} w_0 = \mathrm{sign}\alpha$, which makes it possible to replace the Fresnel's integrals which depend on w_0 by their asymptotic values

$$C\left(\sqrt{\frac{l}{2|\alpha|}} w_0\right) = S\left(\sqrt{\frac{l}{2|\alpha|}} w_0\right) = \frac{\mathrm{sign}\alpha}{2}.$$

Then (3.86), (4.11) determine explicitly the dependence of W_0 on the present values of the variables. Substituting into (4.10) the expression (4.5) determined by (4.11), (3.86), and integrating with respect to w and ξ_n, we arrive at the following formula for γ in the case of a strong inhomogeneity:

$$\gamma = \frac{\omega_c (\pi e|A|\omega)^2}{2mk_\parallel U} \sum_{n=-\infty}^{\infty} \int_0^\infty d\mu f'_{0n} V_n^2 \left[\sum_{l=1}^{\infty} \frac{4\alpha_n^2}{l\beta_n^2} J_l^2 \left(\frac{l\beta_n}{\alpha_n} \right) \right]. \tag{4.12}$$

For $|\beta_n/\alpha_n| \ll 1$, the term $l = 1$, which is then equal to 1, brings the main contribution to the sum over l, and (4.12) turns to

$$\gamma = \sum_{n=-\infty}^{\infty} \gamma_L^{(n)} \equiv \frac{\omega_c (\pi e|A|\omega)^2}{2mk_\parallel U} \sum_{n=-\infty}^{\infty} \int_0^\infty d\mu f'_{0n}(\mu) V_n^2(\mu). \tag{4.13}$$

Since $|\beta_n/\alpha_n| \ll 1$ includes the limit of the wave field amplitude tending to zero, expression (4.13) gives the linear growth rate for whistler-mode waves, which explains the notation γ_L used above. Numerical analysis shows that in the whole range of strong inhomogeneity $|\beta_n/\alpha_n| < 1$ (but not only $|\beta_n/\alpha_n| \ll 1$), the sum over l in (4.12) varies from 1 to 1.67; thus, γ is close to the local linear value. This result is of fundamental importance for understanding wave amplification in an unstable inhomogeneous plasma, in particular, in the magnetosphere. Let us discuss this point in more detail. As is well known (O'Neil 1965), in the nonlinear regime of particle interactions with a monochromatic wave in a homogeneous plasma, the growth rate tends to zero at $t > 1/\omega_{NL}$ due to phase mixing of the resonant particles, after which wave amplification ceases. Although this result has been obtained for the nonlinear regime $\gamma_L \ll \omega_{NL}$, it assumes that the linear wave growth which starts at $\gamma_L \gg \omega_{NL}$ stops when $\gamma_L \sim \omega_{NL}$. As we have seen above, in an inhomogeneous plasma, in particular, in the case of a strong inhomogeneity, the growth rate at $t > 1/\omega_{NL}$ remains of the order of, and even larger than, the linear growth rate due to the permanent renewal of the resonant region which prevents the phase mixing. We may suppose that the wave growth will persist at least until the conditions of strong inhomogeneity, i.e. $\alpha > \omega_{NL}^2$ are fulfilled, leading to much higher wave amplitudes than those which follow from $\gamma_L^2 \sim \omega_{NL}^2$, provided that $\alpha \gg \gamma_L^2$. (We remind the reader that the quantity ω_{NL}^2 is proportional to the wave amplitude.)

We now turn to the case of a weak inhomogeneity $|\beta_n/\alpha_n| \gg 1$. As was shown above, there are now phase trapped particles whose energy varies according to (3.89). Then, the contribution of trapped particles to the growth rate is

$$\gamma_T = -\frac{1}{2U} \int \frac{dW_T}{dt} f_T d\mathbf{v} \equiv -\frac{k_\parallel^2}{2m\omega U} \int \alpha f_T d\mathbf{v}, \tag{4.14}$$

where the integration is performed over the phase volume of trapped particles. (For the sake of brevity, we write the formulas for one resonance, the total growth rate being the sum over all resonances.) We see that the value of (4.14) does not depend on details of the distribution function, but is determined by its average value over the phase volume of trapped particles. We will denote this value of the distribution function by $\bar{f}_T$. Transforming the variables of integration to w, ξ, μ, and introducing the phase volume $\Gamma_T(\mu)$ of trapped particles on the (w, ξ)-plane, we obtain from (4.14)

$$\gamma_T = -\frac{k_\parallel^3 \omega_c}{2\omega^2 m^3 U} \int d\mu \bar{f}_T \alpha \Gamma_T(\mu), \tag{4.15}$$

where $\Gamma_T(\mu)$ is determined by (3.95).

We can now calculate the untrapped particle input to the growth rate, determined by their averaged distribution function. For this objective, we rewrite formula (4.10) for untrapped particles in the following way

$$\gamma_{\mathrm{UT}} = -\frac{e|A|k_{\|}\omega_{\mathrm{c}}}{2m^2U}\int d\mu \bar{f}_{\mathrm{UT}}V_n(\mu)\iint\limits_{\Gamma_{\mathrm{UT}}} d\xi dw \cos\xi. \tag{4.16}$$

A similar expression can be written for trapped particles as well

$$\gamma_T = -\frac{e|A|k_{\|}\omega_{\mathrm{c}}}{2m^2U}\int d\mu \bar{f}_T V_n(\mu)\iint\limits_{\Gamma_T} d\xi dw \cos\xi. \tag{4.17}$$

By comparing (4.17) with (4.15) we find

$$\iint\limits_{\Gamma_T} d\xi dw \cos\xi = \frac{k_{\|}^2}{m\omega^2 e|A|}\frac{\alpha\Gamma_{\mathrm{T}}}{V_n}. \tag{4.18}$$

On the other hand, it is obvious that

$$\iint\limits_{\Gamma_T} d\xi dw \cos\xi + \iint\limits_{\Gamma_{\mathrm{UT}}} d\xi dw \cos\xi = 0;$$

from which we find

$$\iint\limits_{\Gamma_{\mathrm{UT}}} d\xi dw \cos\xi = -\frac{k_{\|}^2}{m\omega^2 e|A|}\frac{\alpha\Gamma_{\mathrm{T}}}{V_n}. \tag{4.19}$$

Substituting (4.19) into (4.16) and summing up with (4.15) we finally have

$$\gamma \equiv \gamma_{\mathrm{T}} + \gamma_{\mathrm{UT}} = \frac{k_{\|}^3\omega_{\mathrm{c}}}{2\omega^2 m^3 U}\int d\mu \alpha \Gamma_{\mathrm{T}}(\mu)(\bar{f}_{\mathrm{UT}} - \bar{f}_{\mathrm{T}}). \tag{4.20}$$

Formula (4.20) is an asymptotic one, and it is valid for sufficiently large times $\tilde{t} - \tilde{t}_0$. Therefore, the averaged untrapped particle distribution is close to an unperturbed one, since the phase space of untrapped particles is permanently renewed. Thus,

$$\bar{f}_{\mathrm{UT}} \simeq f_0(W, \mu). \tag{4.21}$$

For trapped particles [see (3.68), (3.88)]

$$\bar{f}_{\mathrm{T}} \simeq f_0(\bar{W}_0, \bar{\mu}_0) = f_0\left(W - \int\limits_{\tilde{t}_0}^{\tilde{t}} \alpha d\tilde{t}', \quad \mu - \frac{n}{\omega}\int\limits_{\tilde{t}_0}^{\tilde{t}} \alpha d\tilde{t}'\right). \tag{4.22}$$

Using the definition (3.77) of the quantity α, it is not difficult to show that $\bar{f}_T$ coincides with the unperturbed resonance particle distribution at that place of the wave packet where phase trapping takes place, i.e. at the leading front for the resonance $n = 1$, and at the back edge for resonance $n = 0$, provided that the wave packet is entirely in the region of weak inhomogeneity. Otherwise, phase trapping takes place at the corresponding points where $|\beta/\alpha| > rsim1$. Under condition

$$\int\limits_{\tilde{t}_0}^{\tilde{t}} \alpha d\tilde{t}' \ll T_{\mathrm{hot}} \tag{4.23}$$

where $T_{\rm hot}$ is a characteristic temperature (measured in units of energy) of the energetic electron distribution, the difference $\bar{f}_{UT} - \bar{f}_T$ can be expanded and represented in the form:

$$\bar{f}_{\rm UT} - \bar{f}_{\rm T} \simeq f'_{0n}\eta_n, \tag{4.24}$$

η_n denoting the integral:

$$\eta_n = \int\limits_{\tilde{t}_0}^{\tilde{t}} \alpha_n d\tilde{t}' \equiv W_{\rm Rn} - W_{\rm Rn}^{(0)}. \tag{4.25}$$

Substituting (4.24) into (4.20) and introducing the linear growth rate $\gamma_{\rm L}^{(n)}$ corresponding to the n-th resonance contribution [see (4.13)], we obtain

$$\gamma = \sum_{n=-\infty}^{\infty} \gamma_{\rm L}^{(n)} \frac{16}{\pi^2} \frac{k_\parallel^3}{\omega^3 (e|A|m)^{3/2}} \left\langle \frac{\alpha_n \eta_n}{V_n^{3/2}} \right\rangle \equiv \sum_{n=-\infty}^{\infty} \gamma_{\rm L}^{(n)} \frac{16}{\pi^2} \left(\frac{V_n}{\beta_n} \right)^{3/2} \left\langle \frac{\alpha_n \eta_n}{V_n^{3/2}} \right\rangle, \tag{4.26}$$

where $\langle\cdots\rangle$ stands for averaging with the weight $f_{0n}{}'\ V_n^2$. The expression (4.26) is obtained using the asymptotic formula (4.20). It is applicable when, along with (4.23), $\eta_n \gg \sqrt{|\beta_n|}$ is satisfied. Since the quantity η_n has the sign of α_n (see (4.25)), the contribution of the n-th resonance to the growth rate has the sign of $\gamma_{\rm L}^{(n)}$, but its value is completely different, and depends substantially on the time which the phase trapped particles have spent in the trapping region of phase space. Thus, in the case of a weak inhomogeneity, the growth rate is essentially a nonlocal quantity: different parts of the wave packet evolve with different growth rates. This may result in the temporal growth of VLF transmitter signals observed at a receiving station in the geomagnetic conjugate region (Stiles and Helliwell 1977). For details of the explanation of this effect, see Shklyar (1996). We should mention that the case of a weak inhomogeneity, in contrast to the case of a strong inhomogeneity, permits the limit $\alpha \to 0$. In this case, formula (4.26) (which is an asymptotic expression valid at $t > 1/\omega_{\rm NL}$) gives $\gamma \to 0$, in agreement with the fact that, in a homogeneous plasma, the growth rate at sufficiently large t tends to zero.

5 Proton Precipitation Induced by a VLF Transmitter Signal

5.1 Peculiarities of Proton Resonance Interaction with VLF Signals

The interaction between man-made VLF signals and energetic electrons in the magnetosphere and, in particular, the problem of wave-induced electron precipitation, have been a subject of intensive studies during the last few decades (Helliwell and Katsufrakis 1974; Karpman and Shklyar 1977; Matsumoto 1979; Southwood 1983; Albert 2001; Trakhtengerts et al. 2003; Inan et al. 2007). The number of papers on this problem is enormously large, and those mentioned above constitute only a small part of the papers where references to the most important studies may be found. All theoretical work devoted to this problem includes, as the main condition of the interaction, the resonance between the particles and the wave:

$$\omega - k_\parallel v_\parallel = n\Omega; \quad (n = 0, \pm 1, \pm 2, \ldots)$$

We remind the reader that in the VLF wave–electron interaction the first cyclotron ($n = 1$) and Cerenkov ($n = 0$) resonances play the most important role, since they involve particles with the lowest values of kinetic energy which dominate in most kinds of distributions.

However, the same resonance interaction between a VLF wave and energetic protons has not been expected to play any important role, either in the wave evolution or in the evolution of proton distribution. The reason for this is readily apparent. Since the proton cyclotron frequency Ω is much smaller than the wave frequency ω, the resonance velocity for small number resonances is of the order of the electron Cerenkov velocity, which for protons corresponds to very high energies. Clearly, using large number resonances ($n \sim \omega/\Omega$) resolves this problem. However, these resonances are known to be ineffective unless the particle velocity is of the order of the wave phase velocity ($\omega \sim \mathbf{kv}$). This requirement is similar but not identical to the Cerenkov resonance condition ($\omega = k_{\|}v_{\|}$) and is often met more easily. Even in this case, a single resonance wave–proton interaction is much less effective than that for electrons, because of the mass ratio. That is why the observed proton precipitation caused by a VLF wave (Koons 1975; Kovrazhkin et al. 1984) had not been predicted theoretically and even was a surprise. For the same reasons the initially proposed explanation based on the wave interaction at the equator with protons through either Cerenkov resonance or the first cyclotron resonance does not seem to be valid. In this Section, we briefly discuss the mechanism of VLF wave–proton interactions which, in particular, may explain the observed proton precipitation. (A detailed account of this mechanism may be found in Shklyar 1986).

The electron dominance in the wave–particle interactions mentioned above is true only for one single resonance. If we take into account that the distances between resonances in velocity space are $\Delta v_{\mathrm{R}} = \Omega/k_{\|}$, and thus $\Delta v_{\mathrm{Rp}}/\Delta v_{\mathrm{Re}} = m_e/m_p$, while, for equal temperatures, the ratio of the thermal velocities is $v_{\mathrm{Tp}}/v_{\mathrm{Te}} = (m_e/m_p)^{1/2}$, then we find that, for protons, the relative distance between resonances is much smaller than that for electrons. That is why protons are preferred from the viewpoint of the crossing of multiple resonances. The inclusion of multiple cyclotron resonances which makes wave–proton interactions much more efficient is the main important point in the mechanism under discussion.

5.2 Crossing of an Isolated Cyclotron Resonance

As was mentioned above, and will be proved in this Section, the interaction between a wave and energetic protons is only efficient provided that the condition $\omega \sim \mathbf{k.v}$ is fulfilled. Obviously, the larger the wave vector is, the lower is the proton velocity for which this requirement is met. Since, as a rule, the number of particles in a distribution increases with decreasing velocities, for larger $|\mathbf{k}|$ the interaction appears to be more efficient. That is why we restrict our attention to the pararesonance mode of whistler-mode wave propagation ($\omega \simeq \Omega_e \cos\theta$), when the refractive index substantially increases, and the wave is of the potential type (cf. Morgan 1980; Alekhin and Shklyar 1980). In this case, the particle motion is described by the Hamiltonian (3.45) (with the third term omitted) and the corresponding Eqs. (3.53) with the effective amplitudes V_n determined by (3.51), (3.52). These equations have been analyzed in detail in Section 3, where their solution for isolated resonances was found. Under certain restrictions over the wave amplitude and plasma inhomogeneity (which will be specified further on) the approximations of strong inhomogeneity and isolated resonances hold, permitting an analytical treatment of the problem.

In the approximation of isolated resonances we need retain only the n-th resonance term in the Hamiltonian of the interaction, which (the term) appears to be slowly varying. We should stress, however, that in an inhomogeneous plasma, even under the condition (3.59) we cannot be restricted by the n-th term in (3.45) on a long time scale, since due to the change of the quantities $k_{\|}$, Ω and $v_{\|}$, the particle will inevitably pass from one cyclotron resonance to another. Thus, the concept of isolated resonances in an inhomogeneous plasma implies that, on account of the condition (3.59), a particle at any instant of time is either in one particular resonance with the wave, or it moves in a non-resonant region.

We now proceed to consider the particle crossing of the n-th cyclotron resonance described by the Hamiltonian (3.65), and the corresponding Eqs. (3.66), with V_n determined by (3.51), (3.52). The latter show that the amplitude of the interaction at the n-th cyclotron resonance is proportional to the n-th order Bessel function $J_n(\rho)$. Thus, for $n \gg 1$, the wave–particle interaction is significant only for $\rho \equiv k_{\perp}v_{\perp}/\Omega > n$, since otherwise $J_n(\rho)$ is exponentially small. Combining this requirement with the resonance conditions (3.57) we obtain

$$k_{\|}v_{\|} + k_{\perp}v_{\perp} > \omega. \tag{5.1}$$

Hence, in contrast to the case of electrons, the condition for an effective interaction of protons with the wave is not a condition in the form (3.57), which can always be satisfied by choosing an appropriate value of n, but an inequality (5.1), under which the variable ζ (3.40) in the general Hamiltonian (3.39) has stationary-phase points on the cyclotron orbit. Obviously, the larger the refractive index is, the easier it is for conditions (5.1) to be met. This proves in fact the statements on the efficiency of resonance interaction made above. As a remark we will mention that for resonant particles

$$V_n \simeq \frac{\omega}{kc} J_n(\rho), \tag{5.2}$$

which easily follows from the resonance conditions and the expressions (3.51), (3.52).

Returning to Eqs. (3.66) we notice that for a quasi-electrostatic wave in the approximation of isolated resonances, just as in the general case of an electromagnetic wave, the quantity $C_n = nW - \mu\omega$ is a constant of the motion (cf. (3.68). As we have seen above, one of the peculiarities of wave–particle interactions in a homogeneous, or weakly inhomogeneous, plasma is the existence of phase trapped particles. For such particles, the phase ξ oscillates and has a limited variation with time. Clearly, such particles exist only provided that both $d\xi/dt$ and $d^2\xi/dt^2$ may equal zero, which is typical of the weak inhomogeneity case (see Subsection 3.6). For proton interactions with a whistler-mode wave this case takes place in a negligibly small vicinity of the equator, while over the main part of the interaction region the inhomogeneity greatly exceeds the nonlinearity (see below). This permits a simple calculation of the variation of the particle transverse adiabatic invariant μ within the time of crossing the n-th cyclotron resonance. This variation is described by the corresponding equation in (3.66), which we rewrite here for convenience, using expression (5.2) for V_n:

$$\frac{d\mu}{dt} = -\frac{qn\omega}{kc}|A|J_n(\rho)\cos\xi_n. \tag{5.3}$$

We now write the expression $d^2\xi/dt^2$ which follows from (3.66):

$$\frac{d^2\xi}{dt^2} = -a - \frac{q\omega k_{\|}^2}{mkc}|A|J_n(\rho)\cos\xi, \tag{5.4}$$

where a is the renormalized inhomogeneity parameter (cf. (3.77)):

$$a = k_{\|}\left(\frac{\mu}{m}\frac{d\Omega}{ds} + \frac{1}{2}\frac{dv_{\mathrm{R}}^2}{ds}\right). \tag{5.5}$$

When evaluating (5.4), we omitted terms of higher order in the wave amplitude and reciprocal of inhomogeneity scale. In the case of a strong inhomogeneity, when the quantity a exceeds the last term in (5.4), the phase trapped particles do not exist, and the phase ξ in the resonance region may be represented in the form

$$\xi = \xi_{\mathrm{R}} - a(t - t_{\mathrm{R}})^2/2, \tag{5.6}$$

where ξ_{R} and t_{R} are the values of the corresponding quantities at exact resonance, and it is taken into account that at the resonance point, $d\xi/dt = 0$. Putting (5.6) into Eq. (5.3) for μ and integrating over t, we obtain the total change of the particle momentum μ within the time of crossing the n-th cyclotron resonance

$$\begin{aligned} \Delta\mu &= -\int \frac{qn\omega}{kc}|A|J_n(\rho)\cos[\xi_{\mathrm{R}} - a(t - t_{\mathrm{R}})^2/2]dt \\ &\simeq -\frac{qn\omega}{kc}|A|J_n(\rho)\left(\frac{2\pi}{|a|}\right)^{1/2}\cos\left(\xi_{\mathrm{R}} - \frac{\pi}{4}\mathrm{sign}\, a\right). \end{aligned} \tag{5.7}$$

The main input to (5.7) comes from the time interval

$$\Delta t_{\mathrm{R}} = |a|^{-1/2}, \tag{5.8}$$

which is the time of resonance interaction in the strongly inhomogeneous case. As we see, in this case, the time of resonant interaction is small, which permits us to neglect the variations of all quantities, except ξ, on the right hand side of (5.3), replacing them by the values at exact resonance, as used in the evaluation of $\Delta\mu$. From the conservation of the quantity $C_n = nW - \mu\omega$ it follows that the total change of particle kinetic energy W while crossing an isolated resonance is equal to

$$\Delta W = \frac{\omega}{n}\Delta\mu \simeq -\frac{qn\omega}{kc}|A|J_n(\rho)\left(\frac{2\pi}{|a|}\right)^{1/2}\cos\left(\xi_{\mathrm{R}} - \frac{\pi}{4}\mathrm{sign}\, a\right). \tag{5.9}$$

To estimate the time Δt it takes the particle to pass from one cyclotron resonance to another, we notice that, at the n-th resonance, the quantity $d\xi_n/ds = 0$, while at the $(n \pm 1)$-th resonance the same quantity is equal to $\mp\Omega/v_0$. Since in the non-resonant region $d^2\xi_n/dt^2 \simeq -a$ [see (5.4)], we have

$$\Delta t = \frac{\Omega}{|a|}. \tag{5.10}$$

Clearly, the resonances are isolated if $\Delta t_{\mathrm{R}} \ll \Delta t$, i.e.

$$|a|^{1/2} \ll \Omega. \tag{5.11}$$

The inequality (5.11) is the explicit form of the conditions (3.59) in the case of a strong inhomogeneity.

When evaluating (5.7) and (5.9), we substituted t_{R} for t in all slowly varying quantities, and took them outside the integral. Hence, the most rigid condition of use for (5.7) comes from the requirement $\Delta\rho \ll 2\pi$, 2π being the period of $J_n(\rho)$ at $\rho > n \gg 1$. Simple estimates taking into account the definition (5.5) yields the inequality similar to (5.11).

5.3 Distribution Function and Precipitation Flux

We now discuss the mechanism of proton precipitation caused by the interaction with a VLF signal injected into the magnetosphere. For the sake of concreteness, we follow the experimental conditions of Kovrazhkin et al. (1984). In this experiment, a VLF wave of frequency $\omega \sim 10^5$ rad/s was injected by a transmitter in the northern hemisphere at $L = 4$. The precipitating protons were also registered in the northern hemisphere by the spacecraft Aureole 3.

We first recall some geometrical properties of VLF wave propagation in the magnetosphere (see Fig.1) for the case when the wave frequency is of the order of the electron equatorial gyrofrequency at the L-shell of the wave injection (L_{inj}) (for details, see, e.g. Walker 1976; Alekhin and Shklyar 1980, and references therein). If the wave is injected with vertical wave normal angle θ, (i.e. the angle between $\mathbf{k}$ and $\mathbf{B}_0$), the rays first propagate to larger L-shells. When approaching the equator, all the rays injected at various L-shells around L_{inj} start focusing near $L < L_{inj}$. This focusing, which continues on the other side of the equator, is accompanied by the wave passing to lower L-shells, and by the transition to the pararesonance mode of propagation for which

$$\omega \simeq \omega_c \cos\theta.$$

In this regime of propagation the wave is of the potential type, its refractive index increases significantly (being limited by thermal effects, Sazhin 1993), and the group velocity is almost perpendicular to the wave normal vector.

As follows from (5.1), an effective interaction between the wave and protons of energy W starts after the wave vector exceeds the value $k_{\min} = \omega/v(W)$, $v(W)$ being the velocity corresponding to the energy W. For $\omega = 10^5$ rad/s and 100 keV protons ($v(W) \sim 3 \times 10^6$ m/s) one has $k_{\min} \sim 3 \times 10^{-2}$ m^{-1}. The numerical calculations show that k reaches this value in the southern hemisphere at the latitude $|\lambda| \sim 30°$ and then increases continuously. Thus, in the case under discussion, an efficient interaction between protons and the VLF signal takes place outside the equatorial region at latitudes $|\lambda| > rsim 30°$. As an order of magnitude estimation for the wave electric field in this region we will use the value $E \sim 10^{-2}$ V/m. Estimates based on the parameters of the background plasma and the geomagnetic field at $L \sim 4$, and the wave characteristics mentioned above show that all over the interaction region the following conditions are fulfilled: the inhomogeneity greatly exceeds the nonlinearity implying the absence of phase trapped particles, and all cyclotron resonances are isolated (for details, see Shklyar 1986).

We now turn our attention to the proton distribution function. In the absence of the wave, the distribution function depends on the constants of the motion, i.e. on the transverse adiabatic invariant μ and kinetic energy W. When the wave is present, the exact distribution function depends, apart from μ, W also on t, s, and φ. However, to calculate the precipitation flux, it is sufficient to know the "coarse-grained" distribution function, i.e. the distribution function averaged over the time $\sim \Delta t$. Such a distribution function does not depend on gyrophase φ and has only a slow dependence on t and s. To obtain the corresponding equation for this distribution function, we notice that the quantities μ and W change within the time $\sim \Delta t_{\mathrm{R}}$, while during the interval $\Delta t \gg \Delta t_{\mathrm{R}}$ a particle freely streams in the phase space. Thus the action of the wave upon a particle has the character of impacts, which are characterized by changes in the transverse adiabatic invariant and energy determined by (5.7) and (5.9). Since these quantities have an oscillatory dependence on the phase ξ_{Rn} at exact resonance, which in turn depends on the gyrophase φ, the averaged values of $\Delta\mu$ and ΔW are equal to zero. Then, as the resonances are isolated, the quantities ξ_{Rn} at adjacent resonances are non-correlated, and so are the variations $\Delta\mu$ and ΔW. Furthermore, $\Delta\mu \ll \mu$ and $\Delta W \ll W$. Such a character of proton interaction with the wave leads to a diffusive behavior of the particle distribution function in phase space. This diffusion develops with the characteristic time (5.10) and is described by diffusion coefficients:

$$D_{\mu\mu} = \frac{\langle \Delta\mu^2 \rangle}{2\Delta t} \sim \frac{\tilde{n}^2 q^2}{2\rho\Omega}\left(\frac{|E|}{k}\right)^2; D_{\mu W} = D_{W}\mu = \frac{\omega}{\tilde{n}} D_{\mu\mu}; \quad D_{WW} = \frac{\omega^2}{\tilde{n}^2} D_{\mu\mu}, \tag{5.12}$$

where $\Delta\mu$, Δt are determined by (5.7) and (5.10), respectively, and $\langle \cdots \rangle$ means the averaging over the phase volume of the resonant region. The quantity $\tilde{n}$ which appears in (5.12) is the "continuous" number of resonance

$$\tilde{n} = \frac{\omega - k_{\|} v_{\|}}{\Omega}, \quad \left(v_{\|} = \sqrt{2[W - \mu\Omega(s)]} \operatorname{sign} v_{\|} \right) \tag{5.13}$$

as in the diffusion approximation, the "impacts" at exact cyclotron resonances are replaced by continuous diffusion in phase space. In evaluating (5.12), we have used the asymptotic behavior of the Bessel function $J_n(\rho)$ for $\rho > n \gg 1$. Given the diffusion coefficients, the equation for the distribution function takes the form

$$\frac{\partial F}{\partial t} + v_{\|}\frac{\partial F}{\partial s} = |v_{\|}|\left(\frac{\partial}{\partial \mu} + \frac{\partial}{\partial W}\frac{\omega}{\tilde{n}}\right)\frac{D_{\mu\mu}}{|v_{\|}|}\left(\frac{\partial F}{\partial \mu} + \frac{\omega \partial F}{\tilde{n} \partial W}\right). \tag{5.14}$$

The factor $|v_{\|}|^{-1}$ on the r.h.s. of (5.14) is the Jacobian which expresses the phase volume in velocity space through the variables μ and W: $d\mathbf{v} = 2\pi\ \mathrm{m}^{-2}\ \Omega\ |v_{\|}|^{-1} d\mu\ dW$. The Eq. (5.14) does not take into account particle losses through the loss-cone, and is therefore valid out of the loss-cone. From the particle conservation law it follows that the precipitation flux in one hemisphere is equal to the rate of particle decrease in the magnetic tube with unit section at the ionospheric level. The cross section of such a tube varies as $\Omega_0/\Omega(s)$, where Ω_0 is ion cyclotron frequency at that level. Multiplying Eq. (5.14) by

$$\frac{\Omega_0}{\Omega} d\mathbf{v} ds = \frac{2\pi\Omega_0}{m^2|v_{\|}|} dW d\mu ds$$

and integrating over the phase-space corresponding to the particles with the definite sign of $v_{\|}$ out of the loss-cone, we obtain

$$-\frac{dN}{dt} \equiv j_{\text{prec}} = \frac{2\pi}{m^2}\Omega_0 \int ds \int_{W(s)}^{\infty} dW \left\{ \left(1 - \frac{\omega}{\tilde{n}\Omega_0}\right) \frac{D_{\mu\mu}}{|v_{\|}|} \left[\frac{\partial F(\mu, W)}{\partial \mu} + \frac{\omega \partial F(\mu, W)}{\tilde{n} \quad \partial W} \right] \right\}_{\mu = W/\Omega_0}. \tag{5.15}$$

The diffusion coefficient vanishes for low-energy protons. Thus $W(s) = m\omega^2/2\ k(s)^2$, and the integration over s is performed over that part of the geomagnetic line where the wave is a potential wave. As is usual in diffusive processes, the flux is determined by the gradients of the distribution function in velocity space at the boundaries of the loss-cone.

Since transition to the pararesonance mode takes place in the hemisphere opposite to that of the wave injection and is accompanied by the wave moving to lower L-shells, the maximum precipitation flux should be observed at $L < L_{\text{inj}}$, and the delay between the wave injection and the beginning of the precipitation in the same hemisphere should be of the order of the half-sum of the bounce periods of the waves and particles. Regarding the energy of the precipitating protons, for $\omega \sim 10^5$ rad/s, and $k \sim (2-3) \times 10^{-2}\ \mathrm{m}^{-1}$, the condition (5.1) gives the lower energy boundary, which at the same time corresponds to the maximum of the differential precipitation flux, of the order of $100 - 200$ keV. It is important to mention that since ω, $n\Omega \gg k_{\|}v_{\|}$, the diffusion takes place nearly equally for positive and negative $v_{\|}$. Therefore, the precipitation flux in the southern hemisphere should be close to that of the northern hemisphere, but the delay should be shorter. Proton precipitation in the region conjugate to the VLF transmitter has been reported by Koons (1975). As for the magnitude of the precipitation flux, it appears to increase with the number of cyclotron resonances crossed by the energetic protons, $n_{eff} \sim 10^2$. Therefore the efficiency of this mechanism is rather high.

6 Concluding Remarks

We have discussed just two applications of the theory in which oblique wave propagation and the inhomogeneity of the ambient magnetic field are of major importance, namely, the calculation of the whistler-mode wave growth or damping rate, and proton precipitation induced by the signal from a ground-based VLF transmitter. Electron acceleration by the VLF wave related to phase trapping in a weakly inhomogeneous plasma, which we have also discussed, may take place for both ducted and oblique wave propagation. Needless to say the theory presented has many other applications which we have not discussed here, such as the modulation and growth of VLF signals (Likhter et al. 1971; Stiles and Helliwell 1977), emissions of varying frequency, the so-called triggered emissions (Helliwell et al. 1964; Kimura 1968; Helliwell and Katsufrakis 1974) and electron precipitation into the ionosphere (Imhof et al. 1983 and references therein) among others. As was mentioned above, most theoretical work dealing with the phenomena discussed (see, e.g. Ashour-Abdalla 1970; Nunn 1971; Palmadesso and Schmidt 1971; Sudan and Ott 1971; Nunn 1974; Karpman et al. 1974a, b; Karpman and Shklyar 1977; Vomvoridis et al. 1982; Omura and Matsumoto 1982; Inan et al. 1982; Bell 1984; Shklyar et al. 1992, and references therein) was limited to the case of whistler-mode wave propagation along the ambient magnetic field. The theory presented here permits us to remove this limitation and to enrich theoretical studies of various phenomena related to wave–particle interactions in the magnetosphere.

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Appendix A

With the dipole center in the origin of the Cartesian coordinate system x, y, z, the expressions for dipole coordinates through Cartesian ones have the form

$$L(x,y,z) = \frac{1}{R_e}\frac{r^3}{(x^2+y^2)}; \quad \Phi(x,y,z) = \operatorname{atan}(y/x); \quad M(x,y,z) = R_e^2\frac{z}{r^3}; \qquad r \equiv (x^2+y^2+z^2)^{1/2} \tag{A.1}$$

The dipole coordinate L has a constant value on a given field line, and is equal to the distance from the dipole center to the top of the field line at the equator, measured in units of Earth's radius R_e. This coordinate is also called McIlwain's parameter. The curvilinear axes corresponding to coordinate M coincide with the lines of force of the dipole magnetic field. Thus, in this coordinate system, the ambient magnetic field has only a M-component. The coordinate M is measured along the line of force from the equator so that, at the equator, $M = 0$. The coordinate Φ is the azimuth angle, i.e. the angle which the corresponding meridian plane, containing the point (x, y, z) and dipole axis, forms with the (x, z)-plane. The corresponding Lamé coefficients, which are essential quantities characterizing a curvilinear coordinate system, are equal to

$$h_{\mathrm{L}} = \frac{R_e}{(1+\Lambda)(1+4\Lambda)^{1/2}}; \quad h_{\Phi} = \frac{R_e L}{(1+\Lambda)^{3/2}}; \quad h_{\mathrm{M}} = \frac{R_e L^3}{(1+\Lambda)^{5/2}(1+4\Lambda)^{1/2}}. \tag{A.2}$$

In (A.2),

$$\Lambda = \frac{z^2}{x^2+y^2} \equiv \tan^2\lambda, \tag{A.3}$$

where λ is the geomagnetic latitude. In dipole coordinates, the magnitude of the magnetic field is expressed as

$$B_0(L,M) = \frac{B_e}{L^3}(1+\Lambda)^{5/2}(1+4\Lambda)^{1/2} \equiv \frac{B_e R_e}{h_{\mathrm{M}}}, \tag{A.4}$$

where B_e is the magnitude of magnetic field on the Earth's surface at the equator ($\Lambda = \lambda = 0$).

We see that Lamé coefficients and the magnetic field strength are more easily expressed through L, Λ (or L, λ) than through L, M, although, the last two coordinates form the orthogonal dipole coordinate system. The relation that fills this gap has the form

$$M^2 L^4 = \Lambda(1+\Lambda)^3, \tag{A.5}$$

which determines Λ as the function of M and L. Dipole coordinates in meridional plane are shown in Fig. 5.

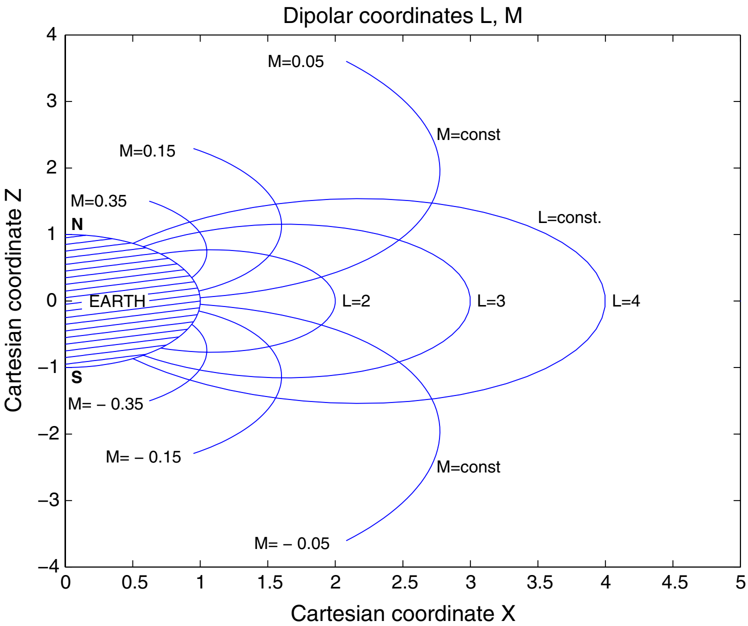

Fig. 5 Dipole coordinates L, M in a meridional plane

Appendix B

To find the relations between the new momenta and the orthogonal projections of the vector $\mathbf{P}$ in a dipole coordinate system we rewrite (3.10) in vector form

$$\mathbf{P} = p_{\rm L}\nabla L(\mathbf{r}) + p_{\Phi}\nabla\Phi(\mathbf{r}) + p_{\rm M}\nabla M(\mathbf{r}). \tag{B.1}$$

Since the dipole coordinate system is an orthogonal one,

$$\nabla L(\mathbf{r}) = |\nabla L(\mathbf{r})|\mathbf{e}_{\rm L}; \quad \nabla\Phi(\mathbf{r}) = |\nabla\Phi(\mathbf{r})|\mathbf{e}_{\Phi}; \quad \nabla M(\mathbf{r}) = |\nabla M(\mathbf{r})|\mathbf{e}_{\rm M}, \tag{B.2}$$

where $\mathbf{e}_{\rm L}$, $\mathbf{e}_{\Phi}$, and $\mathbf{e}_{\rm M}$ are unit vectors along the corresponding coordinate axes. Thus, relation (B.1) gives an orthogonal expansion of the vector $\mathbf{P}$ in dipole coordinates. Equating vector components in (B.1) we find

$$p_{\rm L} = h_{\rm L}P_{\rm L}; \quad p_{\Phi} = h_{\Phi}P_{\Phi}; \quad p_{\rm M} = h_{\rm M}P_{\rm M}, \tag{B.3}$$

where

$$\begin{aligned} h_{\rm L}^{-1} &\equiv |\nabla L(\mathbf{r})| = \sqrt{\left(\frac{\partial L}{\partial x}\right)^2 + \left(\frac{\partial L}{\partial y}\right)^2 + \left(\frac{\partial L}{\partial z}\right)^2}; \\ h_{\Phi}^{-1} &\equiv |\nabla \Phi(\mathbf{r})| = \sqrt{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2}; \\ h_{\rm M}^{-1} &\equiv |\nabla M(\mathbf{r})| = \sqrt{\left(\frac{\partial M}{\partial x}\right)^2 + \left(\frac{\partial M}{\partial y}\right)^2 + \left(\frac{\partial M}{\partial z}\right)^2}. \end{aligned} \tag{B.4}$$

Equations (3.11) and (B.4) determine new momenta $p_{\rm L}$, p_{Φ}, and $p_{\rm M}$ as the functions of old variables, and give the definition of Lamé coefficients.

Lamé coefficients have a simple meaning which is well known from analytical geometry (see, for instance, Davis and Snider 1995): they represent the lengths along the corresponding curvilinear coordinate axes per unit coordinate. For example, the length along the L-coordinate is equal to

$$ds = h_{\rm L}dL \tag{B.5}$$

and similar expressions hold for the Φ and M-axes.

Further on we will need the expression for the gradient of a scalar function in curvilinear coordinates:

$$\nabla\varphi = \frac{1}{h_{\rm L}}\frac{\partial\varphi}{\partial L}\mathbf{e}_{\rm L} + \frac{1}{h_{\Phi}}\frac{\partial\varphi}{\partial\Phi}\mathbf{e}_{\Phi} + \frac{1}{h_{\rm M}}\frac{\partial\varphi}{\partial M}\mathbf{e}_{\rm M}, \tag{B.6}$$

and for orthogonal projections of the *curl* of a vector:

$$\begin{aligned} \nabla\times\mathbf{A} = \frac{1}{h_{\rm L}h_{\Phi}h_{\rm M}}\begin{vmatrix} h_{\rm L}\mathbf{e}_{\rm L} & h_{\Phi}\mathbf{e}_{\Phi} & h_{\rm M}\mathbf{e}_{\rm M} \\ \frac{\partial}{\partial L} & \frac{\partial}{\partial\Phi} & \frac{\partial}{\partial M} \\ h_{\rm L}A_{\rm L} & h_{\Phi}A_{\Phi} & h_{\rm M}A_{\rm M} \end{vmatrix} \equiv \frac{\mathbf{e}_{\rm L}}{h_{\Phi}h_{\rm M}}\left(\frac{\partial(h_{\rm M}A_{\rm M})}{\partial\Phi} - \frac{\partial(h_{\Phi}A_{\Phi})}{\partial M}\right) \\ + \frac{\mathbf{e}_{\Phi}}{h_{\rm L}h_{\rm M}}\left(\frac{\partial(h_{\rm L}A_{\rm L})}{\partial M} - \frac{\partial(h_{\rm M}A_{\rm M})}{\partial L}\right) + \frac{\mathbf{e}_{\rm M}}{h_{\rm L}h_{\Phi}}\left(\frac{\partial(h_{\Phi}A_{\Phi})}{\partial L} - \frac{\partial(h_{\rm L}A_{\rm L})}{\partial\Phi}\right). \end{aligned} \tag{B.7}$$

These expressions will be used in the course of Hamiltonian transformations to the curvilinear dipole coordinates.

Appendix C

In the transformation to a one-dimensional problem, we will systematically neglect the second order terms in wave amplitude, in accordance with the accuracy of our consideration. As the Hamiltonian (3.65), with accepted accuracy, is still equal to the particle kinetic energy, we can write

$$W = \frac{p_{\|}^2}{2m} + \mu|\Omega| + q|A|V_n \sin\xi_n. \tag{C.1}$$

Excluding μ from (C.1) with the help of (3.68), and resolving the obtained relation with respect to $p_{\|}$ to first order in the wave amplitude, we obtain

$$p_{\|} = p_0 - \frac{mq|A|}{p_0} V_n \sin\xi_n, \tag{C.2}$$

where p_0 is equal to the particle parallel momentum expressed through W and C_n to zero order in the wave amplitude

$$p_0 = \sqrt{2m\left[\left(1 - \frac{n\omega_{\mathrm{c}}}{\omega}\right)W + \frac{\omega_{\mathrm{c}}}{\omega}C_n\right]}\,\mathrm{sign}p_{\|}. \tag{C.3}$$

Writing $dW/ds = (dW/dt)/(ds/dt)$ and using (3.67), (3.66) we find that, to accepted accuracy, the equation for W takes the form

$$\frac{dW}{ds} = -\frac{q|A|\omega}{p_0} V_n \cos\xi_n. \tag{C.4}$$

Similarly, relation $d\xi_n/ds = (d\xi_n/dt)/(ds/dt)$, together with the expression (3.46) for ξ_n and Eqs. (3.66), (C.2) give

$$\begin{aligned}\frac{d\xi_n}{ds} &= k_{\|} + \frac{n\omega_{\mathrm{c}} - \omega + nq|A|(\partial V_n/\partial\mu)_{p_{\|}} \sin\xi_n}{p_0/m - (q|A|/p_0)V_n \sin\xi_n + q|A|(\partial V_n/\partial p_{\|})_{\mu} \sin\xi_n} \\ &\simeq k_{\|} + \frac{m(n\omega_{\mathrm{c}} - \omega)}{p_0} + \frac{m^2(n\omega_{\mathrm{c}} - \omega)q|A|}{p_0^2}\left(\frac{V_n}{p_0} - \frac{\partial V_n}{\partial p_{\|}}\right)\sin\xi_n + \frac{mnq|A|}{p_0}\frac{\partial V_n}{\partial\mu}\sin\xi_n.\end{aligned} \tag{C.5}$$

When evaluating the derivatives of the quantity V_n, we can already use relations between W, C_n and $p_{\|}$, μ in the zero approximation

$$\begin{aligned} W &= \frac{p_{\|}^2}{2m} + \mu\omega_{\mathrm{c}}; \\ C_n &= \frac{np_{\|}^2}{2m} + (n\omega_{\mathrm{c}} - \omega)\mu; \end{aligned} \tag{C.6}$$

According to general formulas for changing variables in differential expressions, we obtain from (C.6)

$$\begin{aligned} \left(\frac{\partial}{\partial p_{\|}}\right)_{\mu} &= \frac{np_0}{m}\left(\frac{\partial}{\partial C_n}\right)_W + \frac{p_0}{m}\left(\frac{\partial}{\partial W}\right)_{C_n}; \\ \left(\frac{\partial}{\partial \mu}\right)_{p_{\|}} &= (n\omega_{\mathrm{c}} - \omega)\left(\frac{\partial}{\partial C_n}\right)_W + \omega_{\mathrm{c}}\left(\frac{\partial}{\partial W}\right)_{C_n}; \end{aligned} \tag{C.7}$$

Using (C.7) we can rewrite Eq. (C.5) for ξ_n in the form

$$\frac{d\xi_n}{ds} = k_\parallel + \frac{m(n\omega_c - \omega)}{p_0} + \frac{m^2(n\omega_c - \omega)q|A|}{p_0^3} V_n \sin\xi_n + \frac{m\omega q|A|}{p_0}\left(\frac{\partial V_n}{\partial W}\right)_{C_n} \sin\xi_n. \quad \text{(C.8)}$$

We can easily check that Eqs. (C.8), (C.4) may be derived from the Hamiltonian (3.69).

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