V. L. Ginzburg

THE PROPAGATION OF ELECTROMAGNETIC WAVES IN PLASMAS

ERRATA

Page	for	read
v, line 2	viii	xiii
viii, line 5 from bottom	254	255
110, line 3, equation (11.36)	$\|1-\sqrt{u}\cos\alpha\|$	$\|1-\sqrt{u}\|\cos\alpha\|\|$
111, line 1, equation (11.37)	$1\pm\sqrt{u}\cos\alpha-is$	$1\pm\sqrt{u}\|\cos\alpha\|-is$
228, line 1	an	and
249, line 2, after equation (22.23)		

after $v_{\mathrm{gr}}=d\omega/dk$ insert $(k=\omega n/c)$

V. L. Ginzburg: The Propagation of Electromagnetic Waves in Plasmas

THE PROPAGATION OF ELECTROMAGNETIC WAVES IN PLASMAS

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THE PROPAGATION OF ELECTROMAGNETIC WAVES IN PLASMAS

by

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CONTENTS

PREFACE TO THE ENGLISH EDITION

THE AUTHOR of a book is not, of course, in a position to judge its merits. I trust that I shall not seem to overlook this fact if I express my hope that the appearance of an English edition of my book will prove useful. Perhaps I need only mention that the existence of a translation will make it possible to learn of the results obtained by Soviet physicists, which are still little known outside the U.S.S.R. For an example, it is sufficient to consider the recently published very complete book by Dr. K. G. Budden, *Radio Waves in the Ionosphere*, Cambridge 1961. In this book, out of 243 references to the literature, there is only one to Soviet work, and the date is 1948. In the present edition of my own book, out of a total of over 500 references, some 240 are to the work of Soviet authors. The ratio of the numbers of references may be some reflection of the quantities of scientific results concerned, though it must not be forgotten that many problems have been independently resolved in different countries.

The book certainly contains only a selection of the available material, on account of the inclination, mentioned in the Preface to the Russian edition, to attempt not a complete exposition but an account of those problems with which the author is best acquainted. This has led to some weaknesses in the book, of which I am fully conscious. For example, no discussion is given of such important problems as wave propagation in the presence of various (in particular, statistical) inhomogeneities; in plasmas in the presence of beams, and generally in non-equilibrium conditions; and in bounded systems (in the presence of walls, etc.) Moreover, insufficient attention is given to the allowance for spatial dispersion in the propagation of waves in various media, and particularly in plasmas.

If I were rewriting or revising the book now, I should modify it very considerably, and, I hope, thereby improve it. Lack of time, however, has prevented me from making any extensive changes in the present edition. Apart from a few minor comments and corrections of misprints, therefore, I have merely asked for the addition of translations of three recent papers and appended a further list of references. It should be borne in mind that these added references do not appear in the text itself and are included simply to assist the reader.

Finally, my sincere thanks are due to the translators, Dr. Sykes and Dr. Tayler, for the trouble they have taken and for a number of useful comments.

V. L. Ginzburg

PREFACE TO THE RUSSIAN EDITION

In recent years increasing attention has been given to plasmas, plasma dynamics, and various processes occurring in plasmas. An important problem in this field is the study of the propagation in plasmas of electromagnetic waves of various types (radio waves, plasma waves, hydromagnetic waves, etc.). A particular case of this is the behaviour of a plasma, i.e. an ionised gas, in an electromagnetic field which is uniform in space but variable in time.

The present book deals with such problems, which are of importance in the theory of radio wave propagation in the Earth's ionosphere, in radio astronomy and astrophysics, and in the physics of laboratory plasmas.

The study of wave propagation in plasmas involves a great number of different problems and various forms of these problems. The relevant literature is very extensive, especially if kindred problems of plasma physics are included. It must be emphasised that no attempt is made here to review this literature. The author has tried rather to discuss as simply as possible some of the fundamental results, in particular for problems which he himself has helped to investigate. Thus neither the exposition of the material nor the list of references can lay claim to completeness. However, there seems to be no book, Russian or other, in which wave propagation in plasmas is discussed in even the amount of detail given here. The publication of this book would therefore seem to be justified. In compiling it use has been made, where possible, of the material contained in the author's earlier book *Theory of Radio Wave Propagation in the Ionosphere* (1949) and in the second part of the book by the author, Ya. L. Al'pert and E. L. Feĭnberg, *Radio Wave Propagation* (1953).†

In order to assist the reader and to make the book more useful for reference, certain formulae have been repeated in different sections, and two sections are devoted to the collation of the principal results. Moreover, the list of references includes some original and review articles on subjects which are discussed only briefly or not at all in this book. The most important problem

† Both are in Russian—Translators.

omitted concerns the propagation of waves in the presence of statistical inhomogeneities.

The author's thanks for useful comments and advice are due to E. A. Benediktov, B. N. Gershman, A. V. Gurevich, N. G. Denisov, V. V. Zheleznyakov, N. A. Mityakov, M. S. Rabinovich and V. P. Silin.

V. L. Ginzburg

NOTATION†

N	(number) density of electrons
$N_{\max}$	electron density at maximum of layer
N_m	density of neutral particles (atoms, molecules)
N_+	density of positive ions
N_-	density of negative ions
$N_+ + N_- = N_i$	density of ions
T	temperature of plasma or (where the ion and electron temperatures are not equal) ion temperature (always given in degrees Kelvin)
T_e	electron temperature
T and T_0	length of pulse and period of oscillation (mainly in § 21)
M	mass of heavy particles (ions, atoms and molecules)
λ	wavelength
$2\pi c/\omega = \lambda_0$	wavelength in vacuum
$2\pi c/\omega_H = \lambda_H$	defined by (10.1)
ω	circular frequency
f	frequency
$\sqrt{(4\pi e^2 N/m)} = \omega_0$	plasma frequency
$\omega + i\gamma = \omega'$	complex frequency; $p = i\omega' = i\omega - \gamma$
γ	damping coefficient ($E = E_0 e^{i\omega t} \cdot e^{-\gamma t}$)
$\boldsymbol{E}$	electric field
$\boldsymbol{H}$	magnetic field
$\boldsymbol{B}$	magnetic induction
$\boldsymbol{e}, \boldsymbol{h}$	microscopic fi lds
$\boldsymbol{H}^{(0)}$	external (constant) magnetic field
$\boldsymbol{E} + 4\pi \boldsymbol{P} = \boldsymbol{D}$	electric induction (electric displacement)
$F_\pm$	$= E_x \pm iE_y$ (11.10)
g	Fourier transform of E
$\boldsymbol{P}$	polarisation of the medium
Φ	electric field potential (for a potential field $\boldsymbol{E} = -\mathbf{grad}\,\Phi$)
$\boldsymbol{j}$	conduction current density
$\boldsymbol{j} + \partial \boldsymbol{P}/\partial t = \boldsymbol{j}_t$	total current density due to the motion of charges
$\operatorname{div} \boldsymbol{D}/4\pi = \varrho$	density of "free charges"; also, density of microcharges
$\operatorname{div} \boldsymbol{E}/4\pi = \bar{\varrho}$	mean density of microcharges
$\varepsilon - i \cdot 4\pi\sigma/\omega = \varepsilon'$	complex permittivity
ε	permittivity
σ	conductivity
ε_{ik}, σ_{ik}, ε'_{ik} $(= \varepsilon_{ik} - i \cdot 4\pi\sigma_{ik}/\omega)$	respectively permittivity, conductivity and complex permittivity tensors

† Symbols which occur only rarely are not given. In many cases suffixes also are omitted (for instance, in the text cross-sections are denoted by q with appropriate suffixes m, i, etc., but only the symbol q is given here). The quantities denoted by the same letter do not usually appear in the same section. The charge and mass of the electron, the velocity of light and the quantum constant are denoted, as usual, by e, m, c and $\hbar$.

n	index of refraction; suffixes 1, 2, 3 to n and other quantities pertain to the extraordinary, ordinary and plasma waves respectively
$n - i\varkappa = \tilde{n}$	(used when $\sigma = 0$)
$\varkappa$	index of absorption (damping)
$\varkappa$	Boltzmann's constant $= 1{\cdot}38 \times 10^{-16}$ erg/deg (usually in the expression $\varkappa T$)
$2\omega\,\varkappa/c = \mu$	coefficient of absorption
$\mu_0, \mu_\Omega, \mu_{2\Omega}$	depths of modulation and cross-modulation (only in § 39)
μ	(magnetic) permeability
ξ_i, ξ_e	defined by (8.46)
$\int \mu\, ds = \tau$	optical thickness
τ	relaxation time; free time
ν	collision frequency; suffixes m and i pertain to collisions with molecules and ions
ν_{eff}	effective collision frequency
v	velocity
$\bar{v}$	mean velocity
β	$= v/c$
β_T	$= \sqrt{(\varkappa T/mc^2)}$
W	energy
δ	mean fraction of energy transferred from electron to heavy particle in one collision
$2m/M = \delta_{\text{el}}$	value of δ for elastic collisions
$\frac{1}{2}mv^2 = K$	kinetic energy of electrons
$\tau\bar{v} = \bar{v}/\nu_{\text{eff}} = l$	mean free path
$x \equiv x_1, y \equiv x_2, z \equiv x_3$	Cartesian coordinates
t	time
$\boldsymbol{r}$	radius vector
$\boldsymbol{k}_0 - i\boldsymbol{q} = \boldsymbol{k}$	wave vector ($k_0 = \omega n/c$, $q = \omega\varkappa/c = \frac{1}{2}\mu$)
S	collision integral
$f(t, \boldsymbol{r}, \boldsymbol{v})$	distribution function
$f_0(v), \boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v$	symmetrical and unsymmetrical parts of the distribution function
$f_{00}(v)$	Maxwellian distribution function
$\varphi(t, \boldsymbol{r}, \boldsymbol{v})$	deviation of the distribution from its equilibrium value
D	Debye length ($= \sqrt{(\varkappa T/8\pi e^2 N)}$ for $T_e = T$ and $N_+ = N$)
α	$= 16\pi D^3 N = 2\varkappa TD/e^2$
$d\Omega$	element of solid angle
θ	angle of scattering
θ	angle between normal to wave front and z-axis
θ_0	value of θ at boundary of layer (angle of incidence); in §§ 19 and 20 $\alpha(z) = \sin\theta(z)$ and $\alpha_0 = \sin\theta_0$
q	cross-section
p	$= (c/\omega)\,\partial\Psi/\partial y = ck_y/\omega$
q	$= (c/\omega)\,\partial\Psi/\partial z = (n - i\varkappa)\cos\theta$ (§ 29)
$c/n = v_{\text{ph}}$	phase velocity
v_{gr}	group velocity
α	angle between wave vector $\boldsymbol{k}$ and external magnetic field $\boldsymbol{H}^{(0)}$
χ	angle between $\boldsymbol{H}^{(0)}$ and z-axis (only in § 29)
$\lvert e\rvert H^{(0)}/mc = \omega_H$	electron gyration frequency
$\lvert e\rvert H^{(0)}/Mc = \Omega_H$	ion gyration frequency
Ω	modulation frequency (only in § 39)

u	$= \omega_H/\omega^2$ } dimensionless parameters
v	$= 4\pi\, e^2 N/m\omega^2 = \omega_0^2/\omega^2$ } dimensionless parameters
s	$= \nu_{\text{eff}}/\omega$ } dimensionless parameters
u_M	$= \Omega_H{}^2/\omega^2$ (10.3)
u_L	$= u \cos^2\alpha$ } (11.36)
u_T	$= u \sin^2\alpha$ } (11.36)
A, B, C	coefficients in the wave equations (11.3) for a magnetoactive plasma
$E_{y\,1,2}/E_{x\,1,2} = K_{1,2}$	polarisation coefficients for the normal waves 1 and 2
p_e	electron pressure
p_i	ion pressure
ϱ_M	density of the medium
ϱ_0	unperturbed density of the medium
$\sqrt{(\partial p/\partial \varrho_M)} = u_0$	velocity of sound
$\dfrac{u_0}{H_0/\sqrt{(4\pi\,\varrho_0)}} = \zeta$	(this letter also denotes the variable (17.2))
Ψ	quantity appearing in expressions such as $E = E_0 e^{-i\omega\Psi/c}$
Ψ'	$= d\Psi/dz$, etc.
φ	wave phase
R	amplitude coefficient of reflection
D	amplitude coefficient of transmission
$c\boldsymbol{E} \times \boldsymbol{H}/4\pi = \boldsymbol{S}$	flux of electromagnetic energy
f_{cr}	critical frequency
z_m	half-thickness of parabolic layer
z_a	apparent height
L_0	optical path
L_{gr}	group path
Δt_{ph}	phase delay time
Δt_{gr}	group delay time

CHAPTER I

THE FUNDAMENTAL THEORY OF ELECTROMAGNETIC WAVE PROPAGATION IN PLASMAS

§ 1. GENERAL INTRODUCTION. THE PLASMA PARAMETERS IN VARIOUS CASES

Various cases of wave propagation in plasmas

THE PROPAGATION of electromagnetic waves in plasmas (that is, in partly or completely ionised gases) occurs in a number of problems. The most important of these are the following.

The propagation of radio waves in the upper layers of the Earth's atmosphere (the ionosphere).

The propagation of various low-frequency electromagnetic waves in the ionosphere and the adjoining regions of interplanetary space.

The propagation of radio waves, of cosmic origin, in the Sun's atmosphere, in nebulae, and in interstellar and interplanetary space. These waves are studied by the methods of radio astronomy. Similar cases are the propagation of waves used in obtaining radar echoes from the Moon and planets, and communication with distant artificial Earth satellites, space rockets, etc.

The propagation of low-frequency (hydromagnetic and acoustic) waves in cosmic conditions.

The propagation of plasma waves in cosmic conditions (in the solar corona, etc.) and in the Earth's ionosphere.

The propagation of various kinds of electromagnetic waves in plasmas generated in the laboratory (in studies of gas discharges, in apparatus for work on controlled thermonuclear reactions, etc.).

Since almost all matter in the universe is in the plasma state, the whole of optical astronomy is also concerned with the propagation of electromagnetic waves in plasmas. In the optical region of the spectrum, however, the essential properties specific to plasmas do not appear at the densities occurring in stellar atmospheres. We shall therefore not discuss waves in the infra-red region of the spectrum or at shorter wavelengths.

The concept of a plasma is appropriate and useful not only for a gas but also in considering certain properties of solids (metal optics, discrete energy losses in solids, cyclotron resonance in semiconductors when there are numerous current carriers). However, we shall also leave out of consideration this group of phenomena, which pertain rather to solid-state physics.

Plasma parameters

The plasmas found in Nature or in the laboratory are characterised by parameters which may vary by many orders of magnitude. For example, the electron density N in the interstellar medium is usually between about 10^{-3} and 10 per cm^3, the latter value occurring in interstellar gas clouds.† In the solar corona N is between about 10^4 and 3×10^8, in interplanetary space between about 1 and 10^4, the latter value occurring in the most powerful corpuscular emissions from the Sun. In the Earth's ionosphere N is between about 10^3 and 3×10^6; in apparatus for the utilisation of controlled thermonuclear reactions†† $N \sim 10^{15}$, and for various types of gas-discharge apparatus a typical value is $N \sim 10^{12}$. Finally, the density of conduction electrons in metals is $N \sim 3 \times 10^{22}$, and this is the density which is involved when the concept of a plasma is applied to metals.

Another plasma parameter is the density N_m of neutral particles or the degree of ionisation $r = N/N_m$. In the Earth's ionosphere we have in the lower D layer $N_m \sim 10^{15}$ and $r \sim 10^{-11}$ to 10^{-12}, in the E layer $N_m \sim 10^{12}$ and $r \sim 10^{-7}$, and in the F layer N_m less than about 10^{10} and r less than about 10^{-4}. (The structure of the ionosphere is discussed in § 30.) In the solar corona N_m is practically zero, i.e. $r = \infty$; elsewhere in the universe N_m is sometimes much less than N, but often (except in the neighbourhood of hot stars) $N \ll N_m$, i.e. the gas is only slightly ionised.

If only positive ions are present in the plasma, their density $N_+ = N$ in the conditions of quasineutrality which usually hold. If negative ions also may be present, then $N_+ = N + N_-$ (we assume for simplicity that all ions are singly charged), and N_-/N_+ or N_-/N is a further parameter.

The free path of the particles also varies in accordance with the difference in density, and over about the same range.

The temperature of various plasmas is as follows. All temperatures throughout the book are given in degrees Kelvin.

Earth's ionosphere		$T \sim 300$ to 3000
Interstellar gas	(weakly ionised regions)	$T \sim 100$
	(strongly ionised regions)	$T \sim 10^4$
Solar corona		$T \sim 10^6$

In experimental apparatus for studying controlled thermonuclear reactions†† $T \sim 10^6$ to 10^7. When such machines are used for power generation the temperature will probably be 10^8 to 10^9 degrees.

† All numerical values given are only approximate.

†† *Rendiconti del Terzo Congresso Internazionale sui Fenomeni d'Ionizzazione nei Gas*, Venice, 11–15 June 1957; *Proceedings of the Fourth International Conference on Ionization Phenomena in Gases*, Uppsala, 17–21 August 1959 (North Holland Publishing Co., Amsterdam 1960).

By giving the temperature as a plasma parameter we assume, strictly speaking, that the plasma is in equilibrium (or, more precisely, we assume that the particles in it have a Maxwellian velocity distribution). In fact, however, many formulae to be derived below are within certain limits independent of the form of the particle velocity distribution, or depend on it only slightly. In other cases the ion and molecule velocity distribution is unimportant. In § 8, moreover, we shall consider a two-temperature plasma, in which the electrons and ions both have Maxwellian velocity distributions, but with different temperatures. Nevertheless, we can say that in general the plasmas considered in this book are always assumed to be in equilibrium or quasiequilibrium in velocity space. This restriction is very important since, when the plasma contains various currents and beams of particles (i.e. when the particle velocity distribution function is quite markedly asymmetrical) the propagation of waves in the plasma acquires qualitatively different properties. The most important of these is that, when beams are present, waves in the plasma may be either damped or amplified in the course of propagation. In other words, when beams are present, the plasma is in general unstable: perturbations (waves) generated in it for any reason are amplified with time, in a linear approximation. It is clear from the above that all such phenomena will be ignored in what follows. Moreover, we shall understand by the term *plasma* without further qualification not any plasma, but only one in which (in the absence of an electric field) the velocity distribution either is an equilibrium distribution (a plasma with temperature T) or differs only slightly from an equilibrium or quasiequilibrium state (a two-temperature plasma, etc.).

Plasma properties

The above-mentioned property of plasmas, namely the great range of variation of the parameters, leads to another property, which is important in the study of wave propagation. A plasma is often highly inhomogeneous, so that the propagation of waves takes place in media whose parameters vary from point to point. Of course, inhomogeneous media occur in other problems also, but usually there are sharp boundaries; relatively slight and smooth changes in the properties of the medium are found much more rarely. In plasmas, on the other hand, sharp boundaries seldom appear, and the typical situation is the existence of smooth and very large changes in the properties of the medium. These changes are often so large that the permittivity ε changes sign.

A third characteristic property of plasmas is, indeed, the fact that they furnish without especial difficulty a medium with $\varepsilon \approx 0$ and weak absorption. This leads to the possibility of plasma waves which are only very slightly damped, and to certain other important features.

A fourth property of plasmas is the marked change in their behaviour brought about by the action of a constant magnetic field. As a result of this, even magnetic fields (such as the Earth's field) which are very weak by ordinary standards may considerably alter the propagation of waves in the Earth's ionosphere and elsewhere.

A fifth property of plasmas is the appearance of non-linearity of their electromagnetic properties even in fields which are quite easily obtained. This leads to phenomena of non-linear interaction (cross-modulation, etc.) of waves propagated in a plasma. In other media (except ferromagnetics, ferroelectrics, etc.), non-linear effects occur only in very strong fields.

Of course, the selection of just five properties of plasmas is somewhat arbitrary, but it is undeniable that plasmas have many distinctive properties. It is therefore reasonable to consider various problems of plasma physics independently, as has long been done in practice, and in particular the propagation of electromagnetic waves in plasmas.

§ 2. FUNDAMENTAL EQUATIONS. THE NATURE OF THE APPROXIMATIONS USED

The field equations

Our problem is to give a quantitative discussion of the propagation of electromagnetic waves in plasmas. In most cases the wavelength λ is much greater than the mean distance $\bar{r} \sim N^{-1/3}$ between electrons or ions, even for the shortest radio waves involved. For, in wave propagation in the ionosphere and the solar corona, $N \gtrsim 10^3$ and $\bar{r} \lesssim 0{\cdot}1$ cm, whereas the waves mainly considered have λ greater than 1 metre. Thus the condition $\lambda \gg \bar{r}$ must hold for radio waves almost everywhere except in interstellar space. This case will be discussed in § 37; elsewhere in the book we shall suppose that the above condition is satisfied. For the same reason we can and must discuss the propagation of electromagnetic waves in plasmas on the basis of the usual phenomenological equations of macroscopic electrodynamics:

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} + \frac{i\omega}{c}\boldsymbol{D}, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{D} = 4\pi\varrho, \tag{2.2}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{i\omega}{c}\boldsymbol{H}, \tag{2.3}$$

$$\operatorname{div} \boldsymbol{H} = 0, \tag{2.4}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field vectors, $\boldsymbol{D}$ the electric induction (electric displacement) vector, $\boldsymbol{j}$ the current density and ϱ the density of "free charge". We shall always use the absolute Gaussian system of units and the customary notation.

Equations (2.1) and (2.3) presuppose that all variables are harmonic functions of time, i.e. are proportional to $e^{i\omega t}$. This factor will often be omitted in what follows. Non-harmonic processes in which the principle of superposition holds can be discussed by expanding all quantities as Fourier series or integrals.

Moreover, in equations (2.3) and (2.4) and everywhere below, the magnetic induction $\boldsymbol{B}$ is replaced by the magnetic field $\boldsymbol{H}$, since the magnetic permeability of plasmas is practically equal to unity (see § 3). Taking the curl of equation (2.3), and using (2.1) and the identity $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} \equiv -\triangle \boldsymbol{E} + \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E}$, we have

$$-\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E} + \frac{\omega^2}{c^2}\left(\boldsymbol{D} - \frac{4\pi i}{\omega}\boldsymbol{j}\right) = \triangle \boldsymbol{E} - \mathrm{grad}\,\mathrm{div}\,\boldsymbol{E} + \frac{\omega^2}{c^2}\left(\boldsymbol{D} - \frac{4\pi i}{\omega}\boldsymbol{j}\right) = 0. \tag{2.5}$$

In order to use equations (2.1)–(2.4) or (2.5) to solve problems of electrodynamics, the way in which $\boldsymbol{D}$ and $\boldsymbol{j}$ depend on $\boldsymbol{E}$ and $\boldsymbol{H}$ must be given. If the effect of a constant external magnetic field on the properties of the plasma is neglected, $\boldsymbol{D}$ and $\boldsymbol{j}$ depend only on $\boldsymbol{E}$ and are parallel to it (if the medium is isotropic), i.e.

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{j} = \sigma \boldsymbol{E}, \tag{2.6}$$

where ε is the permittivity (dielectric permeability or dielectric constant) and σ the conductivity. When the effect of an external magnetic field is taken into account the medium becomes anisotropic (magnetoactive), and $\boldsymbol{D}$, $\boldsymbol{j}$ and $\boldsymbol{E}$ are related by the more general linear formulae

$$D_i = \varepsilon_{ik} E_k, \quad j_i = \sigma_{ik} E_k \qquad (i, k = 1, 2, 3), \tag{2.7}$$

where summation over repeated suffixes is understood; $D_1 \equiv D_x$, $D_2 \equiv D_y$, $D_3 \equiv D_z$, etc. Furthermore, the permittivity and conductivity tensors ε_{ik} and σ_{ik} depend on the external magnetic field $\boldsymbol{H}^{(0)}$.

The field equations (2.1)–(2.4) involve the total fields from all sources, and, since ε_{ik} and σ_{ik} in (2.7) depend on the external magnetic field, these equations may formally be regarded as non-linear. However, in the theory of electromagnetic wave propagation the external magnetic field is usually supposed given and independent of time, and consequently the field equations essentially relate only to the field of the wave itself. In the case (2.7) these equations are therefore linear. Sometimes, however, we have "truly" non-linear phenomena, which lead to an interaction of different waves in the plasma; then ε and σ (or ε_{ik} and σ_{ik}) are themselves functions of the electric field. The non-linear phenomena appear only in fairly strong fields, and will be discussed in detail in Chapter VIII. Elsewhere in this book we shall, unless the contrary is specifically stated, assume that the equations are linear, i.e. that ε and σ (or ε_{ik} and σ_{ik}) are independent of the field vectors. In the great majority of cases of wave propagation in the ionosphere and in cosmic conditions this linear approximation is entirely valid.

Inhomogeneity of the medium has the result that ε and σ (or ε_{ik} and σ_{ik}) depend on the coordinates. On account of dispersion, all these quantities are functions of the frequency ω also.

The permittivity and the conductivity may conveniently be combined into one quantity, the complex permittivity

$$\varepsilon' = \varepsilon - i \cdot 4\pi\sigma/\omega. \tag{2.8}$$

Similarly for an anisotropic medium we may conveniently introduce the complex permittivity tensor

$$\varepsilon'_{ik} = \varepsilon_{ik} - i \cdot 4\pi\,\sigma_{ik}/\omega\,. \tag{2.9}$$

The imaginary unit i should not be confused with the suffix i.

In the isotropic case (2.6), equation (2.5) becomes

$$\triangle \boldsymbol{E} - \operatorname{grad}\operatorname{div}\boldsymbol{E} + \frac{\omega^2}{c^2}\,\varepsilon'\,\boldsymbol{E} = 0; \tag{2.10}$$

alternatively, eliminating the field $\boldsymbol{E}$ from equations (2.1) and (2.3) we obtain

$$\triangle \boldsymbol{H} + \frac{1}{\varepsilon'}\operatorname{grad}\varepsilon' \times \operatorname{curl}\boldsymbol{H} + \frac{\omega^2}{c^2}\,\varepsilon'\,\boldsymbol{H} = 0\,. \tag{2.11}$$

Here we have used the fact that (2.1) and (2.6) give

$$\operatorname{curl}\boldsymbol{H} = \frac{i\,\omega}{c}\,\varepsilon'\,\boldsymbol{E}. \tag{2.1 a}$$

In deriving equations (2.10) and (2.11) it has been assumed that the relations (2.6) are valid in all space. This is not quite true, however, since in field sources, for example an aerial, $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{E}_{ex})$, where $\boldsymbol{E}_{ex}$ is the field due to external electromotive forces. Hence, for example, the right-hand side of equation (2.10) should read $i(4\pi\omega/c^2)\,\boldsymbol{j}_{ex}(\omega, x, y, z)$, where $\boldsymbol{j}_{ex} = \sigma\boldsymbol{E}_{ex}$ is the current density due to external sources of e.m.f. This term in (2.10) and the corresponding term in (2.11) have been omitted, since they vanish in the regions of interest to us (outside sources).

One-dimensional problems. Plane waves

In the general case, where ε' depends on all the coordinates, it is impossible to simplify the very involved equations (2.10) and (2.11). It is therefore very important to note that often the dependence of ε' on only one coordinate need be taken into account. For example, in the Earth's ionosphere the most marked dependence of ε' (or ε'_{ik}) is on the height above the Earth. Within a relatively small area of the Earth's surface, for which the Sun's zenith distance may be regarded as constant (i.e. the curvature of the Earth may be neglected), the horizontal variation of ε' is usually random (due to clouds in the ionosphere, etc.) and is superposed on a regular distribution in which ε' depends only on height (the z coordinate). For the Earth as a whole, the regular distribution of ε' depends not only on the distance from the centre of the Earth but also on position on the Earth's surface, on account of the varying zenith distance of the Sun. This latter dependence, however, is much less marked than the dependence on height, and may be either approximately allowed for or neglected altogether. Similarly, the permittivity ε' in the solar corona may, to a certain approximation, be regarded as depending only on the distance from the photosphere.

Thus one-dimensional problems are of importance, and among them especially the propagation of waves in a plane-parallel medium (when ε'_{ik} depends only on one Cartesian coordinate z). The permittivity of a plasma may, in fact, often be regarded as constant in space, so that we have wave propagation in a homogeneous medium. A detailed analysis of this simplest case is necessary in order to solve various more complex problems. In a homogeneous medium the propagation of monochromatic plane waves of the type $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ is, of course, of fundamental importance. The propagation of pulses and, in particular, of quasimonochromatic pulses, can be reduced to the propagation of monochromatic waves by expanding the field as Fourier integrals over frequency and wave vector. In a plane-parallel medium, the propagation of monochromatic plane waves of the form $\boldsymbol{E} = \boldsymbol{E}_0(z) e^{i(\omega t - k_x x - k_y y)}$ is of greatest interest.

The discussion below is therefore based entirely on a consideration of the propagation of plane waves. Here again, however, we may distinguish a particular case of great importance in both theory and practice, namely the propagation of plane waves when they are incident normally on a layer. For an isotropic medium the problem of oblique incidence reduces to that of normal incidence. When anisotropy (the effect of an external magnetic field) is taken into account, however, even the problem of normal incidence is very complex, and for the case of oblique incidence no rigorous solution is yet known. Using the example of normal incidence, we can at least elucidate some of the main properties of wave propagation in a plane-parallel magnetoactive plasma. The problem of normal incidence is also of great practical significance. It is, for instance, approximately realised in vertical probing of the ionosphere, which is of fundamental importance in radio studies of the ionosphere.

In an isotropic† medium with $\varepsilon' = \varepsilon'(z)$, equation (2.10) takes the following form for a plane wave incident normally:

$$\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2} \varepsilon'(\omega, z) E = 0, \tag{2.12}$$

and this equation applies to both the components E_x and E_y (for which reason we have put simply E). In going from (2.10) to (2.12) we have used the fact that the field $\boldsymbol{E}$ can depend only on z for normal incidence of a plane wave.

† We shall use the term "isotropic medium" (or plasma) for one which might more precisely be called a locally isotropic medium. This means that we assume the validity of the relations (2.6) or, more generally, that in a homogeneous and isotropic medium there is assumed to be no physically preferred direction. If the medium is isotropic but not homogeneous, there may be a preferred direction (for example, when $\varepsilon' = \varepsilon'(z)$ the direction of the gradient, i.e. the z-axis, is preferred). In this case the term "isotropic medium" signifies that the relations (2.6) hold good and that there is no preferred direction other than that due to the inhomogeneity of the medium.

Under the same conditions equation (2.11) becomes

$$\frac{d^2 H_{x,y}}{dz^2} + \frac{\omega^2}{c^2}\varepsilon'(\omega, z) H_{x,y} - \frac{1}{\varepsilon'}\frac{d\varepsilon'(\omega, z)}{dz}\frac{dH_{x,y}}{dz} = 0; \qquad (2.13)$$

by (2.4) $dH_z/dz = 0$, i.e. H_z = constant and is of no interest.

Plasma oscillations

The third (z) component of equation (2.10) is equivalent (if $\omega \neq 0$) to

$$\varepsilon'(\omega, z) E_z = 0. \qquad (2.14)$$

If $\varepsilon'(\omega, z) \neq 0$, it follows from (2.14) that $E_z = 0$, and we have pure transverse waves. The case where $\varepsilon'(\omega, z) = 0$ corresponds to the possible existence of longitudinal oscillations in an isotropic plasma. Then $E_x = 0$, $E_y = 0$, $E_z \neq 0$, and the frequency of the oscillations is determined by the condition $\varepsilon'(\omega, z) = 0$, or, for a homogeneous medium,

$$\varepsilon'(\omega) = 0. \qquad (2.15)$$

The frequency ω which satisfies this equation is complex, i.e. the oscillations are damped. (The absence of increasing oscillations in an equilibrium state follows from general properties of the function $\varepsilon'(\omega)$, and is physically quite evident from the fact that equilibrium exists.) This damping is the reason why longitudinal field oscillations in a medium are not usually considered. The plasma is an exception because here the damping is often very slight, since the imaginary part of $\varepsilon'(\omega)$ is small. In such conditions the frequency of longitudinal oscillations, usually called plasma oscillations, can be determined with sufficient accuracy from the equation

$$\varepsilon(\omega) = 0, \qquad (2.16)$$

which has a real root, the plasma frequency ω_0 (see § 8).

For plane plasma oscillations $E_z = E_z(\omega, z)$, and the magnetic field $\boldsymbol{H} = 0$, as follows directly from equation (2.3).

In a more general approach to (longitudinal) plasma oscillations, i.e. not necessarily considering plane waves, we can begin from the condition $\mathbf{curl}\,\boldsymbol{H} = 0$, whence (see (2.1a))

$$\varepsilon'(\omega, \boldsymbol{r}) \boldsymbol{E} = 0; \qquad (2.14\text{a})$$

the same result is, of course, obtained by starting from the condition $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} = 0$ (see (2.5) and (2.10)). Since for plasma oscillations $\mathbf{curl}\,\boldsymbol{H} = 0$ and $\operatorname{div}\boldsymbol{H} = 0$, in the absence of sources and external magnetic fields we have $\boldsymbol{H} = 0$.

If we consider plasma oscillations of the type $E_z = E_{z0}e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, we see that the quantities ω and $\boldsymbol{k}$ are entirely unrelated, since equation (2.16) determines only the frequency ω. Hence we have used the term "plasma oscillations" and not "plasma waves"; if ω is independent of $\boldsymbol{k}$, the group velocity of the waves is zero, i.e. they transfer no energy. The absence of any flux of energy for the plasma oscillations (2.14a) is immediately evident also from the fact that the Poynting vector $\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{H}/4\pi$ is zero. In reality, however, there is a relation between ω and $\boldsymbol{k}$ in this case also, so that we do in fact have plasma waves. To find the corresponding function $\omega = \omega(k)$ we must make the approximate relations (2.6) more accurate. The reason is that the relations (2.6) or (2.7) do not depend on the manner of spatial variation of the field, i.e. they are valid only if we neglect the spatial dispersion, that is, the dependence of ε' (or ε'_{ik}) on wavelength.

Spatial dispersion

The magnitude of the spatial dispersion, when absorption is neglected, is characterised by the parameter $a/\lambda = an/\lambda_0$, where a is a characteristic length for the medium concerned (the size of the molecules, the lattice constant or, in a plasma, the Debye length), $\lambda_0 = 2\pi c/\omega$ is the wavelength in vacuum, $\lambda = \lambda_0/n$ is the wavelength in the medium, and n is the refractive index. In most cases, even in the optical part of the spectrum, not to mention at radio frequencies, the parameter a/λ is extremely small, and the spatial dispersion is negligible. The time dispersion, which leads to the dependence of ε' and ε'_{ik} on the frequency ω, may, however, be large under such conditions, since it is characterised by the parameter ω/ω_j, where the ω_j are the intrinsic frequencies of the medium. In the case of an isotropic plasma ω_j is represented by the plasma frequency ω_0, the root of equation (2.16).

The spatial dispersion may, nevertheless, be of importance even at radio frequencies. This occurs, in particular, for plasma oscillations, where (as already mentioned) there is no relation between ω and $\boldsymbol{k}$ if the spatial dispersion is neglected. This dispersion is also important near resonances, where $n \to \infty$ and so the parameter an/λ_0 becomes large. Such is the case in a magneto-active plasma, that is, one in a constant external magnetic field; see [1] for some other cases where spatial dispersion is important.

Propagation of various types of waves

Plasma waves will be discussed in § 8. In most other sections we shall not need to consider such waves. This is because the frequency of plasma waves is usually different from that of transverse waves which are propagated in the plasma. Moreover, in the linear approximation used here, the longitudinal waves in a homogeneous plasma, or in a plane-parallel plasma with normal incidence, are entirely unconnected with the transverse waves described by equation (2.12) and the condition

$$E_z = 0. \tag{2.17}$$

The plasma waves and other electromagnetic waves (for example, radio waves) can be regarded as independent (apart from non-linear effects and scattering by fluctuations of electron density) except in regions of an inhomogeneous plasma where $\omega \approx \omega_0$, i.e. near the point $\varepsilon(\omega, \boldsymbol{r}) = 0$ (where ω is the frequency of the radio waves and ω_0 the plasma frequency). For a plane-parallel medium, it is further necessary that the incidence should be oblique. The relation between radio waves and plasma waves in such conditions will be discussed in § 20.

In an anisotropic (magnetoactive) plasma we cannot in general separate the longitudinal and transverse waves. For example, even in a homogeneous medium, and in a wave propagated with a definite velocity in the z-direction, all three components E_x, E_y, E_z are non-zero. The situation is simplified only if the angle α between the direction of the external magnetic field $\boldsymbol{H}^{(0)}$ and the z-axis (i.e. the direction of the wave vector $\boldsymbol{k}$) is zero or $\frac{1}{2}\pi$. The plasma waves in an anisotropic medium are also essentially different from those in an isotropic medium, which are in a sense degenerate. The problem of the propagation of electromagnetic waves in a magnetoactive plasma will be discussed in detail in Chapter III. Here we shall merely remark that, for normal incidence of plane waves on a plane-parallel anisotropic medium, equations (2.5) and (2.7) become

$$\left.\begin{aligned} &\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2}\left(D_x - i\frac{4\pi}{\omega}j_x\right) = 0, \\ &\frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2}\left(D_y - i\frac{4\pi}{\omega}j_y\right) = 0, \\ &D_z - i\frac{4\pi}{\omega}j_z = 0, \quad D_i - i\frac{4\pi}{\omega}j_i = \varepsilon'_{ik}(\omega, z)\, E_k. \end{aligned}\right\} \tag{2.18}$$

The study of electromagnetic wave propagation in plasmas amounts to the solution of two problems. First, we must express the permittivity ε' or ε'_{ik} in terms of parameters characterising the plasma, i.e. in terms of the densities N, N_i and N_m of electrons, ions and neutral particles. Here, of course, we take into account also the dependence of ε' on ω and of ε'_{ik} on ω and $\boldsymbol{H}^{(0)}$. The second problem is to solve the wave equations for given functions $\varepsilon'(\boldsymbol{r})$ or $\varepsilon'_{ik}(\boldsymbol{r})$. For example, in the case of normal incidence it is necessary to integrate equation (2.12) or, in the anisotropic case, equations (2.18). However, this division of the problem into two parts is of only limited value, and often is quite unsuitable when spatial dispersion is taken into account. Nevertheless, from the point of view of the exposition in general, and also for the majority of applications, it seems more correct not to attempt the greatest possible generality immediately. We shall therefore select various particular cases in which the spatial dispersion may be neglected, the wave frequency may be regarded as high or low, and so on. This is the basis of the subsequent discussion. At the same time it should not be forgotten that waves of different frequencies and different types propagated in a plasma (e.g. high-frequency waves, plasma waves, acoustic waves, low-frequency waves, hydromagnetic waves) may be considered by a single procedure and in some cases form single branches (i.e. waves of different types pass into one another when the parameters are appropriately varied). For example, in a magnetoactive plasma the high-frequency and plasma waves form a single branch (see § 12). Hydromagnetic waves propagated in the direction of the external magnetic field differ from transverse high-frequency waves only in frequency; that is, the hydromagnetic and high-frequency (radio) waves lie on a single curve (or branch) when the velocity of wave propagation is plotted as a function of frequency. In the general case, hydromagnetic waves are only a particular form of low-frequency electromagnetic waves (see § 14). Thus, besides investigating and taking account of the characteristic properties of waves of various types, we must also notice their common features and interrelations.

In conclusion, it may be noted that the usual statement of the problem of electromagnetic wave propagation in a plasma is based on the assumption that the plasma parameters are known. A different approach also occurs in practice, namely when the study of wave propagation is a means of determining the plasma parameters, e.g. of measuring the electron density, temperature, etc.

CHAPTER II

WAVE PROPAGATION IN A HOMOGENEOUS ISOTROPIC PLASMA

§ 3. THE COMPLEX PERMITTIVITY OF A PLASMA: ELEMENTARY THEORY

Elementary derivation of the expressions for ε and σ

THE PERMITTIVITY ε and conductivity σ of a plasma are usually determined entirely by the motion of the electrons and ions. The contribution to ε and σ due to the presence of neutral particles (atoms and molecules) need be taken into account only if the degree of ionisation of the gas is very small. At radio frequencies and below, we obtain in practice only a small constant increment in ε, and in what follows we shall neglect it.

To calculate ε and σ, it is sufficient to consider a plasma in a uniform electric field. If this is not sufficient, i.e. if spatial dispersion is important, then it is also incorrect to use only the local characteristics $\varepsilon(\omega)$ and $\sigma(\omega)$ of the medium.

In the general case, ε and σ must be calculated from the Boltzmann equation, and this we shall do in § 6. Here we shall discuss the elementary derivation of the fundamental formulae.

Let $\boldsymbol{r}_k$ be the radii vectores of the electrons, and $\boldsymbol{r}_k^{(i)}$ those of the ions. Then the total current density due to the motion of charges is

$$\boldsymbol{j}_t = e \sum_{k=1}^{N} (\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_k^{(i)}),$$

where $e (< 0)$ is the electron charge, and the dot denotes differentiation with respect to time; the ions are, for definiteness, assumed to be singly charged, and negative ions are assumed absent; thus, if the medium is quasineutral, $N = N_i = N_+$. The high conductivity of the plasma has the result that the condition of quasineutrality may be taken to be very nearly valid.† Next,

† It may be mentioned that the accepted term "quasineutrality" signifies that the medium is neutral (i.e. the mean charge density $\bar{\varrho} = e(N - N_i) = 0$), but consists of free charged particles.

by definition we have

$$\begin{aligned} \boldsymbol{j}_t &= \boldsymbol{j} + i\,\omega\,\boldsymbol{P} = \left(\sigma + i\,\frac{\varepsilon - 1}{4\pi}\,\omega\right)\boldsymbol{E} \\ &= i\,\frac{\omega}{4\pi}\,(\varepsilon' - 1)\,\boldsymbol{E} \\ &= e\sum_{k=1}^{N}(\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_k^{(i)}), \end{aligned} \tag{3.1}$$

where $\boldsymbol{j}$ is the conduction current and $\boldsymbol{P}$ the polarisation and, strictly speaking, all quantities should be taken as averaged over a physically infinitesimal volume and over a time $\Delta t \ll 2\pi/\omega$.†

When several species of ions are present, of course, there is no essential difference, and we need only sum over the coordinates of all the ions. The quasineutrality condition becomes $N + \sum N_{-,l} = \sum N_{+,l}$, where $N_{\pm,l}$ is the density of positive and negative ions of species l (with mass M_l).

If the constant magnetic field and the collisions of electrons with one another and with ions and molecules are neglected, the equation of motion of each electron becomes

$$m\,\ddot{\boldsymbol{r}}_k = e\,\boldsymbol{E}_0\,e^{i\omega t} = e\,\boldsymbol{E}, \tag{3.2}$$

where $\boldsymbol{E}_0$ is the electric field amplitude, constant in space and time, and m is the electron mass.

The solution of equation (3.2) is

$$\boldsymbol{r}_k = -\frac{e\,\boldsymbol{E}}{m\,\omega^2} + \boldsymbol{r}_k^{(0)}(t), \tag{3.3}$$

where $\boldsymbol{r}_k^{(0)}$ is the radius vector of the electron in the absence of the field.

For ions of mass M_l, the equation of motion and its solution are the same as (3.2) and (3.3), but with m replaced by M_l. Using (3.3) and the corresponding

† The field equation (2.1) is obtained by averaging the equation of electron theory

$$\operatorname{curl}\boldsymbol{h} = \frac{4\pi}{c}\,\varrho\,\boldsymbol{v} + \frac{1}{c}\,\frac{\partial \boldsymbol{e}}{\partial t},$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ are the microscopic electric and magnetic fields, and ϱ and $\bar{\boldsymbol{v}}$ the microscopic charge density and velocity of the particles. A comparison of this equation with (2.1) shows that $\bar{\boldsymbol{e}} = \boldsymbol{E}$, $\bar{\boldsymbol{h}} = \boldsymbol{H}$ (the magnetic permeability $\mu = 1$), and the current defined above is

$$\boldsymbol{j}_t = \overline{\varrho\,\boldsymbol{v}} = \boldsymbol{j} + \partial\boldsymbol{P}/\partial t = e\sum_k(\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_k^{(i)}),$$

where $\boldsymbol{P} = (\boldsymbol{D} - \boldsymbol{E})/4\pi$, and the bar denotes averaging.

In the case of variable fields in which we are interested, the quantities $\boldsymbol{j}$ and $\boldsymbol{P}$ need not be used, and we may consider only the "total current" $\boldsymbol{j}_t$ and the complex permittivity $\varepsilon' = \varepsilon_1 - i\varepsilon_2 = \varepsilon - i\cdot 4\pi\sigma/\omega$. It seems more suitable, however, to adhere as closely as possible to the terminology and notation of the theory of fields.

expression for $r_k^{(i)}$, it is clear from (3.1) that $\sigma = 0$ and

$$P = \frac{\varepsilon - 1}{4\pi} E = e \sum_k (r_k - r_k^{(i)}) = -\frac{e^2 E}{\omega^2}\left(\frac{N}{m} + \sum_l \frac{N_l}{M_l}\right),$$

since $\sum [r_k^{(0)} - r_k^{(i)(0)}] = 0$, because $P = 0$ in the absence of a field.

Thus

$$\varepsilon = 1 - \frac{4\pi e^2}{\omega^2}\left(\frac{N}{m} + \sum_l \frac{N_l}{M_l}\right), \tag{3.4}$$

where N_l is the density of ions of either sign with mass M_l.

It is evident from (3.4) that, in the absence of magnetic fields and absorption, the ions are equivalent, as regards their effect on ε, to electrons of density $N_{\text{eff}} = \sum m N_l / M_l$. For $O_2^{\pm}$ ions the ratio $m/M = 1{\cdot}7 \times 10^{-5}$, and for $O^{\pm}$ ions $m/M = 3{\cdot}4 \times 10^{-5}$. Thus the effect of the ions is usually very small, and on account of the quasineutrality condition it can become important in the calculation of ε only when a large number of negative ions are present. We shall not usually take explicit account of the effect of ions when considering an isotropic plasma.

For electrons alone, or taking N to be the effective electron density, we have

$$\varepsilon = 1 - \frac{4\pi e^2 N}{m\,\omega^2} = 1 - 3{\cdot}18 \times 10^9 \frac{N}{\omega^2} = 1 - 8{\cdot}06 \times 10^7 N/f^2, \tag{3.5}$$

where $f = \omega/2\pi$, and we have used the values $e = 4{\cdot}80 \times 10^{-10}$ e.s.u. and $m = 9{\cdot}11 \times 10^{-28}$ g.

The reason why σ is zero under the above assumptions is evident. Since there are no collisions, the electrons do not transfer energy to molecules and ions, but merely oscillate in the field. In the elementary theory the effect of collisions, which leads to the appearance of a non-zero conductivity σ and to the absorption of energy, can be taken into account by adding to the right-hand side of equation (3.2) the frictional force $g\dot{r}$, the explicit value of the coefficient g being derived as follows. The expression $g\dot{r}$ is the mean change in momentum per second due to collisions. This change is equal to $m\nu_{\text{eff}}\dot{r}$, where ν_{eff} is the effective number of collisions per second or effective collision frequency (in each collision the electron gives to the molecule or ion an average momentum of the order of $m\dot{r}$, where $\dot{r}$ is the directed velocity given to the electron by the field). This is essentially a definition of $\nu_{\text{eff}} = g/m$. But evidently $\nu = \pi a^2 N_m \bar{v}$, where a is some effective radius of the molecule and $\bar{v}$ some mean electron velocity (here, for simplicity, we ignore collisions with ions, and electron–electron collisions do not lead directly to friction, because of the law of conservation of momentum).

Thus, when collisions are taken into account, the equation of motion becomes

$$m\ddot{r}_k + m\,\nu_{\text{eff}}\,\dot{r}_k = e\,E_0 e^{i\omega t}. \tag{3.6}$$

Proceeding now as before and using (3.1), we easily have

$$\left.\begin{aligned} \varepsilon' &= 1 - \frac{4\pi e^2 N}{m\omega(\omega - i\,\nu_{\mathrm{eff}})},\\ \varepsilon &= 1 - \frac{4\pi e^2 N}{m(\omega^2 + \nu_{\mathrm{eff}}^2)}, \quad \sigma = \frac{1-\varepsilon}{4\pi}\,\nu_{\mathrm{eff}} = \frac{e^2 N \nu_{\mathrm{eff}}}{m(\omega^2 + \nu_{\mathrm{eff}}^2)}. \end{aligned}\right\} \tag{3.7}$$

If, as often happens,

$$\omega^2 \gg \nu_{\mathrm{eff}}^2, \tag{3.8}$$

then

$$\varepsilon \approx 1 - \frac{4\pi e^2 N}{m\,\omega^2}, \quad \sigma \approx \frac{e^2 N\,\nu_{\mathrm{eff}}}{m\,\omega^2} = 2{\cdot}53 \times 10^8 \frac{N\,\nu_{\mathrm{eff}}}{\omega^2}. \tag{3.9}$$

If, on the other hand,

$$\omega^2 \ll \nu_{\mathrm{eff}}^2, \tag{3.10}$$

then

$$\varepsilon \approx 1 - \frac{4\pi e^2 N}{m\,\nu_{\mathrm{eff}}^2}, \quad \sigma \approx \frac{e^2 N}{m\,\nu_{\mathrm{eff}}}. \tag{3.11}$$

It is clear from the above that the exact value of ν_{eff} remains unknown; it can be determined only from a kinetic discussion (§ 6). Moreover, it follows from the kinetic theory that the exact formulae for ε and σ do not reduce to the expressions (3.7) with any one value of ν_{eff} independent of frequency. Thus the formulae of the elementary theory are themselves approximate; but the approximation is a good one in most cases.

The effective field

Let us now consider the validity of using the mean macroscopic field $\boldsymbol{E}$ in (3.2) and (3.6) as the field acting on an electron. The fact that we have indeed taken the mean macroscopic field to be the effective field $\boldsymbol{E}_{\mathrm{eff}}$ which appears in (3.2) and (3.6) is evident from the fact that the relation (3.1) has been used, and here the field $\boldsymbol{E}$ is by definition the mean macroscopic field. Thus it has been assumed that in the plasma

$$\boldsymbol{E}_{\mathrm{eff}} = \boldsymbol{E}. \tag{3.12}$$

In general, however, the field $\boldsymbol{E}_{\mathrm{eff}}$ acting is not equal to $\boldsymbol{E}$, and for an isotropic medium (the only case we consider, for simplicity) it can be written

$$\boldsymbol{E}_{\mathrm{eff}} = \boldsymbol{E} + 4\pi a \boldsymbol{P}, \tag{3.13}$$

where $\boldsymbol{P}$ is the polarisation and a is some coefficient, which may depend on the density, etc.

In the linear theory, formula (3.13) is the most general possible for an isotropic medium. The value of the coefficient a can be calculated only by assuming a certain model. For example, if we suppose that the molecules of the medium are point dipoles arranged in a random manner, then $a = \frac{1}{3}$ and

$$\boldsymbol{E}_{\text{eff}} = \boldsymbol{E} + \tfrac{4}{3}\pi \boldsymbol{P} = \tfrac{1}{3}(\varepsilon + 2)\,\boldsymbol{E}, \tag{3.14}$$

where the term $4\pi \boldsymbol{P}/3$ is often called the Lorentz polarisation term; the derivation of (3.14) on the above assumptions is given, for example, in [2].

Formula (3.14) is not in general applicable to actual bodies. This is reasonable, since in liquids and solids the distances between the molecules are of the same order as the dimensions of the molecules themselves, which consequently cannot be likened to point dipoles. However, even if the value of a is not a universal constant, it nevertheless follows from experiment that usually $a \neq 0$. Here the question arises of the value of the coefficient a for a plasma. This is an important question; for example, if we use the expression (3.14) in (3.2), where the field $\boldsymbol{E}$ must mean the field acting, then instead of (3.5) we obtain

$$\varepsilon = 1 - \frac{4\pi e^2 N}{m\,\omega^2(1 + 4\pi e^2 N/3m\,\omega^2)}. \tag{3.15}$$

The difference between (3.5) and (3.15) may be very considerable. For example, according to (3.5) $\varepsilon = 0$ when $4\pi e^2 N/m\omega^2 = 1$, but according to (3.15) $\varepsilon = 0$ when $4\pi e^2 N/m\omega^2 = 3/2$. As we shall show later, reflection of radio waves from the ionosphere occurs in a region near the point where $\varepsilon = 0$. Thus the electron densities in the reflection region as calculated from formulae (3.5) and (3.15) differ by a factor of 1·5. When the effect of an external magnetic field is allowed for, the difference between formulae based on (3.12) and (3.14) is sometimes even greater. It is therefore obvious why the problem of the effective field in the ionosphere has attracted much attention; the problem, moreover, is complex and there is a diversity of opinions [3–8]. (See [4] for references to earlier literature.) In consequence, the matter has been under continued discussion, and formulae obtained with the Lorentz term and without it (i.e. assuming that $a = 0$) are frequently both given.

There is, however, no reason to proceed in this way. A detailed consideration of the problem [4, 5, 8; 6, § 6] leads to the conclusion that the Lorentz term is unnecessary, and in a plasma $\boldsymbol{E}_{\text{eff}} = \boldsymbol{E}$, the mean macroscopic field. The proof is quite complicated and lengthy. The numerous simpler proofs of this result are, unfortunately, not satisfactory and would equally admit, if desired, a "proof" of the exactly opposite conclusion that the Lorentz term is necessary. We shall therefore omit these proofs, as well as the rigorous analysis of the problem; see in particular [6, § 6; 8]. Although thus avoiding a detailed discussion of the problem of the effective field, we shall make two relevant comments. The fact that $\boldsymbol{E}_{\text{eff}} = \boldsymbol{E}$ in a plasma at a sufficiently low frequency is evident from the following simple, though not rigorous, arguments. When $\omega \to 0$, the electric field may be regarded as a potential field, and the mean macroscopic field $\boldsymbol{E} = -\operatorname{grad}\Phi$, where Φ is a potential, Next, when an electron traverses a macroscopic path $\boldsymbol{L}$ between points A and B, the work done on it is $e(\Phi_A - \Phi_B) = e\boldsymbol{L}\cdot\boldsymbol{E}$. By the definition of the effective field, the force on the electron is $e\boldsymbol{E}_{\text{eff}}$, and the work done by this force along the path $\boldsymbol{L}$ is $e\boldsymbol{L}\cdot\boldsymbol{E}_{\text{eff}}$. Thus $e\boldsymbol{L}\cdot\boldsymbol{E} = e\boldsymbol{L}\cdot\boldsymbol{E}_{\text{eff}}$, i.e. $\boldsymbol{E}_{\text{eff}} = \boldsymbol{E}$.†

The second comment concerns the transition from bound to free electrons [7]. Let us assume that formula (3.14) holds for bound electrons, i.e. that the equation of motion of the electron is

$$m\ddot{\boldsymbol{r}} + m\,\omega_j^2\,\boldsymbol{r} = e\,\boldsymbol{E}_{\text{eff}} = e(\boldsymbol{E} + \tfrac{4}{3}\pi\boldsymbol{P}), \tag{3.16}$$

where ω_j is the intrinsic frequency of the oscillator corresponding to the bound electron considered.

Since $\boldsymbol{P} = eN\boldsymbol{r}$, it follows from (3.16) that, for a harmonic external force $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$, the polarisation is

$$\boldsymbol{P} = \frac{e^2 N\boldsymbol{E}}{m(\omega_j^2 - \omega^2 - 4\pi e^2 N/3m)} = \frac{\varepsilon - 1}{4\pi}\boldsymbol{E},$$

whence

$$\varepsilon = 1 + \frac{4\pi e^2 N}{m(\omega_j^2 - \omega^2 - 4\pi e^2 N/3m)}. \tag{3.17}$$

If we put here $\omega_j = 0$, we obtain the expression (3.15), which (as already stated) is incorrect. On the other hand, it would appear that the transition from bound to free electrons corresponds in fact to the limit $\omega_j \to 0$. This is actually true, however, only for a very rarefied medium, when $N \to 0$ and the Lorentz term necessarily tends to zero. If $N \neq 0$, the electron can never, strictly speaking, be considered completely free, since the greatest electron-ion distance r is of the order of $N^{-\frac{1}{3}}$. The force on the electron at this distance is $e^2/\bar{r}^2 \sim e^2 N^{\frac{2}{3}}$. Equating this to a quasielastic force $m\omega_j^2\bar{r} \sim m\omega_j^2/N^{\frac{1}{3}}$, we obtain $\omega_j^2 \sim e^2 N/m$. The fact that the plasma is a medium which has, in a certain sense, an intrinsic frequency such that $\omega_j^2 \sim \omega_0^2 = 4\pi e^2 N/m$ follows also from other arguments (see, for instance, § 8). If in (3.16) we put $\omega_j^2 = 4\pi e^2 N/3m$, then (3.17) becomes (3.5), as it should. The intuitive arguments given above are not, of course, sufficient for a rigorous proof of formula (3.5), but they show why we cannot put $\omega_j = 0$ in (3.17). Thus there is no paradox in the problem of the transition from a bound to a free electron. We shall henceforward suppose that $\boldsymbol{E}_{\text{eff}} = \boldsymbol{E}$, i.e. that (3.12) is valid.††

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† In the case of a dielectric this argument is invalid, since a test charge (or dipole) is regarded as localised at some point, and the effective field is calculated at that point.

†† It may be noted that the proof of (3.12) given in [6, § 6] assumes that the field $\boldsymbol{E}$ is weak. The same applies to the results of [8], where it is shown that in a weak field (the electron and ion distribution functions being assumed Maxwellian to a first approximation) the effective field is equal to the mean field to within quantities of order $1/\alpha = (16\pi D^3 N)^{-1}$; see § 4 and, in particular, formula (4.24). In the ionosphere and the corona $\alpha \gg 1$, and usually $\alpha \gg 10^3$.

The range of applicability of the formulae

Besides the problem of the effective field, there is also the problem of the validity of applying the classical theory, as we have done above, to the motion of electrons and ions. In the absence of collisions, when formulae (3.4) and (3.5) hold, it is a question of the applicability of the classical theory to the interaction of free charges with a variable electromagnetic field (the radiation field). In this case the classical theory is valid if the inequality

$$\hbar\,\omega \ll m\,c^2 \tag{3.18}$$

is satisfied, where $\hbar = 1{\cdot}05 \times 10^{-27}$ erg sec and m is the mass of the particle, which for definiteness we shall take to be an electron; for ions, the conditions of applicability are certainly less stringent. The scattering of light (electromagnetic waves) by free electrons is described, when (3.18) is satisfied, by Thomson's well-known formula in both quantum and classical theory, and the quantum corrections are of the order of $\hbar\omega/mc^2$ when this ratio is small; see [9]. The condition (3.18) is met not only at radio frequencies, but even for soft X rays ($mc^2 = \hbar\omega_m = 0{\cdot}51 \times 10^6\,\mathrm{eV} = 8{\cdot}2 \times 10^{-7}$ erg; $\omega_m = 7{\cdot}8 \times 10^{20}\,\mathrm{sec}^{-1}$; $\lambda_m = 2\pi c/\omega_m = 2{\cdot}4 \times 10^{-10}$ cm). Moreover, it is well known that the scattering of light entirely determines the refractive index, which in the absence of absorption is equal to $\sqrt{\varepsilon}$ (§ 7). Thus formulae (3.4) and (3.5) are exact when collisions are neglected, and are obtained either by one of the classical derivations (see above and § 6) or by using the quantum theory of dispersion.

The absorption resulting from collisions is due, in quantum terms, to the fact that the quanta of radiation (photons) are absorbed by electrons, whose motion is thereby altered. This absorption process cannot occur with a free electron, since it would violate the laws of conservation of energy and momentum; it therefore takes place only when the effect of molecules and ions on the motion of the electrons is allowed for (i.e. as a result of collisions). At radio frequencies, where $\hbar\omega$ is much less than the ionisation potentials even of highly excited atoms and molecules, the absorption of radiation quanta is not accompanied by bound-free electron transitions; hence only electron transitions between states of the continuous spectrum are involved (free-free transitions, in the language of astrophysics). The opposite process to absorption by such transitions is, of course, electron bremsstrahlung, in which the electron emits radiation quanta as a result of acceleration in collision with a molecule or ion. The probabilities of the direct and inverse processes are connected by Einstein's relation, which shows that one may consider either process (§ 37). The classical theory is applicable to the bremsstrahlung of non-relativistic electrons if the radiation quantum energy is much less than the kinetic energy of the electron, i.e. in our case, if

$$\hbar\,\omega \ll \varkappa T, \tag{3.19}$$

that for a non-degenerate electron gas the susceptibility χ is given by (see [10])

$$\chi = \frac{\mu - 1}{4\pi} = \frac{2}{3}\left(\frac{e\hbar}{2mc}\right)^2 \frac{N}{\varkappa T}. \tag{3.22}$$

Here the spin magnetic moment of the electron has been taken into account; in the absence of spin, $\chi < 0$ and its magnitude is half that of (3.22).

Since $e\hbar/2mc = 9{\cdot}3 \times 10^{-21}$, we see that, even with $N \sim 10^{15}$ and $T \sim 300°$, $\chi \sim 10^{-12}$, i.e. the difference between μ and unity is infinitesimal. It must be emphasised, however, that the above remarks apply in thermodynamic equilibrium; non-equilibrium states of the plasma may have a diamagnetic susceptibility which is appreciable or even large.

§ 4. THE METHOD OF THE BOLTZMANN EQUATION

The distribution function and the Boltzmann equation

The permittivity ε and conductivity σ calculated in § 3 involve as a parameter the effective collision frequency ν_{eff}, whose value was only estimated. To find ν_{eff}, or more precisely to find general expressions for ε and σ in both weak and strong fields, the method of the Boltzmann equation must be used.

In this method the state of the gas is described by the distribution function $f(t, \boldsymbol{r}, \boldsymbol{v})$, which is defined so that the mean number of particles dN in the volume $d\boldsymbol{r}\, d\boldsymbol{v} = dx\, dy\, dz\, dv_x\, dv_y\, dv_z$ is $dN = f\, d\boldsymbol{r}\, d\boldsymbol{v}$, where $\boldsymbol{v}$ is the particle velocity and $\boldsymbol{r}$ the radius vector. By definition

$$\int_{-\infty}^{\infty} f(t, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v} = N, \tag{4.1}$$

where N is the particle density at the point $\boldsymbol{r}$ at time t.

The Boltzmann equation from which the function f must de determined is (see, for example, [11–13])

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \mathbf{grad}_r f + \frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}\, \boldsymbol{v} \times \boldsymbol{H}\right) \cdot \mathbf{grad}_v f + S = 0, \tag{4.2}$$

where e and m are the charge and mass of the particles considered, $\boldsymbol{E}$ and $\boldsymbol{H}$ the electric and magnetic fields,

$$\mathbf{grad}_r f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k},$$

$$\mathbf{grad}_v f = \frac{\partial f}{\partial v_x}\boldsymbol{i} + \frac{\partial f}{\partial v_y}\boldsymbol{j} + \frac{\partial f}{\partial v_z}\boldsymbol{k},$$

and S (called the collision integral) gives the change in f due to collisions of the particles considered (for example, electrons) with all other particles (i.e. electrons, ions and molecules). S may also include terms which allow for changes in f due to various processes such as ionisation, inelastic scattering, etc.

In equilibrium and in the absence of fields, the distribution function is the well-known Maxwellian:

$$f = f_{00}(v) = N\left(\frac{m}{2\pi\varkappa T}\right)^{\frac{3}{2}} e^{-mv^2/2\varkappa T}. \tag{4.3}$$

The Maxwellian distribution (4.3) is easily seen to satisfy the normalisation condition (4.1), since $\int f_{00}\,d\boldsymbol{v} = 4\pi\int f_{00}\,v^2\,dv = N$.

A plasma in a strong electric field

If an ionised gas or plasma is in a variable field $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$, then the function f is, of course, not Maxwellian, and in a sufficiently strong field it may not be even approximately so. This problem will be considered in detail in § 38. Here, in order to ascertain the nature of the deviations of the electron distribution function from the equilibrium form, we shall consider a steady state, in which the mean electron energy does not vary with time. This is so if the energy transmitted to the electrons by the field is equal to the energy transmitted by the electrons to the heavy particles (ions and molecules) by collisions. The former energy is $A = e\dot{\boldsymbol{r}}\cdot\boldsymbol{E}$ per unit time, where $\dot{\boldsymbol{r}}$ is the directed velocity of the electron in the direction of the field, determined by equation (3.6). This equation shows that $\dot{\boldsymbol{r}} = e\boldsymbol{E}_0 e^{i\omega t}/m(i\omega + \nu_{\text{eff}})$, or, using the real form $\boldsymbol{E} = \boldsymbol{E}_0\cos\omega t$ which is more convenient in calculating A,

$$\dot{\boldsymbol{r}} = e\,\boldsymbol{E}_0(\nu_{\text{eff}}\cos\omega\,t + \omega\,\sin\omega\,t)/m(\omega^2 + \nu_{\text{eff}}^2). \tag{4.4}$$

Hence

$$A = e\,\dot{\boldsymbol{r}}\cdot\boldsymbol{E} = e^2\,E_0^2(\nu_{\text{eff}}\cos^2\omega\,t + \omega\,\sin\omega\,t\cos\omega\,t)/m(\omega^2 + \nu_{\text{eff}}^2),$$
$$\bar{A} = e^2\,E_0^2\,\nu_{\text{eff}}/2m(\omega^2 + \nu_{\text{eff}}^2), \tag{4.4a}$$

where the bar denotes averaging with respect to time.

The energy transmitted per unit time by the electrons to the ions and molecules is

$$\bar{U} = W\nu_{\text{eff}} = \delta(\bar{K} - \tfrac{3}{2}\varkappa\,T)\,\nu_{\text{eff}}, \tag{4.5}$$

where $W = \delta(\bar{K} - \frac{3}{2}\varkappa T)$ is the mean energy transmitted in one collision, $\bar{K} = \frac{1}{2}m\,\overline{v^2}$ the mean kinetic energy, and δ the mean fraction of energy which is transmitted per effective collision when $\bar{K} \gg \frac{3}{2}\varkappa T$.

In an elastic collision with a heavy particle the electron loses only a small part of its energy, and for collisions with either ions or molecules we have

$$\delta = \delta_{\text{el}} = 2m/M, \tag{4.6}$$

where m is the mass of the electron and M that of the heavy particle.

The heavy particle is here assumed to be at rest. The corresponding calculation of δ_{el} is given in § 5; the allowance for the motion of the heavy particles is unimportant when $\bar{K} \gg \frac{3}{2}\varkappa T$.† For the atoms, molecules and ions O, $O^{\pm}$, O_2, $O_2^{\pm}$ we obtain from (4.6)

$$\delta_{el.O} = 6.8 \times 10^{-5}, \quad \delta_{el,O_2} = 3.4 \times 10^{-5}. \tag{4.7}$$

When inelastic collisions do occur they are relatively infrequent, and we usually still have $\delta \ll 1$ as the mean over all collisions.

In the steady state assumed, when $\bar{A} = \bar{U}$, we evidently have

$$\bar{K} = \frac{1}{2} m \overline{v^2} = \frac{e^2 E_0^2}{2m(\omega^2 + \nu_{\text{eff}}^2)\delta} + \frac{3}{2}\varkappa T. \tag{4.8}$$

If the frequency ω is small, so that $\omega^2 \ll \nu_{\text{eff}}^2$, the effective time of acceleration of the electron by the field is of the order of the mean free time $\tau = 1/\nu_{\text{eff}} = l/\bar{v}$, where l is the mean free path and $\bar{v}$ the mean velocity. In this case it is clear from (4.4) that $\dot{r} \leqq \dot{r}_{\max} \sim eE_0 l/m\bar{v}$, and if $\bar{K} \gg \frac{3}{2}\varkappa T$, from (4.8)

$$\bar{K} = \frac{1}{2} m \overline{v^2} \sim \frac{1}{2} m \bar{v}^2 \sim \frac{e^2 E_0^2}{m\delta}\left(\frac{l}{\bar{v}}\right)^2 \sim \frac{e E_0 l \dot{r}_{\max}}{\delta \bar{v}},$$

$$\bar{v} \sim e E_0 l/m \sqrt{\delta \bar{v}},$$

or

$$\dot{r}_{\max} \sim \sqrt{\delta}\bar{v}, \quad \bar{K} \sim e E_0 l/\sqrt{\delta}, \quad \bar{K} \gg \tfrac{3}{2}\varkappa T. \tag{4.9}$$

In the opposite limiting case where $\omega^2 \gg \nu_{\text{eff}}^2$, the effective acceleration time is of the order of $1/\omega$, and if $\bar{K} \gg \frac{3}{2}\varkappa T$ then

$$\bar{K} = \frac{1}{2} m \overline{v^2} \sim \frac{1}{2} m \bar{v}^2 \sim \frac{e^2 E_0^2}{2m\omega^2\delta} \sim \left(\frac{e E_0 l}{\sqrt{\delta}}\right)^2 \left(\frac{\nu_{\text{eff}}}{\omega}\right)^2 \frac{1}{\bar{K}} \sim \frac{e E_0 l}{\sqrt{\delta}} \frac{\nu_{\text{eff}}}{\omega},$$

$$\bar{v} \sim e E_0/m\omega\sqrt{\delta}, \quad \dot{r}_{\max} \sim e E_0/m\omega \sim \sqrt{\delta}\bar{v}. \tag{4.10}$$

It is clear from (4.9) and (4.10) that the energy $\bar{K}$ considerably exceeds $\frac{3}{2}\varkappa T$ in fields $E > \varkappa T\sqrt{\delta}/el$ and $E > \varkappa T\omega\sqrt{\delta}/el\nu_{\text{eff}}$ respectively. The distribution function f differs greatly from the equilibrium function $f_{00}(v, T)$, but the velocity $\dot{r}$ of the directed motion is nevertheless small compared with the mean velocity $\bar{v}$, even in a strong field, since δ is small.

The form of the distribution function and the equation for it in a weak field

We can now draw an important conclusion concerning the distribution function f. Let us represent this function as $f = f_0(v) + \varphi(\boldsymbol{v})$, where f_0 depends only on the magnitude v of the velocity, i.e. is the symmetrical part of f.

† The expression for $\bar{U}$ can also be written in other forms, e.g. $\bar{U} = \delta' \bar{K}\nu_{\text{eff}}$, where δ' is the mean fraction of energy which is transmitted in one collision for any $\bar{K}$ and $\frac{3}{2}\varkappa T$. The value of δ', however, must depend on temperature even in elastic collisions, since when $\bar{K} \gg \frac{3}{2}\varkappa T$ we evidently have $\delta' = \delta$, but in thermal equilibrium (when $\bar{K} = \frac{3}{2}\varkappa T$) $\delta' = 0$. It is therefore more convenient to use the parameter δ, which may be regarded as constant for elastic collisions.

The current density $\boldsymbol{j}_t$ is

$$\boldsymbol{j}_t = e N \dot{\boldsymbol{r}} = e\int \boldsymbol{v} f \, d\boldsymbol{v} = e\int \boldsymbol{v}\, \varphi(\boldsymbol{v})\, d\boldsymbol{v}, \tag{4.11}$$

i.e. it is determined only by the asymmetrical part of the function f (since $\int \boldsymbol{v} f_0 \, d\boldsymbol{v} = 0$). On the other hand, the mean velocity $\bar{v} = \int v f \, d\boldsymbol{v}/N$ is in general determined by both the symmetrical and the asymmetrical part of f; but if the condition $\dot{r} \ll \bar{v}$ holds, as in our case, $\bar{v}$ is determined only by the symmetrical part of f, which is considerably greater than the asymmetrical part $\varphi(\boldsymbol{v})$.

Thus, since the mean fraction of energy lost by the electron in a collision with a heavy particle is small (i.e. since $\delta \ll 1$), the electron distribution function may be written

$$\left.\begin{aligned} f(\boldsymbol{v}) &= f_0(v) + \varphi(\boldsymbol{v}) = f_0(v) + \boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v, \\ |\varphi(\boldsymbol{v})| &\sim |\boldsymbol{f}_1(v)| \ll f_0(v), \end{aligned}\right\} \tag{4.12}$$

where the validity of writing $\varphi(\boldsymbol{v})$ in the form $\boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v$ requires explanation. This explanation is as follows. The function $\varphi(\boldsymbol{v})$ may be written as $\varphi(v, \alpha, \beta)$, where α and β are angles determined by the direction of the vector $\boldsymbol{v}$. Expanding φ as a series of spherical harmonics $Y_{lm}(\alpha, \beta)$ and taking the polar axis in the direction of the current $\boldsymbol{j}_t$, we see that the most important term in this expansion is the function Y_{10} = constant $\times \cos\alpha$, whence it follows that $\varphi(\boldsymbol{v})$ may be written as $\varphi_1(v)\cos\alpha = \boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v$, where the vector $\boldsymbol{f}_1(v)$ is parallel to $\boldsymbol{j}_t$. The proof of this statement is given in § 38; see also [11, 13, 34, 258]. The conditions for (4.12) to be valid are discussed in § 38. Here we may note that in a homogeneous plasma these conditions reduce, in the cases of interest to us, to the requirement that $\delta \ll 1$. In a non-homogeneous plasma the condition

$$\left|\frac{\partial f_0}{\partial z}\right| \gg \frac{\bar{v}}{\sqrt{(\omega^2 + \nu_{\text{eff}}^2)}} \left|\frac{\partial^2 f_1}{\partial z^2}\right|,$$

where z is the direction in which the distribution function varies, must also be fulfilled.

The dependence of f on the coordinates has to be taken into account, for example, in considering such problems as thermal conduction and diffusion, which we shall not discuss here. We shall be concerned with distribution functions $f(\boldsymbol{r})$ which depend on the coordinates only when taking account of spatial dispersion. It must also be emphasised that, whereas the expression (4.12) for the electron distribution function is valid in a field of any strength, for heavy particles the distribution function f has the form (4.12) only in weak fields.

The mean kinetic energy of the electron is

$$\bar{K} = \frac{1}{N}\int \frac{1}{2} m v^2 f \, d\boldsymbol{v} = \frac{1}{N}\int \frac{1}{2} m v^2 f_0 \, d\boldsymbol{v}. \tag{4.13}$$

Since the ratio m/M is small, we can draw a further important conclusion: the collision integral S, which takes account of collisions between electrons and heavy particles, can be written not as an integral but as a differential expression. This expression is particularly simple in the case of weak fields, when $\bar{K} \approx \frac{3}{2}\varkappa T$, and the symmetrical part of the distribution function f may be taken to be the Maxwellian function f_{00}. Then

$$S_{em} + S_{ei} = (\nu_{em} + \nu_{ei})\, \varphi(v) = (\nu_{em} + \nu_{ei})\, \boldsymbol{v} \cdot \boldsymbol{f}_1(v)/v, \tag{4.14}$$

where S_{em} and S_{ei} are the collision integrals corresponding to collisions with molecules and ions respectively, and the collision frequencies are

$$\left.\begin{aligned} \nu_{em} &\equiv \nu_m = v/l_m = q_m(v)\, v\, N_m, \\ \nu_{ei} &\equiv \nu_i = v/l_i = q_i(v)\, v\, N_i, \\ q_{m,i}(v) &= 2\pi \int_0^\pi q_{m,i}(v, \theta)\,(1 - \cos\theta) \sin\theta\, d\theta, \end{aligned}\right\} \tag{4.15}$$

where $N_{m,i}$ are the densities of molecules and ions, and $q_{m,i}(v, \theta)$ are the differential cross-sections for elastic collisions of electrons with molecules and ions.†

The relation (4.14) is easily obtained by appropriate calculations (see [11, 13, 258]) but its significance is immediately clear. $S_{em} + S_{ei}$ is the number of electrons leaving a volume element $d\boldsymbol{r}\, d\boldsymbol{v}$ in phase space per unit time as a result of collisions with heavy particles. The equilibrium distribution function f_{00} gives zero collision integral, and S therefore depends only on $\varphi(\boldsymbol{v})$. Finally, since the heavy particles may be considered to be at rest (their velocities being less than those of the electrons by a factor $\sqrt{(M/m)} \sim 100$), $S_{em} + S_{ei}$ must simply be the number of collisions with heavy particles of those electrons which have non-equilibrium velocities. This number is just

$$(\nu_{em} + \nu_{ei})\, \varphi(\boldsymbol{v}) = (q_m N_m + q_i N_i)\, v\, \varphi(\boldsymbol{v}).$$

The only point here which requires a more rigorous proof is why the cross-section q (4.15) is not the total cross-section $q^t = 2\pi\int q(v, \theta) \sin\theta\, d\theta$, but the "transport cross-section" $q^{\mathrm{tr}} = 2\pi \int q(v, \theta)\,(1 - \cos\theta) \sin\theta\, d\theta$. The reason is qualitatively clear: scattering through different angles is not equivalent, since the momentum given to the heavy particle is $v(1 - \cos\theta)$, which is small when the scattering angle θ is small and increases with θ. The appearance of the transport cross-section instead of the total cross-section in the expression for S corresponds to this result, since the greater scattering angles receive greater weight in the transport cross-section than the smaller ones.

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† It may be recalled that the differential cross-section $q(v, \theta)$ for elastic scattering is, by definition, the ratio of the number of particles elastically scattered through an angle θ in the solid angle $d\Omega = 2\pi \sin\theta\, d\theta$ to the number of particles incident in the same time on unit area perpendicular to their velocity. The scattering angle θ is that between the velocities of the incident and scattered electrons.

As stated above, in weak fields we have

$$\left.\begin{aligned} f(v) &= f_{00} + \boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v, \quad |\boldsymbol{f}_1| \ll f_{00}, \\ f_{00} &= N\left(\frac{m}{2\pi\varkappa T}\right)^{\frac{3}{2}} e^{-mv^2/2\varkappa T}. \end{aligned}\right\} \tag{4.16}$$

Substituting this expression in the Boltzmann equation (4.2) and using (4.14), we obtain†

$$\frac{\partial \boldsymbol{f}_1}{\partial t} + \frac{e\boldsymbol{E}}{m}\frac{\partial f_{00}}{\partial v} + \frac{e}{mc}\boldsymbol{H}\times\boldsymbol{f}_1 + (\nu_m + \nu_i)\boldsymbol{f}_1 + \boldsymbol{S}_{1,ee} = 0, \tag{4.17}$$

where $S_{ee} = \boldsymbol{S}_{1,ee}\cdot\boldsymbol{v}/v$ is the part of the collision integral which arises from collisions between electrons. The term $\boldsymbol{v}\cdot\mathbf{grad}_r f$ has been omitted; we have used the fact that $\mathbf{grad}_v f_{00} = (df_{00}/dv)\boldsymbol{v}/v$, and have neglected the asymmetrical part of the distribution function in the term containing the field $\boldsymbol{E}$, since it is much smaller than the symmetrical function f_{00}. From the fact that in (4.17) the magnetic field $\boldsymbol{H}$ is multiplied only by the asymmetrical part of the distribution function it follows that a magnetic field without an electric field does not modify the equilibrium velocity distribution. For this reason the magnetic field in (4.17) need not be supposed weak (since it is multiplied by the small function $\boldsymbol{f}_1$). The condition for the electric field E to be small is obtained by solving the Boltzmann equation in the next approximation (see [22, § 64]) or in a field of any strength (see § 38). Using elementary arguments, the criterion for the field to be weak is clear from (4.8), and is $\overline{K} - \frac{3}{2}\varkappa T \ll \frac{3}{2}\varkappa T$, i.e.

$$E \ll E_p = \sqrt{[3(m\varkappa T/e^2)\,\delta(\omega^2 + \nu_{\text{eff}}^2)]}. \tag{4.8a}$$

The characteristic field E_p is sometimes called the plasma field; we shall return to it in Chapter VIII.

Concerning the field $\boldsymbol{E}$ which appears in equations (4.2) and (4.17), it should also be noted that, in accordance with what was said in § 3, this field must be taken to be the mean macroscopic field of phenomenological electrodynamics.

Transport cross-sections. Debye screening

In order to use equation (4.17) to solve actual problems, only one point must be further clarified, namely, expressions must be given for the transport cross-sections $q_m(v)$ and $q_i(v)$ for collisions of electrons with molecules and ions. When an electron collides with a molecule (by which we mean any neutral particle, i.e. both atoms and molecules proper), the cross-section $q_m(v)$ cannot be calculated exactly. We shall consider this problem further in § 6, and here note that in many cases the molecule may, for our purposes, be replaced by a hard sphere having some effective radius a. For collisions of an electron

† It is easy to see that $\boldsymbol{v}\times\boldsymbol{H}\cdot\mathbf{grad}_v[f_{00}(v) + \boldsymbol{v}\cdot\boldsymbol{f}_1(v)/v] = \boldsymbol{H}\times\boldsymbol{f}_1\cdot\boldsymbol{v}/v$, since, for example, $\mathbf{grad}_v f_{00}(v) = (df_{00}/dv)\,\boldsymbol{v}/v$, etc.

with a hard sphere at rest we have

$$\left.\begin{aligned} q_m(v,\theta) &= \tfrac{1}{4}a^2, \\ q_m(v) &= \pi\, a^2, \\ \nu_m = q_m\, v\, N_m &= \pi\, a^2\, v\, N_m. \end{aligned}\right\} \tag{4.18}$$

For a collision between an electron and an ion, the cross-section is given by Rutherford's well-known formula:

$$\left.\begin{aligned} q_i(v,\theta) &= \tfrac{1}{4}(e^2/m\,v^2)^2 \operatorname{cosec}^4 \tfrac{1}{2}\theta, \\ q_i(v) &= 2\pi(e^2/m\,v^2)^2 \ln(1+\cot^2 \tfrac{1}{2}\theta_{\min}) \\ &= 2\pi(e^2/m\,v^2)^2 \ln(1+p_m^2\, m^2\, v^4/e^4), \\ \nu_i &= q_i(v)\, v\, N_i, \end{aligned}\right\} \tag{4.19}$$

where $\theta_{\min}$ is the minimum angle of deflection, $p_m = (e^2/m v^2) \cot \frac{1}{2}\theta_{\min}$ is the maximum impact parameter, and v the velocity of the electron far from the ion ("at infinity"). The ion is assumed to be singly charged; if its charge is Ze, then in (4.19) e^4 becomes $Z^2 e^4$. The necessity of introducing some maximum impact parameter p_m is due to the fact that the cross-section $q_i(v)$ diverges for large p in a purely Coulomb field. In reality the Coulomb field of a given ion is always screened at large distances by the fields of other ions and electrons surrounding that ion. When this is taken into account, a finite expression is obtained for $q_i(v)$.

Let us consider the screening of the field of a positive ion with charge $e > 0$ at the origin. The potential of the field of the ion and the particles screening it satisfies, in a steady state, Poisson's equation:

$$\triangle \Phi = \frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = -4\pi\,\varrho(r),$$

where we have used the spherical symmetry of the problem, r is the distance from the central ion and ϱ is the charge density, taking into account the contribution of the central ion and the screening particles. In thermal equilibrium the mean density of positive particles at a point where the potential is Φ is, by Boltzmann's formula,

$$N_+(\Phi) = N\, e^{-e\Phi/\varkappa T},$$

where N is the mean density of positive particles in all space; far from the ion when $\Phi = 0$ we must, of course, have $N_+ = N$, since for definiteness we assume that there are no negative ions and that the plasma is quasineutral. The electron density is

$$N_e(\Phi) = N\, e^{e\Phi/\varkappa T},$$

whence the charge density of screening particles is evidently

$$\varrho_{\mathrm{scr}} = e\, N(e^{-e\Phi/\varkappa T} - e^{e\Phi/\varkappa T}) \approx -2e^2 N\, \Phi/\varkappa\, T,$$

where we have used the fact that in our case $e\Phi \ll \varkappa T$.

The charge density of the central ion can be written $\varrho_0 = e\delta(\boldsymbol{r})$, where $\delta(\boldsymbol{r})$ is the delta function $\left(\int \delta(\boldsymbol{r})\, d\boldsymbol{r} = 1,\ \delta(\boldsymbol{r}) = 0 \text{ when } \boldsymbol{r} \neq 0\right)$.

Thus we have for Φ the equation

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} - \frac{8\pi e^2 N}{\varkappa T}\Phi = -4\pi e\,\delta(\boldsymbol{r}). \tag{4.20}$$

The solution of this equation is

$$\Phi = (e/r)\, e^{-r/D},$$

where D is called the Debye length;

$$D(\text{cm}) = \left(\frac{\varkappa T}{8\pi e^2 N}\right)^{\frac{1}{2}} = 4{\cdot}9\sqrt{\frac{T(\text{deg})}{N}}. \tag{4.21}$$

Hitherto we have regarded the ion and electron temperatures as being equal. The calculation can, however, be made without this assumption, and the Debye length is found to be

$$D = \left(\frac{\varkappa T T_e}{4\pi e^2 (T + T_e) N}\right)^{\frac{1}{2}}, \tag{4.22}$$

where T_e is the electron temperature and T the ion temperature. If $T_e \gg T$, the expression for D differs from (4.21) in that the coefficient $\frac{1}{8}$ becomes $\frac{1}{4}$. If the medium is quasineutral, but $N_+ \neq N$, then N in (4.21) must be replaced by N_+.

The use of the concept of a Debye length assumes that the mean number of particles in a sphere with this radius is large, since otherwise the statistical averaging made above is meaningless. This condition is equivalent to the inequality

$$4\pi D^3/3 \gg 1/N, \tag{4.23}$$

since the mean volume per charged particle is $1/N$. Evidently $4\pi D^3 N/3 \sim \alpha$, where

$$\alpha = 16\pi D^3 N = 2\varkappa T D/e^2 = \left(0{\cdot}54\varkappa T/e^2 N^{\frac{1}{3}}\right)^{3/2} = \left(324 T/N^{\frac{1}{3}}\right)^{3/2} \tag{4.24}$$

and the numerical coefficient between α and $4\pi D^3 N/3$ is chosen for future convenience.

In the Earth's ionosphere we always have

$$\alpha \gg 1, \tag{4.25}$$

and the inequality (4.23) holds. (In the most unfavourable conditions, where $T = 200\ ^\circ$K and $N_i = 10^9$, $\alpha \approx 1000$.) In the solar corona, with $N \lesssim 10^9$ and $T \sim 10^6$, the parameter α is much larger still. In the chromosphere also the condition (4.25) is satisfied.

The physical significance of the condition (4.23) or (4.25) becomes evident if we expand the expression for α or D and write the condition in the equivalent form

$$\varkappa T/e^2 N^{\frac{1}{3}} \gg 1.$$

Thus the condition (4.25) holds if the mean kinetic energy $\frac{3}{2}\varkappa T$ of the electrons considerably exceeds the mean energy of their Coulomb interaction, which in order of magnitude is $e^2/\bar{r} \sim e^2 N^{\frac{1}{3}}$. Hence it is clear that, when the condition (4.25) is violated, the plasma cannot be regarded as a gas, and therefore the Boltzmann equation cannot be used.

When screening is taken into account, it would, strictly speaking, be necessary to consider the scattering of electrons not in the "cut-off" Coulomb field, but in the Debye field with potential $(e/r)e^{-r/D}$ from the start. However, in (4.19) the maximum impact parameter appears only in a logarithm, and so this refinement is unimportant. For $T = T_e$ we can use the solution for a Coulomb field, putting

$$p_m = D = (\varkappa T/8\pi\, e^2 N_+)^{\frac{1}{2}}. \tag{4.26}$$

If the electron temperature $T_e \gg T$, then the temperature T again appears in (4.26) but the coefficient $\frac{1}{8}$ must be replaced by $\frac{1}{4}$. The latter change will not be made in what follows, since the argument of the logarithm is, by (4.25), very large, and a factor 2 is of no significance. It should also be borne in mind that the setting up of the screening field requires a finite time, which leads to the possibility of some further relaxation losses in an alternating field. In the ionosphere, according to [15], this effect is unimportant at all frequencies.

Here one further remark must be made. We have used the classical theory to calculate $q_{m,i}(v)$ in (4.18) and (4.19), whereas in general, cross-sections must be calculated on the basis of quantum mechanics. For example, for a hard sphere (i.e. when the potential energy of interaction between the electron and the particle is of the form $U(r) = 0$ for $r > a$ and $U(r) = \infty$ for $r \leqq a$), according to classical theory $q(v) = \pi a^2$ [cf. (4.18)], whereas in the quantum theory for particles of wavelength $\lambda = 2\pi\hbar/mv \gg a$ we find the cross-section $q(v) = 4\pi a^2$; see, for example, [16, p. 406]. Since, however, a and $q_m(v)$ are not calculated but determined from experimental data, we shall use the expression (4.18). For molecules, $q_m(v)$ must be found either from quantum-theory calculations (which are possible in principle, but usually unreliable in practice for cases of interest to us) or, better, from the appropriate experimental data.

For collisions with ions, it is known that Rutherford's formula is rigorously valid in quantum theory also, and so the expression (4.19) for $q_i(v, \theta)$ is always correct. This is not true of (4.19) for $q_i(v)$, however, since it depends on the classical relation $p_m = (e^2/mv^2) \cot \frac{1}{2}\theta_{\min}$. When an electron moves in a Coulomb field, classical theory is valid if

$$e^2 Z/\hbar v \gg 1, \tag{4.27}$$

where Ze is the charge on the nucleus and v the velocity of the electron at infinity.† This condition is easily seen to be equivalent, for $Z = 1$, to

$$v \ll 3 \times 10^8 \text{ cm/sec}, \quad T \sim mv^2/3\varkappa \ll 3 \times 10^5 \text{ deg K}. \tag{4.28}$$

When the inequalities (4.27) and (4.28) hold, the expression (4.19) for $q_i(v)$ is valid. If these inequalities do not hold, however, the relation between $\theta_{\min}$ and p_m can be estimated from the indeterminacy principle. The change in velocity Δv corresponding to an angle $\theta_{\min} \ll 1$ is such that $\Delta v \approx v\theta_{\min} \gtrsim \hbar/mp_m$, whence the impact parameter is $p_m \gtrsim \hbar/mv\theta_{\min}$. Hence, in the non-classical case,

$$\begin{aligned} q_i(v) &= 2\pi(e^2/mv^2)^2 \ln(1 + \cot^2 \tfrac{1}{2}\theta_{\min}) \\ &= 2\pi(e^2/mv^2)^2 \ln(1 + \gamma m v^2 p_m{}^2/\hbar^2), \end{aligned}$$

where γ is a factor of the order of unity and is of no practical significance.

If the opposite inequality to (4.27) holds, the "Born approximation" is valid, and $q_i(v)$ is easily calculated to a higher accuracy than has been used above, directly for a field of potential $\varphi = (e/r)\, e^{-r/D}$; this gives

$$q_i(v) = 2\pi(e^2/mv^2)^2 [\ln(1 + 4m^2 v^2 D^2/\hbar^2) - 1], \tag{4.19a}$$

using the fact that $D \gg \hbar/mv$.

In the Earth's ionosphere, the inequality (4.28) is always satisfied. and we can use the expression (4.19) with $p_m = D$. In the solar corona, on the other hand, formula (4.19a) is more correct. However, if we use the fact that (4.25) holds, it is easily seen that there is very little difference between the two formulae for the ionosphere, the corona or any other similar medium. We shall therefore use formula (4.19) henceforward, unless the contrary is explicitly stated.

The limits of applicability of the kinetic-theory formulae

There is one essential restriction which has not yet been mentioned. It is that, for a Coulomb interaction, the whole of the usual kinetic discussion given above is valid only for fields of sufficiently low frequency. The reason is that, in deriving the Boltzmann equation, it is assumed that the collision time $\Delta\tau$ is much less than the period $2\pi/\omega$ of the alternating field, i.e. that each collision occurs in an external field which is constant in time. In collisions between electrons and neutral particles, this condition is always satisfied at radio frequencies, since $\Delta\tau \sim a/\bar{v} \lesssim 10^{-15}$ sec (the molecular radius $a \sim 10^{-8}$ cm, and the mean electron velocity $\bar{v} \gtrsim 10^7$ cm/sec). In Coulomb interaction the situation is quite different, because in this case the radius of the scattering

† The condition (4.27) is obtained directly from the requirement that, in the classical approximation, the wavelength $\lambda = 2\pi\hbar/mv$ must be much less than the shortest distance $r_{\min} = 2e^2 Z/mv^2$ to which the electron can approach the nucleus.

particle is the Debye length D, and

$$\Delta\tau \sim D/\bar{v} = \left(\frac{\varkappa T}{8\pi e^2 N}\right)^{\frac{1}{2}} \sqrt{\frac{\pi m}{8\varkappa T}} = \frac{1}{4}\sqrt{\pi}\left(\frac{m}{4\pi e^2 N}\right)^{\frac{1}{2}}, \tag{4.29}$$

where $\bar{v} = \sqrt{(8\varkappa T/\pi m)}$ is the arithmetic mean velocity of the electrons.

Thus the condition for the usual kinetic discussion to be strictly valid is $\omega/2\pi \ll 1/\Delta\tau$, or

$$(4\pi e^2 N/m\omega^2)^{\frac{1}{2}} \gg 1. \tag{4.30}$$

If $\omega^2 \gg \nu_{\text{eff}}^2$, then by (3.9) the condition (4.30) holds only for frequencies where $\varepsilon < 0$. The cases of greatest interest, however, are those where $\varepsilon \gtrsim 0$, i.e. $4\pi e^2 N/m\omega^2 \lesssim 1$. In the most important range, therefore, the above discussion of collisions with ions is not rigorous. This fact will be of importance, of course, only when collisions with ions play the determining part, whereas if collisions with neutral particles are mainly involved, the kinetic discussion is valid regardless of whether the condition (4.30) is satisfied or not. Even for collisions with ions, however, the restriction imposed by (4.30) need hardly ever be taken into account. The reason is that the parameter p_m appears only in a logarithm in formula (4.19) and the consequent expressions for the effective collision frequency and the absorption coefficient. Hence the exact value of p_m is relatively unimportant, especially as the logarithmic factor is much greater than unity if (4.25) holds.† In what follows we shall put $p_m = D$ and use the Boltzmann-equation method, bearing in mind that the resulting formulae are in general of only logarithmic accuracy. In the ionosphere this is very high accuracy and in practice it is certainly sufficient; see §§ 6 and 37.

The error resulting from equating p_m and D [see (4.26)] is, however, important when we consider the absorption of radio waves in a very rarefied medium, such as the interstellar electron gas, where the inequality opposite to (4.30) holds. The solution to the problem of absorption of radio waves in this case is given in § 37.

We have not mentioned the effect of a magnetic field. When such a field is present it may be assumed not to affect the nature of the collisions if $r_H \gg D$, where $r_H = v/\omega_H \sim \sqrt{(\varkappa T/m)} \div |e|\, H^{(0)}/mc$ is the radius of curvature of the particle's orbit in the field and $D \sim \sqrt{(\varkappa T/m)} \div \omega_0$ is the Debye length. Hence we derive the condition $\omega_0 \gg \omega_H$; if this does not hold, the collisions take place somewhat differently according as the field is present or absent. This again, however, affects only the logarithmic factor and can therefore be important only if $\omega_H \gg \omega_0$.

† The kinetic calculation, which involves representing the field as being constant over the collision time, is certainly unsuitable even as a first approximation if the condition $e^2/\varkappa T\bar{v} \ll 2\pi/\omega$ is violated; this may also be written

$$\left(\frac{4\pi e^2 N}{m\omega^2}\right)^{\frac{1}{2}} \gg \frac{1}{\alpha} = \left(\frac{N^{\frac{1}{3}}}{324 T}\right)^{3/2}.$$

§ 5. MICROPROCESSES IN PLASMAS

Microprocesses in plasmas. The equations of conservation of particles of each species

In the Boltzmann equation discussed above, we have taken into account only elastic collisions of electrons with molecules and ions. In a plasma, however, a number of other processes may also occur, including ionisation, recombination, attachment and detachment (formation and destruction of negative ions), dissociation and various inelastic collisions.

Here there arises the question of the effect of these microprocesses on the electron distribution function, and furthermore it is necessary to derive equations which describe the occurrence of the microprocesses themselves.

The inclusion of microprocesses in the Boltzmann equation is not difficult, though we shall not pause to do so here, since for the subsequent treatment it is sufficient to derive the equations of conservation of each species of particle. These equations could be obtained from the Boltzmann equations for electrons, molecules and ions by integrating them over all velocities. It is simpler, however, to deduce the resulting expressions directly from obvious arguments concerning conservation of particles. Denoting the densities of electrons, ions of either sign and molecules by N, $N_{\pm}$ and N_m, we have

$$dN/dt = J - \alpha_e N N_+ - \beta N N_m + \gamma N_- N_m + I N_-, \tag{5.1}$$

$$dN_+/dt = J - \alpha_i N_+ N_- - \alpha_e N_+ N, \tag{5.2}$$

$$dN_-/dt = \beta N N_m - \alpha_i N_- N_+ - \gamma N_- N_m - I N_-. \tag{5.3}$$

Here J is the number of electrons formed (i.e. the number of ionisations) per unit volume of gas in unit time by the incident radiation, or by any other agency not allowed for by the remaining terms in the equation. For photoionisation, which is the predominant process in the ionosphere, $J = \overline{q_\varphi S_\varphi} N_m$, where q_φ is the photoionisation cross-section and S_φ the flux of photons (equal to the energy flux S divided by the photon energy $\hbar\omega$). The bar denotes averaging over the spectrum. In (5.1)–(5.3) it is assumed for definiteness that there are ions and molecules of only one species. Also α_e and α_i are the recombination coefficients of positive ions with respect to electrons and negative ions, β the coefficient of attachment of electrons to molecules, γ the coefficient of detachment of electrons from negative ions in collisions with molecules (the corresponding process in collisions of negative ions with other ions of any kind is neglected in these equations), and $I N_-$ is the number of electron detachments caused by the radiation per unit volume and time. The quantity I depends on the intensity and spectrum of the radiation: $I N_- = \overline{q_{\text{rem}} S_\varphi} N_-$, where q_{rem} is the removal cross-section, i.e. the cross-section for a process of the type $O_2^- + \hbar\omega = O_2 + e$ or $O^- + \hbar\omega = O + e$, e being an electron.

In order to see the significance of the coefficients α_e, α_i, β and γ, let us consider in more detail the process of electron recombination, i.e. processes of the type $O_2^+ + e = O_2 + \hbar\omega$, $N^+ + e = N + \hbar\omega$, etc.† The mean number of recombinations between a given ion and electrons per unit time is $\overline{q_{rec}\, v} N$, where q_{rec} is the recombination cross-section, v the relative velocity of the electron and the ion (which is practically equal to the velocity of the electron), and the bar denotes averaging over the velocity distribution; this is necessary, since q_{rec} depends on v. The total number of recombinations per unit volume and time is $\overline{q_{rec}\, v} N N_+$, whence it is clear that $\alpha_e = \overline{q_{rec} v}$. If the velocity distribution is the equilibrium one, then

$$\alpha = \overline{q_{rec}\, v} = \int q_{rec}(v)\, v \cdot 4\pi \left(\frac{m}{2\pi\varkappa T}\right)^{3/2} e^{-m v^2/2\varkappa T}\, v^2\, dv\,. \tag{5.4}$$

The quantity β is determined in an exactly similar manner, viz. $\beta = \overline{q_a v}$, where q_a is the attachment cross-section and v the electron velocity. The coefficients α_i and γ are given by similar expressions, but here v must be taken as the relative velocity of the ions or of the ion and the molecule. It is clear from the above, and in particular from (5.4), that the coefficients α_e, α_i, β and γ need not depend on the pressure, and in equilibrium (more precisely, when the velocity distributions of all particles may with sufficient accuracy be regarded as equilibrium ones) are functions of the temperature T only. This situation, however, may apply only for low pressures, when only binary collisions need be considered. In the ionosphere this condition is usually satisfied; if it is not, then we can go back to equations of the type (5.1)–(5.3), but with the coefficients α_e, α_i, etc., depending on the pressure.

We have not written down so far the equation which gives the variation with time of the molecule density N_m, but have used the fact that N_m may often be regarded as a given quantity. If this is not so, then the equations (5.1)–(5.3) must be augmented by

$$dN_m/dt = -J - \beta N N_m + \alpha_e N N_+ + \alpha_i N_- N_+ + \gamma N_- N_m + I N_-\,. \tag{5.5}$$

Moreover, equations (5.1)–(5.3) and (5.5) do not take account of any possible expansion of the medium, or of motions in it, or of diffusion. The presence of a macroscopic velocity $\boldsymbol{u}$ of the medium is very easily taken into account. It is sufficient in (5.1), for example, to replace dN/dt by $\partial N/\partial t + \operatorname{div}(N\boldsymbol{u})$, where N and $\boldsymbol{u}$ are functions of coordinates and time. The other equations must be modified similarly. The allowance for diffusion in the ionosphere is rendered more difficult because it is necessary to allow also for the electric field produced by diffusion in an ionised gas, and for gravity. In some cases the effect of the Earth's magnetic field is also important. Thus the problem is quite complex, and we shall not enter into it here; cf. [17–19].

† The following discussion is unaffected if the recombination process is of the type $O_2^+ + e = O' + O''$, where O' and O'' are excited oxygen atoms.

Neglecting diffusion, and because of the high conductivity, we may usually assume the plasma quasineutral, so that

$$N + N_- = N_+, \tag{5.6}$$

where for definiteness we assume all ions to be singly charged.

It follows from (5.1)–(5.3) that $d(N + N_-)/dt = dN_+/dt$, i.e. the condition (5.6) continues to hold. Using this relation and a parameter $\lambda = N_-/N$, it is easily shown that (5.1)–(5.3) give

$$\frac{dN}{dt} = \frac{J}{1+\lambda} - (\alpha_e + \lambda\,\alpha_i)\,N^2 - \frac{N}{1+\lambda}\,\frac{d\lambda}{dt}, \tag{5.7}$$

or

$$\left.\begin{aligned} &\frac{dN}{dt} = \frac{J}{1+\lambda} - \alpha' N^2, \\ &N_+ = (1+\lambda)\,N, \quad N_- = \lambda N, \quad \alpha' = \alpha_e + \lambda\,\alpha_i + \frac{1}{N}\,\frac{d}{dt}\ln(1+\lambda). \end{aligned}\right\} \tag{5.8}$$

Of course, the use of an effective recombination coefficient α' affords a simplification only if the ratio λ of the densities of negative ions and electrons is constant or varies only slowly with time. Then

$$\alpha' = \alpha_e + \lambda\,\alpha_i, \tag{5.9}$$

and to a first approximation λ is independent of N. On these hypotheses, equation (5.8) is much simplified and is equivalent to one determining N only from recombination and photoionisation, neglecting the effect of negative ions. For this equation, derived from (5.1) with $N_- = 0$, is

$$dN/dt = J - \alpha_e N^2, \tag{5.10}$$

where we have used the fact that, by (5.6), $N_+ = N$ when $N_- = 0$.

With $\lambda =$ constant, equation (5.10) is of the same form as (5.8). In a steady state, when $dN/dt = 0$, we have

$$J/(1 + \lambda) = \alpha' N^2, \quad \alpha' = \alpha_e + \lambda\alpha_i. \tag{5.11}$$

For $J = 0$ the solution of equation (5.8) with $\lambda =$ constant is

$$N(t) = N_0/(1 + \alpha' N_0 t), \tag{5.12}$$

where N_0 is the density at time $t = 0$.

It must be emphasised, however, that we cannot in general put $\lambda =$ constant. For, if we do this in equation (5.7), which follows from (5.2) and (5.6), it is in general incompatible with (5.1). Hence the relations (5.9), (5.11) and (5.12), which are based on the assumption that the parameter λ is constant, can be used only within certain limits.

The above relations [especially (5.1)–(5.3) and (5.5)] are the starting-point for a consideration of processes occurring in plasmas in the presence of neutral particles, radiation, etc.

The analysis of these processes in various conditions, and particularly in the Earth's ionosphere, is outside the scope of the present discussion; see [20–24]. We shall therefore continue with some comments and estimates which are useful in the subsequent developments. Only applications to the Earth's ionosphere will be under consideration.

Experimental data, when reduced on the basis of equation (5.8) with λ = constant, yield the following values: for the D layer $\alpha' \sim 10^{-6}$ to 10^{-7}; for the E layer $\alpha' \sim 1$ to 3×10^{-8}; for the F_1 layer $\alpha' \sim 10^{-8}$ to 10^{-9}; and for the F_2 layer $\alpha' \sim 10^{-10}$ to 10^{-11}. Taking these values as a rough estimate, we can immediately estimate the lifetime of an electron, i.e. the mean time between the ionisation (or detachment) and recombination (or attachment). This time is evidently [e.g. from (5.12)] of the order

$$\tau_0 \sim 1/\alpha' N, \tag{5.13}$$

since in a time $1/\alpha' N$ the electron density would be halved if there were no ionisation. The relation (5.13) also follows directly from the definition of the coefficient α' (for example the mean number of recombinations of a given electron per unit time is $\alpha_e N = \overline{q_{\text{rec}} v} N$, and the mean lifetime $\tau_0 = 1/\alpha_e N$; if we are using the effective recombination coefficient α', it is clear that in order of magnitude also $\tau_0 \sim 1/\alpha' N$). From the above values we have for the various layers

$$\tau_{0\mathrm{E}} \sim 10^3\,\mathrm{sec}, \quad \tau_{0\mathrm{F}_1} \sim 10^3 \text{ to } 10^4\,\mathrm{sec}, \quad \tau_{0\mathrm{F}_2} \sim 10^4 \text{ to } 10^6\,\mathrm{sec}, \tag{5.14}$$

taking $N_{\mathrm{E}} \sim N_{\mathrm{F}_1} \sim 10^5$, $N_{\mathrm{F}_2} \sim 10^6$.

The slowing-down time of non-equilibrium electrons in a plasma

We can now go on to consider how the production and disappearance of electrons in the gas affects the electron distribution function. This problem arises because the electrons formed will certainly not have a Maxwellian velocity distribution and, in general, are relatively fast; in the ionosphere, they may have energies of the order of a few electron-volts. The electrons which undergo recombination or attachment, on the other hand, are mainly the slowest ones, with energy $\lesssim \varkappa T$. Hence, even when there is no electric field, if processes of ionisation, recombination, etc., take place the distribution function may in principle differ considerably from the Maxwellian. In order to estimate this difference, we must compare the lifetime of an electron in the free state [see (5.14)] with the slowing-down (or relaxation) time, i.e. the time in which a fast electron formed in the gas is retarded to thermal velocities.

When an electron with velocity $\boldsymbol{v}$ collides with a heavy particle, the momentum of the electron is changed by some amount $m\Delta \boldsymbol{v} = m(\boldsymbol{v} - \boldsymbol{v}')$. The heavy particle receives a momentum $-m\Delta \boldsymbol{v}$, and its energy changes by

$$\frac{(\boldsymbol{p} - m\,\Delta \boldsymbol{v})^2}{2M} - \frac{\boldsymbol{p}^2}{2M} = -\frac{m}{M}\boldsymbol{p}\cdot\Delta\boldsymbol{v} + \frac{m^2}{2M}(\Delta\boldsymbol{v})^2,$$

where $\boldsymbol{p}$ is the momentum of the heavy particle before the collision. If the electron velocity is sufficiently great, so that the kinetic energy $K \gg \frac{3}{2}\varkappa T$ (where T is the temperature of the heavy-particle gas), then the heavy particles may be considered to be at rest. Thus, putting $\boldsymbol{p} = 0$ (or averaging over directions on the assumption that $\boldsymbol{p}$ and $\Delta \boldsymbol{v}$ are independent, as they are for $K \gg \frac{3}{2}\varkappa T$), we see that in one collision the electron loses an energy $m^2(\Delta \boldsymbol{v})^2/2M$. Hence the mean energy lost per unit time by electrons of velocity v is

$$\bar{U}(v) = \frac{m^2}{2M}\int (\Delta\boldsymbol{v})^2 q_{m,i}(v,\theta)\, v N_{m,i}\, d\Omega,$$

where $N_{m,i}$ is the density of heavy particles (molecules m or ions i), $q_{m,i}(v,\theta)$ the differential cross-section (4.15), and $d\Omega = 2\pi \sin\theta\, d\theta$. Taking the direction of the initial electron velocity $\boldsymbol{v}$ as the z-axis, we have $\Delta v_z = v(1 - \cos\theta)$ and $(\Delta v_x)^2 + (\Delta v_y)^2 = v^2 \sin^2\theta$. Hence

$$\begin{aligned}\bar{U}(v) &= \frac{m^2 v^3 N_{m,i}}{M}\int q_{m,i}(v,\theta)\,(1-\cos\theta)\,d\Omega \\ &= (2m/M)\,(\tfrac{1}{2}m v^2)\,\nu_{m,i} = (2m/M)\,K\,\nu_{m,i},\end{aligned} \tag{5.15}$$

since $\nu_{m,i} = q_{m,i}(v)\, vN_{m,i}$ and $q_{m,i}(v) = \int q_{m,i}(v,\theta)\,(1-\cos\theta)\,d\Omega$ [see (4.15)].

Thus, if $K \gg \frac{3}{2}\varkappa T$, we have for elastic collisions between electrons and heavy particles $\bar{U}(v) = \delta_{\text{el}} K \nu_{m,i}$, where $\delta_{\text{el}} = 2m/M$ [see (4.6)]. If $\nu_{m,i}$ is independent of the velocity v, then we have formula (4.5) for $\bar{K} \gg \frac{3}{2}\varkappa T$. This is to be expected, since the elementary theory is identical with the kinetic theory for $\nu_{m,i} = \nu_{\text{eff}} = \text{constant}$.

The relative fraction of energy transferred, $\delta(v) = \bar{U}(v)/K\nu_{m,i}(v)$, is equal to δ_{el} only if there are no inelastic collisions. If such collisions are not ruled out on energy grounds, we usually have $\delta \gg \delta_{\text{el}}$. The lower excited electron levels of atoms and molecules are about 1–10 eV above the ground level. For a low-temperature plasma the excitation of these levels is not of great interest; an energy of 1 eV corresponds to a temperature $T_e \approx 10^4$ deg. In atomic gases and in collisions of electrons with ions in a low-temperature plasma, therefore, $\delta = \delta_{\text{el}}$. This equation is valid for a mean electron energy $\bar{K}$ considerably less than the lowest ionisation potential. In molecular gases, however, we have $\delta = \delta(T_e) \gg \delta_{\text{el}}$ even at an electron temperature $T_e = 2\bar{K}/3\varkappa \sim 300°$, i.e. inelastic collisions are possible, with excitation mainly of the rotational levels of the molecules. This is because the distances between rotational levels are very small, being (e.g.) $\sim 10^{-3}$ eV in the molecules O_2 and N_2, corresponding to $T \sim 10°$. Hence, when $T_e \gtrsim 10°$, the electrons can lose energy by exciting rotational levels, whereas excitation of vibrational levels barely occurs in O_2 and N_2, for instance, even for $T_e \sim 300°$. The reduction of various experimental data (see [14], and also [21, 25, 26]) leads to the values of δ shown in Table 5.1 for hydrogen, oxygen, nitrogen, air and various heights in the ionosphere.

TABLE 5.1
Values of $\delta \times 10^3$

T_e	H_2	O_2	N_2	Air	Ionosphere		
					100 km	200 km	300 km
500	2·3	—	—	—	—	—	—
1000	2·5	3·7	0·47	0·89	0·86	0·08	0·06
2000	2·2	6·7	0·36	1·2	1·2	0·12	0·06
3000	2·2	8·6	0·33	1·6	1·5	0·16	0·06
4000	2·5	9·05	0·32	1·7	1·6	0·18	0·06
5000	3·0	8·7	0·34	1·7	1·6	0·22	0·06
6000	3·4	8·2	0·38	1·7	1·6	0·26	0·07
7000	3·9	7·7	0·45	1·7	1·6	0·32	0·07
8000	4·4	7·2	0·60	1·7	1·6	0·43	0·08
9000	4·85	6·8	0·82	1·8	1·7	0·60	0·09
10,000	5·3	6·6	1·15	2·0	2·0	0·85	0·11
12,000	6·1	7·7	2·40	3·2	3·1	1·8	0·23
15,000	7·2	21	9·8	11	10·6	7·7	1·13

For a mixture of gases (air, ionosphere, etc.) the values of δ are obtained from the formula

$$\delta = \delta_1 \frac{\nu_{\text{eff},1}}{\nu_{\text{eff},1} + \nu_{\text{eff},2}} + \delta_2 \frac{\nu_{\text{eff},2}}{\nu_{\text{eff},1} + \nu_{\text{eff},2}},$$

where $\delta_{1,2}$ are the values of δ for gases 1 and 2, and $\nu_{\text{eff},1,2}$ the effective frequencies of collisions between electrons and molecules of gases 1 and 2. The generalisation to more than two components is obvious. The different values of δ in the ionosphere and at various heights therein are due to the change in composition by dissociation, etc. We shall not give here the data used in [14] concerning the composition of the ionosphere, since the actual values of δ are used below only for purposes of estimation. Moreover, even values for gases of definite composition cannot be regarded as firmly established because of certain discordances in the literature. Finally, in Table 5.1 the mean electron energy $\bar{K}$ is expressed in terms of the electron temperature $T_e = \frac{2}{3}\bar{K}/\varkappa$, assuming a Maxwellian electron velocity distribution, but the results do not depend greatly on the form of the distribution function, since δ depends only slightly on T_e. Hence Table 5.1 can be used as a first approximation even for non-Maxwellian distributions, with T_e defined by $\bar{K} = \frac{3}{2}\varkappa T_e$.

With $\bar{K}_0 \sim 1$ eV in the E layer, (5.19) gives $\Delta T \sim \tau'/\varkappa\tau_0 \sim 0{\cdot}1$ °K, and in the F layer (5.20) gives $\Delta T \sim 10$ °K. These estimates are very rough, but they definitely show that the effect of ionisation, recombination and other processes on the electron distribution function in the ionosphere is very slight, on the above assumptions. In the F layer we can further assert that the electron distribution function is Maxwellian to a high degree of accuracy, but with an electron temperature $T_e \neq T$; in other words, the only deviation from equilibrium in this case is that the electron temperature is not equal to the heavy-particle temperature. This follows from a consideration of collisions between electrons, which have not so far been taken into account. The cross-section for such collisions is immediately seen to be of the same order as the cross-section (4.19) for electron-ion collisions, the only difference being that which is due to the equality of mass of the colliding particles. Moreover, in sufficiently close encounters between electrons the average change in their energies is of the same order as the energies themselves. Hence, if we have a pure electron plasma (without molecules), the relaxation time τ_e' for the establishment of equilibrium between electrons is m/M times less than the time τ' for establishment of equilibrium between electrons and ions, which is given in order of magnitude by (5.18).

It will be shown in § 6 that in the F layer the value of ν_{eff} due to electron-ion collisions only is of the order of the experimentally measured value of ν_{eff}. Thus the conditions are similar to those in a pure electron-ion plasma, and $\tau_e' \sim (m/M)\,\tau' \sim (1/\nu_{\text{eff}}) \ln(\bar{K}_0/\varkappa T)$, since δ in (5.18) is of the order of m/M for collisions with ions.

If $\nu_{\text{eff}} \sim 3 \times 10^3\,\text{sec}^{-1}$ and $\bar{K}_0 \sim 1$ eV, we have $\tau_e' \sim 10^{-3}$ sec, $\tau_e'/\tau' \sim m/M \sim$ $\sim 3 \times 10^{-5}$, and $\tau_e'/\tau_0 \sim 10^{-7}$ (with $\tau_0 \sim 10^4$ sec). Thus, in the F layer we can always assume that equilibrium is established between the electrons, and therefore that the distribution function is a Maxwellian with some temperature T_e. As already stated [see (5.21)], since $(\tau'/\tau_0)\,\bar{K}_0$ is relatively small, we also have $T_e \approx T$, where T is the ion temperature. The calculation of T_e requires the use of the Boltzmann equation, including terms which take account of the appearance and disappearance of electrons. This problem is considered in [27] on the basis of certain assumptions, which we shall not discuss here, and the result is in agreement with the above. In a particular case for the F layer given in [27], $T_e - T \sim 50\bar{K}_0$, where $\bar{K}_0$ is measured in electron-volts; according to (5.21), $T_e - T \sim \frac{2}{3}(\tau'/\tau_0)\,\bar{K}_0/\varkappa \sim 8 \times 10^3(\tau'/\tau_0)\,\bar{K}_0 \sim 10\bar{K}_0$ if $\tau'/\tau_0 \sim 10^{-3}$ and $\bar{K}_0$ is measured in electron-volts.

Summarising, we may say that, in the absence of external fields, the assumption of a Maxwellian electron distribution function in the ionosphere is entirely reasonable and must be a good approximation to reality. Since, moreover, it is not usually possible to obtain reliable information about the actual form of the distribution function in the ionosphere or the amount of its deviation from the Maxwellian, the choice of the equilibrium distribution as a basis for discussion is indeed unavoidable in practice. The same may be said of the solar corona. Thus the effect of an electromagnetic wave on the distribution function, which is our fundamental problem, will be examined on the basis of the Boltzmann equation (4.17), neglecting microprocesses in the plasma. The same procedure will be used for strong fields (Chapter VIII). Cases are, of course, conceivable in which this approximation is invalid. For example, if the electrons formed in ionisation undergo recombination or attachment at energies above the thermal (mean kinetic) energy of the molecules and ions, then we naturally cannot regard the electron temperature T_e as being close to the temperature T of the heavy particles. Moreover, in such conditions we cannot always suppose the electron distribution function to be Maxwellian even for a pure electron-ion plasma. It may also be noted that, even when the distribution function is mainly Maxwellian, deviations may quite easily arise at velocities $v \gg \sqrt{(\varkappa T_e/m)}$ (for example, in an electric field; see § 38 and [258]).

§ 6. THE PERMITTIVITY AND CONDUCTIVITY OF A PLASMA: KINETIC THEORY

General relations

Let us now calculate the permittivity ε and the conductivity σ of an isotropic plasma from the Boltzmann equation.

Neglecting for the moment collisions between electrons, we write the initial equations as

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{f}_1}{\partial t} + \frac{e\boldsymbol{E}}{m}\frac{\partial f_{00}}{\partial v} + \nu(v)\boldsymbol{f}_1 = 0,\\ &f = f_{00} + \frac{\boldsymbol{v}\cdot\boldsymbol{f}_1(v)}{v}, \quad f_{00} = N\left(\frac{m}{2\pi\varkappa T}\right)^{3/2} e^{-mv^2/2\varkappa T}.\end{aligned}\right\} \tag{6.1}$$

Here $\nu = \nu_m + \nu_i$ is the collision frequency. In accordance with § 5 and the assumption that the field is weak, the symmetrical part of the distribution function is taken to be Maxwellian, with the electron temperature T_e equal to the heavy-particle temperature T. If the electric field $\boldsymbol{E}$ is zero, then (6.1) shows that $\boldsymbol{f}_1(v, t) = \boldsymbol{f}_1(v, 0)\, e^{-\nu(v)t}$, i.e. the asymmetrical part of the distribution function is damped, and in a steady state $f = f_{00}$.

To calculate ε and σ in a uniform electric field which varies with time, we must put in (6.1) $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$ and seek a solution in the form $\boldsymbol{f}_1 = \boldsymbol{f}_{10} e^{i\omega t}$. This gives directly

$$\boldsymbol{f}_1 = -\frac{e\boldsymbol{E}\,\partial f_{00}/\partial v}{m[i\omega + \nu(v)]}. \tag{6.2}$$

The total current density is [see also (4.11)]

$$\begin{aligned}
\boldsymbol{j}_t &= e\int \boldsymbol{v} f\, d\boldsymbol{v} \\
&= e\int \boldsymbol{v}\frac{\boldsymbol{v}\cdot\boldsymbol{f}_1}{v}\, d\boldsymbol{v} \\
&= e\int \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{f}_1)\, v\, dv\, d\Omega \\
&= \frac{4\pi e}{3}\int\limits_0^\infty \boldsymbol{f}_1 v^3\, dv \\
&= \frac{8e^2 N \boldsymbol{E}}{3\sqrt{\pi}\, m}\int\limits_0^\infty \frac{u^4 e^{-u^2}\, du}{i\omega + \nu(u)} \\
&= \frac{8e^2 N \boldsymbol{E}}{3\sqrt{\pi}\, m}\left\{\int\limits_0^\infty \frac{\nu(u)\, u^4 e^{-u^2}\, du}{\omega^2 + \nu^2(u)} - i\omega\int\limits_0^\infty \frac{u^4 e^{-u^2}\, du}{\omega^2 + \nu^2(u)}\right\},
\end{aligned} \tag{6.3}$$

where

$$u = \sqrt{(m/2\varkappa T)}\, v \tag{6.4}$$

and we have used the fact that

$$\partial f_{00}/\partial v = -mv f_{00}/\varkappa T = -(N/2\pi^{3/2})(m/\varkappa T)^2\, u e^{-u^2}.$$

By the definition of ε and σ we have

$$\boldsymbol{j}_t = \left(\sigma + i\omega\frac{\varepsilon - 1}{4\pi}\right)\boldsymbol{E} = i\omega\frac{\varepsilon' - 1}{4\pi}\boldsymbol{E}. \tag{6.5}$$

Equating (6.3) and (6.5), we find ε and σ.

The effective collision frequency ν_{eff} is also often used; it is defined by

$$\varepsilon = 1 - \frac{4\pi e^2 N}{m(\omega^2 + \nu_{\mathrm{eff}}^2)}, \quad \sigma = \frac{e^2 N \nu_{\mathrm{eff}}}{m(\omega^2 + \nu_{\mathrm{eff}}^2)}. \tag{6.6}$$

The use of this ν_{eff} is, however, not always convenient or justified [28]. If $\nu(v)$ is independent of v, i.e. $\nu(v) = \nu_{\mathrm{eff}} =$ constant, then of course we have at once from (6.3) the expressions (3.7), which are identical with (6.6). This approximation in fact corresponds to the elementary theory of § 3. When the dependence of ν on v is taken into account, we must regard ν_{eff} in (6.6) as a function of the frequency ω, and this function is not the same in the two expressions for ε and σ; furthermore, the function $\nu_{\mathrm{eff}}(\omega)$ in the expression for σ is two-valued [28]. In the kinetic theory, therefore, it is best to use ν_{eff} only in the limiting cases of high and low frequencies ω.

In the limit

$$\omega^2 \gg \nu_{\mathrm{eff}}^2, \tag{6.7}$$

when

$$\varepsilon = 1 - 4\pi e^2 N/m\omega^2, \quad \sigma = e^2 N \nu_{\text{eff}}/m\omega^2, \tag{6.8}$$

the value of ν_{eff} is evidently, from (6.3), (6.5), (6.7) and (6.8),

$$\nu_{\text{eff}} = \frac{8}{3\sqrt{\pi}} \int_0^\infty \nu(u)\, u^4 e^{-u^2} du = \frac{2}{3\sqrt{(2\pi)}} \left(\frac{m}{\varkappa T}\right)^{5/2} \int_0^\infty \nu(v)\, v^4 e^{-mv^2/2\varkappa T} dv, \tag{6.9}$$

where we have taken into account that in the most important range of integration we have, as well as (6.7), $\omega^2 \gg \nu^2(u)$ (see below).

The quantity $\nu(u)$ in (6.9) is to be taken as (4.15), with v replaced by $\sqrt{(2\varkappa T/m)}\, u$.

Collisions with molecules

In collisions with molecules the cross-section $q_m(v, \theta)$ is in general only very slightly dependent on v and θ. In particular, for air at electron energies above about 0·25 eV, the Ramsauer effect (velocity dependence of the cross-section) seems to be almost absent. The data available for oxygen, nitrogen and air are to some extent contradictory. It is therefore best at present to assume the cross-section independent of velocity, especially as we are interested only in thermal velocities, where there is in general no reason to expect a marked Ramsauer effect (see also below).

Regarding the molecule as a hard sphere of radius a and thus taking the expression (4.18) for ν_m, we find from (6.9)

$$\nu_{\text{eff},m} = \frac{4\pi}{3} a^2 \bar{v} N_m = 8{\cdot}3 \times 10^5 \pi a^2 \sqrt{T}\, N_m, \tag{6.10}$$

where $\bar{v} = \sqrt{(8\varkappa T/\pi m)}$ is the arithmetic mean electron velocity. For $T = 300°$, $\bar{v} = 1{\cdot}08 \times 10^7$ cm/sec. From the arguments given below we take $4{\cdot}4 \times 10^{-16}$ cm² as the value of πa^2 for air, or the radius $a = 1{\cdot}2 \times 10^{-8}$ cm, in approximate agreement with the result given by gas kinetics. For this value of a we have, by (6.10),

$$\nu_{\text{eff},m} = 1{\cdot}7 \times 10^{11} \frac{N_m}{2{\cdot}7 \times 10^{19}} \sqrt{\frac{T}{300}}. \tag{6.11}$$

At atmospheric pressure and $T = 300$ °K, therefore, $\nu_{\text{eff}} = 1{\cdot}7 \times 10^{11}\,\text{sec}^{-1}$.

In subsequent estimates we shall use (6.11), although it must be remembered that this may involve an error of several tens per cent when applied to the ionosphere, even neglecting the change in composition and temperature; see below.

It may also be noted that (6.9) has been derived by using the fact that the important range of integration in the integrals (6.3) is that where $u \sim 1$; for $u \gg 1$ the integrand decreases exponentially, and for $u \ll 1$ it is proportional to u^5. The condition (6.7) used is therefore practically equivalent to the condition $\omega^2 \gg \nu^2(u)$ whereby (6.3) becomes (6.9). This is clear also from the fact that, by (6.10), $\nu_{\text{eff},m} \sim \nu(u = 1)$.

Collisions with ions

The situation is similar in calculating ν_{eff} for collisions with ions. In this case we must substitute the expression (4.19) for ν in (6.9). Using also (4.26), the result is

$$\nu_{\text{eff},i} = \frac{2}{3}\pi\frac{e^4}{(\varkappa T)^2}\bar{v}N_i I,$$

$$I = \int_0^\infty \frac{\alpha^2 x e^{-x}\,dx}{1+\alpha^2 x^2},$$

$$\alpha = \frac{2\varkappa T p_m}{e^2} = \frac{2\varkappa T D}{e^2} = \left(0{\cdot}54\frac{\varkappa T}{e^2 N_+^{\frac{1}{3}}}\right)^{3/2} = \left(324\frac{T}{N_+^{\frac{1}{3}}}\right)^{3/2};$$

$$N_i = N_+ + N_-; \quad N_+ = N_- + N, \tag{6.12}$$

where $N_\pm$ is the concentration of positive or negative ions; see [29]. Similar calculations had previously been given in [30], where it was assumed that $p_m \sim 1/N_i^{\frac{1}{3}}$.

The integral

$$I = -\left[Ci\left(\frac{1}{\alpha}\right)\cos\frac{1}{\alpha} + Si\left(\frac{1}{\alpha}\right)\sin\frac{1}{\alpha} - \frac{1}{2}\pi\sin\frac{1}{\alpha}\right],$$

where

$$Ci(x) = -\int_x^\infty \frac{\cos t}{t}\,dt \quad \text{and} \quad Si(x) = \int_0^x \frac{\sin t}{t}\,dt$$

are the cosine and sine integrals. From the condition (4.25), which shows that $1/\alpha$ is small, we have $I \approx -Ci(1/\alpha) \approx \ln\alpha - 0{\cdot}577 = \ln(\alpha/1{\cdot}78)$. Thus

$$\nu_{\text{eff},i} = \pi\frac{e^4}{(\varkappa T)^2}\bar{v}N_i\ln\left(0{\cdot}37\frac{\varkappa T}{e^2 N_+^{\frac{1}{3}}}\right) = \frac{5{\cdot}5 N_i}{T^{3/2}}\ln\left(220\frac{T}{N_+^{\frac{1}{3}}}\right). \tag{6.13}$$

In what follows we shall replace N_+ in the logarithm by $N_i = N_+ + N_-$, since $N_+ \leq N_i \leq 2N_+$ and the error in (6.13) due to replacing N_+ by N_i is negligible. Moreover, in the majority of the cases we can assume that $N_- = 0$, $N_i = N_+ = N$, and thus use the formula

$$\nu_{\text{eff},i} = \pi\frac{e^4}{(\varkappa T)^2}\bar{v}N\ln\left(0{\cdot}37\frac{\varkappa T}{e^2 N^{\frac{1}{3}}}\right) = \frac{5{\cdot}5 N}{T^{3/2}}\ln\left(220\frac{T}{N^{\frac{1}{3}}}\right). \tag{6.14}$$

If the electron temperature T_e is not equal to the ion temperature T, we can calculate similarly and easily derive the formula which replaces (6.13). For example, if $T_e \gg T$, and $N_i = N_+ = N$, then

$$\nu_{\mathrm{eff},i} = \frac{5{\cdot}5N}{T_e^{3/2}}\left[\ln\frac{280T_e}{N^{\frac{1}{3}}} + \frac{1}{3}\ln\frac{T}{T_e}\right]. \tag{6.15}$$

The difference between formulae (6.15) and (6.13), apart from the replacement of T by T_e, is in general outside the limits of accuracy of these formulae. For, as stated in § 4, by putting $p_m = D$ [see (4.26)] we commit an error which affects the argument of the logarithm in (6.13).

However, it must be emphasised that, if $(4\pi e^2N/m\omega^2)^{\frac{1}{2}} \gtrsim 1$, then formula (6.13) can be relied on to within a factor of the order of unity in the argument of the logarithm, which ensures very high accuracy of the calculated value of $\nu_{\mathrm{eff},i}$ (for $T = 300°$ and $N_i = 10^6$, $\ln\left(220T/N_i^{\frac{1}{3}}\right) = 6{\cdot}5$ and, even if we double or halve the argument of the logarithm, the value of $\nu_{\mathrm{eff},i}$ is changed by only 10 per cent). The accuracy of formula (6.13) is insufficient only in the opposite limiting case, i.e. $(4\pi e^2N/m\omega^2)^{\frac{1}{2}} \ll 1$. The calculation of the absorption coefficient μ in such conditions, when $\mu = (1 - n^2)\,\nu_{\mathrm{eff}}/cn \approx 4\pi e^2N\nu_{\mathrm{eff}}/mc\omega^2$ (see § 7), is given in § 37. The results there obtained enable us to assess the accuracy of formula (6.13) under the conditions prevailing in the ionosphere. The accuracy is usually not worse than 5–10 per cent.

The part played by collisions between electrons

The validity of this conclusion is closely related to the fact that in the case (6.7), when $\omega^2 \gg \nu_{\mathrm{eff}}^2$, collisions between electrons, which have been neglected in (6.1), are indeed unimportant [31]. This is not obvious, since the cross-section for collisions of electrons with electrons is of the same order as that for collisions with singly charged ions. In general, therefore, collisions between electrons are important, and it may be shown that at low frequencies, when $\omega^2 \ll \nu_{\mathrm{eff}}^2$ (and, in particular, in a constant field), the conductivity is reduced by a factor 1·73 when $N_i = N$ and collisions between electrons are taken into account (see below, and [32]).

When $\omega^2 \gg \nu_{\mathrm{eff}}^2$, collisions between electrons are unimportant in an isotropic plasma, for a variety of reasons. When such collisions are neglected, the distribution function satisfies equation (6.1), from which we see that the number of electrons which leave a given velocity range as a result of collisions is proportional to ν and to the deviation of the distribution function f from the equilibrium function f_{00}, i.e. is proportional to νf_1. Next, when $\omega^2 \gg \nu^2$, according to (6.2) $f_1 \approx eE(\partial f_{00}/\partial v)(i\omega - \nu)/m\omega^2$; thus the value of f_1 is mainly determined by the acceleration due to the external field, and only in a higher approximation by collisions with ions and molecules (the part

2 a*

depending on collisions is proportional to the number of collisions ν, which is small compared with ω).

A peculiarity of collisions between electrons is that, by the law of conservation of momentum, such collisions cannot in themselves change the mean electron current, which is proportional to the mean electron momentum. Hence, if there were no collisions of electrons with ions and molecules, collisions between electrons would make no contribution to the conductivity. From this it follows that, as regards the effect of collisions between electrons, the only part of the deviation of the distribution function from the equilibrium form which is important is that which is due to collisions with ions and molecules. The above discussion shows that this part is of the order $\nu f_1/\omega \ll f_1$. The part of the collision integral $S_{1,ee}$ which is due to collisions between electrons and must be added to (6.1) [see (4.17)] is equal in order of magnitude to the number of collisions between electrons† $\nu_{ee} \sim \nu_{ei}$, multiplied by that part of the deviation of the distribution function from the equilibrium function which is important as regards the effect of collisions between electrons. As has been mentioned, this latter factor is of the order of $\nu f_1/\omega$, and therefore $S_{1,ee} \sim \nu_{ee}\nu f_1/\omega$, whereas collisions with ions and molecules give a term $\nu f_1 = (\nu_{em} + \nu_{ei}) f_1$ in (6.1). This result, which may be confirmed by a more rigorous calculation [31], indicates that, even in the absence of molecules (when $\nu = \nu_{ei}$), the contribution of collisions between electrons to the conductivity is less than that of electron-ion collisions by a factor of about ν_{ei}/ω. In the F layer of the ionosphere, therefore, where $\nu \sim \nu_{ei}$ (see below) and usually $\nu/\omega \sim 10^{-4}$, collisions between electrons may be entirely neglected. The same applies, of course, to the solar corona. In the lower regions of the Earth's ionosphere this conclusion is still valid, since here, *inter alia*, the frequency of collisions between electrons $\nu_{ee} \sim \nu_{ei} \ll \nu_{\text{eff}} \sim \nu_{\text{eff},m}$.

Before going on to make use of formulae (6.11) and (6.13), which have been shown to be very accurate, we may note that the expression

$$\overline{q(v)} = \frac{\pi e^4}{(\varkappa T)^2} \ln\left(0{\cdot}37 \frac{\varkappa T}{e^2 N^{\frac{1}{3}}}\right)$$

[see (6.13)] for the mean cross-section for electron-ion collisions has an evident physical significance. Let us first consider a collision in which the momentum of the electron is considerably changed, i.e. the electron is deflected through an angle θ of the order of unity. Such a collision occurs if an electron approaching at some impact parameter p has a potential energy of the order of its kinetic energy, i.e. $e^2/p \sim \varkappa T$, or $\pi p^2 \sim \pi e^4/(\varkappa T)^2$. The appearance of a logarithmic factor also in the expression for the cross-section represents an allowance not only for very close but also for more distant collisions, the contribution from which cannot be neglected, because of the slow rate of decrease

† It may be recalled that $\nu_{ei} \equiv \nu_i$ is the number of electron-ion collisions; $\nu_{ee} \sim \nu_{ei}$ only if $N \sim N_i$.

of the Coulomb field. Since the logarithm is large, moreover, most collisions are distant, and we can suppose with logarithmic accuracy that each individual collision involves only a small change of momentum. The effective collisions occurring with frequency ν_{eff} are thus each the result of a large number of individual collisions.

The collision frequency in the ionosphere

The cross-section for collisions with ions at temperatures of the order of hundreds of degrees is fairly large, about a million times that for collisions with molecules. This is clear from Table 6.1, which gives approximately a number of values of $\nu_{\text{eff},m}$ and $\nu_{\text{eff},i}$ calculated from formulae (6.11) and (6.13). The numbers in the first column are N_m or N_i correspondingly.

TABLE 6.1

Values of $\nu_{\text{eff},m}$ and $\nu_{\text{eff},i}$

N_m or N_i	$T = 250$ °K		$T = 300$ °K		$T = 600$ °K	
	$\nu_{\text{eff},m}$	$\nu_{\text{eff},i}$	$\nu_{\text{eff},m}$	$\nu_{\text{eff},i}$	$\nu_{\text{eff},m}$	$\nu_{\text{eff},i}$
10^4	—	111	—	85	—	33
10^5	—	982	—	768	—	297
10^6	—	$8{\cdot}9 \times 10^3$	—	$6{\cdot}9 \times 10^3$	—	$2{\cdot}7 \times 10^3$
10^9	5·5	$5{\cdot}6 \times 10^6$	6	$4{\cdot}4 \times 10^6$	8·5	$2{\cdot}2 \times 10^6$
10^{12}	$5{\cdot}5 \times 10^3$	—	6×10^3	—	$8{\cdot}5 \times 10^3$	—
10^{14}	$5{\cdot}5 \times 10^5$		6×10^5	—	$8{\cdot}5 \times 10^5$	—
10^{15}	$5{\cdot}5 \times 10^6$	—	6×10^6	—	$8{\cdot}5 \times 10^6$	—

In the D layer $N_m \sim 10^{15}$ to 10^{16} and $\nu_{\text{eff},m} \sim 10^7$ to 10^8. In the E layer $N_m \sim 10^{12}$ to 10^{13} and $\nu_{\text{eff},m} \sim 10^4$ to 10^5. The electron density at the maximum of the E layer is $N_{\max} \lesssim 2 \times 10^5$, and if $N_i \approx N$ we have $\nu_{\text{eff},i} \sim 3 \times 10^3$. If $N_i \gg N$, as has sometimes been assumed for the E layer and as may be true for the D layer, then $\nu_{\text{eff},i}$ may exceed $\nu_{\text{eff},m}$; if $N_m \sim 10^{13}$, then $\nu_{\text{eff},i} \sim \sim \nu_{\text{eff},m}$ for $N_i \sim 2 \times 10^7$. However, it seems that actually $N \sim N_i$ in the E layer, and thus collisions with molecules play the dominant part. Here we may note that, when collisions take place both with ions and with molecules, (6.9) evidently gives

$$\nu_{\text{eff}} = \nu_{\text{eff},m} + \nu_{\text{eff},i}, \tag{6.16}$$

where the values of $\nu_{\text{eff},m}$ and $\nu_{\text{eff},i}$ are as before, i.e. are given, for example, by formulae (6.10) and (6.13). In applying (6.16) it is necessary to remember that we have as yet considered only the case of high frequencies [see (6.7)].

When ν_{eff} is determined experimentally from measurements of the absorption of radio waves in the ionosphere, we of course find this total value $\nu_{\text{eff}} = \nu_{\text{eff},m} + \nu_{\text{eff},i}$. In the D and E layers, as already mentioned, we almost certainly have $\nu_{\text{eff},i} \ll \nu_{\text{eff},m}$. In the F layer, or more precisely in its lower part, the situation is more complex and more interesting. The reason is that, even if we assume that $N_i = N_+ = N$ in the F layer, as follows from various arguments, then with $N \lesssim 2 \times 10^6$ we have $\nu_{\text{eff},i} \lesssim 10^4$; the experimental value for the F layer is $\nu_{\text{eff}} \sim 10^3$ to 10^4 (details are given in [22, 23]). Thus in the F layer $\nu_{\text{eff}} \sim \nu_{\text{eff},i}$, but the lack of careful measurements properly reduced makes it impossible to say what is the differenee $\nu_{\text{eff}} - \nu_{\text{eff},i}$ which is to be equated to $\nu_{\text{eff},m}$. (The above comments refer to the lower part of the F layer, since in the higher regions $\nu_{\text{eff}} - \nu_{\text{eff},i} \ll \nu_{\text{eff}}$, and $\nu_{\text{eff},m}$ cannot be reliably determined from radio measurements.) The determination of $\nu_{\text{eff},m}$ is, nevertheless, very important, since the density of molecules in the F layer is not sufficiently well known and varies with time. A measurement of $\nu_{\text{eff},m}$ would make possible a determination of this quantity important in the study of the F layer (see below for details). In order to exhibit the possibilities of this approach, let us write the expression (6.16) in explicit form, substituting the values of $\nu_{\text{eff},m}$ and $\nu_{\text{eff},i}$ from (6.10) and (6.14):

$$\nu_{\text{eff}} = 8{\cdot}3 \times 10^5 \pi a^2 \sqrt{T}\, N_m + \frac{5{\cdot}5 N}{T^{3/2}} \ln\left(220 \frac{T}{N^{\frac{1}{3}}}\right). \tag{6.17}$$

Here the electron density N may be assumed known, since it is easily found by radio methods. If, therefore, we knew the temperature T also, a measurement of ν_{eff} would immediately give $\pi a^2 N_m$ (of course, if $\nu_{\text{eff},m} \ll \nu_{\text{eff},i}$, a measurement of ν_{eff} can give only an upper limit to $\pi a^2 N_m$). In practice, however, the temperature T is unknown, and must be found either from (6.17) itself or by some independent method (see [31]).

If the cross-section πa^2 is independent of T, the latter can be determined from (6.17) by measuring ν_{eff} for various values of T; this ean in principle be done by measuring ν_{eff} at various times of the day and at various altitudes.

Thus, if the temperature dependenee of πa^2 is neglected, radio methods enable us, at least in principle, to find $\pi a^2 N_m$, where πa^2 is the cross-section of the molecule (regarded as a hard sphere) and N_m the density of molecules. More aceurately, $\pi a^2 N_m$ should be replaced by $\sum \pi a_k^2 N_{mk}$, where the suffix k refers to the species of atoms or molecules present.

For oxygen and nitrogen molecules, different determinations of πa^2 have led to values differing by a factor of nearly 2. This wide scatter seems to be due to the fact that the cross-section πa^2 depends on the electron velocity on account of the Ramsauer effect, and this velocity is not the same in different cases. The value of πa^2 for air at $T \sim 300°$, aceording to [26], is fairly accurately known to be $4{\cdot}4 \times 10^{-16}$ cm^2, the value which has been used in (6.11). In discussing the F layer, however, it must be borne in mind that atoms of oxygen and possibly nitrogen are present as well as N_2 and traces of O_2. There are no experimental data concerning the cross-section πa^2 for O and N. Certain theoretical data [33] lead to the conclusion that the oxygen atom may exhibit unusually strong scattering, giving a cross-section as high as 1000×10^{-16} cm^2. Such a value is unlikely, but cannot, apparently, be entirely dismissed, because we do not know certain properties of the oxygen atom which are involved in the calculation of πa^2. Thus, until the value of the O cross-section is decided, we cannot derive N_m from the measured values of $\pi a^2 N_m$. At the same time, despite the progress that has been made in measurements by means of rockets and satellites, a determination of the value of $\pi a^2 N_m$ in the F layer at different times would be a notable step forward in the study of the upper layers of the ionosphere.†

To conclude our discussion of the collision frequency in the F layer, we may note that statements are to be found in the literature (e.g. [20, Chapter VI, § 6]) to the effect that $\nu_{\text{eff}} \approx \nu_{\text{eff},m}$ in the F layer, since collisions with ions cannot be important, on account of the low ion density. It is clear from the foregoing discussion that this view is mistaken, since it is based on an incorrect idea that the cross-sections for electron scattering by ions and neutral particles are about the same. In reality, at $T \sim 300°$ the ion scattering cross-section is about a million times that for molecules.

Low frequencies

So far we have considered only the case of high frequencies (6.7), which is that usually encountered in radio studies of the ionosphere and in radio astronomy. Nevertheless, in practice we do, of course, often meet with frequencies which are less than the collision frequency and, in particular, with fields independent of time. As an example it may be mentioned that, at a height of about 70 km in the Earth's atmosphere, $N_m \sim 2 \times 10^{15}$ and $\nu_{\text{eff},m} \sim 10^7$, while at sea level $\nu_{\text{eff},m} \sim 2 \times 10^{11}$. Thus it is clear that we cannot restrict the investigation to the high frequencies only.

As well as in the high-frequency case (6.7), the formulae for ε and σ can be very simply derived in the opposite case of low frequencies:

$$\omega^2 \ll [\nu_{\text{eff}}^{(0)}]^2. \tag{6.18}$$

The resulting formulae are, of course, valid for a constant field also.

The effective collision frequency $\nu_{\text{eff}}^{(0)}$ which appears in (6.18) is defined by

$$\sigma = e^2 N/m\, \nu_{\text{eff}}^{(0)}. \tag{6.19}$$

The values of $\nu_{\text{eff}}^{(0)}$ and of the effective collision frequency ν_{eff} used previously for the high-frequency case are the same according to the elementary theory (§ 3), but the two are in fact equal only if the collision frequency ν is independent of the velocity v.

† More precisely, we may consider not the whole of the F layer but only the lower part and the region of transition to the E layer. The reason is that, by using formula (6.17) and the data given in Figs. 30.1–30.3 for heights above 200 to 250 km, we obtain the inequality already given, $\nu_{\text{eff}} - \nu_{\text{eff},i} = \nu_{\text{eff},m} \ll \nu_{\text{eff}}$. In such conditions, of course, no reliable determination of $\nu_{\text{eff},m}$ can be made by the method here discussed.

From formulae (6.3) and (6.19), taking into account the possibility of replacing the inequality $\omega^2 \ll \nu^2(u)$ by the condition (6.18), we obtain

$$\frac{1}{\nu_{\text{eff}}^{(0)}} = \frac{8}{3\sqrt{\pi}} \int_0^\infty \frac{u^4 e^{-u^2} du}{\nu(u)} . \tag{6.20}$$

Substituting (4.18) in (6.20) gives

$$\nu_{\text{eff},m}^{(0)} = \frac{3\pi}{8} \pi a^2 \bar{v} N_m . \tag{6.21}$$

The difference between this formula and (6.10) is that the factor $4/3 = 1{\cdot}33$ has been replaced by $3\pi/8 = 1{\cdot}18$. The slightness of this difference, especially in view of our imprecise knowledge of the radius a, means that the elementary theory is in practice adequate to deal with electron-molecule collisions.

In ordinary steady or quasisteady gas-discharge experiments the quantity $\nu_{\text{eff}}^{(0)}$ is found from the values of the conductivity. According to [26], the mean free path in air is $l = \bar{v}/\nu_{\text{eff}}^{(0)} = 5{\cdot}4 \times 10^{-2}$ cm for $\bar{v} = 1{\cdot}08 \times 10^7$ cm/sec and a pressure of 1 mm mercury. Hence $\pi a^2 = 4{\cdot}4 \times 10^{-16}$ cm^2 and

$$\nu_{\text{eff},m}^{(0)} = 1{\cdot}5 \times 10^{11} \frac{N_m}{2{\cdot}7 \times 10^{19}} \sqrt{\frac{T}{300}} . \tag{6.22}$$

In fact, we have used (6.22) (which may be regarded as an experimental value), (6.10) and (6.21) to derive above the value of $\nu_{\text{eff},m}$ for the other limiting case [see (6.11)].

For collisions with ions we have, substituting (4.19) in (6.20),

$$\nu_{\text{eff},i}^{(0)} = \frac{3\pi^2}{32} \left(\frac{e^2}{\varkappa T}\right)^2 \bar{v} N_i \ln\left(0{\cdot}54 \frac{\gamma \varkappa T}{e^2 N_+^{\frac{1}{3}}}\right) = \frac{1{\cdot}6 N_i}{T^{3/2}} \ln\left(324 \gamma \frac{T}{N_+^{\frac{1}{3}}}\right), \tag{6.23}$$

since the condition (4.25) shows that the integral involved is

$$\int_0^\infty \frac{x^3 e^{-x} dx}{\ln(1+\alpha^2 x^2)} \approx \frac{3}{\ln(\gamma\alpha)} ,$$

where $\gamma \sim 1$. Apart from the unimportant difference in the argument of the logarithm, formula (6.23) differs from (6.13) by the factor $3\pi/32$, i.e. $\nu_{\text{eff},i}^{(0)}$ is about one-third of $\nu_{\text{eff},i}$. However, it is not entirely correct to make a direct comparison of formulae (6.13) and (6.23), since the former is valid even when collisions between electrons are taken into account, while (6.23) applies only to electron–ion collisions and is not valid when collisions between electrons are included. The reason is that, as already mentioned, in the low-frequency case (6.18) we cannot neglect collisions between electrons, and if $N_i \approx N$ such collisions certainly affect the conductivity. Thus formula (6.23) holds good only if $N_i \gg N$, in which case collisions between electrons are not important.† If on the other hand (e.g.) $N_i = N_+ = N$, then the results of [32] show that

$$\nu^{(0)}_{\mathrm{eff},i} = 1{\cdot}73 \frac{3\pi^2 e^4}{32(\varkappa T)^2} \bar{v} N \ln\left(0{\cdot}54\gamma \frac{\varkappa T}{e^2 N^{\frac{1}{3}}}\right). \tag{6.24}$$

It is seen from (6.23) and (6.24) that the effect of collisions between electrons in this case is to introduce a factor 1·73. A comparison of formulae (6.13) and (6.24) shows that, other conditions being the same, $\nu_{\mathrm{eff},i}$ is $5{\cdot}5/2{\cdot}8 \approx 2$ times $\nu^{(0)}_{\mathrm{eff},i}$, the frequency of collisions in the case of low frequencies.

It should also be emphasised that for low frequencies, as is clear from (6.20), the relation (6.16) is not valid, i.e. $\nu^{(0)}_{\mathrm{eff}} \neq \nu^{(0)}_{\mathrm{eff},m} + \nu^{(0)}_{\mathrm{eff},i}$. Hence, in comparing formulae (6.13) and (6.24), we have assumed a purely electron-ion plasma, with no molecules, so that $\nu^{(0)}_{\mathrm{eff}} = \nu^{(0)}_{\mathrm{eff},i}$.

The general case of an arbitrary frequency

The permittivity and conductivity of a plasma for any ratio of frequency to collision frequency are conveniently put in the form [28, 34]

$$\left.\begin{aligned} \varepsilon &= 1 - K_\varepsilon\left(\frac{\omega}{\nu_{\mathrm{eff}}}\right)\frac{4\pi e^2 N}{m(\omega^2 + \nu^2_{\mathrm{eff}})}, \\ \sigma &= K_\sigma\left(\frac{\omega}{\nu_{\mathrm{eff}}}\right)\frac{e^2 N \nu_{\mathrm{eff}}}{m(\omega^2 + \nu^2_{\mathrm{eff}})}, \end{aligned}\right\} \tag{6.25}$$

where ν_{eff} is the collision frequency introduced in (6.9) for the case of high frequencies.

By definition K_ε and $K_\sigma \to 1$ when $\omega/\nu_{\mathrm{eff}} \to \infty$. For intermediate values of $\omega/\nu_{\mathrm{eff}}$, the factors K_ε, K_σ give the deviation of the expressions for ε and σ from the formulae obtained in the elementary theory. The functions $K_{\varepsilon,m}$ and $K_{\sigma,m}$ for collisions with molecules are shown in Fig. 6.1 and Table 6.2.

TABLE 6.2

Collisions with molecules

$\omega/\nu_{\mathrm{eff},m}$	$K_{\varepsilon,m}$	$K_{\sigma,m}$	$\omega/\nu_{\mathrm{eff},m}$	$K_{\varepsilon,m}$	$K_{\sigma,m}$
0	1·51	1·13	2·0	0·985	0·95
0·01	1·51	1·13	4·0	1·0	0·98
0·05	1·50	1·13	6·0	1·0	0·99
0·1	1·48	1·12	10·0	1·0	1·0
0·2	1·40	1·09	35·0	1·0	1·0
0·5	1·19	1·02	∞	1·0	1·0
1·0	1·07	0·94			

† Such collisions may also be neglected when multiply-charged ions are predominant ($Z \gg 1$).

The value $K_{\sigma,m}(0) = 1{\cdot}13$ is, as it should be, in agreement with formulae (6.10) and (6.21). The fact that $K_{\varepsilon,m}(0) = 1{\cdot}51 \neq K_{\sigma,m}(0)$ shows that the same value of $\nu_{\text{eff}}^{(0)}$ cannot be derived from the expressions $\sigma = e^2N/m\nu_{\text{eff}}$ and

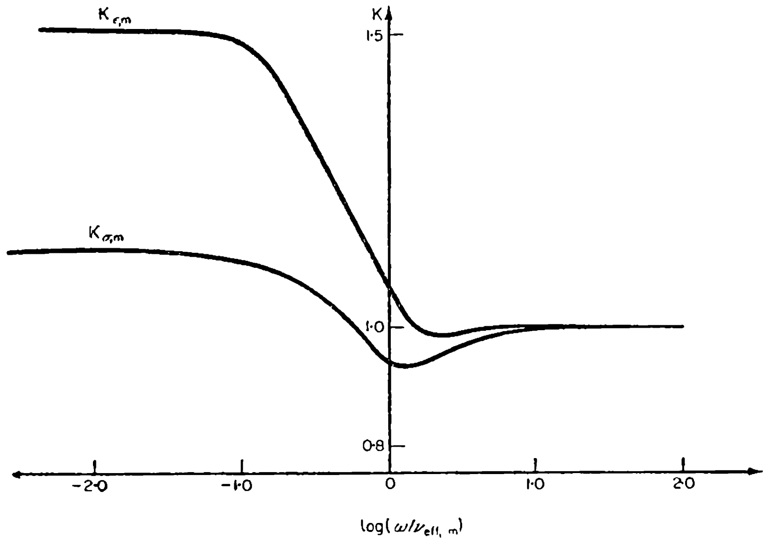

FIG. 6.1. The functions $K_{\varepsilon,m}(\omega/\nu_{\text{eff},m})$ and $K_{\sigma,m}(\omega/\nu_{\text{eff},m})$.

$\varepsilon = 1 - 4\pi e^2N/m\nu_{\text{eff}}^2$ of the elementary theory. In the low-frequency case (6.18) we have therefore used only the expression (6.19) for the conductivity σ. The functions $K_{\varepsilon,i}$ and $K_{\sigma,i}$ for collisions with ions when $N_i = N_+ = N$ are shown in Fig. 6.2a, b and in Table 6.3.

TABLE 6.3

Collisions with ions

$\omega/\nu_{\text{eff},i}$	$K_{\varepsilon,i}$ including electron-electron collisions	$K_{\varepsilon,i}$ neglecting electron-electron collisions	$K_{\sigma,i}$ including electron-electron collisions	$K_{\sigma,i}$ neglecting electron-electron collisions
0	4·59	19·8	1·95	3·39
0·01	4·59	19·5	1·95	3·38
0·05	4·51	15·8	1·92	2·76
0·1	4·34	11·1	1·86	2·12
0·2	3·79	5·47	1·65	1·40
0·5	2·30	2·44	1·07	0·90
1·0	1·41	1·52	0·72	0·68
2·0	1·05	1·15	0·62	0·59
4·0	0·97	1·01	0·73	0·63
6·0	0·98	0·97	0·82	0·72
10·0	0·99	0·96	0·92	0·78
35·0	1·00	0·99	0·99	0·91
∞	1·0	1·0	1·0	1·0

In order to exhibit the significance of collisions between electrons, values are given both with and without allowance for such collisions.† For the limiting cases, these results [34] are in accordance with those obtained in [31, 33, 35]. When collisions with both ions and molecules occur, the above expressions are, strictly speaking, applicable only in the limiting case of high frequencies.

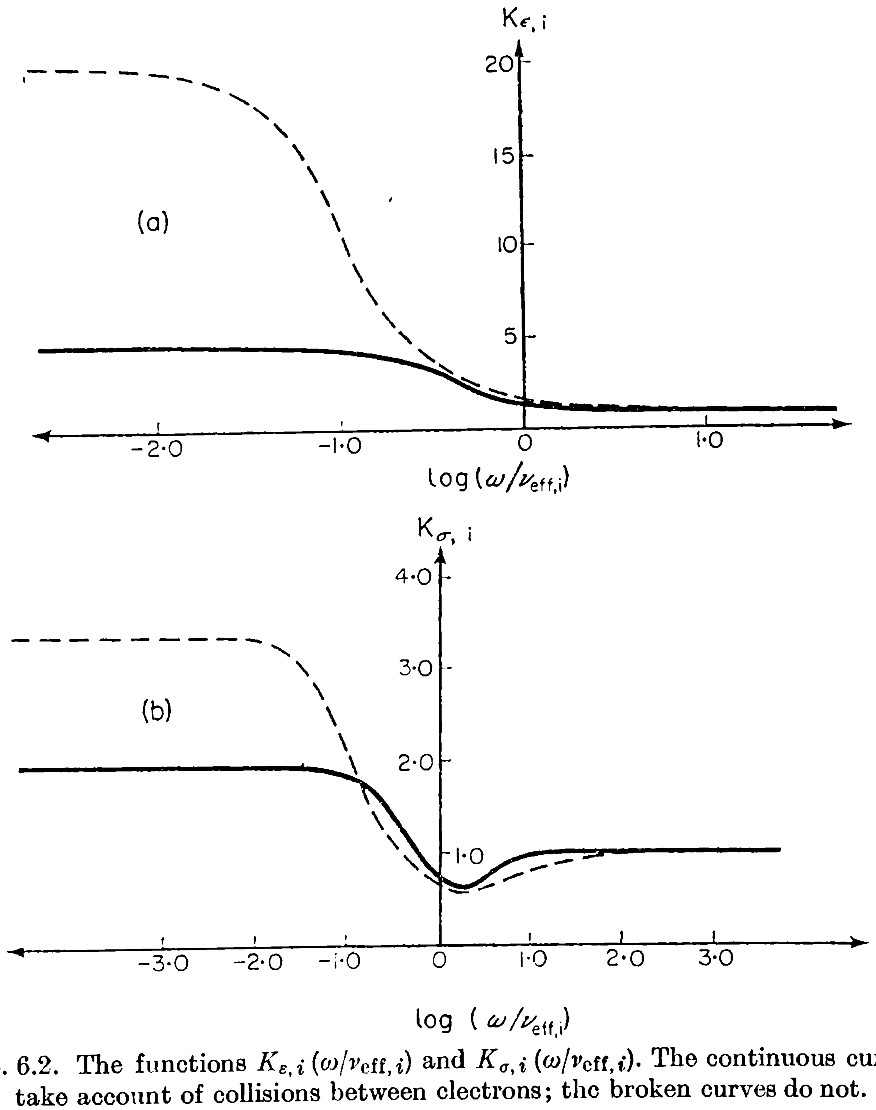

FIG. 6.2. The functions $K_{\varepsilon,i}(\omega/\nu_{eff,i})$ and $K_{\sigma,i}(\omega/\nu_{eff,i})$. The continuous curves take account of collisions between electrons; the broken curves do not.

Collisions of ions with ions and molecules

Let us now consider briefly the calculation of ε and σ in the case where collisions of ions with ions and molecules are predominant. This can, of course,

† The values in Table 6.3 have been derived in the approximation used in the first of [32]; the allowance for collisions between electrons brings in a factor 1·74. Formula (6.24) involves a factor 1·73, which is obtained in higher approximations [32]. This also explains the slight difference between the value of σ from (6.19) and (6.24) and that from (6.25) and Table 6.3 when $\omega/\nu_{eff} \to 0$.

occur (in the absence of a magnetic field) only for a plasma containing very few electrons or none at all.

A rigorous treatment of collisions between heavy particles is in general much more complex than that of collisions between electrons and ions or molecules. However, if we ignore a numerical factor of the order of unity, an expression for the frequency of collisions of an ion with molecules can be derived from formula (6.10) for electron-molecule collisions, if we replace a in (6.10) by the sum of the radii of the two particles and replace the mean velocity $\bar{v}$ by the mean relative velocity $\bar{v}_0 = \sqrt{[8\varkappa T(M_i + M_m)/\pi M_i M_m]}$, where M_i and M_m are the masses of the ion and the molecule respectively. If $M_i = M_m$ and the particles are of equal radius a, then $\bar{v}_0 = \sqrt{2}\,\bar{v}_i$ and, apart from a factor of order unity $(\bar{v}_i = \sqrt{(8\varkappa T/\pi M)})$, we have

$$\nu^{(i)}_{\mathrm{eff},m} \approx \frac{16\sqrt{2}\pi a^2}{3}\bar{v}_i N_m. \tag{6.26}$$

For $O_2^{\pm}$ and $N_2^{\pm}$ ions, taking a as in (6.11), we find

$$\nu^{(i)}_{\mathrm{eff},m} \approx 4\times 10^9 \frac{N_m}{2{\cdot}7\times 10^{19}}\sqrt{\frac{T}{300}}. \tag{6.27}$$

For ion–ion collisions we can use for $\nu^{(i)}_{\mathrm{eff},i}$, with the same proviso as above, formula (6.13) except for the conversion to numerical coefficients, taking $\bar{v}$ to be the mean ion velocity and multiplying the whole expression by $\sqrt{2}$. It must also be borne in mind that the arguments concerning conservation of momentum mentioned previously in connection with electron–electron collisions apply also to ions of like sign. In the expression for $\nu^{(i)}_{\mathrm{eff},i}$, therefore, it is more correct to take N_i as the density of ions of one sign, equal to half the total density. The result is

$$\nu^{(i)}_{\mathrm{eff},i} \approx \frac{\pi}{\sqrt{2}}\frac{e^4}{(\varkappa T)^2}\bar{v}_i N_+ \ln\left(0{\cdot}37\gamma_i\frac{\varkappa T}{e^2 N_+^{\frac{1}{3}}}\right) = \frac{3{\cdot}9 N_+}{T^{3/2}}\sqrt{\frac{m}{M}}\ln\left(220\gamma_i\frac{T}{N_+^{\frac{1}{3}}}\right), \tag{6.28}$$

where $\gamma \sim 1$ and the factor $0{\cdot}37$ in the argument of the logarithm is retained only for convenience of comparison with formula (6.13).

The formula for $\nu^{(i)}_{\mathrm{eff},i}$ thus obtained, like formula (6.26) for $\nu^{(i)}_{\mathrm{eff},m}$, is directly applicable to the case of high frequencies.† However, as we have seen, even in collisions of charged particles, only a factor of order unity appears in the frequency of collisions at low frequencies. The introduction of such a factor in the approximate formulae (6.26) and (6.28) would exaggerate the accuracy

† In the elementary theory $\nu^{(i)}_{\mathrm{eff}} = \nu^{(i)}_{\mathrm{eff},m} + \nu^{(i)}_{\mathrm{eff},i} + \nu^{(i)}_{\mathrm{eff},e}$ for all frequencies, and the effective frequency of ion-electron collisions is $\nu^{(i)}_{\mathrm{eff},e} = N\nu_{\mathrm{eff},i}/N_i$. Here $\nu_{\mathrm{eff},i}$ is given by (6.13). It follows from this and (6.28) that $\nu^{(i)}_{\mathrm{eff},e} \ll \nu^{(i)}_{\mathrm{eff},i}$ if $N \ll N_i\sqrt{(m/M)}$.

of the calculation. In other words, we can use only the formulae of the elementary theory for an ion plasma treated in the above manner.†

In the approximation corresponding to the elementary theory we have for a plasma with an arbitrary ratio of electron and ion densities

$$\left.\begin{aligned} \varepsilon &= 1 - \frac{4\pi e^2 N}{m(\omega^2 + \nu_{\text{eff}}^2)} - \frac{4\pi e^2 N_i}{M\left(\omega^2 + [\nu_{\text{eff}}^{(i)}]^2\right)}, \\ \sigma &= \frac{e^2 N \nu_{\text{eff}}}{m(\omega^2 + \nu_{\text{eff}}^2)} + \frac{e^2 N_i \nu_{\text{eff}}^{(i)}}{M\left(\omega^2 + [\nu_{\text{eff}}^{(i)}]^2\right)}. \end{aligned}\right\} \tag{6.29}$$

These formulae can often be simplified. For example, in the D layer of the ionosphere there seems to be a range of frequencies for which $\omega^2 \gg [\nu_{\text{eff}}^{(i)}]^2$ and $\omega^2 \ll \nu_{\text{eff}}^2$. Then

$$\left.\begin{aligned} \varepsilon &\approx 1 - \frac{4\pi e^2 N}{m\,\nu_{\text{eff}}^2} - \frac{4\pi e^2 N_i}{M\,\omega^2}, \\ \sigma &\approx \frac{e^2 N}{m\,\nu_{\text{eff}}} + \frac{e^2 N_i \nu_{\text{eff}}^{(i)}}{M\,\omega^2}. \end{aligned}\right\} \tag{6.30}$$

Dispersion relations

It is clear from the above discussion that the functions $\varepsilon(\omega)$ and $\sigma(\omega)$ for a plasma are in general quite complicated. It may therefore be useful to bear in mind that for any medium ε and σ are connected by what are called dispersion relations:

$$\left.\begin{aligned} \varepsilon(\omega) - 1 &= 8\int_0^\infty \frac{\sigma(\omega')\,d\omega'}{\omega'^2 - \omega^2}, \\ \frac{4\pi[\sigma(\omega) - \sigma(0)]}{\omega} &= -\frac{2\omega}{\pi}\int_0^\infty \frac{\varepsilon(\omega') - 1}{\omega'^2 - \omega^2}\,d\omega', \end{aligned}\right\} \tag{6.31}$$

where the integral must be taken as a principal value at $\omega = \omega'$. Further details are given in [36, § 62] and [22, § 83].

§ 7. THE PROPAGATION OF ELECTROMAGNETIC (TRANSVERSE) WAVES IN A HOMOGENEOUS PLASMA

The indices of refraction and absorption

When electromagnetic waves are propagated, unlike what happens in the quasistationary case, the permittivity ε and the conductivity σ are of subsidiary

† The variation of $\nu_{\text{eff},m}^{(i)}$ with frequency (as regards the use of the formulae of the elementary theory with ν_{eff} a function of ω), as for electron-molecule collisions, will in most cases be very slight. For ion-ion collisions the frequency effect is larger and it may be that $\nu_{\text{eff},i}^{(i)}$ for low frequencies (i.e. for $\omega^2 \ll [\nu_{\text{eff},i}^{(i)}]^2$) is about half its value for high frequencies (i.e. for $\omega^2 \gg [\nu_{\text{eff},i}^{(i)}]^2$).

importance. A direct physical significance now attaches to the indices of refraction and absorption, and to certain other related quantities.

To introduce these, let us consider the propagation of a plane electromagnetic (transverse) wave,

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}, \tag{7.1}$$

in an infinite homogeneous isotropic medium. The starting point here is equation (2.10) with $\varepsilon' =$ constant. For a transverse wave, where div $\boldsymbol{E} = 0$, this equation becomes

$$\Delta \boldsymbol{E} + (\omega^2/c^2)\,\varepsilon'(\omega)\boldsymbol{E} = 0. \tag{7.2}$$

Substituting the solution (7.1), we obtain the equation

$$k^2 = (\omega^2/c^2)\,\varepsilon'(\omega), \tag{7.3}$$

frequently called the dispersion relation. The relation between the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ is obtained from the field equations (2.1a) and (2.3), which for the plane waves (7.1) give

$$\omega\,\varepsilon'\boldsymbol{E} = -c\,\boldsymbol{k}\times\boldsymbol{H}, \quad \omega\boldsymbol{H} = c\,\boldsymbol{k}\times\boldsymbol{E}. \tag{7.4}$$

Scalar multiplication by $\boldsymbol{k}$ gives the conditions for transverse waves: $\boldsymbol{k}\cdot\boldsymbol{E} = 0$, $\boldsymbol{k}\cdot\boldsymbol{H} = 0$. The former of these follows from (7.4) only when $\varepsilon' \neq 0$; if $\varepsilon' = 0$, longitudinal waves also are possible (see §§ 2 and 8). In general, when the frequency ω is real, the wave vector $\boldsymbol{k}$ is complex, and can be written as $\boldsymbol{k} = \boldsymbol{k}_0 - i\boldsymbol{q}$, where $\boldsymbol{k}_0$ and $\boldsymbol{q}$ are real vectors. The field (7.1) becomes $\boldsymbol{E} = \boldsymbol{E}_0 e^{-\boldsymbol{q}\cdot\boldsymbol{r}} e^{i(\omega t - \boldsymbol{k}_0\cdot\boldsymbol{r})}$, and the difference in the directions of $\boldsymbol{k}_0$ and $\boldsymbol{q}$ corresponds to the difference in the planes of equal phase and amplitude. Such plane waves are said to be inhomogeneous. For homogeneous plane waves the planes of equal phase and amplitude coincide, and the vector $\boldsymbol{k}$ may be written in the form

$$\boldsymbol{k} = (\omega/c)(n - i\varkappa)\boldsymbol{k}/k. \tag{7.5}$$

Taking the direction of the vector $\boldsymbol{k}/k$ as the z-axis, we can write the field (7.1) in the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{\mp\omega\varkappa z/c} e^{i(\omega t \mp \omega n z/c)}, \tag{7.6}$$

where, by (7.3) and (7.5),

$$(n - i\varkappa)^2 = \varepsilon' \equiv \varepsilon - i\cdot 4\pi\sigma/\omega. \tag{7.7}$$

The $\mp$ sign in (7.6) corresponds to waves propagated in the directions of the positive and negative z-axis respectively. The quantities n and $\varkappa$ are the indices of refraction and absorption. The wavelength in the medium is seen from (7.6) to be

$$\lambda = \lambda_0/n, \quad \lambda_0 = 2\pi c/\omega \tag{7.8}$$

for the reasons discussed above, the directions of the phase and group velocities must be the same. Another proof of this statement is given in [36, § 64]. When spatial dispersion is taken into account, the group velocity may be in the opposite direction to the phase velocity; see Appendix A.

Damping of waves in the absence of absorption

When $\sigma = 0$, there is no absorption of electromagnetic energy, since the heat generated per unit time and volume is $\boldsymbol{j}\cdot\boldsymbol{E} = \sigma E^2$. This does not mean, however, that waves cannot be damped. For, if $\sigma = 0$ and $\varepsilon > 0$, then

$$n = \sqrt{\varepsilon}, \quad \varkappa = 0, \tag{7.13}$$

and the solution (7.6) has the form of pure travelling waves; but if $\sigma = 0$ and $\varepsilon < 0$, then

$$n = 0, \quad \varkappa = \sqrt{-\varepsilon} \tag{7.14}$$

and the solution is damped [the minus sign must be taken in (7.6) for a wave propagated in the direction of the positive z-axis].

In this case the damping of the wave signifies that travelling waves cannot be propagated in the medium; the energy flux $\boldsymbol{S} = c\boldsymbol{E}\times\boldsymbol{H}/4\pi$ vanishes on being averaged over time, and the wave is completely reflected from the medium with $\varepsilon < 0$ (total internal reflection).† The term "index of absorption" is unsuitable and inconvenient here, and the imaginary index of refraction $\tilde{n} = -i\sqrt{-\varepsilon} = -i\varkappa$ is therefore often used. Thus for $\sigma = 0$ it is convenient to define

$$\tilde{n}^2 = (n - i\varkappa)^2 = \varepsilon. \tag{7.15}$$

Expressions for n and $\varkappa$ in limiting cases

If $\sigma \neq 0$, we must use the formula (7.12). When

$$|\varepsilon| \gg 4\pi\sigma/\omega \tag{7.16}$$

we have for $\varepsilon > 0$

$$\left.\begin{aligned} &n \approx \sqrt{\varepsilon} = \sqrt{[1 - 4\pi e^2 N/m(\omega^2 + \nu_{\text{eff}}^2)]},\\ &\varkappa \approx 2\pi\sigma/\omega\sqrt{\varepsilon} = 2\pi e^2 N \nu_{\text{eff}}/m\omega(\omega^2 + \nu_{\text{eff}}^2)\sqrt{[1 - 4\pi e^2 N/m(\omega^2 + \nu_{\text{eff}}^2)]},\\ &\mu \approx 2\omega\varkappa/c \approx 4\pi\sigma/c\sqrt{\varepsilon} \end{aligned}\right\} \tag{7.17}$$

and for $\varepsilon < 0$

$$\left.\begin{aligned} &n \approx 2\pi\sigma/\omega\sqrt{-\varepsilon} = 2\pi e^2 N \nu_{\text{eff}}/m\omega(\omega^2 + \nu_{\text{eff}}^2)\sqrt{[4\pi e^2 N/m(\omega^2 + \nu_{\text{eff}}^2) - 1]},\\ &\varkappa \approx \sqrt{-\varepsilon} = \sqrt{[4\pi e^2 N/m(\omega^2 + \nu_{\text{eff}}^2) - 1]}. \end{aligned}\right\} \tag{7.18}$$

Here we have substituted for ε and σ the expressions given by the "elementary theory" for a plasma of the simplest type (see § 3).

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† In writing the Poynting vector $\boldsymbol{S}$ or the energy losses $\boldsymbol{j}\cdot\boldsymbol{E}$ in the usual form, we assume that only the real parts of the fields are taken.

In the opposite limiting case, when

$$|\varepsilon| \ll 4\pi\sigma/\omega, \tag{7.19}$$

we have

$$n \approx \varkappa \approx \sqrt{(2\pi\sigma/\omega)} = \sqrt{[2\pi e^2 N \nu_{\mathrm{eff}}/m\,\omega(\omega^2 + \nu_{\mathrm{eff}}^2)]}, \tag{7.20}$$

$$\mu = 2\omega\varkappa/c \approx \sqrt{(8\pi\omega\sigma/c^2)} = 4\pi\sqrt{(\sigma f)}/c = 4\pi\sqrt{(\sigma/c\lambda_0)}. \tag{7.21}$$

Here the field decreases by a factor of $e = 2{\cdot}72$ over a distance

$$\delta = c/\omega\varkappa \approx \sqrt{(c^2/2\pi\omega\sigma)}. \tag{7.22}$$

The quantity δ is frequently called the thickness or depth of the skin layer, since in metals, where the condition (7.19) is usually satisfied, the length $\delta = c/\sqrt{(2\pi\omega\sigma)}$ is of the order of the depth to which the variable field penetrates into the material.

The values of ε and σ have not yet been made specific in this section, except on the right of formulae (7.17), (7.18) and (7.20). This is natural, since the above discussion relates to any medium except where there is appreciable spatial dispersion. For an isotropic plasma we must use for ε and σ the expressions derived in §§ 3 and 6.

Real and complex frequencies

In conclusion, it may be noted that hitherto the frequency ω has been regarded as real, and we shall usually continue to do so. It must be emphasised that this is not a truism, but a certain manner of stating the physical problem. For there is another possible way of stating the problem, which is in fact sometimes encountered, whereby a real wave vector $\boldsymbol{k}$ is taken, i.e. a wave field harmonic in space is used. The application of this to plasma waves is discussed in § 8. The relation (7.3) between ω and $\boldsymbol{k}$ evidently holds good whether or not these quantities are assumed to be real. It is clear from this (and, of course, from the nature of the problem) that for real $\boldsymbol{k}$ the frequency ω is in general complex (i.e. the field is damped or amplified in the course of time). If $\boldsymbol{k}$ is real and the frequency is $\omega' = \omega + i\gamma$, and $\gamma \ll \omega$, then the relation $ck = (\omega + i\gamma)\,[n(\omega + i\gamma) - i\varkappa(\omega + i\gamma)]$ gives

$$\gamma = \omega\varkappa \div d(n\omega)/d\omega = \omega\varkappa(\omega)\,v_{\mathrm{gr}}/c, \tag{7.23}$$

where $v_{\mathrm{gr}} = c \div d(\omega n)/d\omega$ is the group velocity of the waves (see § 21). The relation (7.23) between the index of damping of the wave with time and the quantity $\omega\varkappa/c$ which determines the spatial damping [see (7.6)] is quite natural, since the energy in the wave travels with the group velocity v_{gr}; the condition $\gamma \ll \omega$ ensures that the damping is small, so that the concept of the group velocity can be used. Equation (7.23) has been derived without using the familiar dispersion relation $c^2k^2 = (n - i\varkappa)^2\omega^2$ for real ω. It is quite clear, however, that the same result is obtained if we first find the relation between real $\boldsymbol{k}$ and complex ω; then (7.23) gives $\varkappa$ for any value of γ.

If $n = \sqrt{(1 - \omega_0^2/\omega^2)}$, $\varkappa = 2\pi\sigma/\omega n = \omega_0^2 \nu_{\text{eff}}/2\omega^3 n = (1 - n^2)\nu_{\text{eff}}/2\omega n$ (cf. (7.17) for $\omega^2 \gg \nu_{\text{eff}}^2$, $\omega_0^2 = 4\pi e^2 N/m$), then $c \div d(\omega n)/d\omega = cn$ and

$$\gamma = \tfrac{1}{2}(1 - n^2)\nu_{\text{eff}} = 2\pi e^2 N \nu_{\text{eff}}/m\omega^2 \equiv \omega_0^2 \nu_{\text{eff}}/2\omega^2. \tag{7.24}$$

§ 8. THE ALLOWANCE FOR SPATIAL DISPERSION. PLASMA WAVES AND ACOUSTIC WAVES

Plasma (longitudinal) waves. Phenomenological allowance for spatial dispersion

The problem of the existence of longitudinal (plasma) electromagnetic waves in an isotropic medium has already been touched upon in § 2. If spatial dispersion is neglected, the frequency ω_0 of these waves is given by the condition $\varepsilon'(\omega_0) = 0$ [see (2.15)] or, when the conductivity is sufficiently low, by

$$\left.\begin{aligned} \varepsilon &= 1 - 4\pi e^2 N/m\omega_0^2 = 0, \\ \omega_0 &= \sqrt{(4\pi e^2 N/m)} = 5{\cdot}64 \times 10^4 \sqrt{N}, \end{aligned}\right\} \tag{8.1}$$

where we now consider the particular case of a plasma and neglect the motion of the ions.

The magnetic field in longitudinal waves is zero, and hence the term "electromagnetic waves" is in this case somewhat arbitrary. It is better to use the term "electric waves" or "charge waves", but we shall use below the more customary term "plasma waves". However, even the term "waves" is strictly correct only when spatial dispersion is taken into account. The allowance for spatial dispersion takes us beyond the merely local relation $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, and we must take into account the dependence of the induction $\boldsymbol{D}$ at a given point on the field not only at that point but also in its neighbourhood.

In a general isotropic medium the allowance for spatial dispersion can be made by assuming that the induction $\boldsymbol{D}$ depends linearly not only on the field $\boldsymbol{E}$ but also on the appropriate derivatives of this field. The result is

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \delta_1 \Delta\boldsymbol{E} + \delta_2 \operatorname{grad}\operatorname{div}\boldsymbol{E}. \tag{8.2}$$

Terms in the first derivatives of the field do not appear in an expansion such as (8.2), on account of symmetry.† There can also be no other second-order terms in (8.2), since $\Delta\boldsymbol{E}$ and **grad** div$\boldsymbol{E}$ are the only second-order invariants. It is in general incorrect, when spatial dispersion is weak, to take account of higher-order derivatives in the expression relating $\boldsymbol{D}$ and $\boldsymbol{E}$ (see below).

† The medium is assumed non-gyrotropic, and therefore has a centre of symmetry, whereas the relation $D_i = \gamma_{ikl}\,\partial E_k/\partial x_l$ is not invariant with respect to inversion (replacement of x_i by $-x_i$). For further details see [1, 36].

The current density $\boldsymbol{j}$ is also related to the field $\boldsymbol{E}$ by an expression of the type (8.2) when spatial dispersion is taken into account. For simplicity this expression will not be used in what follows, i.e. absorption will be neglected.

The field equations (2.1)–(2.4) are, of course, valid independently of the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$. The same applies, therefore, to the wave equation (2.5). Substituting (8.2), this equation gives

$$\triangle \boldsymbol{E} - \operatorname{grad}\operatorname{div}\boldsymbol{E} + (\omega^2/c^2)(\varepsilon + \delta_1\triangle + \delta_2 \,\mathbf{grad}\operatorname{div})\boldsymbol{E} = 0. \tag{8.3}$$

Hence, for homogeneous plane waves $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} = \boldsymbol{E}_0 e^{i(\omega t - \omega n \boldsymbol{k}\cdot\boldsymbol{r}/ck)}$, we have for transverse waves ($\boldsymbol{k}\cdot\boldsymbol{E} = 0$)

$$k^2 = (\omega^2/c^2)(\varepsilon - \delta_1 k^2), \quad \tilde{n}_\perp^2 = c^2k^2/\omega^2 = \varepsilon/(1 + \omega^2\delta_1/c^2); \tag{8.4}$$

for longitudinal waves ($\boldsymbol{k}\cdot\boldsymbol{E} = kE$)

$$(\omega^2/c^2)[\varepsilon - (\delta_1 + \delta_2)k^2] = 0, \quad \tilde{n}_\parallel^2 = c^2k^2/\omega^2 = c^2\varepsilon/\omega^2(\delta_1 + \delta_2). \tag{8.5}$$

It is clear from (8.4) that when the coefficient δ_1 is sufficiently small the allowance for spatial dispersion is unimportant for transverse waves, but (8.5) shows that for longitudinal waves it is important whatever the value of $\delta_1 + \delta_2$. When $\delta_1 + \delta_2 \to 0$, the index of refraction $\tilde{n}_\parallel^2$ can be finite only for $\varepsilon = 0$, and this is also the condition for longitudinal waves to exist when spatial dispersion is neglected [see (8.1)].†

The coefficients δ_1 and δ_2 cannot, of course, be determined by a phenomenological approach, but their values may be estimated as follows. The expression (8.2) represents the first terms in a series expansion, the parameter of the expansion being the ratio of some characteristic length a to the wavelength in the medium $\lambda = \lambda_0/n = 2\pi c/\omega n$ (in practice we use the quantity $\lambdabar = \lambda/2\pi = c/\omega n$). The wavelength appears because the wave field varies considerably over that distance. If the field also varies appreciably over a distance a, i.e. if $\lambdabar \sim a$, then the inhomogeneity of the field is of great importance, and the induction is, roughly speaking, determined equally by the field and by its spatial derivatives. Thus $|\delta_1 \triangle \boldsymbol{E}| \sim \delta_1\omega^2 n^2 E/c^2 = \delta_1 E/\lambdabar^2 \sim E$ for $\lambdabar \sim a$. The same applies to the term $\delta_2 \operatorname{grad}\operatorname{div}\boldsymbol{E}$. Hence

$$\delta_1 \sim \delta_2 \sim a^2. \tag{8.6}$$

In a plasma, the characteristic length a is the distance traversed by an electron in one oscillation, i.e. $a \sim v/\omega_0$, where v is the mean velocity of the electron. For when $\lambdabar \gg a$ the electron oscillates in a nearly uniform field, but if $\lambdabar \lesssim a$ the electron passes through various fields in the course of its oscillations. It is therefore clear why, when allowance is made for the thermal

† Whether or not spatial dispersion is taken into account, the general condition for plasma waves is $\mathbf{curl}\,\boldsymbol{H} = (1/c)(4\pi\boldsymbol{j} + \partial\boldsymbol{D}/\partial t) = 0$, which in the absence of absorption becomes $\partial\boldsymbol{D}/\partial t = 0$, with $\boldsymbol{E} \neq 0$.

motion, the current density at a given point in the plasma depends not only on the field at that point but also on its derivatives. Thus in a plasma we have

$$\delta_1 \sim \delta_2 \sim v^2/\omega_0^2. \tag{8.7}$$

When the electron velocity distribution is Maxwellian, $v \sim v_T = \sqrt{(\varkappa T/m)}$ and $v^2/\omega_0^2 \sim \varkappa T/8\pi e^2 N = D^2$, where D is the Debye length (4.21). Hence in a Maxwellian plasma we have

$$\delta_1 \sim \delta_2 \sim v_T^2/\omega_0^2 \sim D^2 = \varkappa T/8\pi e^2 N. \tag{8.8}$$

This result is readily understandable, since in the plasma the Debye length represents the radius of interaction between molecules or the radius of screening (see § 4).

It is evident from (8.5) and (8.8) that the index of refraction of a plasma wave $\tilde{n}_3 \equiv \tilde{n}_\parallel$ has the form

$$\tilde{n}_3^2 \equiv \tilde{n}_\parallel^2 = \frac{1 - \omega_0^2/\omega^2}{\xi v_T^2/c^2}, \quad \omega_0^2 = 4\pi e^2 N/m, \quad v_T = \sqrt{(\varkappa T/m)}, \tag{8.9}$$

where ξ is a numerical factor of the order of unity. The notation n_3 in place of $n_\parallel$ for longitudinal waves in a plasma is an anticipation of that used in Chapter III and afterwards.

In going from (8.5) to (8.9) we have used the fact that the frequency ω in the denominator may be replaced by ω_0. This is because the index n_3 becomes very large when there is an appreciable difference between the wave frequency ω and the plasma frequency ω_0. The approximation used above is then invalid, since, by taking only the second-order terms in the expansion (8.2), we essentially assume that the next terms are small, i.e. that†

$$a^2/\lambda^2 \sim \delta_{1,2}/\lambda^2 \ll 1. \tag{8.10}$$

Since $\lambda = c/\omega n = \lambda_0/n$, this inequality can be written

$$n^2 \ll c^2/\omega^2 \delta_{1,2} \sim \lambda_0^2/a^2. \tag{8.11}$$

For the plasma wave governed by (8.5) and (8.9) this signifies that

$$|\varepsilon| = |1 - \omega_0^2/\omega^2| \ll 1, \tag{8.12}$$

i.e. the frequencies ω and ω_0 are almost the same.

For transverse waves the condition (8.10) signifies that spatial dispersion is unimportant [see (8.4)]. Equations (8.4), (8.8) and (8.10) show that, for a plasma,

$$\left.\begin{aligned} &\tilde{n}_\perp^2 \equiv \tilde{n}_{1,2}^2 = \varepsilon(1 - \omega^2 \delta_{1,2}/c^2) = \varepsilon + \Delta\varepsilon, \\ &\varepsilon = 1 - \omega_0^2/\omega^2, \quad |\Delta\varepsilon| \sim \varepsilon\, \omega^2 v_T^2/c^2 \omega_0^2 = (\omega^2/\omega_0^2)(v_T/v_{\text{ph}})^2, \\ &v_{\text{ph}} = c/\sqrt{\varepsilon}. \end{aligned}\right\} \tag{8.13}$$

† The third-order terms do not appear in the expansion of the type (8.2), on account of symmetry; the fourth-order terms are less than the second-order terms in the ratio a^2/λ^2, since the parameter of the expansion is a/λ.

Even for $T \sim 10^6$ we have $v_T = \sqrt{(\varkappa T/m)} \sim 3 \times 10^8$ cm/sec and $|\Delta \varepsilon| \lesssim 10^{-4}$.

It might seem at first sight that the expansion (8.2) can be supplemented by higher terms, leading to more general results. In general, however, this is not so. The higher terms are evidently important only when the wavelength λ is close to the characteristic distance a, and in regions whose dimensions are of the order of a the whole macroscopic approach used above becomes invalid.† For solids and liquids this is immediately obvious, since, as has been mentioned in § 2, $a \sim d \sim 10^{-8}$ to 10^{-7} cm. The situation in a plasma is similar, since the macroscopic (quasihydrodynamic) description is impossible for regions whose dimensions are less than the Debye length D. It is sufficient here to note that a discussion of waves of length less than D is possible on the basis of the kinetic theory, and leads to the conclusion that these waves are strongly damped (see below). Thus the range of validity of the expression (8.9) for the index of refraction of plasma waves is in fact limited by the condition (8.12). When this condition is not fulfilled, there can be no plasma waves which are undamped or only slightly damped.

The kinetic theory

The calculation of the coefficient ξ in (8.9) and the determination of the index of absorption require a more detailed description of the plasma, and for a rarefied plasma they can be effected by the use of the Boltzmann equation [37–43]. Neglecting collisions, and in the absence of a magnetic field, we have the linearised Boltzmann equation (4.2) in the form

$$\partial \varphi/\partial t + \boldsymbol{v} \cdot \mathbf{grad}_r \varphi + (e\, \boldsymbol{E}/m) \cdot \mathbf{grad}_v f_0 = 0, \tag{8.14}$$

where φ is the deviation of the distribution function from the unperturbed distribution f_0, and e is the electron charge. In what follows, we shall suppose for definiteness that this unperturbed distribution is the equilibrium distribution, i.e. that the function f_0 is the Maxwellian function f_{00}. When the Boltzmann equation is used, spatial dispersion is automatically taken into account. Formally, this is due to the inclusion in equation (8.14) of the term $\boldsymbol{v} \cdot \mathbf{grad}_r \varphi$, which depends on the spatial derivatives.

Equation (8.14) must be solved together with the field equations, which are obtained by averaging the equations of electron theory:

$$\left.\begin{aligned} &\mathbf{curl}\, \boldsymbol{h} = (1/c)(4\pi\, \varrho\, \boldsymbol{v} + \partial \boldsymbol{e}/\partial t), \quad \operatorname{div} \boldsymbol{e} = 4\pi\, \varrho, \\ &\mathbf{curl}\, \boldsymbol{e} = -(1/c)\, \partial \boldsymbol{h}/\partial t, \quad \operatorname{div} \boldsymbol{h} = 0; \end{aligned}\right\} \tag{8.15}$$

$$\varrho = \sum_k e\, \delta(\boldsymbol{r} - \boldsymbol{r}_k) + \varrho_i, \quad \varrho\, \boldsymbol{v} = \sum_k e\, \boldsymbol{v}_k\, \delta(\boldsymbol{r} - \boldsymbol{r}_k), \tag{8.16}$$

† Here we pass over the fact that, when $a/\lambda \sim 1$, the series of the type (8.2) converges very slowly or at not all and must be replaced by an integral relation between $\boldsymbol{D}$ and $\boldsymbol{E}$.

It may be noted that the terms "macroscopic approach" and "hydrodynamic approach" are employed with the customary meaning that the medium may be regarded as continuous.

where $\boldsymbol{r}_k$ is the radius vector of the kth electron and ϱ_i the ion charge density (the ions being supposed at rest). When the microfields are averaged in the plasma we have, with $\bar{\boldsymbol{e}} = \boldsymbol{E}$, $\bar{\boldsymbol{h}} = \boldsymbol{H}$ (the bar denoting averaging),

$$\left.\begin{aligned} &\operatorname{curl}\boldsymbol{H} = (1/c)(4\pi \boldsymbol{j}_t + \partial \boldsymbol{E}/\partial t), \quad \operatorname{div}\boldsymbol{E} = 4\pi\bar{\varrho}, \\ &\operatorname{curl}\boldsymbol{E} = -(1/c)\,\partial\boldsymbol{H}/\partial t, \quad \operatorname{div}\boldsymbol{H} = 0, \\ &\boldsymbol{j}_t = e\int \boldsymbol{v}\,\varphi\,d\boldsymbol{v}, \quad \bar{\varrho} = e\int \varphi\,d\boldsymbol{v}. \end{aligned}\right\} \tag{8.17}$$

Here it is assumed that the current density $\boldsymbol{j}_t$ and the mean charge density $\bar{\varrho}$ are zero in the equilibrium state with distribution f_{00}.

From (8.17) we obtain in the usual way (see § 2) the wave equation

$$\left.\begin{aligned} \triangle\boldsymbol{E} - \operatorname{grad}\operatorname{div}\boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} &= \frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}_t}{\partial t}, \\ \boldsymbol{j}_t &= e\int \boldsymbol{v}\,\varphi\,d\boldsymbol{v}. \end{aligned}\right\} \tag{8.18}$$

Substituting in (8.14) the solution

$$\varphi(t,\,\boldsymbol{r},\,\boldsymbol{v}) = \varphi_0(\boldsymbol{v})\,e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}, \tag{8.19}$$

we find

$$i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\varphi = -(e/m)\,\boldsymbol{E}\cdot\operatorname{grad}_v f_{00}. \tag{8.20}$$

Ignoring for the moment the possibility that $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ may vanish, we get from (8.20)

$$\varphi = i\,e\,(\boldsymbol{E}\cdot\operatorname{grad}_v f_{00})/m(\omega - \boldsymbol{k}\cdot\boldsymbol{v}). \tag{8.21}$$

Substitution of this expression and a solution of the form $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ in the wave equation (8.18) gives

$$\left.\begin{aligned} &k^2\boldsymbol{E}_0 - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}_0) - (\omega^2/c^2)\,\boldsymbol{E}_0 = -i\cdot 4\pi\,\omega\,\boldsymbol{j}_{t0}/c^2, \\ &\boldsymbol{j}_{t0} = \frac{i\,e^2}{m}\int \frac{\boldsymbol{v}(\boldsymbol{E}_0\cdot\operatorname{grad}_v f_{00})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,d\boldsymbol{v}. \end{aligned}\right\} \tag{8.22}$$

The condition for the homogeneous equations (8.22) to have a non-trivial solution $\boldsymbol{E}_0 \neq 0$ is called the dispersion relation; it involves ω and $\boldsymbol{k}$. Again ignoring the possibility that $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ may vanish and assuming that the value of $\boldsymbol{j}_{t0}$ is determined by the region where $\omega \gg \boldsymbol{k}\cdot\boldsymbol{v}$, we obtain

$$\begin{aligned} \boldsymbol{j}_{t0} &= \frac{i\,e^2}{m}\int \frac{\boldsymbol{v}(\boldsymbol{E}_0\cdot\operatorname{grad}_v f_{00})}{\omega}\left(1 + \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega} + \frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{\omega^2}\right)d\boldsymbol{v} \\ &= \begin{cases} -\dfrac{i\,e^2 N}{m\,\omega}\left(1 + \dfrac{\varkappa T}{m}\dfrac{k^2}{\omega^2}\right)\boldsymbol{E}_0 & \text{for transverse waves,} \\ -\dfrac{i\,e^2 N}{m\,\omega}\left(1 + \dfrac{3\varkappa T}{m}\dfrac{k^2}{\omega^2}\right)\boldsymbol{E}_0 & \text{for longitudinal waves.} \end{cases} \end{aligned} \tag{8.23}$$

Here, in effecting the integration, f_{00} has been assumed to be a Maxwellian function [see (4.3) or (6.1)], so that $\mathbf{grad}_v f_{00} = (\partial f_{00}/\partial v)\boldsymbol{v}/v = -(m\boldsymbol{v}/\varkappa T)f_{00}$. From (8.22) and (8.23) we have the dispersion relation for transverse waves

$$\left.\begin{aligned} \omega^2 &= \omega_0^2 + c^2\left(1 + \frac{\varkappa T}{m c^2}\frac{\omega_0^2}{\omega^2}\right)k^2, \\ \tilde{n}_{1,2}^2 &= c^2 k^2/\omega^2 = \frac{1-\omega_0^2/\omega^2}{1+(\varkappa T/m c^2)\,\omega_0^2/\omega^2}. \end{aligned}\right\} \tag{8.24}$$

The result (8.24) is entirely in agreement with the earlier discussion [see (8.4), (8.8) and (8.13). Thus in (8.4) for a plasma $\delta_1 = \varkappa T\omega_0^2/m\omega^4$]. It should, however, be noted that the correction term $(\varkappa T/mc^2)\omega_0^2/\omega^2$ is of the same order as the relativistic effects. Since the latter have not been taken into account, the temperature correction in (8.24) cannot be regarded as established with accuracy. We shall not give here the corresponding relativistic calculation, since for transverse waves with $T \lesssim 10^6$ the temperature correction is in any case negligible, but merely mention that for $\varkappa T/mc^2 \ll 1$ the result of the relativistic calculation is [267]

$$\tilde{n}_{1,2}^2 = \frac{1-(\omega_0^2/\omega^2)(1-5\varkappa T/2m c^2)}{1+(\varkappa T/m c^2)\,\omega_0^2/\omega^2}.$$

For a plasma (longitudinal) wave we find from (8.22) and (8.23)†

$$\left.\begin{aligned} \omega^2 &= \omega_0^2\left(1+\frac{3\varkappa T}{m}\frac{k^2}{\omega^2}\right) \approx \omega_0^2 + \frac{3\varkappa T}{m}k^2, \\ \tilde{n}_3^2 &= \frac{1-\omega_0^2/\omega^2}{(3\varkappa T/m c^2)\,\omega_0^2/\omega^2} \approx \frac{1-\omega_0^2/\omega^2}{3\varkappa T/m c^2}, \end{aligned}\right\} \tag{8.25}$$

which agrees with (8.9) for $\xi = 3$.

It should be noted that in (8.25), unlike (8.24), the temperature correction is exact for sufficiently short waves. The allowance for relativistic effects (i.e. the dependence of mass on velocity) leads to a change in ω_0^2 by an amount of the order of $(\varkappa T/mc^2)\omega_0^2$. In (8.24) this alters the temperature correction itself, but in (8.25) it merely adds in the numerator a term of the type $(\varkappa T/mc^2)\omega_0^2/\omega^2$, which is small compared with unity. The essential point, therefore, is that for longitudinal waves there is in the dispersion relation no large term c^2k^2, and the term involving the temperature cannot be regarded as a correction; it is the only term which depends on k^2.

† In a longitudinal wave the magnetic field $\boldsymbol{H} = 0$, and so we need not use the wave equation; instead the field equation $\operatorname{div}\boldsymbol{E} = 4\pi e\int\varphi\,d\boldsymbol{v}$, together with the equation $\boldsymbol{E} = -\mathbf{grad}\,\Phi$ which defines the electric potential, can be used. This would not, however, afford any simplification in the problems here discussed.

For $\varkappa T/mc^2 \ll 1$ the relativistic calculation [42] gives the dispersion relation

$$\left.\begin{aligned} \omega^2 &\approx \omega_0^2 + \frac{3\varkappa T}{m} k^2 - \frac{5\varkappa T}{2mc^2}\omega_0^2, \\ \tilde{n}_3^2 &\approx \frac{1-(\omega_0^2/\omega^2)(1-5\varkappa T/2mc^2)}{3\varkappa T/mc^2}, \end{aligned}\right\} \tag{8.25a}$$

confirming the above result. The correction term $-5\varkappa T\omega_0^2/2mc^2$ is small but nevertheless of importance for relatively small values of $\tilde{n}_3^2$, i.e. for sufficiently large values of the wavelength $\lambda = 2\pi/k$. For example, (8.25) gives $\tilde{n}_3^2 = 1$ when $\omega - \omega_0 \approx 3\varkappa T\omega_0/2mc^2$, whereas (8.25a) gives $\tilde{n}_3^2 = 1$ when $\omega - \omega_0 \approx \frac{1}{4}\varkappa T\omega_0/mc^2$. In other words, formula (8.25) is valid only if $|\omega - \omega_0| \gg \varkappa T\omega_0/mc^2$.

The phase and group velocities of plasma waves are, from (8.25),

$$\left.\begin{aligned} v_{\text{ph}} &= \omega/k = c/n_3 = \sqrt{(3\varkappa T/m)} \div \sqrt{(1-\omega_0^2/\omega^2)}, \\ v_{\text{gr}} &= d\omega/dk \approx 3\varkappa Tk/m\omega_0 \\ &\approx \sqrt{(3\varkappa T/m)} \times \sqrt{(1-\omega_0^2/\omega^2)} \\ &= v_{\text{ph}}(1-\omega_0^2/\omega^2). \end{aligned}\right\} \tag{8.26}$$

Here we have regarded the temperature term in the dispersion relation as a correction, and to the accuracy used we can put†

$$\left.\begin{aligned} \omega &= \omega_0 + \frac{3}{2}(\varkappa T/m)k^2/\omega_0^2 = \omega_0(1+3D^2k^2), \\ D &= \sqrt{(\varkappa T/8\pi e^2 N)}. \end{aligned}\right\} \tag{8.27}$$

The condition for the temperature correction to the plasma wave frequency to be small can evidently be written

$$Dk = D/\lambda \ll 1. \tag{8.28}$$

This condition is clearly equivalent to the conditions (8.11) and (8.12) used previously, and implies that, to the accuracy used, we can always replace (8.25) by (8.27) and $1 - \omega_0^2/\omega^2$ by $2(1 - \omega_0/\omega)$. However, we shall for simplicity refrain from making this change.

Cherenkov radiation in a plasma. Absorption of plasma waves

The question of plasma waves whose wavelength is less than those which satisfy the condition (8.28) is closely related to the possibility of satisfying the equation

$$\omega = \boldsymbol{k}\cdot\boldsymbol{v}. \tag{8.29}$$

† It may be noted that an expression frequently used in the literature for the Debye length is $\sqrt{(\varkappa T_e/4\pi e^2 N)}$, which holds for a plasma in which the electron temperature T_e considerably exceeds the ion temperature [see (4.22)].

If this condition may be satisfied then equations (8.21) and onwards are not, strictly speaking, meaningful. This is reasonable, since (8.21) can be derived from (8.20) only if $\omega - \boldsymbol{k}\cdot\boldsymbol{v} \neq 0$. For a transverse wave in a plasma the condition (8.29) cannot hold, since $\omega/k = v_{\text{ph}} = c/n_{1,2} = c \div \sqrt{(1-\omega_0^2/\omega^2)} > c$, and $v < c$. Admittedly the condition $v < c$ is not automatically included when the non-relativistic Maxwellian distribution is used, but the validity of the above statement is evident from physical arguments besides being confirmable by a relativistic calculation. The physical arguments have already been given in essence, but we may make the supplementary comment that equation (8.29) is identical with the condition for the occurrence of Cherenkov radiation. This is usually written

$$v\cos\theta = c/n, \tag{8.30}$$

where $c/n = \omega/k = v_{\text{ph}}$ is the phase velocity and θ is the angle between the velocity $\boldsymbol{v}$ of the particle and the wave vector $\boldsymbol{k}$. Cherenkov radiation (the Vavilov–Cherenkov effect) is possible, evidently, only if $v > c/n$. For transverse waves in a plasma, $n_{1,2} < 1$ and Cherenkov radiation is not possible.

In the case of plasma waves the conditions (8.29) and (8.30) may be satisfied, and this corresponds to Cherenkov emission of plasma waves by a particle (electron). But if the particle can emit a wave of this kind, then it must be able to absorb it, i.e. acquire energy from a similar wave during its propagation. Thus plasma waves must be damped even if there are no collisions.†

The amount of damping is determined principally by the relative number of plasma particles for which the conditions (8.29) and (8.30) can be satisfied. For a plasma wave of phase velocity $v_{\text{ph}} = \omega/k$ propagated in a Maxwellian plasma, the number of absorbing particles is large when $v_{\text{ph}} \sim v_{\text{T}}$ and exponentially small when

$$v_{\text{ph}} = \frac{\sqrt{(3\varkappa T/m)}}{\sqrt{(1-\omega_0^2/\omega^2)}} \gg v_T = \sqrt{(\varkappa T/m)}. \tag{8.31}$$

This condition is obviously the same as (8.12) and (8.28). If the initial distribution function f_0 is not Maxwellian, but differs from it by the absence of very fast particles, there will be no damping. It is clear that there is no physical distinction between very little damping and absence of damping. Thus, when (8.12), (8.28) or (8.31) holds, the above expressions for the frequency and the index of refraction of plasma waves are valid, and the region of Cherenkov emission, where $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$, may be neglected.

† In the absence of collisions the damping of plasma waves increases the energy of electrons for which the condition $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ is satisfied. Since in fact collisions always occur, the energy acquired from the wave is subsequently transformed into the energy of random thermal motion. The collisionless damping of plasma waves was first demonstrated in [38]; its physical interpretation was given in [43], although without explicit reference to Cherenkov radiation. The author is uncertain as to where such reference was first made, but recently the Cherenkov mechanism has been treated as well known (see, e.g., [82, 83]).

The above calculations are, however, insufficient to find the damping coefficient and to analyse the propagation of short plasma waves with $\lambda = 1/k \lesssim D$. To deal with these problems it is necessary to solve simultaneously the Boltzmann equation (8.14) and the wave equation (8.18), e.g. with the following conditions [38]: at the initial instant $t = 0$ there is a given deviation of the distribution function $\varphi(0, \boldsymbol{r}, \boldsymbol{v})$ from the equilibrium distribution f_{00}, and we require the form of the function $\varphi(t, \boldsymbol{r}, \boldsymbol{v})$ at subsequent instants. Without loss of generality we may consider only functions of the form

$$\varphi(t, \boldsymbol{r}, \boldsymbol{v}) = \varphi(t, \boldsymbol{v})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{8.32}$$

where the wave vector $\boldsymbol{k}$ is assumed to have a given real value. At times t close to the initial instant $t = 0$ the function $\varphi(t, \boldsymbol{v})$ is not in general harmonic, and so there is no relation between ω and $\boldsymbol{k}$. In the course of time the initial perturbation $\varphi(0, \boldsymbol{v})$ is "broken down" and transformed, and for sufficiently large t the electric field $\boldsymbol{E}$ depends on the time as $e^{i\omega' t} = e^{i\omega t}e^{-\gamma t}$ (where $\omega' = \omega + i\gamma$ is the complex frequency). We assume that the initial perturbation $\varphi(0, \boldsymbol{r}, \boldsymbol{v})$ is a smooth function and that the unperturbed velocity distribution f_0 also has no singularities (this is, of course, true if f_0 is the Maxwellian distribution f_{00}). Then the solution $E \sim e^{i\omega' t} = e^{i\omega t}e^{-\gamma t}$ becomes valid after a time of the order of $1/\omega_0 \sim D/v_T$, where D is the Debye length (4.21). Thus the relation between ω' and $\boldsymbol{k}$ is set up, as it were, asymptotically, but fairly rapidly in practice.†

In order to indicate how the damping coefficient γ may be calculated without solving a problem with given initial conditions (see above), we shall proceed as follows. If we neglect the possible existence of a pole, (8.22) gives for longitudinal oscillations the dispersion relation

$$\frac{4\pi e^2}{m\omega}\int \frac{u\, d f_{00}(u)/d u}{\omega - k u}\, d u = -1, \tag{8.33}$$

where u is the component of the electron velocity in the direction of $\boldsymbol{k}$, the integration over the components perpendicular to $\boldsymbol{k}$ has been effected, and $f_{00}(u) = N\sqrt{(m/2\pi\varkappa T)}\,e^{-mu^2/2\varkappa T}$. Where it is possible that $\omega = ku$, the relation (8.33) has no precise significance, since the result obtained depends on how the integral is calculated. This means that there is in general no relation between ω and $\boldsymbol{k}$. Such a relation does exist, however, for the waves which are least rapidly damped in the course of time for a given real $\boldsymbol{k}$. The solution of the initial-condition problem stated above [38] shows that this relation is given by equation (8.33) with the integration in the plane of the complex variable u along the contour C shown in Fig. 8.1, i.e. passing above the pole

† The behaviour of the field at times $t \lesssim 2\pi/\omega_0$ has not to the author's knowledge been analysed in detail.

$u_0 = \omega'/k = \omega/k + i\gamma_0/k$:

$$-\frac{4\pi e^2}{m\,\omega'}\int \frac{u\,d f_{00}(u)/du}{\omega' - k u}\,du = \frac{\omega_0^2}{\omega'}\left(\frac{m}{\varkappa T N}\right)\int \frac{u^2 f_{00}\,du}{\omega' - k u} = 1. \tag{8.34}$$

Here ω has been replaced by $\omega' = \omega + i\gamma$, since for real k the equation has a solution only for a complex frequency.†

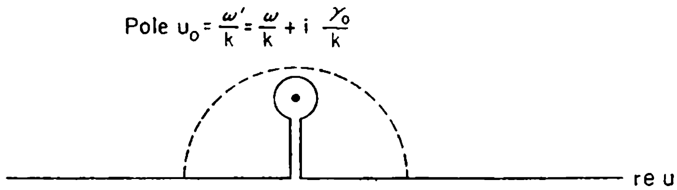

FIG. 8.1. Contour of integration C in the dispersion relation (8.34).

For small k, the pole u_0 is remote [re $u_0 \gg \sqrt{(\varkappa T/m)}$], and $f_{00}(u_0)$ is very small. If we neglect damping, therefore, we need only integrate along the real axis of u, and this gives formula (8.25). To calculate the damping (on the assumption that k is small and $\gamma \ll \omega_0$), the integral in (8.34) is reduced to one along the real axis together with an integral along the semicircle shown by a broken line in Fig. 8.1. The latter integral is equal to πi times the residue at the pole. Thus (8.34) gives

$$\frac{4\pi e^2 N}{m\,\omega'^2} + \frac{4\pi e^2 N}{m\,\omega'^2}\cdot\frac{3\varkappa T}{m}\,\frac{k^2}{\omega'^2} + i\,\frac{4\pi^2 e^2}{m\,k^2}\,\frac{d f_{00}(\omega'/k)}{du} = 1. \tag{8.35}$$

An approximate solution of this equation, using the condition $\gamma \ll \omega$, gives for ω the previous expression (8.25) and for the coefficient γ the value

$$\left.\begin{aligned} \gamma_0 &= \frac{\sqrt{\pi}}{8}\,\omega_0\,e^{-3/2}\,\frac{1}{(k D)^3}\,e^{-1/4(kD)^2} \\ &= \sqrt{\frac{\pi}{8}}\,e^{-3/2}\,\frac{\omega_0^4}{k^3}\left(\frac{m}{\varkappa T}\right)^{3/2} e^{-m\,\omega_0^2/2\varkappa T k^2} \\ &\approx \frac{0{\cdot}05\,\omega_0}{(k D)^3}\cdot 10^{-0.11/(k D)^2}, \\ k D = D/\lambda \ll 1,&\quad D = \sqrt{(\varkappa T/8\pi e^2 N)},\quad \gamma_0 \ll \omega_0. \end{aligned}\right\} \tag{8.36}$$

It is clear from (8.28) and (8.25a) that for $\omega - \omega_0 \lesssim (\varkappa T/mc^2)\omega_0$ the index of refraction $\tilde{n}_3$ is small and may be less than unity. When $\tilde{n}_3 < 1$ Cherenkov absorption of plasma waves is, of course, impossible (see the

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† The same condition (8.34) may be derived by seeking an initial perturbation $\varphi(0, \boldsymbol{v})\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ such that the field is $\boldsymbol{E} = \boldsymbol{E}_0\,e^{i\omega' t}$ for all t [39].

It may be noted that in calculating the coefficient of absorption of plasma waves the question of the limits of applicability of the linear approximation should be given special consideration. This has not yet been done, so far as the author is aware, although it is much to be desired in the light of the results of [40].

corresponding comment above regarding transverse waves). Thus for $\tilde{n}_3 < 1$ the damping $\gamma_0 = 0$. This is not shown in formula (8.36), the calculation being non-relativistic. In the relativistic theory of plasma waves [42] we necessarily have $\gamma_0 = 0$ for $\tilde{n}_3 < 1$. From (8.25), (8.25a) and (8.36), however, it is easy to see (cf. also Table 8.1) that for $\varkappa T/mc^2 \lesssim 10^{-3}$ formula (8.36) ceases to be valid only for values of γ_0 so infinitesimal that they are of no interest whatever.

The problems of wave propagation usually encountered differ from those discussed above in that the wave is regarded as being damped in space and not in time. Equations (8.14) and (8.18) must then be solved for a distribution function φ which is given at every instant t on some plane $z = 0$, to obtain the value of φ for all z (the z-axis being in the direction of the normal to the wave). When there is a relation between ω and k we can go directly from time damping to space damping, because in finding the relation between ω and k both quantities may be supposed complex from the start. The expressions (8.27) and (8.33) are therefore valid for complex k also.

Hitherto we have had

$$p \equiv i\,\omega - \gamma = i\left[\omega_0 + \frac{3}{2}\,(\varkappa\,T/m)\,k^2/\omega_0\right] - \gamma_0, \tag{8.37}$$

where γ_0 is given by formula (8.36). Now let $p = i\omega$ (with ω real) and $\boldsymbol{k} = \boldsymbol{k}_0 - i\boldsymbol{q}$, where $q \ll k_0 \approx k$ (as is certainly true if $\gamma \ll \omega_0$). Then the relation between $k_0 \approx k$ and ω is the same as (8.27), and

$$q \equiv \omega\varkappa_3/c = m\omega_0\gamma_0(k)/3\varkappa T k = \gamma_0(k)/v_{\mathrm{gr}}; \tag{8.38}$$

see (8.26) and (8.36). Formula (8.38) is, as we should expect, of the same form as formula (7.23) for transverse waves.

Collisions in the plasma have not so far been taken into account. When collisions occur, for plasma waves of long wavelength but such that $\omega^2 \gg \nu_{\mathrm{eff}}^2$, the dependence of ω on k is again given by formula (8.27) and there is damping with coefficient

$$\gamma_{\mathrm{coll}} = \tfrac{1}{2}\,\nu_{\mathrm{eff}}, \quad \gamma_{\mathrm{coll}} \ll \omega_0, \tag{8.39}$$

where ν_{eff} is the effective collision frequency (6.9).

Formula (8.39) is obtained from the fundamental condition $\varepsilon'(\omega') = 1 - \omega_0^2/\omega'(\omega' - i\nu_{\mathrm{eff}}) = 0$, which gives the complex plasma frequency $\omega' = \omega + i\gamma$ (we use also the fact that, when $\gamma \ll \omega$, $\omega = \omega_0$ to a good approximation). When damping (8.36) and collision damping (8.39) are both taken into account we have

$$\gamma = \gamma_0 + \gamma_{\mathrm{coll}}, \quad \omega\varkappa_3/c = \gamma/v_{\mathrm{gr}}, \quad \gamma \ll \omega, \quad \omega \gg \nu_{\mathrm{eff}}. \tag{8.40}$$

The specifically plasma damping is small only so long as the wavelength is much greater than the Debye length. If $kD \sim 1$, then $\gamma_0 \sim \omega_0$, and for $kD \gg 1$ we have $\gamma \gg \omega$. In such conditions plasma waves cannot exist. The exponential form of the expression (8.36) has the result that the plasma wave spectrum is very sharply cut off at about $kD = 0{\cdot}1$, i.e. at

$$\lambda_{cr}(\text{cm}) = 2\pi/k_{cr} \approx 63D \approx 300\sqrt{[T(\text{deg})/N]}. \tag{8.41}$$

In the lower part of the solar corona, where $T \sim 10^6$ and $N \sim 10^8, \lambda_{cr} \sim 30$ cm. For $N \sim 10^8$, the frequency is $\omega_0 \sim 5 \times 10^8 \text{ sec}^{-1}$ and the field in a wave of length λ_{cr} is damped in a time $1/\gamma_0$ which is of the order of 1 sec (see Table 8.1). A wave of length even $\frac{1}{2}\lambda_{cr}$ is damped in a time $1/\gamma_0 \sim 10^{-7}$ sec.

When $kD < 0{\cdot}1$ to $0{\cdot}05$, the damping γ_0 is very small (though it may be considerably increased if the unperturbed distribution function differs from the Maxwellian in having more electrons with velocities $v \gg v_T$). In this range collision damping is therefore usually more important. For example, in the lower corona $\gamma_{coll} = \frac{1}{2}\gamma_{eff} \sim 3$ to 10 sec^{-1} and exceeds γ_0 even for $kD = 0{\cdot}1$. In the Earth's ionosphere below the maximum of the F layer $\nu_{eff} \gtrsim 10^3 \text{ sec}^{-1}$ and $\gamma_{coll} \gtrsim 10^3 \sim 10^{-5}\,\omega_0$ (for $N \sim 10^6$ we have $\omega_0 \sim 5 \times 10^7 \text{ sec}^{-1}$). Thus in the ionosphere $\gamma_{coll} \gg \gamma_0$ for $kD \lesssim 0{\cdot}1$ or $\lambda_{cr} \gtrsim 10$ cm, while even for $\gamma \approx \gamma_{coll} \sim 10^3 \text{ sec}^{-1}$ and $v_{gr} \ll \sqrt{(3\varkappa T/m)}$ [see (8.26)] a plasma wave is considerably damped over a distance $v_{gr}/\gamma_0 \ll 10^4$ cm.

TABLE 8.1

Values of ω and γ_0 for plasma waves

kD	ω/ω_0	γ_0/ω_0
0·01	1·0003	10^{-1100}
0·1	1·03	$0{\cdot}5 \times 10^{-9}$
0·15	1·07	$1{\cdot}7 \times 10^{-4}$
0·2	1·12	0·01
0·3	1·27	$\sim$0·16
1	4	$\sim$0·04

Note. In the last two cases formulae (8.27) and (8.36) are not strictly applicable and can be used only to give an estimate of ω/ω_0 and γ_0/ω_0.

The effect of ions. Acoustic waves

The effect of ions has been entirely ignored hitherto except for the compene sation of the electron charges in the equilibrium state. The formulae in th- present section are therefore equally valid for a plasma in which the electron temperature T_e is not equal to the ion temperature T,† except that T must of course be replaced by T_e.

For the plasma waves considered it is legitimate to neglect the effect of the ions, since (when $N_i \sim N_+ \sim N$) they make only a small contribution, of the order of m/M, to the expression for the index of refraction (see § 3). The effect of ions may, nevertheless, be of importance for another branch

† We ignore the fact that in a non-equilibrium plasma the electron distribution function may be appreciably different from the Maxwellian.

of waves in a plasma. For sufficiently long waves in a plasma, as in any quasi-neutral medium, longitudinal acoustic (sound) waves must be propagated. In these waves the electrons and ions move in unison to a first approximation, so that there are no uncompensated charges.† The elasticity of the medium which permits the propagation of an acoustic wave is, of course, related to the pressure. To transmit pressure, the particles must collide sufficiently frequently. Hence acoustic waves in a gas are strongly damped unless the mean free path $l \sim v_T/\nu_{\text{eff}}$ is much less than the wavelength:

$$l \sim v_T/\nu_{\text{eff}} \ll \lambda = v_{\text{ph}}/\omega, \quad \nu_{\text{eff}} \gg \omega\, v_T/v_{\text{ph}}. \tag{8.42}$$

If this condition holds, the ordinary hydrodynamic approximation is applicable, i.e. the equations for the velocity $\boldsymbol{v}$ and plasma density $MN_i + mN \approx MN$ are, in the linear approximation,

$$MN\,\partial\boldsymbol{v}/\partial t = -\operatorname{grad} p, \quad \partial N'/\partial t + N \operatorname{div}\boldsymbol{v} = 0, \tag{8.43}$$

where N' is the difference between the density and the equilibrium value $N_0 \equiv N = N_i = N_+$.

Assuming that the pressure p is the same as that for a perfect gas of electrons and ions ($p = 2\varkappa NT$), and for simplicity regarding the oscillations as isothermal, we find from (8.43) for waves $\boldsymbol{v} = \boldsymbol{v}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, $N' = N_0' e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ the values

$$\omega = \sqrt{(2\varkappa T/M)}k, \quad v_{\text{ph}} = \sqrt{(2\varkappa T/M)}. \tag{8.44}$$

Here, as in any gas, the velocity of sound v_{ph} is of the order of the thermal velocity of the heavy particles. If the plasma contains not only electrons and ions but also molecules of density N_m, then for $N_m \gg N = N_i$ the velocity of sound is given by the usual expression for a gas. For example, in the isothermal approximation, which may serve to give an estimate only, $v_{\text{ph}} = \sqrt{(\varkappa T/M)}$. Formula (8.44) is in general itself only an approximation.

When the condition (8.42) does not hold, acoustic waves in a gas are strongly damped, and when $\lambda \lesssim l$ the damping takes place over a distance of the order of λ. Thus for sound in a gas the free path l is analogous to the Debye length for plasma waves.

The above discussion of the propagation of acoustic waves in gases applies to a plasma also, but only if we neglect the uncompensated space charge. In other words, it is assumed that in an acoustic wave the quasineutrality of the plasma is almost maintained, and that the elasticity of the medium is due only to collisions, not to the Coloumb interaction of macroscopic charges formed during the propagation of the wave. In reality, the quasineutrality of a plasma is to some extent destroyed even by acoustic waves. Hence the

† Acoustic waves in a plasma are, as already stated, analogous to acoustic waves in gases, liquids and solids, while high-frequency longitudinal (plasma) waves are analogous to Born (optical) waves or vibrations in a solid.

condition (8.42) for a plasma is sufficient but not necessary for the existence of relatively slightly damped waves of the acoustic type.

If the condition (8.42) is not satisfied, a quantitative analysis of the manner of propagation of acoustic waves in a plasma can be made only by using the Boltzmann equations for electrons, ions and molecules, together with the field equations. The results will be discussed later (see also [44]); they indicate that, when collisions are absent and the electron and ion temperatures are equal (or nearly equal), the acoustic waves in the plasma are still strongly damped [$\gamma \sim \omega$, $\omega/k \sim \sqrt{(\varkappa T/M)}$].† This conclusion is physically reasonable, since in acoustic waves (unlike plasma waves) the space charge must be small, and therefore the electrical interaction cannot bring about the propagation of a wave without damping. The situation is different only in a plasma where the electron temperature T_e considerably exceeds the ion temperature $T \equiv T_i$ (see below).

Such a plasma can exist only in somewhat special conditions; in most cases the electron and ion temperatures differ only by a small amount, if at all. Thus weakly damped acoustic waves in a plasma must usually be very long in order to satisfy the condition (8.42), i.e. in order that ordinary hydrodynamics should be applicable. (As an example, the free path in the corona $l \gtrsim 10^7$ cm.) At radio frequencies, therefore, only transverse and plasma waves need generally be considered, and for this reason we have on various occasions above, without further discussion, identified longitudinal waves in a plasma, with plasma waves.

It should be noted that longitudinal plasma waves, as well as acoustic waves, can exist not only in a gaseous plasma but also in liquids and solids (metals, semiconductors and dielectrics). In particular, the excitation of plasma waves is related to discrete energy losses undergone by fast particles passing through thin films of various substances. A discussion of these interesting problems is outside the scope of this book; see [45–47, 1], where there are references to other work.

The quasihydrodynamic method

To solve the problem of the propagation of waves in a two-temperature plasma, and with a view to later discussions (see §§ 12 and 14), we shall consider the quasihydrodynamic approach to the study of waves in a plasma.

We have seen that a simple phenomenological allowance for spatial dispersion makes possible a correct qualitative assessment of the effect of the thermal motion on the propagation of sufficiently long waves in the plasma. The Boltzmann-equation method is quantitative, but involves much more

† When $\gamma \sim \omega$ there is evidently "something left" of the wave, and to that extent it is important to take account of the space charge. In a gas of neutral particles without collisions a wave cannot be propagated; formally, $\gamma \gg \omega$.

complex calculations, especially when the effect of ions and a constant magnetic field are taken into account. To describe processes in a plasma it is therefore common to use the quasihydrodynamic approximation, which, while simpler, retains (though inconsistently) some of the advantages of the phenomenological and Boltzmann-equation methods. In the particular case of an isotropic plasma, assuming that molecules are absent and neglecting collisions between particles, the equations used are

$$\left.\begin{aligned} m\,\partial \boldsymbol{v}_e/\partial t &= e\boldsymbol{E} - (1/N)\,\mathbf{grad}\,p_e\,, \\ M\,\partial \boldsymbol{v}_i/\partial t &= -e\boldsymbol{E} - (1/N)\,\mathbf{grad}\,p_i\,, \\ \partial N'/\partial t + N_0 \operatorname{div} \boldsymbol{v}_e = 0\,, &\quad \partial N_i'/\partial t + N_0 \operatorname{div} \boldsymbol{v}_i = 0\,, \\ \operatorname{div} \boldsymbol{E} &= 4\pi e (N' - N_i')\,. \end{aligned}\right\} \tag{8.45}$$

Here the linear approximation has been applied; N' and N_i' are the deviations from the equilibrium values of the electron and ion densities $N_{e0} = N_{i0} \equiv N_0$, $\boldsymbol{v}_e$ and $\boldsymbol{v}_i$ the respective mean velocities, and p_e and p_i the pressures.

When the pressure is neglected, equations (8.45) are, of course, exact but local relations. The terms in **grad** p_e and **grad** p_i contain spatial derivatives and thus take account of spatial dispersion, i.e. the dependence of the particles' motion on the spatial distribution of velocities and field.

By adding the first two equations (8.45) we obtain the hydrodynamic equation (8.43) with $p = p_e + p_i$. It is thus natural to assume that $p_e = \varkappa N T_e$ and $p_i = \varkappa N_i T$. For greater generality we put

$$\left.\begin{aligned} p_e &= \xi_e \varkappa N T_e = \xi_e \varkappa (N_0 + N') T_e\,, \\ p_i &= \xi_i \varkappa N_i T_i = \xi_i \varkappa (N_0 + N_i') T_i\,, \end{aligned}\right\} \tag{8.46}$$

where ξ_e and ξ_i are some constants.

Longitudinal waves in a two-temperature plasma

By solving equations (8.45) for longitudinal plane waves in which all quantities vary as $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ we obtain the dispersion relation:

$$\begin{aligned} \omega^2 = \frac{1}{2}\left[\omega_0^2\left(1 + \frac{m}{M}\right) + \left(\frac{\xi_e \varkappa T_e}{m} + \frac{\xi_i \varkappa T_i}{M}\right)k^2\right] \pm \\ \pm \sqrt{\left\{\frac{1}{4}\left[\omega_0^2\left(1 + \frac{m}{M}\right) + \left(\frac{\xi_e \varkappa T_e}{m} + \frac{\xi_i \varkappa T_i}{M}\right)k^2\right]^2 - \right.} \\ \overline{\left. - \frac{\omega_0^2}{M}(\xi_e \varkappa T_e + \xi_i \varkappa T_i)k^2 - \frac{\xi_e \xi_i \varkappa^2 T_e T_i}{m M}k^4\right\}}\,, \end{aligned} \tag{8.47}$$

where the electron and ion temperatures T_e and T_i are regarded as constant but may be quite different.

If

$$\omega_0^2 \gg \xi_e \varkappa T_e k^2/m\,, \quad \xi_e T_e/m \gg \xi_i T_i/M\,, \tag{8.48}$$

we have

$$\omega_1^2 \approx \omega_0^2 + \xi_e \varkappa T_e k^2/m, \tag{8.49}$$

$$\omega_2^2 \approx \varkappa k^2 (\xi_e T_e + \xi_i T_i)/M; \tag{8.50}$$

here and henceforward we omit correction terms containing the factor $m/M \lesssim 1/2000$. The conditions (8.48) show that, for $T_i \lesssim T_e$, the frequency of the acoustic wave $\omega_2 \ll \sqrt{(m/M)}\,\omega_0 = \omega_{0i}$. The physical significance of the frequency ω_{0i} is discussed below.

The solution (8.49) is the same as the kinetic dispersion relation (8.25) for plasma waves if we take $\xi_e = 3$. It is also evident how the formula (8.49) is related to the phenomenological result (8.9), where again, therefore, we must put $\xi_e = 3$. There are also certain independent arguments in favour of the value $\xi_e = 3$ (see [19]).

The solution (8.50) with $\xi_e = \xi_i = 1$ and $T_e = T_i$ becomes the expression (8.44) for an acoustic wave.

In the opposite limiting case

$$\omega_0^2 \ll \xi_e \varkappa T_e k^2/m, \quad \xi_e T_e/m \gg \xi_i T_i/M, \tag{8.51}$$

we have

$$\omega_2^2 \approx \omega_0^2 \frac{m}{M} + \omega_0^2 \frac{\xi_i T_i}{\xi_e T_e} \frac{m}{M} + \frac{\xi_i \varkappa T_i}{M} k^2, \tag{8.52}$$

while ω_1^2 is again given by (8.49).

If (8.51) are replaced by the more stringent conditions

$$\xi_i \varkappa T_i k^2/m \ll \omega_0^2 \ll \xi_e \varkappa T_e k^2/m, \quad T_e \gg T_i, \tag{8.53}$$

then

$$\omega_2^2 \approx \omega_0^2 \frac{m}{M} + \frac{\xi_i \varkappa T_i}{M} k^2 = \omega_{0i}^2 + \frac{\xi_i \varkappa T_i}{M} k^2, \tag{8.54}$$

the second term on the right being a correction term.

The frequency $\omega_{0i} = \omega_0 \sqrt{(m/M)} = \sqrt{(4\pi e^2 N/M)}$ is evidently the "plasma frequency" for ion oscillations. For an ion plasma consisting of positive and negative ions without electrons such a frequency would replace the electron plasma frequency ω_0. In an electron-ion plasma the appearance of the frequency ω_{0i} when the electron temperature is sufficiently high is also easily understood. For in this case the electrons move quickly and freely, setting up a constant negative background which compensates the mean charge of the ion component. The ions must then oscillate with frequency ω_{0i}, just as the electrons oscillate with frequency ω_0 when the ions merely compensate the mean charge of the electron component of the plasma.

The acoustic waves (8.50) and the ion plasma waves (8.54) lie on one continuous branch of low-frequency oscillations, i.e. on one curve $n_4^2 = c^2 k^2/\omega^2 = n_4^2(\omega_{0i}^2/\omega^2)$, where $\omega_{0i}^2 = 4\pi e^2 N/M$ and $\omega^2 = \omega_2^2$ is the solution (8.47) with the minus sign in front of the root; see [50].

We see that the quasihydrodynamic method of analysing motions in an isotropic plasma is simple and effective. Its relation to the kinetic theory is discussed in § 13. The main disadvantage of the quasihydrodynamic method is that the collisionless damping cannot be derived immediately.

The above results indicate that the plasma wave (8.49) is weakly damped in the case (8.48) and very strongly damped in the case (8.51); the wave (8.50) is also quite strongly damped for $T_e \sim T_i$. For example, when $\omega_2 \ll \omega_{0i}$ and $\gamma_2 \ll \omega_2$ we have

$$\frac{\gamma_2}{\omega_2} \approx \sqrt{\frac{\pi}{8}} \left[\sqrt{\frac{m}{M}} + \left(\frac{T_e}{T_i}\right)^{3/2} e^{-T_e/2T_i}\right]. \tag{8.55}$$

This formula is roughly correct also when $\gamma_2 \sim \omega_2$, and so it follows that when $T_e \sim T_i$ we in fact have $\gamma_2 \sim \omega_2$ and damping is strong.

Now let the stronger condition

$$T_e \gg T_i \tag{8.56}$$

be added to the conditions (8.48). Then a kinetic calculation, assuming a Maxwellian distribution of electrons and ions with temperatures T_e and T_i, gives [48–52]

$$\left.\begin{aligned} \omega_2^2 &\approx (\varkappa T_e/M)(1 + 3T_i/T_e)k^2, \\ \gamma_2/\omega_2 &\approx \sqrt{\left(\frac{1}{8}\pi\right)}\sqrt{(m/M)}; \end{aligned}\right\} \tag{8.55a}$$

the result for γ_2/ω_2 follows directly also from (8.55) with the condition (8.56).

For protons we have $\gamma_2/\omega_2 = \sqrt{\left(\frac{1}{8}\pi m/M\right)} = 1{\cdot}46 \times 10^{-2}$, i.e. the damping over one period is not so great as to prevent the propagation of waves, but the damping over a hundred wavelengths is considerable in terms of the usual formulation of the problem (we have seen that for sufficiently long plasma waves the damping is very much weaker in the absence of collisions).

With the conditions (8.53) the kinetic result is†

$$\left.\begin{aligned} \omega_2^2 &\approx \omega_{0i}^2 + 3\varkappa T_i k^2/M, \\ \gamma_2/\omega_2 &\approx \sqrt{\pi (M/m)}\,(m\omega_2^2/2\varkappa T_e k^2)^{3/2} \approx \sqrt{(\pi m/M)}\,(m\omega_0^2/2\varkappa T_e k^2)^{3/2}. \end{aligned}\right\} \tag{8.57}$$

The expression (8.57) for ω_2^2 agrees with (8.54) for $\xi_i = 3$. In this case the damping (8.53), (8.57) is much less than that given by (8.55a), since the ratio $m\omega_0^2/2\varkappa T_e k^2$ is much less than unity [see (8.53)].

† A more general expression is [50]

$$\left.\begin{aligned} \frac{\gamma_2}{\omega_2} &= \sqrt{\frac{\pi}{8}} \left\{\frac{M}{m}\left(\frac{m}{\varkappa T_e}\frac{\omega_{0i}^2}{k^2}\right)^{3/2} \frac{1}{\beta_i^3 n_2^3} \exp\left[-\frac{1}{2\beta_i^2 n_2^2}\right]\right\}, \\ \beta_i^2 &= \varkappa T_i/Mc^2, \quad n_2^2 = c^2k^2/\omega_2^2 \approx c^2k^2/\omega_{0i}^2. \end{aligned}\right\} \tag{8.57a}$$

Formula (8.57a) is valid when $\omega_{0i}^2 \gg \varkappa T_i k^2/M$ and $T_e \gg T_i$, in which case $\omega_2 \approx \omega_0$ and $\gamma_2 \sim \omega_2$, as is clear from (8.57) and (8.57a).

Using (8.57a) to give a rough estimate for strong damping also, we find that $\gamma_2 \sim \omega_2$ for $\omega_{0i}^2 \sim \varkappa T_i k^2/M$ and $\beta_i n_2 \sim 1$.

§9. SUMMARY OF PRINCIPAL FORMULAE

We collect here for convenience the most important formulae relating to wave propagation in a homogeneous isotropic plasma. The majority of the comments and restrictions made in preceding sections will not be repeated.

The wave equation for any medium is

$$\Delta \boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} + (\omega^2/c^2)(\boldsymbol{D} - i\cdot 4\pi \boldsymbol{j}/\omega) = 0, \tag{2.5}$$

where only the region outside field sources is considered.

In an isotropic medium we have, neglecting spatial dispersion,

$$\boldsymbol{D} = \varepsilon \boldsymbol{E},\quad \boldsymbol{j} = \sigma \boldsymbol{E},\quad \boldsymbol{D} - i\cdot 4\pi \boldsymbol{j}/\omega = \varepsilon' \boldsymbol{E}, \tag{2.6}$$

$$\varepsilon' = \varepsilon - i\cdot 4\pi\sigma/\omega, \tag{2.8}$$

$$\Delta \boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} + (\omega^2/c^2)\,\varepsilon' \boldsymbol{E} = 0, \tag{2.10}$$

where ε', ε and σ are in general functions of the frequency ω and of the co-ordinates.

For transverse waves

$$\left.\begin{aligned} \mathrm{div}\,\boldsymbol{E} &= 0,\\ \Delta\boldsymbol{E} + (\omega^2/c^2)\,\varepsilon'\boldsymbol{E} &= 0. \end{aligned}\right\} \tag{7.2}$$

For longitudinal waves

$$\left.\begin{aligned} &\mathbf{curl}\,\boldsymbol{H} = 0,\\ &\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} = -\Delta\boldsymbol{E} + \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} = 0,\\ &\boldsymbol{D} - 4\pi i \boldsymbol{j}/\omega = \varepsilon'\boldsymbol{E} = 0. \end{aligned}\right\} \tag{2.14a}$$

In a homogeneous medium the complex permittivity $\varepsilon' = \varepsilon'(\omega)$ is independent of position. The following results relate to waves in a homogeneous medium, and only to monochromatic (harmonic) plane waves:

$$\boldsymbol{E} = \boldsymbol{E}_0\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}. \tag{7.1}$$

For such waves the field equations (2.1a) and (2.3) give

$$\omega\varepsilon'\boldsymbol{E} = -c\boldsymbol{k}\times\boldsymbol{H},\quad \omega\boldsymbol{H} = c\boldsymbol{k}\times\boldsymbol{E}. \tag{7.4}$$

We may note that the condition ($\boldsymbol{k}\cdot\boldsymbol{E} = 0$) for the waves to be transverse follows from (7.4) only if $\varepsilon' \neq 0$, in accordance with the relation (2.14a) for longitudinal waves. In longitudinal waves $\boldsymbol{H} = 0$.

Transverse waves

From (7.1) and (7.2) we obtain the dispersion relation between ω and $\boldsymbol{k}$ for transverse waves:

$$\begin{aligned} k^2 = (\omega^2/c^2)\,\varepsilon' &= (\omega^2/c^2)(\varepsilon - i\cdot 4\pi\sigma/\omega)\\ &= (\omega^2/c^2)(n - i\varkappa)^2. \end{aligned} \qquad (7.3)\text{ and }(7.7)$$

This equation is valid for complex $\boldsymbol{k}$ and ω. In general $\boldsymbol{k} = \boldsymbol{k}_0 - i\boldsymbol{q}$. Here we shall give formulae for the case where the frequency ω is real and the plane waves (7.1) are homogeneous, i.e. the vectors $\boldsymbol{k}_0$ and $\boldsymbol{q}$ are parallel and

$$\boldsymbol{k} = (\omega/c)(n - i\varkappa)\,\boldsymbol{k}/k, \tag{7.5}$$

$$\boldsymbol{E} = \boldsymbol{E}_0\, e^{\mp\omega\varkappa z/c}\, e^{i(\omega t \mp \omega n z/c)}, \quad E_z = 0. \tag{7.6}$$

The wavelength λ in the medium and the phase velocity v_{ph} of the wave are

$$\lambda = \lambda_0/n, \quad \lambda_0 = 2\pi c/\omega, \quad v_{\text{ph}} = \omega/k = c/n. \qquad \text{(7.8) and (7.9)}$$

The absorption coefficient μ gives the change in the energy flux due to scattering: $S = S_0 e^{-\mu z}$, with

$$\mu = 2\omega\varkappa/c. \tag{7.10}$$

From (7.3) and (7.7) it follows that

$$\varepsilon = n^2 - \varkappa^2, \quad 4\pi\sigma/\omega = 2n\varkappa, \tag{7.11}$$

$$\left.\begin{aligned} n &= \sqrt{\left\{\tfrac{1}{2}\varepsilon + \sqrt{[(\tfrac{1}{2}\varepsilon)^2 + (2\pi\sigma/\omega)^2]}\right\}}, \\ \varkappa &= \sqrt{\left\{-\tfrac{1}{2}\varepsilon + \sqrt{[(\tfrac{1}{2}\varepsilon)^2 + (2\pi\sigma/\omega)^2]}\right\}}, \end{aligned}\right\} \tag{7.12}$$

where both roots are taken positive (see § 7). In the absence of absorption ($\sigma = 0$) we have

$$\text{for } \varepsilon > 0\colon\ n = \sqrt{\varepsilon}, \quad \varkappa = 0; \tag{7.13}$$

$$\text{for } \varepsilon < 0\colon\ n = 0, \quad \varkappa = \sqrt{-\varepsilon} = \sqrt{|\varepsilon|}. \tag{7.14}$$

In this case the single quantity

$$\tilde{n}^2 = (n - i\varkappa)^2 = \varepsilon \tag{7.15}$$

is frequently used. If

$$|\varepsilon| \gg 4\pi\sigma/\omega \tag{7.16}$$

we have

$$\text{for } \varepsilon > 0\colon\ n \approx \sqrt{\varepsilon}, \quad \varkappa \approx 2\pi\sigma/\omega\sqrt{\varepsilon}, \quad \mu \approx 4\pi\sigma/c\sqrt{\varepsilon}; \tag{7.17}$$

$$\text{for } \varepsilon < 0\colon\ n \approx 2\pi\sigma/\omega\sqrt{-\varepsilon}, \quad \varkappa \approx \sqrt{-\varepsilon}, \quad \mu \approx 2\omega\sqrt{-\varepsilon}/c. \tag{7.18}$$

In the opposite limiting case where

$$|\varepsilon| \ll 4\pi\sigma/\omega \tag{7.19}$$

we find

$$n \approx \varkappa \approx \sqrt{(2\pi\sigma/\omega)}, \quad \mu \approx \sqrt{(8\pi\omega\sigma/c^2)} = 4\pi\sqrt{(\sigma/c\lambda_0)}. \qquad \text{(7.20) and (7.21)}$$

The formulae up to this point are valid for any medium.

For an isotropic plasma (i.e. when there is no external magnetic field) the elementary theory gives

$$\left.\begin{aligned} \varepsilon' &= 1 - 4\pi e^2 N/m\,\omega(\omega - i\,\nu_{\text{eff}}), \\ \varepsilon &= 1 - 4\pi e^2 N/m(\omega^2 + \nu_{\text{eff}}^2), \\ \sigma &= (1 - \varepsilon)\,\nu_{\text{eff}}/4\pi = e^2 N\,\nu_{\text{eff}}/m(\omega^2 + \nu_{\text{eff}}^2). \end{aligned}\right\} \tag{3.7}$$

Using the notation $v = 4\pi e^2 N/m\omega^2 = \omega_0^2/\omega^2$ and $s = \nu_{\text{eff}}/\omega$ in anticipation of Chapter III, we can write

$$\varepsilon' = 1 - v/(1 - is),$$
$$\varepsilon = 1 - v/(1 + s^2),$$
$$\sigma/\omega = vs/4\pi(1 + s^2).$$

The formulae of the elementary theory are strictly valid when the effective frequency ν_{eff} of collisions undergone by an electron is independent of its velocity. The contribution from the ions is not shown in (3.7). For transverse waves in a non-relativistic plasma, spatial dispersion is of negligible importance, being automatically allowed for if the thermal motion is not neglected. The corrections thus obtained are of the order of $\varkappa T/mc^2$ as against unity, i.e. the same as the relativistic effects.

In the high-frequency case

$$\omega^2 \gg \nu_{\text{eff}}^2 \qquad \text{(3.8) and (6.7)}$$

we obtain

$$\left.\begin{aligned} \varepsilon &= 1 - 4\pi e^2 N/m\,\omega^2 = 1 - 3{\cdot}18 \times 10^9 N/\omega^2 \\ &= 1 - 8{\cdot}06 \times 10^7 N/f^2, \\ \sigma &\approx e^2 N\,\nu_{\text{eff}}/m\,\omega^2 = 2{\cdot}53 \times 10^8 N\,\nu_{\text{eff}}/\omega^2 \\ &= 6{\cdot}42 \times 10^6 N\,\nu_{\text{eff}}/f^2 \qquad (f = \omega/2\pi). \end{aligned}\right\} \qquad \text{(3.5) and (3.9)}$$

In the important particular case where $\omega^2 \gg \nu_{\text{eff}}^2$ and $\varepsilon \gg 4\pi\sigma/\omega$ (here $\varepsilon > 0$), we have

$$\left.\begin{aligned} &n \approx \sqrt{(1 - 4\pi e^2 N/m\,\omega^2)}, \quad \varkappa \approx 2\pi\,\sigma/\omega\sqrt{\varepsilon} = (1 - n^2)\,\nu_{\text{eff}}/2\omega\,n, \\ &\mu = 2\omega\,\varkappa/c \approx (1 - n^2)\,\nu_{\text{eff}}/c\,n = \frac{4\pi\,e^2 N\,\nu_{\text{eff}}}{m\,c\,\omega^2\sqrt{(1 - 4\pi\,e^2 N/m\,\omega^2)}} \\ &\qquad = \frac{0{\cdot}106 N\,\nu_{\text{eff}}}{\omega^2\sqrt{(1 - 3{\cdot}18 \times 10^9 N/\omega^2)}}. \end{aligned}\right\} \qquad (7.17)$$

The formulae of the kinetic theory are as follows. In the high-frequency case (3.8), (6.7) we have in formulae (3.5) and (3.9) for σ

$$\nu_{\text{eff}} = \nu_{\text{eff},m} + \nu_{\text{eff},i} \qquad (6.16)$$

(where $\nu_{\text{eff},m}$ and $\nu_{\text{eff},i}$ relate to collisions with molecules and ions respectively), with

$$\nu_{\text{eff},m} = \frac{4}{3}\pi\,a^2\,\bar{v}\,N_m = 8{\cdot}3 \times 10^5 \pi\,a^2 N_m \sqrt{T} = 1{\cdot}7 \times 10^{11} \frac{N_m}{2{\cdot}7 \times 10^{19}} \sqrt{\frac{T}{300}}$$

(6.10) and (6.11)

(the last expression relates to air) and

$$\nu_{\text{eff},i} = \frac{5{\cdot}5 N_i}{T^{3/2}} \ln\left(220 \frac{T}{N_+^{\frac{1}{3}}}\right). \qquad (6.13)$$

When collisions occur, we have for oscillations of the type $e^{pt} \equiv e^{i\omega' t} = e^{-\gamma t} e^{i\omega t}$, with the conditions (3.8) and (6.7),

$$\left.\begin{aligned} \varepsilon' &= 1 - \omega_0^{2\prime}\omega'(\omega' - i\,\nu_{\text{eff}}) \\ &= 1 + \omega_0^{2\prime}p(p + \nu_{\text{eff}}) = 0, \quad p = i\,\omega - \gamma, \\ \omega' &= \omega + i\,\gamma, \quad \omega \approx \omega_0, \quad \gamma_{\text{coll}} \approx \tfrac{1}{2}\nu_{\text{eff}}. \end{aligned}\right\} \tag{8.39}$$

The allowance for thermal motion, which in a plasma is equivalent to an allowance for spatial dispersion, gives the formulae

$$\left.\begin{aligned} \omega^2 &= \omega_0^2 + 3\varkappa T k^2/m, \\ \omega &\approx \omega_0 + 3\varkappa T k^2/2m\omega_0 = \omega_0(1 + 3D^2k^2), \end{aligned}\right\} \tag{8.27}$$

where k is the wave vector,

$$Dk \ll 1, \quad D = \sqrt{(\varkappa T/8\pi e^2 N)} \tag{8.28}$$

or, what is the same thing,

$$|\omega_0 - \omega| \ll \omega_0. \tag{8.12}$$

According to (8.27)

$$\tilde{n}_3^2 = c^2 k^2/\omega^2 = \frac{1 - \omega_0^{2\prime}\omega^2}{(3\varkappa\, T/m\, c^2)\, \omega_0^{2\prime}\omega^2} \approx \frac{1 - \omega_0/\omega}{3\varkappa\, T/2m\, c^2}, \tag{8.25}$$

$$\left.\begin{aligned} v_{\text{ph}} &= \omega/k = c/n_3 = \frac{\sqrt{(3\varkappa\, T/m)}}{\sqrt{(1 - \omega_0^{2\prime}\omega^2)}}, \\ v_{\text{gr}} &= d\,\omega/d\,k \approx 3\varkappa\, T\,k/m\,\omega_0 = \sqrt{(3\varkappa\, T/m)}\,\sqrt{(1 - \omega_0^2/\omega^2)}. \end{aligned}\right\} \tag{8.26}$$

Here, of course, we can also replace $1 - \omega_0^2/\omega^2$ by $2(1 - \omega_0/\omega)$. With the conditions (8.28) and (8.12) and in the absence of collisions we have for a plasma wave $\gamma = \gamma_0$, where

$$\gamma_0 = \frac{\omega_0\sqrt{\pi}\, e^{-3/2}}{8(k\,D)^3}\, e^{-1/4(k\,D)^2} \approx \frac{0{\cdot}05\,\omega_0}{(k\,D)^3}\, 10^{-0{\cdot}11/(k\,D)^2}. \tag{8.36}$$

The specific damping γ_0 usually becomes important for wavelengths $\lambda < \lambda_{\text{cr}}$, where

$$\left.\begin{aligned} \lambda_{\text{cr}}\,(\text{cm}) &= 2\pi/k_{\text{cr}} \approx 300\sqrt{[T\ (\text{deg})/N]}, \\ k_{\text{cr}}\,D &\approx 0{\cdot}1 \end{aligned}\right\} \tag{8.41}$$

(see Table 8.1). The question of the importance of relativistic corrections has been discussed in § 8.

For real ω in a plasma wave of the type $e^{-\omega\varkappa_3 z/c}\, e^{i(\omega t - kz)}$ the relation (8.27) holds between ω and k, and

$$\omega\,\varkappa_3/c = \gamma_0(k)/v_{\text{gr}} = m\,\omega_0\gamma_0(k)/3\varkappa\,T\,k. \tag{8.38}$$

The relation $\omega\varkappa_3/c = \gamma(k)/v_{\text{gr}}$ is general, applying also to transverse waves [see (7.23)]. If collisions and the damping (8.36) are both present, then for weak damping

$$\gamma = \gamma_{\mathrm{coll}} + \gamma_0 \cdot \tag{8.40}$$

When

$$\nu_{\mathrm{eff}}^2 \gg \omega^2 \tag{8.42}$$

[see (8.42) and (8.44)] weakly damped acoustic waves for which

$$\omega = \sqrt{(2\varkappa T/M)}\, k \tag{8.44}$$

can be propagated in the plasma. When the condition (8.42) is not satisfied, acoustic waves in the plasma are strongly damped.† The same is true of plasma waves when (8.42) is satisfied.

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† Here we are considering a plasma in which the electron and ion temperatures are equal. The problem of wave propagation in a two-temperature plasma, where $T_e \neq T$ has been discussed at the end of § 8.

CHAPTER III

WAVE PROPAGATION IN A HOMOGENEOUS MAGNETOACTIVE PLASMA

§ 10. THE COMPLEX PERMITTIVITY TENSOR

The effect of a constant magnetic field on the properties of a plasma

A CONSTANT magnetic field has a great effect on the properties of a plasma. This is true, in particular, of the propagation of various kinds of wave in the plasma.

The effect of a magnetic field $\boldsymbol{H}^{(0)}$ is described in the first place by the ratio of the wave field frequency ω to the gyration frequency of the electrons

$$\left.\begin{aligned} \omega_H &= |e|\,H^{(0)}/m\,c = 1{\cdot}76 \times 10^7 H^{(0)}, \\ \lambda_H\,(\mathrm{cm}) &= 2\pi\,c/\omega_H = 2\pi\,m\,c^2/|e|H^{(0)} = 1{\cdot}07 \times 10^4/H^{(0)} \end{aligned}\right\} \tag{10.1}$$

and that of the ions

$$\Omega_H = |e|\,H^{(0)}/M\,c = 1{\cdot}76 \times 10^7 m\,H^{(0)}/M\,. \tag{10.2}$$

The frequencies ω_H and Ω_H are those with which a free electron or ion (of charge e and mass M) revolves round the lines of force of the field $\boldsymbol{H}^{(0)}$. In addition to ω_H and Ω_H, we shall use the dimensionless parameters

$$u = \omega_H^2/\omega^2, \quad \sqrt{u} = \omega_H/\omega, \quad u_M = \Omega_H^2/\omega^2 = (m/M)^2\,u\,. \tag{10.3}$$

If the electron velocity is comparable with that of light, the frequency ω_H is given by the relativistic formula

$$\omega_H^* = \frac{|e|\,H^{(0)}}{m\,c}\,\frac{m\,c^2}{E} = \frac{|e|\,H^{(0)}}{m\,c}\sqrt{(1-\beta^2)}, \tag{10.4}$$

where $E = mc^2/\sqrt{(1-\beta^2)}$ is the total energy, $\beta = v/c$ and v is the velocity of the electron.

For $T \sim 10^6$ (as in the solar corona) the majority of electrons have a velocity $v_T = \sqrt{(\varkappa T/m)} \sim 3 \times 10^8$ cm/sec and $\beta_T^2 = \varkappa T/mc^2 \sim 10^{-4}$. Although β_T^2 is small, the allowance for effects proportional to β_T^2 is sometimes important in the corona and even in the ionosphere (see § 12). Nevertheless, the condition $\beta_T^2 \ll 1$ means that in general the non-relativistic approximation can be used. For this reason the magnetic field of the wave itself need not usually be taken into account, in accordance with the treatment in Chapter II. The magnetic field H in an electromagnetic wave does not in general exceed the electric field E; for a plane wave $\mathbf{k} \times \mathbf{E} = \omega \mathbf{H}/c$, $k^2 = (\omega^2/c^2)(n - i\varkappa)^2$. Hence the magnetic force $e\mathbf{v} \times \mathbf{H}/c$ is less than the electric force $e\mathbf{E}$ by a factor of the order of v/c if $n \sim 1$. Thus, even if we take v to be the thermal velocity, the magnetic force is less than the electric force by a factor of about 100 for $T \sim 10^6$ and about 3000 for $T \sim 300°$. In reality, the force is determined by the directed velocity of the electrons, which is much less than the thermal velocity.†

In the Earth's ionosphere the magnetic field $H^{(0)} \sim 0{\cdot}2$ to $0{\cdot}5$ oersted, so that $\omega_H \sim 3$ to $9 \times 10^6\,\mathrm{sec}^{-1}$ ($\lambda_H \sim 200$ to 500 m) and, for example, for $O^{\pm}$ ions $\Omega_H \sim 100$ to $300\,\mathrm{sec}^{-1}$. In the corona, the maximum field is $H^{(0)} \sim 5000$, whence $\omega_H \sim 10^{11}$ and $\Omega_H = \omega_H/1836 \sim 10^8$.

The effect of the ions is usually unimportant if

$$\omega \gg \Omega_H, \tag{10.5}$$

and the effect of the magnetic field on the motion of the electrons is unimportant if

$$\omega \gg \omega_H. \tag{10.6}$$

The above values of the gyration frequencies are such that the magnetic field obviously has a considerable effect on the propagation of electromagnetic waves in the ionosphere and corona. To this it should be added that, even when the condition (10.6) holds, the effect of the field is far from being negligible. For example, in the interstellar medium $H^{(0)} \sim 10^{-6}$ to 10^{-5} oersted and $\omega_H \sim 10$ to $100\,\mathrm{sec}^{-1}$ but even for propagation of waves in the metre range ($\omega \sim 10^9$) the effect of the field is important as regards the rotation of the plane of polarisation of cosmic radio waves, on account of the very great distances traversed (see § 37).

The complex permittivity tensor: elementary theory

A plasma in a magnetic field is anisotropic, and its electromagnetic properties are described (spatial dispersion being neglected) by a complex permittivity tensor which is a function of frequency:

$$\varepsilon'_{ik} = \varepsilon_{ik} - i \cdot 4\pi\sigma_{ik}/\omega, \quad j_i = \sigma_{ik} E_k, \quad D_i = \varepsilon_{ik} E_k. \tag{10.7}$$

We shall begin by finding the tensor ε'_{ik} in terms of the elementary theory (§ 3) with the condition (10.5), i.e. neglecting the part played by the ions.

† When the Boltzmann equation (4.17) is used, the possibility of neglecting the magnetic field of the wave in the linear approximation is evident, since the field H is multiplied by the small quantity f_1.

The use of the "elementary theory" essentially signifies that the frequency of collisions between an electron and other particles is assumed to be independent of the electron velocity and equal to some value ν_{eff}. Then we can use the equation

$$m\,\ddot{\boldsymbol{r}} + m\,\nu_{\text{eff}}\,\dot{\boldsymbol{r}} = e\,\boldsymbol{E}_0 e^{i\omega t} + e\,\dot{\boldsymbol{r}} \times \boldsymbol{H}^{(0)}/c, \tag{10.8}$$

which differs from (3.6) in that the action of the magnetic field is taken into account.

The total current density is given, regardless of the presence or absence of a magnetic field, by $\boldsymbol{j}_t = \boldsymbol{j} + i\omega\boldsymbol{P} = eN\dot{\boldsymbol{r}}$, where $\dot{\boldsymbol{r}}$ is the forced component of the solution of equation (10.8) which is proportional to $\boldsymbol{E}_0 e^{i\omega t}$. If we take the direction of the field $\boldsymbol{H}^{(0)}$ as the z-axis and use the combinations $j_{tx} \pm i j_{ty}$, it is therefore easy to see that

$$\left.\begin{aligned} j_{tx} \pm i\,j_{ty} &= e^2 N(E_x \pm i\,E_y)/m(i\,\omega + \nu_{\text{eff}} \mp i\,\omega_H),\\ j_{tz} &= e^2 N\,E_z/m\,(i\,\omega + \nu_{\text{eff}}); \end{aligned}\right\} \tag{10.9}$$

here either the upper or the lower sign must be taken on both sides, and it must be remembered that for electrons $e < 0$, so that $\omega_H = |e|\,H^{(0)}/mc = -eH^{(0)}/mc$.

Since the fundamental equations (2.5) and (2.18) involve the vector $\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega = \boldsymbol{E} + 4\pi\boldsymbol{P} - i\cdot 4\pi\boldsymbol{j}/\omega = \boldsymbol{E} + 4\pi\boldsymbol{j}_t/i\omega$, we shall give the corresponding expressions:

$$\left.\begin{aligned} &(D_x - i\cdot 4\pi\,j_x/\omega) \pm i(D_y - i\cdot 4\pi j_y/\omega)\\ &\qquad\qquad = \left(1 - \frac{\omega_0^2}{\omega^2 \mp \omega\,\omega_H - i\,\omega\,\nu_{\text{eff}}}\right)(E_x \pm i\,E_y),\\ &D_z - i\cdot 4\pi\,j_z/\omega = \left(1 - \frac{\omega_0^2}{\omega^2 - i\,\omega\,\nu_{\text{eff}}}\right)E_z, \end{aligned}\right\} \tag{10.10}$$

where, as everywhere hitherto, $\omega_0^2 = 4\pi e^2 N/m$.

By definition

$$\left.\begin{aligned} j_{ti} &= i\,\omega\,P_i + j_i\\ &= \left(i\,\omega\,\frac{\varepsilon_{ik} - \delta_{ik}}{4\pi} + \sigma_{ik}\right)E_k\\ &= i\,\omega\,(\varepsilon'_{ik} - \delta_{ik})\,E_k/4\pi,\\ D_i - i\cdot 4\pi\,j_i/\omega &= (\varepsilon_{ik} - i\cdot 4\pi\,\sigma_{ik}/\omega)\,E_k\\ &= \varepsilon'_{ik}\,E_k, \end{aligned}\right\} \tag{10.11}$$

where summation from 1 to 3 over repeated suffixes is understood, and $\delta_{ik} = 1$ for $i = k$, $\delta_{ik} = 0$ for $i \neq k$. The suffix i should not be confused with the imaginary unit i which appears as a factor.

From (10.10) and (10.11) we have

$$\left.\begin{aligned}
\varepsilon'_{xx} &= \varepsilon'_{yy} = 1 - \frac{1}{2}\omega_0^2\left(\frac{1}{\omega^2 - \omega\,\omega_H - i\,\omega\,\nu_{\rm eff}} + \frac{1}{\omega^2 + \omega\,\omega_H - i\,\omega\,\nu_{\rm eff}}\right)\\
&= 1 - \frac{\omega_0^2(\omega - i\,\nu_{\rm eff})}{\omega[(\omega - i\,\nu_{\rm eff})^2 - \omega_H^2]},\\
\varepsilon'_{xy} &= -\varepsilon'_{yx} = -i\cdot\frac{\omega_0^2\,\omega_H}{\omega(\omega + \omega_H - i\,\nu_{\rm eff})(\omega - \omega_H - i\,\nu_{\rm eff})},\\
\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} &= \varepsilon'_{yy} \pm i\,\varepsilon'_{yx} = 1 - \frac{\omega_0^2}{\omega^2 \mp \omega\,\omega_H - i\,\omega\,\nu_{\rm eff}},\\
\varepsilon'_{zz} &= 1 - \omega_0^2/(\omega^2 - i\,\omega\,\nu_{\rm eff}),\\
\varepsilon'_{xz} &= \varepsilon'_{zx} = \varepsilon'_{yz} = \varepsilon'_{zy} = 0,\\
\varepsilon'_{xx} &\equiv \varepsilon'_{11},\ \varepsilon'_{yy} \equiv \varepsilon'_{22},\ \varepsilon'_{zz} \equiv \varepsilon'_{33},\ \varepsilon'_{xy} \equiv \varepsilon'_{12},\ \varepsilon'_{xz} \equiv \varepsilon'_{13},\ \varepsilon'_{yz} \equiv \varepsilon'_{23}.
\end{aligned}\right\} \tag{10.12}$$

Properties of the tensor ε'_{ik}

If $\omega_H = 0$, then $\varepsilon'_{ik} = \varepsilon'\delta_{ik} = \delta_{ik}[1 - \omega_0^2/\omega(\omega - i\nu_{\rm eff})]$, as of course it should be. The tensor ε'_{ik} clearly satisfies, according to (10.12), the condition $\varepsilon'_{ik}(\omega_H) = \varepsilon'_{ki}(-\omega_H)$ or, what is the same thing,

$$\varepsilon'_{ik}(\boldsymbol{H}^{(0)}) = \varepsilon'_{ki}(-\boldsymbol{H}^{(0)}). \tag{10.13}$$

This is a general relation valid for any medium in a magnetic field (see, e.g., [36, § 82]). A very important result is that even in the absence of absorption, when $\nu_{\rm eff} = 0$ and $\sigma_{ik} = 0$, the tensor $\varepsilon'_{ik} = \varepsilon_{ik}$ is not real but Hermitian:

$$\varepsilon_{ik} = \varepsilon^*_{ki}, \tag{10.14}$$

where the asterisk denotes the complex conjugate.

The validity of (10.14) for a plasma is immediately evident, since with $\nu_{\rm eff} = 0$ (10.12) give

$$\left.\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{yy} = 1 - \frac{\omega_0^2}{2\omega(\omega - \omega_H)} - \frac{\omega_0^2}{2\omega(\omega + \omega_H)} = 1 - \frac{\omega_0^2}{\omega^2 - \omega_H^2},\\
\varepsilon_{xy} &= -\varepsilon_{yx} = \varepsilon^*_{yx} = -\frac{i\,\omega_0^2}{2\omega(\omega - \omega_H)} + \frac{i\,\omega_0^2}{2\omega(\omega + \omega_H)}\\
&= -i\cdot\frac{\omega_0^2\,\omega_H}{\omega(\omega^2 - \omega_H^2)},\\
\varepsilon_{xx} &\mp i\varepsilon_{xy} = 1 - \omega_0^2/\omega(\omega \mp \omega_H),\\
\varepsilon_{zz} &= 1 - \omega_0^2/\omega^2,\\
\varepsilon_{xz} &= \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0.
\end{aligned}\right\} \tag{10.15}$$

In general the separation of the tensor $\varepsilon'_{ik} = \varepsilon_{ik} - i\cdot 4\pi\sigma_{ik}/\omega$ into ε_{ik} and σ_{ik} may be uniquely effected by requiring that ε_{ik} and σ_{ik} (not $-i\cdot 4\pi\sigma_{ik}/\omega$)

shall be Hermitian tensors. The absorption of energy (Joule heat) in unit volume, averaged over time, is (see [36, §§ 61 and 77])

$$-i\omega(\varepsilon_{ik}'^{*} - \varepsilon_{ki}')E_i E_k^*/16\pi = \tfrac{1}{4}(\sigma_{ik}^* + \sigma_{ki})E_i E_k^* = \tfrac{1}{2}\sigma_{ik}E_i^* E_k,$$

i.e. depends only on σ_{ik}, as it should.

We shall not give here the formulae for ε_{ik} and σ_{ik} separately, since it is usually more convenient to use the tensor ε_{ik}' [but see formulae (10.31) and (10.32)].

A medium which is in a magnetic field is said to be magnetoactive, the word "active" (or "gyrotropic") referring to the fact that even in the absence of absorption the tensor $\varepsilon_{ik}'(\omega) = \varepsilon_{ik}(\omega)$ is not real. When there is no magnetic field, and spatial dispersion is neglected, any medium is inactive, but for crystals or solutions of molecules (e.g. sugar) having no centre of symmetry the allowance for spatial dispersion leads to the appearance of first-order terms in a/λ (see §§ 2 and 8), the effect of which can usually be taken into account by means of a complex tensor ε_{ik}; further details are given in [36, § 83] and [1]. Such media are said to be naturally active. In a medium which is not naturally active but magnetoactive, the tensor ε_{ik} is complex only when an external magnetic field is present.

Thus a plasma in a magnetic field is a magnetoactive medium, and it may exhibit marked activity even in magnetic fields usually regarded as very weak.

In a non-gyrotropic (inactive) anisotropic medium the tensor ε_{ik}' is symmetrical, and in the absence of absorption it is real. There are therefore three principal directions, in which the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ are parallel. In a magnetoactive medium this is not so: in the direction of the magnetic field we have $D_z = \varepsilon_{zz}E_z$, but $D_x \pm iD_y = (\varepsilon_{xx} \mp i\varepsilon_{xy})(E_x \pm iE_y)$ [see (10.10); absorption is assumed absent]. Since the quantity $\varepsilon_{xx} \mp i\varepsilon_{xy}$ is real [see (10.15)] it follows that in the xy-plane the vector $\boldsymbol{D}$ is proportional to $\boldsymbol{E}$ for a field $\boldsymbol{E}$ of constant magnitude rotating clockwise or anticlockwise (for such a field $E_x = E_0 e^{i\omega t}$, $E_y = \mp iE_0 e^{i\omega t}$, re $E_x = E_0 \cos\omega t$, re $E_y = \pm E_0 \sin\omega t$). This relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ is the characteristic physical feature of a gyrotropic medium.

The tensor ε_{ik}' in other coordinate systems

In the coordinate system used above, with the z-axis parallel to the field $\boldsymbol{H}^{(0)}$, the tensor ε_{ik}' takes its simplest form. It is convenient, however, to use other systems of rectangular coordinates also. The tensors ε_{ik}' in two such systems are related by

$$\varepsilon_{ik}'(x_l) = \gamma_{im}\gamma_{kn}\varepsilon_{mn}'(x_l'), \tag{10.16}$$

where $\varepsilon_{ik}'(x_l)$ are the components of the tensor in the system of coordinates x_l, and γ_{im} are the cosines of the angles between the axes x_i and x_m' (x_m' being the old coordinates).

An important particular case is that where the vector $H^{(0)}$ makes an angle α with the z-axis, an angle $\frac{1}{2}\pi - \alpha$ with the y-axis, and a right angle with the x-axis (Fig. 10.1).† This occurs when a wave is incident normally on a

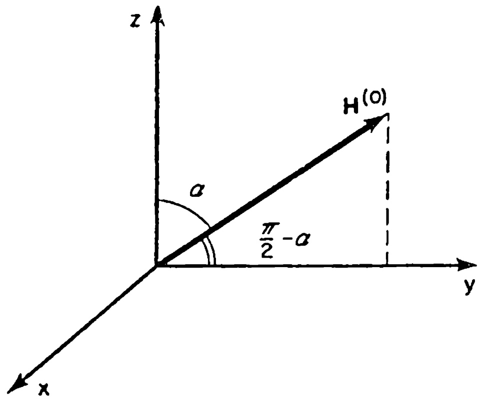

FIG. 10.1. Coordinate system in which the tensor ε'_{ik} has the form (10.17).

layer of plasma (propagation along the z-axis for all z) for an arbitrary orientation of the magnetic field (the x-axis can always be taken perpendicular to the vector $H^{(0)}$). Then $\gamma_{11} = 1, \gamma_{12} = \gamma_{13} = \gamma_{21} = \gamma_{31} = 0, \gamma_{33} = \gamma_{22} = \cos\alpha$, $\gamma_{23} = -\gamma_{32} = \sin\alpha$. In the new coordinate system x, y, z shown in Fig. 10.1 the components of the tensor (10.12) are

$$\left.\begin{aligned}
\varepsilon'_{xx} &= 1 - \frac{v(1-is)}{(1-is)^2 - u}, \\
\varepsilon'_{yy} &= 1 - \frac{v[(1-is)^2 - u\sin^2\alpha]}{(1-is)[(1-is)^2 - u]}, \\
\varepsilon'_{zz} &= 1 - \frac{v[(1-is)^2 - u\cos^2\alpha]}{(1-is)[(1-is)^2 - u]}, \\
\varepsilon'_{xz} &= -\varepsilon'_{zx} = \frac{i\sqrt{u}\,(v\sin\alpha)}{(1-is)^2 - u}, \\
\varepsilon'_{xy} &= -\varepsilon'_{yx} = -\frac{i\sqrt{u}\,(v\cos\alpha)}{(1-is)^2 - u}, \\
\varepsilon'_{yz} &= \varepsilon'_{zy} = \frac{u\,v\cos\alpha\sin\alpha}{(1-is)[(1-is)^2 - u]},
\end{aligned}\right\} \qquad (10.17)$$

where

$$u = \omega_H^2/\omega^2, \quad \sqrt{u} = \omega_H/\omega, \quad v = \omega_0^2/\omega^2 = 4\pi e^2 N/m\,\omega^2, \quad s = \nu_{\text{eff}}/\omega. \qquad (10.18)$$

In the propagation of a plane wave along the z-axis, the field equations (2.18) give

$$D_z - i\cdot 4\pi j_z/\omega = \varepsilon'_{3k} E_k = 0, \qquad (10.19)$$

† In the case of the Earth's magnetic field, if the z-axis is vertical, the angle $I = \frac{1}{2}\pi - \alpha$ is called the magnetic dip.

whence

$$E_z = -(\varepsilon'_{zx} E_x + \varepsilon'_{zy} E_y)/\varepsilon'_{zz}$$
$$= -\frac{i\sqrt{u}\,(1-i\,s)\,v\sin\alpha}{(1-i\,s)\,u-(1-i\,s)^2(1-i\,s-v)-u\,v\cos^2\alpha}\,E_x +$$
$$+\frac{u\,v\cos\alpha\sin\alpha}{(1-i\,s)\,u-(1-i\,s)^2(1-i\,s-v)-u\,v\cos^2 a}\,E_y. \quad (10.20)$$

When equation (10.19) holds,

$$\left.\begin{aligned} D_x - i\cdot 4\pi j_x/\omega &= \varepsilon'_{1k} E_k = A\,E_x + i\,C\,E_y, \\ D_y - i\cdot 4\pi j_y/\omega &= \varepsilon'_{2k} E_k = -i\,C\,E_x + B\,E_y, \end{aligned}\right\} \quad (10.21)$$

where

$$\left.\begin{aligned} A &= \frac{(1-i\,s)\,u-(1-i\,s)(1-i\,s-v)^2-u\,v\cos^2\alpha}{(1-i\,s)\,u-(1-i\,s)^2(1-i\,s-v)-u\,v\cos^2\alpha}, \\ B &= \frac{u(1-i\,s-v)-(1-i\,s)\,(1-i\,s-v)^2}{(1-i\,s)\,u-(1-i\,s)^2(1-i\,s-v)-u\,v\cos^2\alpha}, \\ C &= \frac{\sqrt{u}(1-i\,s-v)v\cos\alpha}{(1-i\,s)\,u-(1-i\,s)^2(1-i\,s-v)-u\,v\cos^2\alpha}. \end{aligned}\right\} \quad (10.22)$$

In two important particular cases we have

for $\alpha = 0$:

$$\left.\begin{aligned} A = B &= \frac{(1-i\,s)^2-u-v(1-i\,s)}{(1-i\,s)^2-u} = 1-\frac{v(1-i\,s)}{(1-i\,s)^2-u}, \\ C &= -\sqrt{u}\,v/[(1-i\,s)^2-u]; \end{aligned}\right\} \quad (10.23)$$

for $\alpha = \frac{1}{2}\pi$:

$$\left.\begin{aligned} A &= \frac{u-(1-i\,s-v)^2}{u-(1-i\,s)(1-i\,s-v)} = 1-\frac{v(1-i\,s-v)}{(1-i\,s)^2-u-(1-i\,s)\,v}, \\ B &= 1-\frac{v}{1-i\,s}, \\ C &= 0. \end{aligned}\right\} \quad (10.24)$$

When (10.19) holds and there is no absorption, with $s = 0$ (i.e. $j = 0$),

$$\left.\begin{aligned} A &= \frac{u-(1-v)^2-u\,v\cos^2\alpha}{u-(1-v)-u\,v\cos^2\alpha}, \\ B &= \frac{u(1-v)-(1-v)^2}{u-(1-v)-u\,v\cos^2\alpha}, \\ C &= \frac{\sqrt{u}(1-v)\,v\cos\alpha}{u-(1-v)-u\,v\cos^2\alpha} \end{aligned}\right\} \quad (10.25)$$

and

$$E_z = -\frac{i\sqrt{u}\,(v\sin\alpha)}{u-(1-v)-u\,v\cos^2\alpha}\,E_x + \frac{u\,v\cos\alpha\sin\alpha}{u-(1-v)-u\,v\cos^2\alpha}\,E_y. \quad (10.26)$$

Kinetic theory

To derive expressions for the components of the tensor ε'_{ik} on the basis of the Boltzmann-equation method we must use the equations [cf. (4.16), (4.11)]

$$\left.\begin{aligned}&\frac{\partial f_1}{\partial t}+\frac{e\boldsymbol{E}}{m}\frac{\partial f_{00}}{\partial v}+\frac{e}{mc}\boldsymbol{H}^{(0)}\times\boldsymbol{f}_1+\nu(v)\boldsymbol{f}_1+\boldsymbol{S}_{1,ee}=0,\\&j_{ti}=e\int v_i\frac{\boldsymbol{f}_1\cdot\boldsymbol{v}}{v}\,d\boldsymbol{v}=\frac{i\omega}{4\pi}(\varepsilon'_{ik}-\delta_{ik})E_k,\ \boldsymbol{E}=\boldsymbol{E}_0e^{i\omega t}.\end{aligned}\right\}\tag{10.27}$$

Neglecting collisions between electrons, i.e. omitting the term $S_{1,ee}$ in (10.27), we find, using a coordinate system with the z-axis in the direction of the field $\boldsymbol{H}^{(0)}$,

$$\left.\begin{aligned}&f_{1z}=-\frac{eE_z\,\partial f_{00}/\partial v}{m(i\omega+\nu)},\quad f_{1x}\pm f_{1y}=-\frac{e(E_x\pm iE_y)\,\partial f_{00}/\partial v}{m[i(\omega\mp\omega_H)+\nu]},\\&j_{tz}=\frac{i\omega}{4\pi}(\varepsilon'_{zz}-1)E_z=\frac{8e^2NE_z}{3\sqrt{\pi}\,m}\int\limits_0^\infty\frac{u^4e^{-u^2}du}{i\omega+\nu(u)},\\&j_{tx}\pm ij_{ty}=\frac{i\omega}{4\pi}[(\varepsilon'_{xx}-1)\mp i\varepsilon'_{xy}](E_x\pm iE_y)\\&\qquad=\frac{8e^2N(E_x\pm iE_y)}{3\sqrt{\pi}\,m}\int\limits_0^\infty\frac{u^4e^{-u^2}du}{[i(\omega\mp\omega_H)+\nu(u)]};\end{aligned}\right\}\tag{10.28}$$

here $u=\sqrt{(m/2\varkappa T)}\,v$ and the calculations are entirely similar to those given in § 6 for an isotropic plasma. It should be noted that the meaning of u and v here is not the same as that used in the preceding sub-section (10.18).

A comparison of (6.3) and (10.28) shows that for a magnetoactive plasma we can use the results of the calculations for an isotropic plasma, replacing ω in some expressions by $\omega\pm\omega_H$. In particular the component ε'_{zz} is equal to ε' for the isotropic plasma, and so all the formulae derived in § 6 can be applied immediately. For the other components of the tensor ε'_{ik} the necessary modifications are evident from the formulae

$$\left.\begin{aligned}&\varepsilon'_{xx}\mp i\varepsilon'_{xy}=\varepsilon'_{yy}\pm i\varepsilon'_{yx}\\&\qquad=1-i\cdot\frac{32\sqrt{\pi}\,e^2N}{3m\omega}\int\limits_0^\infty\frac{u^4e^{-u^2}du}{i(\omega\mp\omega_H)+\nu(u)},\\&\varepsilon'_{zz}=\varepsilon'=1-i\cdot\frac{32\sqrt{\pi}\,e^2N}{3m\omega}\int\limits_0^\infty\frac{u^4e^{-u^2}du}{i\omega+\nu(u)}.\end{aligned}\right\}\tag{10.29}$$

If $\nu(v)$ = constant, i.e. the collision frequency is independent of the velocity, we obtain the formulae of the "elementary theory" given previously, on using the result that

$$\int_0^\infty u^4 e^{-u^2}\,du = \frac{3}{8}\sqrt{\pi}.$$

In the limiting case of high frequencies and outside the resonance region, i.e. if

$$(\omega - \omega_H)^2 \gg \nu_{\text{eff}}^2, \tag{10.30}$$

we have

$$\left.\begin{aligned}
\varepsilon'_{xx} &= \varepsilon'_{yy} = \varepsilon_{xx} - i\cdot 4\pi\,\sigma_{xx}/\omega = \varepsilon_{yy} - i\cdot 4\pi\,\sigma_{yy}/\omega \\
&= 1 - \frac{\omega_0^2}{\omega^2-\omega_H^2} - \frac{1}{2} i\left[\frac{\omega_0^2}{(\omega-\omega_H)^2} + \frac{\omega_0^2}{(\omega+\omega_H)^2}\right]\frac{\nu_{\text{eff}}}{\omega}, \\
\varepsilon'_{zz} &= \varepsilon_{zz} - i\cdot 4\pi\,\sigma_{zz}/\omega = 1 - \frac{\omega_0^2}{\omega^2} - \frac{i\,\omega_0^2}{\omega^2}\frac{\nu_{\text{eff}}}{\omega}, \\
\varepsilon'_{xy} &= -\varepsilon'_{yx} = \varepsilon_{xy} - i\cdot 4\pi\,\sigma_{xy}/\omega, \\
\varepsilon_{xy} &= -i\cdot\omega_0^2\,\omega_H/\omega(\omega^2-\omega_H^2), \\
\sigma_{xy} &= i\,\omega_0^2\,\omega_H\,\omega\,\nu_{\text{eff}}/2\pi(\omega^2-\omega_H^2)^2,
\end{aligned}\right\} \tag{10.31}$$

where ν_{eff} is the effective collision frequency defined in § 6 for the high-frequency case [see (6.9)].

For arbitrary frequencies it is convenient to use the expressions

$$\left.\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{yy} \\
&= 1 - \frac{\omega_0^2}{2\omega}\left[\frac{(\omega-\omega_H)\,K_\varepsilon(|\omega-\omega_H|/\nu_{\text{eff}})}{(\omega-\omega_H)^2+\nu_{\text{eff}}^2} + \right. \\
&\quad \left. + \frac{(\omega+\omega_H)\,K_\varepsilon(\{\omega+\omega_H\}/\nu_{\text{eff}})}{(\omega+\omega_H)^2+\nu_{\text{eff}}^2}\right], \\
\sigma_{xx} &= \sigma_{yy} = \frac{\omega_0^2}{8\pi}\left[\frac{K_\sigma(|\omega-\omega_H|/\nu_{\text{eff}})}{(\omega-\omega_H)^2+\nu_{\text{eff}}^2} + \frac{K_\sigma(\{\omega+\omega_H\}/\nu_{\text{eff}})}{(\omega+\omega_H)^2+\nu_{\text{eff}}^2}\right]\nu_{\text{eff}}, \\
\varepsilon_{xy} &= -i\cdot\frac{1}{2}\omega_0^2\left[\frac{(\omega-\omega_H)\,K_\varepsilon(|\omega-\omega_H|/\nu_{\text{eff}})}{\omega\{(\omega-\omega_H)^2+\nu_{\text{eff}}^2\}} - \right. \\
&\quad \left. - \frac{(\omega+\omega_H)\,K_\varepsilon(\{\omega+\omega_H\}/\nu_{\text{eff}})}{\omega\{(\omega+\omega_H)^2+\nu_{\text{eff}}^2\}}\right], \\
\sigma_{xy} &= i\cdot\frac{\omega_0^2}{8\pi}\left[\frac{K_\sigma(|\omega-\omega_H|)/\nu_{\text{eff}}}{(\omega-\omega_H)^2+\nu_{\text{eff}}^2} - \frac{K_\sigma(\{\omega+\omega_H\}/\nu_{\text{eff}})}{(\omega+\omega_H)^2+\nu_{\text{eff}}^2}\right]\nu_{\text{eff}}, \\
\varepsilon_{zz} &= 1 - \frac{\omega_0^2\,K_\varepsilon(\omega/\nu_{\text{eff}})}{\omega^2+\nu_{\text{eff}}^2}, \quad \sigma_{zz} = \frac{\omega_0^2\,\nu_{\text{eff}}\,K_\sigma(\omega/\nu_{\text{eff}})}{4\pi(\omega^2+\nu_{\text{eff}}^2)}.
\end{aligned}\right\} \tag{10.32}$$

Here the functions $K_\varepsilon(x)$ and $K_\sigma(x)$, both when collisions between electrons are taken into account and when they are not, are the same as the corresponding functions in the isotropic case [see (6.25)]. It may be recalled that for $x \to \infty$ $K_\varepsilon(x) = K_\sigma(x) = 1$. When the condition (10.30) holds, therefore, the formulae (10.32) become (10.31), as they should. Near resonance, when

$$(\omega - \omega_H)^2 \ll \nu_{\text{eff}}^2, \tag{10.33}$$

the behaviour of a magnetoactive plasma is in some respects similar to that of an isotropic plasma at low frequencies, i.e. for $\omega^2 \ll \nu_{\text{eff}}^2$. For example, when $\omega = \omega_H$ and $\omega^2 \gg \nu_{\text{eff}}^2$ we have $\sigma_{xx} = \sigma_{yy} = \omega_0^2 K_\sigma(0)/8\pi\nu_{\text{eff}}^2$, whereas in an isotropic plasma the value of $K_{\varepsilon,\sigma}(0)$ is significant only when $\omega \to 0$. Moreover, in the range (10.33), as at low frequencies for an isotropic electron-ion plasma, collisions between electrons are important [in (10.32) the contribution from such collisions at all frequencies is taken into account by using the corresponding values of K_ε and K_σ given in Tables 6.2 and 6.3]. In the "elementary theory", where $\nu(v)$ = constant, we have of course $K_\varepsilon = K_\sigma = 1$. The function $K_\sigma(x)$, which is of the greatest interest, is changed by not more than 13 per cent in the case of collisions with hard-sphere molecules; for collisions with ions and allowing for collisions between electrons, $K_\sigma(x)$ changes by not more than a factor of two (see Tables 6.2 and 6.3). In most cases, therefore, it is quite correct to use the results of the elementary theory, especially as the collision frequency ν_{eff} is usually known only approximately (for example, as a result of insufficiently exact knowledge of the temperature or composition of the plasma under consideration). The possibility of this simplification is very important, since formulae (10.32) are considerably more complicated than those of the elementary theory (10.12). When other systems of coordinates are used, they become still more complicated.

The thermal motion of the electrons has not yet been taken into account (except, of course, as regards collisions). That is, the part played by spatial dispersion has not been considered. To do so involves the abandonment of the use of local characteristics of the medium, with the components of the tensor ε'_{ik} depending only on the frequency. We shall therefore make allowance for spatial dispersion in a magnetoactive plasma in §§ 12 and 14, when discussing the propagation of waves.

It should be emphasised once more that our whole discussion is restricted to the case of a non-relativistic plasma. A relativistic electron moving in a magnetic field emits not only waves with frequency $\omega_H^* = (|e|\, H^{(0)}/mc)\, mc^2/E$ but also harmonics of that frequency. A relativistic plasma will therefore exhibit resonance absorption at frequencies $s\,\omega_H^*\,(s = 1, 2, 3, \ldots)$, whereas the expressions given above for ε'_{ik} have a resonance only at the frequency ω_H. The relativistic plasma will not be considered in this book, but in § 12 we shall take account of the thermal motion (for $\beta_T^2 = \varkappa T/mc^2 \ll 1$) and, in particular, discuss resonance absorption at frequencies ω_H, $2\omega_H$ and $3\omega_H$.†

† When the thermal motion is neglected, resonance absorption of waves at frequency ω_H occurs only for the extraordinary wave with $\alpha = 0$ (see § 11). When the thermal motion is taken into account, resonance absorption at frequency ω_H occurs for all angles α and for waves of either kind.

The effect of the motion of ions

Let us now consider the effect of ions, and allow for their motion in determining the tensor ε'_{ik}.

Using only the elementary theory, we start from the following equations of motion for electrons, ions and molecules (denoted by suffixes e, i and m respectively):

$$m\,\dot{\boldsymbol{v}}_e = e\,\boldsymbol{E} + e\,\boldsymbol{v}_e \times \boldsymbol{H}^{(0)}/c + m\,\nu_{ei}(\boldsymbol{v}_i - \boldsymbol{v}_e) + m\,\nu_{em}(\boldsymbol{v}_m - \boldsymbol{v}_e), \tag{10.34}$$

$$M\,\dot{\boldsymbol{v}}_i = -e\,\boldsymbol{E} - e\,\boldsymbol{v}_i \times \boldsymbol{H}^{(0)}/c + m\,\nu_{ei}(\boldsymbol{v}_e - \boldsymbol{v}_i) + M\,\nu_{im}(\boldsymbol{v}_m - \boldsymbol{v}_i), \tag{10.35}$$

$$M\,\dot{\boldsymbol{v}}_m = -m\,\nu_{em}(N/N_m)(\boldsymbol{v}_m - \boldsymbol{v}_e) - M\,\nu_{im}(N/N_m)(\boldsymbol{v}_m - \boldsymbol{v}_i). \tag{10.36}$$

Here it is assumed for simplicity that all the ions are singly charged, $N_i = N_+ = N$, and the ions and molecules have the same mass M. The electron charge is denoted by e, and therefore $e < 0$.

In the absence of collisions the form of these equations is self-explanatory. It need only be mentioned that $\boldsymbol{v}_e$, $\boldsymbol{v}_i$ and $\boldsymbol{v}_m$ signify the velocities averaged over a large number of particles. The terms which correspond to collisions are proportional to the relative mean velocities of the colliding particles, since, for instance, when the mean velocities of electrons and ions are equal they cannot be altered by collisions. In the equation (10.8) used previously, the velocity $\boldsymbol{v}_i$ did not appear, simply because the ions were assumed to be at rest. The frictional force exerted by the electrons on the ions is $m\nu_{ei}(\dot{\boldsymbol{r}}_e - \dot{\boldsymbol{r}}_i) = m\nu_{ei}(\boldsymbol{v}_e - \boldsymbol{v}_i)$, since it must be minus the frictional force exerted by the ions on the electrons. A similar argument gives the form of the terms on the right of (10.36). We need only note that (e.g.) the frequency of collisions of an electron with molecules is $\nu_{em} = \overline{q_{em} v} N_m$, and the frequency of collisions of a molecule with electrons is $\nu_{me} = \overline{q_{em} v} N = (N/N_m)\,\nu_{em}$. The collision frequencies ν_{ei}, ν_{em} and ν_{im} in equations (10.34)–(10.36) are some effective values and can depend only on the density and the temperature.

The current density to be substituted in the field equations is

$$\left.\begin{aligned} \boldsymbol{j}_t &= e\,N(\boldsymbol{v}_e - \boldsymbol{v}_i) = e\,N\,\boldsymbol{w}, \\ j_{ti} &= i\,\omega\,(\varepsilon'_{ik} - \delta_{ik})\,E_k/4\pi. \end{aligned}\right\} \tag{10.37}$$

Taking all quantities to be proportional to $e^{i\omega t}$ and the external field $\boldsymbol{H}^{(0)}$ to be along the z-axis, we have from (10.34) and (10.35), neglecting collisions,

$$\left.\begin{aligned} &v_{ez} = e\,E_z/i\,m\,\omega, \quad v_{iz} = -e\,E_z/i\,M\,\omega, \\ &v_{ex} \pm i\,v_{ey} = e(E_x \pm i\,E_y)/i\,m(\omega \mp \omega_H), \\ &v_{ix} \pm i\,v_{iy} = -e(E_x \pm i\,E_y)/i\,M(\omega \pm \Omega_H), \\ &\boldsymbol{v}_m = 0, \quad \boldsymbol{w} = \boldsymbol{v}_e - \boldsymbol{v}_i, \\ &\omega_H = |e|\,H^{(0)}/m\,c = -e\,H^{(0)}/m\,c, \quad \Omega_H = |e|\,H^{(0)}/M\,c, \end{aligned}\right\} \tag{10.38}$$

$$\left.\begin{aligned}
j_{tz} &= eNw_z = i\omega(\varepsilon'_{zz}-1)E_z/4\pi \\
&= -i\left(\frac{e^2N}{m}+\frac{e^2N}{M}\right)E_z/\omega \approx ie^2NE_z/m\omega, \\
j_{tx} \pm i j_{ty} &= i\omega(\varepsilon'_{xx}-1\mp i\varepsilon'_{xy})(E_x \pm iE_y)/4\pi \\
&= eN(w_x \pm i w_y) \\
&= -ie^2N\left[\frac{1}{m(\omega\mp\omega_H)}+\frac{1}{M(\omega\pm\Omega_H)}\right](E_x \pm iE_y) \\
&= \frac{ie^2N\omega(E_x \pm iE_y)}{Mm(\omega\mp\omega_H)(\omega\pm\Omega_H)/(m+M)}.
\end{aligned}\right\} \quad (10.39)$$

The field does not affect the velocity and current components parallel to $\boldsymbol{H}^{(0)}$, and therefore the contribution of the ions (for $N_i = N$) is less than that of the electrons by a factor m/M, as in an isotropic medium.

As regards the velocity and current components perpendicular to the field $\boldsymbol{H}^{(0)}$, the part played by the ions may be neglected if $\omega \gg \Omega_H$ [see (10.5)]. If, however,

$$\omega \ll \Omega_H, \quad (10.40)$$

then the electron and ion velocities perpendicular to the field are approximately equal, and the current is therefore very small. For, when $\omega = 0$,

$$v_{ex} = v_{ix} = cE_y/H^{(0)}, \quad v_{ey} = v_{iy} = -cE_x/H^{(0)}, \quad j_{tx} = j_{ty} = 0, \quad (10.41)$$

since $m\omega_H = M\Omega_H = |e|\,H^{(0)}/c$. The result (10.41) is, of course, evident from the original equations (10.34) and (10.35).

Thus in the low-frequency case (10.40) the ions play a very important part. In an isotropic plasma, $\Omega_H = 0$, and this low-frequency range ceases to exist.

When collisions are taken into account, it is convenient to write equations (10.34) to (10.36) for quantities proportional to $e^{i\omega t}$ as

$$\left.\begin{aligned}
(i\omega+\nu_{ei}+\nu_{em})\boldsymbol{w} &= e\boldsymbol{E}/m + e\boldsymbol{w}\times\boldsymbol{H}^{(0)}/mc + e\boldsymbol{v}_i\times\boldsymbol{H}^{(0)}/mc + \\
&\qquad + \nu_{em}(\boldsymbol{v}_m-\boldsymbol{v}_i), \\
i\omega\boldsymbol{v}_i &= -e\boldsymbol{E}/M - e\boldsymbol{v}_i\times\boldsymbol{H}^{(0)}/Mc + m\nu_{ei}\boldsymbol{w}/M + \nu_{im}(\boldsymbol{v}_m-\boldsymbol{v}_i), \\
\boldsymbol{v}_m &= \frac{\nu_{im}(N/N_m)\boldsymbol{v}_i + (m/M)\nu_{em}(N/N_m)\boldsymbol{w}}{i\omega+\nu_{im}N/N_m}, \\
\boldsymbol{w} &= \boldsymbol{v}_e - \boldsymbol{v}_i,
\end{aligned}\right\} \quad (10.42)$$

where quantities of the order of m/M or $\sqrt{(m/M)}$ in comparison with unity have been neglected in the coefficients of the variables, and we have used the fact that $\nu_{im} \sim \sqrt{(m/M)}\,\nu_{em}$. The same procedure will be used without special mention henceforward.

From (10.42) we easily obtain a general expression for the tensor ε'_{ik}. For the component ε'_{zz} we find in this approximation the previous expression

$\varepsilon'_{zz} = 1 - \omega_0^2/\omega[\omega - i(\nu_{ei} + \nu_{em})]$. Moreover, we again have $\varepsilon'_{xz} = \varepsilon'_{zx} = \varepsilon'_{yz} = \varepsilon'_{zy} = 0$ and $\varepsilon'_{xx} = \varepsilon'_{yy}$, $\varepsilon'_{xy} = -\varepsilon'_{yx}$, with

$$\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} = 1 - \frac{\omega_0^2\,[1 + \nu_{im}/(i\,\omega + \nu_{im}\,N/N_m)]}{A - iB}, \tag{10.43}$$

where

$$A = [\omega \mp \omega_H - i\,(\nu_{ei} + \nu_{em})]\left(\omega \pm \Omega_H + \frac{\omega\,\nu_{im}}{i\,\omega + \nu_{im}\,N/N_m}\right),$$

$$B = \left(\frac{\omega\,\nu_{em}\,m/M}{i\,\omega + \nu_{im}\,N/N_m} \mp \Omega_H\right)\left(\nu_{ei} + \frac{\nu_{em}\,\nu_{im}\,N/N_m}{i\,\omega + \nu_{im}\,N/N_m}\right).$$

When molecules are absent and $\nu_{im} = \nu_{em} = 0$ we have

$$\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} = 1 - \omega_0^2/[(\omega \mp \omega_H)\,(\omega \pm \Omega_H) - i\,\omega\,\nu_{ei}]. \tag{10.44}$$

In the high-frequency case (10.5) the expressions for ε'_{ik} become, of course, (10.12).

A characteristic feature is that, in electron resonance ($\omega = \omega_H$), $\varepsilon'_{xx} - i\varepsilon'_{xy} = 1 - i\omega_0^2/\omega_H\nu_{ei}$; in ion resonance ($\omega = \Omega_H$), $\varepsilon'_{xx} + i\varepsilon'_{xy} = 1 - i\omega_0^2/\Omega_H\nu_{ei}$, i.e. the ion resonance is M/m times "higher" (more precisely, the corresponding components of the conductivity tensor σ_{ik} are M/m times greater).

The accuracy of the expressions (10.43) and (10.44) is less than that of the corresponding formulae derived in the elementary theory neglecting the motion of the heavy particles. In the latter case the elementary theory exactly corresponds to the assumption that the collision frequency is independent of the velocity and to the neglect of collisions between electrons. For collisions between heavy particles this approximation is less well founded, since it is not then permissible to take account of collisions in the Boltzmann equation by means of a term of the form νf_1 [see (6.1)].

The formulae of the elementary theory are, nevertheless, convenient and useful for estimating absorption; more particularly, when spatial dispersion is neglected they are entirely valid for calculating the tensor ε_{ik} for weak absorption.† The latter case is frequently met with in practice. Finally, it may be noted that, in taking account of the motion of the ions, the tensor ε'_{ik} might also be defined in a frame of reference where the mean velocity of all particles is zero. Such a definition leads in general to quite different expressions for ε'_{ik}, although the physical results are naturally independent of the choice of the system of reference.††

† That is, the limiting value of $\varepsilon'_{ik} = \varepsilon_{ik}$ when collisions are completely neglected.

†† Apparent contradictions may arise if this fact is overlooked. For example, it follows from formula (10.44) that for $\omega_H \neq 0$ and $\omega = 0$ the electric conductivity of the medium is zero, $\varepsilon'_{xx} = \varepsilon_{xx} = 1 + \omega_0^2/\omega_H\Omega_H$ and the current density $j_t = 0$. This result is correct, but it must be remembered that in the frame of reference used the whole plasma is in motion with the velocity (10.41). Hence, in a frame of reference fixed in the plasma, the electric field is zero, and it is quite reasonable that the current j_t should vanish [see also (10.39)].

§ 11. HIGH-FREQUENCY WAVE PROPAGATION IN A MAGNETOACTIVE PLASMA

Expressions for the indices of refraction and absorption $n_{1,2}$ and $\varkappa_{1,2}$

Let us consider the propagation of monochromatic waves in a homogeneous magnetoactive plasma, the frequency ω of the waves being much greater than the ion gyration frequency Ω_H [the condition (10.5)]. For such waves, which we shall call high-frequency waves, the effect of the ions may be neglected (for the case where $N_i \sim N$). Hence we can use the expressions (10.12) for ε'_{ik}; these also imply that the velocity dependence of the collision frequency is neglected.

The initial wave equation is [see (2.5), (2.7) and (2.9)]

$$\left.\begin{aligned} &-\mathbf{curl\,curl}\,\boldsymbol{E} + (\omega^2/c^2)(\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega) \\ &\qquad = \triangle\boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} + (\omega^2/c^2)(\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega) = 0, \\ &\qquad D_i - i\cdot 4\pi j_i/\omega = \varepsilon'_{ik}E_k. \end{aligned}\right\} \tag{11.1}$$

Hence we have for plane waves $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) + (\omega^2/c^2)(\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega) = 0. \tag{11.2}$$

For homogeneous plane waves (the only case we shall consider), the planes of equal phase and amplitude coincide, and $k = \omega(n - i\varkappa)/c$. The quantity $(n - i\varkappa)^2$ is found from equations (11.2) as the condition for those equations to have a non-trivial solution. Since there are three equations for the three quantities E_x, E_y, E_z, we might expect the equation for $(n - i\varkappa)^2$ to be of the third degree (i.e. of the sixth degree in $n - i\varkappa$). This is not so, however, since the component of the vector $\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega$ in the direction of $\boldsymbol{k}$ is zero [as is immediately seen by taking the scalar product of (11.2) with $\boldsymbol{k}$]. The expression $\boldsymbol{k}\cdot(\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega)/k$ is independent of $\boldsymbol{k}$, since the tensor ε'_{ik} depends only on ω when spatial dispersion is neglected, and not on $k = \omega(n - i\varkappa)/c$.

Thus there is a linear relation between the components E_x, E_y, E_z in the wave, which does not depend on $(n - i\varkappa)^2$, and so the condition for equations (11.2) to have a solution leads to an equation of only the second degree in $(n - i\varkappa)^2$. This result would also be obtained, of course, by carrying out the calculations. However, with a view to the subsequent discussion it is convenient to take the direction of $\boldsymbol{k}$ immediately as the z-axis and use the condition $D_z - i\cdot 4\pi j_z/\omega = 0$. Then E_z is expressed in terms of E_x and E_y by formula (10.20), and the components $D_{x,y} - i\cdot 4\pi j_{x,y}/\sigma$ are given by (10.21) and (10.22). The equations (11.2) become

$$\left.\begin{aligned} [A - (n - i\varkappa)^2]\,E_x + i\,C\,E_y = 0\,, \\ -\,i\,C\,E_x + [B - (n - i\varkappa)^2]\,E_y = 0\,. \end{aligned}\right\} \tag{11.2a}$$

The same result is more simply reached directly from (11.1), since for a plane wave in which the field depends only on the coordinate z, the vector equation (11.1) gives the equations (2.18) or, using (10.21) and (10.22), the equations

$$\left.\begin{aligned}
&d^2E_x/dz^2 + (\omega^2/c^2)(A E_x + i C E_y) = 0,\\
&d^2E_y/dz^2 + (\omega^2/c^2)(-i C E_x + B E_y) = 0,\\
&A = \frac{(1-is)u - (1-is)(1-is-v)^2 - uv\cos^2\alpha}{(1-is)u - (1-is)^2(1-is-v) - uv\cos^2\alpha},\\
&B = \frac{u(1-is-v) - (1-is)(1-is-v)^2}{(1-is)u - (1-is)^2(1-is-v) - uv\cos^2\alpha},\\
&C = \frac{\sqrt{u}\,v(1-is-v)\cos\alpha}{(1-is)u - (1-is)^2(1-is-v) - uv\cos^2\alpha},\\
&\sqrt{u} = \omega_H/\omega = |e|\,H^{(0)}/mc\omega, \quad v = \omega_0^2/\omega^2 = 4\pi e^2 N/m\omega^2,\\
&s = \nu_{\text{eff}}/\omega.
\end{aligned}\right\} \tag{11.3}$$

The coordinate system used here is evident from Fig. 11.1. Substituting in (11.3) the solution in the form of a plane harmonic wave $E_{x,y} = E_{0x,y}e^{\pm i\omega(n-i\varkappa)z/c}$, we obtain (11.2a).

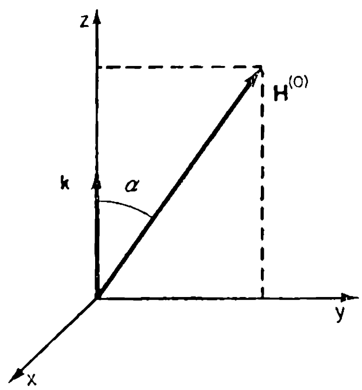

FIG. 11.1. Coordinate system used in § 11.

The condition for the system (11.2a) to have a non-trivial solution gives an equation for $n - i\varkappa$. In determinant form this equation is evidently

$$\begin{vmatrix} A-(n-i\varkappa)^2 & iC \\ -iC & B-(n-i\varkappa)^2 \end{vmatrix} = 0. \tag{11.4}$$

The solution of (11.4) is

$$(n-i\varkappa)^2_{1,2} = n^2_{1,2} - \varkappa^2_{1,2} - 2i\,n_{1,2}\varkappa_{1,2}$$
$$= 1 - \frac{2v(1-is-v)}{2(1-is)(1-is-v) - u\sin^2\alpha \pm \sqrt{[u^2\sin^4\alpha + 4u(1-is-v)^2\cos^2\alpha]}}. \tag{11.5}$$

Taking the upper sign of the root we obtain the quantity $(n - i\varkappa)_2^2 \equiv \equiv (n_2 - i\varkappa_2)^2$ corresponding to the "ordinary" wave; the choice of the lower sign corresponds to the "extraordinary" wave (whose indices of refraction and absorption are n_1 and $\varkappa_1$)†.

In the absence of absorption we have

$$n_{1,2}^2 = 1 - \frac{2v(1-v)}{2(1-v) - u\sin^2\alpha \pm \sqrt{[u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha]}}$$

$$= 1 - \frac{2\omega_0^2(\omega^2-\omega_0^2)}{2(\omega^2-\omega_0^2)\omega^2 - \omega_H^2\omega^2\sin^2\alpha \pm \sqrt{[\omega^4\omega_H^4\sin^4\alpha + 4\omega_H^2\omega^2(\omega^2-\omega_0^2)^2\cos^2\alpha]}}; \tag{11.6}$$

here it is assumed that $n_{1,2}^2 > 0$, since only then is $\varkappa = 0$ in the absence of absorption. If the right-hand side of the expression (11.6) is less than zero, (11.5) shows that it must be regarded as $-\varkappa_{1,2}^2$. In the absence of absorption, however, it is more convenient not to define the index of absorption $\varkappa$ and to use formula (11.6) even when $n_{1,2}^2 < 0$. Then $n_{1,2}^2 = -\varkappa_{1,2}^2$ simply. We shall use this procedure below, avoiding misunderstanding by denoting $n_{1,2}^2$ by $\tilde{n}_{1,2}^2$ (see § 7). In other words, in the absence of absorption $(n - i\varkappa)^2 = \tilde{n}^2$, which is real. The index of refraction is taken as $n_{1,2} = +\sqrt{n_{1,2}^2}$; the solution $n_{1,2} = -\sqrt{n_{1,2}^2}$ corresponds to a wave travelling in the opposite direction. This will be taken into account directly in the expression for the phase of the wave.

Some particular cases

If the magnetic field $H^{(0)} = 0$, i.e. $u = 0$, then

$$(n - i\varkappa)_{1,2}^2 = (n - i\varkappa)_0^2 = 1 - v/(1 - is)$$
$$= 1 - \omega_0^2/\omega(\omega - i\nu_{\text{eff}}), \tag{11.7}$$

as it should be [see (3.7)].

In the important particular case of "longitudinal propagation", when the wave travels along the field, i.e. the angle $\alpha \equiv 0$, we have

$$(n - i\varkappa)_{1,2}^2 \equiv (n - i\varkappa)_\mp^2 = 1 - v/(1 - is \pm \sqrt{u}), \tag{11.8}$$

or, in the absence of absorption,

$$\left.\begin{aligned} \tilde{n}_1^2 \equiv \tilde{n}_+^2 = 1 - v/(1 - \sqrt{u}) = 1 - \omega_0^2/\omega(\omega - \omega_H), \\ \tilde{n}_2^2 \equiv \tilde{n}_-^2 = 1 - v/(1 + \sqrt{u}) = 1 - \omega_0^2/\omega(\omega + \omega_H). \end{aligned}\right\} \tag{11.9}$$

The significance of the notation $\tilde{n}_{1,2}^2 \equiv \tilde{n}_\mp^2$ will become evident when we examine the nature of the polarisation of the waves in longitudinal propagation.

† Sometimes the opposite notation is used, with the suffix 1 for the ordinary wave and 2 for the extraordinary wave. Moreover, the suffix x or e is often used instead of 1, and o instead of 2.

In the general case, the polarisation, i.e. the ratio of the components E_x and E_y, is found directly from one of the equations (11.2a), with $(n - i\varkappa)^2$ replaced by the solution (11.5). For longitudinal propagation, it is more convenient to use from the start the expressions

$$F_\pm = E_x \pm i E_y, \tag{11.10}$$

for which we have from (11.3) the equations

$$\frac{d^2 F_\pm}{d z^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 - i s \mp \sqrt{u}}\right) F_\pm = 0. \tag{11.11}$$

The harmonic solution of this equation is, in agreement with (11.8),

$$\left.\begin{aligned} F_\pm &= F_{0,\pm}\, e^{\pm i\omega(n - i\varkappa)_\pm z/c} \\ (n - i\varkappa)^2_\pm &= 1 - v/(1 - i s \mp \sqrt{u}). \end{aligned}\right\} \tag{11.12}$$

Thus in this case of longitudinal propagation there are two "normal waves" with definite phase velocities $v_\pm = c/n_\pm$, damping (with absorption coefficient $\mu_\pm = 2\omega\varkappa_\pm/c$) and polarisation. For example, if there is a plus wave, then $F_- = 0$, and so $E_x = iE_y$; for a minus wave $F_+ = 0$ and $E_x = -iE_y$. Writing the expression for the wave field with the time factor, i.e. as

$$F_\pm = F_{0,\pm}\, e^{i[\omega t - (n - i\varkappa)_\pm z/c]},$$

and taking real quantities, we easily see that the $\pm$ waves are circularly polarised; in the plus wave the vector $\boldsymbol{E}$ rotates clockwise looking along the field (i.e. along the z-axis), and in the minus wave anticlockwise. In other words, in the $\pm$ waves at $z = 0$ we have $E_x = \mp E_0 \sin \omega t$, $E_y = E_0 \cos \omega t$.

The direction of rotation of the vector $\boldsymbol{E}$ in the plus wave is the same as the direction of revolution of the electron in the magnetic field $\boldsymbol{H}^{(0)}$. It is natural, therefore, that when the frequency of the plus wave approaches the gyration frequency ω_H a resonance occurs [the difference $\omega - \omega_H$ appears in the denominator of the formula for $(n - i\varkappa)^2_+$ in (11.12)]. The plus wave is also called the extraordinary wave or wave 1, and the minus wave is the ordinary wave or wave 2 (see above).

The existence of two normal waves, i.e. waves with a definite velocity, absorption and polarisation, is a property of any anisotropic medium. In an isotropic medium there is a degeneracy, in that transverse waves have the same velocity and absorption for any polarisation. In the case of a magnetoactive medium here considered, the normal waves are in general elliptically polarised. In the particular case of longitudinal propagation (i.e. for $\alpha = 0$) there is circular polarisation, as shown above.

For "transverse propagation", when $\alpha = \frac{1}{2}\pi$, there is another limiting case, and the ellipses described by the vector $\boldsymbol{E}$ in the xy-plane degenerate into straight lines. This result is immediately evident from the expressions (11.3), which show that for $\alpha = \frac{1}{2}\pi$ the equations of propagation separate and take the form

$$\left.\begin{aligned}\frac{d^2 E_x}{d z^2} + \frac{\omega^2}{c^2}\left[1 - \frac{v(1 - i s - v)}{(1 - i s)^2 - u - (1 - i s) v}\right] E_x = 0,\\ \frac{d^2 E_y}{d z^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 - i s}\right) E_y = 0.\end{aligned}\right\} \tag{11.13}$$

From these equations it is clear that the normal waves are those for which the vector $\boldsymbol{E} = \boldsymbol{E}_0 e^{-i\omega(n - i\varkappa)_{1,2} z/c}$ has a zero component along either the x-axis or the y-axis. The wave with $E_x = 0$ and the vector $\boldsymbol{E}$ in the y-direction, i.e. along the field $\boldsymbol{H}^{(0)}$ (see Fig. 11.1, where $\alpha = \frac{1}{2}\pi$ in this case) is called the ordinary wave, since its velocity of propagation is independent of the magnitude of the field $\boldsymbol{H}^{(0)}$. This is quite reasonable, because a magnetic field has no effect on the motion of charges in the direction of the field. In the second normal wave, called the extraordinary wave, $E_y = 0$ and the vector $\boldsymbol{E}$ is in the xz-plane.†

According to (11.13), in these waves

$$\left.\begin{aligned}(n - i\varkappa)_1^2 &= 1 - \frac{v(1 - i s - v)}{(1 - i s)^2 - u - (1 - i s) v}\\ &= 1 - \frac{\omega_0^2(1 - i\,\nu_{\text{eff}}/\omega - \omega_0^2/\omega^2)}{(\omega - i\,\nu_{\text{eff}})^2 - \omega_H^2 - (1 - i\,\nu_{\text{eff}}/\omega)\,\omega_0^2},\\ (n - i\varkappa)_2^2 &= (n - i\varkappa)_0^2 = 1 - v/(1 - i s) = 1 - \omega_0^2/\omega(\omega - i\,\nu_{\text{eff}}).\end{aligned}\right\} \tag{11.14}$$

The same result, of course, follows at once from the general expression (11.5) with $\alpha = \frac{1}{2}\pi$.

The dependence of the indices n and $\varkappa$ on the parameters v, u, s and α is conveniently represented graphically. Figs. 11.2–11.4 show such graphs for $\tilde{n}^2 = (n - i\varkappa)^2$ in the absence of absorption, with $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$, and also for an isotropic plasma. The abscissa is the parameter $v = \omega_0^2/\omega^2$, and the parameter $u = \omega_H^2/\omega^2$ is also given a fixed value. It is sometimes useful to plot the same functions $\tilde{n}_{1,2}^2$ with other coordinates, e.g. $\omega/\omega_0 = 1/\sqrt{v}$ for a given u or ω_H. Fig. 11.5 gives examples of such graphs. In considering wave propagation in a medium where the density and the field both vary in space, i.e. where v and u are both variable, graphs of a third kind are used, which will be given in §§ 35 and 36.

Propagation of waves at an arbitrary angle α to the magnetic field

We shall now examine the nature of wave propagation at an arbitrary angle α to the magnetic field.

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† In the extraordinary wave the vector $\boldsymbol{E}$ is elliptically polarised even for $\alpha = \frac{1}{2}\pi$, and has a component in the z-direction as well as in the x-direction. The vector $\boldsymbol{D}$, however, s in this case along the x-axis (see below).

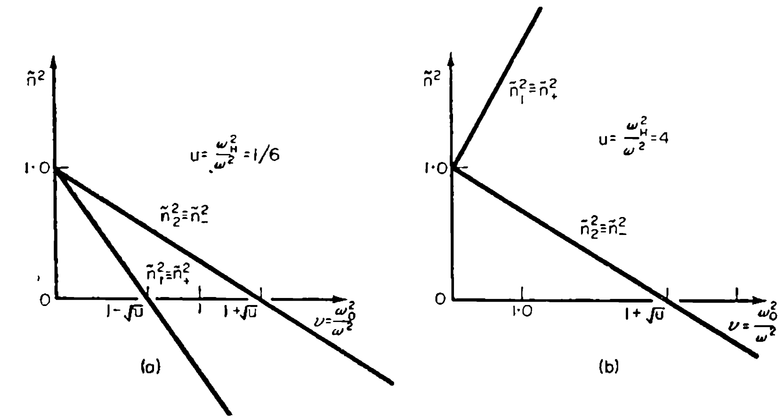

FIG. 11.2. The functions $\tilde{n}^2_{1,2}(v) \equiv \tilde{n}^2_{\pm}(v)$ for u = constant and longitudinal propagation (i.e. $\alpha = 0$).
(a) $u < 1$ (b) $u > 1$

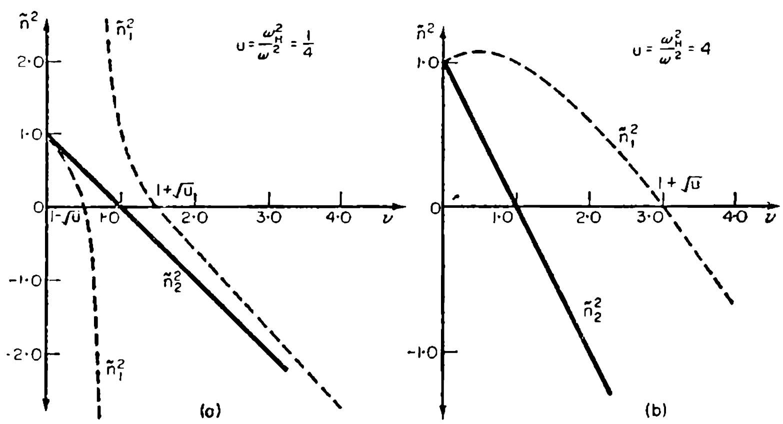

FIG. 11.3. The functions $\tilde{n}^2_{1,2}(v)$ for u = constant and transverse propagation (i.e. $\alpha = 90°$).
(a) $u < 1$ (b) $u > 1$

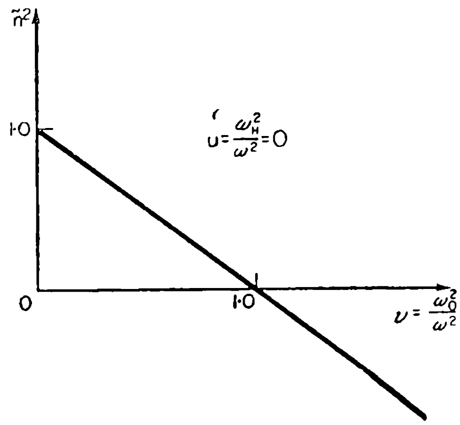

FIG. 11.4. The functions $\tilde{n}^2_{1,2}(v) \equiv \tilde{n}^2_0(v)$ for an isotropic plasma.

Let us assume absorption absent and begin with the case where

$$u = \omega_H^2/\omega^2 < 1. \tag{11.15}$$

For $H^{(0)} = 0{\cdot}5$ oersted the gyration frequency is $\omega_H = 8{\cdot}82 \times 10^6\ \text{sec}^{-1}$ and $\lambda_H = 214$ m. Thus in the Earth's ionosphere the case (11.15) corresponds

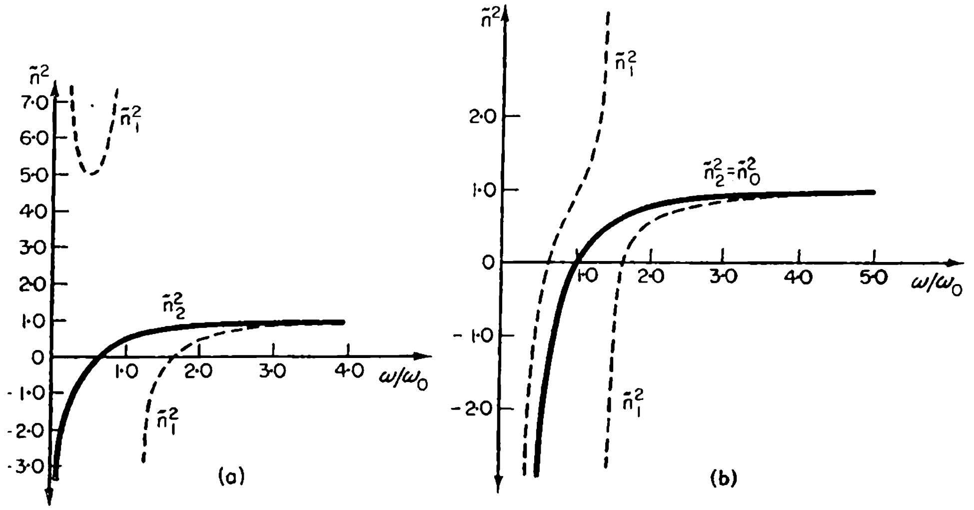

FIG. 11.5. (a) The functions $\tilde{n}_{1,2}^2(\omega/\omega_0) = 1 - \dfrac{1}{(\omega/\omega_0)\,(\omega \pm \omega_H)/\omega_0}$ corresponding to $\alpha = 0$ with $\omega_H/\omega_0 = 1$.
(b) The functions $\tilde{n}_1^2(\omega/\omega_0) = 1 - \dfrac{1 - \omega_0^2/\omega^2}{(\omega/\omega_0)^2 - (\omega_H/\omega_0)^2 - 1}$ and $\tilde{n}_2^2 \equiv \tilde{n}_0^2 = 1 - \omega_0^2/\omega^2$ corresponding to $\alpha = 90°$ with $\omega_H/\omega_0 = 1$. The expression $\tilde{n}_0^2 = 1 - \omega_0^2/\omega^2$ does not depend on ω_H and is valid also for an isotropic plasma.

to short radio waves. When this condition holds, the refractive index $\tilde{n}_2^2$ of the ordinary wave vanishes at the point

$$v_{20} \equiv \omega_0^2/\omega_{20}^2 = 1, \quad \omega_{20}^2 = \omega_0^2 = 4\,\pi e^2 N/m, \tag{11.16}$$

just as when the magnetic field is absent [see (11.8) with $s = 0$]. The function $\tilde{n}_2^2(v)$ is nowhere infinite.

The refractive index $\tilde{n}_1^2$ of the extraordinary wave vanishes at two points:

$$v_{10}^{(\pm)} \equiv \omega_0^2/(\omega_{10}^{\pm})^2 = 1 \pm \sqrt{u} = 1 \pm \omega_H/\omega_{10}^{\pm}. \tag{11.17}$$

The index $\tilde{n}_1^2$ becomes infinite at the point

$$v_{1\infty} \equiv \omega_0^2/\omega_{1\infty}^2 = (1-u)/(1-u\cos^2\alpha) \equiv (1-u)/(1-u_L)$$
$$= (\omega_{1\infty}^2 - \omega_H^2)/(\omega_{1\infty}^2 - \omega_H^2\cos^2\alpha). \tag{11.18}$$

It is evident that $v_{1\infty} \leqq 1$. For $v = 1$, $\tilde{n}_1^2 = 1$ for all u and α. The values of v_{20} and $v_{10}^{(\pm)}$ are independent of the angle α. The angle $\alpha = 0$ (longitudinal propagation) is, however, an exceptional case, which will be discussed presently. Fig. 11.6 shows $\tilde{n}_{1,2}^2$ as functions of v for $u = \frac{1}{4}(\omega = 2\omega_H)$ and $u = 0{\cdot}01$ for $\alpha = 45°$; Fig. 11.7 shows the functions $\tilde{n}_{1,2}^2(\omega/\omega_0)$ for $\omega_H/\omega_0 = 1$ and $\alpha = 45°$.

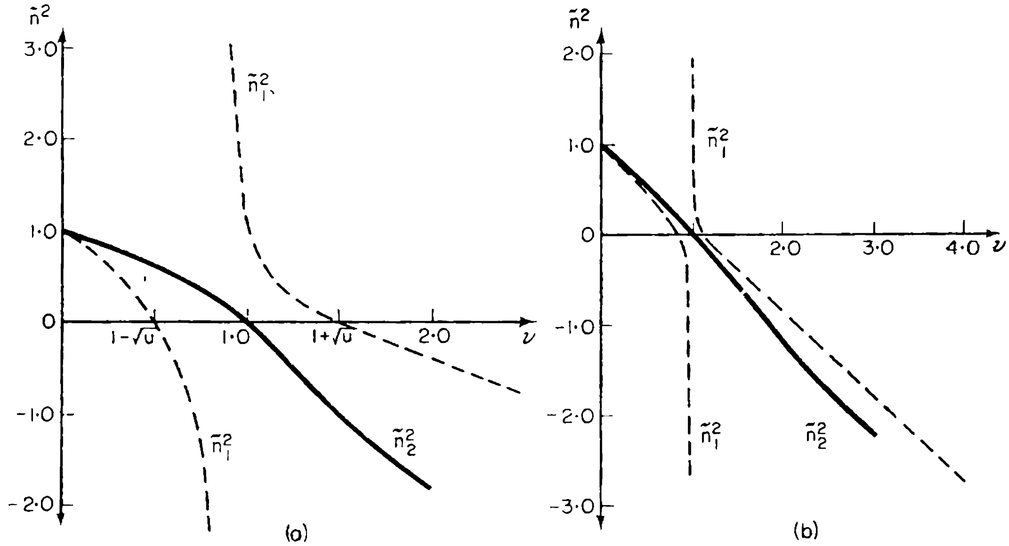

FIG. 11.6. The functions $\tilde{n}^2_{1,2}(v)$ for $\alpha = 45°$.
(a) $u = \frac{1}{4}$ (b) $u = 0{\cdot}01$

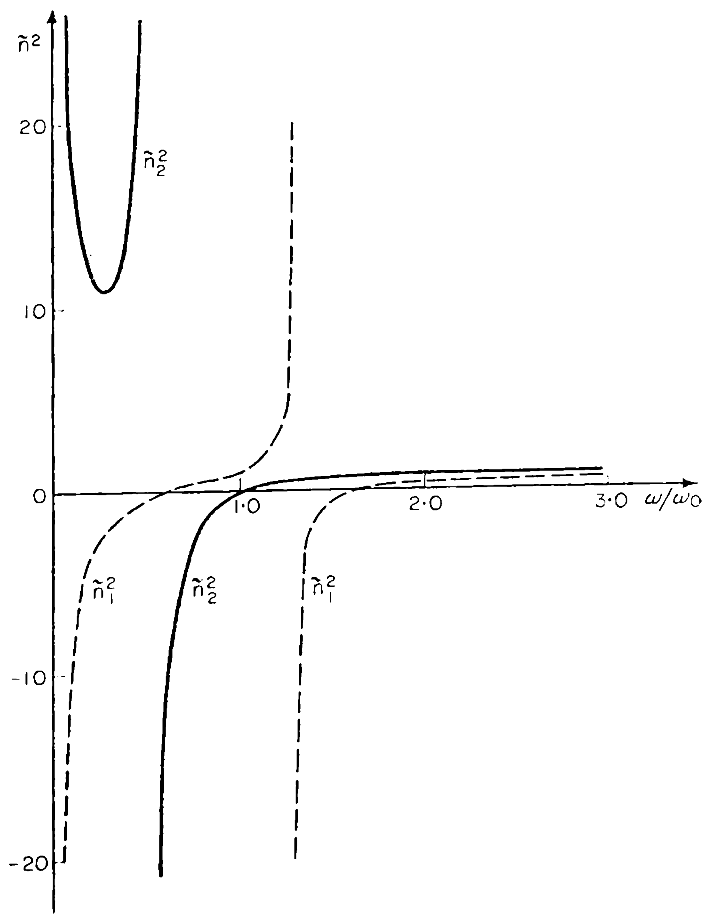

FIG. 11.7. The functions $\tilde{n}^2_{1,2}(\omega/\omega_0)$ for $\omega_H/\omega_0 = 1$, $\alpha = 45°$.

If

$$u = \omega_H^2/\omega^2 > 1, \tag{11.19}$$

the indices $\tilde{n}_2^2$ and $\tilde{n}_1^2$ vanish at the points v_{20} and $v_{10}^{(+)}$ respectively. The root $v_{10}^{(-)} = 1 - \sqrt{u}$ does not exist here for the real values of ω with which we are concerned, since $v_{10}^{(-)} < 0$. Next, if

$$u_L = u \cos^2\alpha \equiv (\omega_H^2/\omega^2)\cos^2\alpha \equiv \omega_L^2/\omega^2 < 1, \tag{11.20}$$

then for $u > 1$ [see (11.19)] neither $\tilde{n}_1^2$ nor $\tilde{n}_2^2$ becomes infinite (for finite v). If, however,

$$u_L = u \cos^2\alpha > 1, \tag{11.21}$$

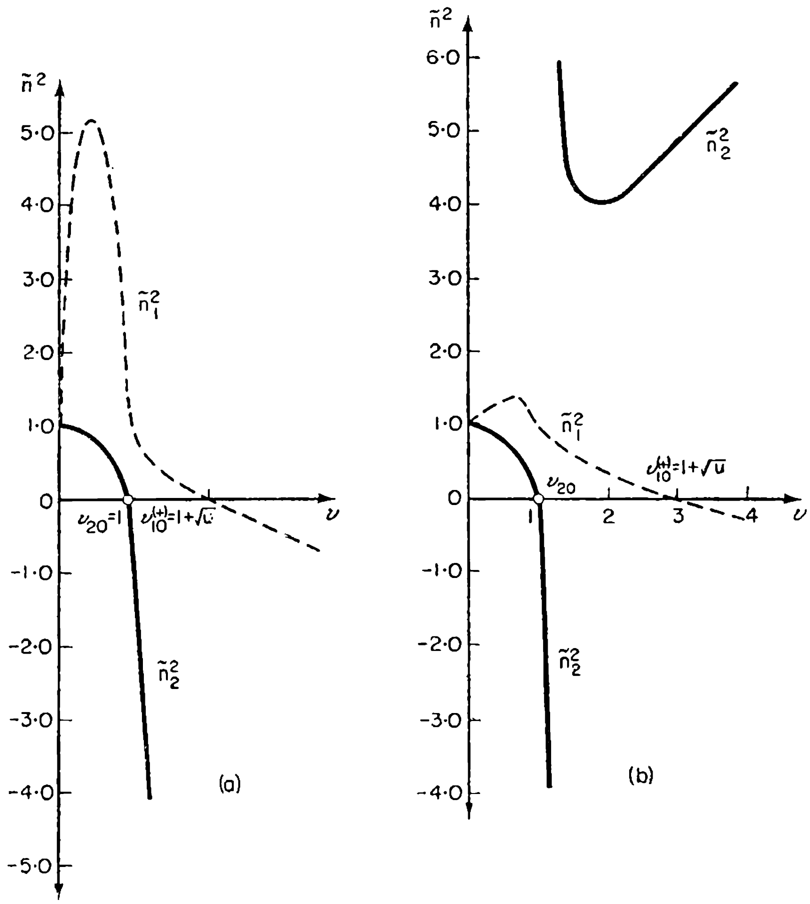

FIG. 11.8. The functions $n_{1,2}^2\ (v)$ for $\alpha = 20°$.
(a) $u = 1{\cdot}08$ (b) $u = 4$

then the index $\tilde{n}_2^2$ becomes infinite at the point

$$v_{2\infty} = (u - 1)/(u \cos^2\alpha - 1). \tag{11.22}$$

This shows that $v_{2\infty} \geqq 1/\cos^2\alpha$, with equality as $u \to \infty$. Figure 11.8 shows $\tilde{n}_1^2$ and $\tilde{n}_2^2$ in the case (11.19) for $u = 1{\cdot}08$ and 4 with $\alpha = 20°$.

The limiting case

$$u \cos^2\alpha = (\omega_H^2/\omega^2)\cos^2\alpha \gg 1, \quad v = \omega_0^2/\omega^2 \gg 1, \quad \omega_0 \gg \omega_H \quad (\text{i.e. } v \gg u), \tag{11.23}$$

which is frequently found in connection with the propagation of whistlers in the ionosphere [53, 54], is also of interest. In the conditions (11.23) we have

$$\tilde{n}_2^2 \approx v/\sqrt{u}\cos\alpha, \quad \tilde{n}_1^2 \approx -v/\sqrt{u}\cos\alpha, \tag{11.24}$$

i.e. wave 1 (extraordinary) cannot be propagated, whereas for wave 2 $\tilde{n}_2^2 = n_2^2 > 0$ and often $n_2^2 \gg 1$. Of course, formula (11.24) can be used only if the high-frequency condition $\omega \gg \Omega_H$ holds in addition to (11.23), so that the effect of the ions can be neglected.

Wave polarisation

The polarisation of the extraordinary and ordinary waves is found from equations (11.2)–(11.4); by (11.4), only one of the equations (11.2a) is independent. The ratio of the components E_y and E_x in waves of the two types is

$$\frac{E_{y\,1,2}}{E_{x\,1,2}} = K_{1,2} = -i\cdot\frac{2\sqrt{u}(1-is-v)\cos\alpha}{u\sin^2\alpha \mp \sqrt{[u^2\sin^4\alpha + 4u(1-is-v)^2\cos^2\alpha]}}, \tag{11.25}$$

where, as before, the upper sign of the root pertains to the ordinary wave (type 2: $n_2, \varkappa_2, E_{x2}, E_{y2}$) and the lower sign to the extraordinary wave (type 1: $n_1, \varkappa_1, E_{x1}, E_{y1}$). The coefficients $K_{1,2}$ are sometimes called polarisation coefficients.

In the absence of absorption

$$\frac{E_{y\,1,2}}{E_{x\,1,2}} = K_{1,2} = \frac{iC}{B-\tilde{n}_{1,2}^2} = -\frac{A-\tilde{n}_{1,2}^2}{iC}$$
$$= -i\frac{2\sqrt{u}(1-v)\cos\alpha}{u\sin^2\alpha \mp \sqrt{[u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha]}}. \tag{11.26}$$

It is clear from (11.25) and (11.26) that, in the general case, waves of either type are elliptically polarised, and in the absence of absorption the axes of the ellipses described by the terminus of the component of the vector $\boldsymbol{E}$ in the xy-plane are parallel to the x and y axes (the axes have been chosen so that the magnetic field lies in the yz-plane). It is also easily seen that

$$K_1 K_2 = 1, \tag{11.27}$$

and $|K_{1,2}|$ is the ratio of the semiaxes of the ellipse.

For $\alpha = 0$, $K_1 = -i$ and $K_2 = +i$, i.e. the two waves are circularly polarised. For $\alpha = \frac{1}{2}\pi$, $K_2 = -i\infty$ and $K_1 = 0$, i.e. $E_{x2} = 0$ and $E_{y1} = 0$, corresponding to linear polarisation of the component of the vector $\boldsymbol{E}$ in the xy-plane. These results are, of course, in agreement with that obtained previously by a direct consideration of longitudinal and transverse propagation.

The field E_z in the waves is given by formula (10.20) or, in the absence of absorption, by (10.26):

$$E_z = -i \cdot \frac{\sqrt{u}\,(v \sin\alpha)}{u-(1-v)-u\,v\cos^2\alpha} E_x + \frac{u\,v\cos\alpha\sin\alpha}{u-(1-v)-u\,v\cos^2\alpha} E_y. \qquad (11.28)$$

Hence it is evident that, in the absence of absorption, the component E_z is in phase with E_y and $\frac{1}{2}\pi$ out of phase with E_x. Thus the vector $\boldsymbol{E}$ describes an ellipse in a plane parallel to the x-axis (the vector $\boldsymbol{k}$, it will be remembered, is along the z-axis, and the field $\boldsymbol{H}^{(0)}$ is in the yz-plane; see Fig. 11.1). The vector $\boldsymbol{D}$ lies in the xy-plane, since $D_z = 0$. Hence the vector $\boldsymbol{D}$ is transverse even in a non-absorbing magnetoactive plasma, but the same is not in general true of the vector $\boldsymbol{E}$. When there is absorption, the vector $\boldsymbol{D} - i \cdot 4\pi\boldsymbol{j}/\omega$ is transverse.

In the cases $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$ the polarisation of the two types of wave is independent of the parameter v, as regards the components $E_{x,y}$. The component E_z is zero for all v when $\alpha = 0$ and (for wave 2) when $\alpha = \frac{1}{2}\pi$; for wave 1 when $\alpha = \frac{1}{2}\pi$, E_z depends on v, and in this case the vector $\boldsymbol{E}$ describes an ellipse in the xz-plane, the ratio of the semiaxes depending on v. For other

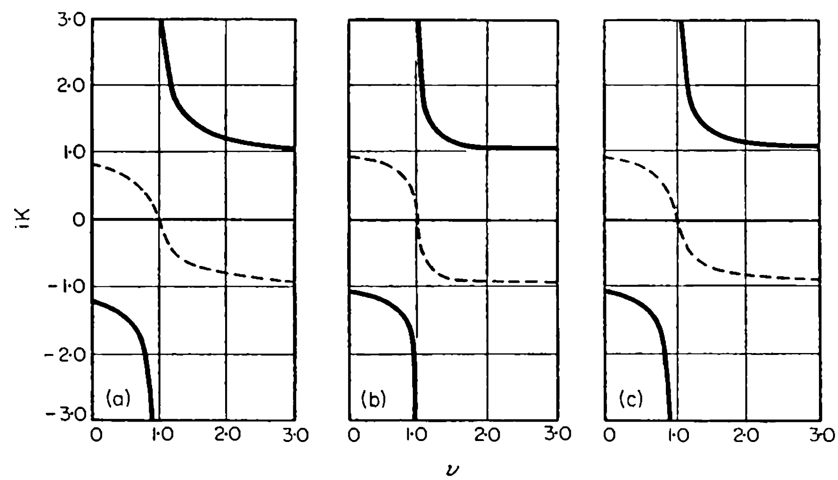

FIG. 11.9. Ratio of semiaxes of the polarisation ellipse iK_1 (broken lines) and iK_2 (continuous lines).
(a) $u = \frac{1}{4}, \alpha = 45°$ (b) $u = 1{\cdot}08, \alpha = 20°$ (c) $u = 4, \alpha = 20°$

values of α the polarisation depends on v; a wave of type 2 with $v = 1$ is always linearly polarised (for $v = 1$, $|K_2| \to \infty$, $E_{x2} = 0$ and the vector $\boldsymbol{E}$ is linearly polarised in the yz-plane). In wave 1 with $v = 1$ the vector $\boldsymbol{E}$ is in general elliptically polarised in the xz-plane (for $v = 1$, $|K_1| \to \infty$, $E_{y1} = 0$). The dependence on v of the quantity $iK_{1,2}$, i.e. the ratio of semiaxes of the ellipses in the xy-plane, is shown in Fig. 11.9 for various values of u and α. The polarisation behaves in a singular manner not only at $v = 1$ but also at the points $v_{1,2\infty}$, where the refractive indices of waves 1 and 2 become infinite [see (11.18) and (11.22)]. It is clear from formula (11.28) that for $v \to v_{1,2\infty}$, $|E_z| \to \infty$ for finite E_x and E_y. In other words, for $v \to v_{1,2\infty}$ the waves are linearly polarised in the direction of the wave vector $\boldsymbol{k}$, i.e. they are longitudinal.

Normal waves. The case of small angles α

Formulae (11.5), (11.25) and (10.20) [or, in the absence of absorption, formulae (11.6), (11.26) and (11.28)] completely determine the nature of waves propagated in a homogeneous ionised gas in a constant magnetic field. Here,

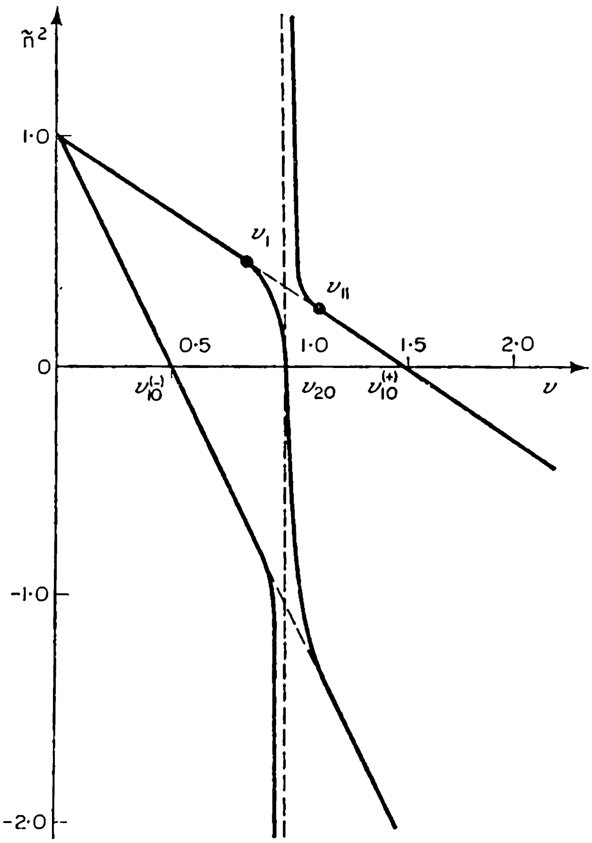

FIG. 11.10a. The functions $\tilde{n}^2_{1,2}(v)$ for $u = \frac{1}{4}$ and $\alpha = 10°$ (continuous lines), $\alpha = 0°$ (broken lines).

as in other anisotropic doubly-refracting media, two types of plane wave can be propagated in any direction defined by the angle α between $\boldsymbol{H}^{(0)}$ and the wave vector $\boldsymbol{k}$. These waves, sometimes called normal waves, are in our case the ordinary wave 2 and the extraordinary wave 1, and differ in their rate of propagation (the phase velocity $v_{\text{ph}} = c/n_{1,2}$), refractive index and polarisation.

In an isotropic medium, as we have seen, there are three high-frequency waves, one being a longitudinal plasma wave. The question thus arises whether

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plasma waves can exist when a magnetic field is present, and what is the nature of the limiting transition from a magnetoactive to an isotropic plasma. These points will be discussed in § 12. Here we shall consider a somewhat different problem, that of the limiting transition from propagation at an angle $\alpha \neq 0$ to longitudinal propagation, where $\alpha = 0$ (we shall see in § 12 that the two problems are in fact closely related). The problem of the limit $\alpha \to 0$ arises

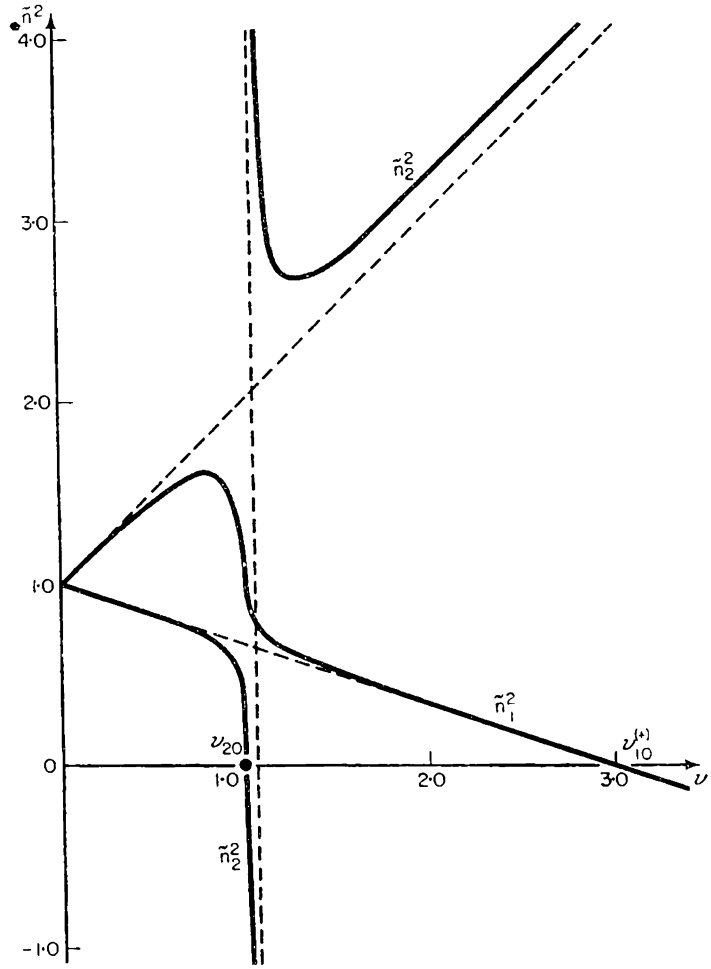

FIG. 11.10b. The same for $u = 4$.

because when $\alpha = 0$ for the ordinary wave $\tilde{n}_2^2 \equiv \tilde{n}_-^2 = 0$ for $v = v_{10}^{(+)} = 1 + \sqrt{u}$, while for the extraordinary wave $\tilde{n}_1^2 \equiv \tilde{n}_+^2 = 0$ for $v = v_{10}^{(-)} = 1 - \sqrt{u}$ [see 11.9)]. For $\alpha \neq 0$, however, as already mentioned, $\tilde{n}_2^2 = 0$ at $v_{20} = 1$, and $\tilde{n}_1^2 = 0$ at $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$. The nature of the singularity when $\alpha \to 0$ becomes clear if we examine the curves of $\tilde{n}_{1,2}^2(v)$ for small angles α. From Fig. 11.10, which shows the case $\alpha = 10°$ and $u = \frac{1}{4}$ and 4, or from a more detailed analysis, it is seen that when $\alpha \to 0$ the curves of $\tilde{n}_{1,2}^2(v)$ become the straight lines (11.9) and the straight line $v = 1$ (see also Fig. 11.11). This also explains the fact that when $\alpha = 0$ the expressions (11.9) for $\tilde{n}^2_{1,2}$ do not become infinite, whereas according to the general formula (11.18) for $\alpha = 0$ we have $v_{1\infty} = 1$ (i.e. there is a point where the quantity $\tilde{n}^2_1$ is infinite in the absence of absorption). The reason is evidently that formula (11.8) has been derived from (11.6), which for $\alpha \to 0$ becomes (11.9) and the line $v = 1$, on which $\tilde{n}^2_1$ takes infinite values. For an inhomogeneous medium this non-triviality of the limiting transition to longitudinal propagation has interesting consequences (see § 28).

The limit $\alpha \to 0$ also has certain special properties as regards the dependence of the polarisation on the parameter v. For small α the polarisation of the waves is almost circular, and as we pass through the point $v = 1$ the direction of rotation is reversed (Fig. 11.12). Hence for $v \approx 1$ the wave of type 2 for $v > 1$ can become the wave of type 1 for $v < 1$, as shown by Fig. 11.10; that is,

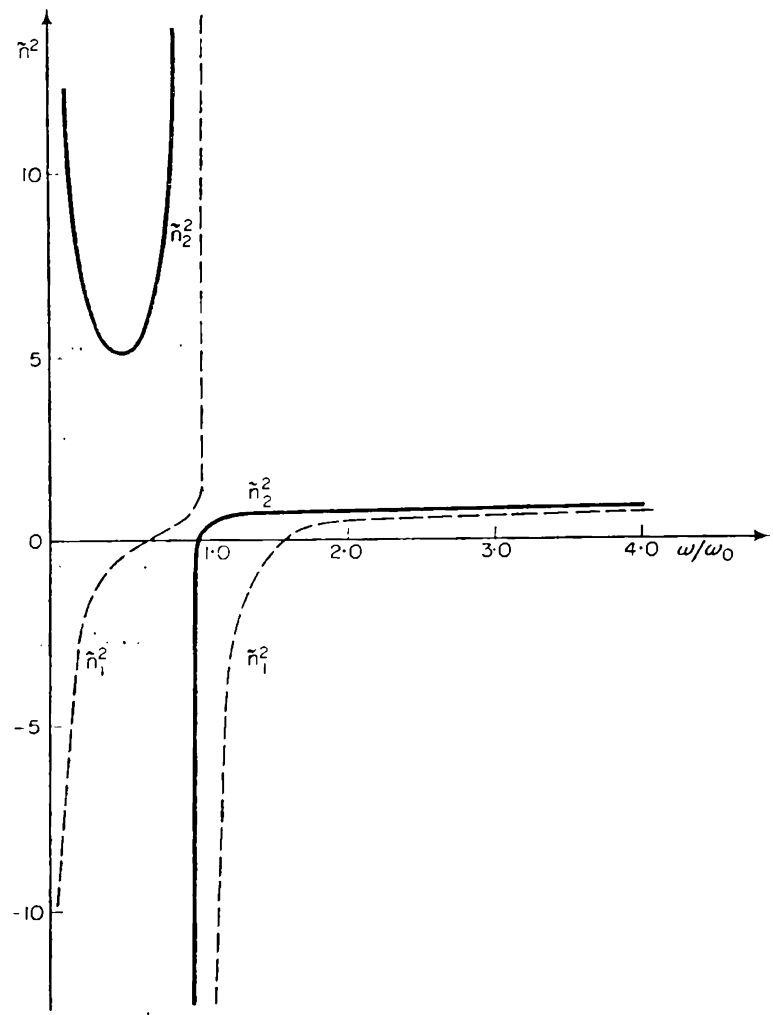

FIG. 11.11. The functions $\tilde{n}^2_{1,2}$ (ω/ω_0) for $\omega_H/\omega_0 = 1, \alpha = 10°$.

as $\alpha \to 0$ and $v \to 1$ the polarisation of wave 2 for $v < 1$ is the same as that of wave 1 for $v > 1$.†

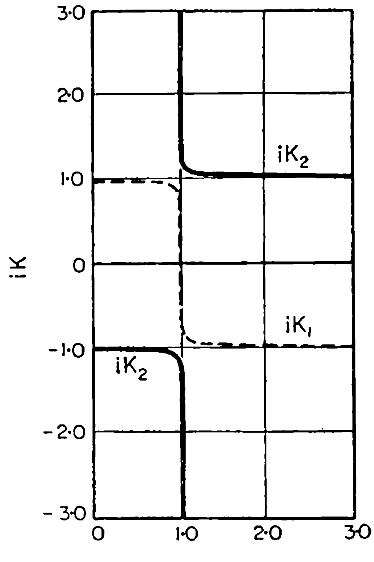

FIG. 11.12. Ratio of semiaxes $iK_{1,2}$ for $\alpha = 10°$ and $u = \frac{1}{4}$.

It may also be noted that both waves are elliptically polarised even in the limit as $v \to 0$:

$$K_{1,2}(v = 0) = -i \frac{2\sqrt{u}\,(\cos\alpha)}{u \sin^2\alpha \mp \sqrt{(u^2 \sin^4\alpha + 4u \cos^2\alpha)}}. \tag{11.29}$$

Evidently for $v = 4\pi e^2 N/m\omega^2 = 0$, i.e. in the absence of electrons (in a vacuum or, practically, in air), there can be no double refraction, and correspondingly (11.6) gives $\tilde{n}^2_{1,2} = 1$ for $v = 0$. Of course there is no contradiction here, since in a vacuum, waves of any polarisation, including elliptical, may be taken as normal waves. For an inhomogeneous medium there arises the problem of how the polarisation of the wave incident from a vacuum varies with distance as it enters the medium (when $v \to 0$), since the wave can have any polarisation. This will be discussed in § 26.

The allowance for absorption

The properties of the curves of $\tilde{n}^2_{1,2}(v, u, \alpha)$ and $K_{1,2}(v, u, \alpha)$ have been discussed above only for the case where absorption is absent. The allowance for absorption involves great complications, and we shall not analyse the

† In considering the limit $\alpha \to 0$ we label the waves 1 or 2 according to their behaviour for $\alpha \neq 0$. The curves of $\tilde{n}^2$ for waves 1 and 2 (or $\pm$) for $\alpha = 0$ are seen from Fig. 11.10 to consist of parts of the curves of $\tilde{n}^2_{1,2}$ for these waves with $\alpha \neq 0$. Thus as $\alpha \to 0$ the waves are to some extent renamed.

problem in full detail (see [55–60]†), but merely consider the most important particular cases and some actual examples.

For longitudinal propagation ($\alpha = 0$) formulae for $(n - i\varkappa)^2_{1,2}$ have already been given which allow for absorption [see (11.8), (11.12)]. The real and imaginary parts separately are

$$\left.\begin{aligned} \varepsilon_{1,2} \equiv n^2_{1,2} - \varkappa^2_{1,2} \equiv n^2_{\mp} - \varkappa^2_{\mp} &= 1 - \frac{v\,(1 \pm \sqrt{u})}{(1 \pm \sqrt{u})^2 + s^2} \\ &= 1 - \frac{\omega_0^2\,(\omega \pm \omega_H)/\omega}{(\omega \pm \omega_H)^2 + \nu^2_{\text{eff}}}, \\ 4\pi\,\sigma_{1,2}/\omega \equiv 2n_{1,2}\,\varkappa_{1,2} \equiv 2n_{\mp}\,\varkappa_{\mp} &= \frac{s\,v}{(1 \pm \sqrt{u})^2 + s^2} \\ &= \frac{\omega_0^2\,\nu_{\text{eff}}/\omega}{(\omega \pm \omega_H)^2 + \nu^2_{\text{eff}}}, \end{aligned}\right\} \tag{11.30}$$

where the upper sign + in front of ω_H corresponds to wave 2(−) and the minus sign to wave 1(+).

The quantities $\varepsilon_{1,2}$ and $\sigma_{1,2}$ are used here in analogy with formulae (7.11) for the isotropic case, where ε and σ are the permittivity and conductivity. In an anisotropic medium these are replaced by the tensors ε_{ik} and σ_{ik}, or $\varepsilon'_{ik} = \varepsilon_{ik} - i\cdot 4\pi\sigma_{ik}/\omega$, and $\varepsilon_{1,2}$ and $\sigma_{1,2}$ are only notations for $n^2_{1,2} - \varkappa^2_{1,2}$ and $2n_{1,2}\varkappa_{1,2}$ respectively. Their use is convenient since $n_{1,2}$ and $\varkappa_{1,2}$ are given in terms of $\varepsilon_{1,2}$ and $\sigma_{1,2}$ in the same way as n and $\varkappa$ in terms of ε and σ, i.e. by formulae (7.12).

In transverse propagation ($\alpha = \frac{1}{2}\pi$) we have for the ordinary wave 2

$$\left.\begin{aligned} &(n - i\varkappa)^2_2 = 1 - v/(1 - is), \\ &\varepsilon_2 \equiv n_2^2 - \varkappa_2^2 = 1 - v/(1 + s^2) = 1 - \omega_0^2/(\omega^2 + \nu_{\text{eff}}^2), \\ &4\pi\,\sigma_2/\omega \equiv 2n_2\,\varkappa_2 = s\,v/(1 + s^2) = \omega_0^2\,\nu_{\text{eff}}/\omega(\omega^2 + \nu_{\text{eff}}^2). \end{aligned}\right\} \tag{11.31}$$

In this case, as when absorption is absent, the ordinary wave 2 behaves exactly as it does when $\boldsymbol{H}^{(0)} = 0$ [see (3.7)].

For the extraordinary wave 1 we have

$$\left.\begin{aligned} (n - i\varkappa)^2_1 &= 1 - \frac{v\,(1 - is - v)}{(1 - is)\,(1 - is - v) - u}, \\ \varepsilon_1 \equiv n_1^2 - \varkappa_1^2 &= 1 - \frac{v\,[(1 - v)^2 - u\,(1 - v) + s^2]}{(1 - v - u - s^2)^2 + s^2\,(2 - v)^2}, \\ 4\pi\,\sigma_1/\omega \equiv 2n_1\,\varkappa_1 &= \frac{s\,v\,[(1 - v)^2 + u + s^2]}{(1 - v - u - s^2)^2 + s^2\,(2 - v)^2}. \end{aligned}\right\} \tag{11.32}$$

† Various authors have studied the curves of $n_{1,2}(v, u, s, \alpha)$, $\varkappa_{1,2}(v, u, s, \alpha)$ and $K_{1,2}(v, u, s, \alpha)$ with allowance for the Lorentz polarisation term, i.e. taking the expression (3.14) for the effective field. As stated in § 3, it is not necessary to take account of this term in a plasma. In this chapter and elsewhere in the book we therefore use the relation (3.12) $\boldsymbol{E}_{\text{eff}} = \boldsymbol{E}$. The curves of n, $\varkappa$ and K with the Lorentz term often (and especially for $u > 1$) differ quite considerably from those without it.

The cumbersomeness of this formula (11.32) for the simple particular case $\alpha = \frac{1}{2}\pi$ gives an idea of the complexity of the corresponding expressions for an arbitrary angle α (it is, of course, not difficult to derive from (11.5) expressions for $\varepsilon_{1,2}$ and $\sigma_{1,2}$ in the general case). It should also be noted that in making allowance for absorption [formula (11.5)] we have used only the elementary theory. If we start from the exact expressions (10.32) for ε_{ik} and σ_{ik} the formulae become considerably more involved. Fortunately it is not usually necessary to use the kinetic theory [formula (10.32)], since the various limiting cases are generally under consideration. For example, if the inequalities

$$(\omega \pm \omega_H)^2 \gg \nu_{\text{eff}}^2 \tag{11.33}$$

or

$$(\omega \pm \omega_H)^2 \ll \nu_{\text{eff}}^2 \tag{11.34}$$

are satisfied for both signs of ω_H [of course, if $(\omega - \omega_H)^2 \gg \nu_{\text{eff}}^2$, $(\omega + \omega_H)^2 \gg \nu_{\text{eff}}^2$ also], then the formulae of the elementary theory with ν_{eff} suitably chosen are valid. At radio frequencies the condition (11.33) is important, since it is usually satisfied in the F layer of the ionosphere and in the solar corona.

When (11.33) holds, the formulae for $n_{1,2}$ and $\varkappa_{1,2}$ are generally much simplified. For example, in (11.31) we can then omit ν_{eff}^2 in the denominator, and formulae (11.32) become

$$\left.\begin{aligned} \varepsilon_1 \equiv n_1^2 - \varkappa_1^2 &= 1 - \frac{v(1-v)}{1-v-u} = 1 - \frac{\omega_0^2(\omega^2 - \omega_0^2)}{\omega^2(\omega^2 - \omega_0^2 - \omega_H^2)}, \\ 4\pi\sigma_1/\omega \equiv 2n_1\varkappa_1 &= \frac{sv[(1-v)^2 + u]}{(1-v-u)^2} = \frac{\nu_{\text{eff}}(\omega_0^2/\omega^2)[(\omega^2 - \omega_0^2)^2 + \omega_H^2\omega^2]}{\omega(\omega^2 - \omega_0^2 - \omega_H^2)^2}. \end{aligned}\right\} \tag{11.35}$$

It must be borne in mind that the condition (11.33) alone is not sufficient for formulae (11.35) to be valid, since in the absence of absorption the denominator in (11.35) becomes infinite at $v_{1\infty} = 1 - u$ [see (11.18) for $\alpha = \frac{1}{2}\pi$]. It is therefore clear that in the neighbourhood of the point $v_{1\infty}$ the absorption is always important, and the quantity s^2 in the denominator of formulae (11.32) cannot be neglected even if the condition (11.33) applies. Away from the point $v_{1\infty}$, the inequality (11.33) is sufficient to give (11.35) from (11.32).

Quasilongitudinal and quasitransverse propagation

The complexity of the expressions for $n_{1,2}$ and $\varkappa_{1,2}$ for an arbitrary angle α means that the possibility of approximating the exact values by formulae such as (11.30) and (11.35) is of practical importance. For example, if

$$\left.\begin{aligned} u_T^2/4u_L &\equiv u\sin^4\alpha/4\cos^2\alpha \\ &= \omega_H^2\sin^4\alpha/4\omega^2\cos^2\alpha \ll (1-v)^2 + s^2, \\ |1 - \sqrt{u}\cos\alpha| &\gg (1+v)\,u\sin^2\alpha/2[(1-v)^2 + s^2], \\ u_T = u\sin^2\alpha, &\quad u_L = u\cos^2\alpha, \end{aligned}\right\} \tag{11.36}$$

then it can be shown from (11.5) (see also [61]) that we have approximately

$$\left.\begin{aligned} (n - i\varkappa)^2_{1,2} &= 1 - v/(1 \pm \sqrt{u}\cos\alpha - is), \\ \varepsilon_{1,2} \equiv n^2_{1,2} - \varkappa^2_{1,2} &= 1 - \frac{\omega_0^2(\omega \pm \omega_L)/\omega}{(\omega \pm \omega_L)^2 + \nu^2_{\mathrm{eff}}}, \\ 4\pi\sigma_{1,2}/\omega = 2n_{1,2}\varkappa_{1,2} &= \frac{\omega_0^2 \nu_{\mathrm{eff}}/\omega}{(\omega \pm \omega_L)^2 + \nu^2_{\mathrm{eff}}}, \end{aligned}\right\} \tag{11.37}$$

where

$$\omega_L = \omega_H \cos\alpha = \sqrt{u}\,(\omega\cos\alpha) = \sqrt{u_L}\,\omega. \tag{11.38}$$

Formulae (11.38) differ from (11.30) for longitudinal propagation only in that ω_H is replaced by ω_L. Hence the case represented by formulae (11.36) and (11.37) is said to be "quasilongitudinal".

When the inequalities

$$\left.\begin{aligned} u_T^2/4u_L = u\sin^4\alpha/4\cos^2\alpha \gg (1-v)^2 + s^2, \quad \tan^2\alpha \gg 1 + v, \\ u\sin^2\alpha \gg [(1-v)^2 + s^2]\cot^2\alpha \end{aligned}\right\} \tag{11.39}$$

hold, we have "quasitransverse" propagation; in this case formulae (11.31) and (11.32) or (11.35) for transverse propagation are approximately valid, but with u replaced by u_T or ω by ω_T, where

$$\left.\begin{aligned} u_T &= u\sin^2\alpha, \\ \omega_T &= \omega_H \sin\alpha = \sqrt{u_T}\,\omega. \end{aligned}\right\} \tag{11.40}$$

The second condition (11.39) pertains only to the ordinary wave, and the third condition only to the extraordinary wave. It should be emphasised that the quasilongitudinal and quasitransverse conditions (11.36) and (11.39) place some restriction not only on the angle α but also on the parameters v, u and s. These inequalities are sufficient but not necessary conditions for the quasilongitudinal or quasitransverse approximation to be applicable. The necessary conditions are complicated, and if the sufficient conditions given above are not satisfied it is better to use formula (11.5) directly.

The critical collision frequency. Graphs of $n_{1,2}(v)$ and $\varkappa_{1,2}(v)$

The analysis of the expression (11.5) for arbitrary v, u, s and α is achieved by the construction of appropriate graphs. The only general remark which we shall make here relates to the introduction of the critical collision frequency $\nu_{\mathrm{eff,cr}}$ or the critical parameter

$$\begin{aligned} s_{\mathrm{cr}} &= \nu_{\mathrm{eff,cr}}/\omega = \omega_T^2/2\omega_L\omega \\ &= \omega_H \sin^2\alpha/2\omega\cos\alpha = \sqrt{u}\,(\sin^2\alpha/2\cos\alpha) = u_T/2\sqrt{u_L}. \end{aligned} \tag{11.41}$$

If angles for which $\cos\alpha < 0$ are included, then in (11.41) $\cos\alpha$ must be replaced by $|\cos\alpha|$. The significance of the critical parameter s_{cr} is that for

$s = s_{cr}$ and $v = 1$ the radicand in (11.5) and (11.25) is zero, and so $(n - i\varkappa)_1 = (n - i\varkappa)_2$ and $K_1 = K_2 = -1$, i.e. at this point (for these values of s and v) the medium ceases to be doubly refracting.†

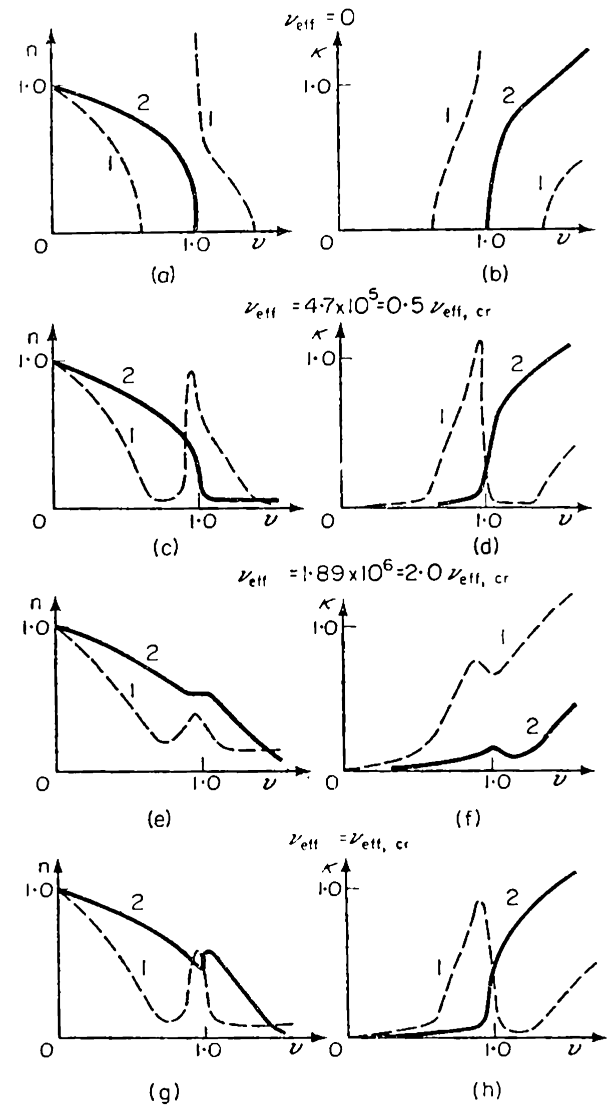

Fig. 11.13. Indices of refraction and absorption for $\lambda = 80$ m and various values of ν_{eff}. The values of α and $u = \omega_H{}^2/\omega^2$ are given in the text. The broken lines refer to waves of type 1, and the continuous ones to those of type 2.

When $\nu_{eff} \ll \nu_{eff,cr}$ the curves of $n_{1,2}(v)$ and $\varkappa_{1,2}(v)$ are similar to those which hold in the absence of absorption, but if $\nu_{eff} \gtrsim \nu_{eff,cr}$ there is a considerable change in the form of the functions $n_{1,2}$ and $\varkappa_{1,2}$. In the Earth's

† It may be noted that in such cases, where there are multiple roots with the same polarisation, we cannot, even in a homogeous medium, restrict ourselves to solutions of the type $E = E_0 e^{i(\omega t - k\cdot r)}$ [351].

ionosphere the latter case can be important only in the lower regions (the D and E layers), except at high latitudes.

To illustrate the behaviour of the curves of $n_{1,2}$ and $\varkappa_{1,2}$ we give a number of graphs from [57] for various particular cases (Figs. 11.13–11.15). In all the examples $H^{(0)}\cos\alpha = 0{\cdot}447$ and $H^{(0)}\sin\alpha = 0{\cdot}218$, i.e. $H^{(0)} = 0{\cdot}497$ G, $\omega_H = 8{\cdot}8 \times 10^6\ \mathrm{sec}^{-1}$ and $\alpha = 25°\ 50'$. The value of $\nu_{\mathrm{eff,cr}}$ is then $9{\cdot}5 \times 10^5\ \mathrm{sec}^{-1}$. Figs. 11.13–15a, b give the values of $n_{1,2}$ and $\varkappa_{1,2}$ for $\nu_{\mathrm{eff}} = 0$ for waves of length $\lambda_0 = 80$ m ($\omega = 2{\cdot}36 \times 10^7\ \mathrm{sec}^{-1}$, $u = 0{\cdot}14$), 225 m ($\omega = 0{\cdot}838 \times 10^7\ \mathrm{sec}^{-1}$, $u = 1{\cdot}10$) and 490 m ($\omega = 0{\cdot}386 \times 10^7\ \mathrm{sec}^{-1}$, $u = 5{\cdot}2$). The difference between these graphs and those given in Fig. 11.6, for example, is not only that $u = \omega_H^2/\omega^2$

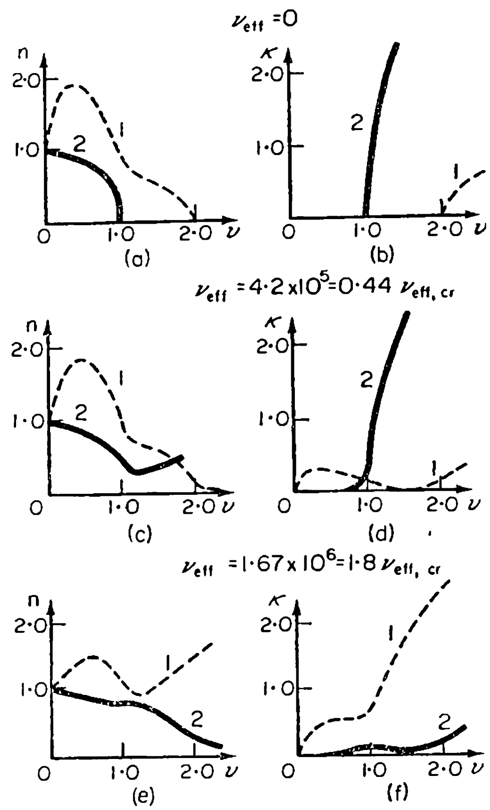

FIG. 11.14. Indices of refraction and absorption for $\lambda = 225$ m and various values of ν_{eff}. The values of α and u are given in the text.

has a different value but also that $n_{1,2}$ and $\varkappa_{1,2}$ are shown in place of $\tilde{n}^2_{1,2}$, and by formula (11.5) we always have $n_{1,2} > 0$, $\varkappa_{1,2} > 0$. Thus the values of $\varkappa^2_{1,2}$ in Figs. 11.13–11.15 correspond to $-\tilde{n}^2_{1,2}$ in Fig. 11.6 and similar figures. Figs. 11.13–11.15c, d give the values of $n_{1,2}$ and $\varkappa_{1,2}$ for the same waves as in Figs. 11.13–11.15a, b, but with $\nu_{\mathrm{eff}} = 4{\cdot}7 \times 10^5 = 0{\cdot}50\ \nu_{\mathrm{eff.cr}}$ for $\lambda_0 = 80$ m, $\nu_{\mathrm{eff}} = 4{\cdot}2 \times 10^5 = 0{\cdot}44\ \nu_{\mathrm{eff,cr}}$ for $\lambda_0 = 225$ m, and $\nu_{\mathrm{eff}} = 3{\cdot}9 \times 10^5 = 0{\cdot}41\ \nu_{\mathrm{eff,cr}}$ for $\lambda_0 = 490$ m.

Figs. 11.13–11.15c,f give similar curves with $\nu_{eff} = 2{\cdot}0\,\nu_{eff,cr}$ for $\lambda_0 = 80$ and 490 m, and $\nu_{eff} = 1{\cdot}8\,\nu_{eff,cr}$ for $\lambda_0 = 225$ m. In addition, Fig. 11.13g,h gives curves of $n_{1,2}(v)$ and $\varkappa_{1,2}(v)$ for $\lambda_0 = 80$ m and $\nu_{eff} = \nu_{eff,cr}$.

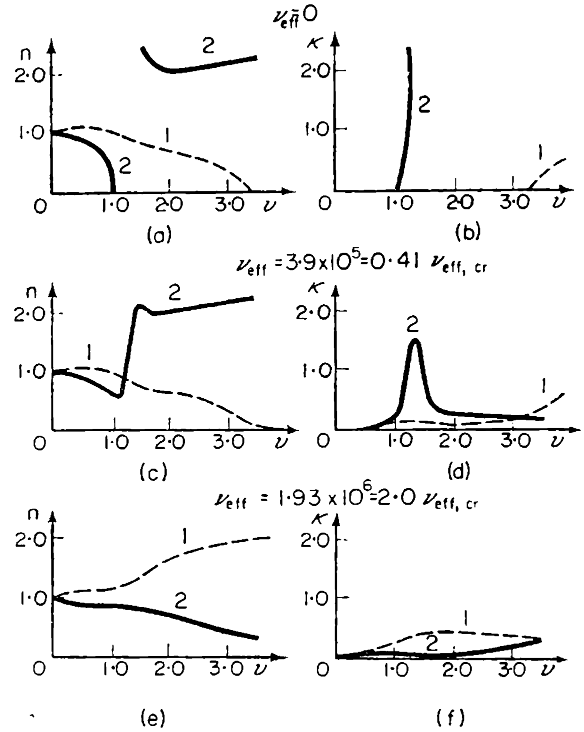

FIG. 11.15. Indices of refraction and absorption for $\lambda = 490$ m and various values of ν_{eff}. The values of α and u are given in the text.

It may be noted that the absorption of the waves is large even for ν_{eff} as small as about $0{\cdot}5\,\nu_{eff,cr}$. In a homogeneous medium the wave amplitude decreases as $e^{-\omega\varkappa_{1,2}z/c} = e^{-2\pi\varkappa_{1,2}z/\lambda_0}$, and for $\varkappa_{1,2} = 0{\cdot}1$ the field attenuation factor is $e^{2\pi} = 540$ at a distance of $10\lambda_0$. Figs. 11.13–11.15 give values of $\varkappa_{1,2} \sim 0{\cdot}1$; an important fact is that the index $\varkappa_{1,2}$ is large where in the absence of absorption $\varkappa_{1,2} = 0$.

In the F layer of the ionosphere, where $\nu_{eff} \sim 10^3$ to $10^4\,\text{sec}^{-1}$, the above values of $H^{(0)}$ and α give $\nu_{eff} \sim 10^{-2}$ to $10^{-3}\,\nu_{eff,cr}$. Evidently the curves of $n_{1,2}$ and $\varkappa_{1,2}$ in this case will be very similar to those shown in Fig. 11.13a,b.

Absorption also affects the polarisation of both waves and in particular has the result that the axes of the ellipses described by the component of $\boldsymbol{E}$ in the xy-plane do not coincide with the x and y axes. For slight absorption the polarisation is, of course, almost the same as for no absorption; the greatest deviation is observed near $v = v_{20} = 1$, where the polarisation varies rapidly with v [see Figs. 11.9 and 11.12 and formula (11.25)].

The complexity of the general expressions (11.5) for $(n - i\varkappa)^2_{1,2}$ and (11.25) for $K_{1,2} = E_{y1,2}/E_{x1,2}$ has the result that it is sometimes convenient to use nomograms to determine n, $\varkappa$ and K. These are given in [58].

The effect of ions

In the lower layers of the ionosphere, especially the D layer, the propagation of radio waves may be affected not only by electrons but also by ions. In the absence of the Earth's magnetic field the allowance for the effect of ions would be as in § 3, where it was shown that, neglecting absorption, ions of density N_i, mass M and unit charge contribute to the expressions for ε in the same manner as electrons of density $N_{\text{eff}} = mN_i/M$ [see (3.4)]. When absorption is taken into account, the position is somewhat more complex, since the effective collision frequencies are different for electrons and ions. Electrons and ions are also not equivalent when the Earth's magnetic field is allowed for, since the effect of this field on electrons and ions is given respectively by the values of the gyration frequencies $\omega_H = |e|H^{(0)}/mc$ and $\Omega_H = eH^{(0)}/Mc = m\omega_H/M$(for O_2^+ ions with $H^{(0)} \sim 0{\cdot}5$ oersted, $\Omega_H \sim 150\,\text{sec}^{-1}$ and $\lambda_H = 2\pi c/\Omega_H \sim 10{,}000$ km). Hence, in the high-frequency case, where the frequency of the radio waves $\omega \gg \Omega_H$ [see (10.5)], the effect of the magnetic field on the ions may be neglected. This fact, which has already been mentioned on various occasions, makes it possible to ignore the ions provided that their density N_i is comparable with the electron density. Usually this case has been assumed to hold, and sometimes without explicit statement to that effect. If, however, $N_i \gg N$, the ions may exert a considerable influence even for high-frequency waves. The expression for the tensor ε'_{kl} is

$$\varepsilon'_{kl} = \varepsilon'^{(e)}_{kl} + \varepsilon'_{(i)}\,\delta_{kl} \tag{11.42}$$

where $\varepsilon'^{(e)}_{ik}$ is the tensor ε'_{ik} in the absence of ions [see, for instance, (10.12)] and

$$\left.\begin{aligned} \varepsilon'_{(i)} &= \varepsilon_{(i)} - i\cdot 4\pi\,\sigma_{(i)}/\omega \\ &= 1 - 4\pi\,e^2 N_i/M\omega\,(\omega - i\,\nu^{(i)}_{\text{eff}}) = 1 - v_i/(1 - is_i)\,, \\ v_i &= 4\pi\,e^2 N_i/M\omega^2, \quad s_i = \nu^{(i)}_{\text{eff}}/\omega\,, \end{aligned}\right\} \tag{11.43}$$

where $\nu^{(i)}_{\text{eff}}$ is the effective frequency of collisions of ions of mass M (assuming for simplicity that all the ions are singly charged and have the same mass M) with all particles present in the medium (electrons, ions and molecules). The expression (11.43) is evidently just the complex permittivity of an isotropic plasma when the motion of the ions alone is taken into account.

From (11.43) we can find the value of $(n - i\varkappa)^2_{1,2}$ in the same way as when the motion of the ions is neglected. Instead of (11.5) we obtain

$$(n - i\varkappa)^2_{1,2} = \varepsilon'_{(i)}\left\{1 - \frac{2\,(v/\varepsilon'_{(i)})\,[1 - (v/\varepsilon'_{(i)}) - is]}{2\,(1 - is)\,[1 - (v/\varepsilon'_{(i)}) - is] - u\sin^2\alpha \pm \sqrt{\{u^2\sin^4\alpha + 4u\,[1 - (v/\varepsilon'_{(i)}) - is]^2\cos^2\alpha\}}}\right\}, \tag{11.44}$$

where v, u and s have the same significance as when ions are absent.

A comparison of (11.44) and (11.5) shows that when ions are taken into account the effect is to replace $v = 4\pi e^2 N/m\omega^2 = \omega_0^2/\omega^2$ in (11.5) by $v/\varepsilon'_{(i)}$ and to multiply the whole expression by $\varepsilon'_{(i)}$.

In the particular cases where $H^{(0)} = 0$ or where the propagation is longitudinal or transverse, we have

$$\left.\begin{aligned}
&\text{for} \quad H^{(0)} = 0, \quad (n - i\varkappa)^2_{1,2} \equiv (n - i\varkappa)^2_0 = \varepsilon'_{(i)} - v/(1 - is) \\
&\qquad\qquad = 1 - \frac{v}{1 - is} - \frac{v_i}{1 - is_i}; \\
&\text{for} \quad \alpha = 0, \quad (n - i\varkappa)^2_{1,2} = \varepsilon'_{(i)} - v/(1 \pm \sqrt{u} - is); \\
&\text{for} \quad \alpha = \tfrac{1}{2}\pi, \quad (n - i\varkappa)^2_1 = \varepsilon'_{(i)} - \frac{v\,[1 - (v/\varepsilon'_{(i)}) - is]}{(1 - is)\,[1 - (v/\varepsilon'_{(i)}) - is] - u}, \\
&\qquad\qquad (n - i\varkappa)^2_2 = (n - i\varkappa)^2_0.
\end{aligned}\right\} \tag{11.45}$$

A detailed analysis of formula (11.44) is given in [57]. Here we shall merely give (Fig. 11.16) curves of $\tilde{n}^2_{1,2} = (n - i\varkappa)^2_{1,2}$ in the absence of absorption for the same case as in Figs. 11.13 and 11.14 ($\lambda_0 = 80$ and $225\,\text{m}$, $\omega_H = 8{\cdot}8 \times 10^6\,\text{sec}^{-1}$, $\alpha = 25°\,50'$) with $\xi = v_i/v = N_i m/NM = 0$, 1 and ∞ ($\xi = 0$ is the case of

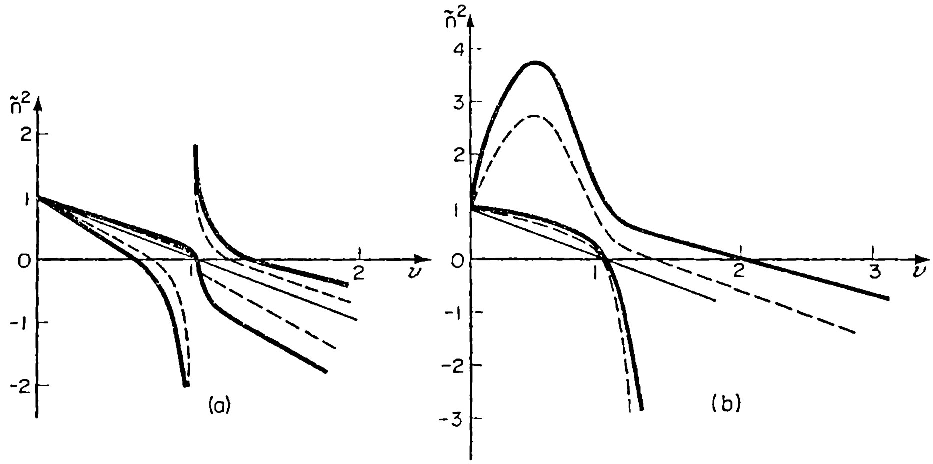

FIG. 11.16. The square of the refractive index for a mixture of electrons and ions, with various values of $\xi = N_i m/NM$, where N and N_i are the electron and ion densities and m and M their masses: (a) $\lambda = 80$ m, (b) $\lambda = 225$ m. The thick line is for $\xi = 0$, the thin line for $\xi = \infty$, and the broken line for $\xi = 1$.

electrons alone, $\xi = \infty$ that of ions alone, and $\xi = 1$ that where electrons and ions make the same contribution to ε in the absence of the magnetic field).

Absorption and emission of electromagnetic waves by a magnetoactive plasma

To conclude, we shall discuss one point which is important in understanding the problem of absorption and emission of electromagnetic waves by a magnetoactive plasma. It is seen from the general expression (11.5) for $(n - i\varkappa)^2_{1,2}$

that for $s^2 = \nu_{\text{eff}}^2/\omega^2 \ll 1$ the absorption is greatest near resonance, i.e. in the region where $\tilde{n}_{1,2}^2$ increases without limit in the absence of absorption; this is especially clear for the particular cases represented by formulae (10.30) and (11.32). In other words, when $s^2 = \nu_{\text{eff}}^2/\omega^2$ is sufficiently small the plasma absorbs appreciably only near the points $v_{1,2\infty}$ [see (11.18) and (11.22)].

For $\alpha = 0$ the point $v_{1\infty}$ corresponds to the frequency $\omega_\infty = \omega_H$, but for all other values of α we have $\omega_\infty \neq \omega_H$ and, for example, at $\alpha = \frac{1}{2}\pi$ we have $\omega_\infty = \sqrt{(\omega_0^2 + \omega_H^2)}$ [see also formula (12.3) below]. If a system absorbs waves at a certain frequency, then it emits waves of that frequency also; in the particular case of thermodynamic equilibrium, this result follows immediately from Kirchhoff's law. Thus a hot magnetoactive plasma must emit waves mainly near the frequency ω_∞. Now a non-relativistic electron in a magnetic field revolves with frequency $\omega_H = |e|H^{(0)}/mc$, and in a vacuum it emits waves of this frequency only. In a rarefied plasma, where $\omega_H \gg \nu_{\text{eff}}$, the electrons will revolve for the greater part of the time with frequency ω_H, as in a vacuum, and must therefore, it would seem, continue to emit only the frequency ω_H or frequencies very close to it.

The resulting paradox is resolved [62] by the fact that the medium affects the emission from a particle moving in it, and in some cases it alters the whole nature of the emission. The simplest example is that of a harmonic oscillator of frequency ω placed in an isotropic plasma with $\varepsilon = \tilde{n}^2 = 1 - \omega_0^2/\omega^2 < 0$. Since the field is then damped, no emission with $\omega < \omega_0$ is possible. For an electron revolving in a magnetoactive plasma the situation is less simple. However, the general formulae (see [63]) show that in the non-relativistic limit the electron does not emit in these conditions when collisions are absent. When collisions occur, the electron of course emits even in a magnetic field, but it emits bremsstrahlung. This bremsstrahlung is also emitted by a non-relativistic plasma, and it is particularly strong near the frequency ω_∞, since the intensity of bremsstrahlung, like that of any dipole radiation, increases with the refractive index.†

The following question may also arise. It apparently follows from the equations of motion (10.8) and their solution (10.9) that resonance occurs in a plasma at a frequency ω_H, but in reality it is shifted to ω_∞. How does this occur? The answer is, of course, given by the discussion in §§ 10 and 11 as a whole, but briefly one may say that the shift of the resonance frequency is due to the allowance for the collective motion of the electrons in the plasma, which leads to a change in the polarisation of normal waves in the medium

† It should be noted that when terms of order $(v_T/c)^2$ and higher orders are taken into account (where $v_T = \sqrt{(\varkappa T/m)}$ is the velocity of the electron) we obtain emission both at the frequency ω_H and at its harmonics. This effect leads to a considerable absorption even at $T \sim 10^6$ deg K ($v_T^2/c^2 = \varkappa T/mc^2 \sim 10^{-4}$, as in the solar corona). The allowance for the effect of the thermal motion on wave propagation in a magnetoactive plasma is discussed in § 12.

as compared with their polarisation in vacuum. To illustrate this, let us consider the behaviour of one electron in a plasma under the action of the extraordinary wave in the case $\alpha = \frac{1}{2}\pi$ (with the field $H^{(0)}$ along the y-axis and collisions neglected). The velocity components of the forced oscillations of the electron are (the derivation may be found in [62])

$$v_y = 0, \quad v_x = (ie/m\omega)(E_x - i\sqrt{u}E_z)/(u-1) = ie(v-1)E_x/m\omega(1-v-u),$$
$$v_z = (ie/m\omega)(E_z + i\sqrt{u}E_x)/(u-1) = -ieE_z/m\omega v,$$

where we have used the fact that in the extraordinary wave with $\alpha = \frac{1}{2}\pi$, we have $E_y = 0$ and $E_x v\sqrt{u} = iE_z(u+v-1)$.

The resonance denominator $u - 1 = (\omega_H^2/\omega^2) - 1$ is here replaced by $1 - v - u = 1 - (\omega_0^2/\omega^2) - (\omega_H^2/\omega^2)$, because in a wave propagated in a magnetoactive plasma along the z-axis we have $E_z \neq 0$, and the effective field is $E_x - i\sqrt{u}E_z$; in an isotropic medium, if the effective and mean fields are not equal, the oscillator absorption eigenfrequency also undergoes a shift.

§ 12. SPATIAL DISPERSION AND PLASMA WAVES IN A MAGNETIC FIELD: THE ALLOWANCE FOR THERMAL MOTION

The passage to the limit of an isotropic plasma

On passing from a magnetoactive plasma to an isotropic one (i.e. as the external magnetic field $H^{(0)}$ tends to zero) we should obtain in the high-

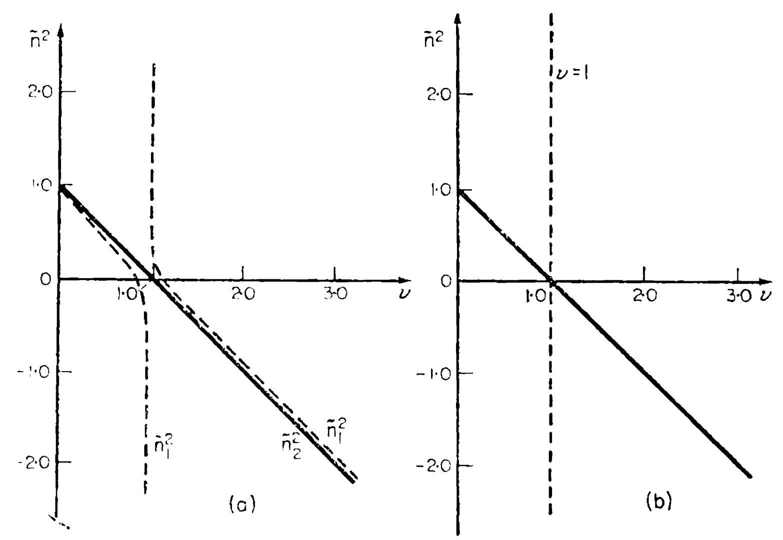

FIG. 12.1. The functions $\tilde{n}_{1,2}^2(v)$.
(a) $\alpha = 45°$, $u = 0{\cdot}01$ (b) $u \to 0$ (transition to an isotropic plasma)

frequency case three normal waves—two transverse and one longitudinal (plasma) wave. In a magnetoactive plasma, however, there are only two waves—ordinary and extraordinary. The nature of the passage to the limit of an isotropic plasma is most simply ascertained by considering the curves of $\tilde{n}^2_{1,2}$ for small values of $u = \omega_H^2/\omega^2$ and for $u \to 0$ (Fig. 12.1; unless otherwise stated, absorption will be neglected). We see that as $u \to 0$ the curves of $\tilde{n}^2_{1,2}$ tend not only to the line $\tilde{n}_0^2 = 1 - v$ but also to the vertical line $v = 1$. When spatial dispersion is neglected, the line $v = 4\pi e^2 N/m\omega^2 = \omega_0^2/\omega^2 = 1$ in Fig. 12.1b corresponds to the plasma wave in the isotropic case [see (8.1)]. The nature of this limiting case is also easily seen from graphs of $\tilde{n}^2_{1,2}$ as a function of the argument ω/ω_0.

However, this property of the curves $\tilde{n}^2_{1,2}$ is clear without recourse to graphs if we use the facts that for $u \to 0$ we have $v_{1,2\infty} = (1 - u)/(1 - u\cos^2\alpha) \to 1$ and $v_{2\,0} = 1$, $v_{1\,0} = 1 \pm \sqrt{u} \to 1$ for all $\alpha \neq 0$. If we consider not only the form of the curves of $\tilde{n}^2_{1,2}(v)$ but also the polarisation of normal waves, then the requirements of the limiting transition are also satisfied (as $v \to v_{1,2\infty}$ the waves 1 and 2 are linearly polarised in the direction of propagation; see § 11).

The same limiting transition (in the qualitative sense) was considered in § 11 for any u and $\alpha \to 0$ (see Fig. 11.10). This coincidence is no accident, since when $\alpha = 0$ the ordinary and extraordinary waves are purely transverse ($E_z = 0$, $E_x = \pm\, iE_y$), and when the thermal motion is neglected (i.e. when spatial dispersion is neglected) the magnetic field cannot affect the directed motion of the particles along it. For this reason, in any field when $\alpha = 0$ there must be a longitudinal wave travelling along the field, i.e. there must exist a plasma wave $v = 1$. This is seen not only from Fig. 11.10 but also from the original equations: when $\alpha = 0$ the condition $D_z = 0$ becomes $D_z = \varepsilon_{zz}E_z = \varepsilon E_z = 0$, which gives the "dispersion relation" for the plasma wave $v = \omega_0^2/\omega^2 = 1$; see § 8.

It may also be noted that these properties of the limiting transitions $u \to 0$ (or $\omega_H^2 \to 0$) and $\alpha \to 0$ are not restricted to a magnetoactive plasma; they are characteristic of any anisotropic medium. The limit $u \to 0$ corresponds to the disappearance of anisotropy, and $\alpha \to 0$ corresponds to an approach to one of the principal axes.

Some confusion is nevertheless possible, because in the isotropic case with spatial dispersion neglected there are at the point $v = 1$ three waves, but in an anisotropic medium there are only two waves for any v. Since the limiting transition is made in the correct manner, as shown above, there can hardly be any grounds for misgivings, but the situation is considerably clarified if we remember that, when spatial dispersion is neglected, plasma waves exist only in a somewhat conventional sense. When spatial dispersion is taken into account, $\tilde{n}_3(v)$ for a plasma has the form

$$\tilde{n}_3^2(v) = (1 - v)/3\beta_T^2 v \approx (1 - v)/3\beta_T^2, \quad \beta_T^2 = \varkappa T/m c^2 \tag{12.1}$$

[see (8.25)]. Hence the three waves (three values of $\tilde{n}^2$) exist not only at $v = 1$ but in a neighbourhood of this point, whose boundaries are determined by the condition that damping should be small; see (8.12) and (8.31).†

The allowance for spatial dispersion in an anisotropic medium

The allowance for spatial dispersion in an anisotropic medium and, in particular, in a magnetoactive plasma gives a similar result. This conclusion is evident without calculation (see [1, 22]). For it was pointed out at the beginning of § 11 that the dispersion relation for $\tilde{n}^2$ should be of degree three in $\tilde{n}^2$, but in fact is of degree two. Thus there is some degeneracy: the coefficient of $\tilde{n}^6$ is zero, and the third root $\tilde{n}_3^2$ becomes infinite. Of course, when a further effect, namely spatial dispersion, is taken into account, the degeneracy disappears and the third root $\tilde{n}_3^2$ is finite. In an isotropic plasma this third root is given by the expression (12.1). For $T \to 0$, $\tilde{n}_3^2(v) \to \infty$, except at the one point $v = 1$. The occurrence of this point is due to another degeneracy, namely the isotropy. Even in an isotropic medium the equation for $\tilde{n}^2$ with $T = 0$ is of degree two and the condition $\varepsilon(v) = 1 - v = 0$ for the existence of a longitudinal wave gives no value of $\tilde{n}_3^2$.

The above arguments also enable us to deduce that the third wave (the third root $\tilde{n}_3^2$) must be important in the region where $\tilde{n}_1^2$ and $\tilde{n}_2^2$ tend to infinity. For, if the coefficient a in the equation $a\tilde{n}^6 + b\tilde{n}^4 + c\tilde{n}^2 + d = 0$ is very small, then the root $\tilde{n}_3^2$ is very large (i.e. $\tilde{n}_3^2 \to \infty$ as $a \to 0$), and has its least value if $b \to 0$ for a given value of a, i.e. if a second root of the equation tends to infinity. Thus in a plasma we can expect that the values of $\tilde{n}_3^2$ will be relatively small only near the point $v_{1,2\infty}$, i.e. when††

$$1 - u - v + u\,v\cos^2\alpha = 1 - \frac{\omega_H^2}{\omega^2} - \frac{\omega_0^2}{\omega^2} + \frac{\omega_0^2\,\omega_H^2}{\omega^4}\cos^2\alpha = 0. \tag{12.2}$$

The frequencies ω_∞^2 which satisfy this condition are

$$\omega_\infty^2 = \frac{1}{2}(\omega_0^2 + \omega_H^2) \pm \sqrt{\left[\frac{1}{4}(\omega_0^2 + \omega_H^2)^2 - \omega_0^2\,\omega_H^2\cos^2\alpha\right]}. \tag{12.2a}$$

The allowance for spatial dispersion in an anisotropic medium can be made phenomenologically in a general form [1], as was done at the beginning of § 8

† As has been mentioned in § 8, the condition that the damping of plasma waves should be small means that we can replace $1 - \omega_0^2/\omega^2$ in the expression for $\tilde{n}_3^2$ by $2(1 - \omega_0/\omega)$. We shall not, however, find it convenient to use such simplifications here and subsequently.

†† In the absence of absorption, equation (11.4) becomes

$$(1 - u - v + u\,v\cos^2\alpha)\,\tilde{n}^4 + [u(2 - v - v\cos^2\alpha) - 2(1 - v)^2]\,\tilde{n}^2 + \\ + (1 - v)\,[(1 - v)^2 - u] = 0. \tag{12.3}$$

The roots of this equation are given by (11.6) and become infinite at

$$v_{1,2\infty} = (1 - u)/(1 - u\cos^2\alpha)$$

[see (11.18) and (11.22)]. This expression for $v_{1,2\infty}$ is most simply obtained from the condition that the coefficient of $\tilde{n}^4$ in (12.3) should be zero, i.e. the condition (12.2).

for longitudinal waves in an isotropic medium. Having in view only the case of a magnetoactive plasma, we shall proceed differently, and consider at once the spatial dispersion due to the thermal motion of the electrons. In a rigorous analysis of this problem it would be necessary to use the kinetic treatment, which leads to quite laborious calculations when a magnetic field is present (see [43, 49, 64–70]). For this reason, before discussing various results of the kinetic theory, we shall treat the problem in the quasihydrodynamic approximation, which has already been used at the end of § 8; see also [19, 69–72] and § 13.

The quasihydrodynamic approximation

In this approximation we start from the following equations (which take account only of the motion of the electrons):

$$\left.\begin{aligned} &m\, d\boldsymbol{v}_e/dt = e(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{H}^{(0)}/c) - (1/N)\,\mathbf{grad}\, p_e - m\,\nu_{\mathrm{eff}}\,\boldsymbol{v}_e, \\ &\partial N/\partial t + \operatorname{div}(N\boldsymbol{v}_e) = 0, \quad p_e = \xi_e \varkappa T_e N, \\ &\triangle \boldsymbol{E} - \mathbf{grad}\,\operatorname{div}\boldsymbol{E} - (1/c^2)\,\partial^2\boldsymbol{E}/\partial t^2 = (4\pi/c^2)\,\partial \boldsymbol{j}_t/\partial t, \quad \boldsymbol{j}_t = eN\boldsymbol{v}_e. \end{aligned}\right\} \tag{12.4}$$

If the pressure term is neglected, we of course obtain the same results as in § 11.

Including the pressure term, linearising (the velocity $\boldsymbol{v}_e$ is small, and the change in density $N' \ll N$), and neglecting damping, we have for monochromatic plane waves, from (12.4), the dispersion relation for $\tilde{n}^2 = c^2k^2/\omega^2$:

$$\begin{gathered} \beta^2(1 - u\cos^2\alpha)\tilde{n}^6 - [1 - u - v + uv\cos^2\alpha + 2\beta^2(1 - v - u\cos^2\alpha)]\tilde{n}^4 + \\ + [2(1-v)^2 - u(2 - v - v\cos^2\alpha) + \beta^2(1 - 2v + v^2 - u\cos^2\alpha)]\tilde{n}^2 + \\ + (1-v)[u - (1-v)^2] = 0, \end{gathered} \tag{12.5}$$

where $\beta = \sqrt{(\xi_e \varkappa T_e/mc^2)} = \sqrt{\xi_e}\,\beta_T$ is of the order of the ratio of the mean thermal velocity of an electron to the velocity of light. In the absence of the thermal motion ($\beta \to 0$), equation (12.5) becomes (12.3). In the non-relativistic plasma here considered,

$$\beta^2 = \xi_e \varkappa T_e/mc^2 \ll 1. \tag{12.6}$$

In the solar corona, for instance, we have $T_e \sim 10^6$ deg K and $\beta_T^2 = \varkappa T_e/mc^2 \sim 10^{-4}$.

The condition (12.6) shows that, throughout the range of values of the parameter v for which the roots $\tilde{n}_1^2$ and $\tilde{n}_2^2$ (calculated for $\beta^2 = 0$) are not very large, the thermal corrections are small. The third root of equation (12.5) is then large, i.e. it is determined by the first two terms of the equation, and thus is given by

$$\left.\begin{aligned} &\tilde{n}_3^2 \approx (1 - u - v + uv\cos^2\alpha)/(1 - u\cos^2\alpha)\,\beta^2, \\ &|\tilde{n}_1^2| \ll |\tilde{n}_3^2|, \quad |\tilde{n}_2^2| \ll |\tilde{n}_3^2|. \end{aligned}\right\} \tag{12.7}$$

The root (12.7) must be large except near the point v_∞ given by the condition (12.2). Thus the quasihydrodynamic model used leads, as we should expect, to results which are in agreement with the general considerations given previously. For $\alpha = 0$ (longitudinal propagation) equation (12.5) leads for the transverse waves to the expressions (11.9) for $\tilde{n}^2_{1,2} \equiv \tilde{n}^2_{\pm}$, and for the longitudinal wave to the expression (12.1) for $\tilde{n}^2_3$. Here we must emphasise that the fact that the expressions for $\tilde{n}^2_{\pm}$ are independent of β^2 is due only to the approximation used and does not hold good in the kinetic calculation (see below).

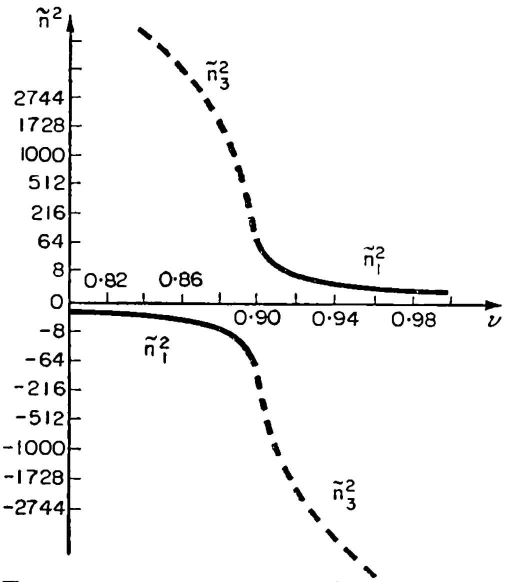

FIG. 12.2. The functions $\tilde{n}^2_1$ (continuous line) and $\tilde{n}^2_3$ (broken line) for $u = 0{\cdot}1$, $\alpha = 90°$ and $\beta^2 = 10^{-5}$ (quasihydrodynamic approximation).

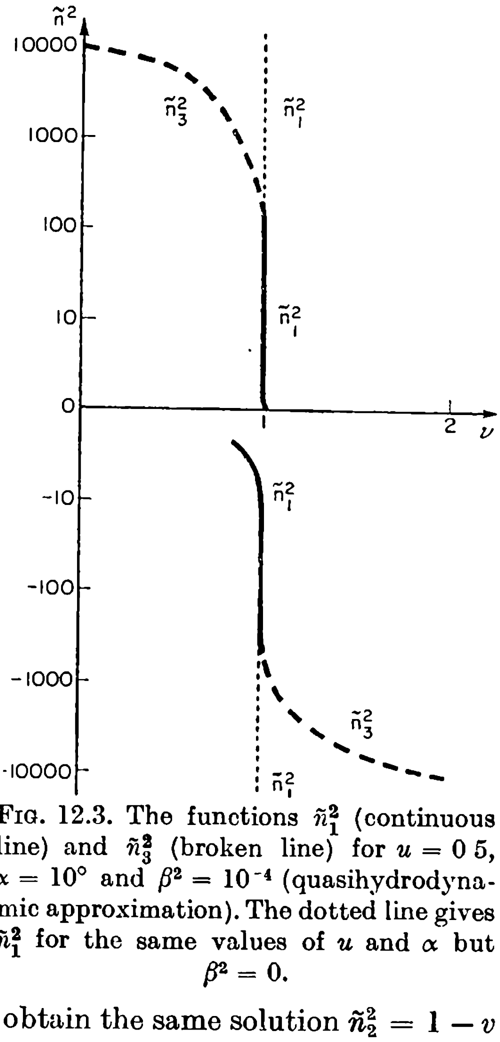

FIG. 12.3. The functions $\tilde{n}^2_1$ (continuous line) and $\tilde{n}^2_3$ (broken line) for $u = 0{\cdot}5$, $\alpha = 10°$ and $\beta^2 = 10^{-4}$ (quasihydrodynamic approximation). The dotted line gives $\tilde{n}^2_1$ for the same values of u and α but $\beta^2 = 0$.

For $\alpha = \frac{1}{2}\pi$ (transverse propagation) we obtain the same solution $\tilde{n}^2_2 = 1 - v$ as before for the ordinary wave and the equation

$$\beta^2\tilde{n}^4 + [(v-1)(1+\beta^2) + u]\tilde{n}^2 + [(v-1)^2 - u] = 0, \qquad (12.8)$$

which corresponds to the solutions $\tilde{n}^2_1$ and $\tilde{n}^2_3$. In accordance with the preceding discussion, the root $\tilde{n}^2_3$ is very large everywhere except near the point $v_{1\infty} = (1-u)/(1-u\cos^2\alpha) = 1 - u$. The root $\tilde{n}_1$ is almost equal to the previous expression [see (11.14) with $s = 0$] except in the neighbourhood of $v_{1\infty}$. Near this point the behaviour of $\tilde{n}^2_{1,3}(v)$ for $u < 1$ is shown by Figs. 12.2 and 12.3, where the broken line refers to $\tilde{n}^2_3$. It may be recalled that for negative values of $\tilde{n}^2$ the corresponding waves are strongly damped according to the law $e^{-\omega|\tilde{n}|z/c}$, since $\tilde{n}^2 = -\varkappa^2$ when $\tilde{n}^2 < 0$.

It is particularly important to emphasise that the solution $\tilde{n}_3^2$ does not form an independent branch of the function $\tilde{n}^2$. The appearance of a third solution is due to the disappearance of the discontinuity of the function $\tilde{n}_1^2$ and to a change in the shape of the curves such that two values of $\tilde{n}^2$ correspond to a given value of v (we are not here considering the ordinary wave). If for $\beta \to 0$ the ordinary wave has a pole, i.e. $\tilde{n}_2^2(v_{2\infty}) \to \infty$ (as happens when $u > 1$ and $u_L = u \cos^2\alpha > 1$), then the plasma wave is a continuation of the branch $\tilde{n}_2^2(v)$. This is clear from Fig. 12.4.

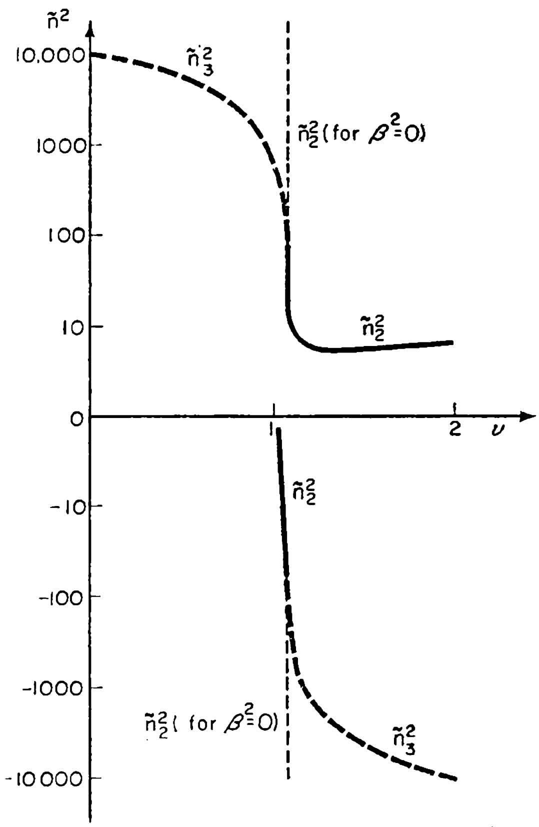

FIG. 12.4. The functions $\tilde{n}_2^2$ and $\tilde{n}_3^2$ for $u = 2$, $\alpha = 10°$ and $\beta^2 = 10^{-4}$ (quasihydrodynamic approximation).

Plasma waves in a magnetoactive plasma

The form of the curves shows that the division of waves into extraordinary and plasma waves (for $u < 1$) or ordinary and plasma waves (for $u > 1$, $u \cos^2\alpha > 1$) is quite arbitrary. In a magnetoactive plasma there are really no particular "plasma waves" and it is only for convenience and with a view to the limiting case of an isotropic plasma that we shall call the corresponding part of the curve of $\tilde{n}_1^2$ or $\tilde{n}_2^2$ the curve for plasma waves. This definition evidently corresponds to giving the name of plasma waves to those waves for which $\tilde{n}_3^2$ does not have finite values as $\beta^2 \to 0$ (see Figs. 12.2–12.4).

In the quasihydrodynamic approach, neglecting collisions, the waves are not damped even if the phase velocity is comparable with the thermal velocities of the electrons. The kinetic treatment, however, leads (as for longitudinal waves in an isotropic plasma) to the possibility of a damping which is not due to collisions. It should also be emphasised that, for the calculation of $\tilde{n}^2$ in a region where damping is slight, the quasihydrodynamic approach is much more restricted for a magnetoactive plasma than in the isotropic case. The reason is that in an isotropic medium the dispersion relation for a longitudinal wave can include, for simple reasons of symmetry, only one unknown coefficient (see § 8 and [1]). For this reason, by taking the value of ξ_e in (8.9) as 3, we were able to obtain complete agreement between the quasihydrodynamic and kinetic results. In an anisotropic medium the allowance for spatial dispersion leads to the appearance of more complex expressions with several coefficients. In equations (12.4), however, we have assumed that the stress tensor reduces to the pressure, i.e. we have again introduced only one unknown constant ξ_e. The result is that for no value of ξ_e does the dispersion relation (12.5) agree with equation (12.52) below, which is derived by the kinetic method. In practice, the difference amounts to a difference in the coefficient of $\tilde{n}^6$, which in both cases is of the order of β_T^2, but depends differently on u, v and α. In particular, with the quasihydrodynamic approach all three roots $\tilde{n}^2_{1,2,3}$ of equation (12.5) are always real both when $u < 1$ and when $u \cos^2\alpha > 1$ (together with the condition $\beta^2 \ll 1$ which is always assumed); in the kinetic theory, as we shall see below, this is not so.

For a homogeneous medium, since a kinetic discussion has been given, the quasihydrodynamic calculation is seen from the above to be certainly redundant. The value of such a calculation is that it can also be effected in cases where the kinetic theory involves great mathematical difficulties: a multicomponent plasma, containing several species of ions or molecules; an inhomogeneous plasma, etc.

The kinetic theory

Let us now consider the kinetic theory of high-frequency waves in a homogeneous magnetoactive plasma.

The initial equations are [see (4.2) and (8.18)]

$$\left.\begin{aligned} &\partial\varphi/\partial t + \boldsymbol{v}\cdot\mathbf{grad}_r\varphi + (e/m)\,\boldsymbol{E}\cdot\mathbf{grad}_v f_0 + (e\,\boldsymbol{v}\times\boldsymbol{H}^{(0)}/m\,c)\cdot\mathbf{grad}_v\varphi = 0, \\ &\triangle\boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\boldsymbol{E} - (1/c^2)\,\partial^2\boldsymbol{E}/\partial t^2 = (4\pi/c^2)\,\partial\boldsymbol{j}_t/\partial t, \quad \boldsymbol{j}_t = e\int\boldsymbol{v}\,\varphi\,d\,\boldsymbol{v}; \end{aligned}\right\} \quad (12.9)$$

here collisions are neglected, the Boltzmann equation has been linearised, the distribution function $f = f_0 + \varphi$, $|\varphi| \ll |f_0|$, and the magnetic field of the wave $|H| \ll H^{(0)}$. If the unperturbed distribution is Maxwellian, as we shall assume below, then $f_0 = f_{00} = N\,(m/2\pi\varkappa T)^{3/2} e^{-mv^2/2\varkappa T}$.

The equations (12.9) differ from those used in § 8 only by the term which involves the external magnetic field $\boldsymbol{H}^{(0)}$. This term makes the calculations very much more complicated even when the problem is solved by Fourier's method, i.e. by putting $\varphi = \varphi_0(\boldsymbol{v})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$. In principle, however, the procedure is the same as in the isotropic case (see § 8), and the result is a dispersion relation between ω and $\boldsymbol{k}$; in the denominators of the integrands in expressions such as (8.33), $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ is replaced by $\omega - s\omega_H - kv_z \cos\alpha$ ($s = 0, \pm 1, \pm 2, \pm 3, \ldots$), where v_z is the component of the velocity of a plasma electron in the direction of the magnetic field $\boldsymbol{H}^{(0)}$ (the z-axis), and α is the angle between $\boldsymbol{H}^{(0)}$ and $\boldsymbol{k}$.

If these denominators vanish, i.e. if

$$\omega = s\omega_H + kv_z \cos\alpha, \tag{12.10}$$

then the solution by Fourier's method is invalid and the problem must be solved together with the initial conditions. To find the complex frequency $\omega' = \omega + i\gamma$ as a function of $\boldsymbol{k}$, it is then convenient to use the dispersion relation given by Fourier's method, but integrating with respect to v_z along a contour which passes above the resonance point $v_z = (\omega - s\omega_H)/k \cos\alpha$.

The nature of the collisionless absorption

Before giving the expressions thus obtained for $\omega(\boldsymbol{k})$ and $\gamma(\boldsymbol{k})$, we shal discuss the physical significance of the condition (12.10), and thus the nature of the collisionless absorption.

In a magnetic field an electron revolves about the lines of force with frequency $\omega_H^* = (mc^2/E)|e|\,H^{(0)}/mc$, which in the non-relativistic limit becomes $\omega_H = |e|\,H^{(0)}/mc$; the electron may also have any velocity $v_z < c$ along the lines of force. An electron (or other charged particle) moving in this way emits electromagnetic waves on account of the acceleration (magnetic bremsstrahlung or synchrotron radiation), and also on account of the Vavilov–Cherenkov effect when the appropriate condition is fulfilled.†

To calculate the spectrum and intensity of the radiation in an arbitrary medium, almost the only suitable method is to use an expansion of the field in normal plane waves which can be propagated in the medium concerned [73–77]. The vector potential of the field is then written as

$$\boldsymbol{A} = \sum_{\lambda, j} \frac{c\sqrt{(4\pi)}}{n_{\lambda j}}\, \boldsymbol{a}_{\lambda j}\, q_{\lambda j}(t)\, e^{i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}} + \text{complex conjugate},$$

where $\boldsymbol{a}_{\lambda j}$ are the complex polarisation vectors corresponding to the normal waves $j = 1$ and $j = 2$ (in a non-gyrotropic medium the vectors $\boldsymbol{a}_{\lambda j}$ are, of course, real, except in a vacuum, where they can be taken to be either real

† When collisions occur, bremsstrahlung due to the acceleration of the electron during collisions is added to the magnetic bremsstrahlung and Cherenkov radiation.

or complex). Substituting the series for $\boldsymbol{A}$ in the field equation for it, multiplying by $[c\sqrt{(4\pi)}/n_{\lambda j}]\,\boldsymbol{a}^*_{\lambda j}\,e^{-i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}}$ and integrating over all space, we obtain for $q_{\lambda j}$ the equation

$$\ddot{q}_{\lambda j} + \omega^2_{\lambda j}\,q_{\lambda j} = \sqrt{(4\pi)}\,(e/n_{\lambda j})\,\boldsymbol{v}_e\cdot\boldsymbol{a}^*_{\lambda j}\,e^{-i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}_e} = f(t)\,, \tag{12.11}$$

where $\omega^2_{\lambda j} = c^2k^2_\lambda/n^2_{\lambda j}$, $\boldsymbol{r}_e$ is the radius vector of the radiating electron and $\boldsymbol{v}_e = d\boldsymbol{r}_e/dt$ its velocity.

Apart from a numerical factor, the form of the "force" $f(t)$ in (12.11) is immediately clear if we recall simply that the current density due to the moving electron is $\boldsymbol{j}_e = e\boldsymbol{v}_e(t)\,\delta(\boldsymbol{r}-\boldsymbol{r}_e)$, $\int e\delta(\boldsymbol{r}-\boldsymbol{r}_e)\,d\boldsymbol{r} = e$, and in the field equations we have on the right-hand side $4\pi\boldsymbol{j}_e/c$. Hence

$$\int \boldsymbol{j}_e\cdot\boldsymbol{a}^*_{\lambda j}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} = e\,\boldsymbol{v}_e\cdot\boldsymbol{a}^*_{\lambda j}\,e^{-i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}_e},$$

which agrees with the right-hand side of (12.11).

The frequencies emitted by the moving particle can be determined at once from the fact that the radiation corresponds to solutions for $q_{\lambda j}$ which increase with time (when radiation is emitted the field energy increases continually, which can happen only if $q_{\lambda j}$ increases). In turn, $q_{\lambda j}$ can increase with time only if there is resonance, the spectrum of the "force" $f(t)$ in (12.11) including frequencies equal to some of the possible values $\omega_{\lambda j} = k_\lambda c/n_{\lambda j}$. Let us consider, for example, a uniform motion of an electron, with $\boldsymbol{r}_e = \boldsymbol{v}t$. Then the spectrum of the force $f(t)$ contains only the frequency $\boldsymbol{k}\cdot\boldsymbol{v}$, and the condition for emission $\omega_{\lambda j} \equiv \omega = \boldsymbol{k}\cdot\boldsymbol{v}$ is just the condition for the existence of Cherenkov waves (this method [78] seems to be one of the simplest ways of obtaining the condition for Cherenkov emission). For an electron in a magnetic field we have

$$\boldsymbol{r}_e = (r_0\cos\omega^*_H t,\quad r_0\sin\omega^*_H t,\ v_z t),$$

$$\boldsymbol{v}_e = (-v_\perp\sin\omega^*_H t,\quad v_\perp\cos\omega^*_H t,\ v_z),\quad v_\perp = r_0\,\omega^*_H,$$

$$f(t) = \text{constant}\times(-a^*_x v_\perp\sin\omega^*_H t + a^*_y v_\perp\cos\omega^*_H t + a^*_z v_z)\times$$
$$\times\, e^{-i(kr_0\sin\alpha\sin\omega_H{}^* t + kv_z t\cos\alpha)},$$

where the coordinate axes are, for simplicity, taken so that $k_x = 0$. Using the expansion of a plane wave in Bessel functions:

$$\exp(-i\,k_\lambda r_0\sin\alpha\sin\omega^*_H t) = \sum_{s=-\infty}^{\infty} J_s(k_\lambda r_0\sin\alpha)\,e^{-is\omega_H{}^* t},$$

we easily see that the resonance condition is

$$\omega = s\omega^*_H + kv_z\cos\alpha \quad (s = 0,\ \pm 1,\ \pm 2,\ \ldots), \tag{12.10a}$$

and in the non-relativistic case this becomes (12.10).

The condition (12.10a) with $s = 0$ is the condition for Cherenkov emission by a particle moving with velocity v_z. For $s \neq 0$, (12.10a) can be replaced as follows, with $k = \omega n(\omega, \cos\alpha)/c$:

$$\left.\begin{aligned} &\text{for} \quad s > 0, \qquad \omega = \frac{s\,\omega_H^*}{1 - (v_z/c)\,n\cos\alpha}\,; \\ &\text{for} \quad s < 0, \qquad \omega = \frac{s\,\omega_H^*}{(v_z/c)\,n\cos\alpha - 1}\,. \end{aligned}\right\} \tag{12.10 b}$$

The frequency ω, as in formulae (12.10), (12.10a) and elsewhere, is always positive.†

If the velocity $v_\perp \ll v_z$, an electron in a magnetic field emits like two suitably chosen dipoles moving along the field with velocity $v_z \approx v$. This case corresponds to $s = \pm 1$ (more precisely, the higher harmonics are unimportant if $kr_0 \sin\alpha = (\omega/c)\,n\,(v_\perp/\omega_H)\sin\alpha \ll 1$). In these conditions formulae (12.10b) are a particular case of that for the Doppler effect in a medium [79, 80]:

$$\omega = \omega_0 \sqrt{(1-\beta^2)}/|1 - \beta n\cos\alpha| = \omega_0'/|1 - \beta n\cos\alpha|, \tag{12.12}$$

where ω_0 is the eigenfrequency of the emitter in a frame of reference where it is at rest, and ω_0' its frequency in the laboratory system of coordinates (for motion in a magnetic field, $\omega_0' = \omega_H^*$); $\beta = v/c$, α is the angle between $\boldsymbol{v}$ and the wave vector $\boldsymbol{k}$, and in general $n = n(\omega, \boldsymbol{k}/k)$.

The range of angles within the Cherenkov cone (on which $\beta n\cos\alpha = 1$) corresponds to the anomalous Doppler effect, and the range $\beta n\cos\alpha < 1$ to the normal Doppler effect. The anomalous Doppler effect and the emission of Cherenkov radiation can, of course, occur only at velocities exceeding that of light, i.e. when $\beta n = vn/c > 1$. For motion in a magnetic field the anomalous effect corresponds to values of $s < 0$ in (12.10) and (12.10a, b).

If the particle emits radiation of some frequency ω, then it will also absorb radiation of that frequency, as is well known from both the classical and the quantum theory.

The nature of the collisionless absorption of waves in a magnetoactive plasma at frequencies satisfying (12.10) is thus quite clear.†† The same result can be obtained by considering, for an electron moving in a magnetic field, the frequency spectrum of the force acting on the electron in the field of the

† The radiation corresponding to the value $s = 0$ is usually called Cherenkov radiation, and that for $s \neq 0$ magnetic bremsstrahlung or synchrotron radiation, but it must be borne in mind that this division is somewhat arbitrary (see [81]). For example, when $v_z = 0$ (motion in a circle) there is no radiation with $s = 0$. From physical considerations it is evident that, when the radius of curvature is sufficiently large, the radiation for $v_z = 0$ and $\beta n > 1$ will be very similar to Cherenkov radiation. When the above terminology is used, the radiation for $v_z = 0$ is purely magnetic bremsstrahlung, and this nomenclature is formally correct, since (when the Doppler effect is neglected or $\alpha = \frac{1}{2}\pi$) the spectrum is discrete ($\omega = s\omega_H$) and the intensity of the radiation does not tend to zero as we approach a vacuum. Thus, when the term "magnetic bremsstrahlung" is used, it must not be forgotten that, when a charge is moving in a medium, such radiation may be of a nature quite different from that in a vacuum.

†† See also [82, 83].

wave. (The frequency of the force is not equal to the frequency of the field $\boldsymbol{E}$, since the electron is in motion and is in fields of different strengths at different times; see [84].) Since the appearance of frequencies $s\omega_H^*$ in addition to ω_H^* is due to the factor $e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$, it is clear that we are taking account of spatial dispersion, i.e. the dependence of the field on the coordinates. In the non-relativistic case and neglecting the term $\boldsymbol{k}\cdot\boldsymbol{r}$ in comparison with unity, the condition for emission becomes $\omega = \omega_H$, in accordance with the theory given in previous sections.

In the absence of collisions, the problem of absorption and refraction (calculation of the refractive index) in a magnetoactive plasma may, as shown by the above discussion, be closely related to the problem of radiation; by effecting the necessary averaging with respect to velocity, we may in this way (see § 37 and [82, 85]) obtain many results without using the Boltzmann equation. However, the Boltzmann-equation method makes possible the most natural averaging over velocities and, what is more important, an allowance without further complications both for the effect of the thermal motion on the refractive index and for the effect of collisions.

In the latter case, the expression $\omega - s\omega_H - kv_z\cos\alpha$ in the denominator in the dispersion relation becomes

$$\omega - s\omega_H - kv_z\cos\alpha - i\nu(v), \tag{12.13}$$

as is immediately clear from (12.9) when the term $\nu(v)\,\varphi$ is added to the equation for φ.

Neglecting the effect of thermal motion corresponds in (12.10) and (12.13) to neglecting the term in v_z and using only the values $s = \pm 1$. This is evident from the fact that when $v \to 0$ the spectrum of the force $f(t)$ retains only the frequency ω_H, which may be either emitted or absorbed†; for a wave whose electric field rotates in the opposite direction to the revolution of the electron, absorption is impossible in the absence of collisions, but the presence of a magnetic field affects the refractive index. This case corresponds to taking $s = -1$ in (12.13). The above results can, of course, be obtained also by direct calculation. Thus, when the thermal motion is neglected, denominators of the form $\omega \pm \omega_H - i\nu(v)$, instead of (12.13), appear in the dispersion relation, and the results are those of §§ 10 and 11.

Far from the frequencies $\omega = s\omega_H$ $(s > 0)$, the effect of the thermal motion is small if $\omega^2 \gg (kv_z)^2\cos^2\alpha$, i.e.

$$(v_z^2 n^2/c^2)\cos^2\alpha \ll 1; \tag{12.14}$$

here v_z may usually be taken as the mean value $v_z \sim \sqrt{(\varkappa T/m)}$, the condition (12.14) thus becoming

$$\beta_T^2 n^2\cos^2\alpha \ll 1, \quad \beta_T = \sqrt{(\varkappa T/mc^2)}, \quad n = n_{1,2}(\omega, \cos\alpha). \tag{12.15}$$

† Here we ignore the points discussed at the end of § 11 which relate to the effect of the surrounding medium on the radiation from an electron.

As $\alpha \to \frac{1}{2}\pi$ this condition is always satisfied, and the thermal motion causes only the appearance of resonance absorption at frequencies $s\omega_H$, with $s > 0$. Physically this is easily understood, since with $\alpha = \frac{1}{2}\pi$ and finite values of n the Cherenkov condition $\beta n \cos\alpha = 1$ cannot be satisfied.† It should also be noted that the inequality (12.15) is a necessary but not sufficient condition for weak damping in the absence of collisions and far from resonance (seebelow).

Even in the solar corona (with $T \sim 10^6$), $\beta_T \sim 10^{-2}$ and the thermal motion has a considerable effect on wave propagation only near the frequencies ω_∞ [see (12.2a)], where the refractive index increases, and near the frequencies $s\omega_H$. At higher temperatures (in stellar interiors and in certain laboratory devices) the situation is, of course, different, and the results for $\beta_T \to 0$ may not serve even as a rough guide. We shall not discuss here this interesting range of very high temperatures††, and so for any α we shall consider only conditions where

$$n^2 \gg 1, \tag{12.16}$$

and the frequency ranges $\omega \approx \omega_H$, $\omega \approx 2\omega_H$ and $\omega \approx 3\omega_H$.

Results of the kinetic theory for longitudinal propagation

Turning now to the results of the kinetic theory, let us begin with the simplest case, that of longitudinal propagation ($\alpha = 0$) when the dispersion relation separates into equation (8.34) for the longitudinal plasma wave and the following equation for the transverse extraordinary $(-)$ and ordinary $(+)$ waves:

$$-\frac{\omega_0\,\omega'}{c^2 k^2 - \omega'^2}\cdot\frac{1}{N}\int\frac{f_{00}(u)\,du}{\omega' \pm \omega_H - ku} = 1, \tag{12.17}$$

---

† The case of a sufficiently weak magnetic field needs special consideration, since it represents the transition to an isotropic plasma. The result of the passage to the limit $\omega_H \to 0$ is, however, known: in an isotropic plasma without collisions, there is only Cherenkov absorption of plasma waves.

†† There is as yet not even an approach to a complete discussion of the plasma with β_T between about 0·1 and 1 (that is, the propagation of waves in a relativistic plasma). Some results on the problem are given in [83, 86, 88]. By using the non-relativistic Boltzmann equation, as we do here, we cannot of course take account of terms of order β_T^2 and higher orders when they are in fact corrections (i.e. for instance, β_T^2 is less than similar terms which are present under the same conditions). Such a non-relativistic treatment with $\beta_T \ll 1$ is, however, sufficient to calculate the collisionless damping or the values of $\tilde{n}_3^2$, these not being "corrections" to any quantities defined for $\beta_T^2 = 0$. Some restriction is necessary here when $\tilde{n}_{1,2,3}^2$ are almost zero (long waves). This has already been discussed (in the absence of a magnetic field) in § 8, where formulae were given for $\tilde{n}_{1,2,3}^2$ accurate as far as terms of order $\beta_T^2 = \varkappa T/mc^2 \ll 1$. When a magnetic field is present and the propagation is longitudinal ($\alpha = 0$), formula (8.25a) for $\tilde{n}_3^2$ is, of course, still valid. For the transverse waves 1 and 2 we have, assuming that $|1 - \sqrt{u}| \gg \beta_T^2$ [267],

$$\tilde{n}_1^2 = \frac{1 - \dfrac{v}{1-\sqrt{u}}\left(1 - \dfrac{5}{2}\beta_T^2\dfrac{1+\sqrt{u}}{1-\sqrt{u}}\right)}{1 + \beta_T^2 v/(1-\sqrt{u})^3}, \qquad \tilde{n}_2^2 = \frac{1 - \dfrac{v}{1+\sqrt{u}}\left(1 - \dfrac{5}{2}\beta_T^2\dfrac{1-\sqrt{u}}{1+\sqrt{u}}\right)}{1 + \beta_T^2 v/(1+\sqrt{u})^3}.$$

where $f_{00}(v_z) = N\sqrt{(m/2\pi\varkappa T)}\, e^{-mv_z^2/2\varkappa T}$, $u = v_z$ and the integration is taken along a contour which passes above the pole of the denominator.

It should be emphasised that equation (12.17) is exact, in the sense that it follows rigorously from equations (12.9). Thus for $\alpha = 0$ there is in fact no value of s other than ± 1 in the dispersion relation. This is entirely reasonable, since for $\alpha = 0$ the spectrum of the "force" $f =$ constant $\times$ $\boldsymbol{v}\cdot\boldsymbol{a}_{\lambda j}^{*} e^{-ikv_z t}$ contains only the frequencies $\omega_H \pm kv_z$. The absence of Cherenkov radiation is formally due to the fact that for $\alpha = 0$ we have $a_z^* = 0$ and therefore $a_z^* v_z = 0$; physically it is due to the fact that the condition $\omega = kv_z$ or $\cos\alpha = 1$ corresponds to the threshold of the effect, when the intensity of the radiation is zero.

On neglecting the thermal motion and formally putting $ku = 0$ in the denominator of (12.17), we have†

$$\left.\begin{aligned} -\frac{\omega_0^2\,\omega'}{(c^2k^2 - \omega'^2)(\omega' \pm \omega_H)} = 1\,, \\ \omega' = \omega, \quad n_{1,2}^2 = c^2k^2/\omega^2 = 1 - \omega_0^2/\omega\,(\omega \mp \omega_H)\,, \end{aligned}\right\} \tag{12.18}$$

in agreement with formulae (11.9).

For the ordinary wave 2 in the non-relativistic theory the thermal corrections may be neglected (and, in a sense, *must* be neglected if the accuracy of the calculation is not to be exaggerated). For the extraordinary wave 1, corresponding to the minus sign in (12.18), we have at a sufficient distance from resonance, when

$$\left(\frac{\omega - \omega_H}{\omega}\right)^2 \gg \frac{\varkappa T}{mc^2}\, n_1^2 \approx \beta_T^2\left(1 - \frac{\omega_0^2}{\omega\,(\omega - \omega_H)}\right), \tag{12.19}$$

the result [65, 68]

$$n_1^2 = \frac{1 - v/(1 - \sqrt{u})}{1 + \beta_T^2 v/(1 - \sqrt{u})^3} = \frac{1 - \omega_0^2/\omega\,(\omega - \omega_H)}{1 + (\varkappa T/mc^2)\,\omega_0^2\,\omega/(\omega - \omega_H)^3}\,. \tag{12.20}$$

The thermal corrections are evidently small if

$$|\omega - \omega_H|^3/\omega_H^3 \gg (\varkappa T/mc^2)\,\omega_0^2/\omega_H^2 = \beta_T^2 v/u \tag{12.21}$$

(we take $\omega \sim \omega_H$), which for $|n_1^2| \gg 1$ is equivalent to (12.19):

$$(\omega - \omega_H)^2/\omega_H^2 \gg \beta_T^2 n_1^2, \quad n_1^2 \gg 1. \tag{12.22}$$

† In this case, for $\omega \neq \omega_H$, the integration is along the whole of the real axis, and

$$\int_{-\infty}^{\infty} f_{00}(u)\,du = N, \quad \text{since} \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}.$$

For $\omega_0^2/\omega_H^2 \sim 1$ in the Earth's ionosphere this gives $|\omega - \omega_H|/\omega_H \gg 5\times10^{-3}$, and for the solar corona $|\omega - \omega_H|/\omega_H \gg 5\times10^{-2}$; the corresponding values are $n_1^2 \ll 200$ and $n_1^2 \ll 20$ but $n_1^2 \gg 1$.

When collisions are taken into account they are unimportant if

$$(\omega - \omega_H)^2 \gg \nu_{\text{eff}}^2. \tag{12.23}$$

For the F layer of the ionosphere ($\nu_{\text{eff}} \sim 10^3$, $\omega_H \sim 10^7$) this means that $|\omega - \omega_H|/\omega_H \gg 10^{-4}$; for the corona, the condition (12.23) becomes $|\omega - \omega_H|/\omega_H \gg 10^{-6}$ even with $\nu_{\text{eff}} \sim 10$ and $\omega_H \sim 10^{-7}$. Thus even in the Earth's ionosphere the effect of thermal motion on the behaviour of $n_1^2(\omega)$ may be greater than that of collisions.†

In the same region (12.22) we have for the damping coefficient γ_1 [91]

$$\gamma_1 = \sqrt{\left(\frac{1}{2}\pi\right)}\frac{1}{\beta_T n_1}\frac{(\omega_H - \omega)^2}{\omega}\exp\{-(\omega-\omega_H)^2/2\beta_T^2 n_1^2\omega^2\}. \tag{12.24}$$

Let us compare the damping γ_1, here due to magnetic bremsstrahlung at frequency $\omega_H + kv_z$, with the damping γ_{coll} due to collisions. To do so, we use formula (11.8):

$$(n - i\varkappa)_1^2 = c^2k^2/\omega^2 = 1 - \omega_0^2/\omega(\omega - \omega_H - i\nu_{\text{eff}}), \tag{12.25}$$

and note that it holds for real k but complex $\omega' = \omega + i\gamma$ (see § 7). Then, with the condition (12.23) and $\gamma \ll \omega$, we have [since in this approximation $c^2k^2/\omega^2 = n_1^2 = 1 - \omega_0^2/\omega(\omega - \omega_H)$]

$$\left.\begin{aligned} \gamma_{1,\text{coll}} &= \omega_0^2\,\omega\,\nu_{\text{eff}}/[2\omega(\omega-\omega_H)^2 + \omega_0^2\omega_H] = \omega\,\varkappa_1 v_{\text{gr}}/c, \\ \varkappa_1 &= \omega_0^2\,\nu_{\text{eff}}/2\omega(\omega-\omega_H)^2 n_1, \quad v_{\text{gr}} = c \div d(n_1\omega)/d\omega. \end{aligned}\right\} \tag{12.26}$$

The more general relation

$$\omega\varkappa/c = \gamma/v_{\text{gr}}\cos(\boldsymbol{k}, \boldsymbol{v}_{\text{gr}}) = \gamma d(n_{1,2}\omega)/c\,d\omega \tag{12.27}$$

(where $(\boldsymbol{k}, \boldsymbol{v}_{\text{gr}})$ is the angle between the two vectors) applies to all waves, whatever the nature of the damping and even when anisotropy is taken into account; the anisotropy has the effect that the group-velocity vector $\boldsymbol{v}_{\text{gr}} = d\omega/d\boldsymbol{k}$ is in general in a different direction from the wave vector $\boldsymbol{k}$. The expression (12.27) is derived in the same way as (7.23); the proof is unaffected by the anisotropy, which means only that n and $\varkappa$ depend on $\boldsymbol{k}/k$ as well as on ω. In (12.27) we have also used the fact that $c \div d(n_{1,2}\omega)/d\omega = v_{\text{gr}}\cos(\boldsymbol{k}, \boldsymbol{v}_{\text{gr}})$, with $\boldsymbol{v}_{\text{gr}} = d\omega/d\boldsymbol{k}$ (see § 24). The relation (12.27) is also directly evident, since a wave packet damped in time as $e^{-\gamma t}$ will move in

† To avoid misunderstandings we should point out that propagation of waves with frequency $\omega \approx \omega_H$ in the F layer is scarcely ever considered; in the lower layers, $\nu \sim 10^5$ to 10^6 and (12.23) gives $|\omega - \omega_H|/\omega_H \gg 10^{-1}$ to 10^{-2}. Moreover, in the F layer with $\omega \approx \omega_H$ the thermal motion is important only in a relatively very narrow frequency band $|\omega - \omega_H| \sim 10^{-2}\,\omega_H \sim 10^5$.

space along $\boldsymbol{v}_{gr}$ (in the z'-direction) and will be damped in space as $e^{-\gamma z'/v_{gr}}$ (Fig. 12.5); in the z-direction (along $\boldsymbol{k}$) this corresponds to a damping of the form $e^{-\omega\varkappa z/c} = e^{-\gamma z/v_{gr}\cos(\boldsymbol{k},\boldsymbol{v}_{gr})}$, since z/z' = cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{v}_{gr}$. For $\alpha = 0$, of course, this cosine is unity and (12.27) gives (12.26)†.

From (12.26), when $(\omega - \omega_H)^2 \ll \omega_0^2$ and $\omega_0^2 \gg \nu_{eff}^2$, we have

$$\gamma_{1,\,coll} \approx \nu_{eff}. \tag{12.28}$$

When $|\omega - \omega_H|/\omega_H \sim \beta_T n_1 \ll 1$ and formula (12.24) can be used as a rough approximation, the magnetic bremsstrahlung damping γ_1 is

$$\gamma_1 \sim \beta_T n_1 \omega_H \sim \left(\frac{\omega_0}{\omega_H}\beta_T\right)^{\frac{2}{3}} \omega_H,$$

$$n_1 \sim (\omega_0/\omega_H)^{\frac{2}{3}} \beta_T^{-\frac{1}{3}}.$$

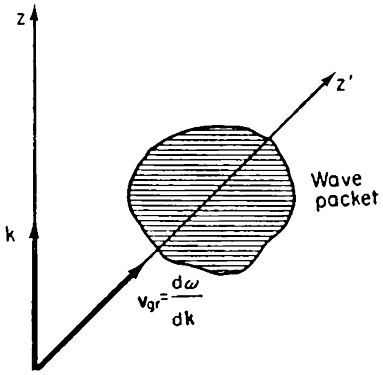

FIG. 12.5. Motion of a wave packet.

In the solar corona, putting $\omega_0 \sim \omega_H$ and $\beta_T \sim 10^{-2}$, we have $\gamma_1 \sim 5 \times 10^{-2}\omega_H \gg \gamma_{1,coll} \sim 10\,\text{sec}^{-1}$. For $|\omega - \omega_H|/\omega_H = 1/10$, $\beta_T = 10^{-2}$ and $\omega_H^2 = \omega_0^2$ we have (in the range $\omega < \omega_H$) $n_1^2 = 10$ and $\gamma_1 = 3 \times 10^{-3}\omega_H$; for $|\omega - \omega_H|/\omega_H = 0{\cdot}2$ and the other conditions unchanged, we have $n_1^2 \approx 5$ and $\gamma_1 \sim 10^{-17}\omega_H \ll \gamma_{1,coll}$. Thus in this example, when $|\omega - \omega_H|/\omega_H > 0{\cdot}15$ only the absorption due to collisions need be taken into account, and when $|\omega - \omega_H|/\omega_H < 0{\cdot}15$ only the magnetic bremsstrahlung absorption, which becomes very large as ω approaches ω_H.

These properties of the absorption near the resonance frequency ω_H when $\alpha = 0$ and in analogous cases ($\alpha \neq 0$, $\omega = s\omega_H$) are characteristic of the absorption lines in any gaseous medium when the Doppler broadening of the line and the broadening by collisions (or natural damping) are taken into account. In the centre of the line the Doppler broadening is the more important,

† For the expression (12.27) to be valid it is necessary that the concept of the group velocity should be meaningful and that the inequalities $\gamma \ll \omega$ and $q \ll k$ should hold. In the particular case of a magnetoactive plasma, formula (12.27) is inapplicable near resonances ($\omega \approx \omega_H$, $\omega \approx 2\omega_H$), when the condition $\gamma/\omega \ll \beta_T n \cos\alpha = \sqrt{(\varkappa T/m)}(k/\omega)\cos\alpha$ is not satisfied (see [93]).

but it decreases exponentially in the "wings" of the line. The collision broadening decreases much less rapidly away from the centre of thc line (according to a power law) and is therefore thc more important in the "wings", i.e. at frequencies fairly far from resonance. The case of a magnetoactive plasma differs in this respect from those occurring in optics only in that polarisation and refraction of the waves undergoing absorption must also be taken into account (the refractive index is not equal to unity).

Resonance absorption for an arbitrary angle α

Near the first resonance $\omega = \omega_H$, and for any angle α, we have [91]

$$\gamma_{1,2} = (A + BC)/D, \tag{12.29}$$

where

$$A = \tfrac{1}{2}\nu_{\text{eff}}[2(v-1)n_{1,2}^4 + 2(v^2-4v+2)n_{1,2}^2 - 3v^2 + 6v - 2],$$

$$B = \sqrt{\left(\frac{1}{2}\pi\right)}\,\frac{\omega(1-\sqrt{u})^2}{\beta_T n_{1,2}|\cos\alpha|}\exp\left[-\frac{(1-\sqrt{u})^2}{2\beta_T^2 n_{1,2}^2\cos^2\alpha}\right],$$

$$C = (v\cos 2\alpha - 1)n_{1,2}^4 + (-\tfrac{1}{4}v^2\tan^2\alpha - v^2\sin^2\alpha - 2v\cos^2\alpha + 2)n_{1,2}^2 + \\ + (v-1)(1-\tfrac{1}{4}v^2) + \tfrac{1}{4}v(2-v)\tan^2\alpha,$$

$$D = (v-1)n_{1,2}^4 + [2(1-v)^2 + v(\cos^2\alpha - 3)]u_{1,2}^2 + (v-2)^2 v + (1-v)(2v-1);$$

here $v = \omega_0^2/\omega^2$, $u = \omega_H^2/\omega^2$, and the values n_1 and n_2 must be taken for waves 1 and 2 respectively. For wave 3 near the first resonance with large values of n_3 it is shown below that $\tilde{n}_3^2 < 0$ [see (12.56)]. Of course, when $\tilde{n}^2 < 0$ it is meaningless to take account of the damping γ.

Near the second resonancc $\omega = 2\omega_H$ wc obtain

$$\gamma = 4\omega\beta_T n\frac{\sin^2\alpha}{|\cos\alpha|}B\left(\frac{1}{4}\right)\exp\left[-\frac{(1-2\sqrt{u})^2}{2\beta_T^2 n^2\cos^2\alpha}\right] \tag{12.30}$$

and near the third resonance $\omega = 3\omega_H$

$$\gamma = \frac{243}{8}\omega\beta_T^3 n^3\frac{\sin^4\alpha}{|\cos\alpha|}B\left(\frac{1}{9}\right)\exp\left[-\frac{(1-3\sqrt{u})^2}{2\beta_T^2 n^2\cos^2\alpha}\right], \tag{12.31}$$

where

$$B(u) = \frac{\sqrt{(\frac{1}{8}\pi)}\,v(u-1)}{\{\ \}}\left\{\frac{1}{2}n^4\sin^2\alpha + \right. \\ \left. + \left[v\left(\frac{1}{2}+\frac{1}{2}\cos^2\alpha + \frac{\sin^2\alpha}{1+\sqrt{u}}\right) - \left(\frac{1}{2}\sin^2\alpha + 1\right)\right]n^2 + (v-1)\left(\frac{v}{1+\sqrt{u}} - 1\right)\right\}, \tag{12.32}$$

with

$$\{\ \} = \{(v+u-2)n^4 + [2(1-v)^2 + uv(1+\cos^2\alpha) - 4u + \\ + 4(1-v)]n^2 + (v-2)[(1-v)^2 - u] + (1-v)(u+2v-2)\}.$$

In equations (12.30)–(12.32) collisions are neglected and n_1 or n_2 must be taken in accordance with the type of wave considered. When $\alpha \to 0$, formula (12.29) becomes the sum of (12.24) and (12.28) as it should (cf. also the condition (12.34) below with $s = 1$).

From (12.29) and (12.31) we see that with $\nu_{\text{eff}} = 0$

$$\gamma(s) \sim \frac{(\beta_T n)^{2s-3} \sin^{2s-2}\alpha}{u^{s-1}} \exp\left[-\frac{(1 - s\sqrt{u})^2}{2\beta_T^2 n^2 \cos^2\alpha}\right], \quad \omega \approx s\,\omega_H. \quad (12.33)$$

This estimate is correct to within a numerical factor even for $s > 3$. For $s = 1$ it must be remembered that the small factor $(1 - \sqrt{u})^2$ appears [cf. (12.29)].

Formula (12.31) is valid both near resonance (in the "wings" of the line) and in the resonance region itself; formulae (12.29) and (12.30) are valid only near resonance, and not at the actual resonance. More precisely, the inequalities

$$\left.\begin{aligned} &|\omega - s\omega_H|/k \gg \sqrt{(\varkappa T/m)} \cos\alpha \\ \text{or} \quad &(1 - s\sqrt{u})^2/\beta_T^2 n^2 \cos^2\alpha \gg 1 \end{aligned}\right\} \quad (12.34)$$

(with $s = 1, 2$) must hold, together with the conditions

$$\left.\begin{aligned} &\delta = \frac{\varkappa T}{m}\,\frac{k^2}{\omega_H^2}\sin^2\alpha = \frac{\beta_T^2 n^2}{u}\sin^2\alpha \ll 1, \\ &(\varkappa T/m)\,k^2\cos^2\alpha \gg \nu_{\text{eff}}^2, \quad \omega^2 \gg (\varkappa T/m)\,k^2\cos^2\alpha, \quad \omega_0^2 \gg \nu_{\text{eff}}^2. \end{aligned}\right\} \quad (12.35)$$

The first two conditions (12.35) are also necessary for formula (12.31) to be valid; formulae (12.29)–(12.31) are invalid also as $\alpha \to \frac{1}{2}\pi$.

In considering wave propagation in a plasma we usually need to know not γ but $q = \omega\varkappa/c$ for a real frequency ω (the wave field varying as $e^{i\omega t - ik_0 z - qz}$). For weak absorption γ and q are related by (12.27), but for strong absorption and considerable dispersion the concept of a group velocity is inapplicable, and consequently so is (12.27). In the case of strong absorption it is therefore necessary to solve the problem directly with the boundary condition as in § 8. For weak damping, this procedure, though no longer obligatory, is nevertheless more convenient than first calculating γ and then deriving q.

In the "wings" of the absorption line $\omega \approx \omega_H$ with the conditions (12.34) and (12.35) we have (here $k \equiv k_0$)

$$\begin{aligned} \frac{q}{k} = \frac{\varkappa}{n} = &\frac{(\nu_{\text{eff}}/\omega)\,\{2(v-1)\,n^4 + 2(v^2 - 4v + 2)\,n^2 - 3v^2 + 6v - 2\}}{2vn^2[2 + \sin^2\alpha - 2v - 2n^2\sin^2\alpha + 4(1 - \sqrt{u})(1 - v)\,n^2/v]} + \frac{(1 - \sqrt{u})^2}{v\,n^2} \times \\ &\times \frac{\{(v\cos 2\alpha - 1)\,n^4 + (-\frac{1}{4}v^2\sin^2\alpha - 2v\cos^2\alpha + 2)\,n^2 + (v-1)(1 - \frac{1}{4}v^2)\}}{2 + \sin^2\alpha - 2v - 2n^2\sin^2\alpha + 4(1 - \sqrt{u})(1 - v)\,n^2/v} \times \\ &\times \frac{\sqrt{(\frac{1}{2}\pi)}}{\beta_T n\cos\alpha} \exp\left[-\frac{(1 - \sqrt{u})^2}{2\beta_T^2 n^2\cos^2\alpha}\right]. \end{aligned} \quad (12.36)$$

Here the damping is assumed to be weak, i.e. $q/k \ll 1$. For $\alpha = 0$ (12.36) gives

$$\frac{q}{k} = \frac{\nu_{\text{eff}}/\omega}{2(\sqrt{u}-1)} + \sqrt{\left(\frac{1}{8}\pi\right)} \frac{\sqrt{u}-1}{\beta_T n_1} \exp\left[-\frac{(1-\sqrt{u})^2}{2\beta_T^2 n_1^2}\right], \tag{12.37}$$

where we have assumed the inequality

$$n_1^2 = 1 - v/(1-\sqrt{u}) \approx v/(\sqrt{u}-1) \gg 1$$

to hold.

Wave 2 is damped only by collisions when $\alpha = 0$. As we have seen in § 11 (see Figs. 11.2b and 11.10b), for $u > 1$, small angles $\alpha \neq 0$ and $n^2 \gg 1$ the extraordinary wave (for $\alpha = 0$) is replaced by the ordinary wave. Accordingly, when $|1-v| \gg (v/|1-\sqrt{u}|)\sin^2\alpha$, $v/(\sqrt{u}-1) \gg 1$, formula (12.37) relates to the ordinary wave in the range $n_2^2 \gg 1$.

Formulae (12.36) and (12.37) can be used to obtain estimates even when

$$\frac{|1-\sqrt{u}|}{\beta n \cos\alpha} = \frac{|\omega - \omega_H|}{\sqrt{(\varkappa T/m)}\, k \cos\alpha} \sim 1.$$

For $\alpha = 0$ we have in this region $n_1^2 \gg 1$ and $q_1/k \sim 1$, but for $\alpha \gg |\sqrt{u}-1|$ and $n_{1,2}^2 \sim 1$ the result $q_{1,2}/k \sim \beta_T \ll 1$ is approximately correct. For the first resonance $\omega = \omega_H$, angles $\alpha \to 0$ are distinctive in that here there is resonance even in the absence of thermal motion ($\beta_T = 0$; see [62] and the end of § 11).

If

$$\left.\begin{aligned} &|\omega - \omega_H|/k \ll \sqrt{(\varkappa T/m)}\cos\alpha, \quad \omega \gg \nu_{\text{eff}}, \\ &\sqrt{(\varkappa T/m)}\, k \cos\alpha \gg \nu_{\text{eff}}, \quad q \ll k, \end{aligned}\right\} \tag{12.38}$$

we obtain in the central part of the line $\omega \approx \omega_H$

$$\begin{aligned} \frac{q}{k} = {} & \frac{\sqrt{(2/\pi)}(\beta_T/n\,v)\cos\alpha}{2v - 2 - \sin^2\alpha + 2n^2\sin^2\alpha} \times \\ & \times \left\{\left[1 - \left(1 - \frac{7}{4}\sin^2\alpha\right)v\right] n^4 - \right. \\ & - \left[2 + v\left(-\frac{5}{2} + \frac{7}{4}\sin^2\alpha\right) + \frac{1}{4}v^2(2\cos 2\alpha - \tan^2\alpha)\right] n^2 + \\ & \left. + \left[1 - \frac{3}{2}v + \frac{1}{2}v^2(1-\tan^2\alpha) + \frac{1}{4}v^3\tan^2\alpha\right]\right\}. \end{aligned} \tag{12.39}$$

For $n_{1,2} \sim 1$ this gives $q/k \sim \beta_T$, in agreement with the above estimate for the range $|\omega - \omega_H|/\sqrt{(\varkappa T/m)}\, k\cos\alpha \sim 1$. When $\alpha = 0$ formula (12.39) is invalid, since $q \sim k$ and the last condition (12.38) is not satisfied. The case $\alpha = 0$ for the centre of the line is discussed in [82]. It should be mentioned that in the centres of the lines $\omega \approx s\omega_H$ ($s = 1, 2, 3, \ldots$), as well as in their "wings", when $\nu_{\text{eff}} \gg \sqrt{(\varkappa T/m)}\, k\cos\alpha$ the absorption is determined by collisions and the formulae of § 11 may be used.

For the second and third resonances (absorption lines) we have for waves 1 and 2

$$\frac{q}{k} =$$

$$= \frac{(\nu_{\text{eff}}/\omega)\{(u-3+2v)n^4+(2v^2-8v-2u+6)n^2+(-3v^2+6v+u-3)\}}{2n^2\{2(1-u-v+uv\cos^2\alpha)n^2-[2(1-v)^2+(1+\cos^2\alpha)uv-2u]\}} +$$

$$+ \frac{[(u-1)/n^2]\,v(\beta_T n/u)\sin^2\alpha}{2\cos\alpha\{2(1-u-v+uv\cos^2\alpha)n^2-[2(1-v)^2+(1+\cos^2\alpha)uv-2u]\}} \times$$

$$\times \left\{\frac{1}{2}n^4\sin^2\alpha + \left[v\left(\frac{1}{2}+\frac{1}{2}\cos^2\alpha+\frac{\sin^2\alpha}{1+\sqrt{u}}\right)-\frac{1}{2}\sin^2\alpha-1\right]n^2 + \right.$$

$$\left. + \left[\frac{v^2}{1+\sqrt{u}} - v\left(\frac{1}{1+\sqrt{u}}+1\right)+1\right]\right\}\sqrt{\left(\frac{1}{2}\pi\right)} \times$$

$$\times \left[\exp\left\{-\frac{(1-2\sqrt{u})^2}{2\beta_T^2 n^2\cos^2\alpha}\right\} + \frac{3\beta_T^2 n^2}{8u}\sin^2\alpha\exp\left\{-\frac{(1-3\sqrt{u})^2}{2\beta_T^2 n^2\cos^2\alpha}\right\}\right]; \qquad (12.40)$$

here, for $\omega \approx 2\omega_H$, we must substitute $u = \frac{1}{4}$ and may omit the second exponential term; for $\omega \approx 3\omega_H$ we must substitute $u = \frac{1}{9}$ and may omit the first exponential term. (In the exponents, of course, the quantity $1 - s\sqrt{u}$ is retained without substituting $u = \frac{1}{4}$ or $\frac{1}{9}$.) Formula (12.40) is valid for both waves and throughout the line (the conditions $\delta \ll 1$, $\sqrt{(\varkappa T/m)}\,k\cos\alpha \gg \nu_{\text{eff}}$ and $\omega \gg \nu_{\text{eff}}$ are assumed to hold). Moreover, the region $\alpha \to \frac{1}{2}\pi$ is excluded.† For $n \sim 1$ we have at the centre of the line (with $\nu_{\text{eff}} = 0$)

$$\left.\begin{aligned} &(q^{(0)}/k)_{\omega=\omega_H} \sim v\beta_T, \quad (q^{(0)}/k)_{\omega=2\omega_H} \sim v\beta_T, \\ &(q^{(0)}/k)_{\omega=3\omega_H} \sim v\beta_T^3, \quad n \sim 1, \quad \alpha \neq 0, \quad \alpha \neq \tfrac{1}{2}\pi, \quad k = \omega n/c. \end{aligned}\right\} \qquad (12.41)$$

The range of angles $\alpha \to 0$ is here excluded for the first resonance $\omega = \omega_H$, for the reason already stated, namely that absorption is present even when $\beta_T \to 0$ (see [62] and the end of § 11). For the resonances $\omega = s\,\omega_H$ $(s > 1)$, the range of angles $\alpha \to 0$ is excluded because when $\alpha = 0$ there is resonance absorption only for the first resonance $\omega = \omega_H$. Finally, for α very close†† to $\frac{1}{2}\pi$ we cannot assume that $n \sim 1$ at resonance. The reason is that for $\alpha = \frac{1}{2}\pi$ there is neither Doppler broadening of the resonance lines nor Cherenkov absorption (see, for example, the expression (12.10), which for $\alpha = \frac{1}{2}\pi$ becomes $\omega = s\omega_H$). Hence these absorption lines are broadened only by collisions, and when these are absent or sufficiently rare the lines are very sharp. The index of refraction n then varies considerably, as in the well-known

† It must be borne in mind that, for a wave propagated at an angle $\pi - \alpha$ to the field, the values of n and γ (or q) are the same as for waves propagated with $\alpha \leqq \frac{1}{2}\pi$. Hence, for example, $\cos\alpha$ in (12.40) is to be understood as $|\cos\alpha|$.

†† Formula (12.40) for $\omega \approx 2\omega_H$ is, roughly speaking, invalid when $\cos\alpha \lesssim \beta$, and for $\omega \approx 3\omega_H$ when $\cos\alpha \lesssim \beta^3$.

case of anomalous dispersion in optics. In particular, near the resonances $\omega = s\omega_H$ "gaps" appear, i.e. regions with $\tilde{n}^2 < 0$, where the waves are damped (see [43, 49, 67, 91] for further details).†

If the angle α is not too close to $\frac{1}{2}\pi$, the Doppler broadening of the lines is considerable, so that $n \sim 1$. More precisely, when $\alpha \neq \frac{1}{2}\pi$ and in the resonance region (but with $s\omega_H \neq \omega_\infty$) we can use the ordinary formulae (11.6) for $\tilde{n}^2_{1,2}$ to give an accuracy of the order of β_T^2. The absorption must be calculated from formulae (12.36)–(12.40). These formulae lead to the approximation (12.41), where the factor v is retained because the absorption must tend to zero with v.

Since $k \equiv k_0 = \omega n/c$, $q = \omega\varkappa/c$, $\mu = 2\omega\varkappa/c = 2q$, we have from (12.41)

$$\left.\begin{aligned} q^{(0)}_{\omega=\omega_H} \sim q^{(0)}_{\omega=2\omega_H} \sim \omega\, v\, \beta_T/c, \\ q^{(0)}_{\omega=3\omega_H} \sim \omega\, v\, \beta_T^3/c \;\; (n \sim 1,\; \alpha \neq 0,\; \alpha \neq \tfrac{1}{2}\pi). \end{aligned}\right\} \tag{12.42}$$

In general [85]

$$q^{(0)}_{\omega=s\omega_H} \sim \frac{s^{2s}}{2^s \cdot s!}\,\frac{\omega}{c}\, v\, \beta_T^{2s-3} \quad (s \geq 2,\; \alpha \sim 1,\; n \sim 1). \tag{12.43}$$

These formulae, which give only the order of magnitude, apply formally to both wave 1 and wave 2. In practice, however, when the numerical factors are taken into account, it is found that formulae (12.41)–(12.43) are valid for the extraordinary wave but give values for the ordinary wave which are one or two orders of magnitude too high (see below).

When collisions only are considered (i.e. $\beta_T \to 0$)

$$q_{\text{coll}} = \omega\,\varkappa/c \sim v\,\nu_{\text{eff}}/c \quad (n \sim 1,\; \alpha \sim 1); \tag{12.44}$$

cf. (11.5) with $\omega^2 \gg \nu_{\text{eff}}$. Comparing (12.42) and (12.44), we see that for the line centre

$$\left.\begin{aligned} q_{\text{coll}}/q^{(0)}_{\omega=\omega_H,\,2\omega_H} \sim \nu_{\text{eff}}/\omega\,\beta_T, \\ q_{\text{coll}}/q^{(0)}_{\omega=3\omega_H} \sim \nu_{\text{eff}}/\omega\,\beta_T^3. \end{aligned}\right\} \tag{12.45}$$

For the F layer of the ionosphere (with $\beta_T \sim 3 \times 10^{-4}$, $\nu_{\text{eff}} \sim 10^3\,\text{sec}^{-1}$, $\omega_H \sim 8 \times 10^6\,\text{sec}^{-1}$),

$$\left.\begin{aligned} q_{\text{coll}}/q^{(0)}_{\omega=\omega_H,\,2\omega_H} \sim 0{\cdot}3, \\ q_{\text{coll}}/q^{(0)}_{\omega=3\omega_H} \sim 10^6, \end{aligned}\right\} \tag{12.46}$$

† By "Doppler broadening" we here mean that which is due to the first-order Doppler effect. If terms of order β^2 are taken into account, there will of course be broadening even at $\alpha = \frac{1}{2}\pi$ on account of the second-order Doppler effect. The latter is automatically included by using the relativistic expression for the frequency $\omega_H^* = (mc^2/E)|e|H/mc \equiv (mc^2/E)\omega_H = \sqrt{(1-\beta^2)}\,\omega_H$. Evidently, when $\alpha = \frac{1}{2}\pi$ and $\beta^2 \ll 1$, the frequencies $s\omega_H^* = s\omega_H - \frac{1}{2}s\beta^2\omega_H$ are emitted and the line width is given by the parameter $s\beta_T^2\omega_H = s(\varkappa T/mc^2)\omega_H$. As a result, the range of angles $\alpha \to \frac{1}{2}\pi$ is actually not distinctive for waves 1 and 2 even for $s = 3$ (though this is not true for $s = 1$ or 2). For the ordinary wave there are also no important singularities for $\omega = 2\omega_H$ (for $\omega = \omega_H$ there is a singularity as $\alpha \to \frac{1}{2}\pi$); for the extraordinary wave with $\alpha \to \frac{1}{2}\pi$ there is a singularity at $\omega = 2\omega_H$ but not at $\omega = \omega_H$. For the plasma wave 3 there are no special properties for $\alpha \to \frac{1}{2}\pi$ if $s \geq 4$; see [86, 327].

i.e. the resonance absorption at the frequencies ω_H and $2\omega_H$ exceeds the absorption due to collisions but at $\omega = 3\omega_H$ the resonance absorption is negligible. For various reasons, however, the ionospheric absorption in the F layer at $\omega = \omega_H$ and $\omega = 2\omega_H$ is difficult to observe; the propagation of such waves in the F layer does not arise in radio communications. In the solar corona (with $\nu_{\text{eff}} \sim 10\,\text{sec}^{-1}$, $\beta_T \sim 10^{-2}$ and $\omega \sim 6\times 10^8\,\text{sec}^{-1}$) we have

$$q_{\text{coll}}/q^{(0)}_{\omega=\omega_H,\,2\omega_H} \sim 10^{-6},$$
$$q_{\text{coll}}/q^{(0)}_{\omega=3\omega_H} \sim 10^{-2},$$
$$q_{\text{coll}}/q^{(0)}_{\omega=4\omega_H} \sim 10^{2}.$$

Thus in the corona the resonance absorption must in general be taken into account for $\omega = s\omega_H$ with $s = 1, 2, 3$ (see § 36). The formulae (12.41)–(12.46), of course, do not enable us to do more than ascertain whether a more exact calculation of the collisionless absorption is necessary or not. To determine the quantity q_1 itself, and still more so q_2, formulae (12.41)–(12.46) are not suitable, and (e.g.) for $\omega \approx 2\omega_H$ and $\omega \approx 3\omega_H$ we must use the cumbersome expression (12.40). Figs. 12.6 and 12.7 show the values of $q_{1,2}/k_{1,2}\beta_T$ for $\omega = 2\omega_H$ and $q_{1,2}/k_{1,2}\beta_T^3$ for $\omega = 3\omega_H$, calculated [92] from formula (12.40) with $\alpha = 45°$. Here $\nu_{\text{eff}} = 0$ and the parameter $v = \omega_0^2/\omega^2$ which is plotted as the abscissa gives the dependence of the absorption only on $\omega_0^2 = 4\pi e^2 N/m$, since the frequency ω is fixed. (It may also be recalled that for wave 1 in the range $v_{10}^{(-)} < v < v_{1\infty}$ we have $n_1^2 < 0$, and the calculation of the absorption would

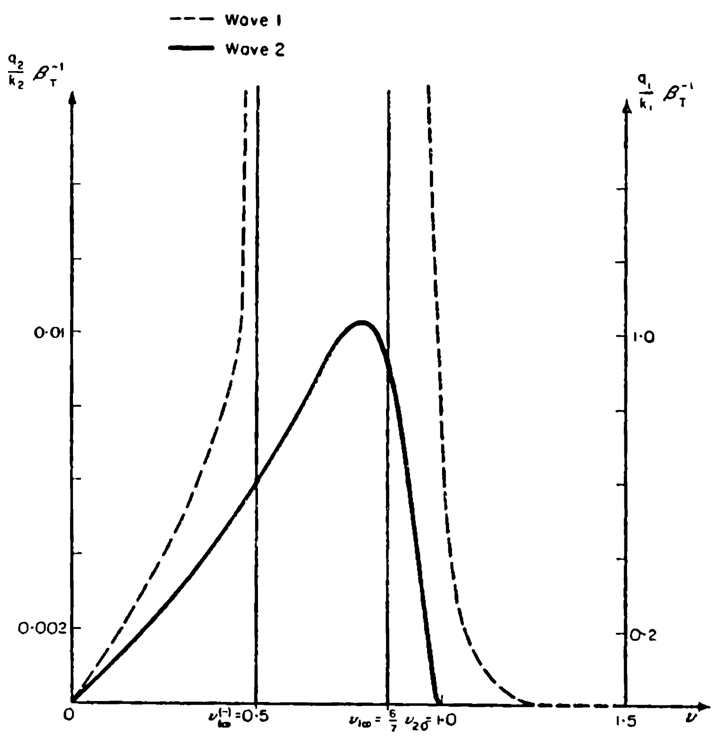

Fig. 12.6. The quantities $q_{1,2}/k_{1,2}\beta_T$ as functions of $v = \omega_0^2/\omega^2$ for $u = \omega_H{}^2/\omega^2 = \frac{1}{4}$ (i.e. $\omega = 2\omega_H$), $\alpha = 45°$ and $\nu_{\text{eff}} = 0$.

then be meaningful only in the problem with boundary conditions.) To find $q_{1,2} = \omega\varkappa_{1,2}/c$, the values given in Figs. 12.6 and 12.7 must be multiplied by $k_{1,2}\beta_T = \omega n_{1,2}\beta_T/c$ and $k_{1,2}\beta_T^3 = \omega n_{1,2}\beta_T^3/c$ respectively. The diagrams show that under these conditions q_2 is usually two orders of magnitude less than q_1.

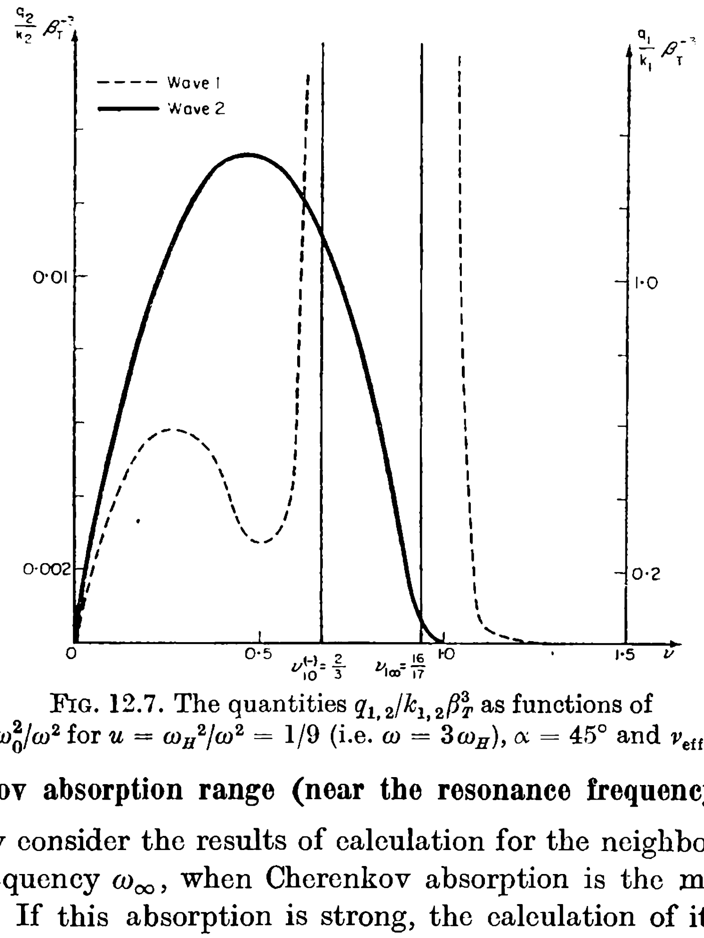

FIG. 12.7. The quantities $q_{1,2}/k_{1,2}\beta_T^3$ as functions of $v = \omega_0^2/\omega^2$ for $u = \omega_H{}^2/\omega^2 = 1/9$ (i.e. $\omega = 3\omega_H$), $\alpha = 45°$ and $\nu_{\text{eff}} = 0$.

The Cherenkov absorption range (near the resonance frequency ω_∞)

Let us now consider the results of calculation for the neighbourhood of the resonance frequency ω_∞, when Cherenkov absorption is the most important (for $\nu_{\text{eff}} \to 0$). If this absorption is strong, the calculation of it is usually of little interest. In practice the determination of the absorption when it is weak and the estimation of the values of the parameters for which the absorption begins to increase rapidly are of importance. For this reason we shall assume the condition (12.15). The physical significance of this condition has in essence already been explained: it is that the Cherenkov absorption (and emission) is possible only when there are relatively few electrons in the "tail" of the Maxwellian velocity distribution. Nevertheless, on account of the great importance of this problem, we shall derive here the conditions for absorption to be small, using a different approach and terminology (see [89]).

A perturbation in the plasma which corresponds to one of the normal waves is damped in a time $t \lesssim 1/\omega$, if the thermal motion of the plasma electrons causes them to move in this time a distance of the order of $\lambda\!\!\!^- = \lambda/2\pi$ along the normal to the wave (i.e. in the direction of $\mathbf{k}$). For with this movement of the electrons during the period of the oscillations they also "transfer" their directed velocity, acquired by the action of the wave field, into a region of space where the phase of the wave differs by $\Delta\varphi \gtrsim 1$ from its value at the initial instant. It is clear that in these conditions we cannot speak of a slightly damped directed motion in the wave. Thus there is strong damping if

$$v_k/\omega \gtrsim \lambda = 1/k, \tag{12.47}$$

where v_k is the mean velocity of the motion of the plasma electrons in the direction of $\boldsymbol{k}$. In other words, if during a time t less than the period $1/\omega$ of the oscillations an electron moves a distance of at least the order of λ along $\boldsymbol{k}$ by virtue of its thermal motion, then it cannot acquire any considerable directed velocity (on account of the rapid decrease of the mean field acting on the electron, because of the averaging along the path). The impossibility of producing a directed velocity signifies that the corresponding waves in which this velocity is non-zero must be strongly damped.

In an isotropic plasma $v_k \sim v_T \sim \sqrt{(\varkappa T/m)}$, and the condition (12.47) becomes $\omega/k = v_{\text{ph}} \ll v_T$, or $\beta_T^2 n_i^2 \gtrsim 1$. The condition for damping to be weak is†

$$\beta_T^2 n_i^2 \ll 1. \tag{12.48}$$

This condition can be violated in a non-relativistic isotropic plasma only for a plasma wave. This criterion (12.48) then coincides with the condition (8.31) for weak damping of a plasma wave.

In a magnetoactive plasma the condition (12.48) is again maintained, provided that the magnetic field does not prevent the movement of the electrons over a distance of the order of λ in the direction of $\boldsymbol{k}$. This is true if the component along $\boldsymbol{k}$ of the radius of revolution of an electron $r_H \sin\alpha \approx \approx (v_T/\omega_H)\sin\alpha \gtrsim \lambda$ (where α is the angle between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$), i.e.

$$\delta = (\beta_T^2 n_i^2 \sin^2\alpha)/u = (v_T k/\omega_H)^2 \sin^2\alpha \gtrsim 1. \tag{12.49}$$

When the conditions (12.48) and (12.49) hold, the absorption of waves in a magnetoactive plasma is slight, but if

$$\delta = (\beta_T^2 n_i^2 \sin^2\alpha)/u = (v_T k/\omega_H)^2 \sin^2\alpha \ll 1 \tag{12.50}$$

the electron can move a distance λ along the normal to the wave only because of the motion along the field $\boldsymbol{H}^{(0)}$, when $v_k \sim v_T \cos\alpha$. Hence it follows that when $\delta \ll 1$ the condition for strong absorption (12.47) becomes

$$\beta_T^2 n_i^2 \cos^2\alpha \gtrsim 1, \tag{12.51}$$

i.e. the condition for weak absorption is (12.15), which is less stringent than (12.48). Thus the condition (12.15) is not only necessary but also sufficient

† Here, as in similar cases, we write an inequality of the form $x \ll 1$ in the form $x^2 \ll 1$ also (in this case, passing from $v_T/v_{\text{ph}} = \beta_T n_i \ll 1$ to $\beta_T^2 n_i^2 \ll 1$). Such a substitution is possible because in quantitative calculations the condition $x^2 \ll 1$ is sufficient, and is more easily satisfied.

for weak damping in the range (12.50) only. In fairly strong magnetic fields the conditions (12.15) and (12.50) arc usually satisfied simultaneously, and we shall now consider this important casc.†

If damping is neglected, the equation for $\tilde{n}^2$ is

$$\left.\begin{aligned}&\beta_T^2 v R \tilde{n}^6 - [1 - u - v + u v \cos^2\alpha]\tilde{n}^4 + \\ &+ [2(1-v)^2 + u v \cos^2\alpha - u(2-v)]\tilde{n}^2 + (1-v)[u - (1-v)^2] = 0, \\ &R = \frac{3\sin^4\alpha}{1-4u} + \left(1 + \frac{5-u}{(1-u)^2}\right)\sin^2\alpha\cos^2\alpha + 3(1-u)\cos^4\alpha,\end{aligned}\right\} \quad (12.52)$$

where terms $O_1(\beta_T^2)$ and $O_2(\beta_T^2)$ proportional to β_T^2 and having denominators $(1 - u)^3$ and $(1 - 4u)$ have been omitted from the coefficients of $\tilde{n}^4$ and $\tilde{n}^2$. These terms are certainly negligible if

$$\left.\begin{aligned}(1 - u)^3 &= (1 - \omega_H^2/\omega^2)^3 \gg \beta_T^2 = \varkappa T/mc^2, \\ (1 - 4u) &= 1 - (2\omega_H)^2/\omega^2 \gg \beta_T^2.\end{aligned}\right\} \quad (12.53)$$

These conditions, which we shall assume to be satisfied, signify that frequency ranges sufficiently far from the resonances $\omega = \omega_H$ and $\omega = 2\omega_H$ are considered. The higher resonances $(\omega = 3\omega_H$, etc.) do not appear, simply because to take account of them would correspond to the inclusion of terms of order β_T^4, β_T^6, etc., which are omitted in (12.52), This is, of course, legitimate when β_T^2 is sufficiently small.

In the range where $\tilde{n}_{1,2}^2 \sim 1$, the corrections to $\tilde{n}_{1,2}^2$ of order β_T^2 are assumed to be negligible, since the non-relativistic theory is used. The thermal corrections are therefore significant only when $\tilde{n}^2 \gg 1$, and we shall considcr this range in what follows.

Apart from the terms $O_1(\beta^2)$ and $O_2(\beta^2)$, which are small in the conditions assumed, equation (12.52) differs from the dispersion relation (12.5) obtained with the quasihydrodynamic approximation in the coefficicnt of $\tilde{n}^6$, which is a different function of u, v and α. For cxample, in equation (12.52) this coefficient is $3\beta_T^2 v/(1 - 4u)$ when $\alpha = \frac{1}{2}\pi$, and has opposite signs for $u > \frac{1}{4}$ and $u < \frac{1}{4}$; in (12.5), this coefficient is $\beta_T^2(1 - u\cos^2\alpha)$, which for $\alpha = \frac{1}{2}\pi$ is independent of u, and is always positive if $u < 1$.

In the range where the third root of cquation (12.52) is large in comparison with the roots $\tilde{n}_1^2$ and $\tilde{n}_2^2$, the value of this root $\tilde{n}_3^2$ is given by the first two terms in (12.52), and is

$$\tilde{n}_3^2 \approx \frac{1 - u - v + u v \cos^2\alpha}{\beta_T^2 v\left[\dfrac{3\sin^4\alpha}{1-4u} + \left(1 + \dfrac{5-u}{(1-u)^2}\right)\sin^2\alpha\cos^2\alpha + 3(1-u)\cos^4\alpha\right]}. \quad (12.54)$$

† If the field $\boldsymbol{H}^{(0)} \to 0$, the propagation of waves is almost the same as for the case of isotropy. It may be noted that the condition (12.50) can also be written $(r_H/\lambda)^2 \sin^2\alpha \ll 1$, where $r_H \sim v_T/\omega_H$ is the radius of curvature of the orbit of a particle in the magnetic field, and $\lambda = 1/k$.

If for simplicity we neglect factors such as $u, v, \cos^2\alpha$, etc., the solution (12.54) is valid when†

$$1 - u - v + uv\cos^2\alpha \gg \beta_T. \tag{12.55}$$

For $\alpha = 0$, (12.54) gives $\tilde{n}_3^2 \approx (1-v)/3\beta_T^2 v$, i.e. we have the expression (12.1) for a plasma wave in an isotropic plasma. This is as it should be, since for $\alpha = 0$, waves 1 and 2 are transverse, and the longitudinal wave cannot differ from a plasma wave in an isotropic medium.

For $0 < \alpha < \frac{1}{2}\pi$, as we approach the resonances $u \to 1$ or $u \to \frac{1}{4}$ [if allowed by the conditions (12.53) and (12.55)], (12.54) gives

$$\left.\begin{aligned} &\tilde{n}_3^2(u \to 1) \approx -(1-u)^2/4\beta_T^2\cos^2\alpha, \\ &\tilde{n}_3^2(u \to \tfrac{1}{4}) \approx [3 - v(4-\cos^2\alpha)](1-4u)/12\beta_T^2 v\sin^4\alpha. \end{aligned}\right\} \tag{12.56}$$

Hence it is clear that, near the first gyromagnetic resonance $u \approx 1$ (i.e. $\omega \approx \omega_H$), the plasma wave cannot be propagated ($\tilde{n}_3^2 < 0$).

For the particular case $\alpha = \frac{1}{2}\pi$ the dispersion relation immediately separates into two factors. The condition for one of these to vanish gives $\tilde{n}^2$ for the ordinary wave. When $\delta = \beta_T^2 n_2^2/u \ll 1$ we have

$$\tilde{n}_2^2 = \frac{1-v}{1 + \beta_T^2 v/(1-u)}. \tag{12.57}$$

Far from the resonance $u = 1$, the thermal corrections are small and are no significant, but as resonance is approached they become important.

The dispersion relation for the other waves when $\alpha = \frac{1}{2}\pi$ and $\delta \ll 1$ is

$$\frac{3\beta^2 v}{1-4u}\tilde{n}^4 - \left[1 - u - v + \frac{4\beta^2 v}{1-4u}(1+2u-v)\right]\tilde{n}^2 + (1-v)^2 - u = 0. \tag{12.58}$$

Far from the resonance $u = \frac{1}{4}$ (i.e. $\omega = 2\omega_H$), and if

$$1 - u - v \gg \beta_T, \tag{12.59}$$

the roots of equation (12.58) are approximately

$$\tilde{n}_1^2 \approx 1 - \frac{v(1-v)}{1-u-v}, \quad \tilde{n}_3^2 \approx \frac{(1-u-v)(1-4u)}{3\beta_T^2 v}. \tag{12.60}$$

This result for $\tilde{n}_1^2$ is the same as when absorption is absent, while that for $\tilde{n}_3^2$ is obtained also from (12.54) when $\alpha = \frac{1}{2}\pi$. Equation (12.58) and the condition (12.59) also follow from (12.52) and (12.55) when $\alpha = \frac{1}{2}\pi$. When the condition (12.59) does not hold, the expressions for $\tilde{n}_1^2$ and $\tilde{n}_3^2$ are more

† This condition is derived from the requirement that, when (12.54) is substituted in (12.52), the terms in $\tilde{n}^2$ and $\tilde{n}^0$ should be small compared with those in $\tilde{n}^6$ and $\tilde{n}^4$. The simplifying assumption made in the derivation is that the coefficients of $\tilde{n}^2$ and $\tilde{n}^0$ are of the order of unity.

complex; as an example, it may be mentioned that for $1 - u - v = 0$ [i.e. $v = v_{1\infty}$ or $\omega = \omega_\infty = \sqrt{(\omega_0^2 + \omega_H^2)}$] the roots of equation (12.58) are

$$\tilde{n}^2 \approx \pm \sqrt{[(1 - 4u)\, u/3\beta^2]}. \tag{12.61}$$

Here we do not add the suffixes 1 and 3, since at the point $v_{1\infty}$ the division into waves of types 1 and 3 is meaningless (see above).

Figs. 12.8–12.12 show the form of the curves of $\tilde{n}^2$ given by (12.58) and (12.52). Each of these figures shows (on a logarithmic ordinate scale) only the region near the points $v_{1,2\infty}$. The curve of $\tilde{n}_2^2$ for $u < 1$ or $\tilde{n}_1^2$ for $u \cos^2\alpha > 1$

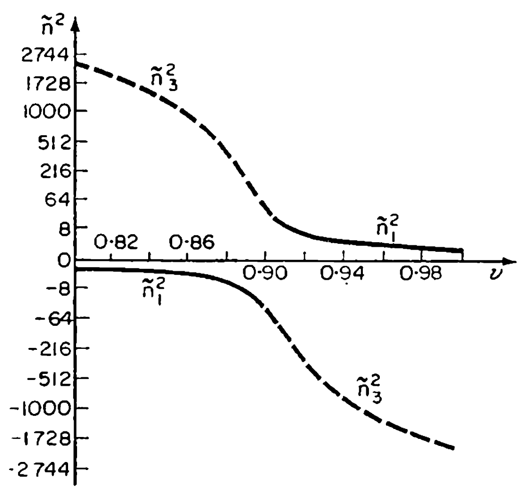

FIG. 12.8. The functions $\tilde{n}_1^2$ and $\tilde{n}_3^2$ for $u = 0{\cdot}1$, $\alpha = 90°$ and $\beta_T^2 = 10^{-5}$. The ordinate scale in Figs. 12.8–12.12 is logarithmic.

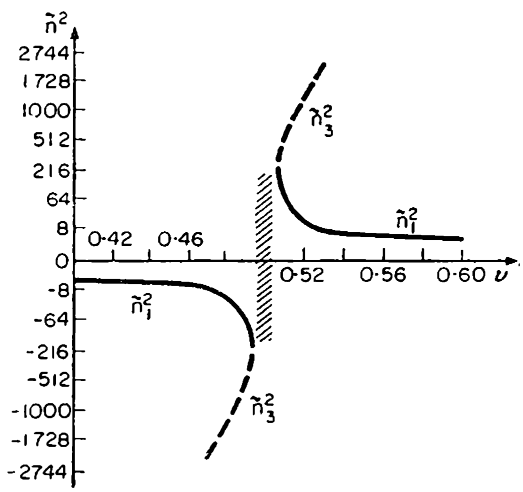

FIG. 12.9. The functions $\tilde{n}_1^2$ and $\tilde{n}_3^2$ for $u = 0{\cdot}5$, $\alpha = 90°$ and $\beta_T^2 = 10^{-5}$.

is not shown. Absorption is neglected. A comparison of Figs. 12.8, 12.11 and 12.12 with Figs. 12.2, 12.3 and 12.4 respectively shows the difference between the results of the quasihydrodynamic and kinetic calculations. For these examples the difference is quantitative only, but sometimes it may become qualitative. For instance, in the quasihydrodynamic approximation, as already noted, the curves always have the form shown in Figs. 12.2 and 12.3. In the kinetic theory, however, the curves of $\tilde{n}_3^2$ may appear either as in Figs. 12.8, 12.10a and 12.11 or as in Figs. 12.9 and 12.10b, depending on the values of u and α. The two types of curve correspond to opposite signs of the denominator in formula (12.54). For the cases represented by Figs. 12.9 and 12.10b there is a range of values of v (i.e. a range of frequencies or densities) in which there is no real value of $\tilde{n}^2$; this range is shown hatched in Fig. 12.9. In these regions the solutions for $\tilde{n}^2$ are complex, i.e. the waves are damped even if, as we have assumed, absorption is absent. The same is true when $\tilde{n}^2$ is real but negative. The only difference is that for $\tilde{n}^2 < 0$ the wave is monotonically damped as it penetrates into the medium ($E \sim e^{-\omega|\tilde{n}|z/c}$), but when $\tilde{n}^2$ is complex the damping is not

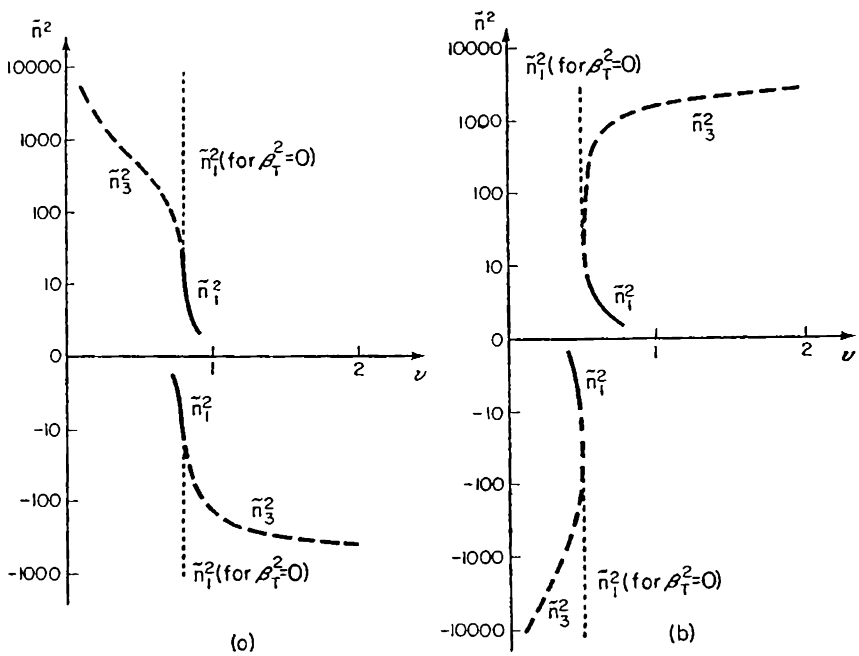

FIG. 12.10. The functions $\tilde{n}_1^2$ and $\tilde{n}_3^2$ for $\alpha = 90°$ and $\beta_T^2 = 10^{-4}$; (a) $u = 0{\cdot}2$, (b) $u = 0{\cdot}5$.

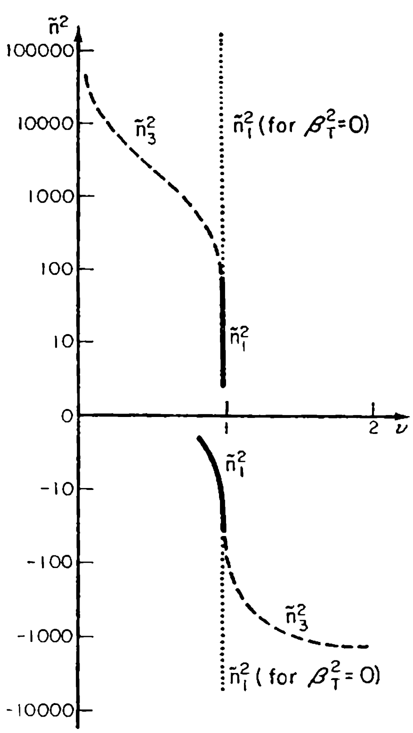

FIG. 12.11. The functions $\tilde{n}_1^2$ and $\tilde{n}_3^2$ for $u = 0{\cdot}5, \alpha = 10°$ and $\beta_T^2 = 10^{-4}$.

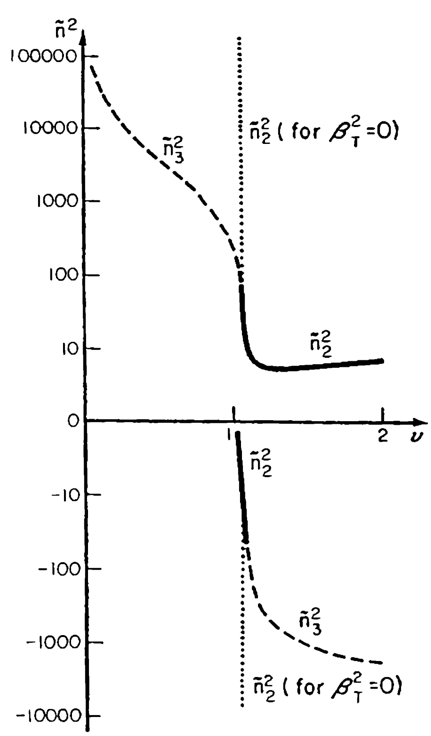

FIG. 12.12. The functions $\tilde{n}_2^2$ and $\tilde{n}_3^2$ for $u = 2, \alpha = 10°$ and $\beta_T^2 = 10^{-4}$.

monotonic. This property (the occurrence of complex $\tilde{n}^2$ even in the absence of absorption) is found near resonances in any medium when spatial dispersion is taken into account and the coefficient of the highest power of $\tilde{n}$ in the dispersion relation is negative (see [1]).

It must be emphasised that all the above diagrams correspond physically to the propagation of waves with a given frequency in media of various densities in a uniform field ($u = \omega_H^2/\omega^2$ = constant, $v = \omega_0^2/\omega^2$ variable). Of course, the problem can also be stated in other ways, with u and v variable and ω constant (see § 36) or ω varying with ω_H and ω_0 constant. The corresponding curves of $\tilde{n}^2$ may also show "gaps" corresponding to the absence of real values of $\tilde{n}^2$. For example, with $\alpha = \frac{1}{2}\pi$, ω_0^2 and ω_H^2 constant, and $\omega = \sqrt{(\omega_0^2 + \omega_H^2)}$, the values of $\tilde{n}_{1,3}^2$ are complex if $\omega < 2\omega_H$ [see (12.61)].

The calculation of the group velocity of plasma waves from the formula $v_{\mathrm{gr}} = d\omega/d\boldsymbol{k}$ shows [268] that this velocity may make either an acute or an obtuse angle with $\boldsymbol{k}$. Their directions coincide when $u = \omega_H^2/\omega^2 \to 0$ or $v \to 0$, $\alpha \to 0$ and $\alpha \to \frac{1}{2}\pi$. For other values of α and u not small, the vectors $\boldsymbol{v}_{\mathrm{gr}}$ and $\boldsymbol{k}$ are nearly at right angles.

Let us now consider the absorption of plasma waves. As has already been mentioned, when $\alpha = \frac{1}{2}\pi$ the Cherenkov absorption does not occur, and so the damping of the waves near the point $v_{1\infty} = 1 - u$ is due only to collisions (it is assumed that the resonance $u = 1$ lies sufficiently far from $v_{1\infty}$, i.e. the parameter $v_{1\infty} = \omega_0^2/\omega_\infty^2$ is not too small; and we shall not consider here the limit $u \to 0$). For angles $\alpha \neq \frac{1}{2}\pi$ there is Cherenkov damping, but near the point v_∞ it is appreciable only when n_i^2 is large, and in particular for the plasma wave 3. As we are, moreover, interested only in weak damping ($\gamma \ll \omega$), we shall also assume the conditions (12.15) and (12.50).† Then, taking account also of damping due to collisions (here and below we assume that $\omega^2 \gg \nu_{\mathrm{eff}}^2$), we have [67, 91]

$$\gamma_3 = \left(1 + \frac{u\,v\sin^2\alpha}{(1-u)^2}\right)^{-1} \left\{\frac{1}{2}\nu_{\mathrm{eff}}\left[1 + \frac{2u\,v\sin^2\alpha}{(1-u)^2}\right] + \right.$$
$$+ \sqrt{\left(\frac{1}{8}\pi\right)}\,\omega\,v\left[\frac{1}{\beta_T^3 n_3^3\cos\alpha}\exp\left(-\frac{1}{2\beta_T^2 n_3^2\cos^2\alpha}\right) + \right.$$
$$\left.\left. + \frac{\sin^2\alpha}{2u\,\beta_T n_3\cos\alpha}\left[\exp\left(-\frac{(1-\sqrt{u})^2}{2\beta_T^2 n_3^2\cos^2\alpha}\right) + \exp\left(-\frac{(1+\sqrt{u})^2}{2\beta_T^2 n_3^2\cos^2\alpha}\right)\right]\right]\right\}. \quad (12.62)$$

† Besides the conditions (12.15), (12.50) and $n_i^2 \gg 1$, the inequalities

$$[\beta_T^2 n^2/(1-\sqrt{u})^2]\cos^2\alpha \ll 1 \quad \text{and} \quad [\beta_T^2 n^2/(1-2\sqrt{u})^2]\cos^2\alpha \ll 1$$

express the remoteness of the resonances $u = 1$ and $u = \frac{1}{4}$. For $u < 1$, the plasma wave 3 is a continuation of the extraordinary waves, and for $u\cos^2\alpha > 1$ it is a continuation of the ordinary waves (when $u > 1$ and $u\cos^2\alpha < 1$, the weakly-damped plasma wave does not exist). It may also be noted that in (12.62) we cannot take $v \gg 1$ (see below).

The refractive index of a plasma wave whose damping is given by this expression is (12.54). If collisions are unimportant and damping is weak, as we have assumed, the use of equations (12.52) and (12.54) is, of course, justified, but if damping is strong ($\gamma \gtrsim \omega$), the expressions (12.52) and (12.54) are invalid. The allowance for collisions also changes the form of the curves of $(n - i\varkappa)^2$, and the change is particularly important near resonance. It is easy to see that the effect of the thermal motion on the refractive index considerably exceeds that of collisions if

$$\nu_{\text{eff}}/\omega\beta_T \ll 1; \tag{12.63}$$

this is discussed in detail in [90].

Thus the results derived from equation (12.52) must be checked by using the criteria $\gamma \ll \omega$, $\nu_{\text{eff}}/\omega\beta_T \ll 1$.

For $\alpha = 0$, (12.62) gives

$$\gamma = \gamma_0 + \gamma_{\text{coll}}, \quad \gamma_{\text{coll}} = \tfrac{1}{2}\nu_{\text{eff}},$$

$$\gamma_0 = \frac{\sqrt{(\frac{1}{8}\pi)}\,\omega v}{\beta_T^3 n_3^3}\exp\left(-\frac{1}{2\beta_T^2 n_3^2}\right)$$

$$\approx \sqrt{\left(\frac{1}{8}\pi\right)}\, e^{-3/2}\frac{\omega_0^4}{k^3}\left(\frac{m}{kT}\right)^{3/2}\exp\left(-\frac{m\,\omega_0^2}{2\varkappa T k^2}\right), \tag{12.64}$$

which agrees with the expression (8.36) for γ_0 for a plasma wave in an isotropic plasma; in (8.36) we have already used the fact that $\omega^2 = \omega_0^2 + 3(\varkappa T/m)\,k^2$, and the same has been done in deriving the last member of (12.64). This result was to be expected, since $v_\infty = 1$ for $\alpha = 0$ and we have the transition from wave 3 to the plasma wave for an isotropic medium, which has already been discussed. The same result (transition to an isotropic plasma) must evidently occur when $u = \omega_H^2/\omega^2 \to 0$. This does not immediately follow from (12.62), because this formula is itself valid only when $\delta = (\beta_T^2 n_3^2/u)\sin^2\alpha \ll 1$, which means that we cannot have $u \to 0$.

When $\beta_T^2 n_3^2 \cos^2\alpha \ll 1$ [see (12.15)] the damping is weak in the sense that $\gamma \ll \omega$, as was assumed in deriving formula (12.62). For rough estimates, however, this formula is applicable even if $\beta_T n \cos\alpha \sim 1$, when the damping may already be strong. Likewise, by means of formula (12.62) we can not only calculate weak damping of plasma waves, but also give conditions for the damping to become strong. In the range of applicability of formula (12.62) for γ we can find q for a plasma wave by means of the relation (12.27):

$$q_3 = \omega\varkappa_3/c = \gamma n_3(2 - v - u)/c(1 - u - v + uv\cos^2\alpha)$$

$$\approx \frac{\gamma n_3}{c}\,\frac{(1-u)\,[1 + (uv\sin^2\alpha)/(1-u)^2]}{(1 - u - v + uv\cos^2\alpha)}, \tag{12.65}$$

since $v \approx (1 - u)/(1 - u\cos^2\alpha)$.

It may be noted that for certain values of the parameters we have $q_3 = \omega \varkappa_3/c < 0$, i.e. the field of a wave in which the vector $\boldsymbol{k}$ is in the z-direction is proportional to $e^{-i\omega(n_3+i|\varkappa_3|)z/c}$; in such cases the component of the group velocity along the wave vector $\boldsymbol{k}$ is negative, and the wave damped along the z-axis is proportional to $e^{i\omega(n_3-i|\varkappa_3|)z/c}$.

The ordinary wave at low frequencies

The refractive index may be very large, not only near the resonances $\omega \approx s\omega_H$ and the point v_∞, but also for the ordinary wave at low frequencies: if

$$v \gg 1, \quad u\cos^2\alpha \gg 1, \quad v \gg u, \quad \omega \gg \Omega_H = |e|H^{(0)}/Mc, \tag{12.66}$$

we have for the ordinary wave

$$n_2^2 \approx v/\sqrt{u}\cos\alpha, \tag{12.67}$$

while $\tilde{n}_1^2 < 0$ [see (11.24); the propagation is quasilongitudinal].

Formula (12.62) is not applicable in this case, since in its derivation the condition $v \gg 1$ was not assumed (see the last footnote). According to (12.66) and (12.67), $n_2^2 \gg 1$ and the allowance for thermal absorption may be important. Calculations [91, 92] show that in the range (12.66) the thermal corrections to n_2^2 are small, and we have

$$\left.\begin{aligned} \gamma_2 &= \frac{\nu_{\text{eff}}}{\sqrt{u}\cos\alpha} + \sqrt{\left(\frac{1}{8}\pi\right)}\,\omega\,\frac{\sin^2\alpha}{\sqrt{u}\,\beta_T^3 n_2^3\cos^4\alpha}\exp\left(-\frac{1}{2\beta_T^2 n_2^2\cos^2\alpha}\right),\\ q_2 &= \omega\varkappa_2/c = (\gamma_2/c)\,d(n_2\omega)/d\omega = n_2\gamma_2/2c, \quad n_2^2 = v/\sqrt{u}\cos\alpha. \end{aligned}\right\} \tag{12.68}$$

The Cherenkov damping given by the second term in γ_2 may be important under certain conditions, for example in the propagation of whistlers in the upper layers of the ionosphere.

Summary

We may summarise the above results relating to the effect of the thermal motion on the propagation of high-frequency waves in a non-relativistic magnetoactive plasma.

The thermal motion gives corrections of order $\beta_T^2 = \varkappa T/mc^2 \ll 1$ which are negligible everywhere except in regions of "plasma resonance" (where $v_{1,2\infty} \approx (1-u)/(1-u\cos^2\alpha)$ and $\tilde{n}_1^2$ or $\tilde{n}_2^2$ tends to infinity as $\beta_T \to 0$ and $\nu_{\text{eff}} \to 0$) and in regions of gyromagnetic resonance $\omega \approx s\omega_H$ ($s = 1, 2, 3, \ldots$). Near plasma resonance the allowance for the thermal motion leads to the appearance of a third root for the squared refractive index, $\tilde{n}_3^2$; waves with $n = \tilde{n}_3$ are conventionally called plasma waves. The form of the curves of $\tilde{n}_{1,2,3}(v)$ and their interrelation (e.g. the fact that $\tilde{n}_1$ and $\tilde{n}_3$ or $\tilde{n}_2$ and $\tilde{n}_3$ correspond to the same branch of the dispersion curve) are evident from the diagrams. The thermal corrections to $\tilde{n}_1^2$ or $\tilde{n}_2^2$ near resonance are important if $\Delta v = |v_{1,2\infty} - v| \lesssim \beta_T$; collisions affect the form of the curves of n^2 for

$\Delta v \sim \nu_{\text{eff}}/\omega$. The collisionless absorption of plasma waves is certainly weak if $\beta_T n_3 \cos\alpha \ll 1$.

For $\alpha \neq 0$ the absorption coefficients in the region of the gyromagnetic resonances $\omega = \omega_H$ and $\omega = 2\omega_H$ are of the order of $\omega v \beta_T/c$, and near $\omega = 3\omega_H$ they are of the order of $\omega v \beta_T^3/c$; see (12.42). When $\alpha \to 0$ the absorption at frequencies which are multiples of ω_H disappears, while that at ω_H increases considerably ($q = \omega\varkappa/c \sim \omega v/c$). Outside the neighbourhood of the point $v_{1,2\infty}$ the refractive indices for $\omega = \omega_H$, $2\omega_H$, $3\omega_H$, etc., have no singularities and may be calculated from formulae (11.6), i.e. the thermal motion may be neglected. An exception is formed by angles $\alpha \to 0$ for the resonance $\omega = \omega_H$ and angles $\alpha \to \frac{1}{2}\pi$ for the first one or two resonances.

§ 13. SOME REMARKS ON PLASMA DYNAMICS

The hydromagnetic approximation

Before going on to investigate the propagation of low-frequency waves in a plasma, where the motion of the ions has to be taken into account, we shall make some remarks concerning plasma dynamics.

In a rigorous analysis of arbitrary motions in a plasma we should start from the field equations together with the Boltzmann equations for the electrons, ions and molecules. Such a complete set of equations is extremely complicated: it is well known that the Boltzmann equation can by no means always be made use of in practice, even for the considerably simpler case of a gas containing only one species of particle (e.g. a monatomic gas). The various approximate methods of solving dynamical problems in a plasma are therefore of great value.

One of the most important of these approximations is to describe the gas by means of the equations of hydrodynamics, or, for a plasma, of magnetic hydrodynamics. This "hydromagnetic" approximation is valid if the free path l of a particle is small compared with the characteristic length L of the problem (the wavelength, or the dimensions of solid bodies or containers). The free time $\tau_{\text{eff}} = 1/\nu_{\text{eff}} = l/v_T$ and the ion revolution period in the magnetic field $2\pi/\Omega_H = 2\pi Mc/|e|H$ must, furthermore, be small compared with the characteristic time $t \sim 2\pi/\omega$ during which the hydrodynamic motion changes appreciably (for example, the period of oscillation). The latter conditions, in the notation used previously, give the inequalities† $\omega \ll \nu_{\text{eff}}$, $\omega \ll \Omega_H$. If $\omega \ll \nu_{\text{eff}}$, then certainly $l \ll L$, provided that the velocity of the macro-

† If $\omega \ll \nu_{\text{eff}}$ and $\omega_H \ll \nu_{\text{eff}}$, the condition $\omega \ll \Omega_H$ becomes unnecessary, since the medium is almost isotropic and the magnetic field plays no part. Thus, as regards the possibility of the hydrodynamic description, the condition $\omega \ll \Omega_H$ is sufficient but not necessary the condition $\omega \ll \sqrt{(\Omega_H^2 + \nu_{\text{eff}}^2)}$ is necessary.

scopic motion v does not greatly exceed the thermal velocity $v_T = \sqrt{(\varkappa T/M)}$. Next, in order that the medium may be regarded as isotropic even in the presence of a magnetic field H, in the frame of reference where its velocity is zero, we must also impose the condition $\omega_H = |e| H/mc \ll \nu_{\text{eff}}$; see equation (13.24) below. Finally, for the low frequencies here considered, we can usually neglect the displacement current in comparison with the conduction current ($4\pi\sigma/\omega \gg \varepsilon$), and assume that the conductivity is independent of the frequency; this we shall do in what follows.

It is not proposed to give here the fundamentals of magnetic hydrodynamics in detail (see [36, Chapter VIII; 94–98]). The initial equations are

$$\operatorname{curl} \boldsymbol{H} = 4\pi \boldsymbol{j}/c, \quad \operatorname{div} \boldsymbol{H} = 0, \quad \operatorname{curl} \boldsymbol{E} = -(1/c)\,\partial \boldsymbol{H}/\partial t, \tag{13.1}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c), \tag{13.2}$$

$$\begin{aligned} \varrho_M\, d\boldsymbol{v}/dt &= -\operatorname{grad} p + \boldsymbol{j} \times \boldsymbol{H}/c \\ &= -\operatorname{grad} p - \boldsymbol{H} \times \operatorname{curl} \boldsymbol{H}/4\pi, \end{aligned} \tag{13.3}$$

$$\partial \varrho_M/\partial t + \operatorname{div}(\varrho_M \boldsymbol{v}) = 0, \tag{13.4}$$

where ϱ_M is the density of the medium, viscosity is neglected†, and we have used the fact that, on the above assumptions and in the non-relativistic case here considered, the density of free charges and the convection current density may be taken as zero (see [95]).

To obtain the complete set of equations we must add to (13.1)–(13.4) the equation of state $p = p(\varrho_M, T)$ and the equation of heat transfer. If we take for simplicity only the isothermal case ($T =$ constant), the latter equation is not needed. Equation (13.2) is Ohm's law for a moving medium; $\boldsymbol{E}\boldsymbol{v} + \times \boldsymbol{H}/c$ is the electric field in a frame of reference moving with the medium.

From (13.1) and (13.2) we obtain for a homogeneous medium ($\sigma =$ constant)

$$\partial \boldsymbol{H}/\partial t = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H}) + (c^2/4\pi\sigma)\,\Delta \boldsymbol{H}. \tag{13.5}$$

In the isothermal case, equations (13.3)–(13.5) and the equation of state form a complete set.

If the conductivity is sufficiently great (formally, $\sigma \to \infty$), the condition for the current density $\boldsymbol{j}$ to be finite and Ohm's law (13.2) give

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}/c. \tag{13.6}$$

The problem of wave propagation in the hydromagnetic approximation will be discussed in § 14. The results of hydromagnetic calculations are sometimes valid even when the general conditions described above for magnetic hydrodynamics to be valid are not satisfied; an example in wave propagation is given in § 14. Apart from such exceptions, however, the hydromagnetic approach is invalid, or at least entirely inadequate, for the analysis of motions

† When viscosity is included the terms $\eta\,\Delta\boldsymbol{v} + (\zeta + \frac{1}{3}\eta)$ grad div $\boldsymbol{v}$ must be added to the right-hand side of equation (13.3), η and ζ being the first and second viscosity coefficients.

in a plasma where the conditions

$$\omega \ll \nu_{\text{eff}}, \quad \omega_H \ll \nu_{\text{eff}} \tag{13.7}$$

do not hold. In most problems of plasma physics at present we are in fact concerned with a range where the inequalities (13.7) are not satisfied. In particular, problems in which collisions may be neglected are of considerable interest.

The quasihydrodynamic approximation

The principal approximate method used in plasma dynamics when magnetic hydrodynamics is not applicable may be called the quasihydrodynamic approximation.† This involves the use of the equations for averaged quantities (equations of transfer), which in form are very similar to the equations of hydrodynamics. This approach has already been used earlier (§§ 10, 12). For convenience and for greater clarity, however, we shall here discuss the derivation of the relevant relations.

For a gas consisting of n species of particle, the necessary initial equations are n Boltzmann equations for the distribution functions $f^{(n)}(t, \boldsymbol{r}, \boldsymbol{v})$ for the particles of each kind:

$$\partial f^{(n)}/\partial t + \boldsymbol{v}\cdot\mathbf{grad}_r f^{(n)} + (\boldsymbol{F}^{(n)}/m^{(n)})\cdot\mathbf{grad}_v f^{(n)} + S^{(n)} = 0, \tag{13.8}$$

where $\boldsymbol{F}^{(n)}(t, \boldsymbol{r}, \boldsymbol{v})$ is the force on particles of the nth kind and $S^{(n)}$ is the collision integral (see § 4).

The average over velocities of any scalar or vector function $G^{(n)}(t, \boldsymbol{r}, \boldsymbol{v})$ is

$$\left.\begin{aligned} \overline{G^{(n)}}(t,\boldsymbol{r}) &= \frac{1}{N^{(n)}(t,\boldsymbol{r})}\int G^{(n)}(t,\boldsymbol{r},\boldsymbol{v})\, f^{(n)}(t,\boldsymbol{r},\boldsymbol{v})\,d\boldsymbol{v}, \\ N^{(n)}(t,\boldsymbol{r}) &= \int f^{(n)}(t,\boldsymbol{r},\boldsymbol{v})\,d\boldsymbol{v}, \end{aligned}\right\} \tag{13.9}$$

where $N^{(n)}$ is the density of particles of the nth kind. Multiplying the Boltzmann equation (13.8) by a function $G^{(n)}(\boldsymbol{v})$ which depends only $\boldsymbol{v}$ and integrating over $\boldsymbol{v}$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(N^{(n)}\overline{G^{(n)}}\right) + \frac{\partial}{\partial x_k}\left(N^{(n)}\overline{G^{(n)}v_k}\right) - \\ &- \frac{N^{(n)}}{m^{(n)}}\overline{\frac{\partial}{\partial v_k}\left(F_k^{(n)}G^{(n)}\right)} + \int S^{(n)}G^{(n)}\,d\boldsymbol{v} = 0, \end{aligned} \tag{13.10}$$

where summation over $k = 1, 2, 3$ is understood, and in the third term we have integrated by parts and used the fact that $f^{(n)} \to 0$ for $v_k \to \pm\infty$. Putting $G_n = 1$ and assuming that $\partial F_k^{(n)}/\partial v_k = 0$, we have the equation of continuity:

$$\partial N^{(n)}/\partial t + \operatorname{div}(N^{(n)}\boldsymbol{v}^{(n)}) = 0. \tag{13.11}$$

† Sometimes called the one-velocity approximation.

Here we have put $\int S^{(n)}\,d\boldsymbol{v} = 0$ because the number of particles cannot be changed by collisions (ionisation, recombination, etc., are ignored). In (13.11), of course,

$$\boldsymbol{v}^{(n)} \equiv \overline{\boldsymbol{v}^{(n)}} = [1/N^{(n)}] \int \boldsymbol{v} f^{(n)}\,d\boldsymbol{v};$$

the bar which indicates averaging will henceforward be omitted where possible. It may be noted that the condition $\partial F_k^{(n)}/\partial v_k = 0$ holds for the Lorentz force $\boldsymbol{F}^{(n)} = e^{(n)}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c)$, and thus involves no limitation in practice. In what follows we shall use this condition without specifically mentioning it.

Next, putting $G^{(n)} = m^{(n)}\boldsymbol{v}$, we have

$$\frac{\partial}{\partial t}\left(m^{(n)} N^{(n)} \overline{\boldsymbol{v}^{(n)}}\right) + \frac{\partial}{\partial x_k}\left(m^{(n)} N^{(n)} \overline{v_k \boldsymbol{v}}\right) - N^{(n)} \overline{\boldsymbol{F}^{(n)}} + \int m^{(n)} \boldsymbol{v} S^{(n)}\,d\boldsymbol{v} = 0. \tag{13.12}$$

In terms of the random velocity $\boldsymbol{w} = \boldsymbol{v} - \overline{\boldsymbol{v}^{(n)}} \equiv \boldsymbol{v} - \boldsymbol{v}^{(n)}$ the second term is

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(m^{(n)} N^{(n)} \overline{v_k \boldsymbol{v}}\right) &= \frac{\partial}{\partial x_k}\left(m^{(n)} N^{(n)} \overline{w_k \boldsymbol{w}}\right) + \frac{\partial}{\partial x_k}\left(m^{(n)} N^{(n)} \overline{v_k^{(n)} \boldsymbol{v}^{(n)}}\right)\\ &= \frac{\partial}{\partial x_k}\boldsymbol{\Pi}_k^{(n)} + m^{(n)} N^{(n)} \boldsymbol{v}^{(n)} \operatorname{div} \boldsymbol{v}^{(n)} +\\ &\quad + v_k^{(n)} \frac{\partial}{\partial x_k}\left(m^{(n)} N^{(n)} \boldsymbol{v}^{(n)}\right),\end{aligned}$$

with the stress tensor

$$\Pi_{ik}^{(n)} = m^{(n)} N^{(n)} \overline{w_i w_k} = m^{(n)} \int w_i w_k f^{(n)}\,d\boldsymbol{v}, \quad \boldsymbol{\Pi}_k^{(n)} = m^{(n)} N^{(n)} \overline{w_k \boldsymbol{w}}. \tag{13.13}$$

Since also, by (13.11),

$$\begin{aligned}\frac{\partial}{\partial t}\left(m^{(n)} N^{(n)} \boldsymbol{v}^{(n)}\right) &= m^{(n)} N^{(n)} \partial \boldsymbol{v}^{(n)}/\partial t + \boldsymbol{v}^{(n)} \partial\left(m^{(n)} N^{(n)}\right)/\partial t\\ &= m^{(n)} N^{(n)} \partial \boldsymbol{v}^{(n)}/\partial t - \boldsymbol{v}^{(n)} \operatorname{div}\left(m^{(n)} N^{(n)} \boldsymbol{v}^{(n)}\right),\end{aligned}$$

we can rewrite equation (13.12) as

$$\begin{aligned}&m^{(n)} N^{(n)} \left[\partial \boldsymbol{v}^{(n)}/\partial t + (\boldsymbol{v}^{(n)} \cdot \operatorname{grad})\, \boldsymbol{v}^{(n)}\right]\\ &= e^{(n)} N^{(n)} \left(\boldsymbol{E} + \boldsymbol{v}^{(n)} \times \boldsymbol{H}/c\right) - \partial \boldsymbol{\Pi}_k^{(n)}/\partial x_k + \boldsymbol{R}^{(n)},\end{aligned} \tag{13.14}$$

where the force $\boldsymbol{F}$ is taken to be the Lorentz force and the "frictional force" is

$$\boldsymbol{R}^{(n)} = -\int m^{(n)} \boldsymbol{v} S^{(n)}\,d\boldsymbol{v} = -\int m^{(n)} \boldsymbol{w} S^{(n)}\,d\boldsymbol{v}.$$

Equation (13.14) is evidently analogous to the law of conservation of momentum (i.e. to the equation of motion in hydrodynamics). By putting $G^{(n)} = \frac{1}{2} m^{(n)} v^2$ we can derive an equation analogous to the law of conservation of energy in hydrodynamics.

All such equations become significant only when specific expressions are substituted for $\boldsymbol{\Pi}_k$ and $\boldsymbol{R}$, and in a consistent treatment this must be done by using the same Boltzmann equations. However, the equations (13.14) may be used to elucidate the nature of various approximations, which are often insufficiently well-founded, or inexact. The simplest such approximation consists in assuming that the stress tensor Π_{ik} reduces to the pressure, i.e. $\Pi_{ik}^{(n)} = p^{(n)}\delta_{ik}$. A somewhat more general approximation is obtained by assuming the tensor Π_{ik} to be diagonal but with unequal "pressures" Π_{xx}, Π_{yy}, Π_{zz}. Finally, viscosity terms may be added to Π_{ik}.

In what follows we shall take $\Pi_{ik}^{(n)} = p^{(n)}\delta_{ik}$ and $p^{(n)} = \varkappa N^{(n)} T^{(n)}$.The "frictional force" $\boldsymbol{R}^{(n)}$ is related to collisions between particles of different kinds and is zero if the mean velocities of all particles are the same. We shall therefore use for $\boldsymbol{R}^{(n)}$ the same approximation as in § 10.

The result for a plasma containing electrons, singly-charged positive ions and molecules, with densities N, N_i and N_m respectively, is

$$\begin{aligned} mN\,[\partial \boldsymbol{v}_e/\partial t + (\boldsymbol{v}_e\cdot\mathbf{grad})\,\boldsymbol{v}_e] &= eN\,(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{H}/c) - \\ &- \mathbf{grad}\,(\varkappa T_e N) + mN\nu_{ei}(\boldsymbol{v}_i - \boldsymbol{v}_e) + mN\nu_{em}(\boldsymbol{v}_m - \boldsymbol{v}_e), \end{aligned} \tag{13.15}$$

$$\begin{aligned} MN_i\,[\partial \boldsymbol{v}_i/\partial t + (\boldsymbol{v}_i\cdot\mathbf{grad})\,\boldsymbol{v}_i] &= -\,eN_i\,(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{H}/c) - \\ &- \mathbf{grad}\,(\varkappa T_i N_i) + mN\nu_{ei}\,(\boldsymbol{v}_e - \boldsymbol{v}_i) + MN_i\nu_{im}\,(\boldsymbol{v}_m - \boldsymbol{v}_i), \end{aligned} \tag{13.16}$$

$$\begin{aligned} MN_m\,[\partial \boldsymbol{v}_m/\partial t + (\boldsymbol{v}_m\cdot\mathbf{grad})\,\boldsymbol{v}_m] &= -\,\mathbf{grad}\,(\varkappa T_m N_m) - \\ &- m\nu_{em}N\,(\boldsymbol{v}_m - \boldsymbol{v}_e) - M\nu_{im}N_i\,(\boldsymbol{v}_m - \boldsymbol{v}_i), \end{aligned} \tag{13.17}$$

$$\left.\begin{aligned} &\partial N/\partial t + \operatorname{div}(N\boldsymbol{v}_e) = 0, \\ &\partial N_i/\partial t + \operatorname{div}(N_i\boldsymbol{v}_i) = 0, \\ &\partial N_m/\partial t + \operatorname{div}(N_m\boldsymbol{v}_m) = 0; \end{aligned}\right\} \tag{13.18}$$

here the charges and masses of the electrons (e), ions (i) and molecules (m) are respectively $e < 0$, $-e, 0$; m, M and M. The frictional forces could also be written in a somewhat different form, for instance by using reduced masses. We shall not do this, since for collisions between electrons and heavy particles $mM/(m + M) \approx m$, and for collisions between heavy particles equations (13.15)–(13.18) are used only to effect an extrapolation.

Equations such as (13.15)–(13.18) have often been used (see, for example, [19, 71, 72, 98, 99, 283]); more rigorous methods have also sometimes been used in plasma dynamics [100, 101].

Equations (13.15)–(13.18) must, of course, be supplemented by the field equations

$$\left.\begin{aligned} &\mathbf{curl}\,\boldsymbol{H} = 4\pi \boldsymbol{j}_t/c + (1/c)\,\partial \boldsymbol{E}/\partial t, \quad \operatorname{div}\boldsymbol{E} = 4\pi\bar{\varrho}, \\ &\mathbf{curl}\,\boldsymbol{E} = -(1/c)\,\partial \boldsymbol{H}/\partial t, \quad \operatorname{div}\boldsymbol{H} = 0, \\ &\boldsymbol{j}_t = e\,(N\boldsymbol{v}_e - N_i\boldsymbol{v}_i) \quad \bar{\varrho} = e\,(N - N_i). \end{aligned}\right\} \tag{13.19}$$

Moreover, to obtain a complete set of equations we must use the equation of conservation of energy or the equation of heat transfer, which depend on the electron, ion and molecule temperatures T_e, T_i and T_m. We shall usually take, for simplicity, the case where $T_e = T_i = T_m = T$.

The propagation of waves in a plasma will be discussed in § 14 on the basis of equations (13.15)–(13.19). Here we shall merely give some general results and some relations which hold good in steady motion of a plasma.

The motion of a pure electron-ion plasma and a weakly ionised gas

We shall use the notation

$$\left.\begin{aligned} &\varrho_e = mN, \quad \varrho_i = MN_i, \quad \varrho_m = MN_m, \quad \varrho_M = \varrho_e + \varrho_i + \varrho_m, \\ &\varrho_p = \varrho_e + \varrho_i, \quad \boldsymbol{v}_p = (\varrho_e \boldsymbol{v}_e + \varrho_i \boldsymbol{v}_i)/(\varrho_e + \varrho_i), \\ &\boldsymbol{v} = (\varrho_e \boldsymbol{v}_e + \varrho_i \boldsymbol{v}_i + \varrho_m \boldsymbol{v}_m)/(\varrho_e + \varrho_i + \varrho_m), \\ &p_p = p_e + p_i, \quad p = p_e + p_i + p_m = \varkappa T (N + N_i + N_m). \end{aligned}\right\} \tag{13.20}$$

Addition of (13.15)–(13.17) gives

$$\varrho_e \, d\boldsymbol{v}_e/dt + \varrho_i \, d\boldsymbol{v}_i/dt + \varrho_m \, d\boldsymbol{v}_m/dt = -\mathbf{grad}\, p + \bar{\varrho} \boldsymbol{E} + \boldsymbol{j}_t \times \boldsymbol{H}/c, \tag{13.21}$$

with the notation $d\boldsymbol{v}^{(n)}/dt = \partial \boldsymbol{v}^{(n)}/\partial t + (\boldsymbol{v}^{(n)} \cdot \mathbf{grad})\, \boldsymbol{v}^{(n)}$.

If the velocities of all particles are approximately equal, i.e. $\boldsymbol{v}_e \approx \boldsymbol{v}_i \approx \boldsymbol{v}_m$, equation (13.21) becomes the fundamental equation of magnetic hydrodynamics (13.3) (in which it is assumed that $\boldsymbol{j}_t = \boldsymbol{j}$ and the charge density $\bar{\varrho} = 0$, as is true with sufficient accuracy in the approximation used); of course, we cannot put $\boldsymbol{v}_e = \boldsymbol{v}_i$ in the expression for $\boldsymbol{j}_t$.

This result (the obtaining of the equation of magnetic hydrodynamics) is physically entirely reasonable, since when collisions between particles are very frequent [see the condition (13.7)] the mean velocities of the different components of the gas must in fact be almost equal. Formally, this is also evident from equations (13.15)–(13.17), where, if ν_{ei}, ν_{em} and ν_{im} are large, the frictional forces can be balanced by the other terms only if the velocity differences $\boldsymbol{v}_i - \boldsymbol{v}_e$, $\boldsymbol{v}_m - \boldsymbol{v}_e$ and $\boldsymbol{v}_m - \boldsymbol{v}_i$ are small.

In using the above equations we take into account the inequalities $m \ll M$, $|N - N_i| \ll N$, $\nu_{em} \sim \sqrt{(M/m)}\,\nu_{im} \gg \nu_{im}$, $M\nu_{im} \gg m\nu_{em}$; we also neglect all terms in derivatives with respect to the coordinates, and take only the limiting cases of either a pure electron-ion plasma ($N_m = 0$, $\varrho_m = 0$) or a weakly ionised gas, where

$$N \approx N_i \ll N_m. \tag{13.22}$$

For $N_m = 0$ and with the above-mentioned simplifications, multiplying (13.15) and (13.16) respectively by e/m and $-e/M$ and adding, we obtain

$$\partial \boldsymbol{j}_t/\partial t = e^2 N \boldsymbol{E}/m + e\boldsymbol{j}_t \times \boldsymbol{H}/mc - \nu_{ei} \boldsymbol{j}_t + e^2 N \boldsymbol{v}_i \times \boldsymbol{H}/mc. \tag{13.23}$$

From (13.20) it is clear that in this case $\boldsymbol{v} = \boldsymbol{v}_p = (\varrho_e \boldsymbol{v}_e + \varrho_i \boldsymbol{v}_i)/(\varrho_e + \varrho_i) \approx$ $\approx (\varrho_e/\varrho_i)\, \boldsymbol{v}_e + \boldsymbol{v}_i \approx \boldsymbol{v}_i$ if $v_i \gg m v_e/M$. For sufficiently low frequencies the latter condition is seen to be satisfied. In this case (which occurs, for example, when

$\omega \ll \Omega_H = |e|H/Mc$, $|\boldsymbol{v}_i \times \boldsymbol{H}| \sim v_i H$), equation (13.23) becomes

$$\frac{1}{\nu_{ei}} \frac{\partial \boldsymbol{j}_t}{\partial t} + \boldsymbol{j}_t + \frac{\omega_H}{\nu_{ei}} \boldsymbol{j}_t \times \boldsymbol{H}/H = \frac{e^2 N}{m \nu_{ei}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c). \tag{13.24}$$

With the conditions (13.7) this becomes the fundamental relation (13.2) of magnetic hydrodynamics, with $\sigma = e^2 N/m\nu_{ei}$. If the term $\boldsymbol{v} \times \boldsymbol{H}/c$ may be neglected, then (13.24) for harmonic motion is equivalent to the expressions (10.9), where the field $\boldsymbol{H} = \boldsymbol{H}^{(0)}$ is in the z-direction. When molecules are also present, we have similarly (with $\nu_e = \nu_{ei} + \nu_{em}$)

$$\partial \boldsymbol{j}_t/\partial t - (e/mc)\boldsymbol{j}_t \times \boldsymbol{H} + \nu_e \boldsymbol{j}_t = (e^2 N/m)(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{H}/c) + eN\nu_{em}(\boldsymbol{v}_m - \boldsymbol{v}_i). \tag{13.25}$$

Addition of equations (13.15) and (13.16) gives, in the approximation considered,

$$\varrho_p \partial \boldsymbol{v}_p/\partial t = \boldsymbol{j}_t \times \boldsymbol{H}/c - m\nu_{em}\boldsymbol{j}_t/e + \varrho_p \nu_{im}(\boldsymbol{v}_m - \boldsymbol{v}_i). \tag{13.26}$$

For a weakly ionised gas we can assume that $\boldsymbol{v}_m \approx \boldsymbol{v}$; this is so under the conditions (13.22) and $mN\nu_e + MN_i\nu_i \ll MN_m\nu_m$. Further, if $\nu_i \gg m\nu_e/M$ we can put $\boldsymbol{v}_i \approx \boldsymbol{v}_p$. Equations (13.25) and (13.26) for a steady state then give

$$\left.\begin{aligned}
&\nu_e \boldsymbol{j}_t - e\boldsymbol{j}_t \times \boldsymbol{H}/mc + e^2 \boldsymbol{H} \times (\boldsymbol{j}_t \times \boldsymbol{H})/mMc^2 \nu_{im} \\
&\quad = (\nu_e + \nu_H)\boldsymbol{j}_t - e\boldsymbol{j}_t \times \boldsymbol{H}/mc - \nu_H (\boldsymbol{j}_t \cdot \boldsymbol{H})\boldsymbol{H}/H^2 \\
&\quad = (e^2 N/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c), \\
&\nu_e = \nu_{ei} + \nu_{em}, \quad \nu_H = \omega_H \Omega_H/\nu_{im}, \quad \omega_H = |e|H/mc, \quad \Omega_H = |e|H/Mc,
\end{aligned}\right\} \tag{13.27}$$

$$\boldsymbol{v} - \boldsymbol{v}_p = -\boldsymbol{j}_t \times \boldsymbol{H}/MNc\nu_{im} + m\nu_{em}\boldsymbol{j}_t/eMN\nu_{im}. \tag{13.28}$$

It is also convenient to write (13.27) in the form

$$\boldsymbol{j}_t = \sigma_{\parallel} \boldsymbol{E}'_{\parallel} + \sigma_{\perp} \boldsymbol{E}'_{\perp} + \sigma_H \boldsymbol{H} \times \boldsymbol{E}'/H, \quad \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c, \tag{13.29}$$

$$\left.\begin{aligned}
\sigma_{\parallel} &= e^2 N/m\nu_e, \\
\sigma_{\perp} &= e^2 N(\nu_e \nu_{im}^2 + \omega_H \Omega_H \nu_{im})/m(\nu_e^2 \nu_{im}^2 + \omega_H^2 \Omega_H^2 + \omega_H^2 \nu_{im}^2), \\
\sigma_H &= e^2 N \nu_{im}^2 \omega_H/m(\nu_e^2 \nu_{im}^2 + \omega_H^2 \Omega_H^2 + \omega_H^2 \nu_{im}^2),
\end{aligned}\right\} \tag{13.30}$$

where $\boldsymbol{E}'_{\parallel}$ and $\boldsymbol{E}'_{\perp}$ are the components of $\boldsymbol{E}'$ parallel and perpendicular to $\boldsymbol{H}$.

If small terms with factors of the order of $\sqrt{(m/M)}$ and smaller are not neglected, we obtain (13.29) with

$$\left.\begin{aligned}
\sigma_{\parallel} &= e^2 N\left(\frac{1}{m\nu_e} + \frac{1}{M\nu_{im}}\right), \\
\sigma_{\perp} &= e^2 N\left(\frac{\nu_e}{m(\omega_H^2 + \nu_e^2)} + \frac{\nu_{im}}{M(\Omega_H^2 + \nu_{im}^2)}\right), \\
\sigma_H &= e^2 N\left(\frac{\omega_H}{m(\omega_H^2 + \nu_e^2)} - \frac{\Omega_H}{M(\Omega_H^2 + \nu_{im}^2)}\right).
\end{aligned}\right\} \tag{13.31}$$

These expressions are compact and just as convenient as the approximate expressions (13.30), which they become when terms of order $\sqrt{(m/M)} \lesssim 1/40$ are neglected.

Steady motion of a weakly ionised gas in a magnetic field
The Earth's ionosphere

Let us now consider steady motion of a weakly ionised gas in an external field $\boldsymbol{H}^{(0)}$. This occurs in the study of motions in the Earth's ionosphere when the Earth's magnetic field is taken into account. It is seen from (13.28) that the mean macroscopic velocity $\boldsymbol{v}$ of the gas (which in this case is practically the same as the mean velocity $\boldsymbol{v}_m$ of the molecules) is equal to the velocity of the ionised component $\boldsymbol{v}_p = (mN\boldsymbol{v}_e + MN\boldsymbol{v}_i)/(m+M)N \approx \boldsymbol{v}_i$ only if the current $\boldsymbol{j}_t$ is zero. In turn, the current is zero, by (13.29), only if

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}^{(0)}/c = 0. \tag{13.32}$$

Let us consider in more detail the relation between $\boldsymbol{v}$, $\boldsymbol{v}_p$ and $\boldsymbol{E}$ [102], assuming that

$$\omega_H \Omega_H \gg \nu_{im}\nu_e, \tag{13.33}$$

which necessarily implies that $\omega_H^2 \gg \nu_e^2$ [in the case considered $\nu_e = \nu_{ei} + \nu_{em} \approx \approx \nu_{em}$, $\nu_{im} \sim \sqrt{(m/M)}\nu_{em}$, $\Omega_H = (m/M)\omega_H$].

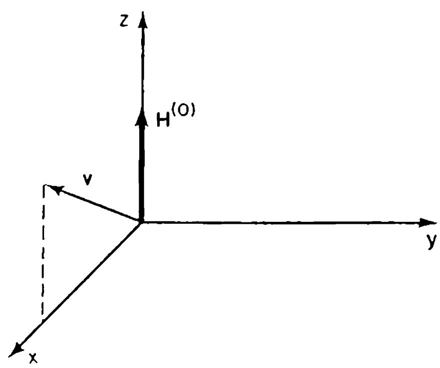

FIG. 13.1. Coordinate system used in (13.34), (13.37) and (13.38).

Taking the z-axis in the direction of the field $\boldsymbol{H}^{(0)}$ and the y-axis perpendicular to $\boldsymbol{v}$ (Fig. 13.1), we obtain from (13.28) and (13.27) or (13.29), on neglecting small terms,

$$\left.\begin{aligned}
v_{p,z} &= -(eE_z'/M\nu_{im})\,\nu_{em}/\nu_e + v_z,\\
v_{p,x} &= -(eE_x'/M)\,\nu_{im}/(\Omega_H^2 + \nu_{im}^2) - e\Omega_H E_y'/M\,(\Omega_H^2 + \nu_{im}^2) + v_x,\\
v_{p,y} &= e\Omega_H E_x'/M\,(\Omega_H^2 + \nu_{im}^2) - e\nu_{im}E_y'/M\,(\Omega_H^2 + \nu_{im}^2),\\
E_x' &= E_x, \quad E_y' = E_y - v_x H^{(0)}/c, \quad E_z' = E_z, \quad v_y = 0,
\end{aligned}\right\} \tag{13.34}$$

where e is the electron charge and $-e$ the positive-ion charge.

In the Earth's ionosphere the condition (13.33) is satisfied at heights exceeding 90 to 100 km. Below this, the medium may be regarded as isotropic.† In the region where formulae (13.34) may be used, the nature of the motion depends considerably on the relation between $\Omega_H = |e|H^{(0)}/Mc$ and ν_{im}; for $O^\pm$ and $N^\pm$ ions in the Earth's ionosphere, $\Omega_H \sim 300$ and $\nu_{im} \sim 10^{-10} N_m \sqrt{(T/300)}$ [see (6.27)]. If $\nu_{im} \gg \Omega_H$, then, in the absence of a field $\boldsymbol{E}$, $v_{p,x} \approx v_x$ and $v_{p,y} \approx \Omega_H v_x/\nu_{im} \ll v_x$. In other words, in this case ($\boldsymbol{E} = 0$) the ionised component is carried along by the neutral component.

If, however,

$$\Omega_H \gg \nu_{im}, \tag{13.36}$$

which is true in the F layer of the Earth's ionosphere, then for $\boldsymbol{E} = 0$ the ionised component is carried along in the direction of the field $\boldsymbol{H}^{(0)}$ but scarcely moves perpendicular to $\boldsymbol{H}^{(0)}$ ($v_{p,x} \approx \nu_{im}^2 v_x/\Omega_H^2$; $v_{p,y} \approx -\nu_{im} v_x/\Omega_H$). Hence the "ionospheric wind" in the F layer, i.e. the occurrence of a velocity $\boldsymbol{v}_p$ there††, is possible only if an electric field is present. When (13.36) holds we have approximately, whatever the value of v_x,

$$v_{p,x} = cE_y/H^{(0)}, \quad v_{p,y} = -cE_x/H^{(0)}. \tag{13.37}$$

More accurate expressions derived from (13.34) for the case (13.36) are

$$\left.\begin{aligned} v_{p,z} &= -(eE_z/M\nu_{im})\,\nu_{em}/\nu_e + v_z, \\ v_{p,x} &= -(eE_x/M)\,\nu_{im}/\Omega_H^2 + cE_y/H^{(0)} + \nu_{im}^2 v_x/\Omega_H^2, \\ v_{p,y} &= -cE_x/H^{(0)} - e\nu_{im}E_y/M\Omega_H^2 - \nu_{im}v_x/\Omega_H. \end{aligned}\right\} \tag{13.38}$$

Knowing $\boldsymbol{v}_p$, i.e. measuring the rate of movement of the ionisation, we can determine $\boldsymbol{E}$ from (13.37). The quantity $\boldsymbol{v} \approx \boldsymbol{v}_m$, i.e. the velocity of the whole gas, which approximately coincides with the velocity of the molecules, remains unknown. In order to determine it, we must know also the current $\boldsymbol{j}_t$ [see (13.28)]. If $\boldsymbol{j}_t = 0$, then the relation (13.32) holds, and when small terms are neglected we have $\boldsymbol{v}_p = \boldsymbol{v}$ (no relative velocity of the ionised component), since (13.37) may be written in the form $\boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{H}^{(0)}/c = 0$. Conversely, if $\boldsymbol{v}_p$ and $\boldsymbol{v}$ can somehow be found independently, the current $\boldsymbol{j}_t$ can be calculated.

A more detailed discussion of this problem is important in the physics of the ionosphere and the theory of variations of the Earth's magnetic field, but is beyond the scope of this book.

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† The condition for isotropy is evident from (13.29) and (13.30), and amounts to the requirement that the conductivities $\sigma_\parallel$ and $\sigma_\perp$ should be almost equal and σ_H should be small. This is so if

$$\omega_H^2 \ll \nu_e^2. \tag{13.35}$$

The inequality $\omega_H \Omega_H \ll \nu_e \nu_{im}$ follows from this.

†† In radio observations we in fact measure the movement of ionisation (e.g. of ionospheric "clouds"), and thus determine the velocity $\boldsymbol{v}_p$.

§ 14. PROPAGATION OF LOW-FREQUENCY AND HYDROMAGNETIC WAVES

Introduction

As has been mentioned previously, the effect of the ions on wave propagation may usually be neglected if

$$\omega \gg \Omega_H = |e| H^{(0)}/Mc; \tag{14.1}$$

see (10.5). (It is assumed that $N \gg m N_i/M$.) The corresponding waves are called high-frequency waves, and have been discussed in §§ 11 and 12. When the frequency satisfies the opposite condition

$$\omega \ll \Omega_H, \tag{14.2}$$

we shall speak of low-frequency waves. In what follows we shall consider both low-frequency and intermediate-frequency waves. In the latter case neither (14.1) nor (14.2) holds, and so the motion of the ions cannot be neglected, although their effect may not be so great as for low-frequency waves.

Hydromagnetic waves

The low-frequency waves include, in particular, hydromagnetic waves, whose frequency satisfies (13.7), so that they may be discussed in terms of the equations of magnetic hydrodynamics [94, 36, 71, 95]. We shall begin with this case, and first consider only undamped waves (i.e. neglect viscosity and thermal conduction, and make $\sigma \to \infty$). Here it may be noted that some of the results thus obtained are of wider significance and are valid even in the absence of collisions, when the conditions (13.7) certainly do not hold.

The initial equations are, on the above assumptions,

$$\left.\begin{aligned} &\partial \boldsymbol{H}/\partial t = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H}), \quad \operatorname{div} \boldsymbol{H} = 0, \\ &d\boldsymbol{v}/dt = -(1/\varrho_M) \operatorname{grad} p - (1/4\pi \varrho_M) \boldsymbol{H} \times \operatorname{curl} \boldsymbol{H}, \\ &\partial \varrho_M/\partial t + \operatorname{div}(\varrho_M \boldsymbol{v}) = 0 \end{aligned}\right\} \tag{14.3}$$

[see (13.1)–(13.5)]. The electric field is given in terms of $\boldsymbol{v}$ and $\boldsymbol{H}$ by (13.6). For a homogeneous unperturbed medium the equation of heat transfer in the absence of dissipation merely states the constancy of the entropy S (the adiabatic approximation).

When considering waves of small amplitude, analogous to sound in ordinary hydrodynamics, we put $\boldsymbol{H} = \boldsymbol{H}^{(0)} + \boldsymbol{H}'$, $\varrho_M = \varrho_0 + \varrho'$, $p = p_0 + p'$, $\boldsymbol{v} \equiv \boldsymbol{v}'$, $\boldsymbol{E} \equiv \boldsymbol{E}'$, the primed quantities being small. Then the linearised equations are (with $\boldsymbol{H}^{(0)}$ constant)

$$\left.\begin{aligned} &\partial \boldsymbol{H}'/\partial t = \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{H}^{(0)}), \quad \operatorname{div} \boldsymbol{H}' = 0, \\ &\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}^{(0)}/c, \quad \partial \varrho'/\partial t + \varrho_0 \operatorname{div} \boldsymbol{v} = 0, \\ &\partial \boldsymbol{v}/\partial t = -(u_0^2/\varrho_0) \operatorname{grad} \varrho' - (1/4\pi \varrho_0) \boldsymbol{H}^{(0)} \times \operatorname{curl} \boldsymbol{H}', \end{aligned}\right\} \tag{14.4}$$

where $u_0^2 = (\partial p/\partial \varrho_M)_S$ is the square of the velocity of ordinary sound in the medium concerned.

When the propagation of waves may be regarded as isothermal instead of adiabatic but dissipation terms are again not explicitly included, the same equations hold, with $u_0^2 = (\partial p/\partial \varrho_M)_T$.

We shall seek solutions of (14.4) which are proportional to $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, and take the z-axis in the direction of $\boldsymbol{k}$, and the x-axis perpendicular to $\boldsymbol{k}$ and to the field $\boldsymbol{H}^{(0)}$; thus $\boldsymbol{H}^{(0)}$ lies in the yz-plane at an angle α to $\boldsymbol{k}$ (see Fig. 11.1). After substituting in (14.4) and eliminating ϱ' we have

$$\left.\begin{aligned} v_{\text{ph}} H'_x = -v_x H_z^{(0)}, \quad v_{\text{ph}} v_x = -H_z^{(0)} H'_x/4\pi\varrho_0, \\ H'_z = 0; \end{aligned}\right\} \tag{14.5}$$

$$\left.\begin{aligned} v_{\text{ph}} H'_y &= -v_y H_z^{(0)} + v_z H_y^{(0)}, \\ v_{\text{ph}} v_y &= -H_z^{(0)} H'_y/4\pi\varrho_0, \\ (v_{\text{ph}} - u_0^2/v_{\text{ph}})\, v_z &= H_y^{(0)} H'_y/4\pi\varrho_0, \quad \varrho' = \varrho_0 v_z/v_{\text{ph}}, \end{aligned}\right\} \tag{14.6}$$

where $v_{\text{ph}} = \omega/k$, $H_y^{(0)} = H^{(0)}\sin\alpha$, $H_z^{(0)} = H^{(0)}\cos\alpha$.

It is clear from (14.5) and (14.6) that two types of perturbation (wave) can be propagated independently. In waves of one type the quantities H'_x and v_x are non-zero; in those of the other type H'_y, v_y, v_z and ϱ' are non-zero. From (14.5) it follows that for waves of the type which may be called properly hydromagnetic the phase velocity is

$$v_{\text{ph},2} = (\omega/k)_2 = H_z^{(0)}/\sqrt{(4\pi\varrho_0)} = (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}. \tag{14.7}$$

These are distinguished by the suffix 2, since they are related to the ordinary normal waves (see below). In this case we have (Fig. 14.1a)

$$\left.\begin{aligned} &v_{x2} = -v_{\text{ph}} H'_{x2}/H_z^{(0)} = -H'_{x2}/\sqrt{(4\pi\varrho_0)}, \\ &H'_{x2} \neq 0, \quad \varrho'_2 = 0, \quad p'_2 = 0, \quad H'_{z2} = 0, \quad H'_{y2} = 0, \\ &v_{y2} = 0, \quad v_{z2} = 0, \quad E_{y2} = v_{x2} H_z^{(0)}/c, \\ &E_{z2} = -v_{x2} H_y^{(0)}/c, \quad E_{x2} = 0. \end{aligned}\right\} \tag{14.8}$$

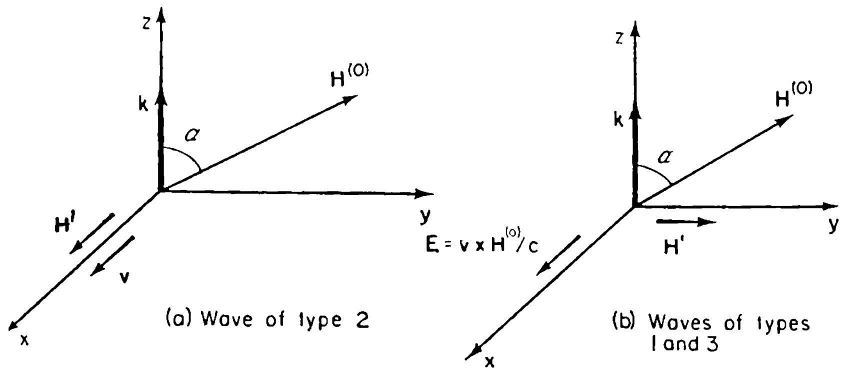

FIG. 14.1. Waves of types 1, 2 and 3 in magnetic hydrodynamics.

According to (14.7) the relation between ω and $\boldsymbol{k}$ in this wave can be written as $\omega = \boldsymbol{H}^{(0)}\cdot\boldsymbol{k}/\sqrt{(4\pi\varrho_0)}$, so that the group velocity is (see § 24)

$$\boldsymbol{v}_{\mathrm{gr},2} = \partial\omega/\partial\boldsymbol{k} = \boldsymbol{H}^{(0)}/\sqrt{(4\pi\varrho_0)}. \tag{14.9}$$

Thus the group velocity in wave 2 is always in the direction of the external field $\boldsymbol{H}^{(0)}$. When $\alpha = 0$ (longitudinal propagation) $v_{\mathrm{ph},2} = v_{\mathrm{gr},2} = H^{(0)}/\sqrt{(4\pi\varrho_0)}$; when $\alpha = \frac{1}{2}\pi$ (transverse propagation) $v_{\mathrm{ph},2} = 0$ and the group velocity is perpendicular to $\boldsymbol{k}$ and has the value $H^{(0)}/\sqrt{(4\pi\varrho_0)}$, as for all other angles (in the hydromagnetic approximation).

The hydromagnetic waves corresponding to (14.6) are often called magneto-acoustic waves. The dispersion relation for these waves is evidently

$$(v_{\mathrm{ph}}^2 - u_0^2)(v_{\mathrm{ph}}^2 - H_z^{(0)2}/4\pi\varrho_0) = v_{\mathrm{ph}}^2 H_y^{(0)2}/4\pi\varrho_0,$$

where $v_{\mathrm{ph}}^2 = \omega^2/k^2$; hence

$$\begin{aligned} v_{\mathrm{ph},1,3}^2 &= \frac{1}{2}\left(u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0}\right) \pm \frac{1}{2}\sqrt{\left\{u_0^4 + \frac{[H^{(0)}]^4}{(4\pi\varrho_0)^2} - 2\frac{u_0^2}{4\pi\varrho_0}(H_z^{(0)2} - H_y^{(0)2})\right\}} \\ &= \frac{1}{2}\left(u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0}\right) \pm \frac{1}{2}\sqrt{\left\{u_0^4 + \frac{[H^{(0)}]^4}{(4\pi\varrho_0)^2} - 2\frac{u_0^2[H^{(0)}]^2}{4\pi\varrho_0}\cos 2\alpha\right\}} \\ &= \frac{1}{4}\left[\sqrt{\left\{u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0} + \frac{2H_z^{(0)}u_0}{\sqrt{(4\pi\varrho_0)}}\right\}} \pm \sqrt{\left\{u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0} - \frac{2H_z^{(0)}u_0}{\sqrt{(4\pi\varrho_0)}}\right\}}\right]^2. \end{aligned} \tag{14.10}$$

For these waves we have (see Fig. 14.1 b)

$$\left.\begin{array}{llll} H_y' \neq 0, & v_z = v_{\mathrm{ph}} H_y^{(0)} H_y'/4\pi\varrho_0(v_{\mathrm{ph}}^2 - u_0^2), & v_x = 0, & \\ H_x' = 0, \quad H_z' = 0, & v_y = -H_z^{(0)} H_y'/4\pi\varrho_0 v_{\mathrm{ph}}, & p' = u_0^2\varrho', & \\ \varrho' = \varrho_0 v_z/v_{\mathrm{ph}}, & E_x = -(v_y H_z^{(0)} - v_z H_y^{(0)})/c, & E_y = 0, & E_z = 0, \end{array}\right\} \tag{14.11}$$

where v_{ph} must be taken to have one of the values $v_{\mathrm{ph},1}$ and $v_{\mathrm{ph},3}$ given by (14.10).

We call wave 1 or + that which corresponds to the plus sign in (14.10), the root itself being always taken as positive; wave 3 or − corresponds to the minus sign in (14.10). The dependence of the velocities $v_{\mathrm{ph},1,2,3}$ on the angle α between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$ is shown by the polar diagrams (Fig. 14.2), where the length of the radius vector from the origin to the curve is $v_{\mathrm{ph}}' = v_{\mathrm{ph}} \div H^{(0)}/\sqrt{(4\pi\varrho_0)}$. The magnetic field $\boldsymbol{H}^{(0)}$ is along the initial line, and the curves are given for values of $\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} = 0{\cdot}2,\ 0{\cdot}8,\ 1{\cdot}0,\ 1{\cdot}2,\ 2{\cdot}0$.

For $\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} < 1$ and $\alpha = 0$ we have from (14.10)

$$\left.\begin{array}{l} v_{\mathrm{ph},1} = H^{(0)}/\sqrt{(4\pi\varrho_0)} = v_{\mathrm{ph},2}(\alpha = 0), \\ v_{\mathrm{ph},3} = u_0, \quad v_{z1} = 0, \quad \varrho_1' = 0, \quad v_{y1} = -H_{y1}'/\sqrt{(4\pi\varrho_0)}, \\ v_{z3} = \varrho_3' v_{\mathrm{ph},3}/\varrho_0, \quad v_{y3} = 0, \quad H_{y3}' = 0, \quad E_{x3}' = 0; \end{array}\right\} \tag{14.12}$$

the values of the variables for wave 3 are most simply found directly from (14.6) with $v_{\mathrm{ph},3} = u_0$. For $\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} > 1$ and $\alpha = 0$, $v_{\mathrm{ph},1} = u_0$ and $v_{\mathrm{ph},3} = H^{(0)}/\sqrt{(4\pi\varrho_0)}$. The appropriateness of the nomenclature of waves 1

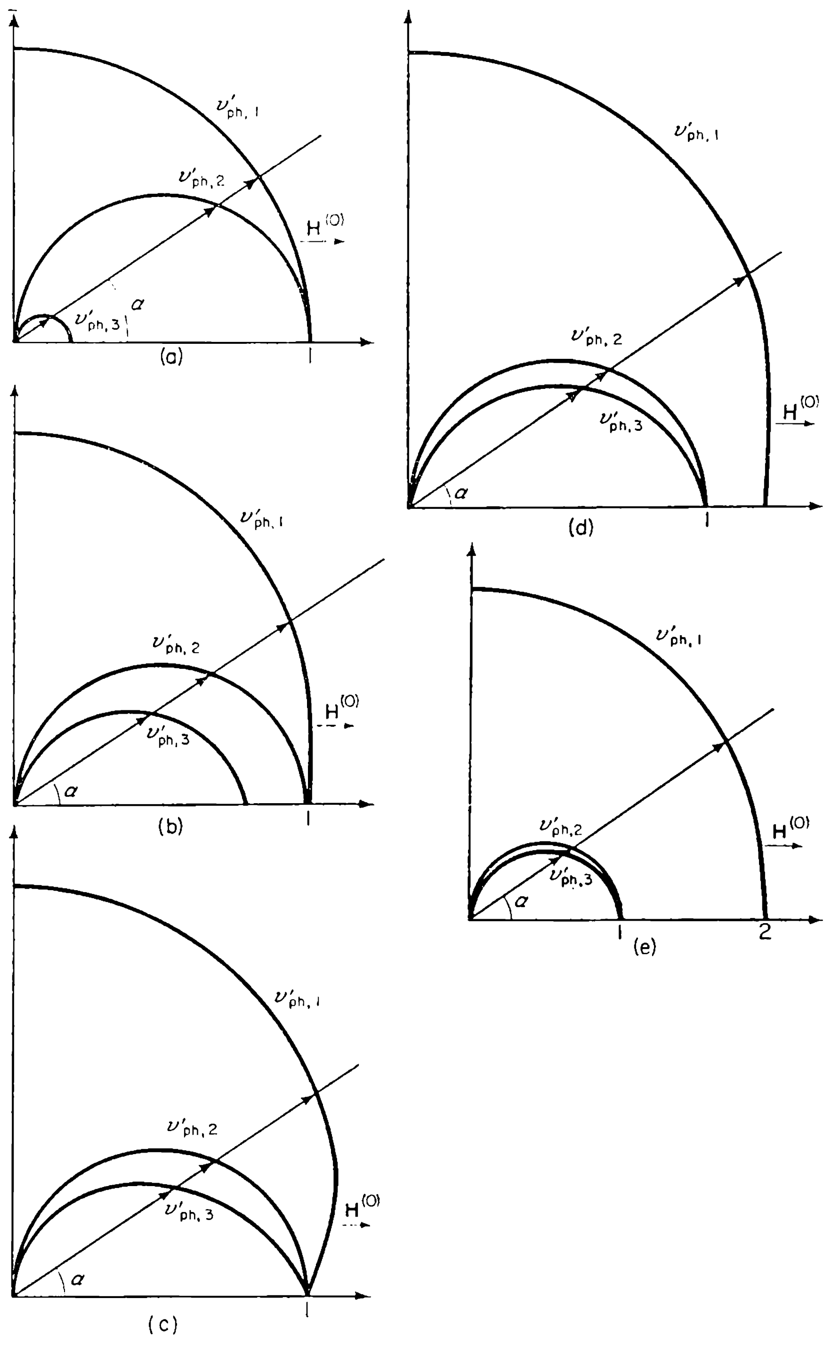

FIG. 14.2. Phase velocities of hydromagnetic waves 1, 2 and 3 (in units of $H^{(0)}/\sqrt{(4\pi\varrho_0)}$): $v'_{ph,1,2,3} = v_{ph,1,2,3} \div H^{(0)}/\sqrt{(4\pi\varrho_0)}$ as functions of the angle α between $\boldsymbol{H}^{(0)}$ (directed along the initial line) and the wave vector $\boldsymbol{k}$.

(a) $\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} = 0{\cdot}2$

(b) $\zeta = 0{\cdot}8$

(c) $\zeta = 1{\cdot}0$

(d) $\zeta = 1{\cdot}2$

(e) $\zeta = 2$ (scale half that of the other diagrams) and 3 is clear from Fig. 14.2. It may also be noted that, when $\zeta = 0$ (i.e. $u_0 \to 0$ or $H_0 \to \infty$), wave 1 passes directly into the extraordinary normal waves (see below), which justifies the choice of the suffix 1. The wave whose velocity is u_0 for $\alpha = 0$ is an ordinary sound wave propagated along the field $\boldsymbol{H}^{(0)}$, which thus has no effect on it.

For $\alpha = \frac{1}{2}\pi$, when $H_z^{(0)} = 0$, we have

$$\left.\begin{aligned} &v_{\text{ph},1} = \sqrt{\{u_0^2 + [H^{(0)}]^2/4\pi\varrho_0\}}, \\ &v_{\text{ph},3} = v_{\text{ph},2}(\alpha = \tfrac{1}{2}\pi) = 0, \\ &v_{y1} = 0, \quad v_{z1} = \sqrt{\left\{u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0}\right\}}\, H'_{y1}/H^{(0)} = v_{\text{ph},1} H'_{y1}/H^{(0)}, \\ &\varrho'_1 = \varrho_0 v_{z1}/v_{\text{ph},1}, \quad E_{x1} = v_{z1} H^{(0)}/c. \end{aligned}\right\} \tag{14.13}$$

For a weak field, i.e. when

$$\zeta^2 = u_0^2 \div [H^{(0)}]^2/4\pi\varrho_0 \gg 1, \tag{14.14}$$

we obtain for wave 1

$$\left.\begin{aligned} &v_{\text{ph},1} \approx u_0, \quad H'_{y1} \approx H_y^{(0)} v_{z1}/u_0, \\ &v_{y1} \approx -H_z^{(0)} H'_{y1}/4\pi\varrho_0 u_0, \quad |v_{y1}| \ll v_{z1}. \end{aligned}\right\} \tag{14.15}$$

When $H^{(0)} \to 0$ these become sound waves. For wave 3 we find with the condition (14.14)

$$\left.\begin{aligned} &v_{\text{ph},3} \approx (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)} = H_z^{(0)}/\sqrt{(4\pi\varrho_0)} = v_{\text{ph},2}, \\ &v_{y3} \approx -H'_{y3}/\sqrt{(4\pi\varrho_0)}, \quad E'_{x3} \approx -v_{y3} H_z^{(0)}/c, \\ &v_{z3} \approx -H_y^{(0)} H_z^{(0)} H'_{y3}/(4\pi\varrho_0)^{3/2} u_0^2 \ll v_{y3}. \end{aligned}\right\} \tag{14.16}$$

Thus this wave is close to the hydromagnetic wave proper (2), and the two coincide (apart from polarisation) for an incompressible fluid ($u_0^2 \to \infty$) or for $H^{(0)} \to 0$; in either case, $\zeta^2 \to \infty$.† Thus in the limit $\zeta^2 \to \infty$ (or better $u_0^2 \to \infty$) there are two hydromagnetic waves proper, with phase velocity $v_{\text{ph}} = H_z^{(0)}/\sqrt{(4\pi\varrho_0)} = (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}$, two independent directions of polarisation, and $\boldsymbol{v} = -\boldsymbol{H}'/\sqrt{(4\pi\varrho_0)}$, $\boldsymbol{k}\cdot\boldsymbol{v} = 0$ and $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}^{(0)}/c$. In strong fields, when

$$\zeta^2 = u_0^2 \div [H^{(0)}]^2/4\pi\varrho_0 \ll 1, \tag{14.17}$$

we obtain

$$\left.\begin{aligned} &v_{\text{ph},1} \approx H^{(0)}/\sqrt{(4\pi\varrho_0)}, \quad v_{\text{ph},3} \approx u_0\cos\alpha, \\ &v_{z1} \approx (H'_{y1}\sin\alpha)/\sqrt{(4\pi\varrho_0)}, \quad v_{y1} \approx -(H'_{y1}\cos\alpha)/\sqrt{(4\pi\varrho_0)}, \\ &v_{z3} \approx (H^{(0)} H'_{y3}\cot\alpha)/4\pi\varrho_0 u_0, \quad v_{y3} \approx -H^{(0)} H'_{y3}/4\pi\varrho_0 u_0. \end{aligned}\right\} \tag{14.18}$$

† The transition to the field $H^{(0)} \to 0$ must here be understood in a somewhat conventional sense, since the condition (14.2) must also hold for hydromagnetic waves in a plasma. It may also be noted that the term "incompressible" is sometimes applied to a fluid for which the pressure term is neglected, i.e. $u_0 \to 0$ instead of $u_0 \to \infty$. The terminology used here is, of course, the more correct.

In the greater part of this book we use the refractive index $n = c/v_{\mathrm{ph}}$ instead of the phase velocity. It should therefore be noted that the refractive index for hydromagnetic waves is usually very large. For example, in a pure electron-ion plasma we find for a hydromagnetic wave proper $n_2 = c/v_{\mathrm{ph},2} = c\sqrt{(4\pi M N)}/H^{(0)}\cos\alpha$.

In the solar corona $N \sim 10^8$, $H^{(0)} \sim 10$ and we have $n_2 \gtrsim 10^2$, $v_{\mathrm{ph},2} \lesssim 3\times 10^8$ cm/sec; in the lower layers of the Sun's atmosphere and in the interior of the Sun, n_2 may of course be very much greater still. In the interstellar medium, with $N \sim 1$ and $H^{(0)} \sim 10^{-5}$, we have $n_2 \gtrsim 10^4$, $v_{\mathrm{ph},2} \lesssim 3\times 10^6$.

Damping of hydromagnetic waves has hitherto been assumed absent. When the viscosity and the finite electrical conductivity are taken into account (still neglecting thermal conductivity), the linearised equations of magnetic hydrodynamics analogous to (14.4) are

$$\left.\begin{aligned} \partial \boldsymbol{H}'/\partial t &= \mathbf{curl}(\boldsymbol{v}\times\boldsymbol{H}^{(0)}) + (c^2/4\pi\sigma)\,\triangle\boldsymbol{H}',\\ \boldsymbol{j} &= c\,\mathbf{curl}\boldsymbol{H}'/4\pi = \sigma(\boldsymbol{E}' + \boldsymbol{v}\times\boldsymbol{H}^{(0)})/c,\\ \partial\boldsymbol{v}/\partial t &= -(u_0^2/\varrho_0)\,\mathbf{grad}\,\varrho' - (1/4\pi\varrho_0)\,\boldsymbol{H}^{(0)}\times\mathbf{curl}\boldsymbol{H}' +\\ &\quad + (\eta/\varrho_0)\,\triangle\boldsymbol{v} + (1/\varrho_0)(\zeta + \tfrac{1}{3}\eta)\,\mathbf{grad}\,\mathrm{div}\boldsymbol{v},\\ \partial\varrho'/\partial t &+ \varrho_0\,\mathrm{div}\boldsymbol{v} = 0. \end{aligned}\right\} \qquad (14.19)$$

The solution of these equation offers no difficulty in principle, but the resulting expressions are in general somewhat complicated. We shall therefore consider only some limiting cases. In an incompressible fluid $u_0^2 \to \infty$, $\varrho' = 0$, $\mathrm{div}\boldsymbol{v} = 0$, and for waves of the type $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ we have the dispersion relation

$$\omega^2 - i\left(\frac{4\pi c^2}{\sigma} + \frac{\eta}{\varrho_0}\right)\omega k^2 - \frac{[H^{(0)}]^2}{4\pi\varrho_0}k^2\cos^2\alpha - \frac{4\pi c^2\eta}{\varrho_0\sigma}k^4 = 0. \qquad (14.20)$$

From this, when $\sigma\to\infty$ and $\eta = 0$, we obtain for the phase velocity of waves with any polarisation $v_{\mathrm{ph}} = (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}$, as it should be in an incompressible fluid. Putting $k = \omega(n - i\varkappa)/c$, with $\varkappa \ll n$ (weak damping), we find

$$\left.\begin{aligned} \omega\varkappa_{2,3}/c = \mathrm{im}\,k &= \frac{\omega^2}{2v_{\mathrm{ph},2,3}^3}\left(\frac{4\pi c^2}{\sigma} + \frac{\eta}{\varrho_0}\right),\\ v_{\mathrm{ph},2,3} &= (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}. \end{aligned}\right\} \qquad (14.21)$$

In the other limiting case we put $u_0^2 = 0$. Then for wave 2 we have (14.21); for wave 3 in this approximation $v_{\mathrm{ph},3} = 0$ [see (14.17) and (14.18)], and for wave 1

$$\omega - i\cdot\frac{4\pi c^2}{\sigma}k^2 - \frac{[H^{(0)}]^2}{4\pi\varrho_0}\left[\frac{\cos^2\alpha}{\omega - i\eta k^2/\varrho_0} + \frac{\sin^2\alpha}{\omega - i(4\eta + 3\zeta)k^2/3\varrho_0}\right]k^2 = 0. \qquad (14.22)$$

Hence, when the absorption is weak,

$$\omega \varkappa_1/c = \frac{\omega^2}{2 v_{\mathrm{ph},1}^3} \left[\frac{4\pi c^2}{\sigma} + \frac{\eta}{\varrho_0} + \left(\frac{\eta}{3\varrho_0} + \frac{\zeta}{\varrho_0} \right) \sin^2\alpha \right], \quad v_{\mathrm{ph},1} = H^{(0)}/\sqrt{(4\pi\varrho_0)}. \tag{14.23}$$

Besides waves which are small perturbations, we may consider in magnetic hydrodynamics, as in ordinary hydrodynamics, waves of large amplitude and various discontinuities. These also exist to some extent beyond the limits of the hydromagnetic approximation. We shall not discuss this interesting problem at all here; it is dealt with from the aspect of magnetic hydrodynamics in [36, 95]. In the more general case, and in particular for a rarefied plasma, the investigation of the nature of the discontinuities is as yet in its initial stages only [83].

Low-frequency waves: the quasihydrodynamic approximation

The hydromagnetic waves discussed above are only a limiting case of the low-frequency waves with $\omega \ll \Omega_H$, for which the conditions $\omega \ll \nu_{\mathrm{eff}}$ and $\omega_H \ll \nu_{\mathrm{eff}}$ hold [see (13.7)]. We shall begin from the quasihydrodynamic equations (13.15)–(13.18) in making the transition to low-frequency and intermediate-frequency waves in the range where magnetic hydrodynamics is not applicable [71, 72, 103, 104]. The waves considered will be those of small amplitude and the linear approximation will be used. The corresponding complete set of equations is, for $T_e = T_i = T_m = T =$ constant, as follows:

$$\left.\begin{aligned}
\frac{\partial \boldsymbol{v}_e}{\partial t} &= \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{H}^{(0)}/c) - \frac{\varkappa T}{m N^{(0)}} \operatorname{grad} N + \\
&\quad + \nu_{ei}(\boldsymbol{v}_i - \boldsymbol{v}_e) + \nu_{em}(\boldsymbol{v}_m - \boldsymbol{v}_e), \\
\frac{\partial \boldsymbol{v}_i}{\partial t} &= -\frac{e}{M}(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{H}^{(0)}/c) - \frac{\varkappa T}{m N^{(0)}} \operatorname{grad} N_i + \\
&\quad + (m/M)\, \nu_{ei}(\boldsymbol{v}_e - \boldsymbol{v}_i) + \nu_{im}(\boldsymbol{v}_m - \boldsymbol{v}_i), \\
\frac{\partial \boldsymbol{v}_m}{\partial t} &= -\frac{\varkappa T}{M N_m^{(0)}} \operatorname{grad} N_m - \frac{m N^{(0)}}{M N_m^{(0)}} \nu_{em}(\boldsymbol{v}_m - \boldsymbol{v}_e) - \\
&\quad - (N^{(0)}/N_m^{(0)})\, \nu_{im}(\boldsymbol{v}_m - \boldsymbol{v}_i), \\
&\partial N/\partial t + N^{(0)} \operatorname{div} \boldsymbol{v}_e = 0, \\
&\partial N_i/\partial t + N^{(0)} \operatorname{div} \boldsymbol{v}_i = 0, \\
&\partial N_m/\partial t + N_m^{(0)} \operatorname{div} \boldsymbol{v}_m = 0.
\end{aligned}\right\} \tag{14.24}$$

$$\left.\begin{aligned}
\Delta \boldsymbol{E} - \operatorname{grad} \operatorname{div} \boldsymbol{E} - (1/c^2)\, \partial^2 \boldsymbol{E}/\partial t^2 &= (4\pi/c^2)\, \partial \boldsymbol{j}_t/\partial t, \\
\boldsymbol{j}_t &= e N^{(0)}(\boldsymbol{v}_e - \boldsymbol{v}_i);
\end{aligned}\right\} \tag{14.25}$$

here $N^{(0)} \equiv N_e^{(0)} = N_i^{(0)}$ and $N_m^{(0)}$ are the unperturbed values of N, N_i and N_m. The index (0) is omitted below where no misunderstanding is possible.

Substituting in (14.24) solutions proportional to $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, we obtain a dispersion relation giving $(n - i\varkappa)^2$ as a function of ω. Since the treatment of the general case is laborious, we shall assume the pressure terms to be negligible (in which case, moreover, equations (14.24) become considerably more exact).† Then equations (14.24) are equivalent to (10.34)–(10.36), and the tensor $\varepsilon'_{ik}(\omega)$ in the form (10.43) may be used. When this is done, we need only use the wave equation (14.25) to determine $(n - i\varkappa)^2$; for a monochromatic plane wave this equation becomes

$$-k^2 E_i + k_i(\boldsymbol{k}\cdot\boldsymbol{E}) + (\omega^2/c^2)\,\varepsilon'_{il}E_l = 0, \quad k^2 = \omega^2(n - i\varkappa)^2/c^2, \tag{14.26}$$

where, in the absence of molecules [see (10.44)],

$$\left.\begin{aligned} &\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} = 1 - \omega_0^2/[(\omega \mp \omega_H)(\omega \pm \Omega_H) - i\omega\,\nu_{ei}],\\ &\varepsilon'_{yy} = \varepsilon'_{xx}, \quad \varepsilon'_{yx} = -\,\varepsilon'_{xy}, \quad \varepsilon'_{zz} = 1 - \omega_0^2/\omega(\omega - i\,\nu_{ei}), \end{aligned}\right\} \tag{14.27}$$

and the remaining components ε'_{ik} are zero.

From (14.26) and (14.27) we obtain, for a wave propagated at an angle α to the direction of the external field $\boldsymbol{H}^{(0)}$ [when we can put $\boldsymbol{k} = (0, k\sin\alpha, k\cos\alpha)$],

$$\left.\begin{aligned} (n - i\varkappa)^2_{1,2} &= \frac{[\varepsilon'^2_{xx}\sin^2\alpha + \varepsilon'_{xx}\varepsilon'_{zz}(1+\cos^2\alpha) + \varepsilon'^2_{xy}\sin^2\alpha] \pm \sqrt{R}}{2(\varepsilon'_{xx}\sin^2\alpha + \varepsilon'_{zz}\cos^2\alpha)},\\ R &= [\varepsilon'^2_{xx}\sin^2\alpha + \varepsilon'_{xx}\varepsilon'_{zz}(1+\cos^2\alpha) + \varepsilon'^2_{xy}\sin^2\alpha]^2 -\\ &\quad - 4(\varepsilon'_{xx}\sin^2\alpha + \varepsilon'_{zz}\cos^2\alpha)(\varepsilon'^2_{xx} + \varepsilon'^2_{xy})\,\varepsilon'_{zz}. \end{aligned}\right\} \tag{14.28}$$

For longitudinal propagation ($\alpha = 0$) this gives

$$\begin{aligned} (n - i\varkappa)^2_{1,2} \equiv (n - i\varkappa)^2_{\mp} &= \varepsilon'_{xx} \pm i\,\varepsilon'_{xy}\\ &= 1 - \omega_0^2/[(\omega \pm \omega_H)(\omega \mp \Omega_H) - i\,\omega\,\nu_{ei}]\\ &\approx 1 - \omega_0^2/\omega(\omega - i\,\nu \pm \omega_H - \omega_H\Omega_H/\omega), \end{aligned} \tag{14.29}$$

where ν in the last expression signifies ν_{ei}, and Ω_H has been neglected in comparison with ω_H. This latter simplification is not permissible near the ion gyromagnetic resonance $\omega \approx \Omega_H$, which will be discussed later.

In the high-frequency case (14.1), (14.29) becomes (11.8); if

$$\omega \ll \Omega_H, \qquad \omega\,\nu \ll \omega_H\,\Omega_H, \tag{14.30}$$

then, neglecting weak absorption, we have from (14.29)

$$\left.\begin{aligned} n^2_{1,2} &= 1 + \omega_0^2/\omega_H\,\Omega_H = 1 + 4\pi M\,Nc^2/[H^{(0)}]^2,\\ v^2_{\mathrm{ph}} &= c^2/n^2_{1,2} = c^2[H^{(0)}]^2/([H^{(0)}]^2 + 4\pi M\,N\,c^2). \end{aligned}\right\} \tag{14.31}$$

† The pressure is $p_e = \varkappa TN$, and the velocity of sound $u_0 \sim \sqrt{(\varkappa T/M)}$. Hence, in a plasma without molecules, when the density $\varrho_0 \approx MN$, the condition (14.17) becomes $[H^{(0)}]^2/8\pi \gg p_e$. Thus the particle pressure can in general be neglected in a sufficiently strong magnetic field, since the magnetic pressure $[H^{(0)}]^2/8\pi$ then plays the principal part.

Usually

$$\omega_0^2/\omega_H \Omega_H = 4\pi c^2 M N/[H^{(0)}]^2 \gg 1 \tag{14.32}$$

and the velocity v_{ph} is, according to (14.31), equal to the velocity of hydromagnetic waves $v_{\text{ph}} = H^{(0)}/\sqrt{(4\pi\varrho_0)}$, since $MN = \varrho_M = \varrho_0$ to within negligible terms of order m/M.

Thus we see that high-frequency waves (e.g. radio waves propagated in the ionosphere or the solar corona) and hydromagnetic waves, though very different in many ways, are nevertheless closely related and are essentially distinguished only by the values of the parameters. The corresponding values of $(n - i\varkappa)^2$ are obtained from general formulae [for instance, (14.29)] [71].

The condition (14.32) signifies that the displacement current $i\omega \boldsymbol{E}/c$ may be neglected in comparison with the "total current" $\boldsymbol{j}_t = \boldsymbol{j} + i\omega\boldsymbol{P}$, multiplied by $4\pi/c$. At the same time (14.31) shows that (14.32) gives $n^2 \gg 1$ and the phase velocity $v_{\text{ph}} = c/n \ll c$. From (14.30) and (14.32) with (14.27) we find

$$\left.\begin{aligned} &\varepsilon'_{xx} = \varepsilon'_{yy} \approx \omega_0^2/\omega_H \Omega_H, \quad \varepsilon'_{zz} \approx -\omega_0^2/\omega(\omega - i\nu), \\ &|\varepsilon'_{zz}| \gg \varepsilon'_{xx}; \end{aligned}\right\} \tag{14.33}$$

the remaining components are small or zero.

Now let the angle α between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$ be not too close to $\frac{1}{2}\pi$, so that we have the condition $|\varepsilon'_{zz}| \cos^2\alpha \gg \varepsilon'_{xx} \sin^2\alpha \sim \varepsilon'_{xx}$, i.e.

$$[\omega_H \Omega_H/\omega \sqrt{(\omega^2 + \nu^2)}] \cos^2\alpha \gg 1. \tag{14.34}$$

Then (14.28) gives

$$\left.\begin{aligned} &n_1^2 = \varepsilon'_{xx} = 4\pi c^2 M N/[H^{(0)}]^2, \quad v_{\text{ph},1} = c/n_1 = H^{(0)}/\sqrt{(4\pi\varrho_0)}, \\ &n_2^2 = \varepsilon'_{xx}/\cos^2\alpha = 4\pi c^2 M N/[H^{(0)}]^2 \cos^2\alpha, \\ &\qquad v_{\text{ph},2} = c/n_2 = (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}. \end{aligned}\right\} \tag{14.35}$$

These expressions are the same as the hydromagnetic formulae (14.7) and (14.10) with $u_0 = 0$ (it should be recalled that in deriving formulae (14.27) we have neglected the pressure, i.e. put $u_0 = 0$).

The range of validity of the hydromagnetic formulae

It is most important to emphasise that the conditions for the hydromagnetic formulae (14.35) to be applicable, i.e. the inequalities (14.30), (14.32) and (14.34), may be considerably less stringent than the general conditions for the hydromagnetic approximation to be valid [see (13.7)].

This is explained as follows. The conditions (13.7) are necessary for the more general relation (13.24) to take the particular form $\boldsymbol{j}_t = (e^2N/m\nu)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c)$, i.e. to reduce to the fundamental equation (13.2) of magnetic hydrodynamics; furthermore, for (13.24) itself to be valid it is necessary that the frequency ω should be sufficiently small (for which the condition $\omega \ll \Omega_H$ is adequate). When the conductivity is made infinite, however, we use only the relation $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}/c$, which holds good even if the displacement current is not neglected:† for a harmonic process, equation (13.24) can be written

$$\boldsymbol{j}_t + \frac{\omega_H \boldsymbol{j}_t \times \boldsymbol{H}}{H(i\omega + \nu)} = \frac{e^2 N}{m(i\omega + \nu)}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c) \tag{14.36}$$

(where $\nu = \nu_{ei}$), whence it is clear that, if ω and ν are sufficiently small or N is sufficiently large, the relation $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}/c$ can in general be used even if the conditions (13.7) are not satisfied. Thus, even if collisions are entirely neglected (a rarefied plasma), the limiting formulae (14.35) hold good with thermal motion neglected (formally $T = 0$) and the conditions

$$\omega \ll \Omega_H, \quad \omega_0^2/\omega_H \Omega_H \gg 1, \quad (\omega_H \Omega_H/\omega^2)\cos^2\alpha \gg 1. \tag{14.37}$$

From the above discussion it is quite understandable why hydromagnetic waves are often considered not only in the range where magnetic hydrodynamics is applicable, but throughout the range where formulae (14.35) are valid, for example with $\nu_{\text{eff}} = 0$ and the conditions (14.37).

Angles α close to $\frac{1}{2}\pi$

Let us now consider the range of angles α close to $\frac{1}{2}\pi$. If $\alpha = \frac{1}{2}\pi$, formula (14.28) gives for any values of the other parameters

$$(n - i\varkappa)_1^2 = (\varepsilon_{xx}'^2 + \varepsilon_{xy}'^2)/\varepsilon_{xx}', \quad (n - i\varkappa)_2^2 = \varepsilon_{zz}'. \tag{14.38}$$

This result for $(n - i\varkappa)_2^2$ is evident *a priori*, since for $\alpha = \frac{1}{2}\pi$ the ordinary wave in the quasihydrodynamic approximation is propagated in exactly the same way as when the magnetic field is absent; the field $\boldsymbol{H}^{(0)}$ does not affect the motion of the particles, since their mean velocities are parallel to the field. For the extraordinary wave 1 we again have from (14.38), with the conditions (14.33), the result (14.35), so that this wave has no exceptional properties for angles $\alpha \to \frac{1}{2}\pi$. For wave 2, on the other hand, (14.35) with $\alpha = \frac{1}{2}\pi$ gives $n_2^2 = \infty$ and $v_{\text{ph},2} = 0$, while (14.38) and (14.33) give $(n - i\varkappa)_2^2 = -\omega_0^2/\omega(\omega - i\nu)$, and the wave is very strongly damped even if $\nu = 0$. Substituting the values (14.33) in (14.28), we obtain, with the condition

$$\cos^2\alpha \ll 1, \tag{14.39}$$

the result

$$(n - i\varkappa)_2^2 = \omega_0^2/(\omega_H \Omega_H \cos^2\alpha - \omega^2 - i\omega\nu). \tag{14.40}$$

When α is sufficiently far from $\frac{1}{2}\pi$ and (14.34) holds, formula (14.40) of course gives (14.35): $n_2^2 = \omega_0^2/\omega_H \Omega_H \cos^2\alpha = 4\pi c^2 M N/[H^{(0)}]^2 \cos^2\alpha$. If $\alpha \to \frac{1}{2}\pi$, or more precisely if (14.34) does not hold, the hydromagnetic expression (14.35)

† For magnetic hydrodynamics [equations (13.1)–(13.4)] to be valid it is, of course, necessary that the displacement current should be negligible.

is invalid; this has already been mentioned, and is evident from (14.40). When $\alpha \to \frac{1}{2}\pi$ and the conductivity is infinite, magnetic hydrodynamics gives incorrect results even for arbitrarily small ω and ν, since for the ordinary wave the vector product $\boldsymbol{v} \times \boldsymbol{H}^{(0)}$ is zero when $\alpha = \frac{1}{2}\pi$. Thus for this wave in magnetic hydrodynamics when $\sigma \to \infty$ we have $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{H}^{(0)}/c = 0$, and therefore the wave exists only when the conductivity is finite or when the displacement current is taken into account. Both possibilities are allowed for in (14.40). It may also be noted that for $\alpha \to \frac{1}{2}\pi$ the polarisation of wave 2 also varies greatly. For a somewhat detailed discussion of low-frequency waves with α near $\frac{1}{2}\pi$, see [105].

The region of ion gyromagnetic resonance

The above formulae make it possible to obtain a very complete idea of the propagation of low-frequency waves in a plasma without molecules. There is no difficulty in going to "intermediate" frequencies; the corresponding results are essentially contained in the general formula (14.28). Qualitatively, the most important property of the frequency range $\omega \sim \Omega_H$ is the appearance of ion gyromagnetic resonance. The existence of a resonance at $\omega = \Omega_H$ when $\alpha = 0$ is immediately evident from formula (14.29) for $(n - i\varkappa)^2_{1,2}$. When $\alpha \neq 0$ the position of the resonance is different; see the end of § 11. We shall not pause to investigate the region of ion gyromagnetic resonance (see [88, 109]), and shall make here only one comment. It appears at first sight that, when small terms of order m/M are neglected in the expression for $(n - i\varkappa)^2$, we can always neglect the frequency Ω_H in comparison with ω_H. This is not so, however, as is particularly evident in the example of formula (14.29). When no quantity is neglected, the denominator in this formula is

$$(\omega \pm \omega_H)(\omega \mp \Omega_H) - i\,\omega\,\nu_{ei} = \omega^2 \pm \omega_H\,\omega \mp \omega\,\Omega_H - \omega_H\,\Omega_H - i\,\omega\,\nu_{ei};$$

for $\omega = \Omega_H$, and taking the upper sign (wave 2), this becomes $-i\omega\nu_{ei}$ (resonance). If, however, the term $\omega\Omega_H$ is neglected in comparison with $\omega\omega_H$, as when we go to the last member of (14.29), then the denominator for $\omega = \Omega_H$ is $\Omega_H^2 - i\omega\nu_{ei}$, and we do not have a true resonance. The above discussion is, of course, quite elementary, and its purpose is merely to emphasise the necessity of carrying out the calculations in the "intermediate" frequency range more accurately than at low frequencies. In consequence the formulae become somewhat more cumbersome.

The effect of molecules

The formulae become considerably more complicated also when the effect of molecules is taken into account. Here we shall give only one example as an illustration. In the presence of molecules, when the tensor ε'_{ik} has the form (10.43), for longitudinal propagation, formula (14.28) gives [71]

$$(n - i\varkappa)^2_{1,2} = 1 - \frac{\omega_0^2}{\omega\left[\omega - i\nu_e \pm \omega_H - \dfrac{\omega_H \Omega_H(i\omega + \nu_{im}N/N_m)}{\omega\{i\omega + \nu_{im}(N + N_m)/N_m\}}\right]}, \tag{14.41}$$

where $\nu_e = \nu_{ei} + \nu_{em}$ and we have used the facts that $M\nu_{im} \gg m\nu_{em}$ and $\nu_{em} \gg \nu_{im}$. Further, (14.41) does not include terms which outside the region of ion gyromagnetic resonance are always small in comparison with the terms retained. At low frequencies, when the inequalities (14.30) hold with $\nu = \nu_e$, we have†

$$(n - i\varkappa)^2_{1,2} = 1 + \frac{4\pi c^2 MN[\omega^2 + \nu_{im}^2 N(N + N_m)/N_m^2]}{[H^{(0)}]^2(\omega^2 + \nu_{im}^2 N^2/N_m^2)} - i\frac{4\pi c^2 MN\omega\nu_{im}}{[H^{(0)}]^2(\omega^2 + \nu_{im}^2 N^2/N_m^2)}. \tag{14.42}$$

In the limit, when

$$\omega^2 \ll \nu_{im}^2 N^2/N_m^2 = (\overline{q_{im}v_{im}}N)^2, \tag{14.43}$$

where q_{im} is the effective cross-section and v_{im} the relative velocity, and neglecting the relatively weak damping, we obtain from (14.42) the hydromagnetic value $n^2 = c^2/v_{\text{ph}}^2 = 4\pi\varrho_0 c^2/[H^{(0)}]^2$, since the density of the medium $\varrho_M = \varrho_0 = M(N + N_m)$. In the opposite limit

$$\omega^2 \gg \nu_{im}^2 N^2/N_m^2 \tag{14.44}$$

we have

$$(n - i\varkappa)^2_{1,2} = 1 + \frac{4\pi c^2 MN}{[H^{(0)}]^2}\left(1 + \frac{\nu_{im}^2 N}{\omega^2 N_m}\right) - i\frac{4\pi c^2 MN\nu_{im}}{\omega[H^{(0)}]^2}$$

$$\approx \frac{4\pi c^2 MN}{[H^{(0)}]^2} - i\frac{4\pi c^2 MN\nu_{im}}{\omega[H^{(0)}]^2}; \tag{14.45}$$

the last expression is valid if

$$4\pi c^2 MN/[H^{(0)}]^2 \gg 1, \quad \nu_{im}^2 N/\omega^2 N_m \ll 1. \tag{14.46}$$

When damping is neglected, (14.45) gives $n^2 = 4\pi c^2 MN/[H^{(0)}]^2$. This may differ greatly from the hydromagnetic value $n^2 = 4\pi c^2 M(N + N_m)/[H^{(0)}]^2$; in other words, in the case (14.44), (14.46) and for weak damping, the wave is propagated as if the molecules were absent. For $q_{im} \sim 10^{-15}$ and $v_{im} \sim 10^5$ the condition (14.43) becomes $\omega \ll 10^{-10}N$, which for the Earth's ionosphere gives the very stringent inequality $\omega \ll 10^{-4}$, since $N \lesssim 10^6$. In the opposite

† In deriving (14.42) from (14.41) the conditions (14.30) are sufficient only if $N/N_m \gtrsim 1$; if $N/N_m \ll 1$, the condition $\Omega_H N/N_m = \omega$ is also necessary for formula (14.42) to be valid.

case (14.44) we may take, for instance, $\omega \sim 10$. Then we have for the F layer ($N \sim 10^6$, $N_m \sim 10^{10}$, $\nu_{im} \sim 1$, $H^{(0)} \sim 0{\cdot}5$)

$$\begin{aligned} n &\approx \sqrt{(4\pi c^2 MN/[H^{(0)}]^2)} \sim 10^3, \\ v_{\text{ph}} &= c/n \sim 3 \times 10^7 \text{ cm/sec}, \\ \lambda &= 2\pi v_{\text{ph}}/\omega \sim 200 \text{ km}, \\ \varkappa &\approx \nu_{im} n/2\omega \sim 50, \\ d &= c/\omega\varkappa \sim 600 \text{ km}; \end{aligned}$$

for the E layer ($N \sim 10^5$, $N_m \sim 10^{13}$ to 10^{14}, $\nu_{im} \sim 10^3$ to 10^4)

$$n \approx \varkappa \approx \sqrt{(2\pi c^2 M N \nu_{im}/\omega[H^{(0)}]^2)} \sim 1 \text{ to } 3 \times 10^{-7} c \sim 3 \times 10^3 \text{ to } 10^4,$$

$$\begin{aligned} v_{\text{ph}} &\sim 3 \times 10^6 \text{ to } 10^7 \text{ cm/sec}, \\ d &= c/\omega\varkappa \sim 3 \text{ to } 10 \text{ km}. \end{aligned}$$

The thermal motion. Some results of the kinetic theory: velocity change, damping in the absence of collisions

The velocity of low-frequency (hydromagnetic) waves is in many cases very small. For example, as already stated, in interstellar space with $H^{(0)} \sim 10^{-5}$ and $\varrho_0 = MN \sim 10^{-24}$ g/cm^3 the velocity is $v_{\text{ph}} = H^{(0)}/\sqrt{(4\pi\varrho_0)} \sim 3 \times 10^6$ cm/sec; in the solar corona with $H^{(0)} \sim 1$ and $\varrho_0 \sim 10^{-16}$ we have $v_{\text{ph}} \sim 10^8$. The thermal velocities of the electrons are in these cases of the order of v_{ph} or greater, and so the allowance for the thermal motion may be important.†

When the pressure terms are retained the equations (14.24)–(14.25) can be used to examine the propagation of waves, taking into account the thermal motion of the particles. However, as in the high-frequency case (see § 12), in the quasihydrodynamic approximation the collisionless absorption is absent, and moreover we cannot assign a quantitative value to the temperature-dependent corrections to the wave velocity. The quasihydrodynamic method is therefore of interest mainly on account of the possibility of effecting a general analysis of the formulae for $n - i\varkappa$ for various values of the parameters, near resonance, etc., the formulae concerned being relatively simple. Since we do not intend to carry out such an analysis here, we shall consider immediately the results of the kinetic theory as applied to a plasma without molecules, and only for the low-frequency case (14.2).

The calculations for this case ([106–108]; see also [51]) differ in nature and procedure from those discussed in § 12 only in that the Boltzmann equation

† The allowance for the effect of the thermal motion, e.g. in an isotropic plasma (see § 8), indicates that this effect cannot be completely determined by the parameter $v_T/v_{\text{ph}} \sim \sqrt{(\varkappa T/m)}\, n/c$ alone. Nevertheless, it is clear from the results of the calculation and from the discussion in §§ 8 and 12 of the significance of Cherenkov radiation that the value of the parameter v_T/v_{ph} is very important; we need only notice that for $v_T \sim v_{\text{ph}}$ the absorption of waves by the inverse Vavilov–Cherenkov effect becomes considerable (see § 8).

is used for ions as well as electrons. The magnitude of the thermal corrections to the rate of propagation and damping of hydromagnetic waves is different for $\alpha \to 0$ (longitudinal propagation), $\alpha \to \frac{1}{2}\pi$ (for the ordinary wave) and $0 < \alpha < \frac{1}{2}\pi$. The case $\alpha \to 0$ is distinguished because the waves 1 and 2 are then both transverse. We have already discussed the singularities of wave 2 for $\alpha \to \frac{1}{2}\pi$; since wave 2 is strongly damped when $\alpha \to \frac{1}{2}\pi$ even when the thermal motion is neglected, it is unnecessary to consider this problem in more detail here.

For $\alpha = 0$ the kinetic calculation [106] gives, in the region of weak damping ($\gamma \ll \omega$) and in the absence of collisions, the following expressions for the velocities of waves 1 and 2, which are circularly polarised with opposite directions of rotation of the electric vector:

$$\left.\begin{aligned} v_{\text{ph},1}^2 &= \frac{[H^{(0)}]^2}{4\pi\varrho_0}\left\{1 + \frac{\varkappa T}{M[H^{(0)}]^2/4\pi\varrho_0}\cdot\frac{\omega}{\Omega_H}\right\} \equiv v_0^2\left\{1 + \frac{v_{iT}^2}{v_0^2}\frac{\omega}{\Omega_H}\right\}, \\ v_{\text{ph},2}^2 &= \frac{[H^{(0)}]^2}{4\pi\varrho_0}\left\{1 - \frac{\varkappa T}{M[H^{(0)}]^2/4\pi\varrho_0}\cdot\frac{\omega}{\Omega_H}\right\} \equiv v_0^2\left\{1 - \frac{v_{iT}^2}{v_0^2}\frac{\omega}{\Omega_H}\right\}; \end{aligned}\right\} \tag{14.47}$$

here it is assumed that

$$\left.\begin{aligned} &[H^{(0)}]^2/4\pi\varrho_0 \ll c^2, \\ &\sqrt{\frac{\varkappa T}{M}} \ll \frac{\Omega_H H^{(0)}/\sqrt{(4\pi\varrho_0)}}{\omega}\sqrt{\left\{1 \pm \frac{\varkappa T}{M[H^{(0)}]^2/4\pi\varrho_0}\cdot\frac{\omega}{\Omega_H}\right\}}. \end{aligned}\right\} \tag{14.48}$$

The plus sign corresponds to the extraordinary wave (1) and the minus sign to the ordinary wave (2). The conditions (14.48) are compatible with the relation

$$(v_{iT}^2/v_0^2)\,\omega/\Omega_H = \frac{\varkappa T/M}{[H^{(0)}]^2/4\pi\varrho_0}\cdot\frac{\omega}{\Omega_H} \sim 1,$$

and so the change in the velocities $v_{\text{ph},1,2}$ due to the thermal motion may be considerable even when formulae (14.47) hold. A still more important fact is that, when the thermal motion is taken into account, the velocities $v_{\text{ph},1}$ and $v_{\text{ph},2}$ are unequal and depend on the frequency. There will therefore be a rotation of the plane of polarisation and a change in the shape of the pulses formed from low-frequency waves of varying frequency and polarisation.

The damping of the field in these waves with time is as $e^{-\gamma t}$, with

$$\left.\begin{aligned} \gamma_1 &= \Omega_H\frac{v_{\text{ph},1}}{v_{iT}}\frac{\Omega_H}{\omega}\sqrt{\frac{\pi}{2}}\left(2 - \frac{v_{iT}^2}{v_{\text{ph},1}^2}\frac{\omega}{\Omega_H}\right)^{-1}\exp\left(-\frac{v_{\text{ph},1}^2\Omega_H^2}{2v_{iT}^2\omega^2}\right), \\ \gamma_2 &= \Omega_H\frac{v_{\text{ph},2}}{v_{iT}}\frac{\Omega_H}{\omega}\sqrt{\frac{\pi}{2}}\left(2 + \frac{v_{iT}^2}{v_{\text{ph},2}^2}\frac{\omega}{\Omega_H}\right)^{-1}\exp\left(-\frac{v_{\text{ph},2}^2\Omega_H^2}{2v_{iT}^2\omega^2}\right). \end{aligned}\right\} \tag{14.49}$$

From (14.47) and (14.49) it is clear that for $\alpha = 0$ the changes in velocity and wave damping due to the thermal motion are determined by the parameters

$$\xi = v_{iT}^2\,\omega/v_0^2\,\Omega_H, \quad \xi\,\omega/\Omega_H = v_{iT}^2\,\omega^2/v_0^2\,\Omega_H^2. \tag{14.50}$$

If $\xi \ll 1$, then certainly $\xi\omega/\Omega_H \ll 1$, and thermal effects may be neglected. We have $\xi \ll 1$ even if $v_0 = H^{(0)}/\sqrt{(4\pi\varrho_0)} \sim v_{iT} = \sqrt{(\varkappa T/M)}$, since $\omega/\Omega_H \ll 1$. This example, by the way, shows that the importance of the thermal corrections is not always determined by the ratio v_T/v_{ph}, but may be represented by a considerably smaller quantity, here ξ for the velocity and $\xi\omega/\Omega_H$ for the damping. Thus, when $\alpha = 0$, the effect of the thermal motion on the propagation of low-frequency transverse waves is very slight under conditions of practical interest.

The situation is in general different for angles α other than zero (more precisely, angles α sufficiently different from zero).† Here we shall give the results (see [108] and also [107]) obtained by assuming that

$$\left.\begin{aligned} &\omega \ll \Omega_H, \quad c^2 k^2/\omega^2 = n^2 \gg 1 \quad (\text{i.e. } v_{\text{ph}}^2 = c^2/n^2 \ll c^2),\\ &(\varkappa T/M)(k^2/\Omega_H^2)\sin^2\alpha = (v_{iT}^2/v_{\text{ph}}^2)(\omega^2/\Omega_H^2)\sin^2\alpha \ll 1. \end{aligned}\right\} \tag{14.51}$$

If also

$$v_{\text{ph}} = c/n \gg \sqrt{(\varkappa T/m)}\cos\alpha, \tag{14.52}$$

we obtain for the velocities $v_{\text{ph},1}$ and $v_{\text{ph},2}$ the hydromagnetic values (14.35). Here we suppose for wave 2 that the last condition (14.37) also holds, i.e. the angle α must not be too close to $\frac{1}{2}\pi$ (as already stated, the range of angles $\alpha \to \frac{1}{2}\pi$ is not of interest as regards the thermal corrections).

The damping decrements under the condition (14.52) are

$$\left.\begin{aligned} \gamma_1 &= \omega\sqrt{\frac{\pi}{8}}\,\beta_T\, n_1 \frac{\sin^2\alpha}{\cos\alpha}\frac{m}{M}\exp\left(-\frac{1}{2\beta_T^2 n_1^2\cos^2\alpha}\right),\\ \gamma_2 &= \omega\sqrt{\frac{\pi}{8}}\frac{\sin^2\alpha}{(\beta_T n_2\cos\alpha)^3}\frac{m\,\omega^2}{M\Omega_H^2\cos^2\alpha}\exp\left(-\frac{1}{2\beta_T^2 n_2^2\cos^2\alpha}\right), \end{aligned}\right\} \tag{14.53}$$

where $\beta_T = \sqrt{(\varkappa T/mc^2)}$, and small terms have been omitted, in particular those of the order of the expressions (14.49) which give $\gamma_{1,2}$ for $\alpha = 0$. For this reason formulae (14.53) with $\alpha = 0$ lead to $\gamma_1 = \gamma_2 = 0$.

According to (14.52), formulae (14.53) are valid only if $\beta_T n\cos\alpha \ll 1$, but they can be used to obtain a rough estimate if $\beta_T n\cos\alpha \lesssim 1$. Even in that case, as well as in the range (14.52), γ_1 and γ_2 are small in the sense that both are always much less than ω. In absolute magnitude the values of γ_1 and γ_2 for $\beta_T n\cos\alpha \sim 1$ may not be small, and certainly not exponentially small; and $\gamma_1 \gg \gamma_2$.

Thus we see that for $\alpha \sim 1$ the condition for appreciable damping is simply $v_{\text{ph}} \sim v_T = \sqrt{(\varkappa T/m)}$; but for $\alpha \to \frac{1}{2}\pi$ in wave 1, where the above formulae are

† Absorption is especially small for $\alpha \to 0$, because for $\alpha = 0$ there is no inverse Cherenkov absorption (see § 12) and the resonance absorption at the frequencies of the ion gyromagnetic resonance $\omega \approx s\Omega_H (s = 1, 2, 3, \ldots)$ is evidently not considered here, since we use the condition $\omega \ll \Omega_H$.

It may also be noted that, for $\alpha = 0$, wave 3 is longitudinal and is strongly damped unless the collision frequency is fairly large (see § 8).

valid, the damping is much reduced and is exponentially small except for $v_{\text{ph}} \sim v_T \cos\alpha$.

Let us now consider the region of smaller wave velocities, for which

$$\sqrt{(\varkappa T/m)} \cos\alpha \gg v_{\text{ph}} \gg \sqrt{(\varkappa T/M)} \cos\alpha. \tag{14.54}$$

Then we have as far as terms of order $\varkappa T/M$

$$\left.\begin{aligned} v^2_{\text{ph},1} &= \frac{c^2}{n_1^2} = \frac{[H^{(0)}]^2}{4\pi\varrho_0} + \frac{3\varkappa T}{M}\sin^2\alpha, \\ v^2_{\text{ph},2} &= \frac{c^2}{n_2^2} = \frac{[H^{(0)}]^2}{4\pi\varrho_0}\cos^2\alpha, \end{aligned}\right\} \tag{14.55}$$

where we have omitted terms of the order $(\varkappa T/M v^2_{\text{ph}})\,\omega/\Omega_H$, which appear in formulae (14.47) but are small under the conditions (14.51) and (14.54). Next, denoting $\varkappa T/Mc^2$ by β_i^2, we have

$$\left.\begin{aligned} \gamma_1 &= \omega\sqrt{\frac{\pi}{8}}\sin^2\alpha\left[\frac{\beta_i^2 n_1^2}{\beta_T n_1\cos\alpha} + \frac{5\beta_i n_1}{\cos\alpha}\exp\left(-\frac{1}{2\beta_i^2 n_1^2\cos^2\alpha}\right)\right], \\ \gamma_2 &= \omega\sqrt{\frac{\pi}{8}}\frac{\omega^2}{\Omega_H^2}\left[\frac{\sin^2\alpha}{\sin^2\alpha + 3(\omega^2/\Omega_H^2)\cos^2\alpha}\right]\times \\ &\times\left[\frac{\beta_i^2 n_2^2}{\beta_T n_2\cos\alpha} + \frac{\beta_T n_2(\sin^2\alpha + 4\cos^2\alpha)}{\cos\alpha}\exp\left(-\frac{1}{2\beta_i^2 n_2^2\cos^2\alpha}\right)\right]. \end{aligned}\right\} \tag{14.56}$$

One of the conditions (14.54) signifies that $\beta_i n\cos\alpha \ll 1$, but formulae (14.56) can be used to give an estimate even if $\beta_i n\cos\alpha \sim 1$, i.e. $v_{\text{ph}} \sim \sqrt{(\varkappa T/M)}\cos\alpha$. In this case, for $\alpha \sim 1$, the damping of wave 1 is very strong ($\gamma \sim \omega$), and as α decreases it is diminished only because of the factor $\sin^2\alpha$. The damping of wave 2 for $\alpha \sim 1$ is less than that of wave 1 by a factor of approximately ω^2/Ω_H^2.

As already stated in §§ 8 and 12, in the presence of the weak damping γ_{coll} due to collisions in addition to the damping $\gamma_0 = \gamma_{1,2}$ already considered, the total damping is $\gamma = \gamma_0 + \gamma_{\text{coll}}$ and the spatial damping is $q = \omega\varkappa/c = \gamma/v_{\text{gr},k}$, where $v_{\text{gr},k}$ is the component of the group velocity in the direction of the wave vector $\mathbf{k}$. Hence, to ascertain the importance of the specific damping, it is sufficient to compare γ_0 with γ_{coll}, and as an estimate we may use the value of γ_{coll} for $\alpha = 0$, since this part of the damping in general depends only slightly on α. According to (14.29), (14.30) and (14.32), $n_{1,2} = \omega_0/\sqrt{(\omega_H\Omega_H)}$, $\omega\varkappa/c = (\omega/2c)\,n_{1,2}\omega\nu/\omega_H\Omega_H$ and $\gamma_{\text{coll}} = \omega^2\nu/2\omega_H\Omega_H$, since in this case $v_{\text{gr},k} = c/n$. In the galactic corona, with $H^{(0)} \sim 10^{-5}$ and a proton density $N \sim 0{\cdot}1$ to $0{\cdot}01$, we have $v_{\text{ph}} = H^{(0)}/\sqrt{(4\pi\varrho_0)} \sim 7\times10^6$ to 2×10^7 cm/sec, whereas $v_i = \sqrt{(\varkappa T/M_p)} \sim 10^6$ and $v_T \sim \sqrt{(\varkappa T/m)} \sim 4\times10^7$ (for $T \sim 10^4$ deg). These values are such that for $\alpha \sim 1$ we can use formulae (14.56); for example, with $v_{\text{ph}} = 7 \times 10^6$, $v_i = 9 \times 10^5$ and $\alpha = 45°$ we have $\gamma_1 = 1{\cdot}3 \times 10^{-3}\,\omega$ and $\omega\varkappa/c = \gamma k/\omega = 1{\cdot}3 \times 10^{-3} \times 2\pi/\lambda = 8{\cdot}2 \times 10^{-3}/\lambda$. This gives a damping factor exceeding unity in a time of the order of hundreds of periods, or a distance of hundreds of wavelengths, so that the effect may be very considerable. For wave 2 in the same conditions, the damping γ_2 is much less on account of the factor ω^2/Ω_H^2; for $\Omega_H \sim 0{\cdot}1$, as for protons in a field $H^{(0)} \sim 10^{-5}$, the factor $\omega^2/\Omega_H^2 < 10^{-8}$ for frequencies $\omega < 10^{-5}$.

For the interstellar plasma with density $N \sim 0{\cdot}1$ to $0{\cdot}01$ and $T \sim 10^4$ the collision frequency $\nu \sim 10^{-5}$ to 10^{-6} ($l = v/\nu \sim 10^{12}$ to 10^{13} cm) and $\gamma_{\text{coll}} = \omega^2\nu/2\omega_H\Omega_H \sim 10^{-7}\omega^2$, i.e. for $\omega \sim 10^{-5}$ it is nine orders of magnitude less than γ_1 (see above) but comparable with γ_2.

In the lower part of the solar corona, with $N \sim 10^8$, $H^{(0)} \sim 10$ oersted and $T \sim 10^6$, we find $v_{\text{ph}} = H^{(0)}/\sqrt{(4\pi\varrho_0)} \sim 3 \times 10^8$, whereas $v_T \sim 4 \times 10^8$ and $v_i \sim 10^7$. In this case we have $v_{\text{ph}} \sim v_T \cos\alpha$ over a certain range of angles and we can use formulae (14.53) to give an estimate. The result is $\gamma_1 \sim m\omega/M \sim$ $\sim 10^{-3}\omega$ and $\gamma_2 \sim (\omega/\Omega_H)^2\gamma_1$, while $\gamma_{\text{coll}} \sim 10^{-12}\omega^2$, since $\nu \sim 10$, $\omega_H \sim 10^8$; even for $\omega \sim 0{\cdot}1\Omega_H \sim 10^4$ we have $\gamma_1 \sim 10$, $\gamma_2 \sim 0{\cdot}1$ and $\gamma_{\text{coll}} \sim 10^{-4}$. In the deeper layers of the Sun and the other stars the velocity of hydromagnetic waves decreases owing to the increased density; for example, if $H^{(0)} \sim 10$ and $\varrho_0 \sim 10^{-5}$ the phase velocity is $v_{\text{ph}} = H^{(0)}/\sqrt{(4\pi\varrho_0)} \sim 10^3$, whereas with $T \sim 10^5$ we have $v_i = \sqrt{(\varkappa T/M)} \sim 3 \times 10^5$ and thus $v_{\text{ph}} \ll v_i$. The latter case has not been calculated quantitatively, but in such cases the damping is probably even stronger than in the case (14.54). In general $\gamma_1 \gg \gamma_2$, and for sufficiently low frequencies $\gamma_{\text{coll}} \ll \gamma_1$.

The above examples have been selected quite arbitrarily, and our purpose is simply to demonstrate the need to bear in mind the possible significance of low-frequency wave damping not due to collisions. This damping may be the more important in that it is very different for the ordinary and extraordinary waves and depends fairly considerably on a number of parameters, and in particular on the angle α between the field $\boldsymbol{H}^{(0)}$ and the wave vector $\boldsymbol{k}$.

§ 15. SUMMARY OF PRINCIPAL FORMULAE

The propagation of waves in a magnetoactive plasma involves a large number of different particular cases in which the very complex general formulae for the refractive indices and other quantities become considerably simpler. For this reason and for purposes of reference we shall give here some of the most important formulae.

In the high-frequency case, where

$$\omega \gg \Omega_H = |e|\,H^{(0)}/Mc, \tag{10.5}$$

the ions may usually be neglected and the tensor ε'_{ik} has the form

$$\left.\begin{aligned}
&\varepsilon'_{xx} = \varepsilon'_{yy} = 1 - \omega_0^2(\omega - i\,\nu_{\text{eff}})/\omega[(\omega - i\,\nu_{\text{eff}})^2 - \omega_H^2],\\
&\varepsilon'_{zz} = 1 - \omega_0^2/\omega(\omega - i\,\nu_{\text{eff}}),\\
&\varepsilon'_{xy} = -\varepsilon'_{yx} = -i\,\omega_0^2\,\omega_H/\omega(\omega + \omega_H - i\,\nu_{\text{eff}})(\omega - \omega_H - i\,\nu_{\text{eff}}),\\
&\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} = 1 - \omega_0^2/(\omega^2 \mp \omega\,\omega_H - i\omega\,\nu_{\text{eff}}),\\
&\varepsilon'_{xz} = \varepsilon'_{zx} = \varepsilon'_{yz} = \varepsilon'_{zy} = 0;
\end{aligned}\right\} \tag{10.12}$$

here the z-axis is in the direction of the external magnetic field $\boldsymbol{H}^{(0)}$, $\omega_0^2 = 4\pi e^2 N/m$, and $\omega_H = |e|\,H^{(0)}/mc = 1{\cdot}76 \times 10^7 H^{(0)}$. In the absence of absorption we must put $\nu_{\text{eff}} = 0$ in (10.12); then $\sigma_{ik} = 0$ and $\varepsilon'_{ik} = \varepsilon_{ik}$. In the coordinate system shown in Fig. 10.1 the components of the tensor ε'_{ik} are given by the expressions (10.17). The problem of calculating ν_{eff} (or, in the kinetic treatment, of finding the tensors ε_{ik} and σ_{ik} as functions of ν_{eff}) has been discussed in § 10; see, in particular, (10.32). When the motion of the ions is taken into account as well as that of the electrons, we have

$$\left.\begin{aligned}
&\varepsilon'_{xx} \mp i\,\varepsilon'_{xy} = 1 - \omega_0^2/[(\omega \mp \omega_H)(\omega \pm \Omega_H) - i\,\omega\,\nu_{ei}],\\
&\varepsilon'_{xx} = \varepsilon'_{yy}, \quad \varepsilon'_{xy} = -\varepsilon'_{yx}, \quad \varepsilon'_{zz} = 1 - \omega_0^2/\omega(\omega - i\,\nu_{\text{eff}});
\end{aligned}\right\} \tag{10.44}$$

the remaining components are zero. Here the molecules are assumed absent, giving a pure electron-ion plasma, with ν_{ei} the effective frequency of collisions of an electron with ions. When molecules are present, the tensor ε'_{ik} is given by formula (10.43).

The wave equation for waves propagated in a magnetoactive plasma is

$$\left.\begin{aligned}
&\triangle\boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} + (\omega^2/c^2)(\boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega) = 0,\\
&D_i - i\cdot 4\pi\,j_i/\omega = \varepsilon'_{ik}\,E_k.
\end{aligned}\right\} \tag{11.1}$$

This equation has the same form in other media, of course, and the choice of a particular medium merely determines the form of the tensor ε'_{ik}.

In the system of coordinates used, with the wave vector $\boldsymbol{k}$ in the z-direction (see Fig. 11.1), we have for plane waves

$$\left.\begin{aligned}
&d^2 E_x/dz^2 + (\omega^2/c^2)(A\,E_x + i\,C\,E_y) = 0,\\
&d^2 E_y/dz^2 + (\omega^2/c^2)(-i\,C\,E_x + B\,E_y) = 0,
\end{aligned}\right\} \tag{11.3}$$

$$\begin{aligned}
E_z = &-\frac{i\sqrt{u}(1 - i\,s)\,v\sin\alpha}{(1 - i\,s)\,u - (1 - i\,s)^2(1 - i\,s - v) - u\,v\cos^2\alpha}\,E_x + \\
&+ \frac{u\,v\cos\alpha\sin\alpha}{(1 - i\,s)\,u - (1 - i\,s)^2(1 - i\,s - v) - u\,v\cos^2\alpha}\,E_y,
\end{aligned} \tag{10.20}$$

$$\left.\begin{aligned}
&\sqrt{u} = \omega_H/\omega = |e|\,H^{(0)}/m\,c\,\omega, \quad v = \omega_0^2/\omega^2 = 4\pi\,e^2\,N/m\,\omega^2,\\
&s = \nu_{\text{eff}}/\omega,
\end{aligned}\right\} \tag{10.18}$$

where the coefficients A, B and C, which depend on u, v, s and α, are given by (10.22); see also (10.23)–(10.25) and (11.3).

Substitution in (11.3) of the solution $E_{x,y} = E_{0x,y} e^{-i\omega(n-i\varkappa)z/c}$ leads to equations (11.2a) and (11.4); hence in the general case $(n - i\varkappa)^2_{1,2}$ is given by formula (11.5), and in the absence of absorption

$$\begin{aligned}\tilde{n}^2_{1,2} &= (n - i\varkappa)^2_{1,2} \\ &= 1 - \frac{2v(1-v)}{2(1-v) - u\sin^2\alpha \pm \sqrt{[u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha]}} \\ &= 1 - \frac{2\omega_0^2(\omega^2 - \omega_0^2)}{2(\omega^2 - \omega_0^2)\omega^2 - \omega_H^2\omega^2\sin^2\alpha \pm \sqrt{[\omega^4\omega_H^4\sin^4\alpha + 4\omega_H^2\omega^2(\omega^2 - \omega_0^2)^2\cos^2\alpha]}}.\end{aligned} \tag{11.6}$$

Here the plus sign corresponds to the ordinary wave 2, and the minus sign to the extraordinary wave 1.

For $H^{(0)} = 0$ (an isotropic plasma) we have

$$(n - i\varkappa)^2_{1,2} = (n - i\varkappa)^2_0 = 1 - v/(1 - is) = 1 - \omega_0^2/\omega(\omega - i\nu_{\text{eff}}). \tag{11.7}$$

For $\alpha = 0$ (longitudinal propagation),

$$(n - i\varkappa)^2_{1,2} \equiv (n - i\varkappa)^2_{\mp} = 1 - v/(1 - is \pm \sqrt{u}), \tag{11.8}$$

and in the absence of absorption ($s = \nu_{\text{eff}}/\omega = 0$)

$$\left.\begin{aligned}\tilde{n}_1^2 &\equiv n_+^2 = 1 - v/(1 - \sqrt{u}) = 1 - \omega_0^2/\omega(\omega - \omega_H), \\ \tilde{n}_2^2 &\equiv \tilde{n}_-^2 = 1 - v/(1 + \sqrt{u}) = 1 - \omega_0^2/\omega(\omega + \omega_H).\end{aligned}\right\} \tag{11.9}$$

For $\alpha = \frac{1}{2}\pi$ (transverse propagation)

$$\left.\begin{aligned}(n - i\varkappa)^2_1 &= 1 - v(1 - is - v)/[(1 - is)^2 - u - (1 - is)v], \\ (n - i\varkappa)^2_2 &= (n - i\varkappa)^2_0 = 1 - v/(1 - is).\end{aligned}\right\} \tag{11.14}$$

In the limit

$$\left.\begin{aligned}u\cos^2\alpha = (\omega_H^2/\omega^2)\cos^2\alpha \gg 1, \quad v = \omega_0^2/\omega^2 \gg 1, \\ \omega_0 \gg \omega_H, \quad \omega \gg \Omega_H\end{aligned}\right\} \tag{11.23}$$

we have

$$\tilde{n}_2^2 \approx v/\sqrt{u}\cos\alpha, \quad \tilde{n}_1^2 \approx -v/\sqrt{u}\cos\alpha. \tag{11.24}$$

This formula for $\tilde{n}_2$ is used, for example, in studying the propagation of whistlers in the ionosphere.

For all $\alpha \neq 0$ we have $\tilde{n}_2^2 = 0$ at $v_{20} = 1$ and $\tilde{n}_1^2 = 0$ at $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$; if $u < 1$, the index $\tilde{n}_2$ nowhere becomes infinite, but $\tilde{n}_1^2(v_{1\infty}) = \infty$, where $v_{1\infty} = (1 - u)/(1 - u\cos^2\alpha)$. In the range where $u > 1$ and $u\cos^2\alpha < 1$, the functions $\tilde{n}^2_{1,2}(v)$ have no poles; for $u > 1$ and $u\cos^2\alpha > 1$, the function $\tilde{n}_1^2(v)$ has no pole, but $\tilde{n}_2^2(v_{2\infty}) = \infty$, where $v_{2\infty} = (u - 1)/(u\cos^2\alpha - 1)$.

The polarisation of the normal waves 1 and 2 (the extraordinary and ordinary waves) is given by the relation

$$\frac{E_{y1,2}}{E_{x1,2}} = K_{1,2} = -i\frac{2\sqrt{u}(1-v)\cos\alpha}{u\sin^2\alpha \mp \sqrt{[u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha]}} \tag{11.26}$$

or, in the presence of absorption, by formula (11.25); the component $E_{z1,2}$ is given by (10.20).

At the critical collision frequency $\nu_{\mathrm{eff,cr}}$ with

$$s_{\mathrm{cr}} = \nu_{\mathrm{eff,cr}}/\omega = \omega_H \sin^2\alpha/2\omega\cos\alpha = \sqrt{u}\sin^2\alpha/2\cos\alpha, \tag{11.41}$$

the medium appears isotropic, since $(n - i\varkappa)_1 = (n - i\varkappa)_2$ and $K_1 = K_2$.

When the thermal motion of the plasma electrons is taken into account, the dispersion relation for a given ω has three roots, $(n - i\varkappa)^2_{1,2,3}$. The appearance of the root $\tilde{n}_3^2$ is closely related to the plasma wave which is propagated in an isotropic plasma. This topic has been examined in detail in § 12. Here we shall merely mention that the third root $\tilde{n}_3^2$ is not very large, and is real, only near the points $v_{1,2\infty}$, where $\tilde{n}_1^2$ or $\tilde{n}_2^2$ becomes infinite if absorption and thermal motion are neglected. The points $v_{1,2\infty}$ and the corresponding frequency ω_∞ are given by the relations (see also above)

$$\left.\begin{aligned} &1 - u - v_\infty + u\,v_\infty\cos^2\alpha = 0,\\ &v_\infty = \omega_0^2/\omega_\infty^2 = (1-u)/(1-u\cos^2\alpha),\end{aligned}\right\} \tag{12.2}$$

$$\omega_\infty^2 = \frac{1}{2}(\omega_0^2 + \omega_H^2) \pm \sqrt{\left[\frac{1}{4}(\omega_0^2+\omega_H^2)^2 - \omega_0^2\,\omega_H^2\cos^2\alpha\right]}. \tag{12.2a}$$

For large values of $n_3^2 \gg n_1^2$ we can use formula (12.54) to find n_3^2. In the general case where the conditions (12.15) and (12.50) hold, the quantities $\tilde{n}^2_{1,2,3}$ satisfy equation (12.52). This formula and others from § 12 are not repeated here, since the discussion in § 12 was itself merely a summary of results relating to the effect of the thermal motion on the values of $\tilde{n}^2_{1,2,3}$ and on the collisionless damping. The allowance for the thermal motion is important not only near $v_{1,2\infty}$ but also near the frequencies $\omega = \omega_H$, $2\omega_H$, $3\omega_H$, etc., and moreover, for the ordinary wave, in the frequency range $\omega \ll \omega_H$, $\omega \ll \omega_0 = \sqrt{(4\pi e^2 N/m)}$, which is of interest, for example, in the study of whistlers.

In § 13 we have given the fundamental equations of magnetic hydrodynamics, and also the quasihydrodynamic equations for a plasma containing molecules. A discussion has also been given of the relation between the velocity $\boldsymbol{v}_p$ of the ionised component of the plasma, the mean velocity $\boldsymbol{v}$ of the whole plasma, and the electric field $\boldsymbol{E}$; it was assumed that $N_m \gg N$, as in the Earth's ionosphere.

At low frequencies, where $\omega \ll \Omega_H = |e|H^{(0)}/Mc$, the investigation of wave propagation in a plasma must take into account the motion of the ions (§ 14).

If the approximation of magnetic hydrodynamics is applicable, then in the absence of dissipation ($\sigma \to \infty$, and viscosity and thermal conduction negligible) three waves are propagated in each direction, with phase velocities

$$v_{\mathrm{ph},2} = (H^{(0)}\cos\alpha)/\sqrt{(4\pi\varrho_0)}, \tag{14.7}$$

$$v^2_{\mathrm{ph},1,3} = \frac{1}{2}\left(u_0^2 + \frac{[H^{(0)}]^2}{4\pi\varrho_0}\right) \pm \frac{1}{2}\sqrt{\left\{u_0^4 + \frac{[H^{(0)}]^4}{(4\pi\varrho_0)^2} - \frac{2u_0^2[H^{(0)}]^2}{4\pi\varrho_0}\cos 2\alpha\right\}}, \tag{14.10}$$

where α is the angle between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$, ϱ_0 the density of the plasma unperturbed by the wave, and u_0 the adiabatic velocity of sound in the medium considered. In the limiting cases we have

$$\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} \gg 1 : v_{\mathrm{ph},1} \approx u_0, \quad v_{\mathrm{ph},3} \approx H^{(0)}/\sqrt{(4\pi\varrho_0)};$$

$$\zeta = u_0 \div H^{(0)}/\sqrt{(4\pi\varrho_0)} \ll 1 : v_{\mathrm{ph},1} \approx H^{(0)}/\sqrt{(4\pi\varrho_0)}, \quad v_{\mathrm{ph},3} \approx u_0 \cos\alpha.$$

In § 14 the relations are also given between various quantities in hydromagnetic waves and the refractive indices of these waves. The relations involve the electric conductivity σ and the viscosities [see (14.21)–(14.23)]. The expression for $(n - i\varkappa)^2$, with allowance for the motion of the ions, is derived not only in the hydromagnetic range but also for the more general case of wave propagation in a medium where the tensor ε'_{ik} is given by (10.44). We shall not repeat here the corresponding general formula (14.28) for $(n - i\varkappa)^2_{1,2}$. For longitudinal propagation ($\alpha = 0$),

$$(n - i\varkappa)^2_{1,2} = 1 - \omega_0^2/\omega(\omega - i\nu \pm \omega_H - \omega_H \Omega_H/\omega). \tag{14.29}$$

At high frequencies ($\omega \gg \Omega_H$) this gives (11.8). If, however,

$$\omega \ll \Omega_H, \quad \omega\nu \ll \omega_H \Omega_H, \quad \omega_0^2/\omega_H \Omega_H \gg 1, \qquad \text{(14.30) and (14.32)}$$

then (14.29) gives the hydromagnetic formula

$$v_{\mathrm{ph},1,2} = H^{(0)}/\sqrt{(4\pi M N)} = H^{(0)}/\sqrt{(4\pi\varrho_0)}. \tag{14.31}$$

When $\alpha \neq 0$, if the conditions (14.30), (14.32) and

$$\frac{\omega_H \Omega_H \cos^2\alpha}{\omega\sqrt{(\omega^2 + \nu^2)}} \gg 1 \tag{14.34}$$

hold, the general expression (14.28) gives the hydromagnetic formulae for the case $u_0 = 0$ (here the thermal motion is neglected, which is formally equivalent to taking the velocity of sound u_0 to be zero). Thus the hydromagnetic formulae for $v_{\mathrm{ph},1,2}$ obtained as $\sigma \to \infty$ are valid even outside the limits of applicability (13.7) of magnetic hydrodynamics. This is discussed in more detail in § 14, together with the following topics: the special case where $\alpha \to \frac{1}{2}\pi$ and the condition (14.34) does not hold; the propagation of low-frequency waves in a plasma containing molecules [the case $\alpha = 0$, formulae (14.41) to (14.45)]; and the effect of the thermal motion on the propagation of low-frequency (hydromagnetic) waves [see formula (14.47) and after].

CHAPTER IV

WAVE PROPAGATION IN AN INHOMOGENEOUS ISOTROPIC PLASMA

§ 16. INTRODUCTION. THE APPROXIMATION OF GEOMETRICAL OPTICS

The wave equations. A medium of plane layers

In the propagation of electromagnetic waves in an inhomogeneous isotropic medium (in particular, a plasma), the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ must satisfy the wave equations (2.10) and (2.11):

$$\triangle \boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{E} + (\omega^2/c^2)\,\varepsilon'(\omega, \boldsymbol{r})\,\boldsymbol{E} = 0, \tag{16.1}$$

$$\triangle \boldsymbol{H} + \frac{1}{\varepsilon'(\omega, \boldsymbol{r})}\,\mathbf{grad}\,\varepsilon'(\omega, \boldsymbol{r}) \times \mathbf{curl}\,\boldsymbol{H} + (\omega^2/c^2)\,\varepsilon'(\omega, \boldsymbol{r})\,\boldsymbol{H} = 0. \tag{16.2}$$

Of course, only one of these equations need be solved, since if $\boldsymbol{E}$ or $\boldsymbol{H}$ is known the other may be found at once from the field equation $\mathbf{curl}\,\boldsymbol{H} = i\omega\varepsilon'\boldsymbol{E}/c$ or $\mathbf{curl}\,\boldsymbol{E} = -i\omega\boldsymbol{H}/c$. However, it is useful to consider both equations (16.1) and (16.2), since either may prove more convenient for use, depending on the nature of the problem (see § 19).

The propagation of waves in inhomogeneous media involves a very wide range of possibilities, which arise mainly from various choices of the function $\varepsilon'(\omega, \boldsymbol{r})$. It is therefore necessary to state the problem more definitely, for example, by considering a medium which consists of plane or spherical layers. In the former case $\varepsilon' = \varepsilon'(\omega, z)$, and in the latter case $\varepsilon' = \varepsilon'(\omega, R)$, where R is the distance from some centre. In what follows we shall be concerned almost exclusively with the very important case of plane-parallel layers. The propagation of waves in media whose properties are constant on spherical or cylindrical surfaces is in many respects similar to propagation in a plane-parallel medium; see §§ 34 to 36 regarding propagation in a medium of spherical layers.

Wave propagation in a plane-parallel medium may conveniently be first considered for the important particular case of a wave incident normally on a layer of an inhomogeneous medium. Here the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ depend only on the coordinate z, and equation (16.1) for $\boldsymbol{E}$ becomes

$$d^2E/dz^2 + (\omega^2/c^2)\,\varepsilon'(\omega, z)\,E = 0, \tag{16.3}$$

where E is either E_x or E_y. The component E_z is taken to be zero. This necessarily follows (for normal incidence) from equation (16.1) if $\varepsilon'(\omega, z) \neq 0$; if $\varepsilon'(\omega, z) = 0$, plasma waves can exist with $E_z \neq 0$, but for normal incidence, in the linear approximation used, they are entirely independent of the transverse waves which satisfy equation (16.3).

For oblique incidence the problem of wave propagation in a plane-parallel medium reduces to that of normal incidence (see § 19) except for one case (see §§ 19 and 20).

An equation like (16.3) occurs in acoustics and indeed in the theory of propagation of waves of any type. In particular, in quantum mechanics Schrödinger's wave equation for one-dimensional motion is

$$d^2\Psi/dz^2 + (2m/\hbar^2)[W - V(z)]\Psi = 0,$$

where Ψ is the wave function, m the mass of the particle, $\hbar = 1{\cdot}05 \times \times 10^{-27}$ erg sec the quantum constant, W the particle energy and $V(z)$ the potential energy.

Exact solutions for a plane-parallel medium

The mathematical results pertaining to equation (16.3) thus have direct reference to problems of acoustics, quantum mechanics, etc. It is therefore understandable that equations such as (16.3) have been the subject of a large amount of investigation. Since this equation for arbitrary $\varepsilon'(\omega, z)$ has no solution which can be written in terms of known functions, the particular cases where this can be done are of considerable interest. For example, in the very important case of a linear form $\varepsilon' = a + bz$, the solution of equation (16.3) can be expressed in terms of Bessel functions of order $\frac{1}{3}$ or Airy functions [110–113]. For a parabolic form $\varepsilon' = a + bz^2$, the solution can be expressed in terms of parabolic cylinder functions (Weber functions) [111, 114]. If ε' has the form $\varepsilon' = a + e^{\gamma z}[b(e^{\gamma z} + 1) + c]/(e^{\gamma z} + 1)^2$ (where, as in the two previous cases, a, b, c and γ are complex constants), equation (16.3) becomes a hypergeometric equation, whose solution has been studied [115, 116]. The solution for $\varepsilon' = (a + bz)^m$ with integral m can be expressed in terms of Bessel functions [117, 118]; for $m = -2$ the solution is a power function. Solutions are also known [119, 120] for certain other forms of $\varepsilon'(z)$. Some of the above solutions are given and discussed also in [121, 122].

Approximate solutions

These solutions as a whole give an idea of the propagation and reflection of waves in layers of quite different properties. The exact solutions and their utilisation will be discussed in §§ 17 and 18 and in Chapter V.

The problem of approximate solutions of equations like (16.3) is of equal interest. This is due, in the first place, to the fact that approximate solutions can be obtained for an arbitrary function $\varepsilon'(\omega, z)$. Secondly, the methods of approximate solution of the very simple equation (16.3) can usually be applied also to more complex cases (e.g. an inhomogeneous magnetoactive plasma), where the exact solutions are either entirely unknown or of little practical value.

The most important approximate method of solving wave equations, and in particular equation (16.3), is based on the use of the approximation of geometrical optics. This method, and a kindred method in quantum mechanics, are often called the quasiclassical method or the Wentzel–Kramers–Brillouin (WKB) method. The method of geometrical optics gives a good approximation to an exact solution of the equation if the properties of the medium change sufficiently slowly with distance. For a plasma, e.g. the ionosphere or the solar corona, the corresponding condition is in general easily satisfied.

The approximation of geometrical optics

The starting point in deriving the approximation of geometrical optics is the solution of equation (16.3) for a homogeneous medium, i.e. for $\varepsilon' = \varepsilon'(\omega)$. This solution is

$$E = E_0 e^{\pm i\omega\sqrt{\varepsilon'}z/c} = E_0 e^{\pm \omega\varkappa z/c} e^{\pm i\omega n z/c}. \tag{16.4}$$

If the medium is inhomogeneous but its properties vary only slightly over the length of the electromagnetic wave, then the propagation of waves in a small region is very similar to propagation in a homogeneous medium whose indices of refraction and absorption are those of the region considered in the inhomogeneous medium. In other words, propagation in the inhomogeneous medium must be approximately the same as in a homogeneous medium with a variable permittivity. This means that the solution must have the form (16.4), but with $\sqrt{\varepsilon'}z = (n - i\varkappa)z$ replaced by $\int\sqrt{\varepsilon'}\,dz = \int(n - i\varkappa)\,dz$.

The above condition that the properties of the medium vary slowly is equivalent to

$$\lambda\, d\varepsilon/dz \ll \varepsilon; \tag{16.5}$$

here $\lambda = \lambda_0/n = \lambda_0/\sqrt{\varepsilon}$ is the wavelength in the medium ($\lambda_0 = 2\pi c/\omega$ being the wavelength in vacuum), and for simplicity, absorption is assumed absent.

These arguments lead to the expression $E = E_0 e^{\pm i\omega\int\sqrt{\varepsilon'}\,dz/c}$, with E_0 = constant, but with the condition (16.5) we can easily obtain a still better approximation to the exact solution of the equation (16.3). Two methods can be used for this purpose. The first, which is the less rigorous, consists in seeking a solution of equation (16.3) in the form

$$E(z) = E_0(z)\, e^{-i\omega\Psi(z)/c}, \tag{16.6}$$

where $E_0(z)$ and $\Psi(z)$ are slowly varying functions of z to be determined. Substituting (16.6) in (16.3), we have

$$E_0'' - 2i\omega\Psi' E_0'/c - i\omega\Psi'' E_0/c + \omega^2(\varepsilon' - \Psi'^2)E_0/c^2 = 0, \tag{16.7}$$

where the primes to E_0 and Ψ denote differentiation with respect to z; the prime to ε has another meaning, but this should not give rise to misunderstanding.

The slow variation of the field signifies, roughly speaking, that $E_0(z)$ and $\Psi'(z)$ vary appreciably only over some characteristic distance $L \gg \lambda_0$. Hence in (16.7) $E_0' \sim E_0/L$, $E_0'' \sim E_0/L^2$, $\Psi'' \sim \Psi'/L$ and therefore, if we multiply throughout by $c^2/\omega^2 = \lambda_0^2/4\pi^2$, we see that the first term is of the order of $c^2E_0''/\omega^2 \sim \lambda_0^2 E_0/4\pi^2L^2$, the second and third terms together are $-2ic\Psi' E_0'/\omega - ic\Psi'' E_0/\omega$ and are of the order of $\lambda_0\Psi' E_0/2\pi L$, and the last term is $(\varepsilon' - \Psi'^2)E_0$ and is independent of $\lambda_0/2\pi L$. The approximate solution of the equation can be found by equating to zero the terms of each order separately, i.e. the terms in ω^2/c^2 and in ω/c in (16.7). This gives

$$(\varepsilon' - \Psi'^2)E_0 = 0, \quad E_0' + \Psi'' E_0/2\Psi' = 0. \tag{16.8}$$

The first of these gives

$$\varepsilon' = \Psi'^2, \; \Psi = \pm \int_{z_0}^{z} \sqrt{\varepsilon'(z)}\, dz = \pm \int_{z_0}^{z} [n(z) - i\varkappa(z)]\, dz, \tag{16.9}$$

where z_0 is some constant.

The expression (16.9) gives the phase of the field in (16.6) in exact agreement with the above discussion of the necessity of replacing, in the solution (16.4), the phase $\pm i\omega\sqrt{\varepsilon'}z/c$ by the integral $\pm i\omega\int\sqrt{\varepsilon'(z)}\,dz/c$.

The second equation (16.8) has the solution

$$E_0(z) = C/\sqrt{\Psi'} = C/[\varepsilon'(z)]^{\frac{1}{4}} = C/\sqrt{[n(z) - i\varkappa(z)]}, \tag{16.10}$$

where C is a constant.

Thus the solution of equation (16.3) in the approximation of geometrical optics is

$$\begin{aligned} E(z) &= \frac{C}{[\varepsilon'(z)]^{\frac{1}{4}}} \exp\left(\pm i\frac{\omega}{c}\int_{z_0}^{z}\sqrt{\varepsilon'}\,dz\right) \\ &= \frac{C}{\sqrt{[n(z) - i\varkappa(z)]}} \exp\left(\pm\frac{\omega}{c}\int_{z_0}^{z}\varkappa(z)\,dz \pm i\frac{\omega}{c}\int_{z_0}^{z} n(z)\,dz\right). \end{aligned} \tag{16.11}$$

In the approximation of geometrical optics the waves propagated in the two directions [corresponding to the $\pm$ signs in (16.11)] are evidently quite independent, as in a homogeneous medium. We can therefore say that the general solution in the approximation of geometrical optics is

$$E(z) = \frac{C_+}{[\varepsilon'(z)]^{\frac{1}{4}}} \exp\left(i\frac{\omega}{c}\int\limits_{z_{0+}}^{z} \sqrt{\varepsilon'}\,dz\right) + \frac{C_-}{[\varepsilon'(z)]^{\frac{1}{4}}} \exp\left(-i\frac{\omega}{c}\int\limits_{z_{0-}}^{z} \sqrt{\varepsilon'}\,dz\right) \quad (16.12)$$

and depends on two arbitrary complex constants C_+ and C_-, since any change in the constants z_{0+} and z_{0-} amounts to a corresponding change in C_+ and C_-.

If the equations (16.8) are satisfied, all the terms in (16.7) except E_0'' are zero. The solution (16.11)–(16.12) can thus be an approximation to the true solution only if the term E_0'' is found to be less than the others when (16.11) or (16.12) is substituted in (16.7). Thus for the validity of the approximation we must require that the term E_0'' should be much less than each of the terms in (16.7) which contain the factor ω/c, i.e. that the inequality

$$|E_0''| \ll (\omega/c)\,|\Psi'' E_0| \sim (\omega/c)\,|\Psi' E_0'| \quad (16.13)$$

should hold.

Substituting (16.9) and (16.10) we obtain the condition for the approximation of geometrical optics to be valid:

$$\left|-\frac{\Psi'''}{2\Psi'} + \frac{3\Psi''^2}{4\Psi'^2}\right| \ll \frac{2\pi\,|\Psi''|}{\lambda_0}.$$

This inequality will certainly hold if (with $\Psi' = n - i\varkappa$)

$$\frac{\lambda_0\sqrt{(n'^2+\varkappa'^2)}}{2\pi(n^2+\varkappa^2)} \ll 1 \quad \text{and} \quad \frac{\lambda_0\sqrt{(n''^2+\varkappa''^2)}}{2\pi\sqrt{(n^2+\varkappa^2)}\sqrt{(n'^2+\varkappa'^2)}} \ll 1, \quad (16.14)$$

or, in the absence of absorption, if

$$(\lambda_0/2\pi)\,|n'|/n^2 \ll 1, \quad (\lambda_0/2\pi)\,|n''|/|n\,n'| \ll 1 \quad (16.15)$$

(where $n' = dn/dz$, etc.)

It is easy to see that, if the inequalities (16.14) or (16.15) hold, we also have $|E_0''| \ll (\omega^2/c^2)\Psi'^2 E_0 = (\omega^2/c^2)\varepsilon' E_0$, i.e. the term E_0'' is in fact much less than all the other terms in equation (16.7).

It may be noted that the first of (16.15) is essentially the same as (16.5), since $\varepsilon = n^2$ and $\lambda = \lambda_0/n$. In (16.14) and (16.15) we have not λ_0 but $\lambda\!\!\!^{-}_0 = \lambda_0/2\pi$; it is a feature common to the whole theory of electromagnetic waves that the true parameter is not λ_0 but $\lambda\!\!\!^{-}_0 = \lambda_0/2\pi = c/\omega$. The difference is usually unimportant, on account of the fact that the theory involves several exponential expressions.

The condition (16.5) or the first inequality (16.15) may also be written (with $\lambda = \lambda_0/\sqrt{\varepsilon}$)

$$d\lambda/dz \ll 2\pi, \quad \text{i.e.} \quad d\lambda\!\!\!^{-}/dz \ll 1. \quad (16.16)$$

The above method of deriving the approximation of geometrical optics and the conditions for its applicability cannot be regarded as entirely satisfactory, since it is based on estimating and neglecting various terms in the original equation (16.3). A small term in the equation may nevertheless considerably affect the solution, and the only completely legitimate procedure is to neglect small terms in the solution but not in the equation.

A more rigorous treatment of the same problem

We shall therefore find an approximate solution by another and considerably more correct method. To do this, we represent the solution of equation (16.3) as

$$E(z) = [E_{(0)} + (c/\omega)\,E_{(1)} + (c^2/\omega^2)\,E_{(2)} + \cdots]\,e^{-i\omega\Psi(z)/c}, \tag{16.17}$$

where Ψ, $E_{(0)}$, $E_{(1)}$, etc., are unknown functions of z.

Substituting (16.17) in (16.3), we obtain an equation of the type

$$A\omega^2/c^2 + B\omega/c + C + Dc/\omega + Fc^2/\omega^2 + \cdots = 0,$$

where $A, B, C, \ldots$, are some expressions involving $E_{(i)}$, Ψ and their derivatives. Since this equation must hold for all ω/c, it can be satisfied only if $A = B = C = D = F = \cdots = 0$. Thus we have after some simple calculations

$$\left.\begin{aligned} (\varepsilon' - \Psi'^2)\,E_{(0)} &= 0, \\ E'_{(0)} + \Psi''\,E_{(0)}/2\Psi' &= 0, \\ E'_{(1)} + \Psi''\,E_{(1)}/2\Psi' &= E''_{(0)}/2i\,\Psi', \text{ etc.} \end{aligned}\right\} \tag{16.18}$$

The solutions of the first two equations are respectively (16.9) and (16.10). The solution of the equation for $E_{(1)}$ is (with $\Psi' = \sqrt{\varepsilon'}$)

$$E_{(1)} = \frac{1}{\sqrt{\Psi'}} \int\limits_{z_0}^{z} \frac{E''_{(0)}}{2i\sqrt{\Psi'}}\,dz = \frac{1}{[\varepsilon'(z)]^{\frac{1}{4}}} \int\limits_{z_0}^{z} \frac{E''_{(0)}}{2i[\varepsilon'(z)]^{\frac{1}{4}}}\,dz, \tag{16.19}$$

where we have taken only the forced term in the solution, since the solution of the homogeneous equation for $E_{(1)}$ has the same form as that of the equation for $E_{(0)}$.

The approximation of geometrical optics† is seen from the preceding discussion to be valid if only the first term need be taken in (16.17), i.e. if

$$\lambda_0\,|E_{(1)}|/2\pi \ll |E_{(0)}|, \tag{16.20}$$

where it is assumed that the higher terms are less than $E_{(1)}$.

The condition (16.20) is shown by (16.19) to be an integral condition; we shall derive only some other inequalities which are merely sufficient to ensure the validity of (16.20).

† More precisely, we are speaking of the *first* approximation of geometrical optics. However, since the higher approximations are not generally used (if λ_0/L is close to unity, the series (16.17) converges only slowly), the condition for geometrical optics to be valid may be equated to that for this first approximation to be valid.

Putting for simplicity $\varepsilon' = \varepsilon = n^2 > 0$ and using the fact that in this case $E_{(0)} = c/\sqrt{\Psi'} = c/\sqrt{n}$, we have from (16.19)

$$\begin{aligned}\frac{\lambda_0}{2\pi}\frac{E_{(1)}}{E_{(0)}} &= \frac{\lambda_0}{2\pi}\int\limits_{z_0}^{z}\left[\frac{3}{8i}\frac{(d\,n/d\,z)^2}{n^3} - \frac{1}{4i}\frac{d^2\,n/d\,z^2}{n^2}\right]d\,z \\ &= -\left[\frac{\lambda_0}{2\pi}\frac{d\,n/d\,z}{4i\,n^2}\right]_{z_0}^{z} - \frac{\lambda_0}{16i\,\pi}\int\limits_{z_0}^{z}\frac{(d\,n/d\,z)^2}{n^3}\,d\,z \\ &= -\left[\frac{\lambda_0}{2\pi}\frac{d\,n/d\,z}{4i\,n^2}\right]_{z_0}^{z} - \frac{\lambda_0}{16i\,\pi}\int\limits_{z_0}^{z}\frac{d\,n/d\,z}{n^2}\,d\ln n.\end{aligned} \tag{16.21}$$

Let us now assume that the function $n(z)$ is monotonic in the range (z_0, z). Then the two factors in the last integral in (16.21) are either both positive or both negative, so that

$$\int\limits_{z_0}^{z}\frac{(dn/dz)^2}{n^3}\,dz = \int\limits_{z_0}^{z}\frac{1}{n^2}\frac{dn}{dz}\,d\ln n \leqq \left(\frac{1}{n^2}\frac{dn}{dz}\right)_{\max}\ln\frac{n(z)}{n(z_0)},$$

where the suffix max signifies the maximum value in the range (z_0, z); it is assumed here that the integral is positive (i.e. $z > z_0$). If it is negative, the minus sign must be inserted at the appropriate point in the estimate. Thus we see that the inequality $\lambda_0|E_{(1)}|/2\pi|E_{(0)}| \ll 1$ is satisfied if throughout the range we have

$$\frac{\lambda_0}{2\pi}\frac{|d\,n/d\,z|}{|n^2|} \ll 1 \tag{16.22}$$

and the value of $|\ln[n(z)/n(z_0)]|$ or simply $|\ln n|_{\max}$ does not become very large. The latter requirement is usually quite unimportant, and so the sufficient condition for the approximation of geometrical optics to be valid has been written as (16.22) and not as

$$\frac{\lambda_0}{2\pi}\left|\frac{d\,n/d\,z}{n^2}\ln n\right| \ll 1.$$

The condition (16.22) is the same as the first inequality (16.15). The second inequality (16.15), which restricts the values of the second derivative $d^2n/dz^2 \equiv n''$, is thus unnecessary for the validity of the approximation of geometrical optics. However, we shall see at the end of this section that, if the correction to the geometrical-optics solution is to be not only small but very small, it is necessary that dn/dz should not be discontinuous (i.e. that d^2n/dz^2 should not become large).

It should once more be emphasised that the condition (16.22) is sufficient but not necessary, since the original condition (16.20) is an integral condition [see (16.19)]. A more important point is that in deriving the sufficient condition (16.22) the function $n(z)$ has been assumed monotonic. For investigations of wave propagation in plasmas such an assumption is usually quite permissible; we shall not discuss here the problem of the reflection of waves from random inhomogeneities. If the function $n(z)$ is not monotonic (for example, if it has the form $n(z) = 1 + a\cos\Omega z$), then the integral condition (16.20) does not in general reduce to a differential condition of the type (16.22), and the ratio $E_{(1)}/E_{(0)}$ may increase with the path traversed by the wave, i.e. with the difference $z - z_0$.†

It may also be noted that the above discussion does not make a careful use of the moduli of $E_{(0)}$, $E_{(1)}$, etc., in the inequalities. Strictly speaking, when moduli are used, inequalities should also be written involving the phases of the respective quantities. We have omitted to do this because in plasma conditions the absorption is usually a secondary effect as regards the nature of wave propagation (the absorption is weak). Consequently we can, as in (16.21), simply take the case where absorption is absent, when $E_{(0)}$ and Ψ' are real for $\varepsilon > 0$. In the inequalities we need therefore only consider whether the expressions concerned may be of unlike sign. For this reason the absolute values are used in (16.15) and (16.22); otherwise the left-hand sides may be negative and the inequalities are no longer meaningful.

Cases where the approximation of geometrical optics is inapplicable. Total internal reflection

The condition (16.22) for the approximation of geometrical optics to be valid is violated in two cases: if the gradient of n is sufficiently steep (i.e. if the derivative dn/dz is large) and if the refractive index n is sufficiently small.††

In a rarefied plasma, steep gradients can occur only sporadically; when the variation of $\varepsilon(z)$ is smooth and regular, the properties of the medium vary negligibly over distances of the order of wavelengths in the radio range. For example, in the E layer of the ionosphere the refractive index $n(z)$ usually varies, for the waves reflected from this layer, by an amount of the order of unity over a path ~ 10 km; thus $dn/dz \sim 10^{-6}$. In the F layer, $dn/dz \sim 10^{-7}$. For $\lambda_0 = 6\times 10^4$ cm $= 600$ m, the condition (16.22) for the E layer therefore becomes $n^2 \gg 10^{-2}$. For the F layer at $\lambda_0 = 6\times 10^3$ cm we have $n^2 \gg 10^{-4}$.

† For example, if $n = 1 + a\cos\Omega z$, where $|a| \ll 1$, then $E_{(1)}/E_{(0)} \sim a^2\Omega^2(z - z_0)$ if the oscillatory term is omitted. A similar result is obtained in the presence of statistical inhomogeneities (see [269]). From (16.21) we can at once obtain a sufficient condition for the ratio $\lambda_0|E_{(1)}|/2\pi|E_{(0)}|$ to be small, in the form (16.22) together with the condition $|n^{-3}(dn/dz)^2|_{\max}\lambda_0(z - z_0)/16\pi \ll 1$, where $z - z_0$ is the path traversed by the wave.

†† Since we are discussing a single condition (16.22), the selection of these two cases is, of course, arbitrary. It is, however, convenient and essentially legitimate.

Thus in these cases geometrical optics is valid except when n is small. In the solar corona the mean gradients of n are considerably smaller than in the Earth's ionosphere. Thus, at radio frequencies in plasmas, geometrical optics fails in general through n being small rather than dn/dz being large.

In the absence of absorption we have in a plasma [see (3.5)]

$$\varepsilon = n^2 = 1 - 4\pi e^2 N(z)/m\omega^2 = 1 - 3{\cdot}18\times10^9\, N(z)/\omega^2, \qquad (16.23)$$

and when $N(z)$ is sufficiently large or the frequency ω is sufficiently small the squared refractive index n^2 is zero. In all other media the region $\varepsilon \approx 0$ is much less important, since the density $N(z)$ in practice varies over a much narrower range than in a gaseous plasma. Moreover, except in a plasma we cannot usually neglect absorption, so that values of $\varepsilon \leqq 0$ are not so interesting as they would be in the absence of absorption.

The inapplicability of geometrical optics for small n has an evident physical significance. For this approximation to be valid it is necessary that the properties of the medium should vary only slightly over the wavelength λ in the medium; but $\lambda = \lambda_0/n$, and as $n \to 0$ the wavelength $\lambda \to \infty$, so that the change in n with distance becomes significant even if $n(z)$ varies smoothly.

It has been mentioned above that in the approximation of geometrical optics the waves travelling upwards and downwards are entirely independent. Thus reflection of waves can occur only in regions where geometrical optics is not valid. In the absence of absorption we have, in agreement with this, that in the approximation of geometrical optics the energy flux $\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{H}/4\pi$ in the travelling wave, averaged over time, is independent of z. For a wave propagated along the z-axis (16.11) gives

$$\left.\begin{aligned}
\boldsymbol{E} &= \frac{\boldsymbol{E}_0}{\sqrt{n(z)}} \exp\left(i\,\omega\,t - \frac{i\,\omega}{c}\int_{z_0}^{z} n(z)\,dz\right), \quad E_z = 0,\\
\boldsymbol{H} &= (i\,c/\omega)\,\mathbf{curl}\,\boldsymbol{E},\\
H_x &= -\frac{i\,c}{\omega}\,\frac{dE_y}{dz} = \left(-\sqrt{n} + \frac{i\,c}{\omega}\,\frac{dn/dz}{n^{3/2}}\right) E_{0y} \exp\left(i\,\omega\,t - \frac{i\,\omega}{c}\int_{z_0}^{z} n\,dz\right),\\
H_y &= \frac{i\,c}{\omega}\,\frac{dE_x}{dz} = \left(\sqrt{n} - \frac{i\,c}{\omega}\,\frac{dn/dz}{n^{3/2}}\right) E_{0x} \exp\left(i\,\omega\,t - \frac{i\,\omega}{c}\int_{z_0}^{z} n\,dz\right).
\end{aligned}\right\} \qquad (16.24)$$

From the condition (16.22) for geometrical optics to be valid it is clear that in this approximation the terms in dn/dz are negligible. Thus the approximation of geometrical optics gives

$$S = \frac{c}{4\pi} E_0^2 \cos^2\left(\omega t - \int_0^z n\, dz\right), \quad \bar{S} = c E_0^2/8\pi = \text{constant}.$$

Even if the terms in dn/dz are retained, the time average of the energy flux $\bar{S}$ is still $cE_0^2/8\pi =$ constant, since those small terms in $\boldsymbol{H}$ are $\frac{1}{2}\pi$ out of phase with the principal term.

Thus the reflection of waves must become stronger as the approximation of geometrical optics becomes less accurate. In particular, reflection occurs in the presence of steep gradients of n, and the coefficient of reflection is fairly large only if the distance of transition from one value of n to another is of the order of $\lambda_0/2\pi$ or less. In the limit as the width of the transition region tends to zero (a boundary between two media) the familiar Fresnel's formulae are obtained (§ 18).

Reflection also occurs from a region where $n = 0$. In the absence of absorption, this reflection is complete provided that the electron density continues to increase over a distance much greater than $\lambda_0/2\pi$ beyond the point $\varepsilon = n^2 = 0$ towards negative ε. Physically, the presence of total reflection is evidently a consequence of the fact that the wave is damped in the region where $\varepsilon < 0$. In that region $n = 0$, $\varkappa = \sqrt{|\varepsilon|}$, $E = E_0 \exp\left(-\omega \int \sqrt{|\varepsilon|}\, dz/c\right)$, and at a distance of a few wavelengths beyond the point where $\varepsilon = 0$ the field is almost zero (if the derivative $d\varepsilon/dz$ is not very small). Since, moreover, energy absorption is assumed absent ($\sigma = 0$), it is clear that all the energy must be reflected, and a standing wave is formed.

The reflection of radio waves from the ionosphere

The above arguments explain the fundamental fact that radio waves may undergo complete reflection from the ionosphere and similar layers of an inhomogeneous plasma even with normal incidence. This reflection is analogous to total internal reflection in optics. For, according to the law of refraction, the angles of incidence and refraction of a wave incident on a boundary surface are related by $\sin\theta_1/\sin\theta_2 = n_2/n_1$, where θ_1, θ_2 are the angles of incidence and refraction, and n_1, n_2 are the indices of refraction of media 1 and 2 (the angle θ_1 being that in medium 1). If $n_1 > n_2$, then for angles $\theta_1 \geqq \sin^{-1}(n_2/n_1)$ we have total internal reflection (since then $\sin\theta_2$ is formally greater than unity). For normal incidence ($\theta_1 = 0$) total internal reflection can occur only when $n_2 = 0$, and this is possible in a plasma (in this case the absence of a sharp boundary is not important).

For a medium with smoothly varying properties we cannot speak of a point of reflection in the literal sense, since reflection occurs over a certain region, but we can somewhat arbitrarily take the point of reflection to be that where

$\varepsilon = n^2 = 0$, since beyond this point the field decreases rapidly and can usually be neglected (see also Chapter VI).

According to (16.23) we have at the point of reflection

$$N = m\omega^2/4\pi e^2 = \omega^2/3{\cdot}18\times 10^9 = 1{\cdot}24\times 10^{-8} f^2. \tag{16.25}$$

This is one of the fundamental relations used in the interpretation of ionosphere data, in radio studies of the Sun's atmosphere, and elsewhere.

The problem of the reflection of radio waves from the ionosphere is discussed in more detail in Chapter VI, which is closely related to the present chapter and contains certain information pertaining not only to the specific problem of the reflection of waves from the ionosphere but also to the general propagation of waves in an inhomogeneous medium. Problems of radio astronomy are analysed in Chapter VII.

A completely non-reflecting layer

The absence of steep gradients of n and of a region where $n = 0$ ensures that there is no strong reflection of waves. Even in these conditions, however, there is in general some weak reflection of waves in an inhomogeneous medium; if geometrical optics is valid, then this reflection is only slight. We can say that the smoother the function $\varepsilon(z)$, the less reflection occurs. Reflection is completely absent only in a homogeneous medium, and in an inhomogeneous medium where $\varepsilon(z)$ has a particular form.

For example, reflection is completely absent for media where solutions of the type (16.11), i.e.

$$E = [E_0/\sqrt{f(z)}]\exp\left(\pm i\,\omega \int_{z_0}^{z} f(z)\,d z/c\right), \tag{16.26}$$

are exact solutions of the wave equation [123]. For it has been shown above [see (16.24)] that, for such solutions with $f(z)$ real, the mean energy flux is independent of z, and so reflection is absent. On the other hand, it is not difficult to find a function $\varepsilon'(z)$ in the wave equation (16.3) such that the solution (16.26) is exact. By direct substitution it is easily seen that the function (16.26) satisfies the equation

$$d^2E/dz^2 + (\omega^2/c^2)\,\varepsilon'(z)E = 0$$

with

$$\varepsilon'(z) = f^2(z) + \frac{c^2}{2\omega^2}\left[\frac{d^2 f/dz^2}{f} - \frac{3(df/dz)^2}{2f^2}\right]. \tag{16.27}$$

It is curious that the requirement of real f, which is necessary if absorption is to be absent, does not ensure that $\varepsilon(z)$ is positive [if f is real, $\varepsilon'(z)$ is of course real and equal to $\varepsilon(z)$]. Thus, even if there is a region where $\varepsilon(z) < 0$, the wave may not be reflected, unlike what happens in the approximation of geometrical optics.

Weak reflection

In some cases it is of interest to consider the weak reflection of waves which occurs under conditions where the first approximation of geometrical optics is a good approximation to the exact solution. To solve this problem, we put the solution of the wave equation (16.3) in the absence of absorption in the form

$$E(z) = \frac{E_0}{\sqrt{n(z)}} \exp\left(-\frac{i\omega}{c}\int_{z_0}^{z} n(z)\,dz\right) + \\ + E_1(z)\exp\left(+\frac{i\omega}{c}\int_{z_0}^{z} n(z)\,dz\right), \quad |E_1| \ll |E_0/\sqrt{n}|. \tag{16.28}$$

The last inequality evidently ensures that the solution of geometrical optics (taken as a wave propagated along the z-axis) is valid as a first approximation.

The function $E_1(z)$ and the amplitude coefficient of reflection R may be found by means of perturbation theory. Substituting (16.28) in (16.3) with $\varepsilon' = \varepsilon(z) = n^2(z)$, we obtain the equation

$$\frac{dE_1}{dz} + \frac{dn/dz}{2n} E_1 = -\frac{ic}{\omega n}\frac{d}{dz}\left(\frac{dn/dz}{4n^{3/2}}\right) E_0 \exp\left(-2\frac{i\omega}{c}\int_{z_0}^{z} n\,dz\right) \equiv \chi(z), \tag{16.29}$$

where the approximation consists in neglecting the term d^2E_1/dz^2. Integration of equation (16.29) gives

$$E_1(z) = \frac{1}{\sqrt{n(z)}}\int_{z_1}^{z} \sqrt{n(z)}\,\chi\,dz$$

$$= -\frac{icE_0}{4\omega\sqrt{n}}\int_{z_1}^{z} \frac{1}{\sqrt{n}}\frac{d}{dz}\left(\frac{dn/dz}{n^{3/2}}\right)\exp\left(-2\frac{i\omega}{c}\int_{z_0}^{z} n\,dz\right)dz$$

$$= -\frac{ic}{4\omega\sqrt{n(z)}}\left[\frac{dn/dz}{n^2}\exp\left(-2\frac{i\omega}{c}\int_{z_0}^{z} n\,dz\right)\right]_{z_1}^{z} +$$

$$+ \frac{E_0}{\sqrt{n}}\int_{z_1}^{z}\frac{dn/dz}{2n}\exp\left(-2\frac{i\omega}{c}\int_{z_0}^{z} n\,dz\right)dz -$$

$$-\frac{icE_0}{8\omega\sqrt{n}}\int_{z_1}^{z}\frac{(dn/dz)^2}{n^3}\exp\left(-2\frac{i\omega}{c}\int_{z_0}^{z} n\,dz\right)dz, \tag{16.30}$$

where z_0 and z_1 are arbitrary constants. We shall now particularise the problem a little and suppose that beyond the layer ($z \to \infty$) the medium is homogeneous and only the transmitted wave is present, and further that we are interested only in the field for values of z in front of the layer, where the medium is likewise homogeneous (n = constant). These boundary conditions hold good when a wave is incident on the layer from the direction of negative z and no wave is incident from the opposite side. Then $E_1(z \to \infty) = 0$ and in (16.30) we must take $z_1 = \infty$, $dn/dz = 0$ for $z = z_1$. In the region in front of the inhomogeneous layer, $dn/dz = 0$ also. Thus the first term in the last member of (16.30) is zero; the third term is considerably less than the second by (16.22), and

$$\left.\begin{aligned} E_1(z) &\approx -\frac{E_0}{\sqrt{n}}\int\limits_z^\infty \frac{dn/dz}{2n}\exp\left(-2\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right)dz,\\ R &\approx \frac{E_1\exp\left(\dfrac{i\omega}{c}\displaystyle\int\limits_{z_0}^z n\,dz\right)}{(E_0/\sqrt{n})\exp\left(-\dfrac{i\omega}{c}\displaystyle\int\limits_{z_0}^z n\,dz\right)}\\ &= -\exp\left(2\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right)\int\limits_z^\infty \frac{dn/dz}{2n}\exp\left(-2\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right)dz. \end{aligned}\right\} \tag{16.31}$$

This expression is the same as that given in [121, § 17], where it was obtained by another and more laborious method, which, however, is valid for strong reflection also.

It should be mentioned that the expression (16.31) is not always convenient; sometimes a more general formula for R, derived from (16.30), is suitable. It is

$$R = \frac{E_1\exp\left(\dfrac{i\omega}{c}\displaystyle\int\limits_{z_0}^z n\,dz\right)}{(E_0/\sqrt{n})\exp\left(-\dfrac{i\omega}{c}\displaystyle\int\limits_{z_0}^z n\,dz\right)} = \frac{-i\lambda_0\exp\left(2\dfrac{i\omega}{c}\displaystyle\int\limits_{z_0}^z n\,dz\right)}{8\pi}\int\limits_{z_1}^z \frac{1}{\sqrt{n}}\frac{d}{dz}\left(\frac{dn/dz}{n^{3/2}}\right)\exp\left(-2\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right)dz. \tag{16.32}$$

Reflection from a discontinuity of the derivative dn/dz

As an interesting and important example, let us apply this formula to the reflection of waves from a discontinuity of the derivative dn/dz. A discontinuity in the value of dn/dz is seen from physical considerations to be equivalent to a large variation of dn/dz over a distance small in comparison with the distance $c/\omega = \lambda_0/2\pi$. In such a region we can regard all the quantities in (16.32) except d^2n/dz^2 as constant. Hence, if the discontinuity is at $z = 0$, we have

$$\left.\begin{aligned} R &= -\frac{i\lambda_0}{8\pi n^2(0)}\left[\left(\frac{dn}{dz}\right)_2 - \left(\frac{dn}{dz}\right)_1\right]\exp\left(2\frac{i\omega}{c}\int\limits_0^z n(z)\,dz\right), \\ |R| &= \frac{\lambda_0}{8\pi}\frac{|(dn/dz)_2 - (dn/dz)_1|}{n^2(0)}, \end{aligned}\right\} \tag{16.33}$$

where $(dn/dz)_{1,2}$ are the values of dn/dz on the two sides of the discontinuity and we have put $z_0 = 0$ to fix the phase factor, since the quantity

$$2\frac{i\omega}{c}\int\limits_0^z n(z)\,dz$$

is evidently the variation of the phase in propagation from the point z to the discontinuity at $z = 0$ and back to z.

Formula (16.33) can also be obtained directly in solving the problem of reflection from a discontinuity of dn/dz in conditions where the approximation of geometrical optics is valid on both sides of the discontinuity and the field may be represented in the form

$$\left.\begin{aligned} &E(1) = \frac{E_0}{\sqrt{n}}\exp\left(-\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right) + \frac{c_1}{\sqrt{n}}\exp\left(\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right) \\ &\text{in medium 1 (the incident and reflected waves),} \\ &E(2) = \frac{c_2}{\sqrt{n}}\exp\left(-\frac{i\omega}{c}\int\limits_{z_0}^z n\,dz\right) \end{aligned}\right\} \tag{16.34}$$

in medium 2 (the transmitted wave).

At the discontinuity of dn/dz (at $z = 0$) the tangential components of the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ (or in this case E and dE/dz) must remain continuous.† For

† It is clear from (16.24), for example, that in this problem the boundary condition $H(2) = H(1)$ (i.e. the equality of the tangential components of the field $\boldsymbol{H}$) is equivalent to the continuity of the derivative dE/dz. This result follows also from formal arguments, since for a second-order equation such as (16.3) the function E and its derivative dE/dz must both be continuous at the boundary.

the fields (16.34) these boundary conditions lead directly to the result (16.33) if we omit the small term which is less than the others by a factor $(\lambda_0/2\pi n^2)dn/dz$. This is, of course, entirely legitimate, since the approximation of geometrical optics is assumed valid outside the discontinuity [see (16.22)]. For this reason the reflection coefficient R in (16.31)–(16.33) is always small, i.e. $|R| \ll 1$.

It has already been mentioned, in connection with the derivation of the conditions for the approximation of geometrical optics to be valid, that this approximation is in fact close to the exact solution for a monotonic function $\varepsilon(z)$ if the single condition (16.22) holds, independently of the values of d^2n/dz^2. However, if the derivative d^2n/dz^2 is also everywhere small, so that the second condition (16.15) holds, the approximation of geometrical optics is still better. This is clear from formulae (16.32) and (16.33), which show that the reflection from the medium decreases with d^2n/dz^2.

The reflection from a discontinuity of the derivative dn/dz is similar to that from a discontinuity of the refractive index n itself (here we dissent from the opinion put forward in [121, § 17.5]). In either case the discontinuity may be regarded as sharp, and its structure† does not (within wide limits) affect the coefficient of reflection if the thickness of the discontinuity is small compared with $\lambda_0/2\pi$. In other words, in either case the reflection effect may be regarded as either occurring at a sharp discontinuity or resulting from reflection throughout a transition layer where the rapid change of dn/dz or $n(z)$ occurs. One difference between the two cases is, admittedly, that the discontinuity of n may occur at the boundary between two homogeneous media, where any reflection is from the region of the actual discontinuity only. If there is a discontinuity of dn/dz, at least one of the two media on either side is inhomogeneous, and the reflection is not in general localised at the discontinuity alone. This is, however, of little significance, since the reflection from the inhomogeneous medium other than at the boundary (i.e. outside the transition layer which is regarded as a discontinuity) may be arbitrarily small or even zero [as in a layer of the type (16.27)].

§ 17. EXACT SOLUTIONS OF THE WAVE EQUATION WITH ε' LINEAR, PARABOLIC, OR EQUAL TO $a/(b+z)^2$

Introduction

In investigating wave propagation in an arbitrary inhomogeneous medium we obviously cannot restrict ourselves to the first approximation of geometrical optics. The same applies to the other general methods: the perturbation method described at the end of § 16, which is essentially the second approximation

† By the structure of the discontinuity we mean, of course, the nature of the variation of the function dn/dz or $n(z)$ within the region, which on a less detailed view is regarded as a point on the z-axis at which dn/dz or n is discontinuous.

of geometrical optics, and the method of phase integrals, in which the approximation of geometrical optics is widely used. (See [124] and the literature there quoted. We shall also discuss this method in § 28.) No such general approximate solution (general in the sense that the function $\varepsilon'(z)$ is not taken to have any particular form) can be used to find the reflection coefficient, still less the wave field in the actual region of reflection from an arbitrary layer. For this reason exact solutions of the wave equation (16.3) are of great value, and they are known for various functions $\varepsilon'(z)$ already specified at the beginning of § 16. We shall now discuss the most important of these exact solutions, namely those for a "linear layer" with $\varepsilon' = a + bz$, a "parabolic layer" with $\varepsilon' = a + bz^2$, and a layer with $\varepsilon' = a/(b+z)^2$. The last of these is of interest mainly because the exact solution is expressible in terms of elementary functions, and is therefore particularly simple.

A linear layer without absorption

Let us first consider the solution for a linear layer of plasma without absorption [110, 111], putting in (16.3) for $z \geqq 0$

$$\varepsilon = \tilde{n}^2 = 1 - 4\pi e^2 N(z)/m\omega^2 = 1 - z/z_1, \quad \sigma = 0. \tag{17.1}$$

For $z < 0$ we can suppose that $\varepsilon = 1$, but it is better not to specify the properties of the medium in that region, considering only the region $z \geqq 0$ and imposing at $z = 0$ the boundary condition in the form of the expression for an incident wave. Here it must be borne in mind that, if the derivative dn/dz is discontinuous at $z = 0$, a wave coming from the region $z < 0$ will be reflected there. We avoid this problem by specifying the wave field at $z = 0+$.

With the change of variables

$$\zeta = (\omega^2/c^2 z_1)^{\frac{1}{3}}(z_1 - z) = (\omega z_1/c)^{2/3}\varepsilon(z), \tag{17.2}$$

it is easily seen that equation (16.3) for the layer (17.1) becomes

$$d^2E/d\zeta^2 + \zeta E = 0. \tag{17.3}$$

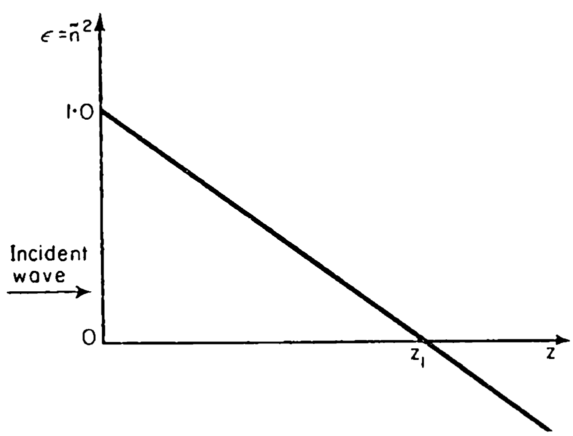

FIG. 17.1. A linear layer.

This can be reduced to Bessel's equation (see, for instance, [125]) and its solution can be expressed in terms of Bessel functions of order $\frac{1}{3}$. We are interested in the solution which corresponds to the case where the wave is incident on the medium in the direction of the positive z-axis (Fig. 17.1). For $z \to \infty$ (i.e. $\zeta \to -\infty$), the wave field must be damped, since at $z = z_1$ ($\zeta = 0$) $\varepsilon = 0$ and for $z > z_1$ ($\zeta < 0$) $\varepsilon < 0$. In the region $z < z_1$ we evidently have a standing wave. The corresponding solution is unique, and is

$$\left.\begin{aligned} E(\zeta) &= A\,\zeta^{\frac{1}{2}}\left[J_{\frac{1}{3}}\left(\frac{2}{3}\zeta^{3/2}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\zeta^{3/2}\right)\right] \quad \text{for} \quad \zeta > 0, \\ E(\zeta) &= A(-\zeta)^{\frac{1}{2}}\left[-I_{\frac{1}{3}}\left(\frac{2}{3}(-\zeta)^{3/2}\right) + \right. \\ &\qquad \left. + I_{-\frac{1}{3}}\left(\frac{2}{3}(-\zeta)^{3/2}\right)\right] \quad \text{for} \quad \zeta < 0, \end{aligned}\right\} \tag{17.4}$$

where A is a constant, J is the Bessel function and $I_\nu(z) = i^{-\nu} J_\nu(iz)$ (see [125]). The necessity for a difference between the explicit forms in which the field E is represented for $\zeta > 0$ and $\zeta < 0$ is due to the presence of a branch point of $\zeta^{3/2}$ at $\zeta = 0$. Of course, the above solutions for E are the same at $\zeta = 0$ (where the field E and its z-derivative are continuous). The solution of equation (17.3) which decreases for $\zeta < 0$, i.e. the solution (17.4), may also be written as an Airy integral (see, e.g., [126]):

$$E(\zeta) = \frac{3A}{\pi}\int_0^\infty \cos\left(\frac{1}{3}x^3 - \zeta x\right) dx. \tag{17.5}$$

The expressions (17.4) and (17.5) are identical (the coefficients having been chosen so as to give the constant A the same value). For $\zeta < 0 \,(z > z_1, \varepsilon < 0)$ the field E is exponentially damped; for $\zeta > 0 \,(z < z_1, \varepsilon > 0)$ the field is oscillatory. The form of the field (17.4)–(17.5) is further discussed in § 32.

For large $\zeta\,(|\zeta| \gg 1)$ we can use the asymptotic representations of the Bessel functions [125] or the Airy integral [126]. The result is

$$E \approx \frac{3A}{\sqrt{\pi}}\,\zeta^{-\frac{1}{4}} \cos\left(\frac{2}{3}\zeta^{3/2} - \frac{1}{4}\pi\right), \quad \zeta \gg 1. \tag{17.6}$$

From (17.2) the condition $\zeta \gg 1$ for the approximation (17.6) to be valid is (with $z_1 = |d\varepsilon/dz|^{-1}$, $\omega/c = 2\pi/\lambda_0$)

$$\zeta = (\omega z_1/c)^{2/3}(1 - z/z_1) = \left(\frac{2\pi}{\lambda_0\,|d\varepsilon/dz|}\right)^{2/3} \varepsilon(z) \gg 1. \tag{17.7}$$

This inequality is the same as the condition (16.22) for geometrical optics to be valid, since $\varepsilon(z) = n^2(z)$. The solution (17.6) is also, of course, a geometrical-optics solution of the type (16.12), since

$$\zeta^{-\frac{1}{4}} = \text{constant}/\sqrt{n(z)},$$

$$\frac{2}{3}\zeta^{3/2} = (2\omega/3c)\, z_1\, \varepsilon^{3/2}(z)$$

$$= \frac{\omega}{c}\int\limits_z^{z_1} \sqrt{\left(1 - \frac{z}{z_1}\right)}\, dz$$

$$= \frac{\omega}{c}\int\limits_z^{z_1} n(z)\, dz. \tag{17.8}$$

We shall suppose that the amplitude E_+ of the incident wave as it enters the layer (at $z = 0$) is equal to unity (i.e. at $z = 0$ the field E_+, including the time factor, is $e^{i\omega t}$). The solution (17.4)–(17.5) is a standing wave, and at $z = 0$ it can be put in the form [omitting the factor $e^{i\omega t}$ and using (17.6)]

$$E = E_+ + E_- = 1 + e^{-i\cdot 4\omega z_1/3c + i\pi/2},$$

$$A = \frac{2}{3}\sqrt{\pi}(\omega z_1/c)^{1/6}\, e^{-i\cdot 2\omega z_1/3c + i\pi/4}. \tag{17.9}$$

The field at any point (with $z \geqq 0$) is determined by the formulae (17.4) and (17.5) with the constant A given by (17.9). The phase shift between the reflected and incident waves is evidently

$$\varphi = 4\omega z_1/3c - \tfrac{1}{2}\pi$$

$$= 2\frac{\omega}{c}\int\limits_0^{z_1} \sqrt{\left(1 - \frac{z}{z_1}\right)}\, dz - \frac{1}{2}\pi$$

$$= 2\frac{\omega}{c}\int\limits_0^{z_1} n\, dz - \frac{1}{2}\pi. \tag{17.10}$$

A linear layer with absorption

The above results can easily be generalised to the case where absorption is present [113]. For a linear layer we have

$$\begin{aligned} \varepsilon' &= \varepsilon - i\cdot 4\pi\sigma/\omega \\ &= (n - i\varkappa)^2 \\ &= 1 - z/z_1 - i(\alpha + \beta z/z_1) \\ &= a - bz/z_1, \end{aligned} \tag{17.11}$$

where α and β are real constants. By the change of variables

$$\begin{aligned}\zeta &= (\omega^2 b/c^2 z_1)^{\frac{1}{3}}(a z_1/b - z)\\ &= (\omega z_1/cb)^{2/3}\varepsilon'(z)\\ &= \xi - i\eta\end{aligned} \tag{17.12}$$

the wave equation (16.3) is again brought to the form (17.3); ζ is, of course, a complex variable, while ξ and η are real. The solution of the equation which satisfies the conditions of the problem ($E \to 0$ for $z \to \infty$) is again of the form (17.4)–(17.5), with the conditions $\xi \gtrless 0$ instead of $\zeta \gtrless 0$. The asymptotic representation of the field for $\xi \gg 1$ is again of the form (17.6), and if $E_+ = 1$, at $z = 0$ we have

$$\left.\begin{aligned}E &= 1 + \exp\left(-i\cdot\frac{4}{3}\zeta^{3/2}_{z=0} + i\cdot\frac{1}{2}\pi\right) = 1 + e^{-i\Psi},\\ A &= \frac{2}{3}\sqrt{\pi}\,\zeta^{\frac{1}{4}}_{z=0}\,e^{-i\Psi/2},\end{aligned}\right\} \tag{17.13}$$

where $\zeta_{z=0}$ is the value of ζ for $z = 0$.

Here, to avoid misunderstanding, it should again be emphasised that the field E_+ of the incident wave is taken to be equal to unity at $z = 0+$, i.e. reflection from the boundary at $z = 0$ is not considered. If the linear layer adjoins at $z = 0$ a homogeneous medium with $\varepsilon' = 1$ (for example), then in the absence of absorption there is a discontinuity of the derivative dn/dz at the boundary $z = 0$, and reflection occurs. This reflection is weak if the gradient $|d\varepsilon/dz| = 1/z_1$ is small. If the layer is absorbing, however, then even for small $|d\varepsilon'/dz|$ the reflection is weak only when the parameter α in (17.11) is small. This is reasonable, since when $\alpha \neq 0$ there is a discontinuity $\omega\alpha/4\pi$ in the conductivity at a boundary with a vacuum. When α is large, therefore, there will be almost total reflection at the boundary (mirror reflection).†

It is not difficult to solve the problem of the reflection of waves from a linear layer with a boundary at $z = 0$, but we shall not pause to do so. This is because, when $\alpha = 0$ and the gradient $d\varepsilon'/dz$ is small, the region $\varepsilon = 0$ lies far from the boundary $z = 0$, and the problem of reflection essentially separates into two, those of reflection from the region $\varepsilon \approx 0$ and from the boundary at $z = 0$. Moreover, by "smoothing" the layer near $z = 0$ we can render the reflection from this boundary entirely negligible.

For a linear layer with $\varepsilon = 1 + z/z_1$ for $z \geq 0$, where ε does not vanish, the waves are not totally reflected from the layer. When the gradient $d\varepsilon/dz = 1/z_1$ is small, this reflection is weak and reduces to reflection from the boundary $z = 0$. (It is assumed that $\varepsilon = 1$ for $z < 0$.) Of course, formula (16.33) may then be

† For a layer where (17.11) holds, only the conductivity σ is discontinuous at a boundary with a vacuum. For a somewhat more general layer, where the permittivity ε is also discontinuous at the boundary, the reflection from the boundary may, of course, be large even if $|d\varepsilon/dz|$ is small.

used. The coefficient of reflection from the layer for any z_1 and oblique incidence is given in [121, § 17.3].

Returning to the problem of reflection from an absorbing layer (17.11), we may point out that by (17.13) the phase shift of the reflected wave is $\varphi = \mathrm{re}\Psi$, and the reflection coefficient is $R = e^{\mathrm{im}\Psi}$, where [see (17.12)]

$$\Psi + \frac{1}{2}\pi = \frac{4}{3}\zeta^{3/2}_{z=0} = 2\int\limits_0^{\zeta_{z=0}} \zeta^{\frac{1}{2}}\, d\zeta$$

$$= 2\left(\frac{\omega z_1}{c b}\right)^{\frac{1}{3}} \int\limits_0^{\zeta_{z=0}} (n - i\varkappa)\, d\zeta = 2\frac{\omega}{c} \int\limits_0^{a z_1/b} (n - i\varkappa)\, dz, \tag{17.14}$$

since $\zeta = 0$ for $z = az_1/b$ and $d\zeta = -(\omega^2 b/c^2 z_1)^{\frac{1}{3}}\, dz$.

In the integral (17.14) the upper limit is

$$az_1/b = (1 - i\alpha)z_1/(1 + i\beta), \tag{17.15}$$

and the variable of integration z is complex. For simplicity we at first put $\beta = 0$. Then (17.11) gives

$$4\pi\sigma/\omega = \alpha = \text{constant}, \tag{17.16}$$

i.e. the conductivity is independent of the coordinates. In this case

$$\Psi + \frac{1}{2}\pi = 2\frac{\omega}{c} \int\limits_0^{(1-i\alpha)z_1} (n - i\varkappa)\, dz$$

$$= 2\frac{\omega}{c}\left\{\int\limits_0^{z_1} (n - i\varkappa)\, dz + \int\limits_{z_1}^{z_1 - i\alpha z_1} \frac{\sqrt{(\alpha + y/z_1)}}{\sqrt{2}} (1 - i)\, dz\right\}$$

$$= 2\frac{\omega}{c}\left\{\int\limits_0^{z_1} n(z)\, dz - \frac{\sqrt{2}}{3}\alpha^{3/2} z_1\right\} -$$

$$- i\cdot 2\frac{\omega}{c}\left\{\int\limits_0^{z_1} \varkappa(z)\, dz + \frac{\sqrt{2}}{3}\alpha^{3/2} z_1\right\}, \tag{17.17}$$

since for $z = z_1 + iy$ (17.16) gives $n = \varkappa = \sqrt{[\frac{1}{2}(\alpha + y/z_1)]}$. From (17.17) and (17.16) we have

$$\varphi = 2\frac{\omega}{c}\left\{\int\limits_0^{z_1} n(z)\, dz - \frac{\sqrt{2}}{3}\left(\frac{4\pi\sigma}{\omega}\right)^{3/2} z_1\right\} - \frac{1}{2}\pi, \tag{17.18}$$

$$-\ln R = 2\frac{\omega}{c}\left\{\int\limits_0^{z_1} \varkappa(z)\, dz + \frac{\sqrt{2}}{3}\left(\frac{4\pi\sigma}{\omega}\right)^{3/2} z_1\right\}. \tag{17.19}$$

The expressions (17.17), (17.18) and (17.19) can, of course, be obtained directly by calculating

$$\varphi = \mathrm{re}\left(\frac{4}{3}\zeta^{3/2}_{z=0}\right) - \frac{1}{2}\pi \quad \text{and} \quad \ln R = \mathrm{im}\left(\frac{4}{3}\zeta^{3/2}_{z=0}\right), \tag{17.20}$$

where

$$\zeta_{z=0} = (\omega^2 b/c^2 z_1)^{\frac{1}{3}}\, a\, z_1/b = [\omega\, z_1/c(1+i\,\beta)]^{2/3}(1-i\,\alpha).$$

If $\alpha = 0$ and $\beta \neq 0$, i.e. $4\pi\sigma/\omega = \beta z/z_1$, then

$$\Psi + \frac{1}{2}\pi = 2\frac{\omega}{c}\int\limits_0^{(1-i\beta)z_1/(1+\beta^2)} (n - i\,\varkappa)\,d\,z.$$

Usually $\beta^2 \ll 1$, and the expressions (17.20) are easily shown to reduce to (17.18) and (17.19) as before (where σ is the conductivity $\omega\beta/4\pi$ for $z = z_1$).

In general, we have to within terms of order α^2, β^2 and $\alpha\beta$

$$-\ln R \approx 2\frac{\omega}{c}\left(\alpha + \frac{2}{3}\beta\right)z_1 = \frac{\omega}{c}\int\limits_0^{z_1}\frac{\alpha + \beta\, z/z_1}{\sqrt{(1 - z/z_1)}}\,d\,z$$

$$= 2\frac{\omega}{c}\int\limits_0^{z_1}\frac{2\pi\,\sigma}{\omega\sqrt{\varepsilon}}\,d\,z. \tag{17.20a}$$

These formulae give the solutions to all problems which may arise concerning waves reflected or propagated in a linear layer. In §§ 30–32 we shall discuss the application of these formulae to the ionosphere.

A parabolic layer without absorption

Let us now consider a parabolic layer without absorption, where for $|z| \leqq z_m$ we have

$$\varepsilon = 1 - 4\pi e^2 N(z)/m\omega^2 = 1 - (f_{\mathrm{cr}}^2/f^2)\,(1 - z^2/z_m^2), \tag{17.21}$$

the coordinate z being measured from the position of maximum density $N = N_{\max}(1 - z^2/z_m^2)$, with $N(0) = N_{\max}$ = the density at the maximum; $f_{\mathrm{cr}} = \omega_{\mathrm{cr}}/2\pi = \sqrt{(e^2 N_{\max}/\pi m)}$ is the critical frequency, $\lambda_{\mathrm{cr}} = c/f_{\mathrm{cr}}$, and z_m is the half-thickness of the layer (Fig. 17.2). We shall not consider reflections from $z = \pm z_m$ (see [127]); they do not occur if the transition to the region where $\varepsilon = 1$ is smoothed so that the derivative $d\varepsilon/dz$ becomes continuous.

In the wave equation (16.3) with ε given by (17.21) we put

$$\left.\begin{aligned} \varrho &= \pi\, z_m(f_{\mathrm{cr}}^2 - f^2)/\lambda_{\mathrm{cr}}\, f_{\mathrm{cr}}^2, \\ v &= (4\pi\, z_m/\lambda_{\mathrm{cr}})^{\frac{1}{2}}\,(z/z_m)\, e^{i\pi/4} = u\, e^{i\pi/4}, \end{aligned}\right\} \tag{17.22}$$

obtaining

$$d^2E/dv^2 + (i\varrho - \tfrac{1}{4}v^2)E = 0. \tag{17.23}$$

This equation is satisfied by the parabolic cylinder functions $D_{i\varrho-\frac{1}{2}}(ue^{i\pi/4})$ and $D_{-i\varrho-\frac{1}{2}}(ue^{-i\pi/4})$, also called Weber functions (see [128, chapter XVI]).

It will be shown in § 38 that the exact solution for a parabolic layer is of very limited interest as regards the theory of the propagation of radio waves in the ionosphere. It is not even necessary to use the exact solution directly,

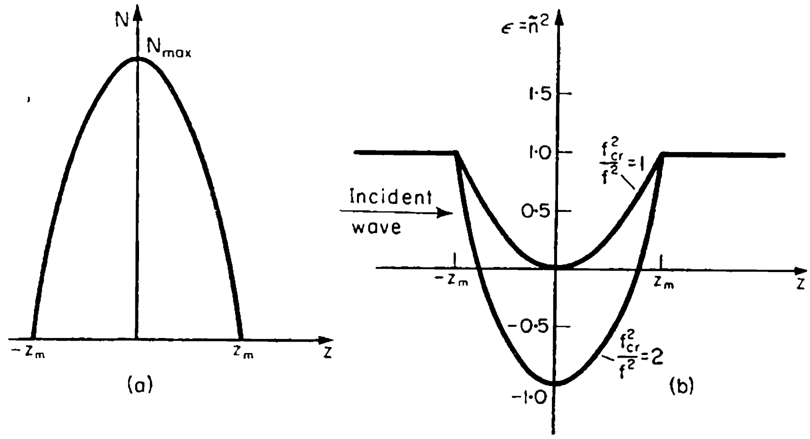

FIG. 17.2. A parabolic layer.
(a) $N(z)$
(b) $\varepsilon(z)$ for $f_{cr}^2/f^2 = 1$ and 2

and it serves only to test various approximate solutions. For this reason and others which will become evident, we shall not examine in detail the solution for a parabolic layer, but simply give the results [114, 127].

At the boundary of the layer ($z = -z_m$), the phase difference between the reflected and incident waves is

$$\varphi = \tfrac{1}{2}u^2 - \varrho \ln 4u^2 + \arg\left[(2i\,\varrho)!/(i\,\varrho)!\right] + \\ + 2\left[1 + \frac{i(\varrho - \frac{1}{2}i)(\varrho - \frac{3}{2}i)}{2u^2} - \cdots\right] + \frac{1}{2}\pi. \tag{17.24}$$

In a parabolic layer the region where $\varepsilon(\omega) < 0$ is always finite; if $f > f_{cr}$ there is no such region [see (17.21) and Fig. 17.2]. Thus in principle the wave always penetrates to some extent through the layer. For a parabolic layer without absorption, the reflection coefficient $|R|^2$ (the ratio of intensities of the reflected and incident waves) is given by

$$\frac{|R|^2}{1 - |R|^2} = e^{2\pi\varrho} = \exp\left(4\pi^2 \frac{z_m}{\lambda_{cr}} \frac{f_{cr}^2 - f^2}{2f_{cr}^2}\right). \tag{17.25}$$

This formula is valid both when $f > f_{cr}$ and when $f < f_{cr}$. In the latter case the reflection coefficient $|R|^2$ is equal to unity for a thick layer, where

$$z_m/\lambda_{cr} \gg 1, \tag{17.26}$$

even in the immediate neighbourhood of the critical frequency f_{cr}. For example, if $z_m/\lambda_{cr} = 10$ the coefficient $|R|^2 = 0{\cdot}999$ for $\Delta f/f_{cr} \equiv (f_{cr} - f)/f_{cr} = 1{\cdot}7 \times 10^{-2}$.

When $f > f_{cr}$, the coefficient $|R|^2$ tends rapidly to zero; for example, if $z_m/\lambda_{cr} = 10$, $|R|^2 = 0{\cdot}001$ for $\Delta f/f_{cr} = -1{\cdot}7\times 10^{-2}$.

It must be borne in mind that formula (17.25) is valid if we neglect reflection of waves at the beginning and end of the layer, as already mentioned above. Such a condition cannot, however, be fulfilled if the layer is very thin ($z_m \lesssim \lambda_{cr}$), since the edges of the layer cannot then be "rounded" without making the whole layer non-parabolic, so that ε cannot vary smoothly with z. Hence formula (17.25) is strictly valid only if the condition (17.26) holds. If this point were overlooked, we might falsely conclude [114] that $|R|^2 \to 0{\cdot}5$ as $z_m \to 0$ for all f, as follows from (17.25). In reality, of course $|R|^2 \to 0$ as $z_m \to 0$ if f_{cr} is finite. Reflection from a smooth layer of arbitrary thickness with a dome-shaped curve of $\varepsilon(z)$ is discussed in § 18.

Since by (17.26) only the region of small Δf, i.e. where

$$\Delta f/f_{cr} \equiv (f_{cr} - f)/f_{cr} \ll 1, \tag{17.27}$$

is of interest, formula (17.25) may be written

$$\left.\begin{aligned} \frac{|R|^2}{1-|R|^2} &= \exp\left(4\pi^2 \frac{z_m}{\lambda_{cr}} \frac{\Delta f}{f_{cr}}\right) \\ &= \exp\left(4\pi^2 \frac{z_m}{c} \Delta f\right), \\ |D|^2 = 1 - |R|^2 &= 1\Big/\left[1+\exp\left(4\pi^2 \frac{z_m}{c}\Delta f\right)\right], \end{aligned}\right\} \tag{17.28}$$

where $|D|^2$ is the transmission coefficient.

The penetration of the waves through the layer is discussed in detail in §33.

A layer with $\varepsilon' = a/(b+z)^2$

At the beginning of § 16 it has already been mentioned that the wave equation (16.3) has an exact solution expressible in terms of known functions not only for linear and parabolic layers but also for certain others. One such fairly general and important case will be considered in § 18. Here we may mention that the wave equation (16.3) has an exact solution in terms of elementary functions also for

$$\varepsilon'(z) = a/(b+z)^2, \tag{17.29}$$

where a and b are any complex numbers [129].

By direct substitution in equation (16.3) with this function $\varepsilon'(z)$ it is easily verified that the solution is

$$\left.\begin{aligned} E &= C_1(b+z)^{r_1} + C_2(b+z)^{r_2}, \\ r_{1,2} &= \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - \omega^2 a/c^2\right)}. \end{aligned}\right\} \tag{17.30}$$

As an example, let us consider reflection from a layer of the type (17.29): let

$$\left.\begin{aligned} \varepsilon &= 1 \quad \text{for} \quad z \leqq \alpha \quad \text{(medium 1)}, \\ \varepsilon &= \alpha^2/z^2 \quad \text{for} \quad z > \alpha \quad \text{(medium 2)} \end{aligned}\right\} \tag{17.31}$$

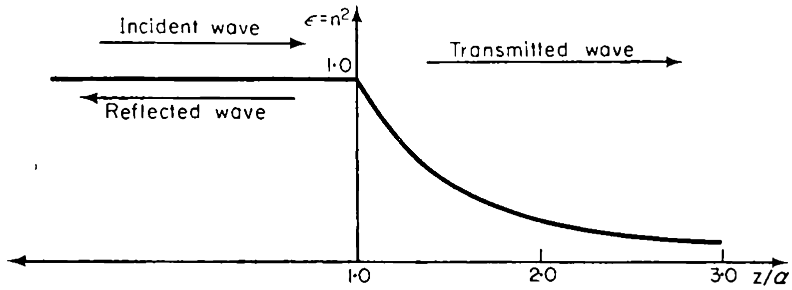

FIG. 17.3. A layer with $\varepsilon = \alpha^2/z^2$ for $z/\alpha > 1$.

(see Fig. 17.3). Let the wave be incident from medium 1, a vacuum, where the field has the form

$$E = e^{-i\omega(z-\alpha)/c} - R\, e^{i\omega(z-\alpha)/c}.$$

In medium 2 the field is

$$E = A\, z^{\frac{1}{2}} z^{-i\sqrt{[(\omega\alpha/c)^2 - \frac{1}{4}]}} = A\, z^{\frac{1}{2}} z^{-i\beta} = A\, z^{\frac{1}{2}} e^{-i\beta \ln z},$$

since in (17.30) we must put $b = 0$, $a = \alpha^2$; we have also used the fact that in medium 2 there is only a wave travelling away from the boundary. It is assumed that the wave can be propagated, which implies that $\omega\alpha/c > \frac{1}{2}$.† At the boundary $z = \alpha$ we must have $E_1 = E_2$ and $(dE/dz)_1 = (dE/dz)_2$, whence

$$\left.\begin{aligned} R &= 1 - A\,\alpha^{\frac{1}{2} - i\beta} \\ &= i\frac{1}{2(\omega\alpha/c + \beta)}, \\ |R| &= \frac{1}{2(\omega\alpha/c + \beta)}, \\ A &= \frac{2\alpha^{-\frac{1}{2}+i\beta}}{1 + i c(\frac{1}{2} - i\beta)/\omega\alpha} \\ &= \alpha^{-\frac{1}{2}+i\beta}\left[1 - i\frac{1}{2(\omega\alpha/c + \beta)}\right], \\ \beta &= \sqrt{\left[(\omega\alpha/c)^2 - \frac{1}{4}\right]}. \end{aligned}\right\} \tag{17.32}$$

If the permittivity ε varies only slightly over the distance of a wavelength, then $\omega\alpha/c \gg 1$ and $\beta \approx \omega\alpha/c$, i.e.

$$R = i\frac{1}{4\omega\alpha/c} = -i\frac{\lambda_0}{8\pi}\left(\frac{dn}{dz}\right)_{2,z=\alpha} \tag{17.33}$$

† If $\omega\alpha/c \leqq \frac{1}{2}$, the time average energy flux $\overline{S} = c\overline{\boldsymbol{E} \times \boldsymbol{H}}/4\pi$ is zero.

since for $z = \alpha$ we have $dn/dz = (d/dz)(\alpha/z) = -1/\alpha$. This formula is in accordance with the general result (16.33), since in (17.33) $n(0) = 1$. From a comparison of (17.32) and (17.33) we can also see the accuracy of the limiting formula (17.33).

§ 18. REFLECTION AND TRANSMISSION OF WAVES BY "SYMMETRICAL" AND "TRANSITION" LAYERS OF ARBITRARY THICKNESS

A smooth layer with four parameters

The parabolic layer (17.21) considered in § 17 cannot serve as a good model of a semi-transparent thin layer, since we must also take into account the reflections at the points $z = \pm z_m$ where the derivative $d\varepsilon/dz$ is discontinuous. Hence the advantage of simplicity is not available for a thin parabolic layer. Moreover, the parabolic layer has only two disposable parameters f_{cr} and z_m [see (17.21)]. It is nevertheless of interest to ascertain the nature of the reflection and transmission of waves by more general layers. The most suitable such layer for which an exact solution is known and has been discussed in detail [115, 116; 121, § 14] is one in which

$$\varepsilon'(z) = a + be^{\gamma z}/(1 + e^{\gamma z}) + ce^{\gamma z}/(1 + e^{\gamma z})^2, \tag{18.1}$$

where a, b, c and γ are complex constants.

In the absence of absorption, putting $a = 1$, we write the expression (18.1) as

$$\varepsilon(z) = 1 - Pe^{\gamma z}/(1 + e^{\gamma z}) - 4Me^{\gamma z}/(1 + e^{\gamma z})^2, \tag{18.2}$$

where P, M and γ are arbitrary real coefficients. The solution of the wave equation for a layer with (18.1)–(18.2) may be expressed in terms of hypergeometric functions. We shall not write out this solution here, but give only the expressions for the reflection coefficient $|R|^2$, i.e. for the ratio of the energy fluxes in the reflected and incident waves; R is the ratio of the amplitudes of the two waves.

A "symmetrical" layer

If we put $P = 0$ in (18.2), and take M and γ to be positive, we have a "symmetrical" layer with a single minimum of the function $\varepsilon(z)$ at $z = 0$, as shown in Fig. 18.1. The value of $\varepsilon(z)$ at the minimum is $\varepsilon_{min} = 1 - M$.

The width of the layer may be characterised by the dimensionless parameter

$$S = 2\omega/c\gamma = 4\pi/\lambda_0\gamma. \tag{18.3}$$

For a "symmetrical" layer, we can put $M = f_{cr}^2/f^2$ to obtain

$$\varepsilon = 1 - \frac{4f_{cr}^2}{f^2}\frac{e^{\gamma z}}{(1 + e^{\gamma z})^2}, \tag{18.4}$$

and the half-width of the layer is $\xi = 0{\cdot}14\, S\lambda_0$, i.e. for $z = \xi$

$$1 - \varepsilon = 0{\cdot}5 f_{\mathrm{cr}}^2/f^2 = 0{\cdot}5(1 - \varepsilon_{\min}).$$

If $M = 0$ with P and γ positive, we have a "transition" layer, as shown in Fig. 18.2; for $z \to -\infty$, $\varepsilon \to 1$ as in a "symmetrical" layer, but for $z \to +\infty$, $\varepsilon \to 1 - P$, whereas in a "symmetrical" layer $\varepsilon \to 1$ also.

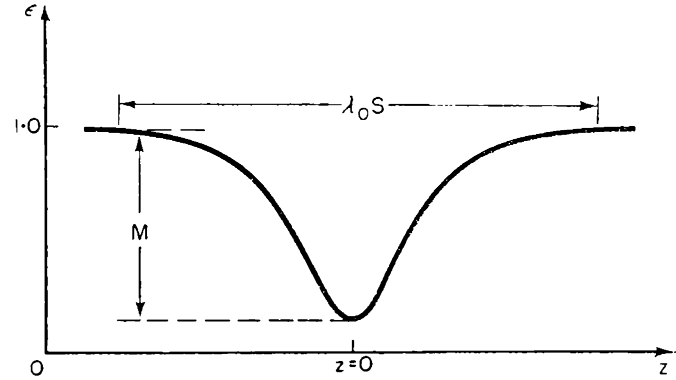

FIG. 18.1. A "symmetrical" layer.

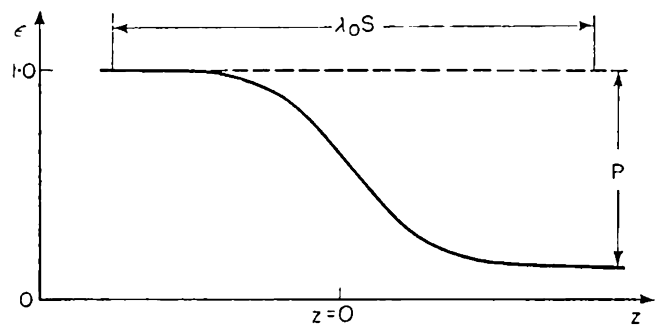

FIG. 18.2. A "transition" layer.

The reflection coefficient for the layer (18.2) is

$$|R|^2 = \left| \frac{\Gamma(iS)\,\Gamma\{\frac{1}{2} - d_2 - i\cdot\frac{1}{2}S[1 + \sqrt{(1-P)}] - i\,d_1\}}{\Gamma(-iS)\,\Gamma\{\frac{1}{2} - d_2 + i\cdot\frac{1}{2}S[1 - \sqrt{(1-P)}] - i\,d_1\}} \times \right.$$
$$\left. \times \frac{\Gamma\{\frac{1}{2} + d_2 - i\cdot\frac{1}{2}S[1 + \sqrt{(1-P)}] + i\,d_1\}}{\Gamma\{\frac{1}{2} + d_2 + i\cdot\frac{1}{2}S[1 - \sqrt{(1-P)}] + i\,d_1\}} \right|^2,$$

where Γ is the gamma function and $2(d_2 + id_1) = \sqrt{(1 - 4S^2M)}$.

For a "symmetrical" layer (18.4), $\sqrt{(1 - 4S^2M)} = \sqrt{(1 - 64\pi^2/\lambda_{\mathrm{cr}}^2\gamma^2)}$, where $\lambda_{\mathrm{cr}} = c/f_{\mathrm{cr}}$ and usually $8\pi/\lambda_{\mathrm{cr}}\gamma > 1$. In that case

$$\left.\begin{aligned} d_2 = 0, \quad \sqrt{(1 - 4S^2M)} = 2i\,d_1, \\ |R|^2 = \cosh^2\pi\,d_1/\cosh\pi(d_1 + S)\cosh\pi(d_1 - S). \end{aligned}\right\} \qquad (18.5)$$

The transmission coefficient—the ratio of the energy fluxes in the transmitted and incident waves—is $|D|^2 = 1 - |R|^2$; the quantity $\sqrt{(|D|^2)}$ is the modulus of the ratio of the amplitudes of the electric field in the transmitted and incident waves, since in a "symmetrical" layer for $z = \pm\infty$ the value of ε is the same, namely unity. The dependence of $|R|^2$ on $M = f_{cr}^2/f^2$ for

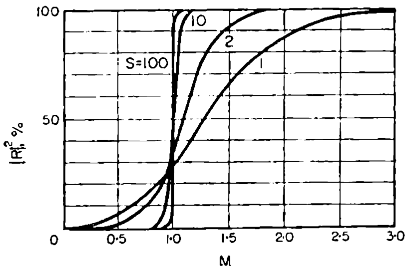

FIG. 18.3. The reflection coefficient for a "symmetrical" layer.

various values of S is shown in Fig. 18.3. Even for $S = 10$ the value of $|R|^2$ for $M \gtrsim 1{\cdot}3$ (i.e. $f \lesssim 0{\cdot}9 f_{cr}$) is almost equal to unity.

A "transition" layer. The limiting transition to a sharp boundary

For a "transition" layer ($M = 0, P > 0, d_2 = \frac{1}{2}, d_1 = 0$) we have

$$|R|^2 = \sinh^2\{\tfrac{1}{2}\pi S[1 - \sqrt{(1-P)}]\}/\sinh^2\{\tfrac{1}{2}\pi S[1 + \sqrt{(1-P)}]\}. \quad (18.6)$$

In this case we have $\varepsilon \to 1 - P$ for $z \to +\infty$, and so the value of $|D|^2 = 1 - |R|^2$ is not the ratio of the squared moduli of the electric field amplitudes in the transmitted and incident waves. For in a medium of permittivity $\varepsilon = n^2$ the energy flux is in absolute magnitude $cEH/4\pi = cnE^2/4\pi$, and thus the ratio of the moduli of amplitudes of the field E in the transmitted and incident waves is $D/\sqrt{n} = D/(1-P)^{\frac{1}{4}}$ (with $\varepsilon \to 1$ for $z \to -\infty$).

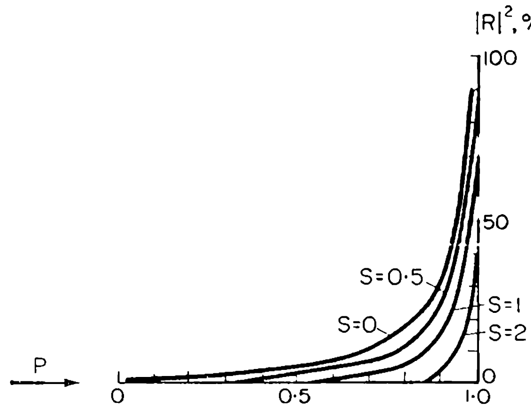

FIG. 18.4. The reflection coefficient for a "transition" layer.

Fig. 18.4 shows the form of the function $|R(P, S)|^2$ for a "transition" layer. It is seen that formula (18.6) is valid only if $P < 1$. If $P > 1$, there is total internal reflection, as is obvious *a priori*, since in this case $\varepsilon < 0$ when $z \to -\infty$.

For $S \to 0$ the "transition" layer tends to a sharp boundary, and Fresnel's formula for $|R|^2$ must be valid. Formula (18.6) shows that this is in fact so, and

$$|R|^2_{S\to 0} = [1 - \sqrt{(1-P)}]^2/[1 + \sqrt{(1-P)}]^2 = (1-n)^2/(1+n)^2, \tag{18.7}$$

where n is the value of the refractive index for $z > 0$.

The simple formulae (18.5) and (18.6) make it possible to obtain an immediate estimate of the reflection coefficient for any layer similar to the fairly typical "symmetrical" and "transition" layers. A more detailed account of the reflection and transmission of waves in the case (18.1)–(18.2) is given in [116, 121].

§19. OBLIQUE INCIDENCE OF WAVES ON A LAYER

General relations. A wave with the electric vector perpendicular to the plane of incidence

Let us now consider harmonic plane waves obliquely incident on a plane-parallel medium. Here we start from equation (16.1) or (16.2) with $\varepsilon' = \varepsilon'(\omega, z)$, the z-axis being normal to the layer.

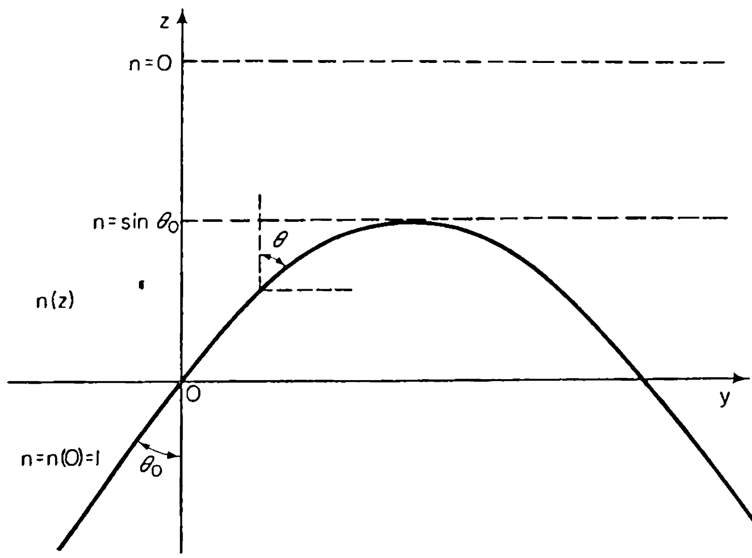

FIG. 19.1. Oblique incidence of a wave on a layer. For $z < 0$ the refractive index is $n = n(0) = 1$, for $z > 0$ it is $n = n(z)$.

The choice of the x and y axes has one degree of freedom, which may conveniently be utilised to make the normal to the wave lie in the yz-plane (Fig. 19.1). Then the incident (plane) wave outside the layer (for $z < 0$ in Fig. 19.1) has the form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-i\omega(y \sin\theta_0 + z\cos\theta_0)/c} = \boldsymbol{E}_0 e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}}, \tag{19.1}$$

where $\boldsymbol{k}_0 = (\omega/c)(0, \sin\theta_0, \cos\theta_0)$, $\boldsymbol{r} = (x, y, z)$, absorption is neglected, and the refractive index is taken to be unity. Evidently such a choice of axes (making $k_{0x} = 0$) is always possible without loss of generality. Since the incident wave is independent of the coordinate x, so is the "refracted" wave in the inhomogeneous medium, whose properties do not depend on x. Consequently the derivatives with respect to x in (16.1) may be taken as zero.

Thus equation (16.1) is, in components,

$$\left.\begin{aligned} \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon'(z)\,E_y - \frac{\partial}{\partial y}\operatorname{div}\boldsymbol{E} = 0\,, \\ \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon'(z)\,E_z - \frac{\partial}{\partial z}\operatorname{div}\boldsymbol{E} = 0\,, \end{aligned}\right\} \tag{19.2}$$

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon'(z)\,E_x = 0\,. \tag{19.3}$$

The wave field $\boldsymbol{E}$ may always be resolved into waves with two mutually perpendicular directions of polarisation. We take one of these directions to be the x-axis. Then (19.2) and (19.3) show that the field component E_x is entirely independent of the components E_y and E_z, i.e. if a wave whose vector $\boldsymbol{E}$ is parallel to the x-axis is incident on the layer, the direction of the vector $\boldsymbol{E}$ remains unchanged. For a wave whose vector $\boldsymbol{E}$ lies in the yz-plane, on the other hand, the components E_y and E_z are related and depend on the coordinates, as is easily understood if we consider the ray paths (see below). It may be noted that in the general case of an inhomogeneous isotropic medium the independence between the two incident waves with different polarisation does not occur, and in our case it is due to the fact that ε' does not vary in the x-direction.

Let us first consider the wave equation (19.3) for the component E_x, which differs from equation (16.3) only by the term $\partial^2 E_x/\partial y^2$. We put

$$E_x = F(z)\, e^{\pm i(\omega/c)y\sqrt{\varepsilon'}\sin\theta} = F(z)\, e^{\pm ik\alpha y}\,, \tag{19.4}$$

where $k(z) = (\omega/c)\sqrt{\varepsilon'(z)}$ and $\alpha(z) = \sin\theta(z)$. Then (19.3) gives

$$d^2F/dz^2 + k^2(1-\alpha^2)\,F$$
$$= \mp 2i\,\frac{d(k\alpha)}{dz}\,y\,\frac{dF}{dz} + \left\{\mp i\,y\,\frac{d^2(k\alpha)}{dz^2} + y^2\left[\frac{d(k\alpha)}{dz}\right]^2\right\}F\,. \tag{19.5}$$

Since this equation must be true for all y, both sides must obviously be equal to zero:

$$d^2F/dz^2 + (\omega^2/c^2)\,\varepsilon'(z)[1-\alpha^2(z)]F = 0 \tag{19.6}$$

and

$$d(k\alpha)/dz = 0. \tag{19.7}$$

The derivation of (19.7) from the vanishing of the right-hand side of (19.5) also follows from the fact that the coordinate y is arbitrary. From (19.7)

we have

$$\begin{aligned} k\alpha &= (\omega/c)\,\alpha\sqrt{\varepsilon'} \\ &= (\omega/c)\sqrt{\varepsilon'(z)}\,\sin\theta(z) \\ &= k_0\alpha_0 = (\omega/c)\sqrt{\varepsilon'(0)}\,\sin\theta_0 = \text{constant}. \end{aligned} \tag{19.8}$$

In the absence of absorption $\sqrt{\varepsilon'} = \sqrt{\varepsilon} = n(z)$, and the relation (19.8) may be written

$$n(z)\sin\theta(z) = \sin\theta_0, \tag{19.9}$$

where θ_0 is the angle of incidence of the wave on the layer, at the boundary of which $n = n(0) = 1$. The identity of the angle θ used above with the angle between the z-axis and the normal to the wave follows from formula (19.14) below.

The relation (19.9) corresponds to the law of refraction at the boundary between two media: $n_1\sin\theta_1 = n_2\sin\theta_2$. Hence formula (19.9) can also be easily obtained in an elementary manner by regarding the inhomogeneous medium as the limiting case of a medium consisting of a large number of homogeneous layers.

Equation (19.6) has the form (16.3) with $(\omega^2/c^2)\,\varepsilon'(z)$ replaced by $(\omega^2/c^2)\,\varepsilon'(z)[1-\alpha^2(z)] = (\omega^2/c^2)\,\varepsilon'(z)\cos^2\theta = (\omega^2/c^2)[\varepsilon'(z) - \varepsilon'(0)\alpha_0^2]$. Thus the solution of equation (19.6) reduces to the solution of equation (16.3) with the same functional dependence of ε' on z (a linear layer remains linear, and so on). For example, for the linear layer (17.1) the coefficient of F in (19.6) is $(\omega^2/c^2)[1 - \varepsilon(0)\sin^2\theta_0 - z/z_1]$, and equation (19.6) is transformed to (17.3) by using the variable

$$\zeta = (\omega^2/c^2 z_1)^{\frac{1}{3}}\,\{z_1[1-\varepsilon(0)\sin^2\theta_0] - z\}.$$

The approximation of geometrical optics

Hence it is clear that for equation (19.3) we can immediately write down the solution in the approximation of geometrical optics by comparing (16.3), (16.10) and (16.11) with (19.3), (19.4), (19.6) and (19.8). The result is

$$\left.\begin{aligned} E_x &= \frac{E_{x0}}{[\varepsilon'(z) - \varepsilon'(0)\,\alpha_0^2]^{\frac{1}{4}}}\times \\ &\quad\times\exp\left[\pm i\frac{\omega}{c}\left\{\int_0^z\sqrt{[\varepsilon'(z)-\varepsilon'(0)\,\alpha_0^2]}\,dz + \sqrt{\varepsilon'(0)}\,\alpha_0\,y\right\}\right], \\ &\varepsilon'(0)\,\alpha_0^2 = \varepsilon'(0)\sin^2\theta_0 = \text{constant}, \end{aligned}\right\} \tag{19.10}$$

or, in the absence of absorption

$$\left.\begin{aligned} E_x = {} & \frac{E_{x0}}{[n^2(z) - n^2(0)\sin^2\theta_0]^{\frac{1}{4}}} \times \\ & \times \exp\left[\pm i\frac{\omega}{c}\left\{\int_0^z \sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}\,dz + n(0)\,y\sin\theta_0\right\}\right]. \end{aligned}\right\} \quad (19.11)$$

This formula, besides depending on y, differs also from that for normal incidence in that $n(z)$ is replaced by $\sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]} = n(z)\cos\theta(z)$. It is clear from this or directly from (19.6) that the wave is reflected from the layer in the neighbourhood of the point z_0, where

$$n(z_0) = n(0)\sin\theta_0. \quad (19.12)$$

Formula (19.9) shows that (19.12) is simply the condition for total reflection, which occurs when $\theta = \frac{1}{2}\pi$; in (19.9) it is assumed that $n(0) = 1$. Since, as already mentioned, the difference between normal and oblique incidence amounts to the replacement of $n(z)$ in the relevant equation by $n(z)\cos\theta_0$, it is evident that the condition (16.22) for the first approximation of geometrical optics to be valid is replaced for oblique incidence by

$$\frac{\lambda_0}{2\pi}\left|\frac{d(n\cos\theta)/dz}{n^2\cos^2\theta}\right| \ll 1, \quad (19.13)$$

where $n(z)\sin\theta(z) = n(0)\sin\theta_0 = \text{constant}$, $\cos\theta = \sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}/n(z)$. This condition is clearly violated when $\cos\theta \to 0$, i.e. near the reflection point (19.12).

The ray treatment

In the approximation of geometrical optics we may work in terms of rays instead of waves. In a homogeneous isotropic medium the direction of the ray (i.e. the direction of the energy flux or the direction of motion of the wave pulse; see § 24) coincides with that of the normal to the wave surface. The same is true where geometrical optics is valid, since the medium may then be regarded as quasihomogeneous, and over relatively short distances the wave is propagated as in a homogeneous medium with the corresponding values of ε and σ (see also § 24).

The direction of the normal is obtained from (19.11) by finding $\mathbf{grad}\, E_x$, and since the denominator depends only slightly on z, only the exponential factor need be differentiated. Thus we find the components of the wave vector $\boldsymbol{k}$:

$$\left.\begin{aligned} & k_x = 0, \quad k_y = (\omega/c)\,n(0)\sin\theta_0 = (\omega/c)\,n(z)\sin\theta, \\ & k_z = (\omega/c)\sqrt{[n^2(z) - n^2(0)\sin\theta_0]} = (\omega/c)\,n(z)\cos\theta, \\ & k^2 = (\omega^2/c^2)\,n^2(z). \end{aligned}\right\} \quad (19.14)$$

The unit vector along the normal to the wave front is $\boldsymbol{k}/k$. In a quasi-homogeneous isotropic medium this vector is, as already stated, also tangential to the ray path.

Near the reflection point z_0 (where $n(z_0) = n(0)\sin\theta_0$) the approximation of geometrical optics is invalid and the direction of the normal to the wave surface does not coincide with the direction of motion of the centre of gravity of the wave pulse. This point is further discussed in § 34. Here we may note that, for the linear layer (17.1), and for any layer far from the critical frequency, there is total internal reflection; and, by comparing (16.3), (17.10) with (19.4), (19.6), we easily see that the phase change of the wave on reflection from a layer with oblique incidence is

$$\varphi = 2\frac{\omega}{c}\int\limits_0^{z_0} \sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}\,dz + \frac{\omega}{c}\,n(0)\,(y_2 - y_1)\sin\theta_0 - \frac{1}{2}\pi$$

$$= 2\int\limits_0^{z_0} k_z\,dz + k_y(y_2 - y_1) - \frac{1}{2}\pi; \tag{19.15}$$

here $n(z_0) - n(0)\sin\theta_0 = 0$, absorption is for simplicity assumed absent, and y_2 and y_1 are the points on the y-axis ($z = 0$) for which the phase difference φ is determined.

In the range of validity of geometrical optics, the phase change of the wave over a distance l is [see (19.11), (19.14)]

$$\varphi = (\omega/c)\int\limits_l n(z)\,ds = \int \boldsymbol{k}\cdot d\boldsymbol{s}, \tag{19.16}$$

where $d\boldsymbol{s}$ is an element of the ray path, whose direction is the same as that of the vector $\boldsymbol{k}$.

The path of a ray incident on the medium at an angle θ_0 at the point $y = y_1$ and $z = 0$ where $n(0) = 1$ (see Fig. 19.1) is evidently determined by the equation

$$y = y_1 + \int\limits_0^z \frac{\sin\theta_0\,dz}{\sqrt{[n^2(z) - \sin^2\theta_0]}}; \tag{19.17}$$

see (19.14), which shows that along the ray path

$$dy/dz = k_y/k_z = \sin\theta_0/\sqrt{[n^2(z) - \sin^2\theta_0]}.$$

Waves with the electric vector in the plane of incidence

Let us now consider the propagation of a wave whose electric vector lies in the yz-plane [see equations (19.2)]. The physical difference from the previous case is that, as the direction of the wave vector changes with increasing depth of penetration into the layer, the vector $\boldsymbol{E}$ must also rotate.

First of all, it is convenient to effect a slight transformation of equation (19.2), using the relation

$$\operatorname{div}(\boldsymbol{D} - i \cdot 4\pi \boldsymbol{j}/\omega) = 0, \tag{19.18}$$

which follows from the field equation (2.1). In the present case we have

$$\begin{aligned}\operatorname{div}(\boldsymbol{D} - i \cdot 4\pi \boldsymbol{j}/\omega) &= \operatorname{div}(\varepsilon' \boldsymbol{E}) \\ &= \boldsymbol{E} \cdot \mathbf{grad}\, \varepsilon' + \varepsilon' \operatorname{div} \boldsymbol{E} \\ &= E_z\, d\varepsilon'/dz + \varepsilon' \operatorname{div} \boldsymbol{E} = 0,\end{aligned} \tag{19.19}$$

whence

$$\mathbf{grad} \operatorname{div} \boldsymbol{E} = -\mathbf{grad}(E_z\, d \ln \varepsilon'/dz).$$

In a homogeneous medium, of course, (19.19) gives $\operatorname{div} \boldsymbol{E} = 0$, and also $\operatorname{div} \boldsymbol{D} = \operatorname{div} \boldsymbol{j} = 0$, since $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{j} = \sigma \boldsymbol{E}$.

Using the above expression for **grad** div $\boldsymbol{E}$ and assuming for simplicity that there is no absorption (i.e. putting $\varepsilon' = \varepsilon$), we can write the equations (19.2) as

$$\left.\begin{aligned}\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon(z) E_y + \frac{\partial E_z}{\partial y}\frac{d \ln \varepsilon(z)}{dz} = 0, \\ \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2}\varepsilon(z) E_z + \frac{\partial}{\partial z}\left[E_z \frac{d \ln \varepsilon(z)}{dz}\right] = 0.\end{aligned}\right\} \tag{19.20}$$

We shall now discuss only the second equation (19.20) for E_z. Substituting the solution

$$\left.\begin{aligned}&E_z = \alpha(z) F_z(z) e^{\pm ik(z)\alpha(z)y}, \\ &k(z) = (\omega/c)\sqrt{\varepsilon(z)} = (\omega/c)\, n(z), \quad \alpha(z) = \sin\theta(z),\end{aligned}\right\} \tag{19.21}$$

and separating the terms in y and y^2 as in (19.5), we obtain for $k\alpha$ (19.7) and for F_z the equation

$$\frac{d^2 F_z}{dz^2} + \left\{\frac{\omega^2}{c^2}\varepsilon(z)[1 - \alpha^2(z)] + \frac{1}{2}\frac{d^2 \ln \varepsilon}{dz^2} - \frac{1}{4}\left(\frac{d \ln \varepsilon}{dz}\right)^2\right\} F_z = 0. \tag{19.22}$$

In (19.22) we have already used the fact that

$$\frac{2\, d\alpha/dz}{\alpha} = -\frac{d \ln \varepsilon}{dz},$$

which follows from (19.7) when absorption is neglected. It is easy to see that when absorption is present we must simply replace ε in (19.22) by ε'.

It may be noted that the transition to normal incidence in formulae (19.2) and (19.3) is obvious, because the derivatives with respect to y vanish, and (since $\operatorname{div} \boldsymbol{E} = dE_z/dz$) the equations (19.2) become (2.12) and (2.14). In (19.6) the transition to normal incidence is also immediate, putting $\alpha = 0$. When $\alpha \to 0$, equation (19.22) is not needed for the calculation of E_z, which tends to zero; this follows from (19.21) and is also obvious from the fact that the field is transverse.

Equation (19.22) differs from (19.6) only by terms involving derivatives of ε. It is easy to see that in the approximation of geometrical optics these terms may be neglected, and so we obtain for E_z the expression (19.11) multiplied by $\alpha(z) = \sin\theta(z)$ [see (19.4) and (19.21)]. The appearance of this factor is readily understandable because, if the derivative $d\varepsilon/dz$ is neglected, (19.19) gives $\operatorname{div} \boldsymbol{E} = 0$ and the vector $\boldsymbol{E}$ is perpendicular to the wave vector $\boldsymbol{k}$. The component E_z must therefore be proportional to $\alpha(z) = \sin\theta$, where θ is the angle between $\boldsymbol{k}$ and the z-axis [see (19.14) and Fig. 19.1].

In the range where geometrical optics is not valid, and in particular near the "reflection point" z_0, the nature of the wave field is different for waves of different polarisation, since the equation (19.22) is equivalent to the wave equation (16.3) for normal incidence with an effective ε given by

$$\varepsilon_{\text{eff}}(z) = \varepsilon(z) - \varepsilon(0)\sin^2\theta_0 + \left(\frac{\lambda_0}{2\pi}\right)^2 \left[\frac{d^2\varepsilon/dz^2}{2\varepsilon} - \frac{3}{4}\frac{(d\varepsilon/dz)^2}{\varepsilon^2}\right]; \qquad (19.23)$$

here we have used the identity

$$\frac{1}{2}\frac{d^2\ln\varepsilon}{dz^2} - \frac{1}{4}\left(\frac{d\ln\varepsilon}{dz}\right)^2 = \frac{d^2\varepsilon/dz^2}{2\varepsilon} - \frac{3}{4}\frac{(d\varepsilon/dz)^2}{\varepsilon^2}.$$

At the same time equation (19.6) for oblique incidence of a wave with the vector $\boldsymbol{E}$ in the x-direction is equivalent to equation (16.3) with $\varepsilon_{\text{eff}} = \varepsilon(z) - \varepsilon(0)\sin^2\theta_0$.

The expression (19.23) tends to infinity as $\varepsilon \to 0$, i.e. the value of the effective index of refraction varies very greatly. Near the point z_0, the difference between ε_{eff} and $\varepsilon(z) - \varepsilon(0)\sin^2\theta_0$ is especially great for small values of $\alpha_0 = \sin\theta_0$, since $\varepsilon(z_0) = \varepsilon(0)\sin\theta_0$, and therefore $\varepsilon(z_0) \to 0$ as $\sin\theta_0 \to 0$.

For a linear layer the coefficient of F_z in (19.22), i.e. the quantity ε_{eff} just defined, is

$$\varepsilon_{\text{eff}} = \varepsilon(z) - \varepsilon(0)\alpha_0^2 - \frac{3(d\varepsilon/dz)^2}{4\varepsilon^2}\left(\frac{\lambda_0}{2\pi}\right)^2$$

and is zero when $z < z_0$, where z_0 is given by the condition $\varepsilon(z_0) = \varepsilon(0)\alpha_0^2$. Thus the wave begins to be damped somewhat before the point where $\alpha = \sin\theta = 1$. In the ionosphere this shift of the "reflection point" is in general negligibly small. As just mentioned, only the range of small angles of incidence requires special investigation. For $\alpha_0 = 0$ we have simply the already familiar case of normal incidence, where $E_z = 0$, and so the result of the passage to the limit $\alpha_0 \to 0$ is known.

The region above the point of reflection for a fairly thick layer might appear to be of no interest: with normal incidence the wave is exponentially damped in that region and we might expect a similar behaviour for oblique incidence. In reality, however, for oblique incidence of a wave whose vector $\boldsymbol{E}$ lies in the plane of incidence, this exponential decrease of the field does not occur near the point where $\varepsilon(\omega, z) = 0$. As we approach this point from below (from larger ε values) the field begins to increase, and in the absence of absorption the components E_y and E_z even tend to infinity at the point $\varepsilon = 0$ itself. (In practice, E_y and E_z remain finite even in the absence of absorption, for reasons to be discussed in § 20.) This noteworthy result is formally related to the singularity of the function ε_{eff} at $\varepsilon = 0$, already noted [see (19.23)].

The equation for the magnetic field of the wave

The problem of the behaviour of the field near $\varepsilon = 0$ for oblique incidence is examined in detail in § 20. Here we may note that, in investigating oblique incidence of a wave with the vector $\boldsymbol{E}$ lying in the plane of incidence, it is more convenient to use the equation for the magnetic field of the wave. In this case it is clear from the equation $\mathbf{curl}\, \boldsymbol{H} = i\omega\varepsilon' \boldsymbol{E}/c$ that we can put†

$$\left.\begin{aligned} &E_y = -(ic/\omega\varepsilon')\,\partial H_x/\partial z, \quad E_z = (ic/\omega\varepsilon')\,\partial H_x/\partial y, \\ &H_y = H_z = 0. \end{aligned}\right\} \tag{19.24}$$

The field $\boldsymbol{H}$ is also governed by the wave equation (16.2), which in this case is

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} - \frac{1}{\varepsilon'}\frac{d\varepsilon'}{dz}\frac{\partial H_x}{\partial z} + \frac{\omega^2}{c^2} H_x = 0. \tag{19.25}$$

Substituting

$$H_x = G(z)\, e^{\pm i(\omega/c) y \sqrt{\varepsilon'} \sin\theta}, \tag{19.26}$$

we arrive, as in the derivation of (19.6) and (19.7) from (19.3), at the law of refraction (19.8), and the equation for G

$$\frac{d^2 G}{dz^2} - \frac{1}{\varepsilon'}\frac{d\varepsilon'}{dz}\frac{dG}{dz} + \frac{\omega^2}{c^2}\left[\varepsilon'(z) - \varepsilon'(0)\sin^2\theta_0\right] G = 0. \tag{19.27}$$

This equation is somewhat more complex than (19.6) for $F(z) = E_x e^{\pm i(\omega/c) y \sqrt{\varepsilon'} \sin\theta}$, but simpler than the equations (19.20). Of course, equation (19.27) and equation (19.22) for $F_z = [E_z/\sin\theta(z)]\, e^{\mp i(\omega/c) y \sqrt{\varepsilon'(z)} \sin\theta}$ are of about the same degree of complexity, but a knowledge of G gives E_y and E_z immediately, whereas the derivation of E_y from F_z is not so simple. Thus in this case, as in other similar cases, the symmetry of the problem makes it more convenient to consider the equation for $\boldsymbol{H}$ rather than that for $\boldsymbol{E}$.

† If H_y or H_z is not zero, then in general $E_x \neq 0$. We are interested in the case where $E_x = 0$, and this corresponds to the field $\boldsymbol{H}$ taken here, with $H_y = H_z = 0$.

§ 20. A PROPERTY OF THE FIELD OF AN ELECTROMAGNETIC WAVE PROPAGATED IN AN INHOMOGENEOUS ISOTROPIC PLASMA. INTERACTION OF THE ELECTROMAGNETIC AND PLASMA WAVES

A physical description of the phenomenon

In order that the clear understanding of the physical picture shall not be obscured by the somewhat lengthy calculations, let us begin by a qualitative description of the results.

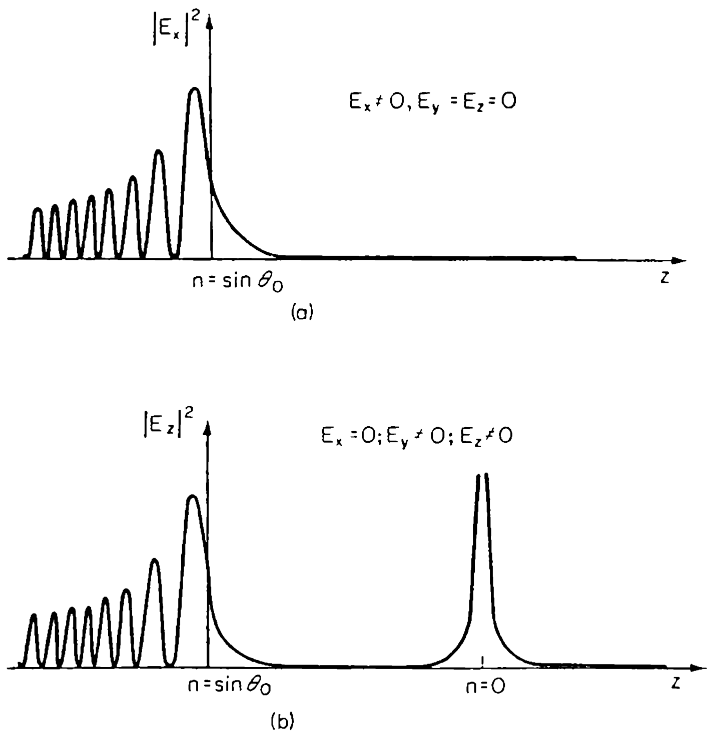

FIG. 20.1. The squared modulus of the electric field component when a wave incident at an angle θ_0 is reflected from an inhomogeneous layer. The refractive index is $n = n(z)$.
(a) A wave with the electric vector perpendicular to the plane of incidence.
(b) A wave with the electric vector in the plane of incidence.

If a wave is incident normally on a layer of inhomogeneous non-absorbing plasma, total reflection of the wave occurs near a point z_1 where $\varepsilon(\omega, z_1) = 0$, and so a standing wave is formed. The amplitude of this wave is oscillatory for $z < z_1$ and decreases exponentially for $z > z_1$, where $\varepsilon < 0$. The dependence of the field on z in the reflection region in this case is discussed in detail in § 32; qualitatively, it is clear from the preceding remarks and from Fig. 20.1a, which shows the squared modulus $|E_x|^2$ of the field for a wave incident obliquely on a layer, with the vector $\boldsymbol{E}$ perpendicular to the plane of incidence ($E_y = E_z = 0$, $E_x \neq 0$; see Fig. 19.1). For normal incidence, the picture is unchanged, as already explained, but $\sin\theta_0 = 0$ and at the "reflection point" $n(\omega, z_1) = \sqrt{\varepsilon(\omega, z_1)} = 0$.

For the case here considered of a wave obliquely incident on the layer, in which the vector $\boldsymbol{E}$ is in the plane of incidence ($E_x = 0$, $E_y \neq 0$, $E_z \neq 0$), the field is of a different type (Fig. 20.1b). Near the "reflection point" z_0, where $n(\omega, z_0) = \sin\theta_0$, the field behaves as for a wave with $E_y = E_z = 0$ or for normal incidence, i.e. as shown in Fig. 20.1a, but further into the layer the field begins to increase near the point where $n(\omega, z_1) = \sqrt{\varepsilon(\omega, z_1)} = 0$, and both E_y and E_z become infinite if we neglect absorption, the formation of plasma waves and non-linear effects (see below).

This singularity of the field at the point where $n = 0$ obviously disappears for normal incidence. Moreover, when the angle of incidence θ_0 is sufficiently large, and if there is any absorption at all, the fields E_y and E_z are everywhere finite, and the increase in the field at the point where $n = 0$ becomes much less marked. This result, which follows from the calculations given below, is physically reasonable, since with increasing angle θ_0 the distance between the "reflection point" $n(\omega, z_0) = \sin\theta_0$ and the point where $n(\omega, z_1) = 0$ increases. Hence for large θ_0 the region of $n = 0$ becomes more "inaccessible" to the wave.

Since this effect disappears when $\theta_0 = 0$ and for large angles θ_0, it is evident that the effect is most marked for some "intermediate" values of the angle of incidence θ_0.

The solution of the wave equation

The problem of oblique incidence of a wave with the vector $\boldsymbol{E}$ in the plane of incidence and of the singularity of the field for $n = 0$ is discussed in [112, 130–133, 70]. Here we shall follow [132], where the most complete solution is given; see also [70].

The basic equation is (19.27):

$$\left.\begin{aligned} &H_x = G(z)\, e^{i\omega t - i\omega\alpha_0 y/c}, \\ &\frac{d^2 G}{d z^2} - \frac{1}{\varepsilon'(z)}\frac{d\varepsilon'}{dz}\frac{dG}{dz} + \frac{\omega^2}{c^2}[\varepsilon'(z) - \alpha_0^2]\, G = 0, \\ &E_y = -\frac{i}{\omega\,\varepsilon'/c}\frac{\partial H_x}{\partial z}, \quad E_z = \frac{i}{\omega\,\varepsilon'/c}\frac{\partial H_x}{\partial y}, \quad \alpha_0 = \sin\theta_0. \end{aligned}\right\} \tag{20.1}$$

For $\varepsilon'(z)$ we take the expression (3.7) for a plasma, and suppose always that $\omega^2 \gg \nu_{\mathrm{eff}}^2$. Then

$$\varepsilon'(z) \approx 1 - \frac{4\pi e^2 N(z)}{m\,\omega^2}\left(1 + \frac{i\,\nu_{\mathrm{eff}}}{\omega}\right) = \varepsilon(z) - i\,(\omega_0^2/\omega^2)\,\nu_{\mathrm{eff}}/\omega\,.$$

For simplicity we shall also assume that the absorption varies only slowly with height, i.e. that ν_{eff} depends only slightly on z. Then the imaginary part of $\varepsilon'(z)$ may be taken as constant and equal to its value for $\omega = \omega_0 = \sqrt{(4\pi e^2 N/m)}$. For a linear layer we thus have†

$$\varepsilon'(z) = -a z - i\, \nu_{\text{eff}}/\omega = -a z - i s, \tag{20.2}$$

where $a > 0$, so that for $z > 0$ the permittivity $\varepsilon(z) < 0$.

The differential equation (20.1) then becomes

$$\frac{d^2 G}{d z^2} - \frac{a}{a z + i s} \frac{d G}{d z} + \frac{\omega^2}{c^2} (-a z - i s - \alpha_0^2)\, G = 0\,.$$

With a new variable $\zeta = az + is$ and the notation $\varrho = \omega/ca$, we have

$$\left.\begin{aligned} &\frac{d^2 G}{d \zeta^2} - \frac{1}{\zeta} \frac{d G}{d \zeta} + \varrho^2(-\zeta - \alpha_0^2)\, G = 0\,, \\ &\varrho = \omega/c\, a = 2\pi/\lambda_0 a, \quad a = |d\,\varepsilon/d\,z|\,. \end{aligned}\right\} \tag{20.3}$$

The equation (20.3) is obviously unchanged in form when $s = 0$; the only difference when absorption is taken into account is that the "mathematical" reflection point $\zeta = -\alpha_0^2$ corresponds to a complex z. For a medium whose properties vary only slowly the quantity $\varrho = \omega/ca$ in (20.3) is much greater than unity. For example, in the F layer of the ionosphere ($a \sim 10^{-7}\,\text{cm}^{-1}$), for a frequency $\omega \sim 10^8\,\text{sec}^{-1}$, we have $\varrho \sim 3 \times 10^4$.

It will be shown below (see [130, 131]) that the solution of equation (20.3) which satisfies the necessary physical conditions has a non-zero value $G(0)$ at the point in the complex plane where $\varepsilon'(z) = 0$. Hence the vertical component of the electric field,

$$E_z = \frac{i}{\omega\, \varepsilon'/c} \frac{\partial H_x}{\partial y} = \frac{\alpha_0}{\varepsilon'(z)}\, G(z)\, e^{i\omega t - i\omega\alpha_0 y/c}, \tag{20.4}$$

is infinite at that point. The nature of the singularity depends on the behaviour of the function $\varepsilon'(z)$; for a linear layer E_z becomes infinite as $1/\zeta = 1/(az + is)$, and the component E_y has a logarithmic singularity, as will be shown below. These singularities are on the real axis only if collisions are absent, when $s = \nu_{\text{eff}}/\omega = 0$. When absorption is taken into account, the maximum value of E_z is

$$|E_z|_{z=0} = \alpha_0\, |G(0)|/s \equiv \omega\, |G(0)| \sin\theta_0/\nu_{\text{eff}}, \tag{20.5}$$

and may be very large when s is sufficiently small. The field then depends considerably on the possible values of the function $G(0)$. This in turn depends

† For any layer the expression (20.2) is usually valid in a small region near a zero of the function $\varepsilon'(z)$, where the approximation of geometrical optics is not valid (see § 30). The choice of a different notation and a different origin in (20.2) as compared with § 17 is dictated by convenience and should not lead to misunderstanding, since all the necessary formulae are given here.

on the angle of incidence θ_0, and thus determines $|E_z|_{z=0}$ throughout the range of variation of the parameter $\alpha_0 = \sin\theta_0$.

It is fairly easy to deduce the form of the function $G(z)$ for large angles of incidence, when the reflection point $\zeta = -\alpha_0^2$ (i.e. $\varepsilon' = \sin^2\theta_0$) and the singular point $\zeta = 0$ (i.e. $\varepsilon' = 0$) are a considerable distance apart. To do this, it is convenient to consider, instead of (20.3), the equation

$$d^2u/d\zeta^2 - [\varrho^2(\zeta + \alpha_0^2) + 3/4\zeta^2]\,u = 0, \tag{20.6}$$

which is satisfied by the function

$$u(\zeta) = G(\zeta)/\sqrt{\zeta}. \tag{20.7}$$

The assumption that the distance between the points $\zeta = -\alpha_0^2$ and $\zeta = 0$ is large signifies in the present case that this distance is much greater than the wavelength. For a medium with slowly varying properties ($\varrho \gg 1$) this is true even when $\alpha_0^2 = \sin^2\theta_0$ is quite small. Then the approximate solution of equa-

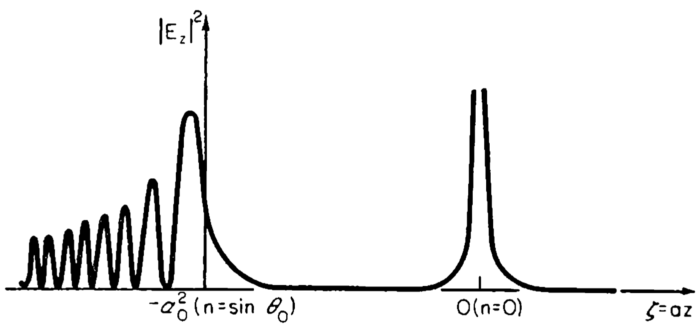

FIG. 20.2. The same as Fig. 20.1b, but with a new coordinate $\zeta = -\varepsilon = az$.

tion (20.6), valid everywhere outside a small neighbourhood of the point $\zeta = 0$ and representing a standing wave to the left of $\zeta = -\alpha_0^2$ (Fig. 20.2), may be written (see [134])

$$u = \sqrt{(\tfrac{1}{2}\pi\,\varrho)}\,e^{-i\pi/12}\sqrt{(S/S')}H_{\frac{1}{3}}^{(1)}(i\,S), \tag{20.8}$$

where

$$\left.\begin{aligned} S &= \varrho \int\limits_{-\alpha_0^2}^{\zeta} \sqrt{(\zeta + \alpha_0^2)}\,d\zeta = \frac{2}{3}\,\varrho(\zeta + \alpha_0^2)^{3/2}, \\ S' &= d\,S/d\,\zeta \end{aligned}\right\} \tag{20.9}$$

and $H_{\frac{1}{3}}^{(1)}$ is the Hankel function of the first kind of order $\frac{1}{3}$. The constant in (20.8) is chosen so that the amplitude of the incident wave field is equal to unity at the boundary of the inhomogeneous layer (where $\varepsilon'(z) = 1$).

Another approximate solution, valid to the right of the reflection point, can be obtained by means of the method proposed in [134]. We use a new independent variable

$$\xi = \varrho \int_0 \sqrt{(\zeta + \alpha_0^2)}\, d\,\zeta = \frac{2}{3} \varrho [(\zeta + \alpha_0^2)^{3/2} - \alpha_0^3]. \tag{20.10}$$

It is easily shown that the function

$$w = A \sqrt{\frac{\xi}{d\,\xi/d\,\zeta}}\, H_1^{(1)}(i\,\xi) \quad (A = \text{constant}) \tag{20.11}$$

satisfies the equation

$$\frac{d^2 w}{d\,\zeta^2} - \left[\varrho^2 (\zeta + \alpha_0^2) + \frac{3}{4}\left(\frac{1}{\xi}\,\frac{d\,\xi}{d\,\zeta}\right)^2 + \frac{5}{16(\zeta + \alpha_0^2)^2}\right] w = 0. \tag{20.12}$$

For small ζ it is clear from (20.10) that $\xi \approx \varrho \alpha_0 \zeta$, and equation (20.12) has exactly the same singularity at $\zeta = 0$ as the original equation (20.6). Moreover, for large values of the parameter ϱ the equations (20.12) and (20.6) are almost the same except in some neighbourhood of the point $\zeta = -\alpha_0^2$, where the term $5/16(\zeta + \alpha_0^2)^2$ becomes very large. Away from this point, therefore, suitably chosen solutions of these equations are almost equal. The function (20.11) is then approximately equal to the solution which tends to zero as $\zeta \to \infty$ [in the region of negative values of $\varepsilon(z)$].

Thus we have the approximate solutions (20.8) and (20.11) which represent the asymptotic behaviour of the desired solution (for $\varrho \gg 1$) in different ranges of the variable ζ: to the left of $\zeta = 0$ (20.8) and to the right of $\zeta = -\alpha_0^2$ (20.11). In the range $-\alpha_0^2 < \zeta < 0$ both approximations are valid, and so the solutions can be correlated in such a way as to represent the behaviour of one particular solution of the problem. This joining of the solutions gives for the constant A the value [132]

$$A = \sqrt{(\tfrac{1}{2}\pi\,\varrho)}\, e^{i\pi/4 - S_0}, \tag{20.13}$$

where

$$S_0 = \varrho \int_{-\alpha_0^2}^{0} \sqrt{(\zeta + \alpha_0^2)}\, d\,\zeta = \frac{2}{3} \varrho\, \alpha_0^3. \tag{20.14}$$

From (20.11) and (20.13) we obtain the final formula which gives the behaviour of the function $G(\zeta)$ in the range $\zeta > -\alpha_0^2$:

$$G(\zeta) = w\sqrt{\zeta} = \sqrt{\left(\frac{1}{2}\pi\,\varrho\right)}\, e^{i\pi/4 - S_0} \sqrt{\frac{\zeta\,\xi}{d\,\xi/d\,\zeta}}\, H_1^{(1)}(i\,\xi). \tag{20.15}$$

If ζ is so small that $\xi = \varrho\alpha_0\zeta \ll 1$, then in calculating the field components we can use the expansion of the function $H_1^{(1)}(i\,\xi)$ as a power series, taking only the leading terms:

$$H_1^{(1)}(i\,\xi) \approx -\frac{2}{\pi\,\xi} - \frac{\xi}{\pi} \ln \xi. \tag{20.16}$$

It is now easy to see that the amplitudes of the field components behave as follows. The amplitude of H_x tends to a constant value as $\zeta \to 0$ [see (20.1)]:

$$|H_x^{(0)}| = |G(0)| \approx 2\,|A|/\pi\,\varrho\,\alpha_0 = \sqrt{(2/\pi\,\varrho)}\,e^{-S_0}/\alpha_0. \tag{20.17}$$

The horizontal component E_y of the electric field as $\zeta \to 0$ is given by (20.1):

$$\left.\begin{aligned} |E_y| &\approx 2\alpha_0\,|A\ln\xi|/\pi, \quad \xi = \varrho\,\alpha_0\,\zeta, \\ \varrho &= \omega/c\,a, \quad a = |d\,\varepsilon/d\,z|; \end{aligned}\right\} \tag{20.18}$$

thus this component has a logarithmic singularity. Finally, the vertical component E_z of the electric field becomes infinite [see (20.4)]:

$$|E_z| \approx \alpha_0\,|G(0)|/|\zeta| = \alpha_0\,|G(0)|/|a\,z + i\,\nu_{\text{eff}}/\omega|. \tag{20.19}$$

Using the expression (20.17) for $|G(0)|$, we have finally

$$\left.\begin{aligned} |E_z| &\approx \sqrt{(2/\pi\,\varrho)}\,e^{-S_0}/|\zeta|, \\ S_0 &= \frac{2}{3}\varrho\,\alpha_0^3 = \frac{2}{3}(\omega/c\,a)\,\alpha_0^3. \end{aligned}\right\} \tag{20.20}$$

For $z = 0$ the field $|E_z|$ takes its maximum value (in a medium with absorption, $\zeta = az + is$)

$$|E_z|_{z=0} = \alpha_0\,|G(0)|/s = \sqrt{(2/\pi\,\varrho)}e^{-S_0}/s. \tag{20.21}$$

It may be recalled that the above formulae are valid only for relatively large angles of incidenee: for $\alpha_0 = \sin\theta_0 \to 0$, formula (20.17) gives a result which is clearly incorrect, since for $\alpha_0 = 0$ (normal incidence) a rigorous solution of the problem shows that $E_z = 0$.

For the upper layers of the Earth's ionosphere, where $\varrho \gg 1$, the approximate formulae (20.17), (20.18), (20.20) and (20.21) are valid up to angles of incidence θ_0 of 4 or 5 degrees, and it is easily seen that the increase in the field near the point $\zeta = 0$ ($n = 0$) is then negligible ($S_0 \gg 1$) even if we neglect the effect of the Earth's magnetic field, which is discussed in § 27. (The electric field can be large only for very small values of ν_{eff}, whereas in the ionosphere $\nu_{\text{eff}} \gtrsim 10^3$ up to the maximum of the F layer.) The presence of a singularity at the point where $\varepsilon' = 0$ does not affect the behaviour of the field in the region below the reflection point, i. e. the reflection of a wave whose E_z component is not zero is in these conditions the same as that of a wave in which the electric vector is perpendicular to the plane of incidence.†

† The whole problem is here considered for the steady-state case, and the question of the time taken to establish the resulting field distribution therefore remains open. From physical arguments, however, it is clear that, when the point $\varepsilon = 0$ is considerably removed from the "reflection point" $\varepsilon = \sin^2\theta_0$, the time to establish the field distribution shown in Fig. 20.1b may be very long. The reason is that the wave can "penetrate" only slowly into the region $\varepsilon \approx 0$, and some time is required for sufficient energy to enter that region. At previous instants the field near the point $\varepsilon = 0$ will be less than its stationary value, and initially the picture will be nearer to that shown in Fig. 20.1a than to the steady state shown in Fig. 20.1b.

Formula (20.19) shows that the value of the field at the point where $\varepsilon = 0$ (i.e. at $z = 0$) is determined not only by ν_{eff} but also by the values of $\alpha_0 G(0)$. For normal incidence, when $\alpha_0 = 0$, the field is $E_z = 0$; when S_0 is large, (20.20) shows that the field E_z decreases with increasing α_0. Consequently, as already noted, for some small angle of incidence the field increase at the point $\varepsilon = 0$ will be greatest. In this connection it is of interest to consider the behaviour of the function $\alpha_0 G(0, \alpha_0)$ for all angles of incidence. An examination of solutions of equation (20.3) shows [132] that the function $\alpha_0 |G(0, \alpha_0)|$ can be approximately represented for all values of the parameter α_0 by

$$\alpha_0 |G(0, \alpha_0)| = \frac{4\tau\, v(\tau^2)}{\sqrt{(2\pi\varrho)}} \sqrt{\frac{v(\tau^2)}{-v'(\tau^2)}} = \Phi(\tau)/\sqrt{(2\pi\varrho)}, \qquad (20.22)$$

where v and v' are the Airy function and its derivative (see [126]) and

$$\tau = \varrho^{\frac{1}{3}} \alpha_0 = (\omega/ca)^{\frac{1}{3}} \sin\theta_0. \qquad (20.23)$$

The dependence of the maximum value of $|E_z|$ on the angle of incidence is thus given by the function $\Phi(\tau)$, which is shown in Fig. 20.3. The scale of angles of incidence is also shown for $a = 10^{-7}$ cm^{-1} and $\omega = 2\pi \times 10^7$ sec^{-1} ($\lambda_0 = 30$ m).

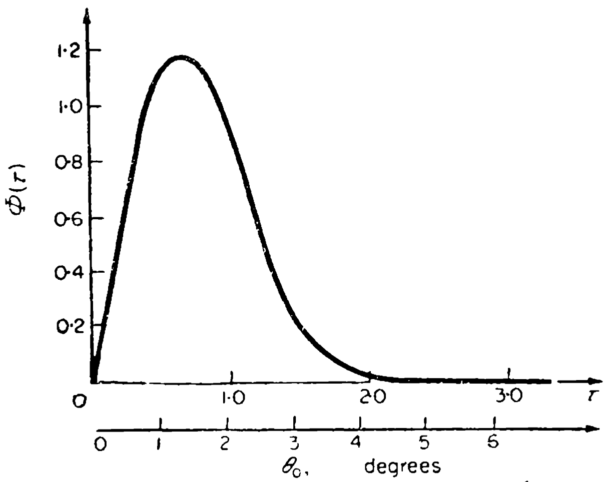

FIG. 20.3. The function $\Phi(\tau) = \sqrt{(2\pi\varrho)}\,|\alpha_0 G(0, \alpha_0)|$; $\tau = \varrho^{\frac{1}{3}}\alpha_0 = (\omega/c\,a)^{\frac{1}{3}} \sin\theta_0$.

An important feature is that the function $\Phi(\tau)$ becomes of the order of unity only for a very narrow range of angles of incidence. Its maximum value is 1·2 and, in this example, corresponds to an angle $\theta_0 = 1{\cdot}5°$. Even for $\theta_0 = 5°$ we have $\Phi(\tau) \sim 10^{-4}$.

On the basis of formulae (20.19) and (20.22) we may estimate the values which the field E_z can attain in various conditions. The maximum value of $|E_z|$ is $1{\cdot}2\omega/\nu_{\text{eff}}\sqrt{(2\pi\varrho)} = 1{\cdot}2(\omega/\nu_{\text{eff}})\sqrt{(ca/2\pi\omega)}$. In the E layer of the ionosphere, therefore, where we can put $a \sim 10^{-6}$, for $\lambda_0 = 100$ m ($\omega = 6\pi \times 10^6$) we have $|E_z|_{z=0} \approx 3{\cdot}6$ for $\nu_{\text{eff}} = 10^5$ and ≈ 36 for $\nu_{\text{eff}} = 10^4$. In the F layer ($a = 10^{-7}$, $\omega = 2\pi \times 10^7$, $\lambda_0 = 30$ m) we have $|E_z|_{z=0} \approx 20$ for $\nu_{\text{eff}} = 10^4$ and ≈ 200 for $\nu_{\text{eff}} = 10^3$. In the solar corona, with $a = |d\varepsilon/dz| = 10^{-10}$,

$\omega = 2\pi \times 10^8 (\lambda_0 = 3\text{ m})$ and $\nu_{\text{eff}} = 10$, the maximum value is $|E_z|_{z=0} = 2000$. However, it must be emphasised that in these estimates we have neglected the effect of the magnetic field, and this is certainly not permissible for the Earth's ionosphere (see below and § 27). It should also be mentioned that by hypothesis $|\boldsymbol{E}| = 1$ and $|E_z| = |\boldsymbol{E}| \sin\theta_0 = \alpha_0$ at the lower boundary of the inhomogeneous layer.

The effective dimension of the region where the field is large is also of interest. From formula (20.19) we easily find that $|E_z|^2$ falls to half its maximum value at a distance

$$\Delta z = \nu_{\text{eff}}/\omega a = \lambda_0 \nu_{\text{eff}}/2\pi c a \tag{20.24}$$

from the point $z = 0$ where the field is a maximum. For $\nu_{\text{eff}} \sim 10^4$ and $a \sim 10^{-7}$ this gives $\Delta z \sim \lambda_0$.

The allowance for spatial dispersion

The rapid increase of the electric field near the point where $\varepsilon(0, z) = 0$ may make it necessary to take account of the spatial dispersion, i.e. it may not be possible to use only the local properties $\varepsilon'(\omega, z)$ of the medium.

In the absence of collisions, spatial dispersion may be neglected if the path $2\pi v/\omega$ traversed by a particle in one period is small compared with the characteristic dimension of the inhomogeneity of the field. In the present problem, in the absence of collisions and using $\varepsilon(z)$, the field $\boldsymbol{E}$ would become infinite at the point where $\varepsilon = 0$. In a sufficiently strong field, however, the electron acquires a large velocity v, so that the quantity $2\pi v/\omega$ increases and the use of the local permittivity $\varepsilon(\omega)$ is no longer admissible. As a result of induction, $\boldsymbol{D}(z)$ differs from $\varepsilon \boldsymbol{E}(z)$ and at the point where $\varepsilon = 0$ $\boldsymbol{D}(z)$ is not zero, but is determined by the field $\boldsymbol{E}$ at a point whose distance is of the order of the amplitude of the oscillations of the electron, i.e. $\sim v/\omega$. Hence, as noted in [131], the denominator in formulae such as (20.19) for $z = 0$ (i.e. $\varepsilon = 0$) is not zero but

$$a \int_0^t v_z \, dt, \tag{20.24a}$$

where v_z is the velocity of the electron in the z-direction, and $t = 0$ is the time when the field is zero. In a strong field the velocity v_z considerably exceeds the velocity of the thermal motion $v \sim \sqrt{(\varkappa T/m)} \sim 10^7$ cm/sec (for $T \sim 300°$K) and is of the order of $eE_z/m\omega$. Hence the expression (20.24a) is of the order of $aeE_z/m\omega^2 \sim 10^{-5} E_z$ (for $a \sim 10^{-7}$ and $\omega = 2\pi \times 10^7$). Since this function of E_z appears in the denominator of expressions such as (20.19) for E_z, the problem is evidently no longer linear. This is reasonable, since the motion of an electron in a field which is harmonic in time but varies rapidly in space is not harmonic.

The non-linear effect just mentioned becomes important in an actual medium if the integral (20.24a) is comparable with ν_{eff}/ω, which appears in the denominator of (20.19) when absorption is taken into account. In the F layer, as we have seen, $\nu_{\text{eff}}/\omega \sim 10^{-4}$, while (20.24a) is $\sim 10^{-5} E_z \sim 10^{-4}$ only for $E_z \sim 10 = 3000$ V/cm. Under these conditions the field at $\varepsilon = 0$ is only 20 times that at the boundary of the layer (see above), and so the non-linear effect would have to be taken into account only in very strong fields, which do not occur in the ionosphere (in such fields, moreover, the linear treatment of radio-wave propagation in plasmas becomes invalid not only near $\varepsilon = 0$ but throughout the layer). Thus, when absorption and the possibility of plasma-wave generation (see below) are taken into account, it is not in general necessary to allow for non-linear effects.

In the problem of the field singularity at the point where $\varepsilon = 0$ there is another complication which is also due to spatial dispersion. When absorption is taken into account but spatial dispersion is to be neglected, the condition derived above that the path $2\pi v/\omega$ should be small is insufficient. This point is of general significance, and we shall therefore consider it in some detail.

In a plasma, neglecting spatial dispersion is equivalent to omitting from the Boltzmann equation the term which involves spatial derivatives. Let us suppose, for example, that the equation†

$$\frac{\partial \varphi}{\partial t} + v_z \frac{\partial \varphi}{\partial z} + \frac{e}{m} E_x \frac{\partial f_{00}}{\partial v_x} + \nu(v)\, \varphi = 0 \tag{20.25}$$

can be used to determine the quantity φ, the deviation of the distribution function from the equilibrium function f_{00}. Then the omission of the term $v_z\, \partial\varphi/\partial z$ gives a local relation between $\boldsymbol{D}$ and $\boldsymbol{j}$ and $\boldsymbol{E}$ (see § 6), i.e. is in fact equivalent to neglecting spatial dispersion. The legitimacy of this procedure can be tested by calculating the term $v_z\, \partial\varphi/\partial z$ from the solution obtained when this term is neglected, i.e.

$$\varphi = -\frac{(e/m)\, E_x\, \partial f_{00}/\partial v_x}{i\,\omega + \nu(v)},$$

where the field E_x is assumed monochromatic. For a wave propagated in the z-direction in a homogeneous medium we have

$$E_x = E_{x0}\, e^{-i\omega\sqrt{\varepsilon'}\, z/c}$$

and

$$v_z\, \partial\varphi/\partial z = -i\,\omega \sqrt{\varepsilon'}\, v_z\, \varphi/c = -(i\,n + \varkappa)\,\omega v_z\, \varphi/c\,.$$

In order that this term should be negligible, it must evidently be small compared with the term $(i\omega + \nu)\,\varphi$ retained in (20.25). Comparing the real and

† Equation (20.25) is obtained from (8.14) by taking account of collisions and assuming that $E_y = E_z = 0$ and that φ and E_x depend only on z.

imaginary parts separately, we therefore have the conditions

$$v\,n/c \ll 1 \quad (\text{or} \quad v/\omega \ll c/\omega\, n = \lambda_0/2\pi\, n = \lambda/2\pi), \tag{20.26}$$

$$v\,\omega\,\varkappa/c \ll \nu_{\text{eff}} \quad (\text{or} \quad l = v/\nu_{\text{eff}} \ll c/\omega\,\varkappa); \tag{20.27}$$

here the collision frequency $\nu(v)$ is replaced by ν_{eff}, and the velocity component by the velocity v characteristic of the problem [for a Maxwellian plasma in a weak field, $v \sim v_T = \sqrt{(\varkappa T/m)}$].

The condition (20.26) is always fulfilled for the transverse waves here considered, in complete agreement with the results of § 8. The condition (20.27) was not discussed in § 8, because absorption was there neglected; it signifies that the free path l must be small compared with the distance $c/\omega\varkappa$ over which the wave amplitude varies appreciably owing to absorption.†

For waves propagated in an inhomogeneous medium, it must be strongly emphasised, the conditions for spatial dispersion to be negligible are more stringent than for a homogeneous medium. This is simply because the amplitude of the field in an inhomogeneous medium may vary considerably even when absorption is absent, i.e. over distances less than $c/\omega\varkappa$. If the wave amplitude varies over a characteristic distance L_a and the phase over L_φ, the conditions (20.26) and (20.27) evidently become

$$v/\omega \ll L_\varphi, \quad l \ll L_a \quad (\text{or} \quad \nu_{\text{eff}} \gg v/L_a). \tag{20.28}$$

In the problem of the singularity of the field at $\varepsilon = 0$ the field amplitude E_z varies considerably over a distance of the order of $\Delta z = \lambda_0 \nu_{\text{eff}}/2\pi c a$ [see (20.24)]. Hence the second condition (20.28) becomes

$$\nu_{\text{eff}} \gg v/\Delta z = 2\pi v c a/\lambda_0 \nu_{\text{eff}},$$

or

$$\nu_{\text{eff}}^2 \gg 2\pi c v a/\lambda_0 = \omega v a. \tag{20.29}$$

For $a = |d\varepsilon/dz| \sim 10^{-7}$, $\omega = 2\pi \times 10^7$ and $v \sim \sqrt{(\varkappa T/m)} \sim 10^7$, this condition becomes $\nu_{\text{eff}} \gg 10^4$. The latter inequality does not hold in the F layer of the ionosphere, and so the Boltzmann equation ought to be used for a rigorous calculation of the field near the point where $\varepsilon = 0$. This is not necessary, however, for applications to the ionosphere. The reason is that we have neglected hitherto the effect of the Earth's magnetic field, and so have been able to regard the ionosphere plasma as isotropic. As we shall see in § 27, the picture is considerably changed when the effect of the magnetic field is taken into account. Consequently, when the ionosphere has been mentioned above in connection with numerical estimates, this was purely conventional and was done only in order to choose parameters of the medium. We shall proceed similarly below in discussing another effect which causes the field at $\varepsilon = 0$ to remain finite.

† For metals the conditions (20.26) and (20.27) correspond to the validity of the theory of the normal skin effect (see, for example, [135]).

Allowance for the generation of plasma waves. The interaction between different normal waves

This effect is as follows. Near the point where $\varepsilon = 0$ a wave incident obliquely on the layer, with the electric vector $\boldsymbol{E}$ in the plane of incidence, is partly converted into a plasma wave. In other words, in this case we cannot neglect the possibility of generation of plasma waves, simply because at the point where $\varepsilon = 1 - 4\pi e^2 N/m\omega^2 = 0$ (absorption being neglected) the wave frequency ω is precisely equal to the frequency of plasma oscillations $\omega_0 = \sqrt{(4\pi e^2 N/m)}$. Hence it may be supposed that the characteristic behaviour of the vertical component E_z near the point $\varepsilon = 0$ is related to the resonance properties of the plasma. The function which gives the dependence of $|E_z|^2$ on z is then of the resonance type (see Figs. 20.1b and 20.2), and near the maximum ($\varepsilon = 0, z = 0$) it has the form $|E_z|^2 = \text{constant} \div [(az)^2 + (\nu_{\text{eff}}/\omega)^2]$ [see (20.19)]. The physical explanation of the appearance of plasma waves when transverse waves are incident on the layer from a vacuum is very simple. In an inhomogeneous medium the incident wave is in general no longer purely transverse, since for a plane-parallel medium we have [see (19.19)]

$$\operatorname{div} \boldsymbol{E} = -(1/\varepsilon')\, \boldsymbol{E} \cdot \operatorname{grad} \varepsilon' = -\frac{E_z\, d\varepsilon'/dz}{\varepsilon'}. \tag{20.30}$$

Evidently $\operatorname{div} \boldsymbol{E} \neq 0$ for precisely the case under consideration, since for normal incidence or for a wave with the vector $\boldsymbol{E}$ perpendicular to the plane of incidence $E_z = 0$ and $\operatorname{div} \boldsymbol{E} = 0$. Next, in a plasma $\operatorname{div} \boldsymbol{E} = 4\pi\bar{\varrho}$, where $\bar{\varrho} = e\Delta N$ and ΔN is the deviation of the electron density from the equilibrium value (the motion of the ions is neglected). Hence, when $E_z \neq 0$ and $d\varepsilon'/dz \neq 0$, charges $\bar{\varrho}$ appear in the wave, and for $\varepsilon = 0$ the density of these charges fluctuates with the plasma frequency ω_0. Thus a wave with $E_z \neq 0$ incident on the layer causes plasma oscillations whose amplitude increases as we approach the resonance point $\varepsilon = 0$. These local oscillations are not independent, because any change in electron density in one part of the medium is transmitted to a neighbouring region by the thermal motion, and in this way plasma waves are generated which carry with them some part of the energy of the standing electromagnetic wave. Ultimately the energy of the plasma waves goes to heat the plasma by dissipation in collisions.

Thus, when the problem is sufficiently general, it is necessary to allow for the possible occurrence of plasma waves, and this leads to the disappearance of the singularity in the solution, the field remaining finite even at the point of resonance. The analysis can be carried out to a first approximation by the quasihydrodynamic method described in §§ 8 and 13. For convenience we shall repeat here the initial equations for the case where collisions are neglected:

$$\left.\begin{aligned}
&m N \ddot{\boldsymbol{r}} = -\varkappa T \operatorname{grad} N + e N \boldsymbol{E},\\
&\partial N/\partial t + \operatorname{div}(N\dot{\boldsymbol{r}}) = 0,\\
&\operatorname{curl} \boldsymbol{H} = i(\omega/c)(\boldsymbol{E} + 4\pi \boldsymbol{P}),\\
&\qquad \boldsymbol{j} = 0, \quad \boldsymbol{P} = e N \boldsymbol{r},\\
&\qquad \boldsymbol{j}_t = \boldsymbol{j} + i\omega \boldsymbol{P} = i\omega \boldsymbol{P},\\
&\operatorname{div}(\boldsymbol{E} + 4\pi \boldsymbol{P}) = 0,\\
&\operatorname{curl} \boldsymbol{E} = -i\omega \boldsymbol{H}/c,\\
&\operatorname{div} \boldsymbol{H} = 0.
\end{aligned}\right\} \tag{20.31}$$

From this we evidently have, for oscillations harmonic in time and after linearisation,

$$\boldsymbol{P} = (e\varkappa T/m\omega^2) \operatorname{grad} N' - (e^2 N/m\omega^2)\boldsymbol{E},$$

$$eN' + \operatorname{div} \boldsymbol{P} = 0, \quad \operatorname{div} \boldsymbol{P} = -\operatorname{div} \boldsymbol{E}/4\pi,$$

where N' is the deviation of the electron density from the equilibrium value N; and hence, when all quantities are independent of x,

$$\left.\begin{aligned}
P_x &= -e^2 N E_x/m\omega^2,\\
P_y &= \frac{\varkappa T}{4\pi m \omega^2}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right) - \frac{e^2 N}{m\omega^2} E_y,\\
P_z &= \frac{\varkappa T}{4\pi m \omega^2}\left(\frac{\partial^2 E_y}{\partial y\,\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right) - \frac{e^2 N}{m\omega^2} E_z.
\end{aligned}\right\} \tag{20.32}$$

The equation $\operatorname{curl} \boldsymbol{H} = i(\omega/c)(\boldsymbol{E} + 4\pi\boldsymbol{P})$ with $\partial H_{y,z}/\partial x = 0$ gives

$$\left.\begin{aligned}
&\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\frac{\omega}{c}(E_x + 4\pi P_x),\\
&\partial H_x/\partial z = i(\omega/c)(E_y + 4\pi P_y),\\
&\partial H_x/\partial y = -i(\omega/c)(E_z + 4\pi P_z).
\end{aligned}\right\} \tag{20.33}$$

For a wave with the vector $\boldsymbol{E}$ perpendicular to the plane of incidence ($E_x \neq 0$, $E_y = E_z = 0$, $H_x = 0$, $H_y \neq 0$, $H_z \neq 0$), (20.32) shows that taking account of the electron pressure makes no difference, but for the wave with $H_x \neq 0$, $H_y = H_z = 0$, $E_x = 0$, $E_y \neq 0$, $E_z \neq 0$ we obtain from (20.32), (20.33) and the field equation $\operatorname{curl} \boldsymbol{E} = -i\omega\boldsymbol{H}/c$

$$\left.\begin{aligned}
&-i\frac{\omega}{c} H_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z},\\
&\frac{\partial H_x}{\partial y} = -i\frac{\omega}{c}\left\{\varepsilon E_z + \frac{\beta_T^2}{\omega^2/c^2}\left(\frac{\partial^2 E_y}{\partial y\,\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right)\right\},\\
&\frac{\partial H_x}{\partial z} = i\frac{\omega}{c}\left\{\varepsilon E_y + \frac{\beta_T^2}{\omega^2/c^2}\left(\frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y\,\partial z}\right)\right\},
\end{aligned}\right\} \tag{20.34}$$

where $\varepsilon = 1 - 4\pi e^2 N/m\omega^2$ is the ordinary permittivity of the plasma and $\beta_T^2 = \varkappa T/mc^2$.

When $\beta_T \to 0$, equations (20.34) are equivalent to those examined previously (19.2) with $\varepsilon' = \varepsilon$. Since in a non-relativistic plasma $\beta_T^2 \ll 1$ (e.g. in the Earth's ionosphere $\beta_T^2 \sim 10^{-7}$) the allowance for the thermal motion (i.e. for spatial dispersion) can be important only in exceptional cases. These include that of the field singularity where $\varepsilon = 0$.

We shall seek a solution of equations (20.34) in the form $H_x = G(z)\, e^{-i\omega\alpha_0 y/c}$, $E_z = F_z(z)\, e^{-i\omega\alpha_0 y/c}$, $E_y = F_y(z)\, e^{-i\omega\alpha_0 y/c}$, where $\alpha_0 = \sin\theta_0$. Then, eliminating $F_y(z)$, we obtain two coupled second-order equations for G and F_z:

$$\left.\begin{aligned} \frac{d^2 G}{dz^2} - \frac{1}{\varepsilon - \beta_T^2\alpha_0^2}\frac{d\varepsilon}{dz}\frac{dG}{dz} + \frac{\omega^2}{c^2}(\varepsilon - \alpha_0^2)\, G \\ = -\frac{\beta_T^2\alpha_0}{\varepsilon - \beta_T^2\alpha_0^2}\frac{d\varepsilon}{dz}\frac{dF_z}{dz}, \\ \beta_T^2\frac{d^2 F_z}{dz^2} + \frac{\omega^2}{c^2}(\varepsilon - \beta_T^2\alpha_0^2)\, F_z = -\frac{\alpha_0\omega^2}{c^2}(\beta_T^2 - 1)\, G. \end{aligned}\right\} \tag{20.35}$$

When $\beta_T = 0$ the first of these equations naturally gives (20.1) with $\varepsilon' = \varepsilon$; in (20.35) absorption is assumed absent, and it may be noted that the result is more accurate when β_T^2 is replaced by $3\beta_T^2$ (see § 8).

Thus the allowance for the thermal motion of the electrons leads to equations of higher order. The solutions of (20.35) correspond to normal waves of two types, which allow a representation of the wave field as a superposition of electromagnetic waves and plasma waves only in certain particular cases, or outside the region where ε is small. For example, if $\alpha_0 = \sin\theta_0 = 0$ (normal incidence), equations (20.35) become independent. The first of these is the same as (20.1) with $\varepsilon' = \varepsilon$ and $\alpha_0 = 0$, and its solutions correspond to electromagnetic waves. The second equation becomes the equation for plasma waves in the quasihydrodynamic approximation:

$$\frac{d^2 F_z}{dz^2} + \frac{\omega^2}{c^2}\frac{\varepsilon(z)}{\beta_T^2} F_z = 0. \tag{20.36}$$

For a homogeneous medium this gives for the refractive index n_3 of plasma waves, in accordance with (8.9), the expression

$$n_3^2 = \varepsilon/\beta_T^2 = \frac{1 - \omega_0^2/\omega^2}{\varkappa T/mc^2}.$$

For oblique incidence ($\alpha_0 \neq 0$) the field cannot, strictly speaking, be divided into electromagnetic and plasma waves. If the normal solution which is damped in the region of negative $\varepsilon(z)$ is chosen, its asymptotic behaviour will therefore represent electromagnetic waves (incident and reflected) and a plasma wave (reflected) below some point of "interaction", where $\varepsilon = \beta_T^2\alpha_0^2$.

It is easy to see that, if $\varepsilon(z)$ has no singular points and $\beta_T^2 \neq 0$, the solutions of equations (20.35) will be analytic functions in the region of interest to us. The singularity of the solution at the point where $\varepsilon(z) = 0$ appears only when the small parameter β_T^2, which is the coefficient of the highest derivative in the fourth-order equation which is equivalent to (20.35), tends to zero.

Thus the allowance for the thermal motion of the electrons in fact removes the singularities of the electromagnetic field. As already mentioned, this occurs because an electromagnetic wave in the resonance region incident on the layer causes a plasma wave whose energy is then converted by collisions (if present) into that of the thermal motion of the electrons. This mechanism of energy dissipation naturally results in a finite value of the energy density in the resonance region.†

If $\alpha_0 = \sin\theta_0$ is not small, the interaction between the electromagnetic and plasma waves is only slight. On solving the equations (20.35) by successive approximations, we can show that the order of magnitude of E_z at the resonance point is given by

$$|E_z|_{z=0} \sim \frac{\alpha_0\,|G(0,\alpha_0)|}{(\beta_T/\varrho)^{\frac{2}{3}}}. \tag{20.36a}$$

The same result can be obtained by the following simple arguments, and we shall not go further than these. The equation of motion of electrons under the action of the component E_z, taking account of collisions and of the pressure gradient, is

$$-\omega^2 N m r + i\omega\nu_{\text{eff}} N m r = eNE_z - \varkappa T\,\partial N/\partial z; \tag{20.37}$$

here the motion has for simplicity already been supposed harmonic, with all quantities proportional to $e^{i\omega t}$, and evidently $r = v/i\omega$ is the displacement of the electron. Using the equation of continuity

$$\partial N/\partial t + \operatorname{div}(N\dot{\boldsymbol{r}}) = 0$$

and putting $\partial N/\partial z \approx kN'$, $\partial N/\partial t = i\omega N'$, $\operatorname{div}(N\dot{\boldsymbol{r}}) \approx i\omega k N|\boldsymbol{r}|$, where N' is the small deviation of the electron density N from its equilibrium value

† The quasihydrodynamic method used here cannot, of course, lay claim to quantitative accuracy. There are, however, reasons for supposing that, at least in an isotropic plasma, this method gives a correct qualitative description and leads to formulae which are correct to within factors of the order of unity. Nevertheless, the problem of the singularity of the field at $\varepsilon = 0$ certainly ought to be examined by the Boltzmann-equation method also. This is really necessary not only when collisions are neglected and the occurrence of plasma waves is taken into account, but also when ν_{eff} is sufficiently small, in which case spatial dispersion cannot be neglected in the region $\varepsilon \approx 0$ (see above).

and $1/k$ a characteristic dimension of the wave field, we can write equation (20.37) as

$$-\omega^2 N m r + i\omega \nu_{\text{eff}} N m r = e N E_z - \varkappa T k^2 N r. \tag{20.38}$$

This shows that the allowance for the pressure gradient (the term $\varkappa T k^2 N r$) is akin to the allowance for collisions (the term $i\omega \nu_{\text{eff}} N m r$). It is therefore reasonable to suppose that, in order to take account of the pressure gradient (i.e. plasma waves), neglecting collisions, we should replace ν_{eff}/ω by $\varkappa T k^2/m\omega^2$ in the formula previously obtained when allowing for collisions:

$$|E_z| \approx \frac{\alpha_0 |G(0, \alpha_0)|}{|a z + i\, \nu_{\text{eff}}/\omega|}. \tag{20.39}$$

The characteristic dimension $1/k$ may be taken to be the distance Δz over which $|E_z|^2$ decreases by a factor of, say, two. This decrease will evidently be obtained if $a\Delta z \sim a/k \sim \varkappa T k^2/m\omega^2$; see (20.19), where ν_{eff}/ω is replaced by $\varkappa T k^2/m\omega^2$. Hence $k \sim (m\omega^2 a/\varkappa T)^{\frac{1}{3}}$, and the value of the field E_z at the resonance point $z = 0$ is given in order of magnitude by

$$|E_z|_{z=0} \sim \frac{\alpha_0 |G(0, \alpha_0)|}{\varkappa T k^2/m\,\omega^2} \sim \frac{\alpha_0 |G(0, \alpha_0)|}{(\beta_T/\varrho)^{\frac{2}{3}}},$$

which is exactly the same as (20.36a).

Thus the effect of plasma waves in our problem may be compared to that of absorption by using an effective collision frequency defined by

$$\left.\begin{aligned} \nu_{\text{eq}}/\omega &= (\beta_T/\varrho)^{\frac{2}{3}} \\ &= (\varkappa\, T/m)^{\frac{1}{3}} (a/\omega)^{\frac{2}{3}}, \\ a &= |d\,\varepsilon/d\,z|. \end{aligned}\right\} \tag{20.40}$$

For the E layer ($\beta_T \sim 2 \times 10^{-4}$, $a = 10^{-6}\,\text{cm}^{-1}$, $\lambda_0 = 100$ m), $\nu_{\text{eq}} \sim 10^3\,\text{sec}^{-1}$; for the F layer ($\beta_T \sim 2$ to 4×10^{-4}, $\lambda_0 = 30$ m), $\nu_{\text{eq}} \sim 4 \times 10^2$ for $a = 10^{-7}$ and $\nu_{\text{eq}} \sim 2 \times 10^3$ for $a = 10^{-6}$.

These figures show that the allowance for the effect of plasma waves might sometimes ($a \sim 10^{-6}$) be as important as the allowance for collisions (with $\nu_{\text{eff}} \sim 10^3$). In the ionosphere, however, even ignoring the influence of the Earth's magnetic field (see § 27), absorption would still play the dominant part.

This effect of interaction between the transverse and plasma waves is, perhaps, of still greater interest when the problem is differently stated: it leads to the possibility of the transformation of plasma waves into electromagnetic waves in an inhomogeneous medium, which is important in the solar corona, for example [133].

The mutual transformation of an interaction between longitudinal and transverse waves in a plasma

Hitherto we have discussed the regular transformation of waves. If there are in a plasma sufficiently small random inhomogeneities of the electron density, the scattering of plasma waves by these inhomogeneities and their conversion into electromagnetic waves and *vice versa* occurs even outside the region where $\varepsilon \approx 0$. The mechanism of this scattering is that, under the action of the field of the incident (plasma or electromagnetic) wave, the plasma inhomogeneity is polarised, i.e. acquires an induced dipole moment; we assume that the inhomogeneity is small in comparison with the wavelength, so that the higher multipole moments are unimportant. Such a dipole in the plasma emits in general both electromagnetic and plasma waves. The mechanism of the transformation of plasma waves into electromagnetic waves by scattering seems to be of importance in the solar corona; see [136] and § 36.

We have used the term "interaction" to describe the transformation of transverse waves into plasma waves and *vice versa*. To avoid misunderstanding it should be emphasised that we refer here, of course, not to the interaction due to the non-linearity of the plasma but to an entirely different concept.

Normal waves in an inhomogeneous plasma are, in general, approximately the same as those in a homogeneous medium only in the approximation of geometrical optics. In cases where this approximation is invalid, wave propagation in an inhomogeneous medium differs considerably from that which occurs in a homogeneous or quasihomogeneous medium. This difference can, if the problem is appropriately formulated, be related to or even reduced to the interaction of waves which are normal in regions where the approximation of geometrical optics is valid. Such an "interaction" has been mentioned above in connection with an inhomogeneous isotropic plasma where the electromagnetic (transverse) and plasma (longitudinal) waves are practically normal everywhere outside the immediate neighbourhood of the point $\varepsilon(\omega, z) = 0$. Near this point, for a wave with $E_z \neq 0$, the divergence of $\boldsymbol{E}$ is non-zero and increases with decreasing ε [see (20.30)]. At the same time the frequency of the incident wave $\omega \approx \omega_0$ and even an approximate separation of the waves into transverse and longitudinal components may here be quite inadmissible. For waves far from the point $\varepsilon(\omega, z) = 0$, the existence of a region where the approximation of geometrical optics is invalid is equivalent to the interaction of the different types of normal wave near the point $\varepsilon = 0$ (in the "interaction region"). In this sense, for example, the occurrence of reflection from the layer at normal incidence may likewise be regarded as a result of the "interaction" of waves travelling in opposite directions in the neighbourhood of the "reflection point" $\varepsilon = 0$. We shall meet this conception of the interaction of waves in an inhomogeneous medium again in Chapter V, for the case of an inhomogeneous magnetoactive plasma.

§21. THE PROPAGATION OF PULSE SIGNALS

The Fourier representation of a pulse field

Up to this point we have discussed the propagation and reflection of waves of a single frequency (monochromatic waves). In practice, however, we often have to deal with pulses or signals formed by a group of waves. In the linear theory this case must be treated by resolving the wave field into a Fourier integral.

We represent the incident plane wave at the boundary of the layer as

$$E_0(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega : \tag{21.1}$$

by Fourier's theorem

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t)\, e^{-i\omega t}\, dt. \tag{21.2}$$

For a monochromatic wave of frequency ω_0 we have

$$E_0 = e^{i\omega_0 t} \quad \text{and} \quad g(\omega) = \delta(\omega - \omega_0),$$

where δ is the delta function:

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, d\omega = 1 \quad \text{and} \quad \delta(\omega - \omega_0) = 0 \quad \text{for} \quad \omega \neq \omega_0.$$

For a wave packet, i.e. a quasimonochromatic group of waves, the function $g(\omega)$ is by definition very "sharp" and appreciably different from zero only in a narrow range of frequencies near the carrier frequency ω_0 of the signal; in other words, the spectral width $\Delta\omega$ of the signal satisfies the inequality

$$\Delta\omega \ll \omega_0. \tag{21.3}$$

The field $E_0(t)$ in a quasimonochromatic group may conveniently be represented in the form

$$E_0(t) = A(t)\, e^{i\omega_0 t}, \tag{21.4}$$

where $A(t)$ is a slowly varying function of t (except at certain points where $A(t)$ may vary rapidly). In the simplest case of a "truncated sine wave" we have

$$\left.\begin{aligned} A(t) &= 1 \quad \text{for} \quad -\tfrac{1}{2}T < t < \tfrac{1}{2}T, \\ A(t) &= 0 \quad \text{for} \quad t > \tfrac{1}{2}T \quad \text{and} \quad t < -\tfrac{1}{2}T, \end{aligned}\right\} \tag{21.5}$$

i.e. $E_0 = e^{i\omega_0 t}$ in the range $-\frac{1}{2}T < t < \frac{1}{2}T$ and $E_0 = 0$ outside this range. From (21.2) we immediately find $g(\omega)$ for the field given by (21.4) and (21.5):

$$g(\omega) = \frac{\sin\frac{1}{2}(\omega - \omega_0)\, T}{\pi(\omega - \omega_0)}. \tag{21.6}$$

In this case we evidently have $\Delta\omega \sim \pi/T$, and the inequality (21.3) holds if the interval T includes many periods $T_0 = 2\pi/\omega_0$, i.e. if $T \gg T_0$.

After reflection of the pulse from a layer, or more generally after the pulse has traversed any path, the field may be written

$$E(t) = \int_{-\infty}^{\infty} R(\omega)\, g(\omega)\, e^{i\omega t - i\varphi(\omega)}\, d\omega, \tag{21.7}$$

where $R(\omega) \equiv |R(\omega)|$ is the amplitude coefficient of reflection (the amplitude attenuation factor) and $\varphi(\omega)$ is the phase shift for a monochromatic wave of frequency ω.

In (21.1) and (21.7) only the dependence of the field on t is explicitly shown, since the waves are assumed to be plane waves depending only on one coordinate (z, say) in addition to t. At each point z we ask what is the form of the pulse in time. Naturally we could equally well consider the form of the pulse in space for a given t, i.e. the dependence of E on z.

The linearity of the field equations has already been used in (21.7), since the waves of different frequencies have been assumed to be propagated independently, i.e. $R(\omega)$ and $\varphi(\omega)$ in (21.7) do not depend on the nature of the pulse.

According to (21.1)–(21.2) and (21.4) the fields E_0 and E can be written

$$E_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\eta)\, e^{i[(\omega_0-\omega)\eta + \omega t]}\, d\omega\, d\eta, \tag{21.8}$$

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\eta)\, e^{i[(\omega_0-\omega)\eta + \omega t - \varphi(\omega)]} R(\omega)\, d\omega\, d\eta. \tag{21.9}$$

In the absence of absorption and with total reflection, $R(\omega) = 1$; this equation will be assumed to hold unless the contrary is specifically stated.

Propagation of a quasimonochromatic pulse without allowance for spreading

For a quasimonochromatic pulse (a group) we have as a first approximation

$$\varphi(\omega) = \varphi(\omega_0) + \Omega\varphi'(\omega_0), \tag{21.10}$$

where $\varphi'(\omega_0) = (d\varphi/d\omega)_{\omega=\omega_0}$ and $\Omega = \omega - \omega_0$. In this approximation

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\eta)\, e^{i\Omega[t - \eta - \varphi'(\omega_0)]}\, d\eta\, d\omega$$
$$= A\left(t - \varphi'(\omega_0)\right) e^{i\omega_0 t - i\varphi(\omega_0)}, \tag{21.11}$$

since by (21.8) and (21.4)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\eta)\, e^{i\Omega(t-\eta)}\, d\eta\, d\Omega = A(t).$$

It should be noted that the phase $\varphi(\omega)$ is directly defined only for positive ω. It always appears in the form $i[|\omega|t - \varphi(|\omega|)]$, and consequently $i[\omega t - \varphi(\omega)]$ in the integral (21.9) and similar integrals must be replaced by $i[|\omega|t - \varphi(|\omega|)] = -i[\omega t + \varphi(|\omega|)]$ when $\omega < 0$. For this reason some caution must be used in dealing with integrals of the type (21.11); on this point see, for instance, [137]. In our case, when the function $g(\omega)$ has a sharp maximum, there are no difficulties at the point $\Omega = 0$ (i.e. $\omega = \omega_0$), since the region of negative ω is unimportant: at $\omega = 0$ the frequency is $\Omega = -\omega_0$, and $\omega_0 \gg \Delta\omega$, so that in the integral (21.11) the lower limit $-\infty$ can be replaced by $-\omega_0$.

If the result (21.11) is compared with (21.4), we see that in the first approximation (21.10) the pulse retains its shape (as a function of t), but on reflection or transmission through the layer the phase of the wave is changed by $\varphi(\omega_0)$, and the whole pulse is delayed by the "group delay time"†

$$\Delta t_{\text{gr}} = (d\varphi/d\omega)_{\omega=\omega_0} \equiv \varphi'(\omega_0). \tag{21.12}$$

We can also define the "phase delay time" [see (21.11)]:

$$\Delta t_{\text{ph}} = \varphi(\omega_0)/\omega_0. \tag{21.13}$$

Instead of the times Δt_{gr} and Δt_{ph} we frequently use the group and optical path lengths:

$$L_{\text{gr}} = c\Delta t_{\text{gr}}, \quad L_{\text{ph}} = L_o = c\Delta t_{\text{ph}}. \tag{21.14}$$

The group path length L_{gr} is evidently the distance traversed by the group in time Δt_{gr} if it moves with the velocity of light in vacuum. The optical path length L_o has a similar significance with Δt_{ph} in place of Δt_{gr}.

Phase and group velocities of waves

In order to relate the times Δt_{gr} and Δt_{ph} to the group and phase velocities, as is usually done, let us consider the propagation of a pulse in a homogeneous isotropic medium. We shall take φ to be the phase shift on traversing a path z, i.e. $E_0(t)$ is here the field at $z = 0$ and $E(t)$ the field at z. Then $\varphi = \omega n(\omega)z/c$ [see (7.6)], and in the approximation (21.10) we have from (21.11)

$$E(t) = A\left(t - \left(\frac{d[\omega n(\omega)/c]}{d\omega}\right)_{\omega_0} z\right) e^{i\omega_0 t - i\omega_0 n(\omega_0) z/c}. \tag{21.15}$$

† The meaning of this is as follows: if the incident pulse appears at some chosen point at time $t = 0$, and $E_0(t) = 0$ for $t < 0$, the reflected pulse appears at that point (in this approximation) at $t = \varphi'(\omega_0)$, before which $E(t) = 0$.

Hence the phase is propagated with the phase velocity

$$v_{\mathrm{ph}}(\omega_0) = z/\Delta t_{\mathrm{ph}} = c/n(\omega_0) = (\omega/k)_{\omega_0}, \tag{21.16}$$

where $k = \omega n(\omega)/c$ is the magnitude of the wave vector.

The whole pulse is propagated without distortion with the group velocity

$$\begin{aligned} v_{\mathrm{gr}}(\omega_0) = z/\Delta t_{\mathrm{gr}} &= \frac{c}{(d[\omega n(\omega)]/d\omega)_{\omega_0}} \\ &= \frac{c}{n(\omega_0) + \omega_0 (dn/d\omega)_{\omega_0}} = (d\omega/dk)_{\omega_0}. \end{aligned} \tag{21.17}$$

If n^2 is defined by (3.5), i.e. $n^2 = 1 - 4\pi e^2 N/m\omega^2$, then

$$v_{\mathrm{gr}} = cn \tag{21.18}$$

and so $v_{\mathrm{gr}} v_{\mathrm{ph}} = c^2$.

The relation $n^2 = c^2k^2/\omega^2 = 1 - \omega_0^2/\omega^2$, incidentally, holds not only for a plasma but also in various other cases, e.g. in simple waveguides (here, of course, the significance of ω_0 is quite different, and moreover no expression for n is usually written down in waveguide theory, the equivalent expression $\omega^2 = \omega_0^2 + c^2k^2$ being used). The results obtained above can be applied in all such cases. Most of the results of this and some other sections are valid whatever the form of the function $n(\omega)$.

In a homogeneous medium

$$\left.\begin{aligned} L_{\mathrm{gr}} &= cz/v_{\mathrm{gr}} \\ &= z\, d(\omega n)/d\omega \\ &= d(\omega L_0)/d\omega \\ &= L_0 + \omega\, dL_0/d\omega, \\ L_0 &= cz/v_{\mathrm{ph}} \\ &= nz, \end{aligned}\right\} \tag{21.19}$$

where z is the group distance traversed.

Since in the ionosphere (neglecting the effect of the magnetic field) we always have $n^2 < 1$, it is clear that $v_{\mathrm{ph}} > c$ and $v_{\mathrm{gr}} < c$. Thus the signals are propagated with a velocity (the group velocity) which is less than that of light, in accordance with the theory of relativity. The group velocity v_{gr} defined by formula (21.17) may, for some forms of $n(\omega)$, exceed the velocity of light in vacuum, c. For example, in a region of anomalous dispersion $dn/d\omega < 0$, and if $n + \omega\, dn/d\omega < 1$ the velocity $v_{\mathrm{gr}} > c$. In such cases, however, we do not have a propagation of signals with a velocity exceeding c; the concept of the group velocity is here inapplicable. The reason is clear from the above calculations: formulae (21.11), (21.12) and (21.17) have been obtained by writing the phase φ in the form (21.10), and this is only a first approximation, corresponding to the linear term in $\omega - \omega_0 = \Omega$ in an expansion of $\varphi(\omega)$ as a Taylor series. The pulse is propagated without change of form in a dispersive medium only with this limitation to the term of order Ω in the series expansion of the phase. When higher powers of Ω are taken into account the form of the pulse changes: it is spread out and the concept of the group velocity is no longer definite without further specification.

The concept of the group velocity as defined above is in general inapplicable also if absorption is present, unless the dependence of R on ω may be regarded as negligible within the spectral width of the signal. In a region of anomalous dispersion this is not possible, since the absorption varies considerably with frequency. Thus in such a region the form of the signal varies considerably during its propagation. Consequently, formula (21.17) has no physical significance in this range of frequencies (i.e. does not determine the velocity of the signal). It should be noted that some meaning may still sometimes be assigned to the concept of the group velocity even when absorption is present, but of course there is no direct relation to formula (21.17). This will be discussed in § 22.

Spreading of pulses

The propagation of a signal in a homogeneous isotropic dispersive medium taking account of the distortion (spreading) of the signal, has been investigated, in [138]. It was shown that the forward front of the signal is always propagated with the velocity of light in vacuum. This result is quite reasonable, since the spectral resolution of a signal of finite width includes arbitrarily high frequencies, for which $n \to 1$ and $v_{\text{gr}} \to v_{\text{ph}} \to c$. However, the contribution of these high frequencies in a long signal is quite negligible. Consequently the part of the signal which adjoins the forward front (called the "precursor") is usually of negligible intensity. The most important part of the signal is that where the intensity is relatively large (the "main part" or "body" of the signal). For a homogeneous medium the manner of variation of the signal intensity has been examined in [138], and for a plasma in more detail in [139]; the case of an inhomogeneous medium and specifically of a plasma (the ionosphere) is discussed in [137, 140, 141].

To calculate the change in form of the main part of a quasimonochromatic pulse when it is propagated in an arbitrary non-absorbing medium, it is usually sufficient to add one more term in the series expansion of the phase [see (21.10)], putting

$$\varphi(\omega) = \varphi(\omega_0) + \Omega\varphi'(\omega_0) + \tfrac{1}{2}\Omega^2\varphi''(\omega_0). \tag{21.20}$$

The field $E(t)$ then becomes [see (21.9)]

$$E(t) = e^{i\omega_0 t - i\varphi(\omega_0)} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} A(\eta)\, e^{i\Omega[t-\eta-\varphi'(\omega_0)] - \frac{1}{2}i\Omega^2\varphi''(\omega_0)}\, d\eta\, d\Omega; \tag{21.21}$$

putting

$$\left(\Omega + \frac{\eta - t + \varphi'}{\varphi''}\right)^2 \varphi'' = \pi \xi^2, \tag{21.22}$$

i.e. with ξ as variable in place of Ω, and using the fact that

$$\int_{-\infty}^{\infty} e^{-i\pi\xi^2/2}\, d\xi = 1 - i,$$

we obtain

$$E(t) = \frac{1-i}{2\pi}\, e^{i\omega_0 t - i\varphi(\omega_0)} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\varphi''(\omega_0)}}\, A(\eta)\, e^{i(\eta - t + \varphi')^2/2\varphi''}\, d\eta.$$

Replacing $(\eta - t + \varphi')^2/2\varphi''$ by $\frac{1}{2}\pi u^2$, we have finally†

$$E = \tfrac{1}{2}(1-i)\, e^{i\omega_0 t - i\varphi(\omega_0)} \int_{-\infty}^{\infty} A\left[t - \varphi'(\omega_0) + \sqrt{\{\pi\varphi''(\omega_0)\}}\, u\right] e^{i\pi u^2/2}\, du. \tag{21.23}$$

If the derivative $\varphi''(\omega_0)$ is sufficiently small, the expression (21.23) of course becomes (21.11), as it should, since

$$\int_{-\infty}^{\infty} e^{i\pi u^2/2}\, du = 1 + i.$$

The above calculations are not sufficiently correct, however, since, as has already been pointed out, the sign of the phase $\varphi(|\omega|)$ must be changed when $\omega < 0$. This is not important if we can to a good approximation replace the limit $-\infty$ by $-\omega_0$ in the integral over Ω in (21.21). After substitution of (21.22) this replacement leads to the appearance in an expression like (21.23) of the following integral over ξ:

$$\begin{aligned} \Psi &= \int_{\Xi}^{\infty} e^{-i\pi\xi^2/2}\, d\xi \\ &= \tfrac{1}{2}(1-i) + F^*(-\Xi), \end{aligned} \tag{21.24}$$

where $\Xi \equiv -\sqrt{(\varphi''/\pi)}\,\omega_0 + [\eta - t + \varphi'(\omega_0)]/\sqrt{[\pi\varphi''(\omega_0)]}$ and F is the Fresnel integral:

$$\left.\begin{aligned} F(u) &= \int_0^u e^{i\pi u^2/2}\, du = C + iS, \\ F^* &= C - iS = \int_0^u e^{-i\pi u^2/2}\, du, \\ F(\infty) &= \tfrac{1}{2}(1+i), \quad F(-u) = -F(u), \quad F(0) = 0, \\ &\left.\begin{aligned} C(u) &\approx \tfrac{1}{2} + (1/\pi u)\sin\tfrac{1}{2}\pi u^2 \\ S(u) &\approx \tfrac{1}{2} - (1/\pi u)\cos\tfrac{1}{2}\pi u^2 \end{aligned}\right\} \text{for } u \gg 1. \end{aligned}\right\} \tag{21.25}$$

† For definiteness we suppose that $\varphi'' > 0$. If $\varphi'' < 0$, we obtain the same results on changing sign in the substitution of variables, but with $\sqrt{[\pi|\varphi''(\omega_0)|]}$ in place of $\sqrt{[\pi\varphi''(\omega_0)]}$. If $\varphi'' = 0$, of course, the subsequent terms in the series expansion of $\varphi(\omega)$ must be taken into account in order to see how the signal is spread out.

The expressions for C and S when $u \gg 1$ are accurate to order $1/\pi^2 u^3$, and so they are valid to within about 1% when $u \gtrsim 3$.

In going from (21.21) to (21.23) we have put $F^* = F^*(\infty) = \frac{1}{2}(1 - i)$ in (21.24). If this is not done, then in (21.23) the factor $1 - i$ in the integrand must be replaced by (21.24) with u instead of $(\eta - t + \varphi')/\sqrt{(\pi\varphi'')}$, i.e. by $\frac{1}{2}(1 - i) + F^*[\sqrt{(\varphi''/\pi)}\omega_0 - u]$. This is equal to $1 - i$, as it should be, if

$$\sqrt{[\varphi''(\omega_0)/\pi]}\,\omega_0 \gg 1 \tag{21.26}$$

and if also in the integral (21.23) the important values of u are $\sim u_0$ such that

$$\sqrt{(\varphi''/\pi)}\,\omega_0 \gg u_0 \tag{21.27}$$

(with $u_0 > 0$). The conditions (21.26) and (21.27) may be replaced by the single requirement that $\sqrt{(\varphi''/\pi)}\omega_0 - u$ should be large throughout the significant range of integration in (21.23).

When the conditions (21.26) and (21.27) hold, the spreading of the signal is determined by formula (21.23), which can be made more specific only if the function $A(t)$ which gives the form of the original (incident) signal is known. For a "truncated sine wave", i.e. the rectangular pulse (21.5), we have

$$\left.\begin{aligned} E(t) &= \frac{1}{2}(1 - i)\, e^{i\omega_0 t - i\varphi(\omega_0)} \int\limits_{-\theta/\sqrt{(\pi\varphi'')}}^{(T-\theta)/\sqrt{(\pi\varphi'')}} e^{i\pi u^2/2}\, du \\ &= \frac{1}{2}(1 - i)\, e^{i\omega_0 t - i\varphi(\omega_0)} \left\{F\left(\frac{T-\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right) - F\left(\frac{-\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right)\right\}, \\ |E(t)| &= \frac{1}{\sqrt{2}}\left|F\left(\frac{T-\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right) + F\left(\frac{\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right)\right|. \end{aligned}\right\} \tag{21.28}$$

Here F is the Fresnel integral (21.25) and

$$\theta = \tfrac{1}{2}T + t - \varphi'(\omega_0). \tag{21.29}$$

Evidently θ is the time measured from the instant $-\frac{1}{2}T + \varphi'(\omega_0)$ when the signal would pass the point considered if there were no spreading.

If the signal is so long that

$$T \gg \sqrt{[\pi\varphi''(\omega_0)]}, \tag{21.30}$$

the distortion of the signal is of interest only in the range $\theta \ll T$ near $\theta = 0$ and the range $T - \theta \ll T$ near $\theta = T$, i.e. at the ends of the signal. In this case we have for the front of the signal

$$\begin{aligned} |E(t)| &= \frac{1}{\sqrt{2}}\left|F(\infty) + F\left(\frac{\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right)\right| \\ &= \frac{1}{\sqrt{2}}\left|\frac{1}{2}(1 + i) + F\left(\frac{\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right)\right| \\ &= \frac{1}{\sqrt{2}}\sqrt{\left\{\left[C\left(\frac{\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right) + \frac{1}{2}\right]^2 + \left[S\left(\frac{\theta}{\sqrt{[\pi\varphi''(\omega_0)]}}\right) + \frac{1}{2}\right]^2\right\}}. \end{aligned} \tag{21.31}$$

For the back of the signal we have the same formula with $-\theta$ replaced by θ_1, the time from the instant $\frac{1}{2}T + \varphi'(\omega_0)$ at which the back of the signal would pass the point considered if there were no spreading.

The expression (21.31) is well known, as it gives the intensity of the wave field in Fresnel diffraction at the edge of a plane screen. Fig. 21.1 shows a graph of the function $|E(u)|$, where

$$u = \theta/\sqrt{[\pi\varphi''(\omega_0)]} = [\tfrac{1}{2}T + t - \varphi'(\omega_0)]/\sqrt{[\pi\varphi''(\omega_0)]}.$$

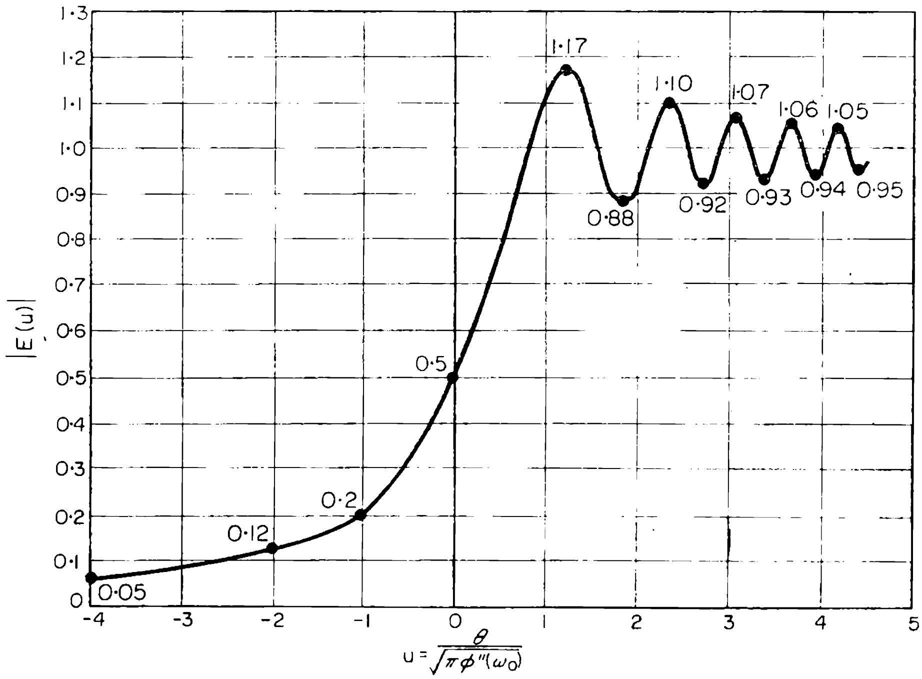

FIG. 21.1. Form of the front of a pulse, originally rectangular, passing through a medium.

The maxima of this function correspond to $\theta/\sqrt{[\pi\varphi''(\omega_0)]} = 1{\cdot}22$, $2{\cdot}35$, $3{\cdot}08$, $3{\cdot}67$, $4{\cdot}18$, etc.; the minima are at $1{\cdot}87$, $2{\cdot}74$, $3{\cdot}39$, $3{\cdot}93$, $4{\cdot}42$, etc. The values of $|E|$ at the maxima and minima are shown in Fig. 21.1. It may be noted that at the first maximum $(\theta/\sqrt{(\pi\varphi'')} = 1{\cdot}22)$ the intensity $|E^2|$ is $1{\cdot}36$, the intensity of the original signal being taken as unity. At $\theta = 0$, i.e. when (in the absence of spreading) the front of the signal would pass the point considered, and the amplitude would change discontinuously from 0 to 1, the amplitude given by (21.31) is

$$|E| = \tfrac{1}{2}. \tag{21.32}$$

The time required to reach a steady signal amplitude is defined, as we see from (21.31), by the parameter

$$\tau_0 = \sqrt{[\pi\varphi''(\omega_0)]}. \tag{21.33}$$

It is clear that this time must be determined by the quantity $\varphi''(\omega_0)$, and the square root in (21.33) follows from dimensional considerations.

The time to establish the steady value can be exactly determined if the limit of permissible deviation is fixed. For example, the time τ between the instants when the amplitude becomes constant to within 5 per cent for $\theta > 0$ and when $|E| = 0{\cdot}05$ for $\theta < 0$ is shown by Fig. 21.1 to be about $8\tau_0$.

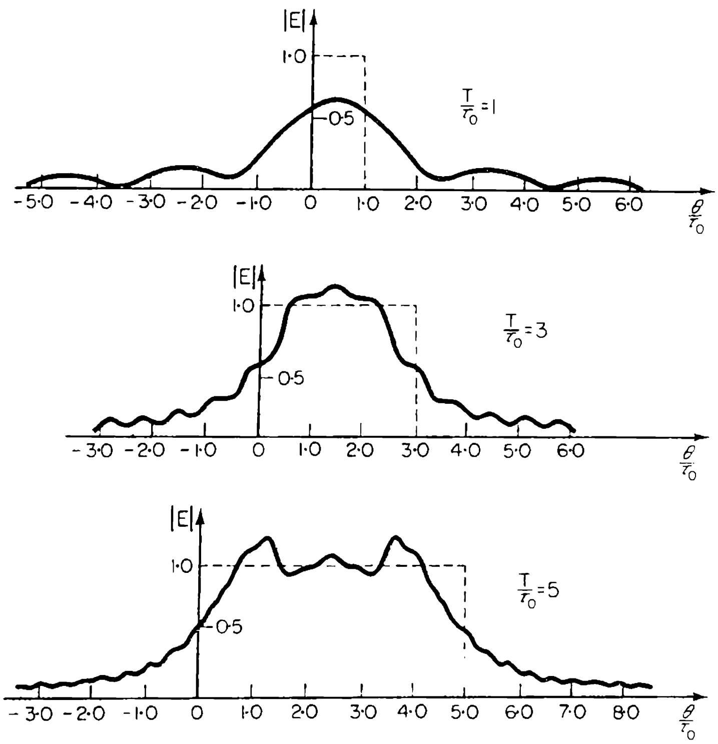

FIG. 21.2. Form of a reflected signal for $T/\tau_0 = 1, 3$ and 5. The broken lines show the original shape of the pulse on the same scale.

If the condition (21.30) is not satisfied, we must use formula (21.28), and the form of the signal is given by the parameter $T/\tau_0 = T/\sqrt{[\pi\varphi''(\omega_0)]}$. The amplitude of the reflected signal $|E|$ given by (21.28) is shown in Fig. 21.2 as a function of the variable θ/τ_0 for $T/\tau_0 = 1$, 3 and 5 (the amplitude of the incident signal being taken as unity).

The condition (21.26) for the expression (21.28) to be valid may be written [see (21.33)]

$$\sqrt{[\varphi''(\omega_0)/\pi]}\,\omega_0 = \tau_0\omega_0/\pi = 2\tau_0/T_0 \gg 1. \tag{21.26a}$$

In the ionosphere, the estimates of τ_0 given in § 30 show that this condition is usually well satisfied. In (21.27) we can take for u_0 the limits of integration $(T - \theta)/\sqrt{(\pi\varphi'')}$ and $-\theta/\sqrt{(\pi\varphi'')}$ in (21.28) if they are positive. Assuming for definiteness that $|\theta| < T$, we have the condition (21.27) in the form

$$\varphi''(\omega_0)\omega_0 \gg T. \tag{21.27a}$$

If (21.3) holds, and $T \sim 1/\Delta\omega \gg 1/\omega_0$, the conditions (21.27) and (21.27a) are less stringent than (21.26) and (21.26a), and therefore only the latter need be considered. More precisely, the condition (21.27) is significant in this case only in the regions $T - \theta \gg T$ and $-\theta \gg T$; these evidently correspond to the remote "precursor" and "tail" of the signal, where the field $|E|$ is negligibly small.

The quantity φ'', like the phase φ itself, increases with the path length traversed by the signal. The conditions (21.26a) and (21.27a) are therefore requirements that this path should be sufficiently long.

When absorption is present the whole problem must in general be re-examined. If, however, as sometimes happens, the amplitude of a wave of frequency ω or the reflection coefficient $R(\omega)$ varies only slightly over the spectral width of the signal, we can put $R(\omega) = R(\omega_0)$, and so all the formulae obtained remain unchanged [except, of course, for the factor $R(\omega_0)$]. It must not be forgotten that the absorption also affects the phase $\varphi(\omega)$, but in the general discussion given above this fact is, of course, unimportant.

The limits of applicability of the approximation used, and some more accurate results

The results so far obtained are limited also by the initial relation (21.20), in which terms proportional to Ω^3, Ω^4, etc., have been omitted.

The condition for these results to be valid may be found by estimating the importance of the next term, $\Omega^3\varphi'''(\omega_0)/6$, in an expansion such as (21.20).

Apart from the remote "precursor" and "tail" of the signal, in which waves are represented whose frequencies differ considerably from ω_0, the range of Ω which is important in all the integrals is of the order of $\Delta\omega$, the spectral width of the signal. Hence the term $\Omega^3\varphi'''(\omega_0)/6$ may be neglected in comparison with the previous term if

$$\frac{\varphi'''(\omega_0)\,\Delta\omega}{3\varphi''(\omega_0)} \sim \frac{\varphi'''(\omega_0)}{\varphi''(\omega_0)\,T} \ll 1\,. \tag{21.34}$$

If, for example, $\varphi(\omega) = \text{constant} \times \omega^m$, where the exponent m is not very large, the inequality (21.34) follows from the condition (21.3) for quasimonochromatic signals; for such signals, i.e. when

$$T \gg T_0, \tag{21.3a}$$

we may therefore neglect the terms in Ω^3, Ω^4, etc., in the main part of the signal. This is confirmed by a quantitative assessment of the importance of the term $\Omega^3\varphi'''(\omega_0)/6$, which can be carried out by calculating the field on the basis of the expansion

$$\varphi(\omega) = \varphi(\omega_0) + \Omega\,\varphi'(\omega_0) + \frac{1}{2}\Omega^2\varphi''(\omega_0) + \frac{1}{6}\Omega^3\varphi'''(\omega_0)\,. \tag{21.35}$$

We then obtain for $E(t)$, instead of (21.28), the expression [141]

$$E(t) = \frac{1-i}{2\sqrt{\pi}} e^{i\omega_0 t - i\varphi(\omega_0)} \int\limits_{-\infty}^{\infty} \left\{ F\left[\frac{T-\theta-[\frac{1}{2}\varphi'''(\omega_0)]^{1/3} y}{\sqrt{[\pi\varphi''(\omega_0)]}}\right] + \right.$$

$$\left. + F\left[\frac{\theta+[\frac{1}{2}\varphi'''(\omega_0)]^{1/3} y}{\sqrt{[\pi\varphi''(\omega_0)]}}\right]\right\} v(y)\, d y, \tag{21.36}$$

where

$$v(y) = \frac{1}{\sqrt{\pi}} \int\limits_0^{\infty} \cos\left(\frac{1}{3}x^3 + x y\right) d x$$

is the Airy function.

When $\varphi'''(\omega_0) = 0$ the expression (21.36) becomes (21.28), as it should, since

$$\int\limits_{-\infty}^{\infty} v(y)\, d y = \sqrt{\pi}.$$

In the approximation (21.35) we see from (21.36) that the signal is characterised by the parameter

$$\beta = [\tfrac{1}{2}\varphi'''(\omega_0)]^{\frac{1}{3}}/\sqrt{[\pi\varphi''(\omega_0)]} \tag{21.37}$$

as well as by $\tau_0 = \sqrt{[\pi\varphi''(\omega_0)]}$.

In the conditions in which radio waves are reflected from the ionosphere, $\beta = 0{\cdot}1$ to $0{\cdot}3$, and calculations [141] based on formula (21.36) show that in the main part of quasimonochromatic signals the accuracy of formula (21.28) is at worst a few per cent. The use of this formula is invalid in practice only for very narrow signals, where the conditions (21.3) and (21.3a) do not hold. Such short pulses, with $T \lesssim T_0 = 2\pi/\omega_0 = 1/f_0$, are not usually of interest in the propagation of radio waves in the ionosphere; in radar probing of the ionosphere we have in most cases $T \sim 10^{-4}$ sec when $T_0 \sim 10^{-6}$ to 10^{-7} sec. However, the discussion in [139] and below shows that formula (21.28) is applicable near the points $\theta = 0$ and $\theta = T$, even for short signals.

The general investigation of the propagation of short pulses and of the "precursor" and "tail" of any pulse offers considerable difficulties. For a homogeneous ionised gas this problem has been considered in detail in [139], where for a rectangular signal (21.4)–(21.5) which has traversed a sufficiently long path the following simple expression was finally obtained:

$$\left.\begin{aligned} |E(t,z)| &= |F(u_1) - F(u_2)|/\sqrt{2}, \\ u_1 &= \sqrt{(\omega_0/\pi)}\,\{\sqrt{(t+\tfrac{1}{2}T - z/c)}\,\sqrt{[1+n(\omega_0)]} - \\ &\quad - \sqrt{(t+\tfrac{1}{2}T + z/c)}\,\sqrt{[1-n(\omega_0)]}\}, \\ u_2 &= \sqrt{(\omega_0/\pi)}\,\{\sqrt{(t-\tfrac{1}{2}T - z/c)}\,\sqrt{[1+n(\omega_0)]} - \\ &\quad - \sqrt{(t-\tfrac{1}{2}T + z/c)}\,\sqrt{[1-n(\omega_0)]}\}, \\ n(\omega_0) &= \sqrt{(1-4\pi e^2 N/m\,\omega_0^2)}. \end{aligned}\right\} \tag{21.38}$$

Here it is assumed that the signal left the point $z = 0$ at time $t = -\frac{1}{2}T$; at $z = 0$ the signal is rectangular and of length T, i.e. the signal ends there at $t = \frac{1}{2}T$. Formula (21.38) is valid only when the signal has travelled so far in the medium that

$$t_{\text{gr}} - \frac{z}{c} = z\left(\frac{1}{v_{\text{gr}}} - \frac{1}{c}\right) = \frac{z(1-n)}{cn} \gg T_0. \tag{21.39}$$

This is evidently equivalent to requiring that the difference between the time t_{gr} needed to traverse the path z with the group velocity $v_{\text{gr}} = cn$ and the time z/c taken for this path by the front ("precursor") of the signal, moving with the velocity of light in vacuum c, should be much greater than the period $T_0 = 2\pi/\omega_0$ of the oscillations. It is also assumed that $t > \frac{1}{2}T + z/c$. This means that only the field at a considerable distance from the front of the signal is considered. The latter is propagated with velocity c and reaches the point z at time $t_1 = z/c - \frac{1}{2}T$; the back of the original signal, if it moved with velocity c, would reach the point z at time $t_2 = z/c + \frac{1}{2}T$.

In the case of a homogeneous ionised gas considered here we have

$$\varphi(\omega_0) = \omega_0 n(\omega_0) z/c,$$

$$\varphi'(\omega_0) = t_{\text{gr}} = z/cn(\omega_0), \text{ and}$$

$$|\varphi''(\omega_0)| = (z/c)(1 - n^2)/n^3\omega_0.$$

Using these relations, we see from (21.38) that, near the point

$$t' = t_{\text{gr}} - \tfrac{1}{2}T = z/cn - \tfrac{1}{2}T,$$

we have to within terms of order $(t - t_{\text{gr}} + \frac{1}{2}T)^2$

$$u_1 = (t - t_{\text{gr}} + \tfrac{1}{2}T)/\sqrt{[\pi\varphi''(\omega_0)]} = \theta/\sqrt{[\pi\varphi''(\omega_0)]}.$$

Similarly, expanding u_2 in series about the point $t'' = t_{\text{gr}} + \frac{1}{2}T$, we obtain $u_2 = (\theta - T)/\sqrt{(\pi\varphi'')}$, where $\theta = t - t_{\text{gr}} + \frac{1}{2}T$.

Substituting in (21.38) these values of u_1 and u_2, we find that it becomes (21.28). This was to be expected for sufficiently long signals, of course, since formula (21.28) must be valid for the particular case of a homogeneous medium. However, we now see that formula (21.28) applies even for short signals (as already mentioned) near the points $\theta = 0$ and $\theta = T$, at least in a homogeneous plasma. At a considerable distance from these points, a case which is of interest for strong spreading of short signals, formula (21.28) cannot be used, but formula (21.38) is valid for a homogeneous ionised gas.

§22. ENERGY DENSITY IN A DISPERSIVE MEDIUM. THE VELOCITY OF SIGNALS IN PLASMAS WHEN ABSORPTION IS PRESENT

Introduction

In order to elucidate certain points, it is necessary to obtain an expression for the energy density in a dispersive medium. The energy density of the electric field in an isotropic medium is often taken to be

$$W_E = \boldsymbol{E}\cdot\boldsymbol{D}/8\pi = \varepsilon E^2/8\pi. \tag{22.1}$$

In a monochromatic wave $E = E_0 \sin\omega_0 t$, and we then have

$$\left.\begin{aligned} W_E &= (\varepsilon E_0^2/8\pi)\sin^2\omega_0 t \\ &= (\varepsilon E_0^2/16\pi)(1-\cos 2\omega_0 t), \\ \overline{W_E} &= (\varepsilon/8\pi)\cdot\tfrac{1}{2}E_0^2 \\ &= \varepsilon\overline{E^2}/8\pi, \end{aligned}\right\} \tag{22.2}$$

where the bar denotes averaging with respect to time.

Now let $\varepsilon(\omega_0) < 0$, as is entirely possible for a non-absorbing ionised gas, where $\varepsilon(\omega) = 1 - 4\pi e^2 N/m\omega^2$. In this case $W_E \leqq 0$, and $\overline{W_E} < 0$. But the density of electric energy cannot take a negative value. As $\omega_0 \to 0$ (a static field) this is clear from thermodynamic considerations: in a state of thermodynamic equilibrium, W_E is the density of free energy (see, for example, [2]), and in the isothermal case the value of W_E is a minimum; if $W_E < 0$, the minimum corresponds to $W_E \to -\infty$ and $E^2 \to \infty$, which is obviously incorrect. Furthermore, when absorption and thermal motion are neglected, in accordance with the above expression for $\varepsilon(\omega)$, the energy density in the medium is the sum of the energy density $W_E^{(0)} = E^2/8\pi$ in vacuum and the energy due to the polarisation of the medium. When there are free charges (as in a plasma) the latter part of the energy is the kinetic energy of the ordered motion of the charges, i.e. it is certainly positive. Thus we see that $W_E \geqq 0$ in a plasma for any frequency when absorption is absent. Thus the expressions (22.1) and (22.2) cannot be valid in the general case.

Energy density in a non-absorbing dispersive medium

To elucidate the question, let us turn to Poynting's original theorem, which expresses the law of conservation of energy and follows from the field equations (see, for instance, [2]; the magnetic permeability $\mu = 1$):

$$\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \frac{1}{4\pi}\boldsymbol{H}\cdot\frac{\partial\boldsymbol{H}}{\partial t} = \frac{\partial}{\partial t}\int^{t}\frac{\boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t}{4\pi}\,dt + \frac{\partial}{\partial t}\left(\frac{H^2}{8\pi}\right)$$
$$= -\boldsymbol{j}\cdot\boldsymbol{E} - c\,\mathrm{div}(\boldsymbol{E}\times\boldsymbol{H})/4\pi. \tag{22.3}$$

From this it follows, at least in the absence of absorption (i.e. when $\boldsymbol{j}\cdot\boldsymbol{E} = 0$), that the electric energy density is

$$W_E = \frac{1}{4\pi}\int^{t}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\,dt. \tag{22.4}$$

For a monochromatic wave $E = E_0 \sin\omega_0 t$ we have $D = \varepsilon(\omega_0) E$ and $W_E(t) = \varepsilon E^2(t)/8\pi$ + constant. The arbitrary constant† included here cannot be determined for a monochromatic wave specified for all t, and so in this case there is no meaning in saying that $\overline{W_E}$ is negative when $\varepsilon < 0$. At first sight, however, it seems that the deduction that $\overline{W_E}$ is negative is valid also for a quasimonochromatic wave: it is sufficient to suppose that the wave is not monochromatic but quasimonochromatic and then that $E \to 0$ as $t \to -\infty$ and the field energy density $W_E(-\infty) = 0$, so that the additive constant is zero. It appears that the change from a signal of infinite duration to one of finite but arbitrarily long duration cannot affect the expression for the energy density; the result is that for a quasimonochromatic wave with $\varepsilon < 0$ we still have $\overline{W_E} < 0$, but this must be incorrect. We shall now show that for a dispersive medium the transition from a monochromatic to a quasimonochromatic wave must be made with greater care, and that the paradox just stated is then resolved.

Let us consider some field

$$E(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, d\omega = \int_0^{\infty} [g(\omega)\, e^{i\omega t} + g^*(\omega)\, e^{-i\omega t}]\, d\omega, \tag{22.5}$$

where the second expression for $E(t)$ is obtained by using the fact that it is real; in discussing energy relations it is convenient to use a real field throughout. Evidently

$$D(t) = \int_0^{\infty} \varepsilon(\omega)\, [g(\omega) e^{i\omega t} + g^*(\omega)\, e^{-i\omega t}]\, d\omega, \tag{22.6}$$

where we have used the fact that $\varepsilon(\omega) = \varepsilon(-\omega)$, a condition which is quite generally satisfied.††

The general expression for the energy density is now obtained by substituting (22.5) and (22.6) in (22.4). It is

$$\begin{aligned} W_E = \frac{1}{4\pi}\int_0^t \int_0^{\infty} \int_0^{\infty} \{g(\omega)\, g(\omega')\, i\,\omega\, \varepsilon(\omega)\, e^{i(\omega+\omega')t} + \text{complex} \\ \text{conjugate}\}\, d\omega\, d\omega'\, dt\, + \\ + \frac{1}{4\pi}\int_0^t \int_0^{\infty} \int_0^{\infty} \{g(\omega)\, g^*(\omega')\, i\,\omega\, \varepsilon(\omega)\, e^{i(\omega-\omega')t} + \text{complex} \\ \text{conjugate}\}\, d\omega\, d\omega'\, dt. \end{aligned} \tag{22.7}$$

† More precisely, we may speak of an arbitrary function of the coordinates, independent of time, which could also be added to (22.4).

†† The equations $\varepsilon(-\omega) = \varepsilon(\omega)$ and $\sigma(-\omega) = \sigma(\omega)$, from which it follows that $\varepsilon'(-\omega) = \varepsilon'^*(\omega)$, are derived from the requirement that D and j are real for any real E; see (22.5), (22.6) and the corresponding expression for j.

Let us now consider the case of a quasimonochromatic pulse, when the function $g(\omega)$ has a sharp maximum near the carrier frequency ω_0. Putting $\omega = \omega_0 + \Omega$ and $\omega' = \omega_0 + \Omega'$, we can write $\varepsilon(\omega)$ as a first approximation in the form $\varepsilon(\omega_0) + \Omega(d\varepsilon/d\omega)_0$. For a sufficiently narrow pulse the term in Ω^2 may be neglected, but the term in Ω may not, as we shall see. Next, in the first term in (22.7) we have factors $\exp(2i\omega_0 t)$ and $\exp(-2i\omega_0 t)$, and these give zero on averaging over time [cf. (22.2)]. The second term in (22.7) is equal to the mean energy density $\overline{W_E}$ and, if terms of order Ω^2 are neglected (the averaging being always over times large compared with $2\pi/\omega_0$), we have

$$\overline{W_E} = \frac{1}{4\pi}\int\limits_0^t \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \{g(\omega_0+\Omega)\, g^*(\omega_0+\Omega') \times$$
$$\times\, i[\omega_0\,\varepsilon(\omega_0) + \Omega\,\varepsilon(\omega_0) + \Omega\,\omega_0(d\,\varepsilon/d\,\omega)_0]\, e^{i(\Omega-\Omega')t} +$$
$$+ \text{ complex conjugate}\}\, d\Omega\, d\Omega'\, dt; \qquad (22.8)$$

here the lower limits of integration have been taken as $-\infty$, since in fact only a narrow range of frequencies near ω_0 is of importance in a quasimonochromatic pulse.

It is easy to see that the part of (22.8) which contains the term $\omega_0\varepsilon(\omega_0)$ is zero; in this case the complex conjugate is the same with the opposite sign. Thus

$$\overline{W_E} = \overline{\frac{1}{4\pi}\int\limits^t \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\, dt} = \frac{1}{4\pi}\left[\varepsilon(\omega_0) + \omega_0\left(\frac{d\varepsilon}{d\omega}\right)_0\right] \times$$
$$\times \int\limits_0^t \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \{i\,\Omega\, g(\omega_0+\Omega)\, g^*(\omega_0+\Omega')\, e^{i(\Omega-\Omega')t} +$$
$$+ \text{ complex conjugate}\}\, d\Omega\, d\Omega'\, dt$$
$$= \frac{1}{8\pi}\left[\varepsilon(\omega_0) + \omega_0\left(\frac{d\varepsilon}{d\omega}\right)_0\right]\overline{E^2} = \frac{1}{8\pi}\left[\frac{d(\omega\,\varepsilon)}{d\omega}\right]_0 \overline{E^2}, \qquad (22.9)$$

the suffix 0 to the brackets signifying that the derivative is taken for $\omega = \omega_0$. If we use the complex field $\boldsymbol{E}$, then $\overline{W_E} = \dfrac{1}{16\pi}[d(\omega\varepsilon)/d\omega]_0 \boldsymbol{E}\cdot\boldsymbol{E}^*$.

The second expression in (22.9) is obtained from the first by means of the result

$$4\pi\, W_E^{(0)} = \int\limits_0^t \boldsymbol{E}\cdot\frac{d\,\boldsymbol{E}}{d\,t}\, dt = \frac{1}{2}\int\limits_0^t \frac{\partial E^2}{\partial t}\, dt = \frac{1}{2}E^2(t);$$

we can put $E(0) = 0$, since there is no field at the beginning of the process, and the choice of the point $t = 0$ is arbitrary. Expressing $4\pi\, W_E^{(0)} = \int \boldsymbol{E}\cdot(\partial \boldsymbol{E}/\partial t)\, dt$

in terms of Fourier integrals, using (22.5) and separating the high-frequency part from the part independent of time, we see that $\frac{1}{2}\overline{E^2}$ is exactly equal to the integral in (22.9). The derivation of (22.9) is also given in [22, 36, 120, 142, 143] (see also Appendix B).

The expression (22.9) shows that when dispersion is present the allowance for non-monochromaticity is important: it leads to the replacement of $\varepsilon(\omega)$ by $\varepsilon + \omega d\varepsilon/d\omega$. When $\varepsilon = 1 - 4\pi e^2 N/m\omega^2$ we have

$$d(\omega\varepsilon)/d\omega = 1 + 4\pi e^2 N/m\omega^2 = 2 - \varepsilon. \tag{22.10}$$

This expression is always positive, and so the paradoxical result described in connection with (22.2) is in fact erroneous, and the energy density in a plasma without absorption is positive, as it should be.

The question may arise of what happens in a non-absorbing medium without dispersion with $\varepsilon < 0$. In this case the derivative $d\varepsilon/d\omega$ plays no part, and $\overline{W_E} < 0$. The answer is simply that such a medium cannot exist; this is proved by the impossible result just mentioned and is, of course, confirmed also by calculating ε for any actual media. The paradox in the conclusion that $\overline{W_E} < 0$ for a quasimonochromatic wave, which could be obtained by starting from (22.2), lay in the fact that the derivation was valid for an actual medium with $\varepsilon = 1 - 4\pi e^2 N/m\omega^2$.

The case of an absorbing medium

The above derivation of formula (22.9) remains valid even when absorption is present, since it has nowhere been assumed absent. Here another problem arises. For a plasma with $\omega^2 \ll \nu_{\text{eff}}^2$ we have $\varepsilon \approx 1 - 4\pi e^2 N/m\nu_{\text{eff}}^2$ [see (3.11)], and ε may be negative for certain values of N and ν_{eff}, even if dispersion is absent, so that by (22.9) we have $\overline{W_E} < 0$.

In this case, however, the expression (22.9) does not represent the total mean density of energy. It must be emphasised that, when absorption is present, it is not in general possible to introduce phenomenologically the concept of the mean electromagnetic energy density. This is formally due to the fact that, when absorption is present, the expression (22.3) which gives the law of conservation of energy contains two volume terms:

$$\frac{\partial}{\partial t}\left(\frac{1}{4\pi}\int_0^t \boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\,dt + \frac{H^2}{8\pi}\right) = \frac{\partial}{\partial t}(W_E + W_H)$$

and $\boldsymbol{j}\cdot\boldsymbol{E}$. For this reason it is not clear, especially as regards the averaging over time, how we can uniquely divide this sum into parts corresponding to the change in energy and to dissipation; the example given above shows that, when absorption is present, it is not permissible to continue to take $\overline{W_E}$

defined by (22.9) as the total mean energy. If a particular model of the medium is used, the problem of the energy in the presence of absorption can be more completely understood.

Energy density in a plasma

Let us consider the motion of electrons in a plasma by means of the Boltzmann equation (4.2), omitting the term $\boldsymbol{v} \cdot \mathbf{grad}_r f$ (the case of spatial homogeneity). Then, multiplying (4.2) by $\frac{1}{2}mv^2$ and integrating with respect to $\boldsymbol{v}$, we easily obtain (for details see Appendix B)

$$\frac{\partial}{\partial t}\left(\mathscr{K} + \frac{E^2}{8\pi}\right) = \frac{\boldsymbol{E}}{4\pi} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j} \cdot \boldsymbol{E} - \int \frac{1}{2} m v^2 S \, d\boldsymbol{v}, \tag{22.11}$$

where $\mathscr{K}$ is the kinetic energy, and we have used the results that for a plasma

$$\left.\begin{aligned} \boldsymbol{j}_t &= \frac{1}{4\pi} \frac{\partial(\boldsymbol{D} - \boldsymbol{E})}{\partial t} + \boldsymbol{j} = e \int \boldsymbol{v} f \, d\boldsymbol{v}, \\ \mathscr{K} &= \int \tfrac{1}{2} m v^2 f \, d\boldsymbol{v}. \end{aligned}\right\} \tag{22.12}$$

The last term in (22.11) is the energy acquired by the heavy particles from the electrons per unit time.

Together with the theorem (22.3), (22.11) gives

$$\frac{\partial}{\partial t}\left(\frac{E^2 + H^2}{8\pi} + \mathscr{K}\right) = -\frac{c}{4\pi} \operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) - \int \frac{1}{2} m v^2 S \, d\boldsymbol{v}. \tag{22.13}$$

In the absence of collisions $\boldsymbol{j} = 0$ and $S = 0$, and for a quasimonochromatic field

$$\overline{\mathscr{K}} = \frac{1}{8\pi}\left[\frac{d(\omega\,\varepsilon)}{d\omega} - 1\right]_0 \overline{E^2}.$$

In the presence of collisions we put for simplicity $d\sigma/d\omega = 0$ for the quasimonochromatic field (see also Appendix B):

$$\overline{\mathscr{K}} + \frac{\overline{E^2}}{8\pi} = \frac{1}{8\pi}\left[\frac{d(\omega\,\varepsilon)}{d\omega}\right]_0 \overline{E^2} + \int_{-\infty}^{t} \sigma \overline{E^2} \, dt - \int_{-\infty}^{t}\!\!\int \frac{1}{2} m v^2 \overline{S} \, d\boldsymbol{v} \, dt. \tag{22.14}$$

Here the bar denotes averaging over a time long compared with $2\pi/\omega_0$, but short compared with the characteristic time of variation of the signal amplitude. Hence all the quantities in (22.14) may depend on t, and it is clear that $\overline{\mathscr{K}}$ depends on the function S, and is not in general of the form constant $\times$ $\overline{E^2}$.

It is also of interest to consider the energy relations obtained by using the equations for the mean electron velocity $\dot{\boldsymbol{r}}$:

$$\left.\begin{aligned} d(m\,\dot{\boldsymbol{r}})/dt &= e\,\boldsymbol{E} - \mathbf{grad}\, U - m\,\nu_{\text{eff}}\,\dot{\boldsymbol{r}}, \\ d(\tfrac{1}{2}\,m\,\dot{\boldsymbol{r}}^2 + U)/dt &= e\,\dot{\boldsymbol{r}} \cdot \boldsymbol{E} - m\,\nu_{\text{eff}}\,\dot{\boldsymbol{r}}^2, \end{aligned}\right\} \tag{22.15}$$

where U is the potential energy corresponding to a certain force, which is zero for free (plasma) electrons. With $\boldsymbol{E} = \boldsymbol{E}_0 \cos(\omega_0 t + \varphi)$ and $U = 0$ we obtain from (22.15) (see, for instance, § 4)

$$\left.\begin{aligned} \overline{W'_E} &= \frac{\overline{E^2}}{8\pi} + N\overline{K} \\ &= \left[1 + \frac{4\pi e^2 N}{m(\omega_0^2 + \nu_{\text{eff}}^2)}\right]\frac{\overline{E^2}}{8\pi} \\ &= \frac{2-\varepsilon}{8\pi}\overline{E^2}, \\ \overline{K} &= \tfrac{1}{2} m \overline{\dot{r}^2} = e^2\overline{E^2}/2m(\omega_0^2 + \nu_{\text{eff}}^2), \end{aligned}\right\} \tag{22.16}$$

since from (22.15) we derive the expressions (3.7) for ε and σ. Evidently the energy $\overline{W'_E}$ is always positive, unlike the expression

$$\frac{1}{8\pi}\left[\frac{d(\omega\varepsilon)}{d\omega}\right]_0 \overline{E^2} = \frac{1}{8\pi}\left[1 + \frac{4\pi e^2 N(\omega_0^2 - \nu_{\text{eff}}^2)}{m(\omega_0^2 + \nu_{\text{eff}}^2)^2}\right]\overline{E^2},$$

with which it coincides when $\nu_{\text{eff}} = 0$. Here it may at first sight seem reasonable to regard $\overline{W'_E}$ as the density of the part of the energy of the plasma which depends on the field (see [22, § 68; 143, 144]). However, a comparison with the result of the kinetic treatment shows that the use of equation (22.15) and the expression (22.16) corresponds to a very definite assumption regarding the form of the integral $\int \frac{1}{2} m v^2 S\, d\boldsymbol{v}$. On averaging over a time long compared with $2\pi/\omega_0$ we must have (Appendix B)

$$m N \nu_{\text{eff}} \overline{\dot{r}^2} = \int \tfrac{1}{2} m v^2 S\, d\boldsymbol{v}. \tag{22.17}$$

This is at least a very particular case, and the equation probably does not hold for the majority of plasmas.

Energy density for an assembly of oscillators

If, instead of a plasma, we have an assembly of independent oscillators with eigenfrequencies ω_i, equations (22.15) give

$$\left.\begin{aligned} \varepsilon' &= 1 - \sum_i \frac{4\pi e_i^2 N_i/m_i}{\omega^2 - \omega_i^2 - i\,\omega\,\nu_{\text{eff},i}}, \\ \overline{W'_E} &= \frac{\overline{E^2}}{8\pi} + N\overline{K} \\ &= \left\{1 + \sum_i \frac{4\pi e^2 N_i(\omega_i^2 + \omega^2)/m_i}{(\omega^2 - \omega_i^2)^2 + \omega^2 \nu_{\text{eff},i}^2}\right\}\frac{\overline{E^2}}{8\pi}, \end{aligned}\right\} \tag{22.18}$$

where N_i is the density of oscillators having frequency ω_i, $\nu_{\text{eff}} = \nu_{\text{eff},i}$, charge e_i and mass m_i [in (22.15) we have $U_i = \frac{1}{2} m_i \omega_i^2 r_i^2$].

If there is no absorption, i.e. $\nu_{\text{eff},i} = 0$, then $\overline{W'_E}$ in (22.18) is given in terms of ε by (22.9). When absorption is present, however, $\overline{W'_E}$ cannot be expressed in terms of ε or ε' only, even for this simple oscillator model. A similar result occurs in electrical circuits (bipolar), where ε' is represented by the impedance, and the energy in different circuits with the same impedance $Z = R + iX$ may have almost any value (if $R \neq 0$). Hence it is clear that, when absorption is present, we cannot give an expression for the mean energy density which would be obtained phenomenologically, i.e. would be uniquely expressed in terms of $\varepsilon'(\omega)$. Thus to calculate the field energy in an absorbing medium we must use the microscopic theory and a particular model of the medium. When the energy of interaction between the particles is neglected, and equations (22.15) and the elementary formula $\varepsilon' = 1 - \omega_0^2/\omega(\omega - i\nu_{\text{eff}})$ are used, a plasma forms an exception, since even in the presence of absorption the mean energy density $\overline{W'_E}$ can be expressed in terms of ε'; from (22.18) with $\omega_i = 0$ we see that $\overline{W'_E} = (2 - \varepsilon)\overline{E^2}/8\pi = (2 - \text{re}\,\varepsilon')\overline{E^2}/8\pi$ even for a plasma consisting of several species of particle.

Energy density in plasma waves

The above expression $\overline{W'_E} = (2 - \varepsilon)\overline{E^2}/8\pi$ for the energy density in a plasma is a local quantity, i.e. it is independent of the way in which the field varies in space. In other words, spatial dispersion is neglected and the field may equally well be regarded as longitudinal or transverse (the polarisation of the field in a wave $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ is given by $\operatorname{div}\boldsymbol{E} = -i\boldsymbol{k}\cdot\boldsymbol{E}$ and becomes indeterminate as $k \to 0$). Hence the time average energy density in plasma waves (oscillations) is

$$\overline{W_E} = \overline{E^2}/4\pi, \tag{22.16a}$$

as follows from (22.9) and (22.10) or (22.16) with the condition $\varepsilon = \varepsilon(\omega_0) = 0$ which corresponds to plasma waves (neglecting collisions).

The result (22.16a) does not contradict the fact that the expression (22.4), i.e. $(1/4\pi)\int \boldsymbol{E}\cdot(\partial\boldsymbol{D}/\partial t)\,dt$, is zero for plasma waves in which $\partial\boldsymbol{D}/\partial t = 0$. The reason is that Poynting's theorem [see (22.3)] is no longer meaningful if we impose initially the condition $\operatorname{curl}\boldsymbol{H} = (1/c)\,\partial\boldsymbol{D}/\partial t = 0$. If, however, we start from the energy density $W_E = E^2/8\pi + \int \frac{1}{2}mv^2(f - f_0)\,d\boldsymbol{v}$, it becomes

$$\frac{1}{4\pi}\int\limits_{-\infty}^{t} \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\,dt,$$

where

$$\frac{\partial\boldsymbol{D}}{\partial t} = \frac{\partial\boldsymbol{E}}{\partial t} + 4\pi e\int \boldsymbol{v}f\,d\boldsymbol{v},$$

only if two conditions hold. Firstly, we must assume that as $t \to -\infty$ we have $f \to f_0$ (the equilibrium distribution function) and $E \to 0$, i.e. $W \to 0$; secondly, we use a Boltzmann equation of the type $df/dt = -(e\boldsymbol{E}/m)\cdot\mathbf{grad}_v f$. The former condition means that initially ($t = -\infty$) there are no plasma oscillations, as must be assumed in considering quasimonochromatic waves. These waves, however, cannot appear spontaneously; they must be produced either by non-electric forces or by the motion of external charges, of density ϱ. In the first case (for example, when electrons and ions are separated by gravity) the above equation for df/dt, in which non-electric forces were ignored, is not valid. In the second case, which is of much greater practical importance (for example, the plasma waves emitted by a moving charge), we have $\operatorname{div}\boldsymbol{D} = 4\pi\varrho$, and we cannot put $\partial\boldsymbol{D}/\partial t = 0$ throughout the interval of time considered. Thus the apparent contradiction due to the use of the expression (22.4) for plasma waves is resolved, and formula (22.16a) in fact gives the energy density in plasma waves, and also in transverse waves with a carrier frequency ω_0 for which $\varepsilon(\omega_0) = 0$. In this case we may start from the expression (22.4), but $\partial\boldsymbol{D}/\partial t$ is not identically zero, since a quasimonochromatic wave is being considered. For this reason the spectrum contains frequencies, other than ω_0, for which ε and $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ are not zero. It was this fact which led to the replacement of (22.1) and (22.2) by (22.9).

Velocity of signals in an absorbing medium. Application to a plasma

In conditions where the expression for the energy density in an absorbing medium can be used, it is possible to define also the velocity of pulse signals propagated in that medium [143, 144].

The group velocity has been defined in § 21 essentially from kinematic considerations, and the expression $v_{\text{gr}} = d\omega/dk$ is in fact valid for any kind of quasimonochromatic wave. This group velocity, being the rate of propagation of signals, must also have a significance as the rate of energy transfer. This is easily seen to be so by defining the rate of energy transfer as the ratio of the time average energy flux density $\overline{S}$ to the mean energy density $\overline{W}$:

$$v_{\text{en}} = \overline{S}/\overline{W}. \tag{22.19}$$

For a non-absorbing dispersive medium

$$\begin{aligned}\overline{W} = \overline{W_E} + \overline{W_H} &= \frac{1}{8\pi}\left[\frac{d(\omega\,\varepsilon)}{d\omega}\right]\overline{E^2} + \frac{\overline{H^2}}{8\pi}\\ &= \frac{1}{16\pi}\left[\frac{d(\omega\,\varepsilon)}{d\omega}\right]\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{16\pi}\boldsymbol{H}\cdot\boldsymbol{H}^*,\end{aligned} \tag{22.20}$$

where in the last expression we use the complex fields $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$ and $\boldsymbol{H} = \boldsymbol{H}_0 e^{i\omega t}$; the quasimonochromaticity of the field has already been allowed for in (22.20) by replacing ε by $d(\omega\varepsilon)/d\omega$, and so in this equation the fields may be considered monochromatic.

In any medium, as in a vacuum, the electromagnetic energy flux is

$$\left.\begin{aligned} \boldsymbol{S} &= c\,\boldsymbol{E} \times \boldsymbol{H}/4\pi, \\ \overline{S_z} &= c(\overline{E_x H_y} - \overline{E_y H_x})/4\pi \\ &= c\,\mathrm{re}\,(E_x H_y^* - E_y H_x^*)/8\pi, \end{aligned}\right\} \tag{22.21}$$

for a transverse wave travelling in the z-direction, the fields in the last expression being complex.

For a transverse monochromatic homogeneous plane wave

$$\left.\begin{aligned} &\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}, \quad \boldsymbol{H} = \sqrt{\varepsilon'}\,\boldsymbol{k} \times \boldsymbol{E}/k, \\ &\overline{S_z} = c\,n\,\boldsymbol{E}\cdot\boldsymbol{E}^*/8\pi, \quad \boldsymbol{k}\cdot\boldsymbol{E} = 0, \\ &k^2 = \omega^2 \varepsilon'/c^2 = \omega^2 (n - i\varkappa)^2/c^2, \quad \mathrm{re}\sqrt{\varepsilon'} = n. \end{aligned}\right\} \tag{22.22}$$

In the absence of absorption $\sqrt{\varepsilon'} = \sqrt{\varepsilon} = n$, $k = \omega n/c$ and

$$\left.\begin{aligned} \overline{S_z} &= c\,n\,\boldsymbol{E}\cdot\boldsymbol{E}^*/8\pi, \\ \overline{W} &= \frac{n}{8\pi}\left(n + \omega\frac{dn}{d\omega}\right)\boldsymbol{E}\cdot\boldsymbol{E}^* \\ &= \frac{n}{8\pi}\,\frac{d(\omega n)}{d\omega}\,\boldsymbol{E}\cdot\boldsymbol{E}^*, \\ v_{\mathrm{en}} &= \overline{S}/\overline{W} = c \div d(\omega n)/d\omega = d\omega/dk, \end{aligned}\right\} \tag{22.23}$$

i.e. $v_{\mathrm{en}} = v_{\mathrm{gr}}$, as stated.

When absorption is present, $v_{\mathrm{gr}} = d\omega/dk$ is in general no longer meaningful, and may, for example, give values exceeding the velocity of light in vacuum c (see § 21). If, however, the energy density $\overline{W_E'}$ in the presence of absorption is known, the velocity v_{en} given by formula (22.19), which no longer equals $d\omega/dk$, must in some sense represent the rate of propagation of the signal.

For a plasma, using the elementary formula for ε' and formula (22.16) for $\overline{W_E'}$:

$$\left.\begin{aligned} &\varepsilon' = 1 - \frac{\omega_0^2}{\omega(\omega - i\,\nu_{\mathrm{eff}})} = \varepsilon - i\frac{4\pi\sigma}{\omega} = (n - i\varkappa)^2, \quad \overline{S_z} = c\,n\,\boldsymbol{E}\cdot\boldsymbol{E}^*/8\pi, \\ &\overline{W} = [(2-\varepsilon)\,\boldsymbol{E}\cdot\boldsymbol{E}^* + \boldsymbol{H}\cdot\boldsymbol{H}^*]/16\pi \\ &\quad = (2 - \varepsilon + |\varepsilon'|)\,\boldsymbol{E}\cdot\boldsymbol{E}^*/16\pi, \end{aligned}\right\} \tag{22.24}$$

we obtain

$$v_{\mathrm{en}} = \overline{S}/\overline{W} = 2c\,\mathrm{re}\sqrt{\varepsilon'}/(2 - \mathrm{re}\,\varepsilon' + |\varepsilon'|) = c\,n/(1 + \varkappa^2). \tag{22.25}$$

In the absence of absorption $\varkappa = 0$ and $v_{\mathrm{en}} = cn = d\omega/dk$ [see (21.18)]; if $\sigma = 0$ and $\varepsilon < 0$, then $n = 0$ and $v_{\mathrm{en}} = 0$, as it should be for total internal reflection without absorption. From this and from other examples it is clear that formula (22.25) gives entirely reasonable results for the velocity of signals in a plasma. It must be borne in mind, however, that the expression (22.16) used for $\overline{W'_E}$ is a particular case and is not generally valid. Moreover, formula (22.25) may still be entirely inapplicable if the wave is appreciably damped over the length of the quasimonochromatic pulse considered (this point is discussed further in [144]). The latter statement is readily understandable, since the concept of the signal velocity in a medium is itself essentially an approximate one, describing the propagation of the signal while neglecting its change and distortion in the course of time. In the general case, any information about the propagation and distortion of a signal in a linear medium can, of course, be obtained by analysis of the corresponding Fourier integral (see § 21).

CHAPTER V

WAVE PROPAGATION IN AN INHOMOGENEOUS MAGNETOACTIVE PLASMA

§ 23. INTRODUCTION. THE APPROXIMATION OF GEOMETRICAL OPTICS

The wave equations

IN THE discussion of electromagnetic wave propagation in an inhomogeneous anisotropic medium and, in particular, in a magnetoactive plasma the basic equation is [see (2.5) and (2.9)]

$$\triangle E_i - \frac{\partial}{\partial x_i} \operatorname{div} \boldsymbol{E} + \frac{\omega^2}{c^2} \varepsilon'_{ik}(\omega, \boldsymbol{r}) E_k = 0. \tag{23.1}$$

For an isotropic medium, where $\varepsilon'_{ik} = \varepsilon' \delta_{ik}$, the equations represented by (23.1) naturally reduce to (16.1). The corresponding equations for the magnetic field $\boldsymbol{H}$ in the wave are in general inconvenient because they are so cumbersome; they are obtained by eliminating the field $\boldsymbol{E}$ from the equations

$$\triangle \boldsymbol{H} + (i\omega/c) \operatorname{curl}(\boldsymbol{D} - 4\pi i \boldsymbol{j}/\omega) = 0, \qquad \operatorname{curl} \boldsymbol{E} = -i\omega \boldsymbol{H}/c,$$

$$D_i - 4\pi i j_i/\omega = \varepsilon'_{ik}(\boldsymbol{r}) E_k.$$

In the particular case of a plane-parallel medium, where $\varepsilon'_{ik} = \varepsilon'(\omega, z)$, and for plane waves, the equations (23.1) are greatly simplified, but even when the waves are incident normally on the layer we have two second-order equations [cf. (2.18) and (11.3)]:

$$\left.\begin{aligned} \frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2}(A E_x + i C E_y) = 0, \\ \frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2}(-i C E_x + B E_y) = 0; \end{aligned}\right\} \tag{23.2}$$

here A, B, C are given by formulae (11.3), in which the parameters $v = 4\pi e^2 N/m\omega^2$, $u = \omega_H^2/\omega^2$ and $s = \nu_{\text{eff}}/\omega$ depend on z. [In particular cases, of course, only one or two of these parameters may depend on z; a dependence of u on z occurs when the plasma is in a non-uniform external magnetic field $\boldsymbol{H}^{(0)}(z)$.]

The equations (23.2) are equivalent to a fourth-order equation for E_x or E_y, whereas when waves are normally incident on an isotropic layer the components E_x and E_y satisfy the second-order equation (16.3). The complications resulting from the use of (23.2) are mathematically so great that the exact solution of the latter has not yet been studied. It is, of course, assumed here that the magnetic field $\boldsymbol{H}^{(0)}$ is at an angle α to the z-axis (i.e. to the normal to the wave) which is neither zero nor $\frac{1}{2}\pi$. When $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$, equations (23.2) separate rigorously into two independent second-order equations, and so these particular cases are of especial importance both in themselves and when used (with certain modifications) as approximate solutions sometimes applicable for other values of α.

The above discussion explains why approximate solutions are of even greater significance in the propagation of waves in an inhomogeneous magnetoactive plasma than they are for an isotropic medium. They include first of all the approximation of geometrical optics, and also the phase-integral method and the method of perturbation theory.

The approximation of geometrical optics

The wide range of applicability of the approximation of geometrical optics is due to the fact that the properties of the plasma usually vary only slowly in space in the ionosphere, the solar corona and some other regions. The variation is said to be slow if the indices $n_{1,2}$ and $\varkappa_{1,2}$ change very little over distances of the order of the wavelength in the medium. Evidently the propagation in any relatively small region of the layer may then be regarded as being the same as in a homogeneous medium, with the corresponding values of $n_{1,2}$ and $\varkappa_{1,2}$.

The nature of the propagation and reflection of waves in a layer of inhomogeneous plasma in the approximation of geometrical optics is clear from the curves of $n_{1,2}(v)$ for given u and α; see Figs. 11.2, 11.3, 11.6, etc. (for simplicity, absorption is ignored). At the boundary of a layer, with $v = 0$ (i.e. $N(z) = 0$), the wave is split into an ordinary and an extraordinary component, which are thereafter propagated independently. If, for example, $u < 1$, then for $v = v_{10}^{(-)} = 1 - \sqrt{u}$ the index $n_1 = 0$, and the extraordinary wave cannot be propagated further (being damped when $n_1^2 < 0$). Hence, as in an isotropic medium (see § 16), we may conclude that in the region $n_1^2 \approx 0$ there is total reflection of the extraordinary wave from the layer. The ordinary wave is seen from similar arguments to be reflected at the point $v = v_{20} = 1$. An exception occurs when the angle α is small. The region of reflection must then "jump" from the neighbourhood of $v_{20} = 1$ to $v_{10}^{(+)} = 1 + \sqrt{u}$. This "jump" cannot be explained on the basis of the approximation of geometrical optics, and is discussed in § 28. The fact that geometrical optics is invalid near $v_{20} = 1$ for small angles α is immediately evident from Fig. 11.10, which shows that the properties of the medium vary rapidly with v in this case.

On the basis of the identification, in § 16 and above, of the approximation of geometrical optics with the possibility of regarding the medium as quasi-homogeneous, it is clear that in this case the change in the wave phase during propagation must be given by an expression of the type $(\omega/c)\int n_{1,2}(z)\,dz$, as in an isotropic medium. For a complete determination of the fields $E_{x,y1,2}$ in the approximation of geometrical optics, however, it is necessary to consider the problem in more detail, as in § 16 (see [145]).

We shall seek a solution of equations (23.2) in the form

$$\boldsymbol{E}(\boldsymbol{r}) = \left[\boldsymbol{E}^{(0)}(\boldsymbol{r}) + \frac{c}{\omega}\boldsymbol{E}^{(1)}(\boldsymbol{r}) + \frac{c^2}{\omega^2}\boldsymbol{E}^{(2)}(\boldsymbol{r}) + \cdots\right] e^{-i\omega\Psi(\boldsymbol{r})/c}, \tag{23.3}$$

where $\boldsymbol{r}$ may be replaced by z, since the only differentiations in (23.2) are with respect to z. Substituting (23.3) in (23.2) and equating to zero the coefficient of each power of the ratio ω/c, we obtain

$$\left.\begin{aligned} (A - \Psi'^2)\,E_x^{(0)} + i\,C\,E_y^{(0)} &= 0\,, \\ -i\,C\,E_x^{(0)} + (B - \Psi'^2)\,E_y^{(0)} &= 0\,, \end{aligned}\right\} \tag{23.4}$$

$$\left.\begin{aligned} (A - \Psi'^2)\,E_x^{(1)} + i\,C\,E_y^{(1)} &= i(\Psi''\,E_x^{(0)} + 2\,\Psi'\,E_x^{(0)\prime})\,, \\ -i\,C\,E_x^{(1)} + (B - \Psi'^2)\,E_y^{(1)} &= i(\Psi''\,E_y^{(0)} + 2\,\Psi'\,E_y^{(0)\prime})\,, \end{aligned}\right\} \tag{23.5}$$

$$\left.\begin{aligned} (A - \Psi'^2)\,E_x^{(2)} + i\,C\,E_y^{(2)} &= i(\Psi''\,E_x^{(1)} + 2\,\Psi'\,E_x^{(1)\prime}) - E_x^{(0)\prime\prime}\,, \\ -i\,C\,E_x^{(2)} + (B - \Psi'^2)\,E_y^{(2)} &= i(\Psi''\,E_y^{(1)} + 2\,\Psi'\,E_y^{(1)\prime}) - E_y^{(0)\prime\prime}\,, \end{aligned}\right\} \tag{23.6}$$

etc., where the prime denotes differentiation with respect to z ($\Psi' \equiv d\Psi/dz$, etc.).

Equations (23.4) for $E_{x,y}^{(0)}$ are the same as (11.2a) for a homogeneous medium if n^2 is replaced by Ψ'^2 (for simplicity, absorption is assumed to be absent). Thus the condition for the existence of a non-trivial solution for $E_{x,y}^{(0)}$ gives the function Ψ'^2:

$$\Psi'^2_{1,2} = n^2_{1,2}\,, \tag{23.7}$$

where $n^2_{1,2}$ are given by equation (11.6). Hence

$$\Psi_{1,2} = \pm \int_{z_0}^{z} n_{1,2}(z)\,dz\,, \tag{23.8}$$

the plus and minus signs corresponding to propagation in the positive and negative z-directions respectively. In what follows we shall always use the plus sign in (23.8), taking the sign of Ψ into account (if necessary) in the expression for the field itself.

Next, by (23.4),

$$\frac{E^{(0)}_{y1,2}}{E^{(0)}_{x1,2}} = \frac{i\,C}{B - \Psi'^2_{1,2}} = -\frac{A - \Psi'^2_{1,2}}{i\,C} = K_{1,2}\,, \tag{23.9}$$

i.e. the polarisation of the wave in the first approximation of geometrical optics is at every point the same as in a homogeneous medium with the corresponding values of u, α and $v(z)$ [see (11.26)]. This is, of course, to be expected.

The condition for the existence of a non-trivial solution of (23.5) gives equations for $E^{(0)}_{x1,2}$ and $E^{(0)}_{y1,2}$ as functions of the coordinates:

$$\left.\begin{aligned} E_x^{(0)\prime} + \left(\frac{\Psi''}{2\Psi'} - \frac{K K'}{1-K^2}\right) E_x^{(0)} = 0\,, \\ E_y^{(0)\prime} + \left(\frac{\Psi''}{2\Psi'} - \frac{K'}{K(1-K^2)}\right) E_y^{(0)} = 0\,, \end{aligned}\right\} \tag{23.10}$$

where Ψ and K for waves 1 and 2 are given by (23.8) and (23.9) with the suffixes 1 and 2 respectively. According to (23.10)

$$E^{(0)}_{x1,2} = \frac{\text{constant}}{\sqrt{[\Psi'_{1,2}(1-K^2_{1,2})]}} = \frac{\text{constant}}{\sqrt{[n_{1,2}(z)\{1-K^2_{1,2}(z)\}]}}\,, \tag{23.11}$$

and the component $E_y^{(0)}$ is $K_{1,2}E_x^{(0)}$, by (23.9).

Thus, in the first approximation of geometrieal optics,

$$\left.\begin{aligned} E_{x1,2} &= \frac{\text{constant}}{\sqrt{[n_{1,2}(z)\{1-K^2_{1,2}(z)\}]}} \exp\left[\pm i\frac{\omega}{c}\int\limits_{z_0}^{z} n_{1,2}(z)\,dz\right], \\ E_{y1,2} &= K_{1,2}E_{x1,2}. \end{aligned}\right\} \tag{23.12}$$

If we take account of absorption we obtain a similar result with $\Psi' = n_{1,2}$ replaced by $\Psi' = n_{1,2} - i\varkappa_{1,2}$, the latter expression being given by (11.5). In this case $K_{1,2}$ is given by (11.25).

The general solution of equations (23.2) in the approximation of geometrical optics is

$$\begin{aligned} E_x = {} & \frac{C_{1+}}{\sqrt{[(n_1 - i\varkappa_1)(1-K_1^2)]}} \exp\left[i\frac{\omega}{c}\int\limits_{z_0}^{z}(n_1 - i\varkappa_1)\,dz\right] + \\ & + \frac{C_{1-}}{\sqrt{[(n_1 - i\varkappa_1)(1-K_1^2)]}} \exp\left[-i\frac{\omega}{c}\int\limits_{z_0}^{z}(n_1 - i\varkappa_1)\,dz\right] + \\ & + \frac{C_{2+}}{\sqrt{[(n_2 - i\varkappa_2)(1-K_2^2)]}} \exp\left[i\frac{\omega}{c}\int\limits_{z_0}^{z}(n_2 - i\varkappa_2)\,dz\right] + \\ & + \frac{C_{2-}}{\sqrt{[(n_2 - i\varkappa_2)(1-K_2^2)]}} \exp\left[-i\frac{\omega}{c}\int\limits_{z_0}^{z}(n_2 - i\varkappa_2)\,dz\right]. \end{aligned} \tag{23.13}$$

The expression for E_y differs from (23.13) in that C_{1+} and C_{1-} are multiplied by K_1, and C_{2+} and C_{2-} by K_2. The component E_z is given in terms of E_x and E_y by formula (10.20). The solution (23.13) depends on four arbitrary constants $C_{1\pm}$ and $C_{2\pm}$ (as it should), since any change in z_0 is equivalent to some change in these constants.

Formula (23.12) differs from (16.11) for an isotropic medium only in that it depends on the quantity $K_{1,2}$ which characterises the polarisation. In the isotropic case with normal incidence [equation (16.3)] the polarisation of the wave does not vary with z, i.e. $K_{1,2}$ = constant and (23.12) becomes (16.11).

The limits of applicability of the approximation of geometrical optics

These we take to be given by the conditions for the first approximation (23.12) to be valid, and they may be found by determining the field in the second approximation, i.e. by calculating $E_{x,y}^{(1)}$ from (23.5) and (23.6).

From (23.5) we find

$$\begin{aligned} E_y^{(1)} &= K\,E_x^{(1)} + (\Psi''\,E_x^{(0)} + 2\Psi'\,E_x^{(0)\prime})/C \\ &= K\,E_x^{(1)} + 2K\,K'\,\Psi' E_x^{(0)}/C(1-K^2), \end{aligned} \tag{23.14}$$

where C is one of the coefficients in (23.2).

The condition for the existence of a solution of (23.6) gives an equation for $E_x^{(1)}$:

$$\left.\begin{aligned} &E_x^{(1)\prime} + \left(\frac{\Psi''}{2\Psi'} - \frac{K\,K'}{1-K^2}\right) E_x^{(1)} = f(z), \\ &f(z) = \frac{1}{2i\,\Psi'(1-K^2)}\Big\{(1-K^2)\,E_x^{(0)\prime\prime} - 2K\,K'\,E_x^{(0)\prime} + \\ &\quad + \frac{i\,\Psi''\,K}{C}\left(\frac{2K\,K'}{1-K^2}\,\Psi'\,E_x^{(0)}\right) + \\ &\quad + 2i\,\Psi'\,K\,\frac{d}{dz}\left(\frac{2K\,K'}{C(1-K^2)}\,\Psi'\,E_x^{(0)}\right)\Big\}. \end{aligned}\right\} \tag{23.15}$$

The approximation of geometrical optics is valid if

$$\frac{\lambda_0\,|E_x^{(1)}|}{2\pi} = \frac{\lambda_0}{2\pi}\left|\frac{1}{\sqrt{\chi}}\int_{z_0}^{z} f(z)\sqrt{\chi}\,dz\right| \ll |E_x^{(0)}| = \text{constant}/\sqrt{\chi}, \tag{23.16}$$

where $\chi = \Psi'(1-K^2) = n(1-K^2)$, $\lambda_0 = 2\pi c/\omega$, and for definiteness we have taken the component E_x.

As in § 16, the condition (23.16) may be replaced by a series of sufficient conditions. Here, for the sake of brevity, we shall not perform the integration by parts in (23.16) as we did in (16.21), but merely make a simple estimate.

If the function $\chi(z)$ is monotonic and the inequality

$$\lambda_0 |f(z)|/2\pi \ll |E_x^{(0)} d \ln \chi/dz| \tag{23.17}$$

holds, then the condition (23.16) is satisfied, provided that the value of $|\ln \chi|$ is not too large throughout the range (z, z_0) (for, example, the requirement $|\ln \chi| < 10$ corresponds to $10^{-4} < \chi < 10^4$). In order to prove this, we need only remember that $E_x^{(0)} = \text{constant}/\sqrt{\chi}$ and substitute (23.17) in (23.16). The condition (23.17) is easily seen, by using the explicit expression for $f(z)$, to be certainly satisfied if

$$\left.\begin{aligned} &\lambda_0 \chi'/2\pi n \chi \ll 1, && \frac{\lambda_0}{2\pi} \frac{K}{C} \frac{n n' K K'}{\chi'(1-K^2)} \ll 1, \\ &\lambda_0 K K'/2\pi \chi \ll 1, && \lambda_0 \chi''/2\pi n \chi' \ll 1, \\ &\lambda_0 K K''/2\pi \chi' \ll 1, && \frac{\lambda_0}{2\pi} \frac{n K \sqrt{\chi}}{\chi'} \frac{d}{dz}\left[\frac{n K K'}{C(1-K^2)\sqrt{\chi}}\right] \ll 1; \end{aligned}\right\} \tag{23.18}$$

here all the quantities must be taken as moduli (see § 16) and the values of n, K and χ must be substituted for each wave (1 and 2). Some of these inequalities could be simplified or omitted in the same way as was done in § 16 for the second inequality (16.15). We shall not pause to do this, since it is easily seen that when v varies smoothly and slowly with z the conditions (23.18) reduce in practice (with an exception to be described below) to the single inequality

$$\frac{\lambda_0}{2\pi} \left| \frac{n'_{1,2}}{n^2_{1,2}} \right| \ll 1, \tag{23.19}$$

which is entirely similar to the condition (16.22) and whose physical meaning is quite clear.

In order to show that the conditions (23.18) may legitimately be replaced by (23.19), we must use the fact that the functions C and C' tend to infinity only as $v \to v_{1,2\infty}$ and tend to zero as $v \to 1$ and $v \to 0$; $|K_2| \to \infty$ and $|K'_{1,2}| \to \infty$ as $v \to 1$. The parameter K_1 does not tend to infinity even when $v \to 1$ (see Fig. 11.9). From the discussion in § 11 of the behaviour of the curves $n_{1,2}(v)$ (see Fig. 11.3, etc.) it is evident that at the points $v = v_{20} = 1$ and $v_{1,2\infty}$ geometrical optics is inapplicable to the corresponding waves because the condition (23.19) is not satisfied.

In the approximation of geometrical optics the waves of the two types 1 and 2 are entirely independent, and so are waves of the same type propagated in opposite directions. Hence, in particular, the reflection of waves can be observed only in regions where geometrical optics is not valid. This happens when $n^2_{1,2}$ is small or $dn_{1,2}/dz$ is large. A wave of type 1, therefore, can be reflected in the neighbourhood of the points $v_0^{(\pm)}$ or, as we shall conventionally say, at the points $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$ where $n_1^2 = 0$ (for simplicity we shall analyse only the case where $u < 1$). Geometrical optics is, however, applicable to wave 2 at the points $v_{10}^{(\pm)}$, and so the reflection of wave 1 does not affect wave 2.

Geometrical optics is inapplicable to wave 2 (the ordinary wave) at $v = v_{20} = 1$, where $n_2^2 = 0$. In turn, geometrical optics is valid for wave 1 at this point. An exception is formed by the case of small angles α, when, as we see from Fig. 11.10, geometrical optics is applicable to waves of neither type near the point $v = 1$; as $\alpha \to 0$ with $v \approx 1$, $n_2 \approx 0$ and $dn_{1,2}/dz = n'_{1,2} \to \infty$. Hence, as $\alpha \to 0$, waves of type 2 may become of type 1, and this actually occurs, thus explaining the peculiarity, mentioned in § 11, of the passage to the limiting case of longitudinal propagation (see § 28).

We have hitherto considered the case where v varies smoothly and slowly with z. If the density N has steep gradients (or, what is the same thing, v has steep gradients), the derivatives $dn_{1,2}/dz$ may be large for both waves for any v. In other words, the diagrams such as Figs. 11.2, 11.3, etc., where the function $n_{1,2}(v)$ is plotted, show the behaviour of the function $n_{1,2}(z)$ only if v is, over the whole range, a linear or at least a smooth monotonic function of z. Only with this condition does the smallness of the derivative dn/dv imply that of $n' \equiv dn/dz$, which appears in (23.19). When steep gradients $dn_{1,2}/dz$ are present, the waves of different types and directions may in general change into one another. In the limit of a sharp discontinuity of the properties of the medium at some boundary, reflection and refraction of waves are described by the familiar Fresnel's formulae for an anisotropic medium.

The region near the boundary of the layer and the interaction of normal waves there

At the boundary of the layer ($v \to 0$) the condition (23.19) is not, however, a sufficient one for the approximation of geometrical optics to be valid; this is the exceptional case mentioned in connection with the derivation of (23.19).

For small v such that

$$v \ll 1, \quad v \ll |(u - 1)/(u \cos^2\alpha - 1)| \tag{23.20}$$

we have as far as first-order terms [see (11.3), (11.6) and (11.26)]

$$\left.\begin{aligned} A &= 1 + v/(u-1), \\ B &= 1 + v(1 - u\sin^2\alpha)/(u-1), \\ C &= \sqrt{u}\,(v\cos\alpha)/(u-1), \\ n_{1,2}^2 &= 1 - 2v/\{2 - u\sin^2\alpha \pm \sqrt{(u^2\sin^4\alpha + 4u\cos^2\alpha)}\}, \\ \Delta n &= n_2 - n_1 = \sqrt{(u^2\sin^4\alpha + 4u\cos^2\alpha)}v/2(1-u), \\ K_{1,2} &= -i\cdot\frac{2\sqrt{u}(1-v)\cos\alpha}{u\sin^2\alpha \mp \sqrt{(u^2\sin^4\alpha + 4u\cos^2\alpha)}}. \end{aligned}\right\} \tag{23.21}$$

Let us now consider the second condition (23.18):

$$\frac{\lambda_0}{2\pi}\,\frac{K}{C}\,\frac{n\,n'\,K\,K'}{\chi'(1-K^2)} \ll 1.$$

Using (23.21), and especially the fact that C is proportional to the parameter v, we see that $C \to 0$ when $v \to 0$, and thus the second condition (23.18) is not satisfied. The inequality (23.19) *is* satisfied, provided that the derivative dv/dz is not too large, as is usually true in the ionosphere and the solar corona.

If we write v at the boundary of the layer as

$$v = az, \tag{23.22}$$

then, ignoring for simplicity the factors $u - 1$, $\cos^2\alpha$, $\sin^2\alpha$, etc. (i.e. assuming that these factors are neither very large nor very small), we obtain instead of the second inequality (23.18) the condition

$$\frac{\lambda_0}{2\pi z} \sim \frac{\lambda_0}{2\pi} \frac{d\Delta n/dz}{\Delta n} \ll 1; \tag{23.23}$$

here $n_2 - n_1 = \Delta n \sim v \approx az$, and the coordinate z is measured from the boundary of the layer. If instead of (23.22) we put $v = az^m$, where the exponent m is neither very small nor very large, the condition obtained is practically the same as (23.23). For an arbitrary dependence of v on z, if we again ignore factors such as $u - 1$, (23.23) is replaced by

$$\frac{\lambda_0}{2\pi} \frac{dv/dz}{v} \ll 1.$$

It is clear from (23.23) that geometrical optics is invalid when z is sufficiently small. More precisely, geometrical optics cannot then be used to discuss the polarisation of the waves which are propagated, since for $v \to 0$ the indices $n_{1,2} \to 1$ and the medium approaches a vacuum as regards the velocity of propagation and rate of change of phase. The inapplicability of the approximation of geometrical optics at the boundary of the layer ($v \to 0$) is a particular case of its inapplicability to the consideration of polarisation in an inhomogeneous medium for vanishingly small anisotropy. The reason is that in an anisotropic medium the polarisation of normal waves remains fixed [see, for example, (11.29)] even when the anisotropy (that is, for instance, the difference $n_2 - n_1$) tends to zero. In an isotropic medium, however, we have degeneracy, since the velocities of propagation of the two normal waves are the same, and these waves may themselves be chosen arbitrarily within certain limits: they may be taken to be linearly, circularly or elliptically polarised. Next, in an isotropic medium with the wave incident normally on the layer (i.e. if the wave normal is parallel to the gradient of ε) there is no change in polarisation [in (16.3) the components E_x and E_y are independent].† For a magnetoactive medium, on the other hand, (11.25) and the above discussion

† In the general case for an isotropic medium there is, of course, a change in polarisation (see, in particular, § 19) due to the fact that the vector equation (16.1) does not separate into independent equations for E_x, E_y and E_z. If the ray is twisted, i.e. its path does not lie in a plane, there is also a rotation of the plane of polarisation.

show that the polarisation depends on $v(z)$ and, according to the approximation of geometrical optics, should begin to vary at the boundary of the layer. However, it is evident that the presence of extremely weak anisotropy cannot have any significant effect, and the approximation of geometrical optics is therefore inapplicable.

The situation is especially clear for the example of propagation of light in a crystalline medium with slight anisotropy in which the directions of the principal axes of the permittivity ellipsoid vary along the ray [146]. Let the propagation be along the z-axis, with two principal axes in the xy-plane and rotating as z varies, in such a way that over a distance Δz the angle between the principal axes and some fixed direction in the xy-plane varies by $\Delta\Psi = a\Delta z$. In the approximation of geometrical optics the normal waves, which are linearly polarised in the direction of the principal axes of the permittivity ellipsoid, are entirely independent of each other, and at every point their polarisation is the same as in the corresponding homogeneous medium. Thus in the approximation of geometrical optics the plane of polarisation of waves propagated in the medium considered must rotate, over a distance Δz, through the same angle $\Delta\Psi = a\Delta z$ as the principal axes. But if the difference Δn of the refractive indices of the waves is sufficiently small, the rotation of the plane of polarisation must evidently tend to zero, since for $\Delta n = 0$ there is no rotation of the plane of polarisation (in this case the rotation of the axes is purely formal).

This contradiction is resolved if we take account of the fact that geometrical optics is invalid when $\Delta n \to 0$. The condition for the approximation of geometrical optics to be valid in the present case is [146]

$$\frac{d\Psi}{dz}\,\frac{\lambda_0}{2\pi\,\Delta n} = \frac{a\,\lambda_0}{2\pi\,\Delta n} \ll 1\,. \tag{23.24}$$

If $\Delta n \sim 1$, this condition takes the form of the usual requirement for the applicability of geometrical optics, that the change in the properties of the medium (in this case the angle Ψ) over a wavelength should be small, but when $\Delta n \ll 1$ the condition (23.24) has independent significance.

In a magnetoactive medium, $d\Psi/dz$ is replaced by $dK_{1,2}/dz$; at the edge of the ionosphere, $dK_{1,2}/dz \sim dv/dz \sim a$, $\Delta n \sim v \sim az$, and the condition (23.24) becomes (23.23). The propagation of waves at the boundary of a layer of magnetoactive plasma, where geometrical optics is invalid, is discussed in § 26. Using the terminology explained in § 20, we can say that there is an interaction of the normal waves of types 1 and 2 at the boundary of the layer. In other words, the normal waves 1 and 2 obtained in the approximation of geometrical optics do not represent, at the boundary of the layer, the exact solution of equations (23.2); nevertheless, a certain approximation to a correct solution can be obtained from a certain combination of interrelated ("interacting") solutions of the geometrical-optics type. The interaction of the waves in this sense occurs also in the region $v \approx 1$ in the case of small angles α mentioned above; this is considered in more detail in § 28.

Finally, essentially the same type of interaction is responsible for the reflection of waves of type 1 or type 2 from the points $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$ or $v_{20} = 1$, where n_1^2 or n_2^2 vanishes. Here, however, an important simplification already discussed above is possible for sufficiently extensive layers of inhomogeneous plasma: if the points $v_{10}^{(\pm)}$ and v_{20} are far apart, the inapplicability of geometrical optics affects only one type of wave at each point. In such cases we go beyond the limits of geometrical optics by investigating not the equations (23.2) but one second-order equation analogous to (16.3) for an isotropic medium. The corresponding calculations are given in § 25.

§ 24. PROPAGATION OF PULSES

The group-velocity vector in a magnetoactive medium

In Chapter III and § 23 we have considered only the propagation of monochromatic plane waves $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ in a homogeneous magnetoactive medium, and that of waves with $\boldsymbol{E} = \boldsymbol{E}_0(z)\, e^{i\omega t}$ in an inhomogeneous magnetoactive plasma. We must now examine also the problem of the propagation of pulses (i.e. wave packets or groups bounded in space and time) in a magnetoactive medium. The direction of motion of such a packet is, by definition, the direction of the ray, and the velocity of the packet is called the ray velocity or group velocity. In an anisotropic medium the direction of the ray and that of the wave vector $\boldsymbol{k}$ (i.e. the normal to the wave) do not in general coincide.

The direction of the ray, which is the direction of motion of the wave group, in an arbitrary non-absorbing homogeneous medium may be found by expanding the wave field as a Fourier integral of "normal" plane waves propagated in the medium:

$$E(\boldsymbol{r}, t) = \int g(\boldsymbol{k})\, e^{i[\omega(\boldsymbol{k})t - \boldsymbol{k}\cdot\boldsymbol{r}]}\, d\boldsymbol{k}; \tag{24.1}$$

here $d\boldsymbol{k} = dk_x\, dk_y\, dk_z$ and E may be taken as any component of the electric field. We could, of course, equally well consider components of the vectors $\boldsymbol{H}$ and $\boldsymbol{D}$, and if the function $g(\boldsymbol{k})$ is taken to be a vector, we can take E in (24.1) to be the field vector $\boldsymbol{E}$ itself.

As already stated, the waves $E = \text{constant} \times e^{i[\omega(\boldsymbol{k})t - \boldsymbol{k}\cdot\boldsymbol{r}]}$ are assumed to satisfy equations (11.1) and (11.2) with $\boldsymbol{j} = 0$ (absence of absorption) and the equation $D_i = \varepsilon_{ik}(\omega)\, E_k$ (a linear homogeneous dispersive medium). Such waves have been referred to as "normal" waves. In an anisotropic medium which is arbitrary (except for the restrictions just mentioned) two waves (apart from the plasma wave, which is not considered here) can be propagated in each direction characterised by the unit vector along the wave normal $\boldsymbol{k}_1 = \boldsymbol{k}/|\boldsymbol{k}| = \boldsymbol{k}/k$. These waves differ in polarisation and in phase velocity v_{ph}:

$$v_{\text{ph}}(\boldsymbol{k}) = c/n(\boldsymbol{k}) = \omega(\boldsymbol{k})/|\boldsymbol{k}| = \omega/k, \tag{24.2}$$

where $n(\boldsymbol{k}) = c/v_{\text{ph}}$ is, by definition, the refractive index, and writing the frequency ω in the form $\omega(\boldsymbol{k})$ signifies that for any given frequency ω and direction $\boldsymbol{k}_1 = \boldsymbol{k}/|\boldsymbol{k}|$ the quantity k [or v_{ph} or $n(k)$] is entirely determinate for each of the normal waves. Thus

$$k = k(\boldsymbol{k}_1, \omega) = \omega n(\boldsymbol{k}_1, \omega)/c, \tag{24.3}$$

and conversely $\omega = \omega(\boldsymbol{k}_1, k) = \omega(\boldsymbol{k})$. In the particular case of an ionised gas in a magnetic field the value of $n = n_{1,2}$ is given by formula (11.6), where $n_{1,2}$ depends on ω through $v \equiv \omega_0^2/\omega^2$ and $u \equiv \omega_H^2/\omega^2$ and on the direction of the wave normal through the angle α between $\boldsymbol{k}_1$ and $\boldsymbol{H}^{(0)}$.

Assuming that the pulse is quasimonochromatic, and therefore sufficiently extensive in all directions, so that the function $g(\boldsymbol{k})$ has a sharp maximum near the "carrier wave vector" $\boldsymbol{k}_0$, we can expand the frequency $\omega(\boldsymbol{k})$ in (24.1) in series. Taking only two terms of the expansion, we have

$$E(\boldsymbol{r}, t) = e^{i[\omega(\boldsymbol{k}_0)t - \boldsymbol{k}_0 \cdot \boldsymbol{r}]} \int g(\boldsymbol{k}) \exp\left[i \frac{\partial \omega}{\partial \boldsymbol{k}} \cdot \Delta \boldsymbol{k}\, t - \Delta \boldsymbol{k} \cdot \boldsymbol{r}\right] d\boldsymbol{k}, \tag{24.4}$$

where
$$\Delta \boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}_0$$
and
$$\frac{\partial \omega}{\partial \boldsymbol{k}} \equiv \text{grad}_k\, \omega \equiv \frac{\partial \omega}{\partial k_x} \boldsymbol{i} + \frac{\partial \omega}{\partial k_y} \boldsymbol{j} + \frac{\partial \omega}{\partial k_z} \boldsymbol{k}',$$

$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}'$ being unit vectors along the x, y, z axes, and the value of $\partial\omega/\partial \boldsymbol{k}$ being taken for $\boldsymbol{k} = \boldsymbol{k}_0$. It is clear from (24.4) that in this approximation the pulse is propagated as a whole with the group velocity†

$$\boldsymbol{v}_{\text{gr}} = \partial\omega/\partial \boldsymbol{k}, \tag{24.5}$$

the value of $\partial\omega/\partial \boldsymbol{k}$ being taken for $\boldsymbol{k} = \boldsymbol{k}_0$. When higher derivatives in the series expansion of $\omega(\boldsymbol{k})$ are taken into account, the pulse is spread out; the simplest case of this process has been discussed in § 21.

The direction of the vector $\boldsymbol{v}_{\text{gr}}$ is that of the ray, and its magnitude is the velocity of motion of the group along the ray.

In an isotropic medium, by definition $k = \omega n(\omega)/c$, i.e. k is independent of direction. Hence we have, for example,

$$\partial\omega/\partial k_x = (d\omega/dk)\, \partial k/\partial k_x = (d\omega/dk)\, k_x/k \left(\text{since } k = \sqrt{(k_x^2 + k_y^2 + k_z^2)}\right)$$

and

$$\boldsymbol{v}_{\text{gr}} = \frac{d\omega}{dk} \frac{\boldsymbol{k}}{|\boldsymbol{k}|} = \frac{d\omega}{dk} \boldsymbol{k}_1, \tag{24.6}$$

† The integral in (24.4) is constant if $(\partial\omega/\partial \boldsymbol{k})t - \boldsymbol{r}$ is constant, and this gives formula (24.5) for the velocity of the packet.

i.e. the group velocity is, as it should be, equal in magnitude to (21.17) and parallel or antiparallel to the vector $\boldsymbol{k}$. In the absence of spatial dispersion, $d\omega/dk > 0$, i.e. the vectors $\boldsymbol{v}_{\mathrm{gr}}$ and $\boldsymbol{k}$ are parallel; when spatial dispersion is present, $d\omega/dk$ may have either sign (see Appendix A).

In the case of a magnetoactive medium at present under consideration, the index $n = n_{1,2}$ depends only on the angle α between the vector $\boldsymbol{k}_1 = \boldsymbol{k}/|\boldsymbol{k}|$ and the field $\boldsymbol{H}^{(0)}$ (as well as on ω). Hence, taking the direction of $\boldsymbol{H}^{(0)}$ as the z'-axis and $\cos\alpha = \gamma_3 \equiv \gamma$, we have

$$k = (\omega/c)\, n(\omega, k_{z'}/k) = (\omega/c)\, n(\omega, \gamma), \tag{24.7}$$

from which it is clear that $\omega = \omega(k, k_{z'})$. Differentiating (24.7), we obtain

$$\frac{dk}{dk_{z'}} = \gamma = \frac{1}{c}\frac{\partial\omega}{\partial k_{z'}} n + \frac{\omega}{c}\frac{\partial n}{\partial\omega}\frac{\partial\omega}{\partial k_{z'}} + \frac{\omega}{c}\frac{\partial n}{\partial(k_{z'}/k)}\frac{\partial(k_{z'}/k)}{\partial k_{z'}}$$

$$= \frac{1}{c}\left(n + \omega\frac{\partial n}{\partial\omega}\right)\frac{\partial\omega}{\partial k_{z'}} + \frac{1}{n}\frac{\partial n}{\partial\gamma}(1-\gamma^2), \text{ etc.,}$$

and thus

$$\left.\begin{aligned}
v_{\mathrm{gr},x'} &= \frac{\partial\omega}{\partial k_{x'}} = \frac{\gamma_1 c[1 + (\gamma/n)\,\partial n/\partial\gamma]}{n + \omega\,\partial n/\partial\omega} \\
&= \frac{\gamma_1\,\partial(\gamma n)/\partial\gamma}{(n/c)\,\partial(\omega n)/\partial\omega}, \\
v_{\mathrm{gr},y'} &= \frac{\partial\omega}{\partial k_{y'}} = \frac{\gamma_2\,\partial(\gamma n)/\partial\gamma}{(n/c)\,\partial(\omega n)/\partial\omega}, \\
v_{\mathrm{gr},z'} &= \frac{\partial\omega}{\partial k_{z'}} = \frac{c[\gamma - \{(1-\gamma^2)/n\}\,\partial n/\partial\gamma]}{n + \omega\,\partial n/\partial\omega} \\
&= \frac{\gamma\,\partial(\gamma n)/\partial\gamma - \partial n/\partial\gamma}{(n/c)\,\partial(\omega n)/\partial\omega}, \\
v_{\mathrm{gr},r'} &= \sqrt{(v_{\mathrm{gr},x'}^2 + v_{\mathrm{gr},y'}^2)} \\
&= \frac{\sqrt{(1-\gamma^2)}\,\partial(\gamma n)/\partial\gamma}{(n/c)\,\partial(\omega n)/\partial\omega} \\
&= -\frac{\partial(n\cos\alpha)/\partial\alpha}{(n/c)\,\partial(\omega n)/\partial\omega},
\end{aligned}\right\} \tag{24.8}$$

where γ_1 and γ_2 are the cosines of the angles between the vector $\boldsymbol{k}$ and the x' and y' axes (and, as before, $\gamma \equiv \gamma_3 = \cos\alpha$, the cosine of the angle α between the vector $\boldsymbol{k}$ and the z'-axis, which is in the direction of the field $\boldsymbol{H}^{(0)}$). Also

$$\left.\begin{aligned}
v_{\mathrm{gr}} &= \left|\frac{\partial\omega}{\partial\boldsymbol{k}}\right| = \frac{c\sqrt{[1 + (1/n^2)(\partial n/\partial\gamma)^2(1-\gamma^2)]}}{n + \omega\,\partial n/\partial\omega}, \\
\cos(\boldsymbol{k}, \partial\omega/\partial\boldsymbol{k}) &= \frac{1}{\sqrt{[1 + (1/n^2)(\partial n/\partial\gamma)^2(1-\gamma^2)]}},
\end{aligned}\right\} \tag{24.9}$$

$$\left.\begin{aligned}\tan(\boldsymbol{k}, \partial\omega/\partial\boldsymbol{k}) &= (1/n)(\partial n/\partial\gamma)\sqrt{(1-\gamma^2)}\\ &= (1/n)\sin\alpha\,\partial n/\partial\cos\alpha\\ &= -(1/2n^2)\,\partial n^2/\partial\alpha,\\ v_{\mathrm{gr},k} = v_{\mathrm{gr}}\cos(\boldsymbol{k}, \partial\omega/\partial\boldsymbol{k}) &= c \div d(\omega n)/d\omega.\end{aligned}\right\}\tag{24.10}$$

These formulae (24.9) and (24.10) are evidently valid for any choice of coordinates, with γ always the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$. The value of $n(\omega,\gamma)$ in (24.8) and (24.9) must be that given by (11.6), i.e. n_1 for the extraordinary wave and n_2 for the ordinary wave (with $\gamma = \cos\alpha$); and we take $n_{1,2}$ to be real, thus ignoring values of $n_{1,2}^2 < 0$ (in the latter case

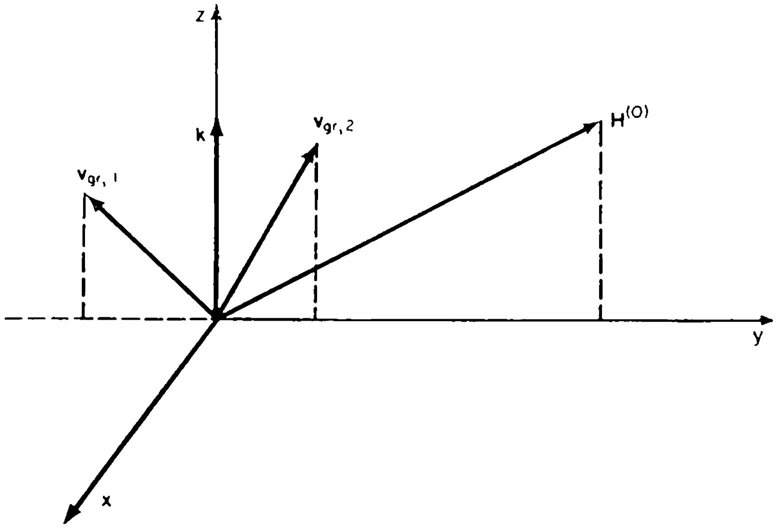

FIG. 24.1. Directions of the group-velocity vectors $\boldsymbol{v}_{\mathrm{gr},1,2}$.

the waves are usually strongly damped and the concept of group velocity is in general inapplicable). The presence of absorption, as in § 21, does not alter the above results if the absorption coefficient varies sufficiently slowly with frequency within the spectral width of the signal.†

The preceding discussion shows that in general the group-velocity vectors of the waves of types 1 and 2, i.e. the vectors $\boldsymbol{v}_{\mathrm{gr},1}$ and $\boldsymbol{v}_{\mathrm{gr},2}$, are not parallel to the normal vector $\boldsymbol{k}$ or to each other, but lie in the plane of $\boldsymbol{H}^{(0)}$ and $\boldsymbol{k}$ (Fig. 24.1). We shall not pause here to determine the direction and magnitude of the vectors $\boldsymbol{v}_{\mathrm{gr},1}$ and $\boldsymbol{v}_{\mathrm{gr},2}$ in various cases (see § 29). For a wave normally incident on an inhomogeneous medium the question of the direction of $\boldsymbol{v}_{\mathrm{gr}}$ is not of great importance, since the vector $\boldsymbol{k}$ is always in the z-direction, and the component of $\boldsymbol{v}_{\mathrm{gr}}$ in that direction, which in this case determines the group delay time [see formula (24.15) below], is, by (24.10),

$$v_{\mathrm{gr},z} = \frac{c}{n_{1,2} + \omega\,\partial n_{1,2}/\partial\omega} = \{\partial[\omega\, n_{1,2}(\omega,\gamma)/c]/\partial\omega\}^{-1}.$$

† For an absorbing magnetoactive plasma we can obtain expressions for the mean energy density $\overline{W'_E}$ and the rate of energy flow as in § 22 for an isotropic plasma.

In other words, the component of the group velocity in the direction of $\boldsymbol{k}$ (in this case, along the z-axis) has the same form as the group velocity in an isotropic medium [see (21.17)], with, of course, n replaced by $n_{1,2}$.

On account of the complexity of (11.6), the expressions for $v_{\mathrm{gr},1,2z}$ in terms of v, u and α are fairly involved. Here we shall merely mention that, for "quasi-transverse" (and, in particular, transverse) propagation of an ordinary wave, the value of $v_{\mathrm{gr},2z}$ is, by (11.14), equal to the group velocity

$$v_{\mathrm{gr}} = cn = c\sqrt{(1 - 4\pi e^2 N/m\omega^2)}$$

obtained when the effect of the field is neglected. For "quasilongitudinal" propagation we have [see (11.37) with $\nu_{\mathrm{eff}} = 0$]

$$\begin{aligned} v_{\mathrm{gr},1,2z} &= \frac{c\,n_{1,2}}{1 \mp 2\pi e^2 N\,\omega_L/m\,\omega(\omega \pm \omega_L)^2} \\ &= \frac{c\sqrt{[1 - 4\pi e^2 N/m\,\omega(\omega \pm \omega_L)]}}{1 \mp 2\pi e^2 N\,\omega_L/m\,\omega(\omega \pm \omega_L)^2}. \end{aligned} \tag{24.11}$$

It may also be noted that for low frequencies, when the formula $n_2^2 = v/\sqrt{u}\cos\alpha$ holds [see (11.24)], (24.9) gives $\tan(\boldsymbol{k}, \partial\omega/\partial\boldsymbol{k}) = -\frac{1}{2}\tan\alpha$. From this it follows that with $n^2 = \mathrm{constant}/\cos\alpha$ the angle between $\boldsymbol{k}$ and $\partial\omega/\partial\boldsymbol{k}$ depends only on α. It is also easily seen that in this case the angle between $\partial\omega/\partial\boldsymbol{k}$ and the field $\boldsymbol{H}^{(0)}$ cannot exceed 19° 28′.

The group-velocity vector, the direction of the ray and the energy-flux vector

The above derivation of the direction of the ray as that of the group velocity is equivalent to determining the ray on the basis of Huygens' principle, according to which the ray surface is the envelope of a family of wave planes. For the family of wave planes is in parametric form (with parameters k_x, k_y, k_z) $\boldsymbol{k}\cdot\boldsymbol{r}' = \omega(\boldsymbol{k}) = ck/n(\omega, \boldsymbol{k})$, and the equation of the envelope in parametric form is $\boldsymbol{r}' = \partial\omega/\partial\boldsymbol{k}$. In other words, for a given $\boldsymbol{k}$ the ray is in the direction of $\partial\omega/\partial\boldsymbol{k}$, which vector in turn depends on ω (on account of dispersion) when $\boldsymbol{k}/|\boldsymbol{k}|$ is given.

In an isotropic medium the direction of the ray evidently coincides not only with $\boldsymbol{k}$ but also with the energy-flux vector $\boldsymbol{S} = c\boldsymbol{E}\times\boldsymbol{H}/4\pi$. In non-gyrotropic crystals, where the tensor ε_{ik} is real and symmetrical, the direction of the ray (i.e. $\boldsymbol{v}_{\mathrm{gr}}$) does not coincide with that of $\boldsymbol{k}$, but is parallel to the vector $\boldsymbol{S}$ (see, for example, [36, § 77]). In gyrotropic crystals, i.e. those which possess natural optical activity, and in a magnetoactive medium (in particular, in an ionised gas in a magnetic field), the tensor ε_{ik} is Hermitian but not real, and the normal waves are in general elliptically polarised. Hence the direction of the vector $\boldsymbol{S} = c\boldsymbol{E}\times\boldsymbol{H}/4\pi$ varies with time, and its terminus describes a closed curve in half a period, i.e. in a time π/ω. We shall not give expressions for the components of the vector $\boldsymbol{S}$, since they are not needed, but it may be mentioned that the components of the vector $\boldsymbol{E}$ are given by the formulae of § 11, while the magnetic field of a plane wave $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$ is

$$\boldsymbol{H} = (ic/\omega)\operatorname{curl}\boldsymbol{E} = c\boldsymbol{k} \times \boldsymbol{E}/\omega = n\boldsymbol{k} \times \boldsymbol{E}/|\boldsymbol{k}|.$$

In calculating the energy-flux vector $\boldsymbol{S}$ we must use real vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, i.e. take re $\boldsymbol{E}$ and re $\boldsymbol{H}$. The corresponding formulae are given, for example, in [147]; the nature of the rotation of the vector is shown diagrammatically in Fig. 24.2. Since the vector $\boldsymbol{S}$ rotates, its instantaneous value has, of course, no particular physical significance. It may be assumed, however, that the time-average vector $\overline{\boldsymbol{S}}$ is in the direction of the energy flux; Fig. 24.2 shows that

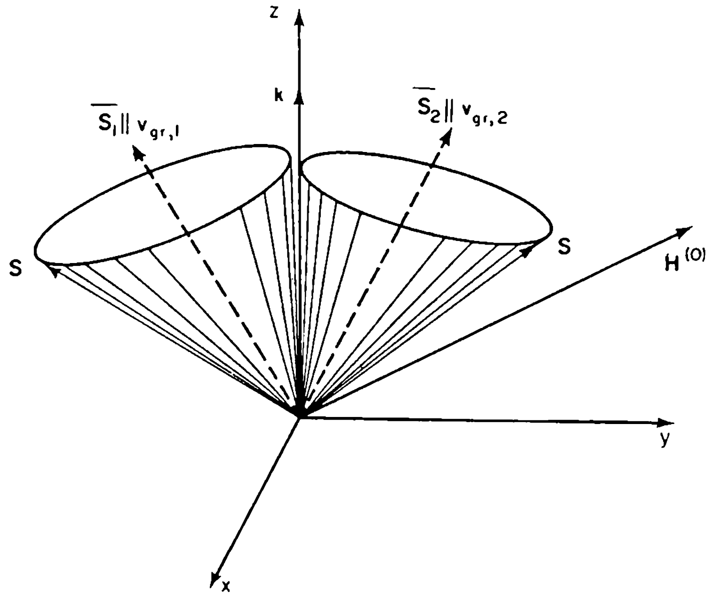

FIG. 24.2. Poynting vector S, the time average vector $\overline{S}$, and the group-velocity vector $\boldsymbol{v}_{gr}$ (suffixes 1 and 2 refer to the extraordinary and ordinary waves respectively).

this vector is along the axis of the cone described by the vector $\boldsymbol{S}$, and so the latter never coincides with $\overline{\boldsymbol{S}}$ unless the cone degenerates to a sector or a line.

In the case of a magnetoactive plasma the group-velocity vector $\boldsymbol{v}_{gr}$ is parallel to $\overline{\boldsymbol{S}}$, as has been shown in [147]; this result holds good in an arbitrary non-absorbing linear homogeneous medium (see [142; 36, § 77]). The calculation of $\boldsymbol{v}_{gr}$ in a homogeneous medium is simpler than that of $\overline{S}$. Thus there is no particular reason or need to use the vector $\boldsymbol{S}$ in a homogeneous medium, and so we have not discussed the matter in greater detail. There is, however, no objection to finding the direction of the ray by calculating $\overline{S}$. Here the most convenient procedure is to calculate immediately what is called the complex energy-flux vector, which is equal to $\overline{S}$. This method was used in [58] and will be discussed in that connection in § 34.

Propagation of pulses in an inhomogeneous medium

In the approximation of geometrical optics, the propagation of pulses in an inhomogeneous medium is the same as in a homogeneous medium with varying properties (a quasihomogeneous medium). In other words, in any region of the medium the pulse is propagated with the group velocity $\boldsymbol{v}_{gr}$ corresponding to a homogeneous medium with the same values of v, u and α as that region of the inhomogeneous medium. The ray path is a curve at every point of which the direction of the tangent is that of the vector $\boldsymbol{v}_{gr}$.

These results are clear from the significance of the approximation of geometrical optics, and may be formally demonstrated as follows. In an inhomogeneous medium the expansion (24.1) with an arbitrary function $g(\boldsymbol{k})$ is inadmissible, because in this case the plane waves do not satisfy the wave equation. However, if the approximation of geometrical optics is valid, we can, at least in a restricted region of space, expand the field $\boldsymbol{E}(\boldsymbol{r}, t)$ in terms of solutions of the type (23.12). Neglecting the coefficients of the exponentials, which vary only slowly with the coordinates, we can write the field as

$$E(\boldsymbol{r}, t) = \int g(\boldsymbol{k}) \exp\left\{i\left[\omega(\boldsymbol{k}(\boldsymbol{r}))\, t - \int^{\boldsymbol{r}} \boldsymbol{k}(\boldsymbol{r}) \cdot d\,\boldsymbol{r}\right]\right\} d\,\boldsymbol{k}. \tag{24.12}$$

Now expanding $\omega(\boldsymbol{k})$ in series, we obtain

$$E(\boldsymbol{r}, t) = \exp\left\{i\left[\omega(\boldsymbol{k}_0(\boldsymbol{r}))\, t - \int^{\boldsymbol{r}} \boldsymbol{k}_0 \cdot d\,\boldsymbol{r}\right]\right\} \times$$
$$\times \int g(\boldsymbol{k}) \exp\left\{i\left[\frac{\partial\omega}{\partial\boldsymbol{k}} \cdot \Delta\,\boldsymbol{k}\, t - \int^{\boldsymbol{r}} \Delta\,\boldsymbol{k} \cdot d\boldsymbol{r}\right]\right\} d\,\boldsymbol{k}, \tag{24.13}$$

where $\boldsymbol{k}_0(\boldsymbol{r})$ is the wave vector of the carrier wave of the pulse.

Hence it follows that the equation of motion of the pulse is

$$\frac{\partial\omega}{\partial\boldsymbol{k}} \cdot t\,\Delta\,\boldsymbol{k}(\boldsymbol{r}) - \int^{\boldsymbol{r}} \Delta\,\boldsymbol{k}(\boldsymbol{r}) \cdot d\,\boldsymbol{r} = \text{constant},$$

and the velocity of the pulse is $\boldsymbol{v}_{gr} = d\,\boldsymbol{r}/dt = \partial\omega/\partial\,\boldsymbol{k}$ if the derivatives of $\boldsymbol{k}$ with respect to $\boldsymbol{r}$ are sufficiently small. The expansion used in (24.12) shows that the derivative is taken for $\boldsymbol{k} = \boldsymbol{k}_0(\boldsymbol{r})$.

If the approximation of geometrical optics is inapplicable, the field $\boldsymbol{E}$ must be expanded as an integral of eigenfunctions of the wave equation in the inhomogeneous medium concerned; for a linear layer in the isotropic case, these are Bessel or Airy functions, as shown in § 17. Such a treatment is necessary, for example, in order to determine the direction of the group velocity at the vertex of the ray path on reflection from a layer, but it has not yet been directly carried out. In the ionosphere and the solar corona geometrical optics is invalid only in small regions, and so the discussion of ray paths beyond the scope of this approximation is not of great importance (see also § 34).

In the approximation of geometrical optics with normal incidence (the wave vector $\boldsymbol{k}$ parallel to the z-axis) the change in the wave phase over a path z is, by (23.12),

$$\varphi = \frac{\omega}{c} \int_0^z n_{1,2}(z)\, d z . \tag{24.14}$$

In the same approximation the group delay time along a ray path ds is [see (24.10)]

$$\begin{aligned} \Delta t_{\mathrm{gr}} &= \int_0^z \frac{d s}{v_{\mathrm{gr},1,2}} = \int_0^z \frac{d z}{v_{\mathrm{gr},1,2} \cos(\boldsymbol{v}_{\mathrm{gr}}, \boldsymbol{k})} = \int_0^z \frac{d z}{v_{\mathrm{gr},1,2z}} \\ &= \int_0^z \frac{1}{c}\, \frac{\partial[\omega\, n_{1,2}(\omega, z)]}{\partial \omega}\, d z = \partial \varphi / \partial \omega , \end{aligned} \tag{24.15}$$

where it is assumed that the integration along the ray takes place at the same altitude z as in (24.14). In particular, the point z may be the point at which the wave is reflected from the layer. Formula (24.15) differs from that in the isotropic case [see, for instance, (30.10)] only in that n is replaced by $n_{1,2}$. The difference in the directions of $\boldsymbol{v}_{\mathrm{gr}}$ and the normal vector $\boldsymbol{k}/|\boldsymbol{k}|$, and that in the values of $\boldsymbol{v}_{\mathrm{gr}}$ and the group velocity in an isotropic medium, have no effect in (24.15) except to replace n by $n_{1,2}$; this is clear from (24.10). It must be emphasised, however, that this statement is valid only on the assumption that the properties of the medium are independent of the coordinates x and y (here see § 35). We may also mention that the relation $\Delta t_{\mathrm{gr}} = \partial\varphi/\partial\omega$ does not depend on the particular form of the function φ, but is general, since the proof given in § 21 is independent of the properties of the medium. The same is true of the spreading of signals, and so the results of § 21 are entirely valid in the present case also.

§ 25. REFLECTION OF WAVES FROM AN INHOMOGENEOUS LAYER

Reflection from a layer. Angles $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$

The approximation of geometrical optics is inadequate even in the simplest case where one of the waves is reflected from the layer when it reaches the region of negative $\tilde{n}^2$. In such cases, however, it is not usually necessary to investigate the general solution of equations (23.2). As has already been noted in § 23, and as is clear from qualitative arguments, the solution of the problem of reflection of one of the waves can be reduced, as a good approximation, to that of one second-order equation of the form

$$d^2 g_{1,2}/dz^2 + (\omega^2/c^2)\tilde{n}^2_{1,2} g_{1,2} = 0, \tag{25.1}$$

where the function $g_{1,2}$ is in some way related to E_x and E_y. The generalisation of the results thus obtained to the case where absorption is present offers no difficulty in principle. We shall therefore consider only a non-absorbing magnetoactive plasma.

If $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$ (longitudinal and transverse propagation), equations (23.2) separate rigorously into two independent second-order equations. For example, in longitudinal propagation we use new variables

$$F_\pm = E_x \pm iE_y. \tag{25.2}$$

Since, for $\alpha = 0$ (see § 11),

$$\left.\begin{aligned} \tilde{n}^2_{1,2} \equiv \tilde{n}^2_\mp = A \mp C = 1 - v/(1 \pm \sqrt{u}), \\ K_{1,2} = \pm i, \end{aligned}\right\} \tag{25.3}$$

it is easy to see that in terms of the variables $F_\pm$ equations (23.2) become

$$\frac{d^2 F_\pm}{dz^2} + \frac{\omega^2}{c^2} n^2_\pm F_\pm = \frac{d^2 F_\pm}{dz^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 \mp \sqrt{u}}\right) F_\pm = 0. \tag{25.4}$$

Each of the equations (25.4) is of the form (25.1) with $g_{1,2} = F_\pm$; in particular, if v is a linear function of z, i.e.

$$v = az + b, \tag{25.5}$$

each equation (25.4) is equivalent to (25.1) for a linear layer, with all the consequences resulting from this (see § 16).

For transverse propagation ($\alpha = \frac{1}{2}\pi$) we have

$$\left.\begin{aligned} \tilde{n}^2_1 = A = 1 - v(1-v)/(1-u-v), \quad K_1 = 0, \\ \tilde{n}^2_2 = \tilde{n}^2_0 = B = 1 - v, \quad C = 0, \quad K_2 = i\infty, \end{aligned}\right\} \tag{25.6}$$

and equations (23.2) become

$$\left.\begin{aligned} \frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2} n^2_1 E_x = \frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2}\left[1 - \frac{v(1-v)}{1-u-v}\right] E_x = 0, \\ \frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2} n^2_2 E_y = \frac{d^2 E_y}{dz^2} + \frac{\omega^2}{c^2}(1-v) E_y = 0, \end{aligned}\right\} \tag{25.7}$$

i.e. the equations separate and have the form (25.1) with $g_1 = E_x$ and $g_2 = E_y$.

The second equation (25.7), moreover, is the same as the equation for an isotropic plasma, since $n^2_2 = n^2_0 = 1 - v$. The equation for E_x is non-linear in z even if $v = az$, but for the validity of certain formulae it is sufficient that the function $n^2_1(z)$ may be replaced by a linear function in the region where geometrical optics is inapplicable. Taking the origin where $n^2_1 = 0$, we put

$$v = 1 \pm \sqrt{u} + az, \tag{25.8}$$

where the + or − sign is chosen depending on which of the two points $n_1^2 = 0$ is in the region considered.

Then, if

$$|az/(1 \pm \sqrt{u})| \ll 1, \tag{25.9}$$

we have

$$n_1^2 \approx -2az/(1 \pm \sqrt{u}), \tag{25.10}$$

and the first equation (25.7) reduces to (25.1) with $\tilde{n}^2$ a linear function of z.

The use of the condition (23.19) and of the expression (25.10) shows that geometrical optics is valid if

$$|z| \gg \left(\frac{1 \pm \sqrt{u}}{8a} \cdot \frac{\lambda_0^2}{4\pi^2}\right)^{\frac{1}{3}}. \tag{25.11}$$

Thus, if the conditions (25.9) and (25.11) both hold, the function $n_1^2(z)$ in the first equation (25.7) may be taken to be linear throughout the region where the approximation of geometrical optics is invalid; from this we can deduce some important general formulae (§ 30). For the F layer of the ionosphere we have $a \sim 10^{-6}$ to $10^{-7}\,\mathrm{cm}^{-1}$, $\lambda_0 \sim 6 \times 10^3$ cm, and when $1 \pm \sqrt{u} \sim 1$ the condition (25.9) signifies that $z \ll 10^6$ to 10^7 cm, while the condition (25.11) gives $z \gg 10^4$ cm, so that the two inequalities are entirely compatible.

The approximate solution for an arbitrary angle α

The separation of equations (23.2) into two independent second-order equations for $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$ occurs because in these cases the polarisation of the wave is independent of z (i.e. $K_{1,2}$ = constant). In general, however, $K = K(z)$, equations (23.2) do not separate, and they can be reduced only to one fourth-order equation for E_x or E_y. Here it is convenient, therefore, to use an approximate solution (see [148]; the problem has been discussed somewhat differently in [149]).

If we use the variables

$$F_\pm = E_y \pm KE_x, \tag{25.12}$$

where $K = K_1$ or K_2 according to the nature of the problem, equations (23.2) become, with $K' = dK/dz$, $K'' = d^2K/dz^2$,

$$\begin{aligned}\frac{d^2F_+}{dz^2} - \frac{K'}{K}\frac{dF_+}{dz} + \frac{\omega^2}{c^2}\left\{\frac{A+B}{2} + \frac{iC(K^2-1)}{2K}\right\}F_+ +\\ + \left\{\left(\frac{K'}{K}\right)^2 - \frac{K''}{2K}\right\}F_+ + \frac{K'}{K}\frac{dF_-}{dz} +\\ + \frac{\omega^2}{c^2}\left\{\frac{B-A}{2} + \frac{iC(K^2+1)}{2K}\right\}F_- - \left\{\left(\frac{K'}{K}\right)^2 - \frac{K''}{2K}\right\}F_- = 0,\end{aligned} \tag{25.13}$$

$$\frac{d^2F_-}{dz^2} - \frac{K'}{K}\frac{dF_-}{dz} + \frac{\omega^2}{c^2}\left\{\frac{A+B}{2} - \frac{iC(K^2-1)}{2K}\right\}F_- +$$
$$+\left\{\left(\frac{K'}{K}\right)^2 - \frac{K''}{2K}\right\}F_- + \frac{K'}{K}\frac{dF_+}{dz} +$$
$$+\frac{\omega^2}{c^2}\left\{\frac{B-A}{2} - \frac{iC(K^2+1)}{2K}\right\}F_+ - \left\{\left(\frac{K'}{K}\right)^2 - \frac{K''}{2K}\right\}F_+ = 0. \qquad (25.14)$$

Since by (11.3) and (11.26)

$$\frac{A+B}{2} + \frac{iC(K^2-1)}{2K} - n^2 = \frac{B-A}{2} - \frac{iC(K^2+1)}{2K} = 0,$$

an estimate of the order of magnitude of the various terms in (25.13) and (25.14) makes it natural to seek a solution of the form

$$F_\pm = F_\pm^{(0)} + F_\pm^{(1)}, \qquad (25.15)$$

where

$$|F_\pm^{(1)}| \ll |F_\pm^{(0)}| \qquad (25.16)$$

and $F_\pm^{(0)}$ satisfy the equations

$$\left.\begin{aligned} &\frac{d^2F_+^{(0)}}{dz^2} - \frac{K'}{K}\frac{dF_+^{(0)}}{dz} + \frac{\omega^2}{c^2}n^2F_+^{(0)} + \frac{\omega^2}{c^2}\frac{iC(K^2+1)}{K}F_-^{(0)} = 0,\\ &\frac{K'}{K}\frac{dF_+^{(0)}}{dz} - \frac{\omega^2}{c^2}\frac{iC(K^2-1)}{K}F_-^{(0)} = 0.\end{aligned}\right\} \qquad (25.17)$$

Thus we use essentially the method of perturbation theory.

It is easy to see that in the approximation of geometrical optics the solutions of (25.13)–(25.14) and (25.17) which correspond to the normal waves are the same as the results given in § 23; in this approximation $F_+ = 2E_y = 2KE_x$ and $F_- = 0$.

Using (25.15)–(25.17), we obtain from (25.13) and (25.14) the following equations for $F_\pm^{(1)}$:

$$\left.\begin{aligned} &\frac{d^2F_+^{(1)}}{dz^2} - \frac{K'}{K}\frac{dF_+^{(1)}}{dz} + \frac{\omega^2}{c^2}n^2F_+^{(1)} + \frac{\omega^2}{c^2}\frac{iC(K^2+1)}{K}F_-^{(1)}\\ &\qquad = \left\{\frac{K''}{2K} - \left(\frac{K'}{K}\right)^2\right\}(F_+^{(0)} - F_-^{(0)}) - \frac{K'}{K}\frac{dF_-^{(0)}}{dz},\\ &\frac{K'}{K}\frac{dF_+^{(1)}}{dz} - \frac{\omega^2}{c^2}\frac{iC(K^2-1)}{K}F_-^{(1)}\\ &\qquad = -\frac{d^2F_-^{(0)}}{dz^2} + \frac{K'}{K}\frac{dF_-^{(0)}}{dz} - \frac{\omega^2}{c^2}n^2F_-^{(0)} +\\ &\qquad\quad + \left\{\frac{K''}{2K} - \left(\frac{K'}{K}\right)^2\right\}(F_-^{(0)} - F_+^{(0)}).\end{aligned}\right\} \qquad (25.18)$$

Eliminating $F_{-}^{(0)}$ and $F_{-}^{(1)}$ from (25.17) and (25.18) respeetively, we have, after some simple transformations

$$L(F_{+}^{(0)}) \equiv \frac{d^2 F_{+}^{(0)}}{dz^2} + \frac{2K'}{K(K^2-1)} \frac{dF_{+}^{(0)}}{dz} + \frac{\omega^2}{c^2} n^2 F_{+}^{(0)} = 0, \tag{25.19}$$

$$\begin{aligned} L(F_{+}^{(1)}) = f(z) \equiv &- \frac{2}{K^2+1}\left\{\frac{K''}{2K} - \left(\frac{K'}{K}\right)^2\right\}(F_{+}^{(0)} - F_{-}^{(0)}) + \\ &+ \frac{2K'}{K(K^2-1)} \frac{dF_{-}^{(0)}}{dz} - \frac{\omega^2}{c^2}\left(\frac{K^2+1}{K^2-1}\right) F_{-}^{(0)} - \\ &- \frac{K^2+1}{K^2-1} \frac{d^2 F_{-}^{(0)}}{dz^2}. \end{aligned} \tag{25.20}$$

If the condition (25.16) holds, the approximate solution is $F_{\pm}^{(0)}$ and is obtained from (25.17) or (25.19).

It is now important to examine the behaviour of the functions $K_{1,2}$ [see (11.26) and (11.27)] near the reflection points $v_{20} = 1$ and $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$. As far as the first-order terms in $v - v_{20}$ and $v - v_{10}^{(\pm)}$ we have

$$\left.\begin{aligned} &K_1(v \approx 1 \pm \sqrt{u}) = \pm i\cos\alpha + \frac{i\sin^2\alpha\cos\alpha}{\sqrt{u}(1+\cos^2\alpha)}(v - 1 \mp \sqrt{u}), \\ &K_2(v \approx 1) = i \cdot \sqrt{u}\sin^2\alpha/(1-v)\cos\alpha, \\ &\left(\frac{dK_1}{dz}\right)_{v_{10}} = \frac{i\sin^2\alpha\cos\alpha}{\sqrt{u}(1+\cos^2\alpha)}\left(\frac{dv}{dz}\right)_{v_{10}}, \\ &\left(\frac{d^2K_1}{dz^2}\right)_{v_{10}} = \frac{i\sin^2\alpha\cos\alpha}{\sqrt{u}(1+\cos^2\alpha)}\left(\frac{d^2v}{dz^2}\right)_{v_{10}} + \\ &\qquad + \frac{i\cdot 2\sin^2\alpha\cos\alpha(3+3\cos^2\alpha+\cos^4\alpha)}{u(1+\cos^2\alpha)^3}\left(\frac{dv}{dz}\right)^2_{v_{10}}. \end{aligned}\right\} \tag{25.21}$$

Let us now consider the propagation of a wave of type 1 near the points $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$. In this ease we must put $n^2 = n_1^2$ and $K = K_1$ everywhere in (25.13)–(25.20); for a wave of type 2 we should have to put $n^2 = n_2^2$ and $K = K_2$, but this case is of no interest, sinee for wave 2 near the points $v_{10}^{(\pm)}$ the approximation of geometrieal opties is valid by hypothesis.

Putting in (25.19)

$$F_{+}^{(0)} = K_1 g_{+}^{(0)}/\sqrt{(1-K_1^2)}, \tag{25.22}$$

we obtain for $g_{+}^{(0)}$

$$\frac{d^2 g_{+}^{(0)}}{dz^2} + \left\{\frac{\omega^2}{c^2} n_1^2 + \frac{(3K_1^2-2)K_1'^2}{K_1^2(K_1^2-1)^2} - \frac{K_1''}{K_1(K_1^2-1)}\right\} g_{+}^{(0)} = 0. \tag{25.23}$$

This equation is of the form (25.1) but differs by the terms in K_1. These terms are of the order of $(dv/dz)^2$ and (d^2v/dz^2), i.e. are of the order of 10^{-12} to 10^{-14} in the ionosphere (where $v \sim az$ with $a \sim 10^{-6}$ to 10^{-7}). At the same time

$\omega^2/c^2 \sim 10^{-6}$, and so the terms in K_1 are important only when $n_1^2 \lesssim 10^{-5}$ to 10^{-7}, while the value corresponding to the first maximum of the field is

$$n_1^2 \approx \left|\frac{d n_1^2}{dz}\right| \left(\frac{\lambda_0^2}{4\pi^2 \, |dn_1^2/dz|}\right)^{\frac{1}{3}} = \left(\left|\frac{dn_1^2}{dz}\right| \frac{\lambda_0}{2\pi}\right)^{\frac{2}{3}} \sim 10^{-2}$$

[see (32.2), etc.]. Thus the effect of the terms in K_1 is merely to shift the zero of the coefficient of $g_+^{(0)}$ by $\Delta z \sim 0{\cdot}1$ to 1 cm (as an estimate we take $n_1^2 \sim az$) and to change the phase by a negligible amount $\Delta\varphi \sim 10^{-6}$.

Thus the solution of the equations of propagation near the reflection point reduces to equation (25.1); the fields themselves have an amplitude factor $K_1/\sqrt{(1 - K_1^2)}$ [see (25.22)], the necessity of which is already seen in the approximation of geometrical optics (23.11).

This derivation is, of course, valid only if the condition (25.16) holds, i.e. if the quantities $F_\pm^{(0)}$ are in fact an approximate solution of equations (25.13) and (25.14).

The solution of equation (25.20) is

$$F_+^{(1)} = F_0 \int \left\{ \frac{K^2}{F_0^2(K^2-1)} \int^z \frac{f(z')\, F_0}{K^2} \,(K^2 - 1)\, dz' \right\} dz, \tag{25.24}$$

where F_0 is the solution of the homogeneous equation, i.e. the expression (25.22). According to (25.17)

$$F_-^{(0)} = \frac{c^2}{\omega^2} \frac{K'}{i\,C(K^2-1)} \frac{dF_+^{(0)}}{dz}\,; \tag{25.25}$$

near the points $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$, C does not tend to zero, and K_1 tends neither to zero nor to infinity [see (25.21)]. In the present case, therefore,

$$f(z) < \frac{\omega^2}{c^2} F_-^{(0)} \sim \left(\frac{dv}{dz}\right)_{v_{10}} \frac{dF_+^{(0)}}{dz} \lesssim \left(\frac{dv}{dz}\right)_{v_{10}} \frac{2\pi}{\lambda_0} F_+^{(0)}; \tag{25.26}$$

here, for simplicity, we omit the factors involving u and $\sin^2\alpha$, which are usually of the order of unity. To prove (25.26) we use the facts that in the approximation of geometrical optics

$$\frac{d^2 F_-^{(0)}}{dz^2} + \frac{\omega^2}{c^2} n_1^2 F_-^{(0)} \sim \frac{\omega}{c} \frac{dn_1}{dz} F_-^{(0)},$$

and in the region where $n_1^2 \approx 0$ the deviation from that approximation is of the order of the field itself, i.e.

$$\frac{d^2 F_-^{(0)}}{dz^2} + \frac{\omega^2}{c^2} n_1^2(z) F_-^{(0)} \sim \frac{\omega^2}{c^2} n_1^2 \left(z \sim \frac{\lambda_0}{2\pi}\right) F_-^{(0)} < \omega^2 F_-^{(0)}/c^2,$$

where $n_1^2(z \sim \lambda_0/2\pi)$ is the value of n_1^2 for $z \sim \lambda_0/2\pi$ (see the results of § 32). Also $dF_+^{(0)}/dz \sim 2\pi n_1 F_+^{(0)}/\lambda_0 < 2\pi F_+^{(0)}/\lambda_0$, and by (25.17) $\omega^2 F_-^{(0)}/c^2 \sim (dv/dz)_{v_{10}} \times$ $\times\, dF_+^{(0)}/dz$, whence we have (25.26).

Substituting in (25.24) the value of $f(z)$ from (25.26), it is seen that

$$F_+^{(1)} < (dv/dz)_{v_{10}}\, z F_+^{(0)}. \tag{25.27}$$

In the region under consideration near the point $n_+^2 = 0$ we have $(dv/dz)z \sim az \ll 1$. Thus the condition (25.16) is satisfied, and the solution $F_1^{(0)}$ is a good approximation, especially as in practice the $<$ sign in (25.27) may be replaced by $\ll$.

For a wave of type 2 near the reflection point $v_{20} = 1$ where $n_2^2 = 0$ the above approximation is *not* valid. It follows from (11.3) and (25.21) that near this point $C \sim 1 - v$, $K_2 \sim 1/(1-v)$, in (25.25) $F_-^{(0)} \sim [1/(1-v)]dF_+^{(0)}/dz$ and in $f(z)$ the principal term is $\sim F_+^{(0)}/(1-v)^3$. Hence the condition (25.16) is not satisfied, and the whole approximation is invalid. This is because, near the point $v = 1$, the polarisation of wave 2 varies rapidly (see Fig. 11.9). The region $v \approx 1$ must therefore be considered separately.

In this region we put

$$v = 1 + az, \tag{25.28}$$

where

$$|az| \ll 1. \tag{25.29}$$

Then equations (23.2) become, to within terms of higher order [see (11.3)],

$$\left.\begin{aligned} &\frac{d^2E_x}{dz^2} + \frac{\omega^2}{c^2}\left\{\left(1 - \frac{az}{u\sin^2\alpha}\right)E_x - \frac{iaz\cos\alpha}{\sqrt{u}(\sin^2\alpha)}E_y\right\} = 0,\\ &\frac{d^2E_y}{dz^2} + \frac{\omega^2}{c^2}\left\{\frac{iaz\cos\alpha}{\sqrt{u}(\sin^2\alpha)}E_x - \frac{az}{\sin^2\alpha}E_y\right\} = 0. \end{aligned}\right\} \tag{25.30}$$

Also $$n_2^2 = -az/\sin^2\alpha, \quad K_2 = -i\sqrt{u}(\sin^2\alpha)/az\cos\alpha. \tag{25.31}$$

In the approximation of geometrical optics, if the condition

$$\left|\frac{1}{K_2}\right| = \left|\frac{az\cos\alpha}{\sqrt{u}(\sin^2\alpha)}\right| \ll 1 \tag{25.32}$$

holds, we have also $|E_{y2}/E_{x2}| = |K_2| \gg 1$; for $v = 1$, wave 2 is linearly polarised in a homogeneous medium [see (11.26) and Fig. 11.9]. This makes it reasonable to use the approximation

$$E_y = E_y^{(0)} + E_y^{(1)}, \quad E_x = E_x^{(1)}, \tag{25.33}$$

where

$$|E_y^{(1)}| \ll |E_y^{(0)}|, \quad |E_x^{(1)}| \ll |E_y^{(0)}| \tag{25.34}$$

and $$\frac{d^2E_y^{(0)}}{dz^2} - \frac{\omega^2}{c^2}\frac{az}{\sin^2\alpha}E_y^{(0)} = \frac{d^2E_y^{(0)}}{dz^2} + \frac{\omega^2}{c^2}n_2^2E_y^{(0)} = 0. \tag{25.35}$$

For $E_{x,y}^{(1)}$ we have the equations [see (25.30), (25.34) and (25.35)]

$$\frac{d^2E_x^{(1)}}{dz^2} + \frac{\omega^2}{c^2}\left(1 - \frac{az}{u\sin^2\alpha}\right)E_x^{(1)} = i\frac{\omega^2}{c^2}\frac{az\cos\alpha}{\sqrt{u}(\sin^2\alpha)}E_y^{(0)} \equiv f_1(z), \tag{25.36}$$

$$\frac{d^2E_y^{(1)}}{dz^2} + i\frac{\omega^2}{c^2}\frac{az\cos\alpha}{\sqrt{u}(\sin^2\alpha)}E_x^{(1)} - \frac{\omega^2}{c^2}\frac{az}{\sin^2\alpha}E_y^{(1)} = 0. \tag{25.37}$$

The solution of equation (25.36) is

$$E_x^{(1)} = E_{x0}\int\left\{\frac{1}{E_{x0}^2}\int^z f_1(z')\,E_{x0}\,dz'\right\}dz, \tag{25.38}$$

where E_{x0} is the solution of the homogeneous equation corresponding to (25.36). To estimate the magnitude of the expression (25.38) we may put $E_{x0} = e^{\pm i\omega z/c}$; using also the fact that, in the region considered near $v = 1$, $n_2^2 \ll 1$ and therefore $E_y^{(0)}$ depends much less than E_{x0} on z [see (25.35)], we have

$$E_x^{(1)} \sim \frac{i\,a\cos\alpha}{\sqrt{u}\sin^2\alpha}\left(z + 2i\frac{c}{\omega}\right)E_y^{(0)}. \tag{25.39}$$

When the condition (25.32) holds (and, strictly speaking, if it holds also for $z \sim \lambda_0$), the second inequality (25.34) follows from (25.39). The first inequality (25.34) follows from the equation (25.37). Thus the approximation used above is valid when the conditions (25.29) and (25.32) hold. In the case here considered, the condition for the validity of the approximation of geometrical optics is seen from (23.19) and (25.31) to be equivalent to

$$|z| \gg \left(\frac{\lambda_0^2}{4\pi^2}\,\frac{\sin^2\alpha}{a}\right)^{\frac{1}{3}}. \tag{25.40}$$

For example, when $a \sim 10^{-6}\,\text{cm}^{-1}$, $\lambda_0/2\pi \sim 10^3\,\text{cm}$ and $\sqrt{u} \sim \sin\alpha \sim \cos\alpha \sim 1$, the conditions (25.29) and (25.32) give $z \ll 10^6$ cm, while (25.40) gives $z \gg 10^4$ cm. Thus in this case the approximation is valid even in the region where geometrical optics is already valid. For small α the position is less favourable, but even when $\sin\alpha \sim \alpha \sim 1/10$ the conditions (25.32) and (25.40) give $z \ll 10^4$ cm and $z \gg 2 \times 10^3$ cm. For smaller values of α the approximation must be invalid owing to the "tripling" effect due to the interaction of waves 1 and 2 in the region $v \approx 1$ (§ 28).

Equation (25.35) has the form (25.1) with $g_2 = E_y$, i.e. the results obtained for an isotropic medium can be applied *in toto* to this solution.

In the preceding discussion we not only have endeavoured to show that an equation of the type (25.1) can be used for a reflected wave in the region of the reflection points, but also have made this region a linear layer. This is because in the great majority of cases for thick layers we need only consider the exact solution of the wave equation for a linear layer (see § 30). The linear approximation for the function $\tilde{n}_{1,2}^2(z)$ near the reflection point is in practice inadmissible only near the critical frequencies, when the point $\tilde{n}_{1,2}^2(z) = 0$ approaches the maximum of the layer. In that case the waves 1 and 2 may be regarded as independent under conditions which are evident from the foregoing. The propagation of each wave can therefore again be discussed on the basis of an equation of the form (25.1).

The results derived in this section will be used in § 35 in dealing with the reflection of radio waves from the ionosphere, taking into account the Earth's magnetic field.

§ 26. THE LIMITING POLARISATION OF WAVES LEAVING A LAYER OF INHOMOGENEOUS MAGNETOACTIVE PLASMA

Introduction. Some estimates

The approximation of geometrical optics is inapplicable to the propagation of electromagnetic waves in an inhomogeneous magnetoactive medium, in particular, at the boundary of the layer, where $v = 4\pi e^2 N(z)/m\omega^2 \to 0$. The reason for this has been explained in § 23: it arises, essentially, from the "polarisation degeneracy" which occurs in a vacuum and in any isotropic medium, where all waves are normal waves, whatever their polarisation.

Let us now ascertain the effect of the region of low density (i.e. the boundary of the layer) on the polarisation and phase of waves leaving the layer (e.g. waves returning to the Earth after reflection from the ionosphere).

It is simplest to estimate the contribution to the phase of the wave and its polarisation coefficients $K_{1,2}$ which results from propagation in the region of low density. To do this, we can consider [148] directly the equations (23.2), substituting for A, B, C the values (23.21) which pertain to the range $v \ll 1$. The approximate solution of the resulting equations can be found by taking as the zero-order approximation either the solution $E^{(0)}_{x,y} = \text{constant} \times e^{\pm i\omega z/c}$ or the more exact solution of geometrical optics. We shall not go through the calculations, but merely give the result for the case where $v = az$; the same result is obtained if $v = az^m$ with m not too large.

The corrections to the zero-order approximation (or, more precisely, the ratio of the first-order and zero-order terms) are of order az and $2\pi az^2/\lambda_0$. In the region where geometrical optics is valid, i.e. where the condition

$$\frac{\lambda_0}{2\pi z} \sim \frac{\lambda_0}{2\pi}\,\frac{d|\Delta n|/dz}{\Delta n} \ll 1 \qquad (\Delta n = n_2 - n_1 \sim a z)$$

holds [see (23.23)†], the principal term is $\sim 2\pi az^2/\lambda_0$. In the upper layers of the ionosphere this ratio is much less than unity: with $a \sim 10^{-6}$ to $10^{-7}\,\text{cm}^{-1}$ and $\lambda_0/2\pi \sim 10^3$ cm we have $2\pi az^2/\lambda_0 \sim 10^{-9}$ to $10^{-10} \times z^2$, and so with $z \sim 10^4$ cm we obtain $2\pi az^2/\lambda_0 \sim 0{\cdot}1$ to $0{\cdot}01 \ll 1$, and geometrical optics is still applicable, since $\lambda_0/2\pi z \sim 0{\cdot}1 \ll 1$. Thus the corrections to the phase and to the amplitude ratio $K_{1,2} = E_{y\,1,2}/E_{x\,1,2}$ are much less than unity, and so the phase and, in particular, the polarisation of waves reflected from the

† It may be recalled that the condition (23.23) has been derived in § 23 on the assumption that factors such as $u - 1$, $\cos^2\alpha$, $\sin^2\alpha$ are of the order of unity. Hence, as $u \to 1$ (region of the gyration frequency) and as $\alpha \to 0$ or $\alpha \to \frac{1}{2}\pi$, the estimates must be more carefully made. Evidently geometrical optics may continue to be valid at the boundary of the layer when $\alpha = 0$ or $\frac{1}{2}\pi$, since the polarisation is independent of v in these cases, i.e. $K' = 0$. From this we see that when $\alpha \to 0$ or $\alpha \to \frac{1}{2}\pi$ the inequality (23.23) is unnecessarily stringent.

ionosphere are given to a good approximation by using geometrical optics throughout the layer and neglecting the region where the condition (23.23) is violated and geometrical optics is inapplicable. We can therefore use formula (11.29) for the polarisation of waves reflected from the layer.

This result is simply explained by the fact that, at the boundary of the layer ($v \to 0$) in the upper-ionosphere conditions considered, the phase difference of the two waves and the change in their polarisation are so small as to be of no interest, whether or not they are accurately given by geometrical optics. For example, the phase difference at the boundary of the layer is of the order of $2\pi a z^2/\lambda_0$, i.e. usually very small, as shown above. The phase itself for each wave is determined by the value of $n_{1,2}$, which at the boundary of the layer is close to unity, so that the phase $\varphi_{1,2} \approx \omega z/c = 2\pi z/\lambda_0$ in the first approximation involves no error.

The inapplicability of the approximation of geometrical optics at the boundary of the layer might become significant for $\lambda_0/2\pi \sim 10^3$ cm with $a \gtrsim 10^{-5}\,\text{cm}^{-1}$; see the above-mentioned estimates of the quantity $2\pi a z^2/\lambda_0$.

Since $v = 4\pi e^2 N/m\omega^2 = 3{\cdot}18 \times 10^9 N(z)/\omega^2$, the parameter a for $\lambda_0/2\pi = c/\omega = 10^3$ cm is of the order of $a \sim 3{\cdot}18 \times 10^9 (1/\omega^2)(dN/dz)_0 \sim 3 \times 10^{-6}(dN/dz)_0$. In order to have $a \sim 10^{-5}$ in this example we must take $dN/dz \sim 3$, whereas in the F layer we usually have $dN/dz \sim 0{\cdot}1$ (the half-thickness of the layer $z_m \sim 100$ km, and $N_{\max} \sim 10^6$). At the boundary of the layer we should naturally expect a still smaller value of dN/dz. In the E layer the quantity

$$\frac{2\pi a z^2}{\lambda_0} = \frac{3{\cdot}18 \times 10^9 \lambda_0}{2\pi c^2}\left(\frac{dN}{dz}\right)_0 z^2$$

is usually greater than in the F layer, since the waves reflected from the F layer are longer. In the E layer, however, for the same reason, the condition $\lambda_0/2\pi z \ll 1$ for geometrical optics to be valid is more stringent than in the F layer. The same applies even more strongly to the region of the ionosphere lying below the E layer. This region reflects long and very long waves, for which geometrical optics is inapplicable not only because of the "polarisation degeneracy" at the boundary of the layer but also because the indices $[n(z) - i\varkappa(z)]_{1,2}$ vary too greatly over distances of the order of a wavelength. Thus for long waves the problem of the limiting polarisation must be regarded as only one aspect of a more general problem. We shall not discuss here this problem of the reflection of long and very long waves from the ionosphere; discussions both with and without allowance for the effect of the Earth's magnetic field may be found in [22, 23, 121, 122, 150–157].

The approximate solution

For not-too-long waves the limiting polarisation may evidently be calculated on the basis of the approximation of geometrical optics. More precisely, the solution may be sought in the form of "interacting" normal waves of the geometrical-optics type. In estimating the importance of the boundary of the layer in the calculation of the coefficients $K_{1,2}$ and the phases $\varphi_{1,2}$ we have essentially proceeded in this way. The method of perturbations used makes it possible to find immediately a criterion for the deviation of $K_{1,2}$ and $\varphi_{1,2}$ from their geometrical-optics values to be small. In order to elucidate more fully the nature of the interaction of normal waves at the boundary of the layer it is necessary to solve the problem under conditions where this interaction is fairly large.

The problem of the limiting polarisation of waves leaving a layer has been discussed by several authors [158–162, 148]. The calculations given below [160] are based on the use of certain "coupled wave equations", which have been widely applied to solve other problems also [149, 163]. These equations are identical with the fundamental equations (23.2), but are sometimes more convenient. However, such a statement about convenience is somewhat arbitrary and is often simply the result of habit. These equations should, nevertheless, be quoted, if only because they have been used by several authors.

Instead of E_x and E_y we use new functions E_{x1} and E_{x2}, defined by

$$\left.\begin{aligned} &E_x = E_{x1} + E_{x2}, \quad E_y = E_{y1} + E_{y2}, \\ &E_{y1} = K_1 E_{x1}, \quad E_{y2} = K_2 E_{x2}, \\ K_{1,2}(z) = -i \cdot &\frac{2\sqrt{u}(1 - i s - v)\cos\alpha}{u \sin^2\alpha \mp \sqrt{[u^2 \sin^4\alpha + 4u(1 - i s - v)^2 \cos^2\alpha]}}. \end{aligned}\right\} \tag{26.1}$$

The lower sign corresponds to wave 1 and the upper sign to wave 2. The significance of this replacement is that for a homogeneous medium the ratios E_{y1}/E_{x1} and E_{y2}/E_{x2} are the same as for the extraordinary and ordinary waves; the coefficients $K_{1,2}$ in (26.1) are the coefficients (11.25) which determine the polarisation of normal waves in a homogeneous medium. For this reason we have used the suffixes 1 and 2 in the notation for the new functions, although in the general case of an inhomogeneous medium they may be quite unrelated to the normal waves 1 and 2 in a homogeneous medium.

For the functions $E_{x,y1}$ and $E_{x,y2}$ or combinations of them we can derive a system of equations equivalent to the original equations (23.2). Such equations, in particular, are (25.13) and (25.14) for $F_\pm = E_y \pm K_{1,2} E_x$. In [149, 160, 163], however, a further transformation is made by using the functions

$$\Pi_1 = E_{x1}\sqrt{(1 - K_1^2)}, \quad \Pi_2 = E_{x2}\sqrt{(1 - K_2^2)}, \tag{26.2}$$

where $E_{x1,2}$ are the functions defined by (26.1). From (23.2) we have the equivalent equations for $\Pi_{1,2}$

$$\left.\begin{aligned} \frac{d^2\Pi_1}{dz^2} + \left[\frac{\omega^2}{c^2}(n - i\varkappa)_1^2 - \Psi'^2\right]\Pi_1 &= \frac{d\Psi'}{dz}\Pi_2 + 2\Psi'\frac{d\Pi_2}{dz}, \\ \frac{d^2\Pi_2}{dz^2} + \left[\frac{\omega^2}{c^2}(n - i\varkappa)_2^2 - \Psi'^2\right]\Pi_2 &= -\frac{d\Psi'}{dz}\Pi_1 - 2\Psi'\frac{d\Pi_1}{dz}, \end{aligned}\right\} \tag{26.3}$$

where the functions $(n - i\varkappa)^2_{1,2}$ are given by formula (11.5) and

$$\begin{aligned}\Psi &= \frac{1}{2} i \frac{d}{dz} \ln \frac{K_2 - 1}{K_2 + 1} = \frac{i}{K_2^2 - 1} \frac{dK_2}{dz} \\ &= \frac{1}{4} i \frac{d}{dz} \ln \left[\frac{1 - v - is + i\sqrt{u}\,(\sin^2\alpha)/2\cos\alpha}{1 - v - is - i\sqrt{u}\,(\sin^2\alpha)/2\cos\alpha} \right]. \end{aligned} \tag{26.4}$$

It must be noted that the transformation (26.2) is inadmissible when $K_1^2 = 1$ or $K_2^2 = 1$. Since we always have $K_1K_2 = 1$ [see (11.27)], either of these conditions implies the other. Furthermore, $K_1^2 = K_2^2 = 1$ only when $v = 4\pi e^2 N(z)/m\omega^2 = 1$, $s = \nu_{\text{eff}}/\omega = s_{\text{cr}} = \sqrt{u}\sin^2\alpha/2|\cos\alpha|$ [see (11.4)], so that the equations (26.3) are inapplicable only in this case

If the external magnetic field $\boldsymbol{H}^{(0)}$ may be regarded as uniform, we need differentiate only v and s with respect to z in (26.4), and

$$\Psi = -\frac{\sqrt{u}\,(\sin^2\alpha)/4\cos\alpha}{(1 - v - is)^2 + u^2\sin^4\alpha/4\cos^2\alpha}\left(\frac{dv}{dz} + i\frac{ds}{dz}\right). \tag{26.5}$$

It is noteworthy that the function Ψ is not zero, even in the complete absence of ionisation $(v = 0)$, if $ds/dz = (1/\omega)d\nu_{\text{eff}}/dz \neq 0$. This result is quite intelligible, since when $v = 0$ we have

$$K_{1,2} = -i \cdot \frac{2\sqrt{u}(1 - is)\cos\alpha}{u\sin^2\alpha \mp \sqrt{[u^2\sin^4\alpha + 4u(1 - is)^2\cos^2\alpha]}}, \tag{26.6}$$

i.e. the polarisation coefficients depend on z through $s(z)$ even when $v = 0$.

Equations (26.3) are exact, and in the absence of ionisation they give, of course, the obvious result that the polarisation is constant (i.e. the ratio $E_y/E_x = (E_{y1} + E_{y2})/(E_{x1} + E_{x2})$ is constant), although the auxiliary quantities $\Pi_{1,2}$ depend on z. However, even this example of the passage to the limit of a non-ionised medium, and the above-mentioned requirement that $K_{1,2}^2 \neq 1$, indicate that equations (26.3) can by no means be considered as invariably convenient in application.

The problem of the limiting polarisation is solved from equations (26.3) on the assumption that geometrical optics is applicable at the boundary of the layer for the equations

$$\frac{d^2\Pi_{1,2}}{dz^2} + \frac{\omega^2}{c^2}(n - i\varkappa)^2_{1,2}\Pi_{1,2} = 0. \tag{26.7}$$

The corresponding particular solutions are [see (16.11)]

$$\left.\begin{aligned}\Pi_{1\pm}^{(0)} &= \frac{1}{\sqrt{(n - i\varkappa)_1}}\exp\left[\pm i\frac{\omega}{c}\int (n - i\varkappa)_1\,dz\right], \\ \Pi_{2\pm}^{(0)} &= \frac{1}{\sqrt{(n - i\varkappa)_2}}\exp\left[\pm i\frac{\omega}{c}\int (n - i\varkappa)_2\,dz\right]. \end{aligned}\right\} \tag{26.8}$$

In the absence of absorption these expressions give a good approximation to the exact solutions of equations (26.7) if the conditions

$$\frac{\lambda_0}{2\pi}\,\frac{|dn_{1,2}/dz|}{|n_{1,2}|^2} \ll 1$$

hold [see (16.22)]; in the presence of absorption, the corresponding conditions may be written, in a somewhat symbolic form,

$$\frac{\lambda_0}{2\pi}\,\frac{d(n-i\varkappa)_{1,2}/dz}{(n-i\varkappa)^2_{1,2}} \ll 1.$$

Equations (26.3) may be written

$$\left.\begin{aligned}
&\frac{d^2\Pi_1}{dz^2}+\frac{\omega^2}{c^2}(n-i\varkappa)_1^2\,\Pi_1 = f_1(z),\\
&\frac{d^2\Pi_2}{dz^2}+\frac{\omega^2}{c^2}(n-i\varkappa)_2^2\,\Pi_2 = f_2(z),\\
&f_1 = \Psi^2\Pi_1 + \Pi_2\,d\Psi/dz + 2\Psi\,d\Pi_2/dz,\\
&f_2 = \Psi^2\Pi_2 - \Pi_1\,d\Psi/dz - 2\Psi\,d\Pi_1/dz.
\end{aligned}\right\} \tag{26.9}$$

Using as the solutions of the homogeneous equations (26.7) the expressions (26.8), we can now find the solutions of equations (26.9) from the familiar formulae obtained by "variation of the parameters": if Π_{1+} and Π_{1-} are solutions of the homogeneous equation (26.7), a particular solution of the inhomogeneous equation (26.9) for Π_1 is

$$\left.\begin{aligned}
&\Pi_1 = -\Pi_{1+}\int(\Pi_{1-}f_1/W_1)\,dz + \Pi_{1-}\int(\Pi_{1+}f_1/W_1)\,dz,\\
&W_1 = \Pi_{1+}\,d\Pi_{1-}/dz - \Pi_{1-}\,d\Pi_{1+}/dz.
\end{aligned}\right\} \tag{26.10}$$

The solution for Π_2 is obtained simply by substituting the suffix 2 for 1. The Wronskians for the solutions (26.8) are $W_1 = W_2 = -2i\omega/c$. Moreover, in the problem of the limiting polarisation for a smooth layer we need consider only waves travelling in one direction (namely, out of the layer). Consequently, only the terms proportional to $\Pi^{(0)}_{1,2\pm}$ need be retained as $\Pi_{1,2\pm}$ in solutions of the type (26.10) with the functions (26.8). The result is

$$\left.\begin{aligned}
&\Pi_1 = -\frac{i\,\Pi^{(0)}_{1+}}{2\omega/c}\int^z f_1\,\Pi^{(0)}_{1-}\,dz,\\
&\Pi_2 = -\frac{i\,\Pi^{(0)}_{2+}}{2\omega/c}\int^z f_2\,\Pi^{(0)}_{2-}\,dz.
\end{aligned}\right\} \tag{26.11}$$

These expressions are still not solutions of the problem, since the functions $f_{1,2}$ themselves depend on $\Pi_{1,2}$. Nevertheless, they can be used when the interaction between waves 1 and 2 is weak and so the functions $f_{1,2}$ are small.

This is most simply done by the method of successive approximations, also called the method of perturbation theory. As applied to the expressions (26.11), this method consists in substituting the solutions $\Pi^{(0)}_{1,2+}$ for $\Pi_{1,2+}$ in $f_{1,2}$. This procedure has, in essence, been used already at the beginning of this section in treating the original equations (23.2). A more complete solution is given in [160], but only for a particular model of the ionospheric layer. The calculations are quite laborious, though straightforward. We shall therefore describe only their general form and the result.

The solutions $\Pi_{1,2}$ are sought in the form

$$\left.\begin{aligned} \Pi_{1,2} &= A_{1,2}(z)\,\Pi^{(0)}_{1,2+},\\ A_1 &= u_1(z)\exp\left[-i\frac{\omega}{2c}\int\limits_{z_0}^{z}(n_2-n_1)\,dz\right],\\ A_2 &= u_2(z)\exp\left[i\frac{\omega}{2c}\int\limits_{z_0}^{z}(n_2-n_1)\,dz\right], \end{aligned}\right\} \tag{26.12}$$

where the point z_0 is taken outside the layer, so that $n_2(z_0) = n_1(z_0)$. In the presence of absorption, $n_{1,2}$ should simply be replaced by $(n - i\varkappa)_{1,2}$ here and henceforward.

The functions $A_{1,2}(z)$ in the case of weak interaction may be regarded as varying slowly in comparison with $\Pi^{(0)}_{1,2+}$.

For the particular conditions prevailing at the boundary of the E layer of the ionosphere, estimates show that the factors such as $\Psi^2\Pi^{(0)}_{2+}\Pi^{(0)}_{2-}$ and $\Pi^{(0)}_{2-}\Pi^{(0)}_{1+}\,d\Psi/dz$ may be neglected in comparison with $2\Psi\Pi^{(0)}_{2-}\,d\Pi^{(0)}_{1+}/dz$ and $2\Psi\Pi^{(0)}_{1-}\,d\Pi^{(0)}_{2+}/dz$. This is evident directly, since the derivatives $d\Pi^{(0)}_{1,2+}/dz$ contain the large factor ω/c, which is not present in the neglected terms. Thus we obtain for the functions $u_{1,2}$ the equation

$$\frac{d^2u_{1,2}}{dz^2} + \left\{\Psi^2 + \frac{1}{4}\frac{\omega^2}{c^2}(n_2-n_1)^2 - \frac{1}{2}i\frac{\omega}{c}\frac{d}{dz}(n_2-n_1)\right\}u_{1,2} = 0. \tag{26.13}$$

If

$$|\Psi^2| \gg \left|\frac{\omega^2}{4c^2}(n_2-n_1)^2 - \frac{1}{2}i\frac{\omega}{c}\frac{d}{dz}(n_2-n_1)\right|, \tag{26.14}$$

equation (26.13) is almost the same as when $n_1 = n_2$. For an isotropic medium, however, the solution of the equations is already known to be that the polarisation of the wave remains unchanged (and so we shall omit the proof, which involves simply converting $u_{1,2}$ to $\Pi_{1,2}$ and then to the field $E_{x,y}$). In the opposite limiting case

$$|\Psi^2| \ll \left|\frac{\omega^2}{4c^2}(n_2-n_1)^2 - \frac{1}{2}i\frac{\omega}{c}\frac{d}{dz}(n_2-n_1)\right| \tag{26.15}$$

we can neglect the term in Ψ^2 in equation (26.13), and, using the fact that $u_{1,2}$ vary only slowly with z, find

$$u_{1,2} = \text{constant} \times \exp\left[\pm i \frac{\omega}{2c} \int (n_2 - n_1)\, dz\right].$$

Hence we find, in particular, $A_{1,2}$ = constant and $\Pi_{1,2}$ = constant $\times\ \Pi^{(0)}_{1,2+}$ [see (26.12)], i.e. the waves Π_{1+} and Π_{2+} do not interact. Thus the region of interaction in the determination of the limiting polarisation corresponds to the condition

$$|\Psi^2| \sim \left|\frac{1}{4}\frac{\omega^2}{c^2}(n_2 - n_1)^2 - \frac{1}{2} i \frac{\omega}{c} \frac{d}{dz}(n_2 - n_1)\right|. \tag{26.16}$$

Usually one of the terms on the right of this equation is small compared with the other. For example, with the model used in [160], for $f > 1$ Mc/s ($\lambda_0 < 300$ m) we can neglect the term $(\omega/c)d(n_2 - n_1)/dz$ in comparison with $(\omega^2/c^2)(n_2 - n_1)^2$, while for $f < 0{\cdot}5$ Mc/s ($\lambda_0 > 600$ m) the second term is dominant.

Results of the calculation

Equation (26.13) has the form of a wave equation in an isotropic medium [see (16.3)] and it can be analysed fully. This has been done in [160] for the case where $n_2 - n_1$ = constant $\times\ e^{\alpha z}$, Ψ = constant and $\lambda_0 \lesssim 300$ m. The result of the calculations is that the limiting polarisation of a wave 1 or 2 leaving the layer is

$$\left.\begin{aligned} \left(\frac{E_y}{E_x}\right)_{1,2} &= K_{1,2} = \frac{K^{(0)}_{1,2}(F-g) - (F+g)}{(F-g) - K^{(0)}_{1,2}(F+g)}, \\ F/g &= -2^{-2r} r^{2r} e^{i\pi r}\, \Gamma(1-r)/\Gamma(1+r), \quad r = i\Psi/\alpha. \end{aligned}\right\} \tag{26.17}$$

Here $\Gamma(x)$ is the gamma function, and $K^{(0)}_{1,2}(z_b)$ are the usual polarisation coefficients [see (11.25) or (26.1)], in which we must substitute the electron density at a level z_b given by the condition

$$\Psi^2(z_b) = (\omega^2/4c^2)[n_2(z_b) - n_1(z_b)]. \tag{26.18}$$

For the particular example [160] of the boundary of the E layer and $\lambda_0 = 300$ m, we have in the region of interaction $|\Psi| \approx 2 \times 10^{-8}$ cm^{-1}, $r \approx 2 \times 10^{-3}$ and $F/g \approx -0{\cdot}973$; for shorter waves, $|\Psi|$ is still less and $|F/g|$ is still closer to unity. If $F/g = -1$, then

$$(E_y/E_x)_{1,2} = K_{1,2} = K^{(0)}_{1,2}(z_b). \tag{26.19}$$

Thus in this example the limiting polarisation for waves shorter than 300 m is practically the same as in the approximation of geometrical optics for a medium whose parameters correspond to the point z_b given by the condition (26.18). This point is complex, since we cannot in general neglect absorption at the boundary of the layer, and in practice $n_2 - n_1$ must everywhere be replaced by $(n - i\varkappa)_2 - (n - i\varkappa)_1$. However, for sufficiently short waves (here, $\lambda_0 < 300$ m) we again have a simplification, and the coordinate z_b may be regarded as real. It is also found that the point z_b corresponds to very small values of the parameters v and s. This is reasonable, since at the boundary of the layer $v \ll 1$ and $s \ll 1$. We have finally an almost trivial result: for $K_{1,2} = K_{1,2}^{(0)}$ we must take the limiting value

$$K_{1,2} = -i \cdot \frac{2\sqrt{u}\,(\cos\alpha)}{u\sin^2\alpha \mp \sqrt{(u^2\sin^4\alpha + 4u\cos^2\alpha)}} \tag{26.20}$$

which follows from (11.25), (26.1) and (26.6) when $v = s = 0$.

In other words, in the particular conditions which have been analysed fully in [160] the approximation of geometrical optics is in fact applicable even to calculate the polarisation at the boundary of the layer. The same conclusion could have been reached on the basis of the estimates made at the beginning of this section. We have nevertheless indicated the procedure for a more detailed treatment, on account of the methodological aspect and the possibility of discussing other examples in which the interaction is more important. However, in the ionosphere, in the region of interaction (26.18), we apparently always have $v = 4\pi e^2 N/m\omega^2 \ll 1$, $s = \nu_{\text{eff}}/\omega \ll 1$, and the value of the limiting polarisation is determined only by the Earth's magnetic field, i.e. by the values of u and α. Hence it is clear that an experimental determination of the limiting polarisation cannot provide any further information regarding the ionosphere (the field $\boldsymbol{H}^{(0)}$ in the lower ionosphere is, of course, sufficiently accurately known).

In the solar corona, on the other hand, the magnetic field is not accurately known, and the measurement of the limiting polarisation might be very useful. Here, however, we encounter another difficulty because we have no information on the nature of the polarisation or the type of the wave before it reaches the "edge" of the corona, or more precisely the region of interaction. Nevertheless, it is not impossible that other considerations may sometimes allow us to regard a wave leaving the corona as purely ordinary or purely extraordinary. From a measurement of the polarisation of such a wave we could derive valuable information regarding the magnetic field in the corona at the "level of interaction".

§ 27. THE BEHAVIOUR OF THE WAVE FIELD AND THE COEFFICIENTS OF REFLECTION AND TRANSMISSION WHEN THE REFRACTIVE INDEX HAS SINGULARITIES

Introduction. Singularities (poles) of the refractive index

One of the characteristic features of an inhomogeneous magnetoactive plasma is the possibility that the refractive indices $\tilde{n}_{1,2}$ may become infinite while the electron density $N(z)$ remains finite. For, as already mentioned, with a given value of $u = \omega_H^2/\omega^2 < 1$ we have $\tilde{n}_1^2 \to \infty$ when $v \to v_{1\infty} = 4\pi e^2 N_\infty/m\omega^2 = (1-u)/(1-u\cos^2\alpha)$; if $u\cos^2\alpha > 1$, then $\tilde{n}_2^2 \to \infty$ when $v \to v_{2\infty} = 4\pi e^2 N_\infty/m\omega^2 = (u-1)/(u\cos^2\alpha - 1)$. The form of the curves of $\tilde{n}_{1,2}^2(v)$ near the poles is seen from Figs. 11.3, 11.6, 11.8 and 11.10. In particular, $\tilde{n}_1^2$ may have a pole in the simple case of transverse propagation ($\alpha = \frac{1}{2}\pi$), when the wave equation for wave 1 has the form [see (25.7)]

$$\left.\begin{aligned} d^2E_x/dz^2 + (\omega^2/c^2)\,\tilde{n}_1^2 E_x = 0, \\ \tilde{n}_1^2 = 1 - v(1-v)/(1-u-v). \end{aligned}\right\} \tag{27.1}$$

Hence we see from the above general expression for $v_{1\infty}$ that for $\alpha = \frac{1}{2}\pi$

$$v_{1\infty} = 1 - u, \quad \omega_\infty^2 = \omega_0^2 + \omega_H^2, \tag{27.2}$$

where ω_∞ is the frequency at which $\tilde{n}_1^2 \to \infty$ for given ω_0 and ω_H. The two expressions given are, of course, identical, since $v = \omega_0^2/\omega_\infty^2$ and $u = \omega_H^2/\omega_\infty^2$. When $\alpha = 0$ and $u \neq 1$ the functions $\tilde{n}_{1,2}^2 = \tilde{n}_\pm^2$ have no poles; as $\alpha \to 0$, $v_\infty = (1-u)/(1-u\cos^2\alpha) \to 1$, but the straight line $v = 1$ thus obtained corresponds to a plasma wave (§ 12). If $\alpha \neq 0$ and $\alpha \neq \frac{1}{2}\pi$, the wave equations (23.2) do not strictly separate into two second-order equations, but for a sufficiently extended layer this separation can usually still be carried out to a good approximation (see § 25). One exception is the region of low densities ($v \to 0$) discussed in § 26. The problem of the poles of the functions $\tilde{n}_{1,2}^2$ is here of little interest, since $v_{1\infty} \to 0$ only as $u \to 1$. Moreover, at the boundary of the layer, collisions are usually important and the pole lies far from the real axis (see below). Another exceptional case where we must go beyond the second-order equation occurs when $\alpha \to 0$. This corresponds to the "tripling" of signals discussed in § 28. At present we need only consider an equation of the type (16.3):

$$d^2F/dz^2 + (\omega^2/c^2)\varepsilon'_{\text{eff}}(\omega, z)F = 0, \tag{27.3}$$

where F is some function related to the wave field $\boldsymbol{E}$; see, for instance, (25.22), (25.23), and (25.4).

In a magnetoactive plasma with $\alpha = \frac{1}{2}\pi$ we have $\varepsilon'_{\text{eff}} = (n - i\varkappa)_1^2$ or, in the absence of collisions, $\varepsilon'_{\text{eff}} = \tilde{n}_1^2$; for other values of α we can still usually put $\varepsilon'_{\text{eff}} = (n - i\varkappa)_{1,2}^2$, as shown in § 25. Equation (27.3) with a function $\varepsilon'_{\text{eff}}$ which has a pole is of interest also for an isotropic plasma when a wave having the vector $\boldsymbol{E}$ in the plane of incidence is obliquely incident on the layer. In this case we have [see (19.22) and (19.23)]

$$\left.\begin{aligned} \varepsilon'_{\text{eff}} &= \varepsilon'(z) - \varepsilon'(0)\sin^2\theta_0 + \frac{c^2}{\omega^2}\left[\frac{d^2\varepsilon'/dz^2}{2\varepsilon'} - \frac{3}{4}\frac{(d\varepsilon'/dz)^2}{\varepsilon'^2}\right], \\ E_z &= F\sin\theta(z)\exp[\pm i(\omega/c)\sqrt{\varepsilon'(0)}\sin\theta_0]. \end{aligned}\right\} \tag{27.4}$$

In the absence of collisions this function $\varepsilon'_{\text{eff}} = \varepsilon_{\text{eff}}$ has a pole at the point where $\varepsilon = 0$. For example, for a linear layer $\varepsilon' = -az - is$ with $\varepsilon'(0) = 1$

we have

$$\varepsilon'_{\text{eff}} = -az - is - \sin^2\theta_0 - \frac{3c^2}{4\omega^2}\frac{a^2}{(az+is)^2}. \tag{27.5}$$

In an isotropic plasma with normal incidence

$$\varepsilon'_{\text{eff}} = \varepsilon' = 1 - 4\pi e^2 N(z)/m\omega[\omega - i\nu_{\text{eff}}(z)]$$

and so ε' has no pole if the density N is finite and z is real. For other media the permittivity ε' in the absence of absorption has a pole even in the isotropic case [see, for example, (22.18)].

Thus, in order to solve some problems of wave propagation, we must examine equation (27.3) with a function $\varepsilon'_{\text{eff}}(z)$ which has a pole. The pole evidently lies on the real axis, and so is of interest here, only when $\varepsilon'_{\text{eff}} = \varepsilon_{\text{eff}}$, i.e. when the function $\varepsilon'_{\text{eff}}$ is real.

If collisions in the plasma are taken into account, however, the pole of $\varepsilon'_{\text{eff}}$ corresponds to complex values of z [see (27.5)]. For a magnetoactive plasma in the presence of collisions, $(n - i\varkappa)^2_{1,2} \to \infty$ when†

$$\begin{aligned} v_{1,2\infty} &= 4\pi e^2 N(z)/m\omega^2 \\ &= (1 - is)[(1 - is)^2 - u]/[(1 - is)^2 - u\cos^2\alpha], \end{aligned} \tag{27.6}$$

where, in the most general case of a plane-parallel medium, $u = \omega_H^2(z)/\omega^2$ and $s = \nu_{\text{eff}}(z)/\omega$.

Since, when collisions are taken into account, all quantities are finite on the real z-axis, it may seem that this problem has no essential singularities. This is not so, however. When the function $\varepsilon'_{\text{eff}}(z)$ has a first-order pole, a wave passing near the singularity is found to be attenuated even when $s = \nu_{\text{eff}}/\omega \to 0$. Moreover, for small s the wave field in the region of large values of $|\varepsilon'_{\text{eff}}|$ behaves in a very curious manner, as we have seen in § 20. Thus the problem of wave propagation when there is a "resonance region", i.e. a pole of the function $\varepsilon'_{\text{eff}}$, in fact merits separate analysis. Various aspects of this problem are discussed in [22, § 79; 130–132, 167–169].

The rigorous solution for a layer with $\varepsilon'_{\text{eff}} = g/(z + is)^2$

For certain functions $\varepsilon'_{\text{eff}}(z)$ which have a pole, equation (27.3) has a rigorous solution expressible in terms of known functions. For instance, if

$$\varepsilon'_{\text{eff}} = g/(z + is)^2, \tag{27.7}$$

the general solution of equation (27.3) has already been given at the end of § 17, and is

$$\left.\begin{aligned} F &= C_1(z + is)^{r_1} + C_2(z + is)^{r_2}, \\ r_{1,2} &= \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - \omega^2 g/c^2\right)}. \end{aligned}\right\} \tag{27.8}$$

† See the general expression (11.6) for $(n - i\varkappa)^2_{1,2}$. Formula (27.6) can also be obtained directly by using the fact that the value $(n - i\varkappa)^2_{1,2} = \infty$ corresponds to the vanishing of the denominators in the expressions for A, B and C in the initial equations (11.3).

If the constant g is real and $\omega^2 g/c^2 > \frac{1}{4}$, the exponents $r_{1,2}$ are complex and we have waves travelling in opposite directions. These are independent, and so there is no reflection at or near the point $z = 0$. The function F for $z = 0$ is finite and tends to zero as $s \to 0$. For an isotropic medium and normal incidence, F in (27.3) may be taken to be the field E itself, with $\varepsilon'_{\text{eff}}$ equal to the complex permittivity ε' of the medium. For such a medium with $g > 0$ we see that the presence of a second-order pole of ε' leads to no singularity of the wave field. The situation is different if $g < 0$ in (27.8) and one of the roots $r_{1,2}$ is negative. The function F then has a singularity at $z = 0$ when $s \to 0$; if $r_1 > 0$ and $r_2 < 0$, the singularity of course exists only when $C_2 \neq 0$. The peaking of the field components E_z and E_y for oblique incidence of a wave on an isotropic layer (see § 20) is an instance of this case. For in § 20 the function $\varepsilon'_{\text{eff}}$ had the form (27.5), i.e. was similar to (27.7) when $z \to 0$ and $s \to 0$: $\varepsilon'_{\text{eff}} = -3c^2a^2/4\omega^2(az + is)^2$. From this we have, when $s \to 0$, $r_1 = -\frac{1}{2}$, $r_2 = 3/2$ and $F = C_1/z^{\frac{1}{2}} + C_2 z^{3/2}$; also $|E_z| = |F \sin\theta| = |F \sin\theta_0/\sqrt{\varepsilon'(z)}| = |F \sin\theta_0/\sqrt{(-az - is)}|$ [see (19.21)]. Thus, as $s \to 0$, $E_z = \text{constant}/z$; the constant $C_2 = 0$, on account of the fact that the field must vanish as $z \to \infty$. This result for E_z agrees with that obtained in § 20 [see (20.19)], but the constant can, of course, be determined only by examining the solution throughout the region (as was done in § 20).

The rigorous solution for a layer with $\varepsilon'_{\text{eff}} = g^2/(z + is)$. The physical interpretation

When the permittivity $\varepsilon' = (a + bz)^m$ with $m \neq -2$, equation (27.3) has a solution in terms of Bessel functions [117, 118]. In particular, if

$$\varepsilon' = g^2/(z + is), \tag{27.9}$$

the general solution of equation (27.3) is [125, 167]

$$E = (z + is)^{\frac{1}{2}} \{C_1 H_1^{(1)}[2g(z + is)^{\frac{1}{2}}] + C_2 H_1^{(2)}[2g(z + is)^{\frac{1}{2}}]\}, \tag{27.10}$$

where $H_1^{(1)}$ and $H_1^{(2)}$ are Hankel functions, and we have already put $\varepsilon'_{\text{eff}} = \varepsilon'$ and $F = E_{x,y} = E$, i.e. we are considering wave propagation normal to the layer in an isotropic medium.

We shall choose the sign of $\sqrt{z}$ such that $\sqrt{z} > 0$ for $z > 0$ and $\sqrt{z} = i\sqrt{|z|}$ for $z < 0$ (i.e. $0 < \arg(z + is) < \pi$). Then the requirement that the field is finite as $z \to -\infty$ necessarily means that $C_2 = 0$ in (27.10). For $|\zeta| \gg 1$ we have the familiar results

$$H_1^{(1)}(\zeta) \approx \sqrt{(2/\pi\zeta)}\, e^{i(\zeta - 3\pi/4)}, \quad H_1^{(2)}(\zeta) \approx \sqrt{(2/\pi\zeta)}\, e^{-i(\zeta - 3\pi/4)}.$$

Thus for $z < 0$ only the damped solution remains, as is always the case in the region where $\varepsilon < 0$ (Fig. 27.1). For large positive z, when $2g|(z + is)^{\frac{1}{2}}| \gg 1$, we obtain

$$E \approx C_1 \sqrt{(1/\pi g)}(z + is)^{\frac{1}{4}} \exp\left\{i\left(\omega t + 2g\left[(z + is)^{\frac{1}{2}} - \frac{3}{4}\pi\right]\right)\right\}, \tag{27.10a}$$

i.e. we have a wave travelling from the region of $z = \infty$ to the origin $z = 0$. There is here no reflected wave; for $z \to -\infty$ the field is zero, and the wave is

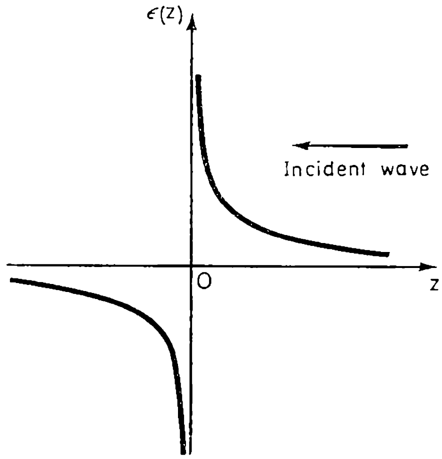

FIG. 27.1. A layer with $\varepsilon(z) = g^2/z$.

therefore completely absorbed near the origin $z = 0$, where for $s = 0$ the permittivity $\varepsilon'(z)$ has a pole.

When $2g|(z + is)^{\frac{1}{2}}| \ll 1$ we have

$$E = C_1 (z + i s)^{\frac{1}{2}} H_1^{(1)}[2g(z + i s^{\frac{1}{2}})]\, e^{i\omega t} \approx (-i\, C_1/\pi g)\, e^{i\omega t}, \qquad (27.10\text{b})$$

which is finite.

If $s \neq 0$ the absence of reflection is not particularly surprising: the wave travels to the region $z \approx 0$ and is absorbed there.† The absorption in the region of small z is in fact especially large, being equal to σE^2 with $\sigma = -(\omega/4\pi)\operatorname{im}\varepsilon' = g^2\omega s/4\pi(z^2 + s^2)$ [see (27.9)], and the real field $E = (C_1/\pi g)\sin\omega t$ [see (27.10b)]. If $s \to 0$, however, there is again no reflection, because the absorption does not vanish in the limit: for small s the total losses are

$$\int \sigma E^2\, dz \approx E^2 \int_{-\infty}^{\infty} \sigma\, dz = \frac{g^2\,\omega\, E^2}{4\pi} \int_{-\infty}^{\infty} \frac{s}{z^2 + s^2}\, dz$$

$$= \frac{1}{4} g^2\,\omega\, E^2 = (C_1^2\,\omega/4\pi^2)\sin^2\omega\, t,$$

† Near a pole of the refractive index n, the derivative dn/dz is large, and so it might appear at first sight that geometrical optics becomes invalid and reflection must occur. In fact, however, the condition

$$\frac{\lambda_0}{2\pi}\frac{|dn/dz|}{n^2} \ll 1$$

[see (16.22)] is satisfied near the pole even better than far from it. This is, of course, due to the fact that the wavelength $\lambda = \lambda_0/n \to 0$ as $n \to \infty$.

i.e. the absorption is independent of s and so naturally does not vanish as $s \to 0$.†

The question still remains of what happens if we put $s = 0$ *ab initio*. There can then be no absorption of energy, and it is difficult to understand what becomes of the energy of the incident wave. The answer is that when $s = 0$ it is not physically justifiable to treat wave propagation as a steady process: in such a process, when $s = 0$, an infinite amount of energy must accumulate in the neighbourhood of the pole, the influx of energy per unit time being finite. Thus we must consider a transition process in which the energy of the incident wave continually accumulates near the pole, and the absence of absorption again involves no paradox.

In physical problems the transition to the case where absorption is absent may usually be conveniently regarded as a passage to the limit obtained when the absorption tends to zero, and we shall interpret in this sense the statement that collisions are absent. The presence of non-zero absorption in the limit is a consequence of a certain resonance and is quite clear from the above discussion (see also below).

A layer with $\varepsilon'_{\text{eff}} = g_1^2 + g_2^2/(z + is)$

In [167] a solution is given of the problem of wave transmission and reflection in a layer with

$$\varepsilon'_{\text{eff}} = g_1^2 + g_2^2/(z + is), \tag{27.11}$$

where as $s \to 0$ we have both the pole $z_\infty = 0$ and the zero $z_0 = -g_2^2/g_1^2$ (Fig. 27.2).

The solution of equation (27.3) with this form of the permittivity $\varepsilon'_{\text{eff}} = \varepsilon'$ can be expressed in terms of a certain kind of hypergeometric functions, sometimes known as Whittaker functions [128]. Investigation shows that, when the wave is incident on the pole from the left, i.e. from the zero of $\varepsilon(z)$ (as shown by the continuous arrows in Fig. 27.2), the moduli of the amplitude coefficients of reflection $|R_0|$ and transmission $|D_0|$ are (for $s \to 0$)

$$\left.\begin{aligned} |R_0| &= 1 - \exp(-\pi\omega g_2^2/c g_1), \\ |D_0| &= \exp(-\pi\omega g_2^2/2c g_1). \end{aligned}\right\} \tag{27.12}$$

† This result corresponds to the well-known fact that the total energy absorbed by a damped harmonic oscillator under the action of a force with a continuous spectrum (i.e. the area of the resonance curve) is independent of the damping decrement when the damping is weak. The condition for the absorption to be small enters into the above derivation when the limits of integration are taken as $\pm\infty$ and the field E is replaced by the constant $E(z = 0)$. This procedure is permissible, since

$$\int_{a<0}^{b>0} \frac{s}{z^2+s^2}\,dz = [\tan^{-1}(z/s)]_{a/s}^{b/s} \to \pi \quad \text{when} \quad s \to 0.$$

Near the pole a relative amount of energy $|A_0|^2 = 1 - |R_0|^2 - |D_0|^2 \geqq 0$ is thus absorbed. If the distance between the zero $z_0 = -g_2^2/g_1^2$ and the pole $z_\infty = 0$ is large, so that $\pi\omega g_2^2/cg_1 = 2\pi^2 g_2^2/\lambda_0 g_1 \gg 1$, then $|R_0| = 1$, $|D_0| = 0$ and $|A_0| = 0$. This is evidently correct, since between the zero and the pole

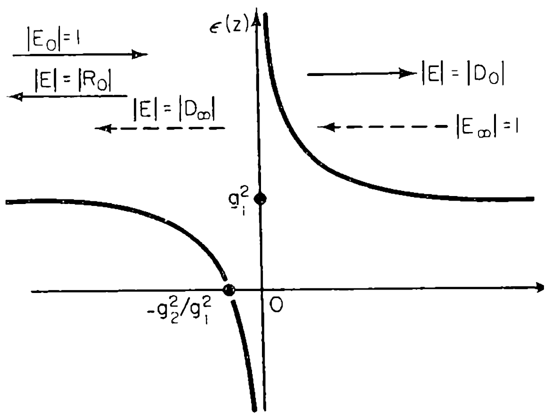

FIG. 27.2. A layer with $\varepsilon(z) = g_1^2 + g_2^2/z$.

the function $\varepsilon(z) < 0$ and the wave is damped. Hence the absorption, which occurs only at the pole when $s \to 0$, cannot be appreciable unless the pole and the zero are close together. When $g_2 \to 0$, the pole and the zero disappear, and $|R_0| \to 0$, $|D_0| \to 1$, $|A_0| \to 0$. Finally, (27.12) shows that the fraction of absorbed energy is

$$|A_0|^2 = 1 - |R_0|^2 - |D_0|^2 = e^{-\delta_0}(1 - e^{-\delta_0}), \quad \text{where} \quad \delta_0 = \pi\omega g_2^2/c g_1. \tag{27.13}$$

This is a maximum when $\delta_0 = \ln 2 = 0{\cdot}6931$, in which case one quarter of the incident energy is absorbed in the resonance region.

Now let the wave be incident from the right, i.e. from the pole (as shown by the broken arrows in Fig. 27.2). Then

$$\left.\begin{aligned} &|R_\infty| = 0, \quad |D_\infty| = \exp(-\pi\omega g_2^2/2c g_1) = |D_0|, \\ &|A_\infty|^2 = 1 - |D_\infty|^2 > 0. \end{aligned}\right\} \tag{27.14}$$

As $g_1 \to 0$ the function (27.11) becomes (27.9), and so in this limiting case we have $|D_\infty| = 0$ as well as $|R_\infty| = 0$, i.e. all the energy is absorbed in the region of the pole. The transmission coefficient is, of course, very small even if $g_1 \neq 0$ but the distance between the pole and the zero of $\varepsilon(z)$ is sufficiently large. On the other hand, when the pole is very close to the zero ($\pi\omega g_2^2/2cg_1 \ll 1$), the wave is "unable to be absorbed" near the pole and passes freely through this region ($|D_\infty| \to 1$). It may also be noted that the result $|D_\infty| = |D_0|$ derived above is no accident, but is common to all such problems for an isotropic medium; this follows from the reciprocity theorem.†

† In a magnetoactive medium the reciprocity theorem in its usual form is not in general valid. In § 29 we shall discuss a generalised reciprocity theorem which holds in such a medium.

The pole of the function $(n - i\varkappa)^2_{1,2}$ in a magnetoactive plasma

Finally, let us consider the problem of principal interest to us, namely the effect of the pole of the function $(n - i\varkappa)^2_{1,2}$ on the propagation of waves in a magnetoactive plasma. As already mentioned, if we exclude from consideration the region of small angles α, the pattern of wave propagation in the presence of the pole can be ascertained from the example of transverse propagation [168, 169]. The corresponding equation (27.1) for a linear layer $v = az$ with no absorption is

$$\frac{d^2 E_x}{d\zeta^2} + \varrho^2\left(2u - \zeta + \frac{u(1-u)}{\zeta}\right)E_x = 0, \tag{27.15}$$

where $\varrho = \omega/ca = 2\pi/\lambda_0 a$ and $\zeta = v + u - 1$ ($\zeta = 0$ at $v_{1\infty} = 1 - u$, where $\tilde{n}_1^2 \to \infty$; see Fig. 27.3). A comparison of (27.11) and (27.15), and of Figs. 27.2 and 27.3, shows that the functions $\varepsilon'_{\text{eff}}$ in these two cases are very similar if

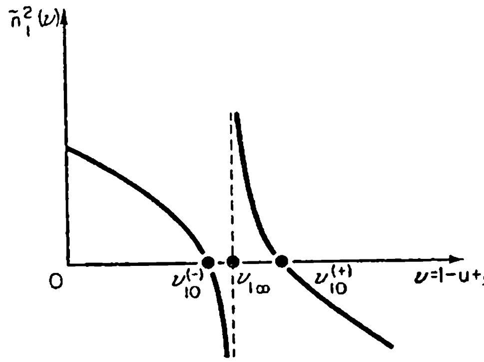

FIG. 27.3. The function $\tilde{n}_1{}^2(v)$ for $\alpha = \frac{1}{2}\pi$.

we may ignore the second (right-hand) zero $v_{10}^{(+)}$ of the function $\tilde{n}_1^2$. This is permissible if the distance between this zero $v_{10}^{(+)} = 1 + \sqrt{u}$ and the pole $v_{1\infty} = 1 - u$ is sufficiently large compared with the wavelength $\lambda_0/2\pi$; for a linear layer $v = az$, this distance is $\Delta z = \sqrt{u}(1 - \sqrt{u})/a = \omega_H(\omega - \omega_H)/a\omega^2$.

Let us therefore begin by neglecting the zero $v_{10}^{(+)}$, and thereby assuming that the wave can pass into the region of large v in Fig. 27.3. (The same result can be achieved if the layer is not linear and the maximum density $N_{\max}$ in it is less than the corresponding point $v_{10}^{(+)} = 4\pi e^2 N_{10}^{(+)}/m\omega^2 = 1 + \sqrt{u}$.) Then we can use, as a certain approximation, the results (27.12)–(27.14) for the layer (27.11), replacing g_2^2 by $u(1-u)/a^2$, g_1^2 by $\sqrt{u}(1+\sqrt{u})/a^2$ and z by ζ, so that the pole $v_{1\infty}$ of n_1^2 and its left-hand zero $v_{10}^{(-)} = 1 - \sqrt{u} = \zeta_{10}^{(-)} + 1 - u$, $\zeta_{10}^{(-)} = -\sqrt{u}(1 - \sqrt{u})$ coincide with the pole and zero of (27.11). The result is

$$\left.\begin{aligned} &|R_0| = 1 - e^{-\delta_0}, \quad |D_0| = |D_\infty| = e^{-\delta_0/2},\\ &|R_\infty| = 0, \quad \delta = \pi\omega u^{\frac{3}{4}}(1-u)/ca\sqrt{[1+\sqrt{u}]},\\ &|A_0|^2 = 1 - |R_0|^2 - |D_0|^2 = e^{-\delta_0}(1 - e^{-\delta_0}),\\ &|A_\infty|^2 = 1 - e^{-\delta_0}. \end{aligned}\right\} \tag{27.16}$$

The energy absorbed in the region of the pole when the wave is incident from the left (i.e. from small v) again, of course, reaches its maximum value of $\frac{1}{4}$ when $\delta_0 = \ln 2$.

The moduli of the coefficients of reflection and transmission for sufficiently extended layers, and usually to a good approximation for the more general case, can be obtained by the method of phase integrals. Some comments on this method will be given in § 28. Here we shall simply mention the result [169] as applied to equation (27.15).

If reflection from the second zero $v_{10}^{(+)}$ is neglected, the expressions for $|R|$ and $|D|$ are given by (27.16), with

$$\left.\begin{aligned}\delta_0 &= -\tfrac{1}{2} i\,\varrho \oint n_1\, d\zeta, \quad \varrho = \omega/c\,a, \\ n_1 &= \sqrt{[2u - \zeta + u(1-u)/\zeta]},\end{aligned}\right\} \tag{27.17}$$

where the integral is taken in the complex plane of ζ along a contour which passes twice round each of the singularities $\zeta_{10}^{(-)} = -\sqrt{u}(1-\sqrt{u})$ and $\zeta_\infty = 0$

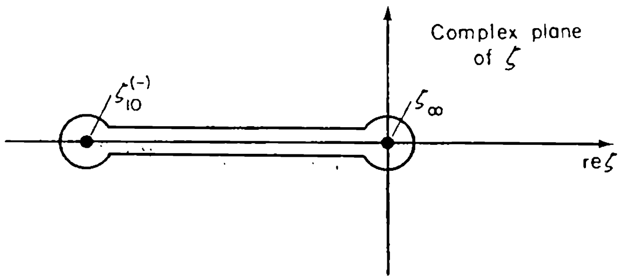

FIG. 27.4. Contour of integration in (27.17). The contour must be traversed twice, although the fact that the surface has two sheets is not shown in the diagram.

(Fig. 27.4).† By distorting the contour into a segment of the real axis between $\zeta_{10}^{(-)}$ and ζ_∞, we can also write (substituting $\zeta = -\sqrt{u}(1-\sqrt{u})t^2$)

$$\begin{aligned}\delta_0 &= -2i\varrho \int_{\zeta_{10}^{(-)}}^{\zeta_\infty} n_1\, d\zeta \\ &= 4u^{3/4}\varrho(1-\sqrt{u})(1+\sqrt{u})^{1/2}\int_0^1 \sqrt{\left[(1-t^2)\left(1+\frac{1-\sqrt{u}}{1+\sqrt{u}}t^2\right)\right]}\, dt.\end{aligned} \tag{27.18}$$

If $\delta_0 \gg 1$, the coefficient $|R_0| = 1 - e^{-\delta_0}$ is very close to unity, and so the case where $\delta_0 \lesssim 1$ is of interest. When $\varrho = \omega/ca \gg 1$, i.e. for a thick layer,

† The necessity of passing twice round $\zeta_{10}^{(-)}$ and ζ_∞ arises from the fact that $n_1(\zeta)$ is analytic only on a surface of two sheets. This is not shown in Fig. 27.4.

this can occur only if $u = \omega_H^2/\omega^2 \ll 1$. Then (27.18) becomes

$$\delta_0 = 4\frac{\omega}{ca}u^{\frac{3}{4}}\int_0^1 \sqrt{(1-t^4)}\,dt = 3{\cdot}5(\omega/c\,a)\,u^{\frac{3}{4}}, \quad u \ll 1, \tag{27.19}$$

since

$$\int_0^1 \sqrt{(1-t^4)}\,dt = \frac{1}{4}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{4}\right)\Big/\Gamma\left(\frac{7}{4}\right) \approx 0{\cdot}875,$$

where $\Gamma(x)$ is the gamma function. From (27.16), on the other hand, we have $\delta_0 = \pi(\omega/ca)u^{\frac{8}{4}}$ for $u \ll 1$.

The result (27.19) is apparently the more exact, despite the approximations used in deriving formulae (27.18) and (27.19) when the points $\zeta_{10}^{(-)} = -\sqrt{u}(1-u)$ and ζ_∞ move together. This may be deduced as follows. Formula (27.16) for a layer of the type (27.11) has been obtained by an exact solution of the wave equation, but the application of the general expression (27.18)

$$\delta_0 = -2i\varrho\int_{\zeta_{10}^{(-)}}^{\zeta_\infty} n_1\,d\zeta$$

to the layer (27.11) gives the same formula (27.16). Thus it seems that the method of phase integrals in practice leads to exact formulae for $|R|$ and $|D|$ in the problems under consideration and in similar problems.

A second conclusion which may be drawn is that the replacement of the layer (27.15) by the corresponding layer of the type (27.11) leads (at least for $u \ll 1$) to almost the same expression for δ_0, the coefficient 3·5 in (27.19) being replaced by $\pi = 3{\cdot}14$ in (27.16) for $u \ll 1$.

When reflection from the second zero $v_{10}^{(+)} = 1 + \sqrt{u} = \zeta_{10}^{(+)} + 1 - u$, $\zeta_{10}^{(+)} = \sqrt{u}(1+\sqrt{u})$, is taken into account, the method of phase integrals gives

$$\left.\begin{aligned} R_0 &= e^{i\pi/2}[1 - e^{-\delta_0}(1 - e^{-2iS})],\\ |R_0|^2 &= 1 - 4e^{-\delta_0}(1 - e^{-\delta_0})\sin^2 S,\\ S &= \varrho\int_{\zeta_\infty}^{\zeta_{10}^{(+)}} n_1\,d\zeta\\ &= 2\varrho\,u^{\frac{3}{4}}(1+\sqrt{u})(1-\sqrt{u})^{\frac{1}{2}}\int_0^1 \sqrt{\left[(1-t^2)\left(1+\frac{1+\sqrt{u}}{1-\sqrt{u}}t^2\right)\right]}\,dt, \end{aligned}\right\} \tag{27.20}$$

where δ_0 is again defined by (27.18).

If the distance between the pole $\zeta_\infty = 0$ and the zeros $\zeta_{10}^{(\pm)}$ is large, we of course obtain from (27.20) the correct result $R_0 = e^{i\pi/2}$. This is also obtained when $u \to 0$ and the medium becomes isotropic, with total reflection from the layer with $\varepsilon = 1 - v$. The result is not completely trivial, since the method of phase integrals cannot in general be regarded as applicable when the singular points are close together.

When $u \ll 1$ [see (27.19)] we have

$$\left.\begin{aligned} S = \tfrac{1}{2}\delta_0 &= 1{\cdot}75(\omega/ca)\,u^{\frac{3}{4}}, \\ |R_0|^2 &= 1 - 4e^{-\delta_0}(1 - e^{-\delta_0})\sin^2\tfrac{1}{2}\delta_0. \end{aligned}\right\} \qquad (27.21)$$

The reflection coefficient $|R_0|^2$ as a function of δ_0 in this approximation is shown in Fig. 27.5. The minimum value of $|R_0|^2$ is about 65 per cent, i.e. a maximum of 35 per cent of the incident energy is lost in the resonance

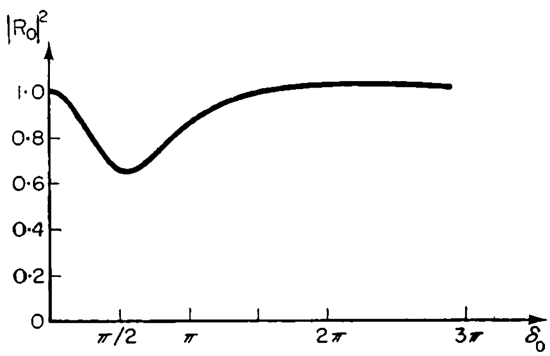

FIG. 27.5. Reflection coefficient $|R_0|^2$ as a function of the parameter δ_0.

region. When reflection from the second zero is neglected, as above, not more than 25 per cent of the incident energy is absorbed. This relates to the limiting case $s \to 0$. When $s \neq 0$ the calculation can still be carried out, and in the method of phase integrals with $s \ll 1$ the total absorption is simply the sum of the resonance absorption calculated previously and the absorption undergone by the wave on the way to the reflection point $\zeta_{10}^{(-)}$ and back.

For angles $\alpha \neq \frac{1}{2}\pi$ but such that the interaction between the normal waves is negligible, the problem of the reflection and transmission of waves when a resonance region is present varies with α only as regards the form of the functions $n_{1,2}^2(v, u, s, \alpha)$ in the wave equation of the type (27.1). Here the method of phase integrals leads to the results (27.17), (27.20) with $n_1(v, u, \alpha = \frac{1}{2}\pi)$ replaced by $n_{1,2}(v, u, \alpha)$,

The mechanism of resonance. The "peaking" of the field in a magnetoactive plasma

The absorption of energy near a pole of the function $\varepsilon'_{\text{eff}}$ has here been called "resonance absorption" essentially only by analogy with the resonance absorption of a harmonic oscillator. In reality, for actual media we have more than an analogy; it is the same phenomenon considered in different conditions. This is particularly easily seen for the case of an isotropic medium consisting of an assembly of harmonic oscillators of eigenfrequency ω_i, when $\varepsilon' = \varepsilon - i \cdot 4\pi\sigma/\omega = 1 + 4\pi e^2 N/m(\omega_i^2 - \omega^2 + i\omega\nu_{\text{eff}})$ [see (22.18)]. If the frequency ω of the incident radiation is varied, the energy σE^2 absorbed by the oscillators is given, as a function of ω, by the resonance curve constant $\times\, E^2/[(\omega - \omega_i)^2 + (\frac{1}{2}\nu_{\text{eff}})^2]$ (it is assumed that $\omega - \omega_i \ll \omega_i$). Now let the frequency ω be constant but the eigenfrequency ω_i vary as a function of the coordinate z. It is then clear that the result will be the same as in the previous case, except that the resonance curve will "spread" over the coordinate z instead of the frequency ω. For oscillators this statement of the problem is somewhat artificial, but it corresponds exactly to the state of affairs in an inhomogeneous magnetoactive plasma. The eigenfrequency is here represented by the frequency $\omega_0 = \sqrt{[4\pi e^2 N(z)/m]}$, and in complete accordance with this we find plasma waves exactly at the points $v_{1,2\infty}$ where $\tilde{n}_{1,2}^2 \to \infty$ (§ 12).

When we go to the isotropic case, $v_{1,2\infty} \to 1$ and the plasma wave exists only when $v = 1$. For normal incidence on an isotropic layer the plasma wave is not excited, however, and so the resonance effect occurs in an isotropic medium only for oblique incidence of a wave with $E_z \neq 0$ (see § 20).

In a magnetic field, the plasma wave in the neighbourhood of the point $v_{1,2\infty}$ can be excited even at normal incidence if $\alpha \neq 0$, because [see (11.28)]

$$E_z = -i \cdot \frac{\sqrt{u}\,(v \sin\alpha)}{u - (1 - v) - u\,v\cos^2\alpha}\,E_x + \frac{u\,v\cos\alpha\sin\alpha}{u - (1 - v) - u\,v\cos^2\alpha}\,E_y,$$

and consequently $E_z \neq 0$ in most cases even for normal incidence; an exception is formed by both normal waves for $\alpha = 0$ and by the ordinary wave for $\alpha = \frac{1}{2}\pi$. In particular, for the extraordinary wave with $\alpha = \frac{1}{2}\pi$ we have $E_x \neq 0$, $E_y = 0$ and

$$E_z = -\frac{i\sqrt{u}\,(v)}{u - (1 - i s)(1 - i s - v)}\,E_x, \tag{27.22}$$

where we have taken account of absorption [see (10.20)].

Thus it is evident from physical considerations that the "peaking" of the wave field, which occurs in an isotropic plasma near the point $\varepsilon = 0$ (i.e. for $v \approx 1$) for oblique incidence of a wave with $E_z \neq 0$, must occur in a magnetoactive plasma near the points where $\tilde{n}_{1,2}^2 \to \infty$ (i.e. for $v \approx v_{1,2\infty}$) for both normal and oblique incidence. Moreover, it is evident that, other conditions being equal, the "peaking" becomes greater as the point $\tilde{n}_{1,2}^2 \to \infty$ approaches the reflection point. For normal incidence of wave 1 we have at the reflection point $\tilde{n}_1^2 = 0$ and $v_{10}^{(-)} = 1 - \sqrt{u}$. For oblique incidence the reflection point given by the condition $n_{1,2} = \sin\theta_0$ lies below its position for normal incidence, and so the "peaking" of the field is less marked.† In the ionosphere and the

† When $\alpha \neq \frac{1}{2}\pi$ and $\alpha \neq 0$ this statement needs some refinement, as the reflection points for the ray and the wave normal do not coincide (§ 29).

corona, i.e. for thick layers, the points $v_{1,2\infty}$ and v_{10} or $v_{20} = 1$ are close together only for small values of u. When $u \to 0$, however, the effect disappears for normal incidence, since $E_z \to 0$. For oblique incidence with $u \to 0$, the effect of course becomes the same as for an isotropic plasma. Thus the picture is fairly clear from the physical point of view.

The derivation of quantitative results in the general case, where a magnetic field is present, is a problem as yet unsolved. For normal incidence and $\alpha = \frac{1}{2}\pi$ the problem of the "peaking" of the field near the point $v_{1\infty} = 1 - u$ can be treated for a linear layer by solving equation (27.15). In the immediate neighbourhood of the pole this equation is equivalent to the wave equation for the layer (27.9), i.e. we can put $2u - \zeta + u(1-u)/\zeta \approx u(1-u)/\zeta$. From (27.10) we see that the field E_x is finite at the pole itself, and then (27.22) shows that at the pole

$$E_z = C/[u + v - 1 + is(1+u)] = C/[az' + is(1+u)], \tag{27.23}$$

where absorption is assumed small ($s^2 \ll 1$) and we put $u + v - 1 = az'$ (for $s = 0$, $az' = \zeta$).

This result [168] corresponds exactly to formula (20.19) for an isotropic plasma. The constant C in (27.23) tends to zero as $u \to 0$, as is immediately evident from (27.22), which shows that $E_z = 0$ when $u = 0$. Further, it follows from the corresponding calculation for an isotropic plasma [see (20.14)], and from more general arguments, that the constant C must be given by the factor

$$C \sim \exp\left(-\left|\frac{\omega}{ca}\int_{\zeta_{10}^{(-)}}^{\zeta_\infty} n_1\, d\zeta\right|\right) = e^{-\delta_0/2} \tag{27.24}$$

which characterises the damping of the wave on the path from the reflection point $\zeta_{10}^{(-)}$ to the pole $\zeta_\infty = 0$.

The Earth's ionosphere

For the layers of the ionosphere, where $\nu_{\text{eff}} \gtrsim 10^3$ and the magnetic field $H^{(0)} \sim 0{\cdot}5$ oersted, the "peaking" of the field is negligible. The reason is that the quantity $u = \omega_H^2/\omega^2$ cannot be regarded as very small, since the frequency ω cannot exceed the critical frequency ω_{cr} of the layer (otherwise the function $\tilde{n}_1^2$ would have no pole). For the F layer $2\pi f_{\text{cr}} = \omega_{\text{cr}} \lesssim 10^8$ ($\lambda_{\text{cr}} \gtrsim 20$ to 30 m), and for $\omega_H \sim 8 \times 10^6$ ($H^{(0)} \sim 0{\cdot}5$ oersted) we have $u \gtrsim 10^{-2}$. For $u \ll 1$, equations (27.19) and (27.24) give

$$C \sim \exp(-1{\cdot}75\,\omega u^{\frac{3}{4}}/ca). \tag{27.25}$$

Hence for $\omega = 10^8$, $a \sim 10^{-7}$ and $\omega_H \sim 8 \times 10^6$ we obtain $C \sim e^{-1000}$. In an isotropic plasma the "peaking" of the field might be greater only because by reducing the angle of incidence θ_0, which is equivalent to reducing u, we could bring the reflection point $n(z_0) = \sin\theta_0$ nearer to the pole of the function $\varepsilon'_{\text{eff}}$ at $n(z_\infty) = 0$.

Thus for the Earth's ionosphere, at least when ionisation is normal, the peaking of the field near the points $\tilde{n}_{1,2} \to \infty$ is of no importance. In the solar corona, where the frequencies ω are greater and the field $H^{(0)}$ may be very weak, the effect is perhaps sometimes more significant.

The allowance for spatial dispersion

The component E_z of the wave field and the field energy at the point $\tilde{n}^2_{1,2} \to \infty$ become finite not only when collisions are taken into account [i.e. when $s \neq 0$ in formula (27.23)] but also when allowance is made for spatial dispersion. Here the situation is essentially the same as in an isotropic plasma (§ 20). The only difference is that, when a magnetic field is present, plasma waves appear not as an isolated branch but as a direct continuation of the branches of the wave for which $\tilde{n}^2 \to \infty$. For this reason it is physically evident that, when spatial dispersion is taken into account, the wave travelling to the pole simply becomes a plasma wave (§ 12), and the plasma waves are damped even when collisions are absent. However, if even a small number of collisions occur, all the absorbed energy is ultimately converted into heat.

§ 28. THE "TRIPLING" OF REFLECTED SIGNALS BY THE INTERACTION OF NORMAL WAVES FOR SMALL α

The range of small angles α between the magnetic field and the wave vector. Description of the phenomenon

When the angle α between the direction of the external magnetic field $\boldsymbol{H}^{(0)}$ and the wave vector $\boldsymbol{k}$ is small (for normal incidence, which is considered here, the vector $\boldsymbol{k}$ is in the z-direction), there is an interaction between the ordinary and extraordinary waves near the point $v_{20} = 4\pi e^2 N(z)/m\omega^2 = 1$. The nature of wave propagation for small α has already been described (§§ 11, 12, 23, 25). Here, therefore, we shall only give a brief recapitulation of the important points, using the example of radio signals reflected from an inhomogeneous layer (the ionosphere, for instance).

In longitudinal propagation ($\alpha = 0$) the radio signals are reflected from the points $v_{10}^{(\pm)} = 1 \pm \sqrt{u}$; if the angle α is sufficiently large, reflection occurs from the points $v_{10}^{(-)} = 1 - \sqrt{u}$ and $v_{20} = 1$ (see Fig. 28.1a, where the case $u < 1$ is taken and the frequencies are assumed to be below critical). Accordingly the relative delay of the two signals in longitudinal propagation is considerably greater than when $\alpha \neq 0$. Formally, in terms of the $n_{1,2}(v)$ curves, there is no continuous transition from the case $\alpha \neq 0$ to $\alpha = 0$, i.e. reflection of the ordinary signal must jump from v_{20} to $v_{10}^{(+)}$. Physically, it is clear that the transition must be continuous, and the observed phenomena will be as follows.

For $v < 1$, wave 1 (the extraordinary wave) is always propagated up to the point $v_{10}^{(-)}$ only, and undergoes total reflection there. If the angle α is sufficiently large, the ordinary wave 2 is totally reflected at $v_{20} = 1$. When the angle α decreases, however, geometrical optics becomes inapplicable to both types of wave in the region $v \approx 1$; we cannot use the curves of $n_{1,2}(v)$, and there is only partial reflection of wave 2 from the region $v \approx 1$. The wave is partly transmitted as a wave of type 1, which can be propagated for $v \geqq v_{1\infty}$. The possibility of this change is seen from Fig. 28.1a, which shows that as $\alpha \to 0$ the curves of n_1 and n_2 come very close together, so that in the transition region waves 1 and 2 have almost the same properties. (These regions are

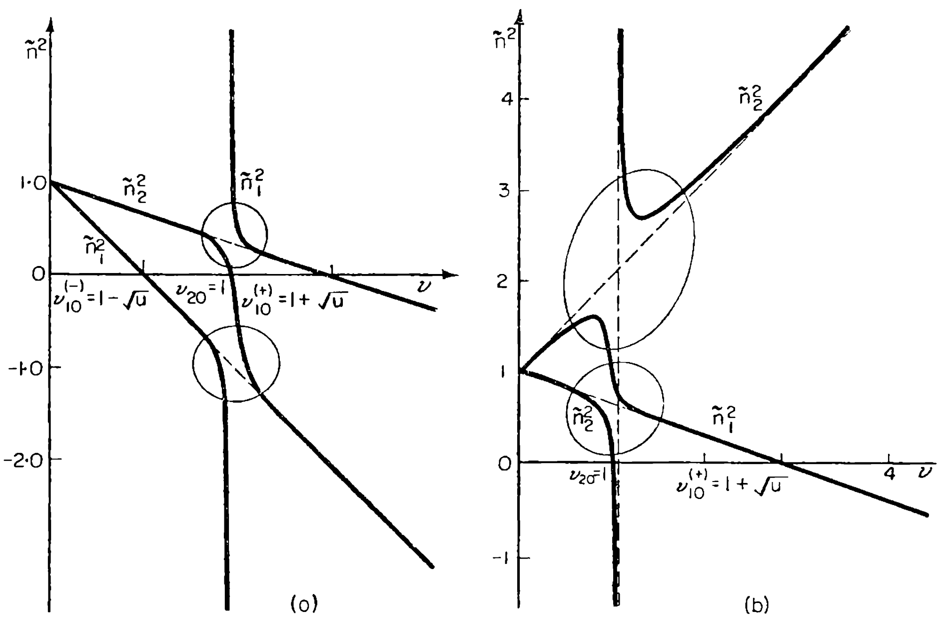

FIG. 28.1. The functions $\tilde{n}_{1,2}^2$ for small angles α (diagrammatic). The broken lines are for $\alpha = 0$.
(a) $u < 1$ (b) $u > 1$

encircled in the diagram.) This applies not only to the phase velocity, which is determined by the refractive index $n_{1,2}$, but also to the state of polarisation, since by Fig. 11.12 for small α the polarisations of wave 2 for $v \lesssim 1$ and of wave 1 for $v \gtrsim 1$ are almost the same.

The wave of type 1 propagated upwards (for $v > 1$) is totally reflected at the point $v_{10}^{(+)} = 1 + \sqrt{u}$ and returns downwards (we assume that the layer extends for a sufficient distance in the direction of values of v exceeding $v_{10}^{(+)}$). The wave 1 travelling downwards in the region $v \approx 1$ is partly transmitted and changed into a wave of type 2, and partly absorbed in the resonance region. The wave 1 travelling downwards will not be reflected in this region, as we see from the solution of the equations and the entirely similar absence of reflection in the kindred examples discussed in § 27.

Consequently, in radar probing of the ionosphere, not two but three reflected signals will be observed .This is the "tripling" effect.† As will be shown below, when absorption (other than resonance absorption) is neglected the amplitudes of the second and third signals $|E_2|$ and $|E_3|$ are together equal to that of the incident wave $|E_0|$. The corresponding pattern is shown diagrammatically in Fig. 28.2, where $|E_1|$ is the amplitude of the first reflected signal (reflection from the point $v_{10}^{(-)} = 1 - \sqrt{u}$), which is taken to be equal to the amplitude $|E_0|$ of the incident ordinary wave.††

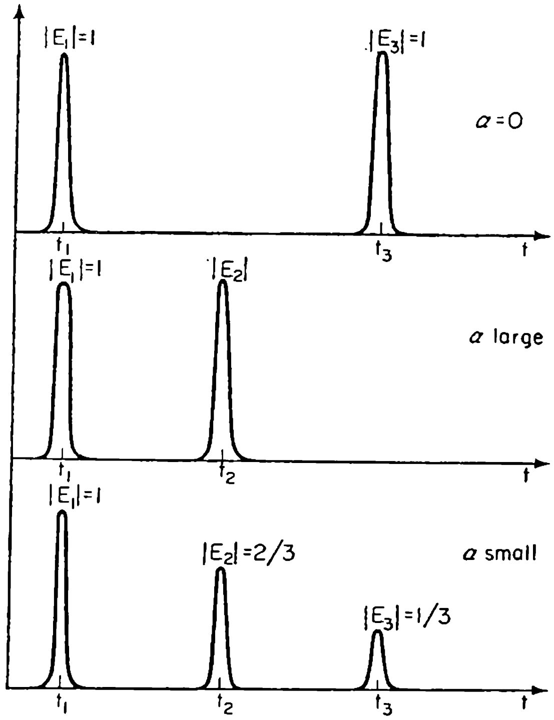

FIG. 28.2. The "tripling" of reflected signals for $u < 1$ and various angles α (diagrammatic).

If $u > 1$, there is only one signal when $\alpha = 0$, which is reflected from the point $v_{10}^{(+)} = 1 + \sqrt{u}$ (see Fig. 28.1 b). When α is sufficiently large there are two signals of equal strength (absorption is neglected, and the amplitude of the incident waves of types 1 and 2 is assumed to be the same). As the angle α decreases there are, as we shall see at the end of this section, several reflected

† In [22, § 79] the "multiplication" of signals was also discussed for $u < 1$, since it was assumed that wave 1 travelling downwards can be reflected from the region $v \approx 1$.

†† By amplitude we here mean, strictly, the modulus of the amplitude. The abscissa in Fig. 28.2 is the time taken for the signal to reach the ground, and for the first reflected signal this time is arbitrarily given the same value in every case. The diagram shows the particular case where $t_3 - t_2 = t_2 - t_1$ and for a small angle α the amplitudes are $|E_2| = \frac{2}{3}$ and $|E_1| = \frac{1}{3}$.

signals and the strength of all except one decreases with α and tends to zero as $\alpha \to 0$. When $\alpha = 0$, one of the waves passes freely through the layer, and when α is small this wave still penetrates to some extent. This effect is very important as regards the possibility of the passage of long waves through the ionosphere, when we must have $u = \omega_H^2/\omega^2 > 1$, and the frequency ω is much less than the critical frequency $\omega_{cr} = \sqrt{(4\pi e^2 N_{max}/m)}$ for the ordinary wave.

The "tripling" and "penetration" of radio waves in a layer of magneto-active plasma are of great interest not only in the Earth's ionosphere but also in those of other planets and in the solar corona. This effect has been the subject of much investigation [145, 22, 70, 124, 148, 164, 170–174]; experimental data are given in [164, 175–177], and the significance of oblique incidence is discussed in [177–183].

In the case of "tripling", as already mentioned, geometrical optics is not applicable to either of the two waves in the region $v \approx 1$, and approximate solutions of the type considered in § 25 cannot be valid, as we see from the condition (23.19). Thus to calculate the reflection coefficient $|R|$ it is necessary to use equations (23.2). No general solution of these can be obtained (more precisely, it would have to be very complex), and so we must have recourse to approximate methods.

Solution by the perturbation method for very small α

Let us begin with the case where the amplitude coefficient of reflection R is very small, as it is for small angles α (see [22, § 79]):

$$|R| \ll 1. \tag{28.1}$$

As the zero-order approximation we take the solution $F_\pm^{(0)}$ of the equations for longitudinal propagation when $\alpha = 0$:

$$\left.\begin{aligned} &d^2 F_\pm^{(0)}/dz^2 + (\omega^2/c^2)(n_\pm - i\varkappa_\pm)^2 F_\pm^{(0)} = 0, \\ &(n_\pm - i\varkappa_\pm)^2 = 1 - v/(1 - i s \mp \sqrt{u}). \end{aligned}\right\} \tag{28.2}$$

If absorption is absent, $s = \nu_{eff}/\omega = 0$. In the present case, however, we cannot immediately neglect absorption when using this method, but must take it into account until the final expression for $|R|$ is reached.† If the angle α is sufficiently small, i.e. if

$$\alpha \ll 1, \tag{28.3}$$

† This necessity is due to the fact that when $s = 0$ a divergent integral appears in the expressions (28.11) and (28.13) below. The inclusion of absorption (which afterwards is allowed to tend to zero) is equivalent to specifying the path along which we must pass round the singularity in effecting the integration. In [148] this was overlooked, and an error was committed (in particular, a factor 2 was missing from the expression for R) which was corrected in [22, § 79].

the condition (28.1) must be satisfied and it is natural to seek a solution of equations (23.2) in the form

$$F_{\pm} = E_x \pm i\, E_y = F_{\pm}^{(0)} + F_{\pm}^{(1)}, \tag{28.4}$$

where

$$|F_{\pm}^{(1)}| \ll |F_{\pm}^{(0)}|. \tag{28.5}$$

It can be shown from (23.2) that $F_{\pm}^{(1)}$ satisfy the equations

$$\begin{aligned} &\frac{d^2 F_{\pm}^{(1)}}{dz^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 - i s \mp \sqrt{u}}\right) F_{\pm}^{(1)} \\ &= \frac{\alpha^2 v}{[(1 - i s)^2 - u](1 - i s - v) - \alpha^2 u v} \times \\ &\times \left[u \frac{d^2 F_{\pm}^{(0)}}{dz^2} + \frac{\omega^2}{c^2}\frac{\alpha^2}{2}\sqrt{u}(v - 1 + i s) F_{\pm}^{(0)} + \frac{\omega^2}{2c^2} u F_{\pm}^{(0)}\right]; \end{aligned} \tag{28.6}$$

here, using (28.3), we have put in (23.2) $\cos\alpha = 1 - \frac{1}{2}\alpha^2$ and $\cos^2\alpha = 1 - \alpha^2$, i.e. we have expanded in series as far as terms of order α^2.

To make the problem specific, let us assume that an inhomogeneous medium occupies some range of values of v near the point $v = 1$, such that

$$v = 1 + az, \quad -z_0 \leqq z \leqq z_0, \quad az_0 \ll 1, \tag{28.7}$$

while for $|z| > z_0$ the medium is homogeneous with no absorption and a refractive index

$$n_2^2 = n_2^2(v = 1) \equiv n_-^2(v = 1) = 1 - 1/(1 + \sqrt{u}). \tag{28.8}$$

In the zero-order approximation we shall suppose, in accordance with the nature of the problem considered for $u < 1$, that a wave of type 2 $(-)$ is propagated upwards in the medium, so that

$$F_-^{(0)} = \exp(-i\,\omega\, n_2 z/c). \tag{28.9}$$

The wave $F_-^{(0)}$ is of the type which in a homogeneous medium is reflected from the point $v_{10}^{(+)} = 1 + \sqrt{u}$. A wave which has been reflected from this point is of no interest here, especially as the waves of the "minus" type propagated in different directions near $v = 1$ are independent. Equation (28.9) also does not take account of absorption, which we assume to be small and which we include in the equation for $F_{\pm}^{(1)}$ only because we cannot there neglect even small absorption. Substituting (28.9) in (28.6) and neglecting terms of the order of s in comparison with unity, and in particular putting on the left-hand side

$$1 - \frac{v}{1 - i s + \sqrt{u}} = 1 - \frac{1}{1 + \sqrt{u}} = n_2^2,$$

we have for the wave $F_-^{(1)}$

$$\begin{aligned} \frac{d^2 F_-^{(1)}}{dz^2} + \frac{\omega^2}{c^2} n_2^2 F_-^{(1)} &= \frac{\alpha^2 u\, \omega^2 \exp(-i\,\omega\, n_2 z/c)}{2c^2(1 + \sqrt{u})^2[a z + i s + \alpha^2 u/(1 - u)]} \\ &= \frac{Q \exp(-i\,\omega\, n_2 z/c)}{a z + i s + \alpha^2 u/(1 - u)}. \end{aligned} \tag{28.10}$$

The general solution of this equation, in the region $-z_0 \leqq z \leqq z_0$ where it is valid, is (if s is independent of z)

$$F_-^{(1)} = C_1 \exp(-i\,\omega\, n_2\, z/c) + C_2 \exp(i\,\omega\, n_2\, z/c) + $$
$$+ \frac{Q \exp(-i\,\omega\, n_2\, z/c)}{2 i\,\omega\, n_2\, a/c} \left\{ \tfrac{1}{2} \ln \left[\frac{\{-a\, z_0 + \alpha^2 u/(1-u)\}^2 + s^2}{\{a\, z + \alpha^2 u/(1-u)\}^2 + s^2} \right] + \right.$$
$$\left. + i \tan^{-1} \frac{a\, z + \alpha^2 u/(1-u)}{s} + i \tan^{-1} \frac{a\, z_0 - \alpha^2 u/(1-u)}{s} \right\} +$$
$$+ \frac{Q \exp(i\,\omega\, n_2\, z/c)}{2 i\,\omega\, n_2/c} \int\limits_{-z_0}^{z} \frac{\exp(-2 i\,\omega\, n_2\, z/c)}{a\, z + \alpha^2 u/(1-u) + i\, s}\, dz. \qquad (28.11)$$

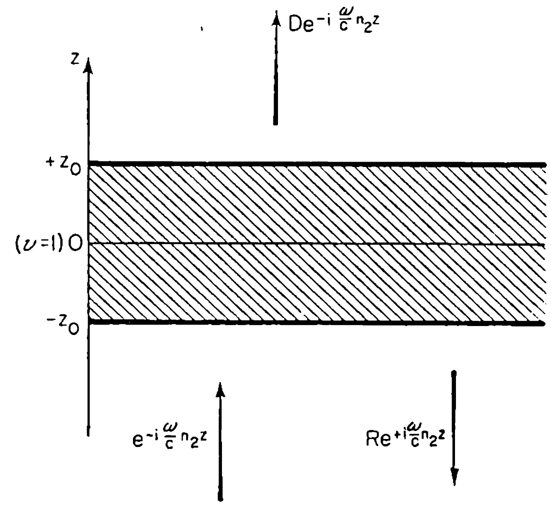

FIG. 28.3. Reflection from a layer of homogeneous plasma with $\alpha \ll 1$.

The coefficients C_1 and C_2 must be determined from the boundary conditions, using the facts that for $z > z_0$ we must have only the transmitted wave and for $z < -z_0$ the incident and reflected waves (Fig. 28.3), i.e.

$$\left.\begin{aligned} z > z_0: &\quad F_- = D \exp(-i\,\omega\, n_2\, z/c)\,, \\ z < -z_0: &\quad F_- = \exp(-i\,\omega\, n_2\, z/c) + R \exp(i\,\omega\, n_2\, z/c)\,. \end{aligned}\right\} \qquad (28.12)$$

For $z = \pm z_0$ the field $F_- = F_-^{(0)} + F_-^{(1)}$ and its first derivative must be continuous, so that

$$\left.\begin{aligned} C_2 = R &= -\frac{Q}{2 i\,\omega\, n_2/c} \int\limits_{-z_0}^{z_0} \frac{\exp(-2 i\,\omega\, n_2\, z/c)}{a\, z + i\, s}\, dz, \\ C_1 &= 0, \\ D &= 1 - \frac{Q}{\omega\, n_2\, a/c} \tan^{-1}(a\, z_0/s)\,, \end{aligned}\right\} . \qquad (28.13)$$

where the term $\alpha^2 u/(1-u)$ has been neglected throughout, as is permissible if $\alpha^2 u/(1-u) \ll a z_0$. When $a z_0/s \to \infty$, i.e. when the absorption s is sufficiently small and the thickness z_0 of the inhomogeneous layer is not too small, $\tan^{-1}(a z_0/s) \to \frac{1}{2}\pi$, and we have

$$D = 1 - \frac{\pi Q}{2\omega n_2 a/c} = 1 - \frac{\pi \omega u \alpha^2}{4 c n_2 a (1+\sqrt{u})^2}$$

$$= 1 - \frac{\pi \omega u^{\frac{3}{4}} \alpha^2}{4c(1+\sqrt{u})^{3/2}(N^{-1}\, dN/dz)_{v=1}}; \tag{28.14}$$

here we have used the results that

$$n_2^2(v=1) = \sqrt{u}/(1+\sqrt{u})$$

and

$$a = \left(\frac{4\pi e^2}{m\omega^2}\frac{dN}{dz}\right)_{v=1} = \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1},$$

since near $v = 1$

$$v = \frac{4\pi e^2 N}{m\omega^2} = 1 + \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1} z = 1 + az.$$

For a sufficiently thick layer $(-z_0, z_0)$, when $2\omega n_2 z_0/c \gg 1$ and $a z_0 \gg s$ (while at the same time, by the hypothesis (28.7), $a z_0 \ll 1$), the amplitude coefficient of reflection R tends to an upper limit given by

$$|R|_M = 2(1-D) = \frac{\pi \omega u^{\frac{3}{4}} \alpha^2}{2c(1+\sqrt{u})^{3/2}(N^{-1}\, dN/dz)_{v=1}}, \tag{28.15}$$

since, as $s \to 0$,

$$\int_{-\infty}^{\infty} \frac{\exp(-2i\omega n_2 z/c)}{z + i s/a}\, dz \to -2\pi i.$$

In practice we can always suppose that $|R| \approx |R|_M$; for example, when $z_0 \sim 10^5$ cm and $\lambda_0 = 2\pi c/\omega \sim 6 \times 10^3$ cm, we have $2\omega n_2 z_0/c \sim 10$ to 100, and for $a \sim 10^6$ to 10^7 the condition (28.7) is satisfied but it may be shown that $|R|_M - |R| \sim |R|_M \div 2\omega n_2 z_0/c \ll |R|_M$. Hence we shall use only the quantity $|R|_M$ in what follows, and omit the suffix M.

Even when $s \to 0$ the total absorption in the layer is finite: the energy absorbed in the layer is proportional to $|A|^2 = 1 - (|D|^2 + |R|^2) \approx |R|$, since by hypothesis $|R| \ll 1$. This result, derived in [22, § 79], is similar to that discussed in § 27, and is, of course, of the same resonance type; in equation (28.10) the right-hand side has a first-order pole. However, in the present case of interaction of waves at small angles α the situation is somewhat more complex, and will be further discussed below.

The amplitude coefficient of transmission D determines directly the field strength E_3 of the signal (the third when $u < 1$) reflected from the point $v_{10}^{(+)} = 1 + \sqrt{u}$:

$$E_3 = |D|^2 \approx 1 - |R| \approx 1 - \frac{\pi\,\omega\, u^{\frac{3}{4}}\,\alpha^2}{2c(1+\sqrt{u})^{3/2}(N^{-1}\,dN/dz)_{v=1}}$$
$$= 1 - \frac{1{\cdot}64 \times 10^{-20}\,\omega^3\,u^{\frac{3}{4}}\,\alpha^2}{(1+\sqrt{u})^{3/2}(dN/dz)_{v=1}}; \qquad (28.16)$$

here we have used the fact that the coefficients $|D|$ for the passage of the wave in either direction are the same, as may be seen by a calculation similar to the above, or from the more general result discussed previously. The amplitude of the incident signal is taken to be unity. As an example, for $u = \frac{1}{4}$ and $\omega = 2\omega_H = 1{\cdot}76 \times 10^7\,\mathrm{sec}^{-1}$ we have

$$|D|^2 \approx 1 - \frac{17\alpha^2}{(dN/dz)_{v=1}}. \qquad (28.16\text{a})$$

The amplitude of the signal reflected from the point $v_{20} = 1$ is

$$E_2 = |R| = 1 - E_3. \qquad (28.17)$$

This is the second signal for $u < 1$ and the first for $u > 1$. Thus absorption essentially changes the distribution of intensity among the signals as compared with that which would occur if absorption were completely (and unjustifiably) neglected; in the latter case $E_3 = 1 - |R|^2$ if $E_2 = |R|$.

For a parabolic layer $N = N_{\max}(1 - z^2/z_m^2)$, we have at the point $v = 4\pi e^2 N/m\omega^2 = 1$

$$\left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1} = \frac{4\pi e^2}{m\,\omega^2}\left(\frac{dN}{dz}\right)_{v=1} = \frac{4\pi e^2 N_{\max}}{m\,\omega^2 z_m^2}\cdot 2\,|z(v=1)|$$
$$= \frac{2\,\omega_{\mathrm{cr},0}^2}{z_m\,\omega^2}\sqrt{\left(1 - \frac{\omega^2}{\omega_{\mathrm{cr},0}^2}\right)} = \frac{2 f_{\mathrm{cr},0}^2}{z_m f^2}\sqrt{\left(1 - \frac{f^2}{f_{\mathrm{cr},0}^2}\right)}. \qquad (28.18)$$

Here $\omega_{\mathrm{cr},0} = 2\pi f_{\mathrm{cr},0} = \sqrt{(4\pi e^2 N_{\max}/m)}$ is the critical circular frequency for the ordinary wave, and we have used the fact that reflection of the ordinary wave with frequency ω at the point $v = 1$ occurs when

$$|z(v=1)| = z_m\sqrt{(1 - f^2/f_{\mathrm{cr},0}^2)}.$$

According to (28.18) the formula (28.16) for a parabolic layer becomes

$$E_3 = |D|^2 = 1 - \frac{\pi^2\,u^{\frac{3}{4}}(z_m/\lambda_0)\,\alpha^2\,f^2/f_{\mathrm{cr},0}^2}{2(1+\sqrt{u})^{3/2}\sqrt{(1 - f^2/f_{\mathrm{cr},0}^2)}}. \qquad (28.19)$$

For $f = 2{\cdot}8 \times 10^6 (\omega = 1{\cdot}76 \times 10^7, \lambda_0 \approx 10^4\,\mathrm{cm}), f_H = 1{\cdot}4 \times 10^6, f_{\mathrm{cr},0} = 3f = 8{\cdot}4 \times 10^6$ and $z_m = 10^7 = 100\,\mathrm{km}$, corresponding to $u = \frac{1}{4}, N(v=1) = 10^5$ and $(dN/dz)_{v=1} \approx 0{\cdot}17$, we have

$$1 - |D|^2 = |R| \approx 10^2\alpha^2. \qquad (28.20)$$

The condition (28.1) shows that formula (28.20) is valid if $\alpha^2 \ll 10^{-2}$, i.e. if $\alpha < 2$ to 3 degrees. In order of magnitude $|R| \sim 1$ when $\alpha \sim 5°$, and so in this example the "tripling" appears at angles $\alpha \sim 5°$. For values of $dN/dz \sim 1$, however, which may very well appear sporadically, this region may reach 10 to 20 degrees, and thus be easily accessible to observation (see also below).† Here it should be noted that, as is clear from the qualitative pattern of the "tripling" effect (see, in particular, Fig. 28.1a), the region in which wave 2 passes into wave 1 is, roughly speaking, that between the points $v_{20} = 1$ and $v_{1\infty} = 1 - u\sin^2\alpha/(1 - u\cos^2\alpha)$. The width of this region is $\Delta v = v_{20} - v_{1\infty} = u\sin^2\alpha/(1 - u\cos^2\alpha)$, or $\Delta z = \Delta v/a = u\sin^2\alpha/a(1 - u\cos^2\alpha)$. When $u = \frac{1}{4}$ and α is even as large as 20° we have $\Delta z \approx 0{\cdot}03/a = 0{\cdot}03\, N(v = 1) \div (dN/dz)_{v=1}$, or, for a parabolic layer (28.18),

$$\Delta z \approx \frac{0{\cdot}03 z_m f^2}{2 f_{\mathrm{cr},0}^2 \sqrt{(1 - f^2/f_{\mathrm{cr},0}^2)}} \sim 10^4 = 100\,\mathrm{m}$$

(for $z_m \sim 100$ km and $f/f_{\mathrm{cr},0} = \frac{1}{3}$). Hence it is clear that the variation of $(dN/dz)_{v=1}$ over a distance of only a few hundred metres leads to a considerable change in the transmission coefficient D.

The variational method: second limiting case

In another more interesting limiting case, where

$$|D|^2 \ll 1, \tag{28.21}$$

the problem with $u < 1$ can be solved by taking as the zero-order approximation a standing wave of type 2, which is completely reflected near the point $v_{20} = 1$. The explicit form of the corresponding zero-order solution can be obtained immediately from the results of § 25 and the solutions for the isotropic case (§§ 30–32). To find the coefficient $|D|^2$ a variational method of the Ritz type is used [145; 22, § 79].

Here we shall consider this problem only very briefly. The equations (23.2) can be derived from the variational principle, i.e. the condition for the first variation of the integral I to be zero, where

$$I = \int_{z_1}^{z_2} L\, dz$$

† If the magnetic latitude of the place of observation is θ (i.e. the angle between the Earth's magnetic axis and the vertical is $\frac{1}{2}\pi - \theta$), then, assuming the Earth's magnetic field to be that of a dipole at its centre, we have the following relation between the angle α (between the vector $\boldsymbol{H}^{(0)}$ and the vertical) and the angle θ:

$$\cos\alpha = \frac{2\sin\theta}{\sqrt{(3\sin^2\theta + 1)}}, \quad \sin\theta = \frac{\cos\alpha}{\sqrt{(4 - 3\cos^2\alpha)}}.$$

An angle $\alpha = 5°$ corresponds to a geomagnetic latitude $\theta = 80°$, and $\alpha = 20°$ to $\theta = 54°$.

and L is the "Lagrangian":

$$L = -\tfrac{1}{2}(E_x' E_x'^* + E_y' E_y'^*) + \tfrac{1}{2} E_x^* (\omega^2/c^2)(A E_x + i C E_y) + \\ + \tfrac{1}{2} E_y^* (\omega^2/c^2)(-i C E_x + B E_y) + \text{complex conjugate.} \tag{28.22}$$

In this expression for L, the prime denotes differentiation with respect to z and the asterisk denotes the complex conjugate; the coefficients A, B, C are, of course, the same as in (23.2). When the function (28.22) is varied, the variations $\delta E_{x,y}$ and $\delta E_{x,y}^*$ must all be regarded as independent. For example, the variation of L with respect to E_x^* is

$$\delta L = -E_x' \delta E_x'^* + (\omega^2/c^2)(A E_x + i C E_y)\, \delta E_x^* \\ = \{E_x'' + (\omega^2/c^2)(A E_x + i C E_y)\}\, \delta E_x^* - d(E_x' \delta E_x^*)/dz. \tag{28.23}$$

The necessary condition for an extremum of the integral I is that the variation δI should vanish. When E_x^* is varied, we have

$$\delta I = \int_{z_1}^{z_2} \{E_x'' + (\omega^2/c^2)(A E_x + i C E_y)\}\, \delta E_x^*\, dz - [E_x' \delta E_x^*]_{z_1}^{z_2}.$$

Since the variation δE_x^* is arbitrary the condition $\delta I = 0$ leads to the equation $E_x'' + (\omega^2/c^2)(A E_x + i C E_y) = 0$, i.e. the first equation (23.2). Similarly, by taking the variation of L with respect to E_y^*, we obtain the second equation (23.2). In addition, however, the variations at the limits must also vanish, i.e. the quantities

$$[E_{x,y}']_{z_1}^{z_2} = E_{x,y}'(z_2) - E_{x,y}'(z_1)$$

must be zero, as we see from the expression for δI. Thus, if these quantities do vanish, equations (23.2) are sufficient to give

$$\delta I = \int_{z_1}^{z_2} \delta L\, dz = 0,$$

where the function L is given by the expression (28.22).

From the above discussion and the principle of the direct methods of the calculus of variations (see, for example, [184]), it is clear that the selection of functions which cause the integral I to approach the extremum is also a means of finding an approximate solution of equations (23.2).

We shall seek a solution of these equations, with the condition (28.21), in the form

$$E_{x,y} = E_{x,y2} + D E_{x,y1}, \tag{28.24}$$

where $E_{x,y2}$ is a standing wave of type 2, i.e. the solution which is obtained by neglecting the penetration of the wave far into the region $v > 1$, and $E_{x,y1}$ is a wave of type 1 propagated in the region $v > v_{1\infty}$. The unknown quantity in (28.24) is the parameter D, which represents a transmission coefficient.

Substituting (28.24) in (28.22) and choosing the parameter D in such a way that the integral $I = \int L\,dz$ is an extremum, we obtain, as shown above, an approximate value for the transmission coefficient. Since it is clear from physical considerations that the initial function (28.24) exhibits the main features of the true solution of the problem when the condition (28.21) holds, we must suppose that the resulting approximation is a good one. This could be rigorously demonstrated only by taking, as in Ritz's method, a trial function like (28.24) but with two unknown parameters, and so on.

In this method of solution it is essential that the variation of the integral I at the limits z_1 and z_2 should be zero for any change in the parameter D; for it is clear that only in this case are the equations (23.2) necessary conditions for the integral I to be an extremum. At the lower limit z_1, which may be taken as the beginning of the layer, the Earth's surface, etc., we have $E_{x,y1} = 0$ and so the variation is certainly zero. At the upper limit with $v \sim 1 + \sqrt{u}$ we may suppose that $E_{x,y1} \sim \exp\left[-i(\omega/c)\int n_1\,dz\right]$ and $E_{x,y2} = 0$. The variation of I at the upper limit is $\delta I = -\{E'_x \delta E^*_x + E'^*_x \delta E_x + E'_y \delta E^*_y + E'^*_y \delta E_y\}$, and is easily seen to be proportional to $i(\omega/c)\,n_1(z_2)(\delta D^* - \delta D)$. Thus, if this variation is to be zero, the parameter D must be real, i.e. the phase of the transmitted wave cannot be calculated. This assumption is convenient, and we shall make it.

The integral I, with (28.24) substituted, is an extremum if $dI/dD = 0$, i.e. if

$$\begin{aligned} D = &-\int\limits_{z_1}^{z_2} \{E'^*_{y1} E'_{y2} + E'_{y1} E'^*_{y2} + E'^*_{x1} E'_{x2} + E'_{x1} E'^*_{x2} + \\ &+ E''_{x2} E^*_{x1} + E''^*_{x2} E_{x1} + E''_{y2} E^*_{y1} + E''^*_{y2} E_{y1}\}\,dz \div \\ &\div 2 \int\limits_{z_1}^{z_2} \{E'_{y1} E'^*_{y1} + E'_{x1} E'^*_{x1} + E''_{x1} E^*_{x1} + E''^*_{x1} E_{x1} + \\ &+ E''_{y1} E^*_{y1} + E''^*_{y1} E_{y1}\}\,dz. \end{aligned} \tag{28.25}$$

The value of D given by (28.25) is real, so that the above assumption that D is real is not inconsistent with our procedure.

For the functions $E_{x,y1,2}$ in (28.24) and (28.25) it would be most correct to take the approximate solutions of equations (23.2) discussed in § 25 [for wave 2, for instance, we have equations (25.35)–(25.37)]. In this case, however, the solution $E_{x,y1,2}$ for a linear layer is expressed in terms of Bessel functions and makes the subsequent treatment fairly complicated. In (28.25) the most important point is not $v = 1$, where the approximation of geometrical optics is invalid, but one corresponding to a complex value of v (see below). It is therefore reasonable to make a further approximation by deriving $E_{x,y1,2}$ from the solutions of geometrical optics (23.13). The wave of type 2 reflected from the point $v = 1$ is of no interest in this approximation, and as the field

$E_{x,y2}$ we may take simply a wave of type 2 travelling upwards. In addition, it must be remembered that, in the presence of absorption, the intensity of the wave of type 1 produced by that of type 2 has to increase with z (or v) in the region here considered. Thus we have

$$\left.\begin{aligned} E_{x2} &= \frac{\exp[-i(\omega/c)\int\limits_0^z (n_2 - i\varkappa_2)\,dz]}{\sqrt{[(n_2 - i\varkappa_2)(1-K_2^2)]}}, \quad E_{y2} = K_2 E_{x2}, \\ E_{x1} &= \frac{\exp[-i(\omega/c)\int\limits_0^z (n_1 + i\varkappa_1)\,dz]}{\sqrt{[(n_1 + i\varkappa_1)(1-K_1^2)]}}, \quad E_{y1} = K_1 E_{x1}, \end{aligned}\right\} \tag{28.26}$$

where $n_{1,2}$, $\varkappa_{1,2}$ and $K_{1,2}$ are given by formulae (11.5) and (11.25), and the lower limit of the integrals ($z = 0$) corresponds to the reflection point $v_{20} = 1$. The choice of this lower limit is the most convenient and does not affect the final result: the substitution of a different limit would simply give an additional phase factor, and this cannot affect $|D|^2$.

The only feature of (28.26) which needs special explanation is the appearance of $n_1 + i\varkappa_1$ (instead of $n_1 - i\varkappa_1$ as we might expect) in the expression for E_{x1}. The reason for this is that we have taken the wave E_{x1} to be travelling upwards, but with an amplitude which increases away from the point $z = 0$. This choice, already mentioned above, is dictated by the physical nature of the problem. As the absorption increases, the transmission coefficient does not decrease, but increases, because the curves of n_1^2 and n_2^2 move still closer together (for $\nu = \nu_{cr}$ these curves touch at $v = 1$). This effect of absorption is taken into account by replacing $n_1 - i\varkappa_1$ by $n_1 + i\varkappa_1$. There is no objection to this in the method under consideration, since the approximate solutions of the type (28.24) are to some extent arbitrary, and the best functions are those which lead to values of the integral I closest to its exact extremum value.

Without pausing to consider other arguments in favour of choosing the functions (28.26), we may note that the values of the transmission coefficient D obtained from them coincide with the values calculated by an entirely different method (see below).

Substituting (28.26) in (28.25), we derive an expression for D in which the integrand in the numerator involves rapidly oscillating factors. The calculation of an accurate value of D from (28.25) is therefore difficult, but it is very easy to find D apart from the coefficient of the exponential. The reason is that the numerator of (28.25) is re J, where

$$J = \int\limits_{z_1}^{z_2} f(z)\, e^{-i\omega\varphi(z)/c}\,dz, \quad \varphi(z) = \int\limits_0^z [(n_2 - i\varkappa_2) - (n_1 - i\varkappa_1)]\,dz,$$

with $f(z)$ a slowly varying function of z. The integral J is of such a form that it may be computed by the saddle-point method, continuing the integrand into the complex domain. At the saddle point the derivative of the function φ in the exponent, i.e. the derivative

$$\frac{d}{dz}\int_0^z [(n_2 - i\varkappa_2) - (n_1 - i\varkappa_1)]\, dz,$$

must be zero. Hence it follows that at the saddle point v_s we have

$$(n_2 - i\varkappa_2)_{v_s} = (n_1 - i\varkappa_1)_{v_s}. \tag{28.27}$$

Using the expression (11.5) for $n_{1,2} - i\varkappa_{1,2}$, we find

$$v_{s\pm} = 1 - is \pm i\sqrt{u}\,(\sin^2\alpha)/2\,|\cos\alpha| = 1 - is \pm i s_{\mathrm{cr}}; \tag{28.28}$$

here $s_{\mathrm{cr}} = \nu_{\mathrm{cr}}/\omega = \sqrt{u}\,(\sin^2\alpha)/2\,|\cos\alpha|$ is the critical parameter used in § 11 [see (11.41)]. For $s = s_{\mathrm{cr}}$ and $v = 1$ we have $n_2 - i\varkappa_2 = n_1 - i\varkappa_1$ and $K_1 = K_2$, i.e. there is a point at which the medium becomes isotropic (in other words, for $s = s_{\mathrm{cr}}$ the saddle point v lies on the real axis and can be reached by the wave).

When absorption is taken into account, the points v_{20}, $v_{10}^{(\pm)}$ and $v_{1\infty}$ at which $(n - i\varkappa)_{1,2}$ become zero or infinite are

$$\left.\begin{aligned} v_{20} &= 1 - is,\\ v_{10}^{(\pm)} &= 1 - is \pm \sqrt{u},\\ v_{1\infty} &= 1 - is - \frac{(1-is)\,u\sin^2\alpha}{(1-is)^2 - u\cos^2\alpha}. \end{aligned}\right\} \tag{28.29}$$

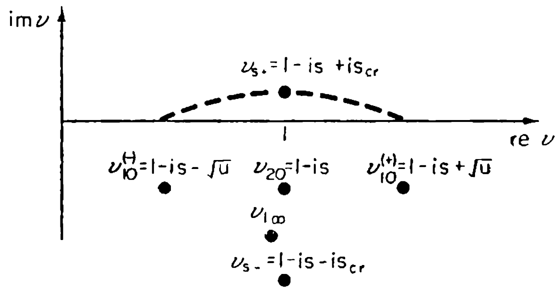

FIG. 28.4. The complex plane of v.

Here it is important to note that the point $v_{1\infty}$ at which the function $(n_1 - i\varkappa_1)$ becomes infinite lies in the lower half-plane of the complex variable v (see Fig. 28.4). Hence, by transforming the path of integration in the integral J, we can make this path pass through the saddle point $v_{s+} = 1 - is + is_{\mathrm{cr}}$, which, for $s < s_{\mathrm{cr}}$ (i.e. for

$$\nu \equiv \nu_{\mathrm{eff}} < \nu_{\mathrm{cr}} = \omega_H \sin^2\alpha/2\cos\alpha, \tag{28.30}$$

where we now write $\cos\alpha$ in place of $|\cos\alpha|$), lies in the upper half-plane; but we cannot make the path pass through the saddle point $v_{s-} = 1 - is - is_{\mathrm{cr}}$, which lies in the lower half-plane. Thus the saddle point which determines the value of the integral J is†

$$\left.\begin{aligned} v_{s+} = v_s &= 1 + i\,(s_{\text{cr}} - s) \\ &= 1 + i\,\frac{\nu_{\text{cr}}}{\omega}\left(1 - \frac{\nu}{\nu_{\text{cr}}}\right) \\ &= 1 + i\left(\frac{\sqrt{u}\,(\sin^2\alpha)}{2\cos\alpha} - \frac{\nu}{\omega}\right). \end{aligned}\right\} \tag{28.31}$$

The use of the saddle-point method involves the replacement of integration along the real v-axis (we assume that $v = 1 + az$, so that the complex planes of v and z are equivalent) by integration along a contour shown diagrammatically by the broken line in Fig. 28.4. The value of J, and therefore of D, is given, apart from the coefficient of the exponential, by the real part of the integrand at the saddle point, so that

$$|D|^2 \sim |J|^2 \sim \exp\left(-2\,\frac{\omega}{c}\,\text{re}\left\{i\int_0^{z_s} [(n_2 - i\,\varkappa_2) - (n_1 - i\,\varkappa_1)]\,dz\right\}\right)$$

$$= \exp\left(2\,\frac{\omega}{c}\,\text{im}\int_0^{z_s} [(n_2 - i\,\varkappa_2) - (n_1 - i\,\varkappa_1)]\,dz\right) = e^{-2\delta}, \tag{28.32}$$

where z_s is the value of z at the point v_s.

We have now merely to calculate the quantity

$$\begin{aligned} 2\delta &= 2\,\frac{\omega}{c}\,\text{im}\int_0^{z_s} [(n_1 - i\,\varkappa_1) - (n_2 - i\,\varkappa_2)]\,dz \\ &= 2\,\frac{\omega}{c}\,\text{im}\int_1^{v_s} [(n_1 - i\,\varkappa_1) - (n_2 - i\,\varkappa_2)]\,\frac{dv}{a} \\ &= \frac{2\omega}{c\,a}\,\text{im}\,i\int_0^{w_s} [(n_1 - i\,\varkappa_1) - (n_2 - i\,\varkappa_2)]\,dw \\ &= \frac{2\,\omega\,w_s}{c\,a}\,\text{re}\int_0^1 [(n_1 - i\,\varkappa_1) - (n_2 - i\,\varkappa_2)]\,d\xi, \end{aligned} \tag{28.33}$$

† It may be noted that, if absorption is neglected as in [145], i.e. if $\varkappa_1 = \varkappa_2 = 0$, then the point $v_{1\infty}$ lies on the real axis, and the problem of deciding between the points $v_{s\pm}$ becomes more difficult. The correct result was obtained in [145] by taking the point v_{s-}, but the function $n_2^* - n_1^*$ was used instead of $n_2 - n_1$. The procedure given here of including damping is more convenient, and the result is (28.31).

where

$$\left.\begin{aligned} &v = 1 + a z = 1 + i w, \quad \xi = w/w_s, \quad dv/dz = a, \\ &a = \frac{4\pi e^2}{m\omega^2}\left(\frac{dN}{dz}\right)_{v=1} = \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1}, \\ &v_s = 1 + i\frac{\nu_{cr}}{\omega}\left(1 - \frac{\nu}{\nu_{cr}}\right) = 1 + i w_s, \\ &\nu_{cr} = \omega_H \sin^2\alpha/2\cos\alpha = \omega\sqrt{u}\sin^2\alpha/2\cos\alpha. \end{aligned}\right\} \tag{28.34}$$

Thus

$$\begin{aligned} 2\delta &= \frac{2\nu_{cr}(1-\nu/\nu_{cr})\beta}{c(4\pi e^2/m\omega^2)(dN/dz)_{v=1}} \\ &= \frac{\omega\sqrt{u}\,\beta(1-\nu/\nu_{cr})\sin^2\alpha}{c(N^{-1}dN/dz)_{v=1}\cos\alpha} \\ &= \frac{m\omega^3\sqrt{u}\,\beta(1-\nu/\nu_{cr})\sin^2\alpha}{4\pi e^2 c(dN/dz)_{v=1}\cos\alpha} \\ &= 1{\cdot}06\times 10^{-20}\beta\,\omega^3\sqrt{u}\left(1-\frac{\nu}{\nu_{cr}}\right)\frac{\sin^2\alpha}{(dN/dz)_{v=1}\cos\alpha}, \end{aligned} \tag{28.35}$$

where

$$\left.\begin{aligned} \beta &= \mathrm{re}\int_0^1 [(n_1 - i\varkappa_1) - (n_2 - i\varkappa_2)]\,d\xi \\ &= \mathrm{re}\,\frac{1}{1-\nu/\nu_{cr}}\int_{\nu/\nu_{cr}}^1 [(n_1 - i\varkappa_1) - (n_2 - i\varkappa_2)]\,d\eta, \\ v &= 1 + i w, \quad \xi = w/w_s, \\ w_s &= \frac{\nu_{cr}}{\omega}\left(1-\frac{\nu}{\nu_{cr}}\right) = s_{cr}\left(1-\frac{s}{s_{cr}}\right), \\ s_{cr} &= \nu_{cr}/\omega = \sqrt{u}\,(\sin^2\alpha)/2\cos\alpha, \quad s = \nu/\omega, \\ \eta &= \left(1-\frac{\nu}{\nu_{cr}}\right)\xi + \frac{\nu}{\nu_{cr}} = \frac{(s_{cr}-s)\,\xi + s}{s_{cr}} \\ &= \frac{w_s\,\xi + s}{s_{cr}}, \\ (n - i\varkappa)_{1,2} &= \sqrt{\left[1 - \frac{2v(1-v-is)}{2(1-is)(1-v-is) - u\sin^2\alpha \pm \sqrt{\{u^2\sin^4\alpha + 4u(1-v-is)^2\cos^2\alpha\}}}\right]} \\ &= \sqrt{\left[1 - \frac{2i(1+iw_s\xi)(w_s\xi+s)}{2i(1-is)(w_s\xi+s) + u\sin^2\alpha \mp \sqrt{\{u^2\sin^4\alpha - 4u(w_s\xi+s)^2\cos^2\alpha\}}}\right]} \\ &= \sqrt{\left[\frac{1 \mp \sqrt{(1-\eta^2)} + \frac{1}{2}\eta^2\tan^2\alpha}{1 \mp \sqrt{(1-\eta^2)} + i(1-is)\,\eta/\sqrt{u}\cos\alpha}\right]}; \end{aligned}\right\} \tag{28.36}$$

here the upper sign of the root corresponds to wave 2 and the lower sign to wave 1.

In the absence of absorption, when $\nu \equiv \nu_{\text{eff}} = 0$, we obtain

$$\left.\begin{aligned} 2\delta = 2\delta_0 &= \frac{2\,\nu_{\text{cr}}\,\beta_0}{c(N^{-1}\,dN/dz)_{v=1}} \\ &= \frac{\omega\,\sqrt{u}\,\beta_0 \sin^2\alpha}{c(N^{-1}\,dN/dz)_{v=1}\cos\alpha}, \\ \beta = \beta_0 &= \operatorname{re}\int_0^1 (n_1 - n_2)\,d\eta, \\ n_1 - n_2 &= \sqrt{\left[\frac{1+\sqrt{(1-\eta^2)}+\frac{1}{2}\eta^2\tan^2\alpha}{1+\sqrt{(1-\eta^2)}+i\,\eta/\sqrt{u}\cos\alpha}\right]} - \\ &\quad - \sqrt{\left[\frac{1-\sqrt{(1-\eta^2)}+\frac{1}{2}\eta^2\tan^2\alpha}{1-\sqrt{(1-\eta^2)}+i\,\eta/\sqrt{u}\cos\alpha}\right]}. \end{aligned}\right\} \tag{28.37}$$

Since for $\eta = 0$ the difference $n_1 - n_2 = 1$, and for $\eta = 1$, $n_1 - n_2 = 0$, it is easy to see that $\beta_0 < 1$. For small angles α (28.37) shows that $n_1 - n_2$, and therefore β_0, are independent of α and are determined only by $\sqrt{u} = \omega_H/\omega$. The value of β_0 may be found by numerical integration. As an example, for $u = \frac{1}{4}$ and small α (so that $\cos\alpha \approx 1$), $\beta_0 \approx 0{\cdot}6$. However, if we assume that the curves $n_1(v)$ and $n_2(v)$ are close together, so that we can put $n_1 + n_2 = 2n_2 \;\; (v = 1,\;\; \alpha = 0) = 2\sqrt{u}/(1+\sqrt{u})$, the integral β_0 may be calculated [124], giving

$$\left.\begin{aligned} 2\,\delta_0 &= \frac{\pi\,(\omega/c\,a)\,u^{\frac{3}{4}}\,(1+\sqrt{u})^{\frac{1}{2}}}{2(1+\sqrt{u}\cos\alpha)^2}\sin^2\alpha, \\ a &= \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1} = dv/dz; \end{aligned}\right\} \tag{28.38}$$

that is, $\beta_0 = \pi u^{\frac{1}{4}}(1+\sqrt{u})^{\frac{1}{2}}\cos\alpha/2(1+\sqrt{u}\cos\alpha)^2$ and, as stated above, for $u = \frac{1}{4}$ and $\cos\alpha = 1$ we have $\beta_0 = 0{\cdot}6$. The derivation given here of the formula for $|D|^2$ can be expected to lead to the correct result only when the angle α is not too small, and only to within some coefficient of the exponential. It is found, however, that for all angles we have with fairly high accuracy

$$|D|^2 = e^{-2\delta_0}, \tag{28.39}$$

where $2\delta_0$ is given by (28.37) and (28.38). This follows from the fact that, for small δ_0 (i.e. small α), formula (28.39) gives $|D|^2 = 1 - 2\delta_0$, which coincides with the result obtained from (28.14). For, according to (28.14) and (28.38),

$$\left.\begin{aligned} |D|^2 &\approx 1 - \frac{\pi\,(\omega/c\,a)\,u^{\frac{3}{4}}\,\alpha^2}{2(1+\sqrt{u})^{\frac{3}{2}}} = 1 - 2\delta_0, \\ a &= \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1}, \end{aligned}\right\} \tag{28.40}$$

since when $\alpha \ll 1$ we can put in (28.38) $\cos\alpha = 1$ and $\sin^2\alpha = \alpha^2$ (and, moreover, formula (28.38) itself has been derived on the assumption that $n_1 + n_2 = 2n_2$, which, we know, is valid for small α). In other words, the coefficient of the exponential in (28.39) is determined (its value being unity) from considerations of the passage to the limit of small α, when the value of $|D|^2$ is calculated in an independent manner.

When α is small, (28.16) shows that the modulus of the amplitude coefficient of reflection of the ordinary wave from the interaction region is $|R| = 1 - |D|^2 = 2\delta_0$. By analogy with the relation between (28.40) and (28.39) we can deduce that in the more general case

$$|R| = 1 - |D|^2 = 1 - e^{-2\delta_0}. \tag{28.41}$$

The method of phase integrals

Formulae (28.41) and (28.39) are confirmed by a calculation using the method of phase integrals [124, 173].

This method is based on a representation of the solution of one or more differential equations as the general solution in the approximation of geometrical optics. For example, for the equation

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\varepsilon'(z)\,E = 0$$

the solution is taken as

$$E = \varepsilon'^{-1/4}\left[C_1\exp\left(i\frac{\omega}{c}\int^z\sqrt{\varepsilon'}\,dz\right) + C_2\exp\left(-i\frac{\omega}{c}\int^z\sqrt{\varepsilon'}\,dz\right)\right]. \tag{28.42}$$

If it were possible to determine correctly the coefficients C_1 and C_2 in this solution, it could be regarded as simply an asymptotic approximation to the exact solution of the problem. A knowledge of the asymptotic form is sufficient to calculate the reflection and transmission coefficients for a layer. To find C_1 and C_2 without solving the equation itself, we may use the fact that the solution $E(z)$ must be an analytic function if $\varepsilon'(z)$ is one.

This leads to some conditions on C_1 and C_2, since the expression (28.42) is a combination of many-valued functions; $\sqrt{\varepsilon'}$ has a branch point at $\varepsilon' = 0$, and we assume that ε' has zeros and consider the problem of reflection from a layer, for example. The solution (28.42) can be an approximation to the exact solution only if the coefficients C_1 and C_2 take different values in different regions of the complex z-plane. At the boundaries between these regions the values of C_1 and C_2 change discontinuously (sometimes referred to as Stokes's phenomenon). The boundaries themselves (Stokes's lines) are determined by the requirement that the functions $\exp\left[\pm i(\omega/c)\int\sqrt{\varepsilon'}\,dz\right]$ should have real values on them.

After a complete circuit round a branch point the solution must be unchanged, and this gives us the ratio of the coefficients C_1 and C_2 on the real

z-axis. This is the basis of the method of phase integrals. The practical application of this method and the calculations involved are described in [124, 185–187]; see also [169, 170, 172]. In [124] this method is used to solve, as an example, the well-known problem of the reflection and transmission of waves by a parabolic layer.

The method of phase integrals is so called because there appear in the calculations, on account of the passage round the zeros of the function ε' or other special points, integral expressions for the phase $i(\omega/c)\oint\sqrt{\varepsilon'}\,dz$ [see, for example, (27.17), etc.]. It is difficult to give a general assessment of the

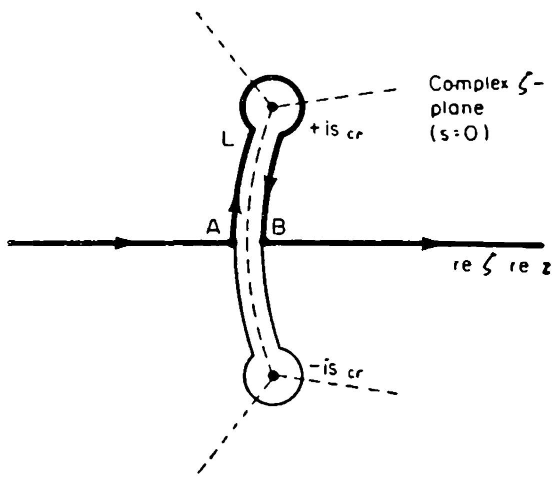

Fig. 28.5. Contour of integration in (28.43).

accuracy of the method. Certainly the correct (rigorous) results for $|R|$ and $|D|$ can in general be obtained by using this method not only for thick layers but also for thin ones [124]. In the problem of reflection from an isotropic layer with a maximum [in the absence of any points to be avoided except the two zeros of $\varepsilon'(z, \omega)$] this is to be expected. In problems where several points have to be avoided the position is less clear, since Stokes's phenomenon occurs at a series of intersecting straight lines and in practice it is impossible to take account of all of these [124, 174].

In the problem of "tripling" of signals (with $u < 1$) the Stokes's lines start from the saddle points $v_{s\pm}$ [see (28.28)], and the equations for these lines may be written

$$\arg\left(\pm i \int\limits_{\pm i s_{cr}}^{\zeta} \tfrac{1}{2}\left[(n_1 - i\varkappa_1) - (n_2 - i\varkappa_2)\right] d\zeta\right) = \pi + 2\pi m, \quad m = 0, 1, 2, \ldots;$$

here $\zeta = v - 1 + is = az + is$ (the layer being supposed linear in the interaction region), and the saddle points $v_{s\pm}$ correspond to $\zeta_{s\pm} = \pm i s_{\text{cr}}$ in the complex ζ-plane (see Fig. 28.5, in which the Stokes's lines are shown by broken lines).

General results for $u = \omega_H^2/\omega^2 < 1$

If an ordinary wave is incident on the interaction region from the left (i.e. from negative values of z and ζ, or from smaller values of v), a reflected ordinary wave of amplitude R_2 and a transmitted extraordinary wave of amplitude D_1 are formed, with

$$\left.\begin{aligned} &|R_2| = 1 - e^{-2\delta_0}, \quad |D_1| = e^{-\delta_0}, \\ &\delta_0 = \left|\frac{\omega}{c\,a}\oint\limits_L \frac{n_1 - n_2}{4}\,d\zeta\right| = \left|\frac{\omega}{c\,a}\int\limits_A^B \frac{n_1 - n_2}{2}\,d\zeta\right|, \end{aligned}\right\} \tag{28.43}$$

where absorption is neglected. The integration in the first integral is along a closed contour L which encloses both the singular points $\zeta_{s\pm} = \pm i s_{cr}$ of the function $n_1 - n_2$; the integration in the second integral is between the points A and B along a half-loop enclosing the point $+ i s_{cr}$ (see Fig. 28.5). It is important to note that the broken line joining the points $\pm i s_{cr}$ in Fig. 28.5 passes between the pole of n_1 at $\zeta_{1\infty} = (1-u)/(1-u\cos^2\alpha) - 1$ and the zero of n_2 at $\zeta_{20} = 0$. No account is taken of these two points in the calculation, which is approximate for that reason.

By changing the path of integration in (28.43) to a line joining the points $\pm i s_{cr}$ (this is justified by the absence of singularities in the result), we can see that the value of $2\delta_0$ is determined by the expression (28.38), and so formulae (28.43) agree with (28.39) and (28.41). For the reflection coefficient $|R_2|$ this is a new result when δ_0 is not small, since essentially no derivation of formula (28.41) has been given above. A more important point is that the analysis by the method of phase integrals demonstrates the existence of a further wave, namely an extraordinary wave reflected from the interaction region downwards, i.e. towards the pole of the function n_1. Of course, the presence of such a wave would be expected on general grounds; if the existence of any wave is possible without violating the boundary conditions of the problem, it must appear except in special cases. But in addition its amplitude R_1 is given by the method of phase integrals [124]:

$$|R_1| = e^{-\delta_0}\sqrt{(1 - e^{-2\delta_0})}. \tag{28.44}$$

Evidently

$$|R_1|^2 + |R_2|^2 + |D_2|^2 = 1, \tag{28.45}$$

i.e. in the absence of collisions ($s = 0$) there is no absorption. This conclusion is entirely reasonable, since in the above calculation the pole of n_1 was unimportant and no account was taken of it. The resonance absorption, which occurs even when $s \to 0$, can take place only in the neighbourhood of a pole. Nevertheless, the above deduction that there is a wave 1 travelling downwards does not contradict the result of the calculation by perturbation theory for angles

$\alpha \ll 1$ (see above). The reason is that the wave 1 travelling towards the pole is not reflected, but completely absorbed (§ 27). Thus the energy

$$|R_1|^2 = 1 - |R_2|^2 - |D_2|^2 = e^{-2\delta_0}(1 - e^{-2\delta_0}) \tag{28.46}$$

is finally converted into heat. (The maximum value of $|R_1|^2$ is $\frac{1}{4}$, when $2\delta_0 = \ln 2$.) This conclusion remains valid even when spatial dispersion is taken into account and the wave considered becomes, for $v < v_{1\infty}$, a "plasma wave" (§ 12), which is then absorbed. When $2\delta_0 \ll 1$ the absorbed energy $|R_1|^2$ is equal to the absorbed energy $|A|^2 = 1 - |D|^2 - |R|^2 = 2\delta_0$ calculated by the perturbation method. Moreover, the interpretation of the energy $|A|^2$ as that absorbed in the resonance region when $\alpha \ll 1$ is correct in itself, since in the perturbation method we consider only waves propagated with $\alpha = 0$ and the perturbation due to the presence of the pole. The extraordinary wave travelling downwards is entirely concentrated near the pole, and need not be discussed, when spatial dispersion is neglected and $\alpha \to 0$.

Here we may give the result of a calculation [169] for an extraordinary wave incident on the interaction region from the right, i.e. from the zero $v_{10}^{(+)} = 1 + \sqrt{u}$ (see Fig. 28.1 a). This wave splits into two: an ordinary wave transmitted to the region of small v with amplitude D_2', and an extraordinary wave travelling to the pole of n_1^2 with amplitude D_1'. No reflected wave 1 occurs, in agreement with the discussion in § 27 and above. That is, the wave 1 coming from the right (from above in the case of the ionosphere) partly penetrates downwards and partly goes to the region of the pole and is absorbed there. Quantitatively we have

$$\left.\begin{aligned} |D_2'| &= e^{-\delta_0}, \\ |D_1'| &= \sqrt{(1 - e^{-2\delta_0})}, \\ |R_1'| &= 0, \\ |D_1'|^2 + |D_2'|^2 &= 1, \end{aligned}\right\} \tag{28.46a}$$

where δ_0 is defined as in (28.43).

Comparing formulae (28.43) and (28.46a), we see that $|D_2'| = |D_1|$, i.e. the transmission coefficient of the interaction region is the same in both directions. (This result is not obvious, because the usual reciprocity theorem is not valid for a magnetoactive medium.) Hence it follows that the amplitude of the third reflected signal transmitted through the interaction region upwards and downwards is $|D_1|\,|D_2'| = |D_1|^2 = e^{-2\delta_0}$, i.e. is given by formula (28.39). The absence of reflection of the wave 1 travelling downwards leads to the appearance of just three reflected signals, whereas in the presence of multiple reflection from the interaction region there would be more than three.

As regards certain possible applications (e.g. to the generation of radio waves in the solar corona) a third statement of the problem is of interest. Let an extraordinary wave travelling from the pole of n_1^2 be incident on the interaction region. This wave may, for example, be produced in the region of the pole by a beam of particles. Then, after the interaction, an extraordinary wave with $|D_1''| = \sqrt{(1 - e^{-2\delta_0})}$ enters the region of larger values of v, while for the reflected extraordinary wave $|R_1''| = e^{-2\delta_0}$. In addition an ordinary wave with $|D_2''| = e^{-\delta_0}\sqrt{(1 - e^{-2\delta_0})}$ is propagated downwards (towards smaller values of v).

Formulae for δ_0. Allowance for collisions

The formulae given above for $|D|$ and $|R|$ have been derived by neglecting collisions, and are therefore valid if

$$s \equiv \nu_{\mathrm{eff}}/\omega \ll s_{\mathrm{cr}} \equiv \nu_{\mathrm{cr}}/\omega = \omega_H \sin^2\alpha/2\omega \cos\alpha; \tag{28.47}$$

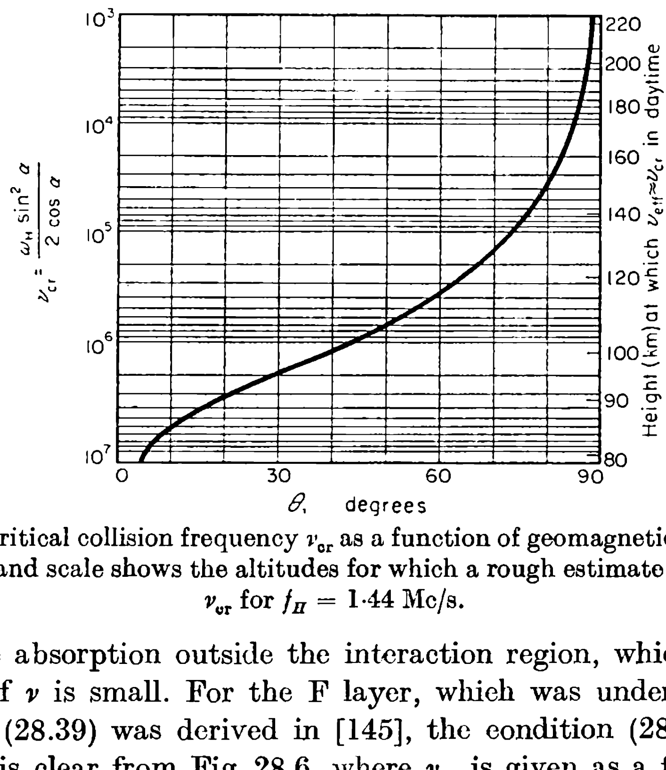

FIG. 28.6. Critical collision frequency ν_{cr} as a function of geomagnetic latitude θ. The right-hand scale shows the altitudes for which a rough estimate gives $\nu_{\mathrm{eff}} = \nu_{\mathrm{cr}}$ for $f_H = 1{\cdot}44$ Mc/s.

we here ignore absorption outside the interaction region, which may be important even if ν is small. For the F layer, which was under consideration when formula (28.39) was derived in [145], the condition (28.47) is usually satisfied. This is clear from Fig. 28.6, where ν_{cr} is given as a function of the geomagnetic latitude θ. On the right are shown the altitudes for which $\nu_{\mathrm{eff}} \approx \nu_{\mathrm{cr}}$. We see that, at heights above 200 km, ν_{eff} is comparable with ν_{cr} only for geomagnetic latitudes $\theta > 85°$. For example, when $\nu_{\mathrm{eff}} = 10^3$, $\nu_{\mathrm{eff}} = \nu_{\mathrm{cr}}$ for angles $\alpha \approx 0{\cdot}8°$ and $\theta \approx 88.4°$.

Formulae (28.38) for $2\delta_0$ may also be written

$$2\delta_0 = \frac{m\,\omega^3 u^{\frac{3}{4}} (1 + \sqrt{u})^{\frac{1}{2}} \sin^2\alpha}{8e^2 c(1 + \sqrt{u}\cos\alpha)^2 (dN/dz)_{v=1}}$$
$$= \frac{1{\cdot}65 \times 10^{-19}\, \omega^3 u^{\frac{3}{4}} (1 + \sqrt{u})^{\frac{1}{2}} \sin^2\alpha}{(1 + \sqrt{u}\cos\alpha)^2 (dN/dz)_{v=1}}, \tag{28.48}$$

since $N = m\omega^2/4\pi e^2$ when $v = 1$.

For $u = \frac{1}{4}$, $\omega = 2\omega_H = 1{\cdot}76 \times 10^7$ sec^{-1} ($\lambda_0 = 107$ m), putting $\cos\alpha \approx 1$, we have $2\delta_0 \approx 17 \sin^2\alpha \div (dN/dz)_{v=1}$; the corresponding value of $|D|^2 = e^{-2\delta_0}$ is shown in Fig. 28.7. From this diagram or directly from formula (28.48) it is clear that in the favourable instance considered ($|D|^2$ decreasing with increasing ω) we have $|D|^2 \approx 0{\cdot}1$ for $\sin^2\alpha \div (dN/dz)_{v=1} \approx 0{\cdot}125$; if $(dN/dz)_{v=1} \approx 0{\cdot}1$ this corresponds to $\alpha \approx 6°$, i.e. geomagnetic latitude $\theta = 80°$. As α increases, the field $|E_3| = |D|^2$ of the third signal diminishes rapidly.

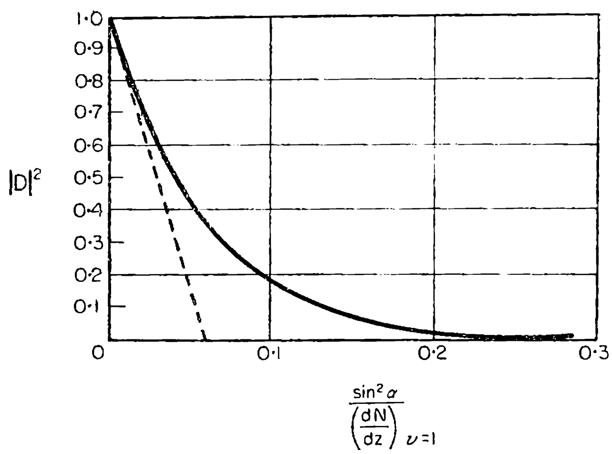

FIG. 28.7. Transmission coefficient $|D|^2$ as a function of $\sin^2\alpha \div (dN/dz)_{v=1}$ for $u = \frac{1}{4}$ and $\omega = 1{\cdot}76 \times 10^7$ ($\lambda_0 = 107$ m).
Continuous curve: $|D|^2 = e^{-2\delta_0}$
Broken curve: $|D|^2 = 1 - 2\delta_0$

For more detailed estimates it is convenient to write formulae (28.38) and (28.39) for the particular case of a parabolic layer (28.18). In this case

$$\left.\begin{aligned} |D|^2 &= e^{-2\delta_0}, \\ 2\delta_0 &= \frac{\pi^2 (z_m/\lambda_0)\, u^{3/4} (1+\sqrt{u})^{1/2} (f^2/f^2_{\mathrm{cr},0}) \sin^2\alpha}{2(1+\sqrt{u}\cos\alpha)^2 \sqrt{(1-f^2/f^2_{\mathrm{cr},0})}} = \frac{v_{\mathrm{cr}}\, \beta_0\, z_m\, f^2/f^2_{\mathrm{cr},0}}{c\sqrt{(1-f^2/f^2_{\mathrm{cr},0})}}. \end{aligned}\right\} \tag{28.49}$$

For $u = \frac{1}{4}$, $\cos\alpha \approx 1$ and $\lambda_0 = 107$ m ($\omega_H = 8{\cdot}82 \times 10^6$ sec^{-1}) we have

$$2\delta_0 \approx \frac{9 \times 10^{-5} z_m (f^2/f^2_{\mathrm{cr},0}) \sin^2\alpha}{\sqrt{(1-f^2/f^2_{\mathrm{cr},0})}} \sim \frac{10^3 (f^2/f^2_{\mathrm{cr},0}) \sin^2\alpha}{\sqrt{(1-f^2/f^2_{\mathrm{cr},0})}}, \tag{28.50}$$

where in the second expression we have taken $z_m \sim 100$ km. From (28.50) it is clear that for a parabolic layer with (e.g.) $f = \frac{1}{3} f_{\mathrm{cr},0}$ the coefficient $2\delta_0 \sim 10^2 \sin^2\alpha$, and $2\delta_0 \sim 1$ for $\alpha \sim 0{\cdot}1 \sim 6°$, in agreement with the estimate made above. Thus for a parabolic F layer, and in fact for any regular layer of the same size, the "tripling" of the signals at normal incidence can be systematically observed only at high latitudes ($\theta \gtrsim 75$ to $80°$). At medium latitudes, where $\alpha \sim 20°$ ($\sin^2\alpha \sim 0{\cdot}1$), the "tripling" of the signals in the F layer can be observed only sporadically, when $(dN/dz)_{v=1} \sim 1$ instead of its usual value $\sim 0{\cdot}1$. The value of $(dN/dz)_{v=1}$ need be anomalously large only in a layer a few hundred metres or a few kilometres thick (see above).† Hence the sporadic effect may well occur at medium latitudes ($\alpha \sim 20°$).

In most cases, however, the appearance of three or more signals reflected from the F layer, observed at medium and low latitudes, is apparently explained by the sporadic occurrence of various types of layer [23]. There is also observed [177, 178] a peculiar "tripling" of the signals for oblique incidence when scattering inhomogeneities are present at the level of reflection (§ 29). Experimentally, the "tripling" effect can be distinguished from reflections from sporadic layers by (in particular) polarisation measurements: in "tripling" the signals E_2 and E_3 in Fig. 28.2 must be ordinary, and E_1 extraordinary. At higher latitudes the "tripling" effect should be observed more and more frequently, and this is confirmed by experiment [164, 176].

In the F layer, as already mentioned, the effect of absorption is usually unimportant. The situation is different in the E layer [164], since the amplitude of the third signal $E_3 = |D|^2$ increases as the collision frequency $\nu_{\text{eff}} \equiv \nu$ approaches the critical frequency ν_{cr}. As we see from Fig. 28.6, at geomagnetic latitudes exceeding 55 to 60° but not very high we in fact have $\nu \sim \nu_{\text{cr}}$ in the E layer, and the "tripling" effect may be very marked. The increase of $|D|^2$ with ν is determined not only by the factor $1 - \nu/\nu_{\text{cr}}$ in (28.35) but also by the fact that $\beta \to 0$ when $\nu/\nu_{\text{cr}} \to 1$; this is clear from (28.36), since $(n_1 - i\varkappa_1) - (n_2 - i\varkappa_2) \to 0$ when $\eta \to 1$. It is shown in [164] that for $\alpha \approx 13°$ we have

$$\beta \approx \beta_0 \sqrt{(1 - \nu/\nu_{\text{cr}})}. \tag{28.51}$$

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† If the gradients of N could be so large that the density N in the region $v \sim 1$ varied appreciably over distances of at most the order of $\lambda_0/2\pi$, the appearance of the extraordinary wave in the region $v > 1$ would be "trivial", as it would then follow from Fresnel's formulae for the coefficients of reflection and transmission of waves at a sharp boundary between two anisotropic (here magnetoactive) media. However, it is not possible for such steep gradients of N (i.e. $dN/dz \sim 10^2$, since $\lambda_0/2\pi \sim 10^3$ and $N \sim 10^5$ to 10^6), which would be needed to make Fresnel's formulae at all applicable, to exist for any considerable time in the ionosphere. For in the F layer the electron free path is $l = v/\nu_{\text{eff}} \sim 10^7/10^3 = 10^4 \gg \lambda_0/2\pi \sim 10^3$, and therefore a steep gradient of the above value, even if formed (which there is no reason to suppose), would be smoothed out in a time less than the free time $\tau = 1/\nu_{\text{eff}} \sim 10^{-3}$ sec. For electrons in the E layer $\nu_{\text{eff}} \sim 10^5$, $l \sim 10^2$, $\tau = 1/\nu_{\text{eff}} \sim 10^{-5}$ and the time for the boundary to diffuse over a distance $\lambda_0/2\pi \sim 10\,l$ is again of the order of $\Delta t = 100\,l^2/6D = 50\,l/v = 50\tau \sim 10^{-3}$ sec; here $D = \frac{1}{3} l v$ is the electron diffusion coefficient. If the boundary is removed by diffusion of ions, the corresponding estimate gives $\Delta t \sim 0{\cdot}1$ sec.

This comment on the significance of diffusion amounts to the result that even a relatively slight gradient $dN/dz \sim 1$ in a layer of thickness ~ 1 km cannot be observed over a long period of time. For the usual still smaller values of $(dN/dz)_{v=1}$, as already stated, the "tripling" effect in the F layer cannot be observed at medium latitudes. (This refers to normal ionosphere probing without allowance for scattering by inhomogeneities (§ 29).)

The expressions (28.35), (28.37), (28.49) and (28.51) give for the parabolic layer (28.18)

$$\left.\begin{aligned} |D|^2 &= e^{-2\delta}, \\ 2\delta &\approx \frac{\nu_{\rm cr} z_m \beta_0 (1 - \nu/\nu_{\rm cr})^{3/2} f^2/f^2_{\rm cr,0}}{\sqrt{(1 - f^2/f^2_{\rm cr,0})}} = \left(1 - \frac{\nu}{\nu_{\rm cr}}\right)^{3/2} \cdot 2\delta_0, \end{aligned}\right\} \quad (28.52)$$

where it is assumed that $\nu < \nu_{\rm cr}$. If $\nu > \nu_{\rm cr}$, then $|D|^2 = 1$ (in the present approximation $|D|^2 \to 1$ as $\nu \to \nu_{\rm cr}$ and $|D|^2 = 1$ for $\nu > \nu_{\rm cr}$). Here it must not be forgotten that we are ignoring the damping of the wave due to absorption in the region $v < 1$. Fig. 28.8 shows $|D|^2$ as a function of $\nu/\nu_{\rm cr}$ given by formula (28.52) for the specific conditions of observation used in [164]. When

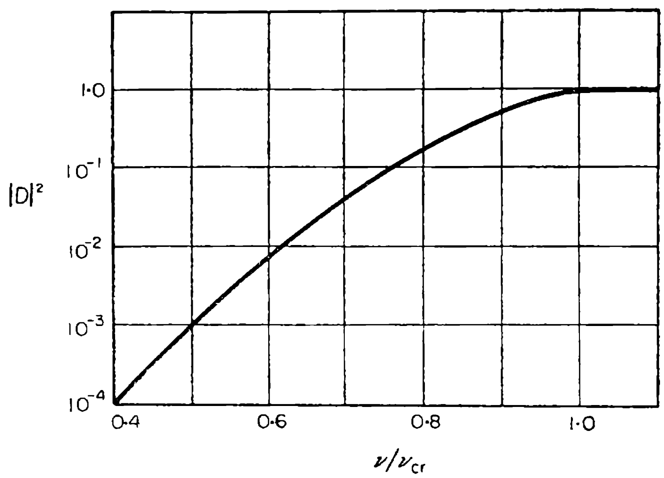

FIG. 28.8. Transmission coefficient $|D|^2$ as a function of $\nu/\nu_{\rm cr}$ for $u = \frac{1}{4}$ and $\omega = 1{\cdot}76 \times 10^7 (\lambda_0 = 107$ m). The curve is calculated for a parabolic layer ($z_m = 30$ km) and $\nu_{\rm cr} = \omega_H \sin^2\alpha/2 \cos\alpha = 2{\cdot}3 \times 10^5$; the critical frequency for wave 2 is $f_{\rm cr,2} \equiv f_{\rm cr,0} = 4$ Mc/s, $f = 3{\cdot}47$ Mc/s, $f_H = \omega_H/2\pi = 1{\cdot}44$ Mc/s, and $\alpha = 13°$.

the calculation is carried out by the method of phase integration, we again obtain (28.52), with [70]

$$\left.\begin{aligned} \delta &= \left| \frac{\omega}{ca} \int\limits_{A'}^{B'} \frac{(n_1 - i\varkappa_1) - (n_2 - i\varkappa_2)}{2} \, d\zeta \right|, \\ a &= \left(\frac{1}{N}\frac{dN}{dz}\right)_{v=1}, \end{aligned}\right\} \quad (28.53)$$

the integral being taken along a loop round the point $+is_{\rm cr}$ (Fig. 28.9). When $s = \nu/\omega = 0$, this formula becomes (28.43). By changing the contour to the broken line joining the point $+is_{\rm cr}$ to the axis of re z, we obtain formulae (28.35) and (28.36) and, in the particular case, (28.52). It is clear, however, from (28.53) and Fig. 28.9 that, for $s > s_{\rm cr}$ (i.e. $\nu > \nu_{\rm cr}$), $\delta = 0$ in this approximation, since the contour of integration does not enclose the point is_{cr}. When $s > s_{cr}$ we therefore have quasilongitudinal propagation with reflection from the point $v_{10}^{(+)} = 1 + \sqrt{u}$.

We have essentially assumed in the foregoing that ν = constant. If ν depends appreciably on z but less so than N, the formulae all remain valid, but ν must be taken to be the effective collision frequency at the point $v = 1$.

It may also be noted that in the interaction region the seattering of radio waves by inhomogeneities in the ionosphere must be especially great, since when $v \to 1$ and $\alpha \to 0$ the values of $dn_{1,2}/dv$, and therefore of $dn_{1,2}/dN$, must be particularly large. In consequence, even relatively small changes in N bring about considerable changes in the indices $n_{1,2}$.

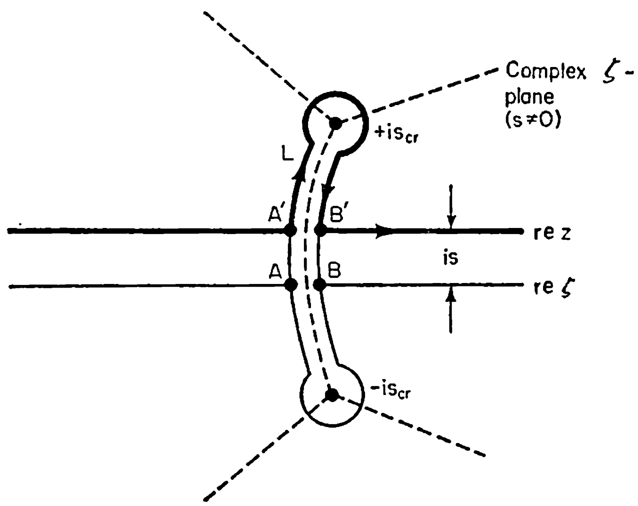

FIG. 28.9. Contour of integration in (28.53).

The results for $u = \omega_H^2/\omega^2 > 1$

We have discussed the "tripling" effect for normal incidence and mainly for the case $u = \omega_H^2/\omega^2 < 1$ only. The case of oblique incidence will be considered in § 29; here we shall briefly treat the case where $u > 1$. In the Earth's ionosphere at medium and high latitudes this corresponds to waves with $\lambda > 214$ m. For very small angles α, where the perturbation method is applicable, the result for the ordinary wave with $u > 1$ has already been obtained above [see (28.14), (28.16)]. The calculation by this method is valid for the ordinary wave with any value of u because the zero-order approximation (28.9) is the same for $u < 1$ and $u > 1$, and the first approximation is derived without any assumption as to the magnitude of u. A similar calculation can be made for the extraordinary wave with $u > 1$, assuming that only the extraordinary wave exists in the zero-order approximation, i.e. $F_+^{(0)} = e^{-i\omega n_1 z/c}$, $n_1^2 = 1 +$

$+ v/(\sqrt{u} - 1)$. Then the reflection coefficient $|R_1|^2$ and the transmission coefficient $|D_1|^2$ are†

$$\left.\begin{aligned} |R_1|^2 &= 4\delta_{0,2}^2, \quad |D_1|^2 = 1 - 2\delta_{0,2}, \\ 2\delta_{0,2} &= \frac{\pi\omega u^{\frac{3}{4}}\alpha^2}{2ca(\sqrt{u}-1)^{3/2}} = \frac{\pi\omega\alpha^2}{2ca(1-\omega/\omega_H)^{3/2}}. \end{aligned}\right\} \tag{28.54}$$

For comparison, we may repeat the result of the corresponding calculation for the ordinary wave [see (28.14)–(28.16)]:

$$\left.\begin{aligned} a &= (dv/dz)_{v=1}, \\ |R_2|^2 &= 4\delta_{0,1}^2, \quad |D_2|^2 = 1 - 2\delta_{0,1}, \\ 2\delta_{0,1} &= \frac{\pi\omega u^{\frac{3}{4}}\alpha^2}{2ca(1+\sqrt{u})^{3/2}} = \frac{\pi\omega\alpha^2}{2ca(1+\omega/\omega_H)^{3/2}}. \end{aligned}\right\} \tag{28.55}$$

(Previously $\delta_{0,1}$ was denoted by δ_0 simply; cf. (28.38) with $\alpha \ll 1$.)

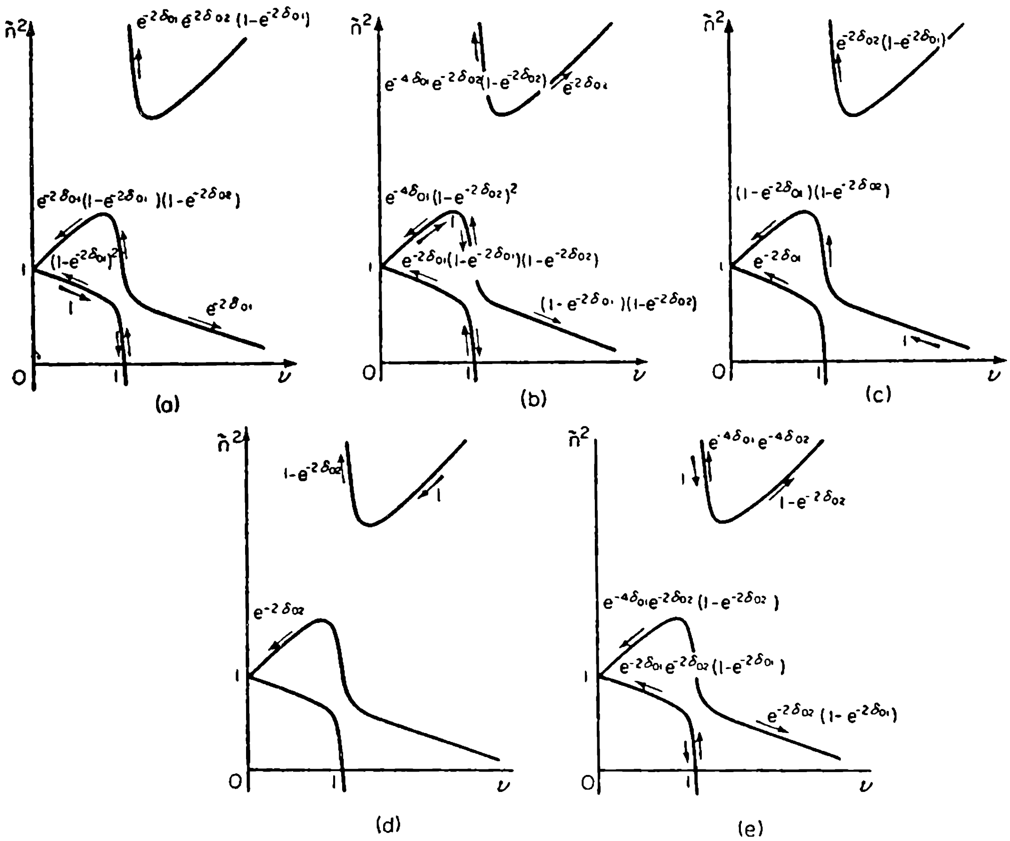

FIG. 28.10. Various cases of interaction of waves with $u_L = u\cos^2\alpha > 1$. The diagrams show the squared moduli of the wave amplitudes, with the amplitude of the incident wave (shown by the thick arrow) taken as unity. Reflection from the point $v_{10}^{(+)}$ is neglected.

† The suffixes 1 and 2 in formulae (28.54) and (28.55) relate to the unperturbed wave with $\alpha = 0$, and are therefore different from those used in (28.43), etc.

For $u_L = u\cos^2\alpha > 1$ there are five principal cases of interaction of waves, while for $u < 1$ there are three.† The values of $|D|^2$ and $|R|^2$ for $u_L > 1$, obtained by the method of phase integrals in [174], are shown in Fig. 28.10. Instead of (28.54) and (28.55), expressions involving $\exp(-2\delta_{0,1,2})$, $1 - \exp(-2\delta_{0,1,2})$, etc., are obtained. In this respect the situation is similar to that for $u < 1$. For convenience of comparison, the results previously discussed for frequencies $\omega > \omega_H$ (i.e. for $u < 1$) are shown in Fig. 28.11.

Strictly speaking, the expressions for $\delta_{0,1,2}$ for arbitrary angles α are different from those given in (28.54) and (28.55), which pertain to small α, but in practice the interaction is important only for small α, and the formulae shown in

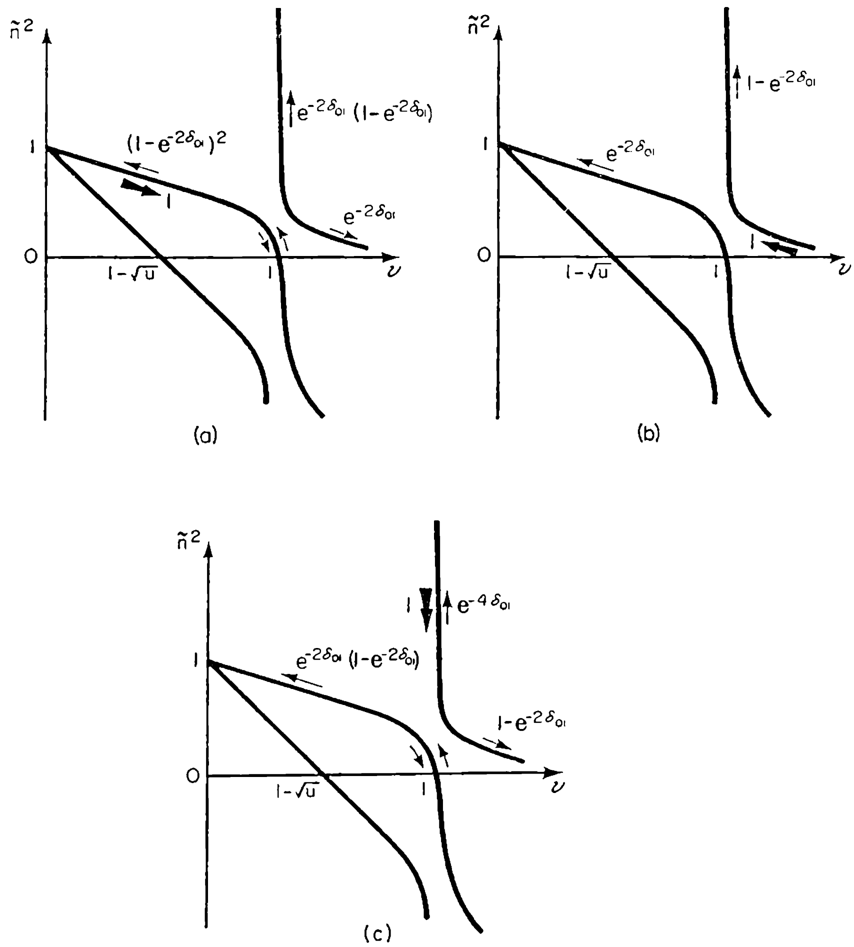

FIG. 28.11. Various cases of interaction of waves with $u < 1$. The notation and conditions are the same as in Fig. 28.10.

Figs. 28.10 and 28.11 may be used, with the values of $\delta_{0,1,2}$ given by (28.54) and (28.55). The allowance for thermal motion does not greatly affect $\delta_{0,1,2}$,

† The range where $u > 1$, $u_L = u\cos^2\alpha < 1$ requires special consideration, but has not yet received it. In practice the interaction occurs only for small α, so that usually $u_L > 1$ if $u > 1$.

provided that $\beta_T^2 = \varkappa T/mc^2 \ll 1$. The absorption of waves does not have an important effect on the results if $\nu_{\text{eff}} \ll \nu_{\text{cr}} = \omega_H \sin^2\alpha/2 \cos\alpha$ in the interaction region.

A very important property of this interaction of waves is that it does not of itself lead to the formation of reflected waves (see Fig. 28.10c,d and Fig. 28.11b). The occurrence of reflection is therefore always due to the fact that some wave reaches a zero of n_2^2.

In all the cases shown in Figs. 28.10 and 28.11, reflection from the zero $v_{10}^{(+)} = 1 + \sqrt{u}$ is ignored. It is clear from Fig. 28.10a,b that when an ordinary signal is sent upwards an extraordinary signal also returns, and *vice versa*. Thus when $u_L > 1$ and α is small, so that the interaction is important, four reflected signals should appear when the ionosphere is probed by an unpolarised signal, in the absence of reflection from the point $v_{10}^{(+)} = 1 + \sqrt{u}$. If there is reflection from this point, however, the incident ordinary signal gives two ordinary and two extraordinary reflected signals. The same is true for an incident extraordinary signal, since the extraordinary signal returning after reflection from the point $v_{10}^{(+)} = 1 + \sqrt{u}$ is converted into one extraordinary and one ordinary signal on passing through the interaction region; see Fig. 28.10c. As a result, an unpolarised signal incident on the ionosphere may give as many as eight reflected signals. In the range $u > 1$, however, the "multiplication" of reflected signals is of less interest than the possibility that the extraordinary wave may pass through the whole layer at normal incidence, even if $\alpha \neq 0$ and the frequency is below the critical frequency corresponding to the point $v_{10}^{(+)}$. This effect occurs to some extent for oblique incidence also, and is even intensified in certain conditions (§ 29).

§ 29. WAVES OBLIQUELY INCIDENT ON A LAYER. THE RECIPROCITY THEOREM

Introduction

In solving the problem of oblique incidence of waves on a layer of magnetoactive plasma, we start from the general wave equation (2.5) for an arbitrary medium:

$$\text{curl curl}\,\boldsymbol{E} - \frac{\omega^2}{c^2}\left(\boldsymbol{D} - i\frac{4\pi}{\omega}\boldsymbol{j}\right) = 0; \tag{29.1}$$

for a plane layer of magnetoactive plasma, this equation becomes

$$\triangle E_i - \frac{\partial}{\partial x_i}\operatorname{div}\boldsymbol{E} + \frac{\omega^2}{c^2}\varepsilon'_{ik}(\omega, z)\,E_k = 0 \tag{29.2}$$

[see (23.1)]. The tensor ε'_{ik} is defined by the formulae of § 10.

Equation (29.2) has an important particular solution of the type

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{F}(z)\,e^{-ik_y y} = \boldsymbol{F}(z)\,e^{-i\omega p y/c}, \tag{29.3}$$

corresponding to waves with normals in the yz-plane. (The magnetic field $\boldsymbol{H}^{(0)}$ in a plane-parallel medium is uniform or depends only on z, but it may have any direction. Thus by choosing a solution of the form (29.3) we do not assume that the field $\boldsymbol{H}^{(0)}$ lies in the yz-plane, and the coordinate system used is consequently different from that shown in Fig. 10.1.)

Substituting (29.3) in (29.2), we obtain

$$\left.\begin{aligned} d^2F_x/dz^2 + (\omega^2/c^2)\,[(\varepsilon'_{xx} - p^2)\,F_x + \varepsilon'_{xy}F_y + \varepsilon'_{xz}F_z] = 0\,, \\ d^2F_y/dz^2 + i\,(\omega/c)\,p\,dF_z/dz + (\omega^2/c^2)\,[\varepsilon'_{yx}F_x + \varepsilon'_{yy}F_y + \varepsilon'_{yz}F_z] = 0\,, \\ i\,(\omega/c)\,p\,dF_y/dz + (\omega^2/c^2)[\varepsilon'_{zx}F_x + \varepsilon'_{zy}F_y + (\varepsilon'_{zz} - p^2)\,F_z] = 0\,, \end{aligned}\right\} \quad (29.4)$$

where we have assumed that $p = ck_y/\omega$ = constant. The correctness of this important point may be seen by at first supposing that $p = p(z)$. Then we obtain equations for $F_{x,y,z}$ which are satisfied for all y only if p is constant (cf. the analogous calculations in § 19 for an isotropic medium). For normal incidence of waves on the layer, $p = 0$ and equations (29.4) simplify in an obvious way, becoming two second-order equations already discussed [see (23.2) etc.]. Equations (29.4) also become two second-order equations at the magnetic pole (where the external field $\boldsymbol{H}^{(0)}$ is in the z-direction) and when the waves are propagated at the equator in the plane of the magnetic meridian [where the field $\boldsymbol{H}^{(0)}$ is in the y-direction and we have waves of the type (29.3)] [166].† However, even in these particular cases, still less in the general case, equations (29.4) have not only not been solved rigorously for any one model of the layer, but have not even been used for an approximate allowance for the interaction of normal waves obliquely incident. Consequently, the analysis of the solution for oblique incidence on a layer of magnetoactive plasma has been carried out only in the approximation of geometrical optics. Nevertheless, the nature of the interaction of waves which leads to their penetration through the layer and also to the peculiar "tripling" of signals at oblique incidence (an effect which differs considerably from that occurring at normal incidence) is qualitatively clear, and it is these aspects which we shall discuss. In conclusion we shall also prove a generalised reciprocity theorem valid in a magnetoactive medium and of possible usefulness in solving various specific problems.

The approximation of geometrical optics

The solution in the approximation of geometrical optics is obtained in a general form for an arbitrary medium. To do this, we must seek a solution of the basic equation (29.1) in the form

$$\boldsymbol{E}(\boldsymbol{r}) = \left[\boldsymbol{E}^{(0)}(\boldsymbol{r}) + \frac{c}{\omega}\,\boldsymbol{E}^{(1)}(\boldsymbol{r}) + \frac{c^2}{\omega^2}\,\boldsymbol{E}^{(2)}(\boldsymbol{r}) + \cdots\right] e^{-i\omega\Psi(\boldsymbol{r})/c} \quad (29.5)$$

† Another special case, for reasons of symmetry, is that of wave propagation at the equator in a plane perpendicular to that of the magnetic meridian.

and similar series for $\boldsymbol{D}$ and $\boldsymbol{j}$. Substituting these series in (29.1) and equating to zero the terms in each power of ω/c, we have

$$\left.\begin{aligned}
&\mathbf{grad}\,\Psi \times (\mathbf{grad}\,\Psi \times \boldsymbol{E}^{(0)}) + \boldsymbol{D}^{(0)} - i\cdot(4\pi/\omega)\,\boldsymbol{j}^{(0)} = 0,\\
&\mathbf{grad}\,\Psi \times (\mathbf{grad}\,\Psi \times \boldsymbol{E}^{(1)}) + \boldsymbol{D}^{(1)} - i\cdot(4\pi/\omega)\,\boldsymbol{j}^{(1)}\\
&\qquad = -i\,\mathbf{curl}(\mathbf{grad}\,\Psi \times \boldsymbol{E}^{(0)}) - i\,\mathbf{grad}\,\Psi \times \mathbf{curl}\,\boldsymbol{E}^{(0)},\\
&\mathbf{grad}\,\Psi \times (\mathbf{grad}\,\Psi \times \boldsymbol{E}^{(2)}) + \boldsymbol{D}^{(2)} - i\cdot(4\pi/\omega)\,\boldsymbol{j}^{(2)}\\
&\qquad = -i\,\mathbf{curl}(\mathbf{grad}\,\Psi \times \boldsymbol{E}^{(1)}) - i\,\mathbf{grad}\,\Psi \times \mathbf{curl}\,\boldsymbol{E}^{(1)} +\\
&\qquad\quad + \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^{(0)},\ \text{etc.}
\end{aligned}\right\} \tag{29.6}$$

Putting $D_i - i\cdot(4\pi/\omega)j_i = \varepsilon'_{ik}(\boldsymbol{r})E_k$ and comparing the first equation (29.6) with equations (11.2)–(11.5), we see that

$$(\mathbf{grad}\,\Psi)^2_{1,2} = [n(\boldsymbol{r}) - i\varkappa(\boldsymbol{r})]^2_{1,2}; \tag{29.7}$$

here $(n - i\varkappa)^2_{1,2}$ is given for a magnetoactive plasma by the expressions (11.5), which can now depend on the coordinates (that is, v, u, s and α in (11.5) may be functions of $\boldsymbol{r}$). It is important to note that the angle α in (11.5) is that between the external field $\boldsymbol{H}^{(0)}$ and $\mathbf{grad}\,\Psi$, i.e. the direction of the normal to the wave. The right-hand side of equation (29.7) therefore depends on $(\mathbf{grad}\,\Psi)_{1,2}$, although this is not explicitly shown in (29.7).

The expression (29.7) is derived from the conditions for the existence of a non-trivial solution to the first equation (29.6). The condition for the second of these equations to be resolvable then gives an equation for the field $\boldsymbol{E}^{(0)}$, and so on.

The solution (29.3) in the approximation of geometrical optics is obtained by substituting in equations (29.4) the series

$$\boldsymbol{F}(z) = \left[\boldsymbol{F}^{(0)}(z) + \frac{c}{\omega}\boldsymbol{F}^{(1)}(z) + \cdots\right] e^{-i\omega\psi(z)/c}. \tag{29.8}$$

The result may be written in the following form, where x, y, z correspond to 1, 2, 3 respectively and summation over repeated suffixes is understood:

$$\left.\begin{aligned}
&a_{ik}F_k^{(0)} = 0,\quad a_{ik}F_k^{(1)} = i\,g_i,\quad \text{etc.},\\
&\quad a_{11} = \varepsilon'_{xx} - p^2 - (d\psi/dz)^2,\quad a_{12} = -a_{21} = \varepsilon'_{xy} = -\varepsilon'_{yx},\\
&\quad a_{13} = -a_{31} = \varepsilon'_{xz} = -\varepsilon'_{zx},\\
&\quad a_{22} = \varepsilon'_{yy} - (d\psi/dz)^2,\quad a_{33} = \varepsilon'_{zz} - p^2,\\
&\quad a_{23} = \varepsilon'_{yz} + p\,d\psi/dz,\quad a_{32} = \varepsilon'_{zy} + p\,d\psi/dz\\
&\qquad\qquad\qquad\qquad\qquad\quad = -\varepsilon'_{yz} + p\,d\psi/dz,\\
&g_1 = 2\frac{d\psi}{dz}\frac{dF_x^{(0)}}{dz} + F_x^{(0)}\frac{d^2\psi}{dz^2},\\
&g_2 = 2\frac{d\psi}{dz}\frac{dF_y^{(0)}}{dz} + F_y^{(0)}\frac{d^2\psi}{dz^2} - p\frac{dF_z^{(0)}}{dz},\\
&g_3 = -p\,dF_y^{(0)}/dz.
\end{aligned}\right\} \tag{29.9}$$

The condition for the equations for $F_k^{(0)}$ to have a solution, i.e. the vanishing of the determinant $\Delta \equiv |a_{ik}|$, must be (and, of course, is) identical with (29.7), which for the solution (29.3) and (29.8) takes the form

$$(d\psi/dz)^2 + p^2 = (n - i\varkappa)^2_{1,2}. \tag{29.10}$$

The vector $\mathbf{grad}\,\Psi = (0, p, d\psi/dz)$ is in the direction of the normal to the wave front; we neglect the derivatives of $F_i^{(0)}$, which are small in the approximation of geometrical optics. The quantities $\Psi(\boldsymbol{r})$ and $\psi(z)$ appear in (29.5) and (29.8) respectively.

It is therefore reasonable to put

$$\left.\begin{aligned}(\partial\Psi/\partial z)^2 &= (d\psi/dz)^2 = q^2 = (n - i\varkappa)^2_{1,2}\cos^2\theta(z)\,,\\ (\partial\Psi/\partial y)^2 &= p^2 \equiv (c\,k_y/\omega)^2 = (n - i\varkappa)^2_{1,2}\sin^2\theta(z)\,,\end{aligned}\right\} \tag{29.11}$$

where θ is the angle between $\mathbf{grad}\,\Psi$ and the z-axis, and n and $\varkappa$ depend on p and $q = d\psi/dz$.

Since p is constant, this gives the law of refraction:

$$[n_{1,2}(z, \theta) - i\varkappa_{1,2}(z, \theta)]\sin\theta(z) = \sin\theta_0; \tag{29.12}$$

by hypothesis we have at the boundary of the layer the angle of incidence $\theta = \theta_0$, $n_{1,2} = 1$ and $\varkappa_{1,2} = 0$.

In (29.12) we have shown also the argument θ of $n_{1,2}$ and $\varkappa_{1,2}$, since $(n - i\varkappa)_{1,2}$ depends on the angle α between the field $\boldsymbol{H}^{(0)}$ and the normal to the wave, which is in the direction of $\mathbf{grad}\,\Psi$. The angle α varies with θ, and so the indices $n_{1,2}$ and $\varkappa_{1,2}$ in (29.11) and (29.12) also vary with θ. The paths described by the wave normal and also by the rays (i.e. by the wave packets) in an inhomogeneous magnetoactive medium are therefore in general very complex. Much work has been done on the determination of these paths under various conditions [58, 147, 156, 158, 177–183, 188–190].

The field in the first approximation of geometrical optics

Before discussing the nature of the paths of the wave normals and rays, let us determine the form of the functions $d\psi/dz = q(z)$ and $\boldsymbol{F}^{(0)}(z)$ in (29.8), i.e. the field in the first approximation of geometrical optics.

To find $d\psi/dz = q$ it is evidently necessary to make explicit the right-hand side of equation (29.10) as a function of q. This means that in the expression (11.5) for $(n - i\varkappa)^2_{1,2}$, where α is the angle between $\boldsymbol{H}^{(0)}$ and $\mathbf{grad}\,\Psi$, we must put

$$\cos^2\alpha = (H_y^{(0)}p + H_z^{(0)}q)^2/[H^{(0)}]^2(p^2 + q^2). \tag{29.13}$$

Of course, the same result can be obtained directly from the condition for the first equation (29.9) to have a non-trivial solution.

Thus we obtain from (29.10) the following quartic equation for q:

$$\alpha_p q^4 + \beta_p q^3 + \gamma_p q^2 + \delta_p q + \varepsilon_p = 0, \tag{29.14}$$

where the coefficients depend on $v = 4\pi e^2 N/m\omega^2$, $u = \omega_H^2/\omega^2$, $s = \nu_{\text{eff}}/\omega$, p and the direction of the field $\boldsymbol{H}^{(0)}$ in space.

In the coordinate system used in the present section, with the wave vector $\boldsymbol{k} = (\omega/c)(0, p, q)$ in the yz-plane and the vector $\boldsymbol{H}^{(0)} = (H_x^{(0)}, H_y^{(0)}, H_z^{(0)})$ directed arbitrarily, we have [189]

$$\left.\begin{aligned}
\alpha_p &= (1-is)[(1-is)^2-u]-v[(1-is)^2-u_z],\\
\beta_p &= 2pv\sqrt{(u_y u_z)},\\
\gamma_p &= -2(1-is)\{[(1-p^2)(1-is)-v](1-is-v)-\\
&\quad-(1-p^2)u\}+v[p^2u_y-(1-p^2)u_z-u],\\
\delta_p &= -2(1-p^2)pv\sqrt{(u_y u_z)},\\
\varepsilon_p &= [(1-p^2)(1-is)-v]\{[(1-p^2)(1-is)-v](1-is-v)-\\
&\quad-(1-p^2)u\}-(1-p^2)p^2u_y v,
\end{aligned}\right\} \tag{29.15}$$

where

$$\sqrt{u_y} = eH_y^{(0)}/mc\omega, \quad \sqrt{u_z} = eH_z^{(0)}/mc\omega,$$
$$u = \omega_H^2/\omega^2 = e^2[H^{(0)}]^2/m^2c^2\omega^2 = u_x + u_y + u_z.$$

It may be noted that, in a coordinate system with the vector $\boldsymbol{H}^{(0)}$ lying in the yz-plane and the wave vector $\boldsymbol{k} = (\omega/c)(p_1, p_2, q)$, with $p_1 = \sin\theta_0\cos\varphi_0$, $p_2 = \sin\theta_0\sin\varphi_0$, $q = \cos\theta_0$, the coefficients in equation (29.14) are [183]

$$\left.\begin{aligned}
\alpha_p &= (1-is)[(1-is)^2-u]-v[(1-is)^2-u_L],\\
\beta_p &= 2p_2v\sqrt{(u_L u_T)},\\
\gamma_p &= -2(1-is)\{[(1-p^2)(1-is)-v](1-is-v)-\\
&\quad-(1-p^2)u\}+v[p_2^2u_T-(1-p^2)u_L-u],\\
\delta_p &= -2(1-p^2)p_2v\sqrt{(u_L u_T)},\\
\varepsilon_p &= [(1-p^2)(1-is)-v]\{[(1-p^2)(1-is)-v](1-is-v)-\\
&\quad-(1-p^2)u\}-(1-p^2)p_2^2u_T v,
\end{aligned}\right\} \tag{29.15a}$$

where $p^2 = p_1^2 + p_2^2 = \sin^2\theta_0 = 1 - q^2(z=0)$, $1 - p^2 = \cos^2\theta_0 = q^2(z=0)$, $u_L = u\cos^2\chi$, $u_T = u\sin^2\chi$, and χ is the angle between $\boldsymbol{H}^{(0)}$ and the z-axis; for normal incidence $\chi = \alpha$, the angle between $\boldsymbol{H}^{(0)}$ and the normal to the wave (i.e. the z-axis; cf. § 11, etc.). For wave propagation in the plane of the magnetic meridian we have in (29.15a) $p_1 = 0$, $p_2 = p = \sin\theta_0$. The change from (29.15a) to (29.15) is made by replacing p_2 by p, u_L by u_z and u_T by u_y.

In a homogeneous medium, q is independent of the coordinates, and the axes may always be chosen so that $p = 0$ and $q^2 = (n - i\varkappa)^2$; cf. (29.11) with $\cos\theta = \cos\theta_0 = 1$. Equation (29.14) becomes a quadratic in q^2, and its solution is (11.5).

The ratios of the components $F_x^{(0)}$, $F_y^{(0)}$ and $F_z^{(0)}$ are found from any two of the equations $a_{ki}F_k^{(0)} = 0$ [see (29.9)]:

$$\left.\begin{aligned} \frac{F_x^{(0)}}{F_y^{(0)}} &= \frac{T_{31}}{T_{32}} = \frac{a_{12}a_{23} - a_{22}a_{13}}{a_{21}a_{13} - a_{11}a_{23}} = K\,, \\ \frac{F_z^{(0)}}{F_y^{(0)}} &= \frac{T_{33}}{T_{32}} = \frac{a_{11}a_{22} + a_{12}^2}{a_{21}a_{13} - a_{11}a_{23}} = B\,. \end{aligned}\right\} \tag{29.16}$$

Here T_{ij} are the cofactors of the elements a_{ij} in the determinant $\Delta \equiv |a_{ik}|$. (By definition $\Delta \equiv |a_{ik}| = T_{i1}a_{i1} + T_{i2}a_{i2} + T_{i3}a_{i3}$, i.e. $T_{ik} = (-1)^{i+k}M_{ik}$, where M_{ik} is the minor of the element a_{ik}.) The quantities K and B in (29.16) evidently depend on p and q, and are therefore different for waves 1 and 2, where q takes values q_1 and q_2. Finally, the dependence of $F_i^{(0)}$ on z is given by the conditions for equation (29.9) to be soluble for $F_i^{(1)}$. The result is [183]

$$F_y^{(0)} = \text{constant}/\sqrt{[q(K^2 - 1) + pB]}, \tag{29.17}$$

and the field $\boldsymbol{E}$ in the first approximation of geometrical optics is

$$\left.\begin{aligned} &E_x = KE_y\,, \quad E_z = BE_y\,, \\ &E_y = \frac{\text{constant}}{\sqrt{[q(K^2-1)+pB]}}\exp\left[\pm i\,\frac{\omega}{c}\left(py + \int_{z_0}^{z} q\,dz\right)\right], \\ &q = d\psi/dz = (n - i\varkappa)\cos\theta\,, \end{aligned}\right\} \tag{29.18}$$

where we have made explicit the $\pm$ signs of the phase which were previously concealed in p and q.

Graphs of the functions $q_{1,2}(v)$

An idea of the nature of wave propagation for oblique incidence in the absence of absorption in the approximation of geometrical optics is most

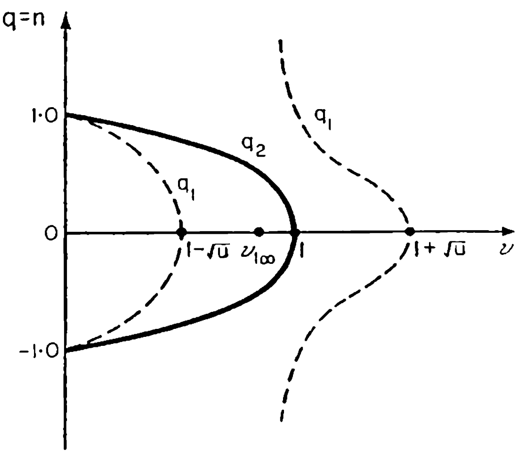

FIG. 29.1. The functions $q_{1,2}(v)$ for normal incidence ($u = \frac{1}{4}$, $\alpha = 45°$).

easily obtained by means of graphs showing $q = n \cos \theta$ as a function of $v = 4\pi e^2 N/m\omega^2$ (see, for example, [158, 189, 183]). For oblique incidence these graphs correspond to those of $n^2_{1,2}(v)$ for normal incidence. In the latter case the two types of graph are essentially equivalent, since in the absence of absorption $q_{1,2} = n_{1,2}$ for a wave travelling upwards ($\cos \theta = 1$) and $q_{1,2} = -n_{1,2}$ for a wave travelling downwards ($\cos \theta = -1$). Thus for normal incidence the graphs of $q(v)$ are simply those of $n(v)$ and their reflection in the v-axis (Fig. 29.1). Symmetrical curves are also obtained, in particular, at the pole (where the vector $\boldsymbol{H}^{(0)}$ is in the z-direction; Fig. 29.2) and at the equator for wave propagation in the plane of the magnetic meridian (where the vector $\boldsymbol{H}^{(0)}$ is in the y-direction; Fig. 29.3). If the magnetic field $\boldsymbol{H}^{(0)}$ is in an arbitrary direction, or even lies in the yz-plane (propagation in the plane of the magnetic meridian), however, the curves of $q(v)$ are in general unsymmetrical (Figs. 29.4

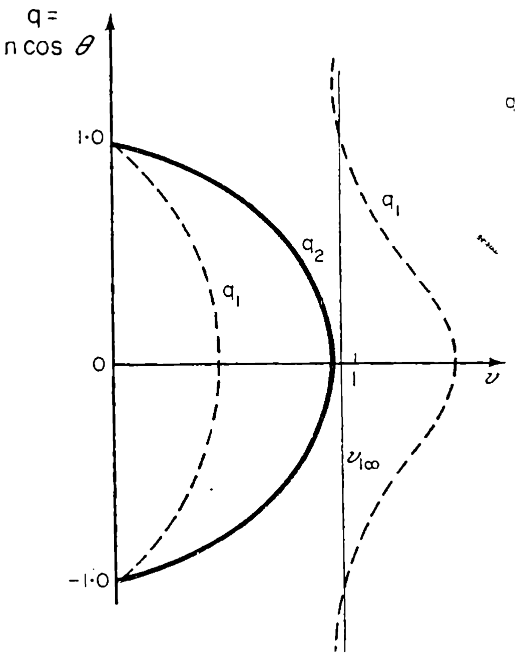

FIG. 29.2. The functions $q_{1,2}(v)$ for propagation at the pole (vertical magnetic field $\boldsymbol{H}^{(0)}$). The values used are $\theta_0 = 20°$, $u = \frac{1}{4}$.

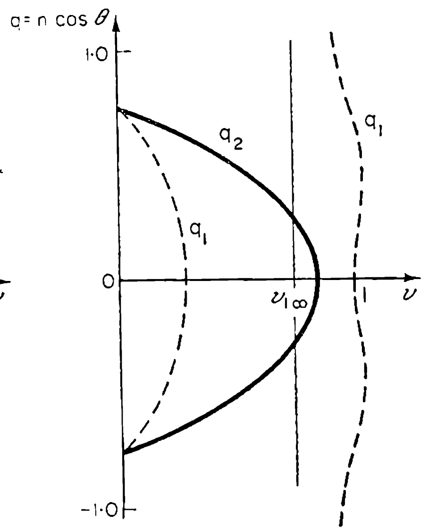

FIG. 29.3. The functions $q_{1,2}(v)$ for propagation at the equator in the plane of the magnetic meridian ($\theta_0 = 42°$, $u = \frac{1}{4}$).

and 29.5).† In the general case the curves of $q_{1,2}(v)$ go to infinity at the point $v_{1,2\infty} = (1 - u) \div (1 - u \cos^2 \chi)$, where χ is the angle between the field $\boldsymbol{H}^{(0)}$ and the z-axis. This result is quite natural, since the quantity $q = n \cos \theta$

† In all these curves and in those given below, $n_{1,2} = 1$ and $q_{1,2} = \pm \cos \theta_0$ at the boundary of the layer (θ_0 being the angle of incidence of the wave on the layer). Absorption is everywhere neglected. It may be noted that symmetrical curves of $q(v)$ occur when the coefficients β_p and δ_p in equation (29.14) are zero. Hence the curves of $q(v)$ are symmetrical not only in the particular instances mentioned in the text but wherever $u_y = 0$ or $u_z = 0$ [see (29.15)]. The case $u_z = 0$ corresponds to propagation at the magnetic equator.

becomes infinite as $n \to \infty$, when the law of refraction (29.12) gives $\sin\theta = (1/n)\sin\theta_0 \to 0$. If $\sin\theta = 0$, the normal to the wave is in the z-direction, and the angle χ is equal to the angle α between the normal and the field $\boldsymbol{H}^{(0)}$, used in previous sections. The above expression for $v_{1,2\infty}$ is for this reason identical with that obtained previously [see (11.18) and (11.22)]. The zeros of $q_{1,2}(v)$ are the roots of a certain cubic equation [189]. It is important to note that, when the angle of incidence θ_0 and the direction of the field $\boldsymbol{H}^{(0)}$ change, the zeros of $q_{1,2}(v)$ vary only within certain limits, frequently quite narrow. This is seen from Fig. 29.6, where the vertical hatching corresponds to the zeros of q_2 and the horizontal hatching to those of q_1. (For any given value

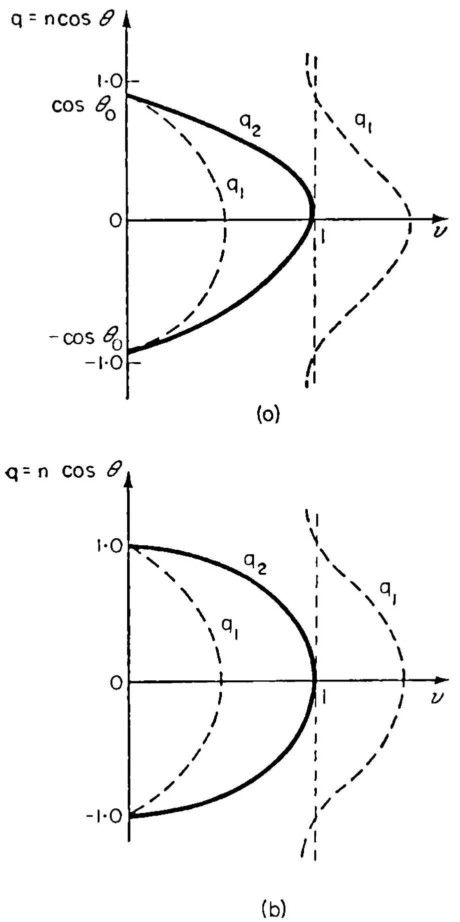

FIG. 29.4. The functions $q_{1,2}(v)$ for propagation in the plane of the magnetic meridian when the vector $\boldsymbol{H}^{(0)}$ is at an angle $\chi = 45°$ to the vertical (z-axis).
(a) $\theta_0 = 10°$, $u = \frac{1}{4}$
(b) $\theta_0 = 0°$, $u = \frac{1}{4}$

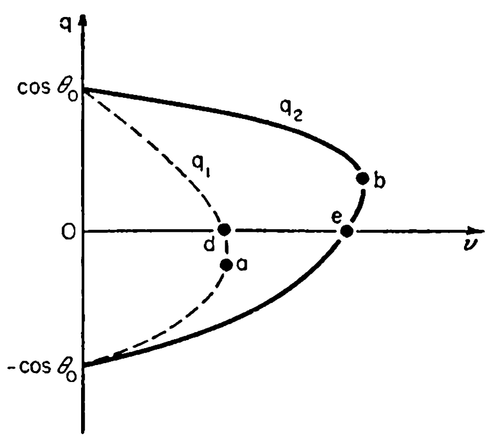

FIG. 29.5. Diagrammatic form of the functions $q_{1,2}(v)$ for $u < 1$ (only one branch of the curve of $q_1(v)$ is shown).

of $\sin\theta_0$, the zeros of $q_{1,2}$ for any direction of $H^{(0)}$ lie in the hatched regions, i.e. in a certain range of values of v.) The curves T which are boundaries of the hatched regions give the zeros of $q_{1,2}$ for propagation in a plane perpendicular to that of the magnetic meridian; in this case $q = 0$ for $v = \cos^2\theta_0$ and for $v = \frac{1}{2}[1 + \cos^2\theta_0 \pm \sqrt{(\sin^4\theta_0 + 4u \sin^2\theta_0)}]$, which gives the curves of $T(v)$. For propagation in a meridian plane at the magnetic equator, $q_{1,2} = 0$ for $v = 1$ and

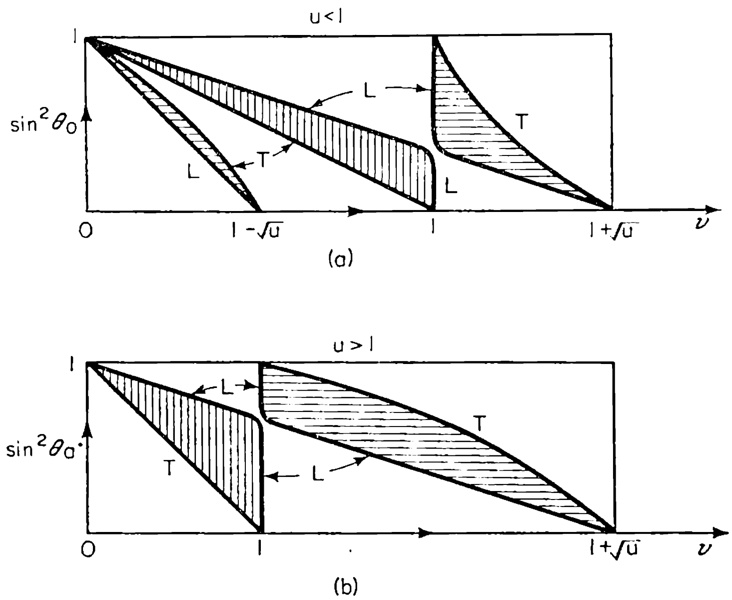

FIG. 29.6. Ranges of variation of the zeros of $q_{1,2}(v)$.
(a) $u = \frac{1}{4}$ (b) $= 4$

for $v = (1 \pm \sqrt{u}) \cos^2\theta_0$, which gives the curves L in Fig. 29.6a. In Fig. 29.6b, i.e. for $u > 1$, the boundary curves L do not always correspond to propagation in a meridian plane at the equator.

The paths of the wave normals and rays

The asymmetry of the curves of $q(v)$ corresponds to the fact that the incident wave ($q > 0$) and the reflected wave ($q < 0$) have different directions of the wave normal at the same level v. In other words, the path described

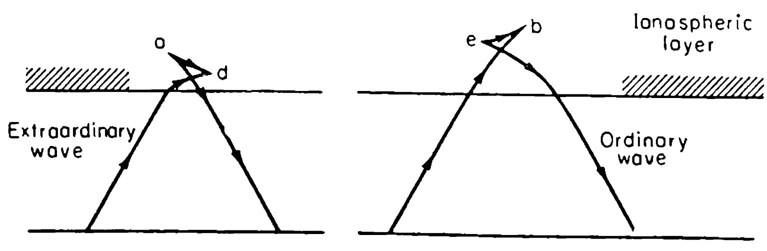

FIG. 29.7. Paths described by the wave normal in the case represented in Fig. 29.5, to which the points a, b, d, e correspond.

by the wave normal (for which $dz/dy = \cot\theta = q(\theta)/p$) is very complex. As an example, Fig. 29.7 gives the paths of the normal for the case shown in Fig. 29.5. The peculiar loop at the top of the path is due to the fact that, for the extraordinary wave, after the value $q = 0$ (where the normal is horizontal) is reached, a further decrease in q to negative values at first corresponds to an increase in v. For the ordinary wave, the maximum value of v is reached with $q > 0$, and q becomes zero only at the point e (corresponding to e in Fig. 29.5).

The curious form of the curves in Fig. 29.7 and in similar diagrams (cf. [188, 189, 147]) does not in general signify the presence of any physical peculiarities in the region of reflection. This conclusion follows, first of all, from the fact that the curves of $q(v)$ shown in Figs. 29.1–29.5 have no singularities (in the range of v corresponding to the loop in Fig. 29.7). Secondly, it is not the paths of the normals but the paths of the rays (wave packets or signals) which are of direct physical significance. The latter have no "loops" and behave smoothly except in special cases.

The ray path is determined by the equations

$$dx/v_{\mathrm{gr},x} = dy/v_{\mathrm{gr},y} = dz/v_{\mathrm{gr},z}, \tag{29.19}$$

where $v_{\mathrm{gr}} = d\omega/d\boldsymbol{k}$ is the group-velocity vector (see § 24). Whereas the normal to the wave always lies in the plane of incidence (the yz-plane in the above discussion), the group-velocity vector has, in general, a component in the x-direction. Thus the signal describes a three-dimensional curve of the kind shown in Fig. 29.8. The projections of the ray paths on the coordinate planes

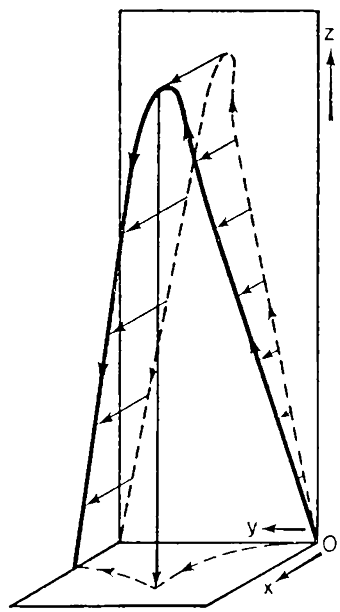

Fig. 29.8. Path of ray in plane layer of magnetoactive plasma (angle of incidence $\theta_0 = 5^0$, ordinary wave, angle between y-axis and horizontal component of magnetic field $\varphi_H = 27°$).

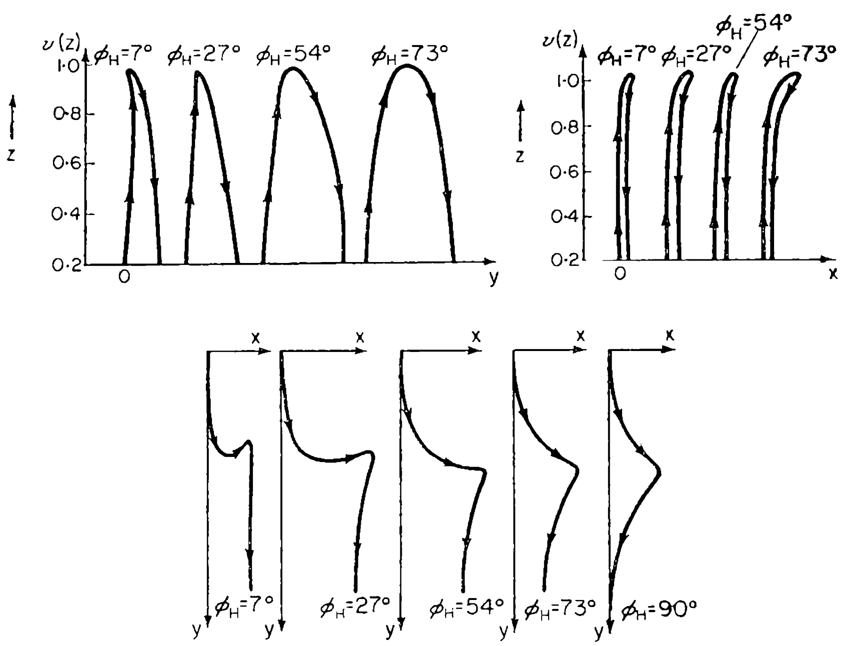

FIG. 29.9. Projections of ray paths on coordinate planes for various angles φ_H and $\theta_0 = 5^0$ (ordinary wave).

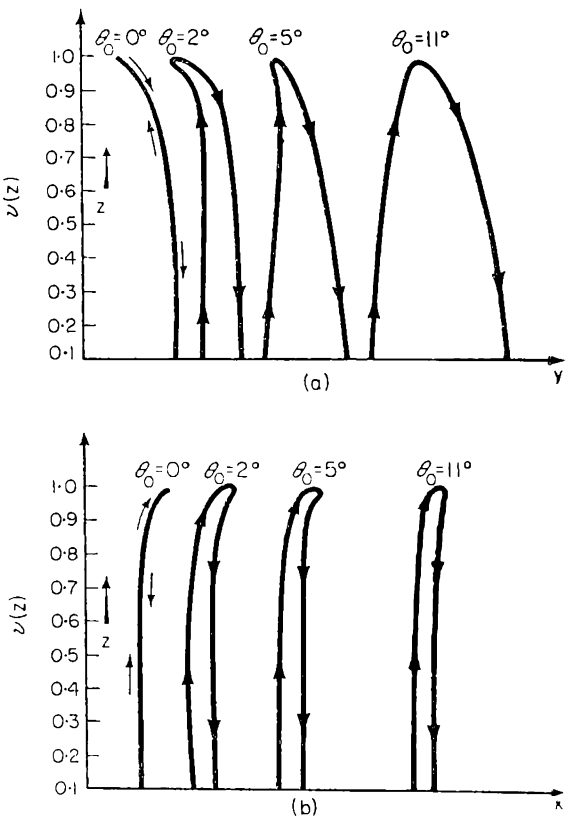

FIG. 29.10. Projections of ray paths for various angles θ_0 and $\varphi_H = 27°$ (ordinary wave).

are shown in Figs. 29.9 and 29.10; Fig. 29.11 gives the paths of the wave normal for the same angle ($\theta_0 = 5°$) as in Figs. 29.7 and 29.8. In Figs. 29.7 to 29.10, which pertain to the ordinary wave [147], the layer is assumed linear, and φ_H is the angle between the y-axis and the projection of the magnetic field on the xy-plane. A number of ray paths, values of the group velocity of radio waves in the ionosphere, and other relevant data are given in [58, 147, 156, 158, 178–182, 190]; we shall also discuss further in § 35 the ray paths for normal incidence. The calculation of the paths of whistlers is a cognate subject [53, 54]; in this case $u \gg 1$ and we can generally use formula (11.24) for $\tilde{n}_2^2$.

Let us now consider some properties of the ray paths. It is clear from symmetry that, for wave propagation in the plane of the magnetic meridian, the ray path lies in that plane. The vertex of the ray path (the point of

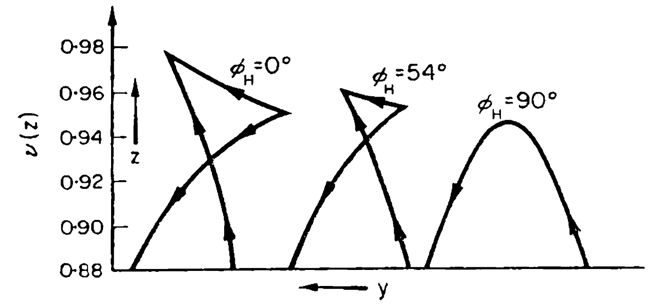

FIG. 29.11. Paths of wave normal for $\theta_0 = 5°$ and various values of φ_H (ordinary wave).

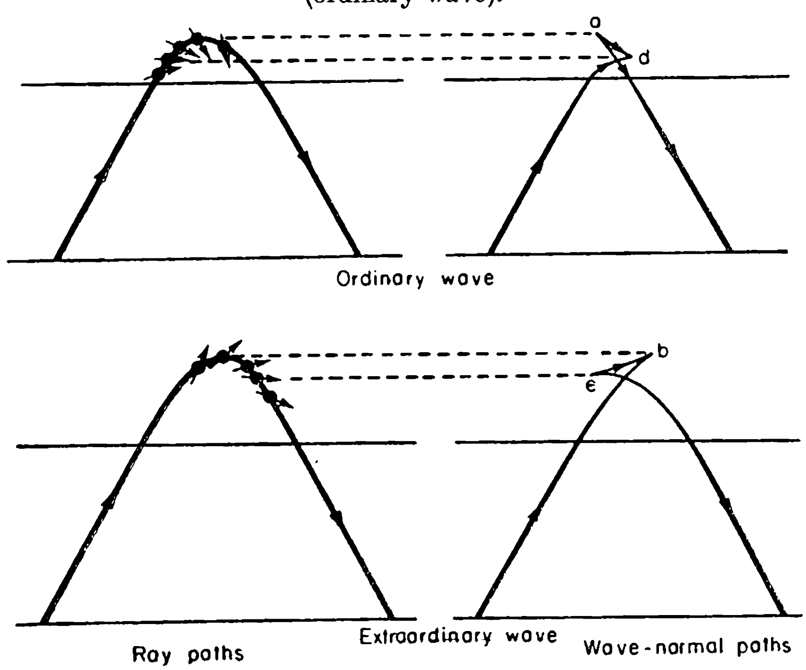

FIG. 29.12. Relation between projections of ray paths on the plane of incidence and projections of wave normals in the case corresponding to Figs. 29.5 and 29.7. The arrows on the ray paths give the directions of the normals.

reflection of the ray) in this (and in every) case corresponds, not to the point $q = 0$ (where the wave normal is horizontal), but to the point where the group-velocity vector $\boldsymbol{v}_{gr}$ is horizontal, i.e.

$$v_{gr,z} = \partial\omega/\partial k_z = \beta_1 v_{gr,x'} + \beta_2 v_{gr,y'} + \beta_3 v_{gr,z'} = 0. \tag{29.20}$$

Here $v_{gr,x'y'z'}$ are the components of $\boldsymbol{v}_{gr}$ in a coordinate system where the vector $\boldsymbol{H}^{(0)}$ is in the z'-direction [see (24.8)], and $\beta_1, \beta_2, \beta_3$ are the cosines of the angles between the axes x', y', z' and the axes x, y, z used here (the z-axis being along the normal to the inhomogeneous layer or, as we assume, vertical). From (24.8) we see that the condition $v_{gr,z} = 0$ gives $\gamma'\partial(n\gamma)/\partial\gamma = \beta_3\partial n/\partial\gamma$, or

$$n = \frac{\beta_3 - \gamma\gamma'}{\gamma'}\frac{\partial n}{\partial\gamma}, \tag{29.21}$$

where $\gamma' = \cos\theta$ is the cosine of the angle between the normal to the wave and the normal to the layer (the z-axis), γ is the cosine of the angle between the normal to the wave and the field $\boldsymbol{H}^{(0)}$, and β_3 is the cosine of the angle between the z-axis and $\boldsymbol{H}^{(0)}$ (i.e. the z'-axis). For normal incidence $\gamma' = 1, \gamma = \beta_3$

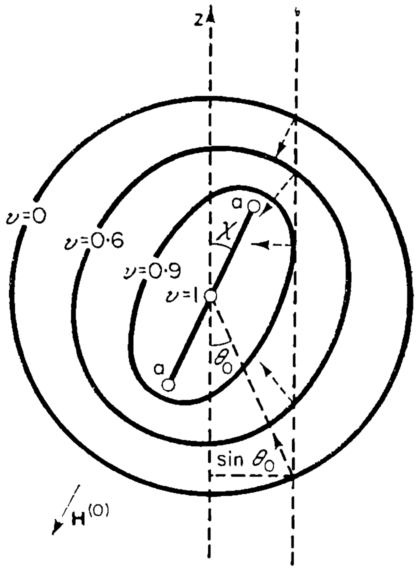

FIG. 29.13. Refractive index n_2 for ordinary wave as a function of the angle θ and the parameter v ($H^{(0)} = 0{\cdot}5$ oersted, $\alpha = 25^0$, $\lambda_0 = 80$ m).

$= \cos\alpha$, and we have the familiar reflection condition $n = 0$. In an isotropic medium, where $\boldsymbol{H}^{(0)} = 0$ and $\partial n/\partial\gamma = 0$, the expression before (29.21) gives the obvious reflection condition $\gamma' n = n\cos\theta = 0$; hence, for oblique incidence, $\cos\theta = 0$ and, using the law of refraction, we find the familiar condition $n \sin\theta = n = \sin\theta_0$. Figs. 29.4 and 29.5 suggest that the ray is reflected at the points where the curves turn downwards and the maximum value of v is reached. These evidently correspond to the condition

$$dq/dv = \infty. \tag{29.22}$$

This assumption is confirmed by calculation [189]; in other words, the conditions (29.21) and (29.22) are equivalent. Of course, in those particular cases where the curves of $q_{1,2}(v)$ are symmetrical, the vertex of the ray path is at $q_{1,2} = 0$. The interrelation between the normal paths and the projections of the ray paths on the plane of incidence is seen from Fig. 29.12.

Some special cases

The graphs given above of the functions $q_{1,2}(v)$ and of the paths of the rays and wave normals correspond to typical conditions. There are also certain cases where the situation is more complex. These complications are due, where the layers have smooth monotonic variations, to the peculiar behaviour of the refractive indices $n_{1,2}$ in a magnetoactive plasma for $v \approx 1$ and small angles α between the normal to the wave and the field $\boldsymbol{H}^{(0)}$.

For oblique incidence the situation may be elucidated by means of graphs of $q_{1,2}(v)$ and also by constructing the wave-vector surface. To do this, we draw from some centre a radius vector of length $n_{1,2}$ at an angle θ to the vertical (the z-axis), for a fixed direction of $\boldsymbol{H}^{(0)}$ and fixed values of the parameters v and u. The cross-section of the resulting surface $n_{1,2}(\theta, \boldsymbol{H}^{(0)}, v, u)$ in the plane of incidence is some plane curve. Fig. 29.13 shows such curves for an ordinary wave propagated in the plane of the magnetic meridian, for various values of v (see [180]; the z-axis is upwards, $n_2 = 1$ for $v = 1$, $\lambda_0 = 2\pi c/\omega = 80$ m, $H^{(0)} = 0{\cdot}5$ oersted, and $\chi = 25°$, i.e. the magnetic dip is $I = \frac{1}{2}\pi - \chi = 65°$).†

The whole of the surface of n_2 for a given v is obtained from the corresponding curve in Fig. 29.13 by rotating it about the field $\boldsymbol{H}^{(0)}$; this is legitimate, of course, only when the plane of the diagram, i.e. the plane of incidence, coincides with the magnetic meridian plane. A noteworthy feature of Fig. 29.13 is that, as $v \to 1$, the curves of $n_2(\theta)$ tend to a straight line. At the ends of this line (the points a) $n_2 = \sqrt{[1 - 1/(1 + \sqrt{u})]} = \sqrt{[\sqrt{u}/(1 + \sqrt{u})]}$, corresponding to the value of n_2 for $v = 1$ and $\alpha = 0$ [see (11.9)].

It is easily seen from the graphs of $\tilde{n}^2_{1,2}$ in § 11 for $u < 1$ that, for values of v close to $v = 1$, the curves $n_2 =$ constant in fact cluster round the line aa in Fig. 29.13. The values of $n(v, \theta)$ which satisfy the law of refraction (29.12) correspond to the points of intersection of the curves $n(v, \theta)$ in Fig. 29.13 with the vertical broken line at a distance $\sin\theta_0$ from the centre of the figure. At the points of intersection of the curve of n_2 with the vertical line we have $n_2 \sin\theta = \sin\theta_0$. Thus at these points of intersection the direction of the wave normal coincides with that of the radius vector, and that of the ray coincides

† In the northern hemisphere the Earth's magnetic field is directed downwards, as shown by the arrow in Fig. 29.13. For normal incidence this means that $\cos\alpha < 0$ in the formulae of the previous sections, and in some cases we must take $|\cos\alpha|$.

with the normal to the curve $n_2(v, \theta)$. In Fig. 29.13 the direction of the rays is shown by the broken arrows. If the plane of incidence does not coincide with that of the magnetic meridian, the broken arrows correspond to the projections of the ray directions.

Fig. 29.13 exhibits one interesting feature. For small angles θ_0, when $\sin\theta_0 < \sqrt{[\sqrt{u}/(1 + \sqrt{u})]}\sin\chi$, the vertical intersects the line aa. At the point of intersection the ordinary ray is evidently perpendicular to the line aa, and therefore to the direction of the magnetic field, and the direction of the ray is reversed on crossing the line aa. Thus for $\theta_0 < \theta_{cr,2}$ the reflection of the ordinary rays takes place at $v = 1$, i.e. always at the same height, and the angle $\theta_{cr,2}$ is determined by

$$\sin\theta_{cr,2} = \sqrt{\frac{\sqrt{u}}{1+\sqrt{u}}}\sin\chi = \sqrt{\frac{\omega_H}{\omega_H + \omega}}\sin\chi, \tag{29.23}$$

χ being the angle between the field $\boldsymbol{H}^{(0)}$ and the z-axis.

For $\theta_0 < \theta_{cr,2}$ the ray path has a curious "beak" at the point of reflection, as shown in Fig. 29.14b. This diagram gives the paths of rays reflected from

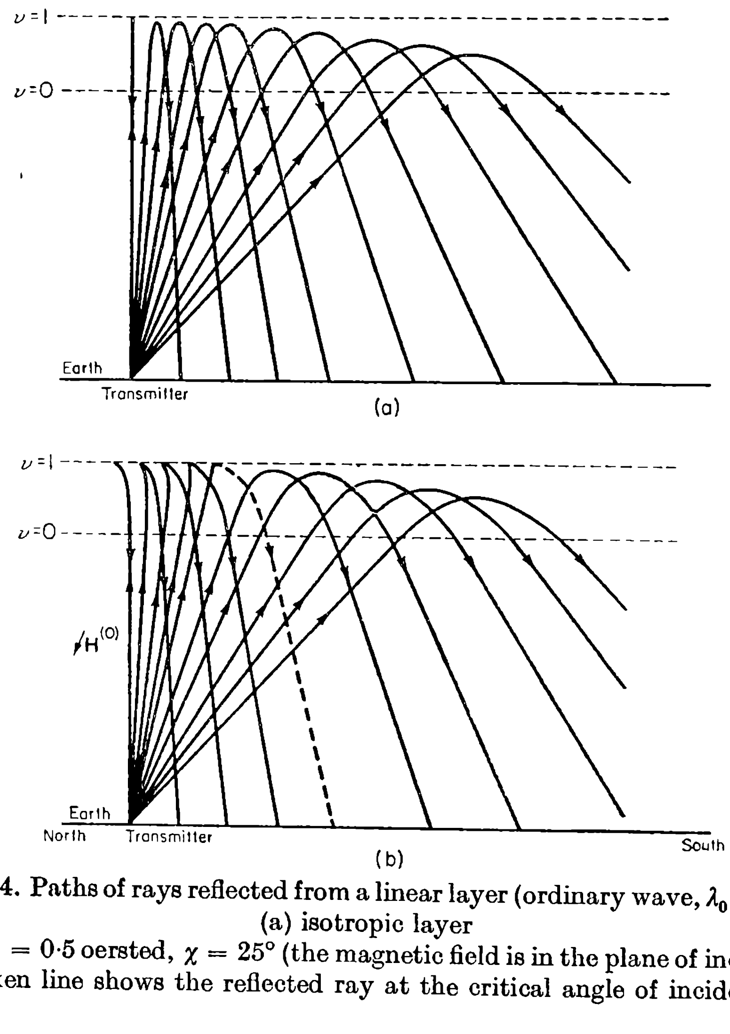

FIG. 29.14. Paths of rays reflected from a linear layer (ordinary wave, $\lambda_0 = 80$ m).
(a) isotropic layer
(b) $H^{(0)} = 0{\cdot}5$ oersted, $\chi = 25°$ (the magnetic field is in the plane of incidence)
The broken line shows the reflected ray at the critical angle of incidence $\theta_{cr,2}$.

a linear layer with $H^{(0)} = 0{\cdot}5$ oersted, $\chi = 25°$, $\lambda_0 = 80$ m; Fig. 29.14a gives paths for the same layer but with $H^{(0)} = 0$. For angles of incidence $\theta_0 > \theta_{\mathrm{cr},2}$ the reflection is qualitatively similar to that which occurs in the isotropic case. This discussion relates to wave propagation in the plane of the magnetic meridian, when the rays remain in that plane. For propagation outside the plane of the magnetic meridian it is easy to see that the rays cannot have "beaks" at the point of the reflection, since the line aa is nowhere reached. In this case the plane of incidence in Fig. 29.13, if the vector $\boldsymbol{H}^{(0)}$ is taken to lie in the plane of the diagram, does not coincide with the plane of the diagram, and so the wave normal also does not lie in that plane. Physically, of course, there can be no essential difference between the "beak" and a ray path near the point of reflection for propagation in a plane very close to the meridian plane. The smoothing-out of the difference is due not only to the closeness of the paths in these cases but also to the fact that geometrical optics, and consequently the whole of the ray treatment, are inapplicable near the point of reflection. Nevertheless, the ray paths make it possible to obtain some idea of wave propagation in the reflection region also, since for a smoothly varying layer the region where geometrical optics is valid is usually very wide and excludes only a very narrow range near the "reflection point" itself.

Penetration of waves and the "tripling" of signals for oblique incidence

The critical angle of incidence $\theta_{\mathrm{cr},2}$ [see (29.23)] is the angle for which, at the reflection point ($v = 1$), the wave normal for the ordinary wave is parallel to the magnetic field $\boldsymbol{H}^{(0)}$. In accordance with the previous discussion, the normal $\mathbf{grad}\,\Psi$ can be parallel to $\boldsymbol{H}^{(0)}$ only for propagation in a meridian plane, since the normal always lies in the plane of incidence. From the consideration of wave propagation at normal incidence we know that, if the direction of the wave normal approaches that of the field, the point of reflection "jumps" from $v = 1$ to $v = 1 + \sqrt{u}$; this is the "tripling" effect for the case $u = \omega_H^2/\omega^2 < 1$ (§ 28). Thus it is clear that at the critical angle of incidence $\theta_{\mathrm{cr},2}$ the ray is not reflected at the level $v = 1$ but proceeds further. However, after reflection at some higher level the ray cannot turn back in regular propagation, since it takes another path, the wave normal does not become parallel to $\boldsymbol{H}^{(0)}$ for $v = 1$, and the wave energy is absorbed in the resonance region. In order to see more clearly the nature of propagation near the critical angle of incidence, when geometrical optics is invalid, it is convenient to return to the curves of $q_{1,2}(v)$.

Fig. 29.15 shows such curves [183] for the case $u = \frac{1}{4}$ and $\chi = 22°$, when $\theta_{\mathrm{cr},2} = 12{\cdot}5°$. It is evident from Fig. 29.15b that for $\theta_0 = \theta_{\mathrm{cr},2}$ the ordinary branch 2 becomes the second extraordinary branch 1 at the point $v = 1$; subsequently this wave is reflected and enters the region of large q below the point $v_1^{(+)}$, and is absorbed at $v_{1\infty}$. At angles θ_0 close to $\theta_{\mathrm{cr},2}$ the ordinary wave need not, according to the graphs of $q_{1,2}(v)$, become extraordinary, but it is physically clear that such a transition will in fact occur in a certain range of angles near $\theta_{cr,2}$. In this region geometrical optics is invalid for both waves near $v = 1$; they interact and the penetration effect occurs. This effect is essentially the same as in normal incidence (§ 28). No quantitative theory has yet been given for the case of oblique incidence.

A curious feature is that, in probing of the ionosphere, the ordinary wave which has penetrated through the region $v \approx 1$ can still be observed. The reason is that hitherto we have always considered the regular phenomena occurring for smoothly varying layers. In reality, the ionosphere contains

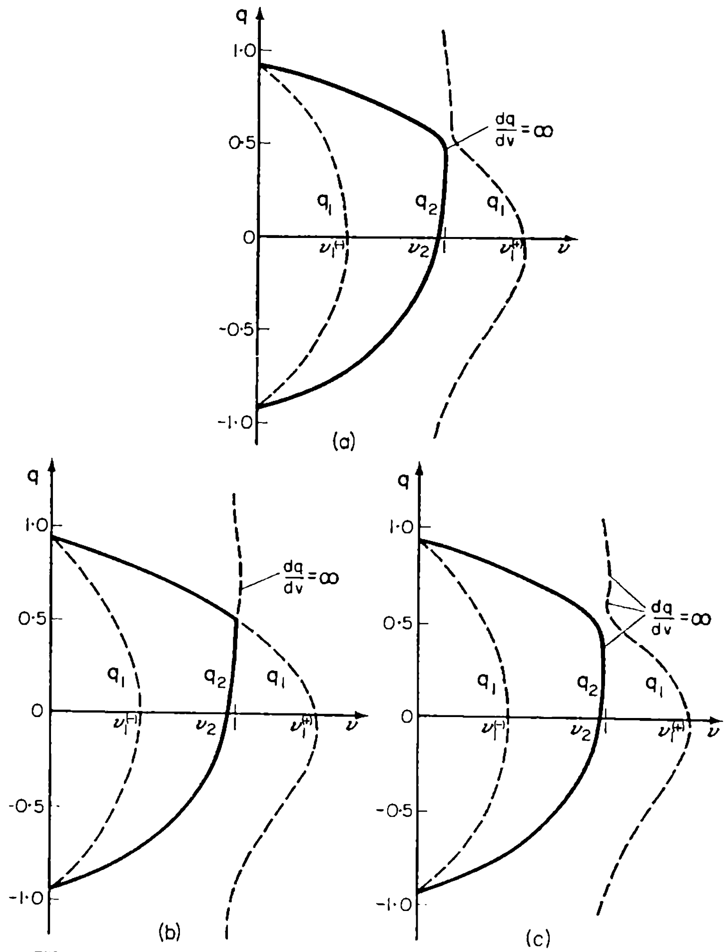

FIG. 29.15. The functions $q_{1,2}(v)$ for propagation in a meridian plane with $u = \frac{1}{4}$ and $\chi = 22^0$.
(a) $\theta_0 = 10°$ (b) $\theta_0 = \theta_{cr,2} = 12° \, 30'$ (c) $\theta_0 = 15°$

inhomogeneities which lead to scattering of the waves. For this reason the wave after penetration is partly returned along the same path by scattering near the reflection point (i.e. where $dq/dv = \infty$ on the right-hand q_1 curve in Fig. 29.15, and in practice near the point $v_1^{(+)}$), and so comes back to Earth. The signal which has made this journey arrives after the signals 1 and 2 reflected at lower levels, so that we have a "tripling" of signals for oblique incidence. This effect was mentioned in § 28. It is related to the effect for normal incidence, but differs fundamentally in that it is observed only when wave scattering takes place. Moreover, this effect is of course observed only when the transmitter sends out waves at angles $\theta_0 \approx \theta_{cr,2}$, and when the reflected waves are received in that direction. Such an effect has been observed [177, 178] in the reflection of radio waves from the F layer with $\chi = 18°$, $\omega = 2\pi \times 4{\cdot}65 \times 10^6$, $\omega_H = 2\pi \times 1{\cdot}55 \times 10^6$ (i.e. $\sqrt{u} = 0{\cdot}334$), $\theta_{cr,2} = 8\cdot 7°$. The amplitude of the third signal has a quasi-Gaussian form as a function of $\theta - \theta_0$, and experimentally [178] the squared amplitude (the power) was found to be halved at $\theta_0 - \theta_{cr,2} = 0{\cdot}42°$.

Thus, whatever its subsequent fate, for $\theta_0 \approx \theta_{cr,2}$ the ordinary signal passes through a "hole" in the layer at the level $v = 4\pi e^2 N/m\omega^2 = 1$. Besides the two possibilities mentioned above, namely absorption of the signal near $v_{1\infty}$ and partial scattering of it (including scattering into the opposite direction), the signal may pass right through the layer; this occurs if the electron density does not reach the value $N \approx m(\omega^2 + \omega\omega_H)/4\pi e^2$ needed for reflection of the extraordinary signal from the region near the point $v_1^{(+)}$ (Fig. 29.15a). In other words, the signal passes through the layer if the critical frequency for third reflection $f_{cr,3} = f_{cr,z}$ is less than the carrier frequency f of the signal (see also § 35).

Penetration of waves with $u = \omega_H^2/\omega^2 > 1$

The penetration of waves can also occur when $u > 1$, i.e. for waves of length greater than $\lambda_H = 2\pi c/\omega_H$. As in the case $u < 1$ discussed above, penetration is possible if the direction of the wave normal in the region $v \approx 1$ is sufficiently close to that of the magnetic field $\boldsymbol{H}^{(0)}$. These directions are parallel if the plane of incidence is that of the magnetic meridian and

$$q/p = n \cos\theta/\sin\theta_0 = \cot\theta = \cot\chi,$$

i.e. the angle θ between the wave normal and the z-axis is equal to the angle χ between the field $\boldsymbol{H}^{(0)}$ and the z-axis. As shown above, and as is clear from graphs of the functions $\tilde{n}_{1,2}^2(v)$ for $u > 1$ (see, for example, Fig. 28.1b), the interaction of waves with $\chi \to 0$ occurs when $v \approx 1$. For $v = 1$ and $\chi = 0$ we have

$$n_{1,2}^2(v = 1) = p^2 + q^2 = 1 - 1/(1 \pm \sqrt{u}) = \sqrt{u}/(\sqrt{u} \pm 1).$$

Thus the critical angle θ_{cr} is given by

$$n_{1,2}(v = 1)\cos\chi/\sin\theta_{cr} = \cot\chi$$

or

$$\sin\theta_{cr,1,2} = \sqrt{\frac{\sqrt{u}}{\sqrt{u} \pm 1}}\sin\chi = \sqrt{\frac{\omega_H}{\omega_H \pm \omega}}\sin\chi, \tag{29.24}$$

where the + sign refers to the ordinary wave 2 and the − sign to the extraordinary wave 1. For the ordinary wave $\theta_{\mathrm{cr},2} < \chi$ and for the extraordinary wave $\theta_{\mathrm{cr},1} > \chi$. When $u < 1$ and the values of $\tilde{n}_{1,2}^2$ are positive (as we assume), the critical angle exists only for the ordinary wave. Formula (29.24) is then identical with (29.23) obtained by a geometrical construction. For $u = \omega_H^2/\omega^2 > 1$, the critical angle again always exists for the ordinary wave. For the extraordinary wave there is interaction and the critical angle exists only if $[\sqrt{u}/(\sqrt{u} - 1)] \sin^2\chi \leqq 1$, i.e. if

$$\sqrt{u} \geqq 1/\cos^2\chi. \tag{29.25}$$

For very low frequencies $\sqrt{u} \to \infty$ and $\theta_{\mathrm{cr}} \to \chi$, i.e. the interaction occurs when the propagation is along the lines of force of the magnetic field. This is the case in whistlers [53, 54].

For angles of incidence θ_0 close to $\theta_{\mathrm{cr},1,2}$, the curves of $q_{1,2}(v)$ for $u > 1$ have various features similar to those shown in Fig. 29.15 for the region $u < 1$. We shall not pause to discuss this problem in detail; the investigation of it is as yet far from complete [180, 189]. The possibility of passage of the wave through the layer is particularly important as regards applications. For normal incidence and $u_L = u \cos^2\alpha > 1$, this problem has been considered at the end of § 28. The significant property of oblique incidence is that the penetration effect may be greatly increased and may be appreciable even when the angle χ between the field and the vertical is not small, i.e. not only at high latitudes. The physical situation is simply that, because of refraction in the layer, the wave normal may approach the direction of the field $\boldsymbol{H}^{(0)}$ in the interaction region $v \approx 1$. This is what occurs for angles of incidence close to $\theta_{\mathrm{cr},1,2}$.

It is thus evident from Fig. 28.10b that for $\theta_0 \approx \theta_{\mathrm{cr},1}$ the extraordinary wave (for $v < 1$, that is) will pass through the whole layer, however thick. The ordinary wave with $\theta_0 \approx \theta_{\mathrm{cr},2}$ can pass through the region $v \approx 1$ and be propagated further as an extraordinary wave.† This wave passes through the layer only if it does not reach the point $v_{10}^{(+)} = 1 + \sqrt{u}$, i.e. if the maximum electron density in the layer is $N_{\max} < m(\omega^2 + \omega\omega_H)/4\pi e^2$.††

Proof of the reciprocity theorem

The solution of the problem of wave propagation and reflection in layers of magnetoactive plasma usually involves the use of graphical or numerical methods, even when geometrical optics is applicable. The picture is still more complicated if allowance must be made for the interaction of waves, their intensity must be calculated, etc. Here it is particularly useful to know the

† More precisely, as stated in § 11, the wave reflected from the point $v_{10}^{(+)} = 1 + \sqrt{u}$ is called extraordinary for $v > 1$ if $\alpha \neq 0$ and ordinary if $\alpha = 0$.

†† For oblique incidence the extraordinary wave is reflected somewhat lower than the region where $v = v_{10}^{(+)}$; this is not taken into account here, since for $\theta_{\mathrm{cr},2} \ll 1$ the difference from the case of normal incidence is not large.

general properties of solutions of the field equations. An example of such general properties is the well-known reciprocity theorem in electrodynamics (see, for example, [22, § 9; 36, § 69; 143, § 77]). In its usual form, the reciprocity theorem is not generally valid when a magnetoactive medium is present, and for this very reason a light valve or radio valve may be constructed so as to transmit radiation in only one direction [192]. However, a generalised reciprocity theorem is valid even in a magnetoactive medium; moreover, in some particular cases the usual reciprocity theorem or its consequences may be used even when such a medium is present.

For convenience we shall give the proof of the reciprocity theorem *ab initio*. To do so, let us consider two field sources 1 and 2, in which the external current densities are respectively $\boldsymbol{j}_{\mathrm{ex}}^{(1)}(\boldsymbol{r})$ and $\boldsymbol{j}_{\mathrm{ex}}^{(2)}(\boldsymbol{r})$. The resulting fields $\boldsymbol{E}^{(1)}$, $\boldsymbol{H}^{(1)}$ and $\boldsymbol{E}^{(2)}$, $\boldsymbol{H}^{(2)}$ satisfy the equations

$$\left.\begin{aligned} \operatorname{curl} \boldsymbol{H}^{(1)} &= i\,\omega\,\boldsymbol{D}^{(1)}/c + 4\pi \boldsymbol{j}_{\mathrm{ex}}^{(1)}/c, \\ \operatorname{curl} \boldsymbol{E}^{(1)} &= -i\,\omega\,\boldsymbol{B}^{(1)}/c, \end{aligned}\right\} \tag{29.26}$$

$$\left.\begin{aligned} \operatorname{curl} \boldsymbol{H}^{(2)} &= i\,\omega\,\boldsymbol{D}^{(2)}/c + 4\pi \boldsymbol{j}_{\mathrm{ex}}^{(2)}/c, \\ \operatorname{curl} \boldsymbol{E}^{(2)} &= -i\,\omega\,\boldsymbol{B}^{(2)}/c; \end{aligned}\right\} \tag{29.27}$$

here all quantities are assumed to vary as $e^{i\omega t}$, and in the presence of absorption $\boldsymbol{D}$ signifies $\boldsymbol{D} - i \cdot 4\pi \boldsymbol{j}/\omega$, where $\boldsymbol{D}$ is the electric induction (electric displacement) and $\boldsymbol{j}$ the conduction current density. So far the medium is entirely arbitrary, and we have therefore used the magnetic induction $\boldsymbol{B}$.

Multiplying equations (29.26) respectively by $\boldsymbol{E}^{(2)}$ and $\boldsymbol{H}^{(2)}$, and (29.27) by $-\boldsymbol{E}^{(1)}$ and $-\boldsymbol{H}^{(1)}$, adding and using the formula

$$\operatorname{div}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \operatorname{curl} \boldsymbol{A} - \boldsymbol{A} \cdot \operatorname{curl} \boldsymbol{B},$$

we find

$$\begin{aligned} \operatorname{div}(\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)} - \boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}) &= (4\pi/c)(\boldsymbol{j}_{\mathrm{ex}}^{(1)} \cdot \boldsymbol{E}^{(2)} - \boldsymbol{j}_{\mathrm{ex}}^{(2)} \cdot \boldsymbol{E}^{(1)}) + \\ &+ (i\,\omega/c)(\boldsymbol{D}^{(1)} \cdot \boldsymbol{E}^{(2)} - \boldsymbol{E}^{(1)} \cdot \boldsymbol{D}^{(2)} + \boldsymbol{H}^{(1)} \cdot \boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)} \cdot \boldsymbol{H}^{(2)}). \end{aligned}$$

When this expression is integrated over the volume, the first term becomes a surface integral and is zero (the field is assumed to decrease suitably at infinity, and it is easy to see that surfaces of discontinuity also make no contribution to the integral). Hence the reciprocity theorem in its usual form

$$\int \boldsymbol{j}_{\mathrm{ex}}^{(1)}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(2)}(\boldsymbol{r})\, d\boldsymbol{r} = \int \boldsymbol{j}_{\mathrm{ex}}^{(2)}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(1)}(\boldsymbol{r})\, d\boldsymbol{r} \tag{29.28}$$

is valid if

$$\int (\boldsymbol{D}^{(1)} \cdot \boldsymbol{E}^{(2)} - \boldsymbol{E}^{(1)} \cdot \boldsymbol{D}^{(2)} + \boldsymbol{H}^{(1)} \cdot \boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)} \cdot \boldsymbol{H}^{(2)})\, d\boldsymbol{r} = 0. \tag{29.29}$$

For layers at rest, neglecting spatial dispersion and assuming a linear layer, we have $D_i - i \cdot 4\pi j_i/\omega = \varepsilon'_{ik} E_k$ and $B_i = \mu'_{ik} H_k$, where the complex tensors ε'_{ik} and μ'_{ik} may depend on ω and on the coordinates. The expression (29.29) takes the form

$$\int [(\varepsilon'_{ik} - \varepsilon'_{ki})\, E_k^{(1)}\, E_i^{(2)} + (\mu'_{ik} - \mu'_{ki})\, H_i^{(1)} H_k^{(2)}]\, d\boldsymbol{r} = 0, \tag{29.30}$$

where summation over repeated suffixes is understood, and we have used the identity $\varepsilon'_{ik}E_i^{(1)}E_k^{(2)} \equiv \varepsilon'_{ki}E_k^{(1)}E_i^{(2)}$, etc. Hence it is evident that the reciprocity theorem (29.28) is valid for media with symmetrical tensors ε'_{ik} and μ'_{ik}. For media with unsymmetrical tensors the reciprocity theorem (29.28) is in general not valid, as is clear from the possibility already mentioned of constructing a valve which transmits waves in only one direction.

The generalisation to the case of a magnetoactive medium

The generalisation of the reciprocity theorem to the case of a magnetoactive medium (with the tensor ε'_{ki} unsymmetrical; $\mu'_{ik} = \mu\delta_{ik}$, and in a plasma μ is almost unity) is obtained if we use the fact that the generalised principle of the symmetry of the kinetic coefficients gives the general relation

$$\varepsilon'_{ik}(\boldsymbol{H}^{(0)}) = \varepsilon'_{ki}(-\boldsymbol{H}^{(0)}); \tag{29.31}$$

see (10.13) and [36, § 82]. For a magnetoactive plasma, the relation (29.31) follows from the expressions (10.12) for the tensor ε'_{ik}. From (29.31) and the condition (29.30) we derive the generalised reciprocity theorem:

$$\int \boldsymbol{j}_{\text{ex}}^{(1)}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(2)}(\boldsymbol{r}, \boldsymbol{H}^{(0)})\, d\boldsymbol{r} = \int \boldsymbol{j}_{\text{ex}}^{(2)}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(1)}(\boldsymbol{r}, -\boldsymbol{H}^{(0)})\, d\boldsymbol{r}. \tag{29.32}$$

Here the field $\boldsymbol{E}^{(1)}(\boldsymbol{r}, -\boldsymbol{H}^{(0)})$ is that due to the source 1 when the magnetic field constant in time (external relative to the fields considered) is $-\boldsymbol{H}^{(0)}(\boldsymbol{r})$, i.e. everywhere has the opposite sign to that which occurs when the field $\boldsymbol{E}^{(2)}(\boldsymbol{r}, \boldsymbol{H}^{(0)})$ from source 2 is being found. If the medium is not magnetoactive, the tensor ε'_{ik} and the fields $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ do not depend on $\boldsymbol{H}^{(0)}$, which of course leads to a symmetrical tensor ε'_{ik} and the old form (29.28) of the theorem. The relation (29.32), which is more general, is naturally weaker than the usual reciprocity theorem (29.28), since it relates the fields $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ under different conditions, namely for opposite directions of the field $\boldsymbol{H}^{(0)}$. Nevertheless, the theorem (29.32) is useful, and shows, for example, that the reciprocity theorem is valid in its original form if by symmetry or from the nature of the problem we have

$$\boldsymbol{E}^{(1)}(\boldsymbol{r}, -\boldsymbol{H}^{(0)}) = \boldsymbol{E}^{(1)}(\boldsymbol{r}, \boldsymbol{H}^{(0)}) \text{ or } \boldsymbol{E}^{(2)}(\boldsymbol{r}, -\boldsymbol{H}^{(0)}) = \boldsymbol{E}^{(2)}(\boldsymbol{r}, \boldsymbol{H}^{(0)}).$$

The discussion in [193, 194] amounts to seeking such cases.

For sources which are electric and magnetic point dipoles we must put in (29.32)

$$\boldsymbol{j}_{\text{ex}}^{(1,2)} = i\,\omega\,\boldsymbol{P}^{(1,2)} + c\,\mathbf{curl}\,\boldsymbol{M}^{(1,2)}$$

with $\boldsymbol{P}^{(1,2)} = \boldsymbol{p}^{(1,2)}\,\delta(\boldsymbol{r} - \boldsymbol{r}_{1,2})$ and $\boldsymbol{M}^{(1,2)} = \boldsymbol{m}^{(1,2)}\,\delta(\boldsymbol{r} - \boldsymbol{r}_{1,2})$, δ being the delta function. Then

$$\begin{aligned}\boldsymbol{p}^{(1)} \cdot \boldsymbol{E}^{(2)}(1, \boldsymbol{H}^{(0)}) - \boldsymbol{m}^{(1)} \cdot \boldsymbol{H}^{(2)}(1, \boldsymbol{H}^{(0)})\\ = \boldsymbol{p}^{(2)} \cdot \boldsymbol{E}^{(1)}(2, -\boldsymbol{H}^{(0)}) - \boldsymbol{m}^{(2)} \cdot \boldsymbol{H}^{(1)}(2, -\boldsymbol{H}^{(0)}).\end{aligned} \tag{29.33}$$

In transforming the term in $\boldsymbol{m}$ we have used the equation $\mathbf{curl}\,\boldsymbol{E} = -i\omega\boldsymbol{B}/c$; the permeability μ is taken as unity and, for example, $\boldsymbol{E}^{(2)}(1, \boldsymbol{H}^{(0)})$ is the field of the electric dipole $\boldsymbol{p}^{(2)}$ and the magnetic dipole $\boldsymbol{m}^{(2)}$ at the position of the dipoles $\boldsymbol{p}^{(1)}$ and $\boldsymbol{m}^{(1)}$, the external field being $\boldsymbol{H}^{(0)}$.

Media with an unsymmetrical tensor μ'_{ik} and with spatial dispersion

For media with an unsymmetrical tensor μ'_{ik} it follows from the generalised principle of symmetry of the kinetic coefficients that $\mu'_{ik}(\boldsymbol{B}^{(0)}) = \mu'_{ki}(-\boldsymbol{B}^{(0)})$, where $\boldsymbol{B}^{(0)}$ is the magnetic induction, here regarded as "external" (independent) with respect to the fields of frequency ω under consideration. This result is evident from [36, § 88], since the mean macroscopic magnetic field is $\boldsymbol{B}$ and can be replaced by $\boldsymbol{H}$ only in a non-magnetic medium.

Hence it follows that the theorem (29.32) is valid also for a medium with an unsymmetrical tensor μ'_{ik} if $\boldsymbol{H}^{(0)}$ is replaced by $\boldsymbol{B}^{(0)}$, which we may do from the start. For ferrites, however, when the tensor μ'_{ik} must be used, the sign of $\boldsymbol{B}^{(0)}$ is usually reversed by changing the sign of the external field $\boldsymbol{H}^{(0)}$, and the expression (29.32) may be used immediately.

When spatial dispersion is taken into account, the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ (we shall assume for simplicity that $\boldsymbol{B} = \boldsymbol{H}$) are related by an integral or differential relation instead of the algebraic relation $D_i = \varepsilon'_{ik}E_k$. The principle of symmetry of the kinetic coefficients shows (see [36, § 83]) that in the absence of an external magnetic field $\boldsymbol{H}^{(0)}$ the equation (29.29) holds, and so the reciprocity theorem is valid in its ordinary form. The same conclusion is reached if spatial dispersion is taken into account by using a relation of the type [1]

$$D_i = \varepsilon'_{ik}\,E_k + \gamma_{ikl}\,\partial E_k/\partial x_l + \delta_{iklm}\,\partial^2 E_k/\partial x_l\,\partial x_m. \qquad (29.34)$$

The symmetry properties of the tensors γ_{ikl} and δ_{iklm} ($\gamma_{ikl} = -\gamma_{kil}, \delta_{iklm} = \delta_{kilm}$) then enable us to derive (29.29) by integrating by parts. This derivation is equivalent to the preceding one, since the symmetry properties of the tensors γ_{ikl} and δ_{iklm} themselves follow from the principle of symmetry of the kinetic coefficients.

Naturally-active media are those where the term in γ_{ikl} in (29.34) must be taken into account. In such media, therefore, the ordinary reciprocity theorem is valid when $\boldsymbol{H}^{(0)} = 0$, and consequently, as is well known, they cannot be used to construct an optical valve.

When spatial dispersion (thermal motion) is taken into account in a magnetoactive medium, the generalised reciprocity theorem (29.32) is valid, i.e. spatial dispersion itself causes no effect here also.

As an example of the application of the reciprocity theorem in a magnetoactive medium, we may mention the following result [194]. In reflection of waves from an ionospheric layer in the plane of the magnetic meridian (i.e. with the magnetic field in the plane of incidence) the reciprocity theorem in its usual form (i.e. not replacing $\boldsymbol{H}^{(0)}$ by $-\boldsymbol{H}^{(0)}$) is valid for aerials emitting and receiving a field $\boldsymbol{E}$ lying in the plane of incidence. The same happens if both aerials emit and receive a field $\boldsymbol{E}$ perpendicular to the plane of incidence. For two aerials, of which one receives a field $\boldsymbol{E}$ in the plane of incidence and the other emits a field $\boldsymbol{E}$ in a perpendicular direction (or *vice versa*), the ordinary reciprocity theorem is again valid, but only for the moduli of the field (i.e. neglecting the phase).

In §§ 27 and 28 we have also seen that in some cases, in accordance with the usual reciprocity theorem, the moduli of the coefficients for transmission of waves in opposite directions are equal even in a magnetoactive plasma. As already noted, it is clear from the theorem (29.32) that, to demonstrate the complete validity of the ordinary reciprocity theorem in any particular problem for a magnetoactive medium, it is sufficient to show that the electric field of the wave is unchanged when $\boldsymbol{H}^{(0)}$ is replaced by $-\boldsymbol{H}^{(0)}$. This is in fact true in those cases where the usual reciprocity theorem is valid in a magnetoactive plasma.

CHAPTER VI

REFLECTION OF RADIO WAVES FROM IONOSPHERIC LAYERS

§ 30. INTRODUCTION. REFLECTION FROM AN ARBITRARY SMOOTH LAYER

Propagation of radio waves in the ionosphere

THE Earth's ionosphere was historically the first region which gave rise to the study of problems of the theory of electromagnetic wave propagation in an inhomogeneous plasma. It is therefore understandable that in the foregoing sections the ionospheric layers have frequently provided examples of applications. Below we shall discuss a number of important topics concerning reflection of waves from an inhomogeneous medium, again mainly in terms of applications to the ionosphere. It must be emphasised, however, that most of the results have a more general significance. Moreover, the whole subject is closely related to what has been treated previously, and to place it in a separate chapter is largely an arbitrary procedure.

There has been an exceedingly large amount of work on the propagation and reflection of radio waves in the ionosphere and on kindred problems. There are numerous special features concerning the propagation of waves of various lengths, at various times and at various latitudes, together with a variety of ionospheric disturbances, sporadic phenomena, and so on. The majority of these interesting topics are ignored in what follows, and we shall be mainly concerned with a single problem, the propagation and reflection of radio waves by a smooth layer of plasma. Discussions of the remaining subjects may be found, in particular, in [20, 22, 23, 190, 191]. A thorough bibliography of work published up to 1947 is given in [195], and the history of ionosphere studies in the Soviet Union is dealt with in [22, 197]. Here particular attention should be drawn to work on the propagation of waves with allowance for random inhomogeneities.

In recent years it has become entirely clear that, both in the Earth's ionosphere and in the solar corona, some inhomogeneities are present and give rise to various effects. Of these we may mention the fluctuations in amplitude and phase of radio waves reflected from the ionosphere, and those in amplitude and direction of the cosmic radio waves caused by passing through the ionosphere. The scattering of radio waves from the Crab Nebula when they pass through the solar corona is a cognate topic. The scattering and refraction of cosmic radio waves propagated in a statistically inhomogeneous interstellar and interplanetary medium may also be significant.

The propagation of waves of various types, with allowance for statistical inhomogeneities, is at present a large and to some extent independent department of research, being of interest in connection with radio physics, radio communications, acoustics, radio astronomy and optical astronomy. This field will not be touched on here. The reader is referred, in the first instance, to the original and review articles on the subject [23, 156, 198–214, 269].

Parameters of the ionosphere

Until quite recently the principal method, and in practice the only reliable method, of determining the electron density in the ionosphere was by radar probing from the Earth's surface. This has now been supplemented by the

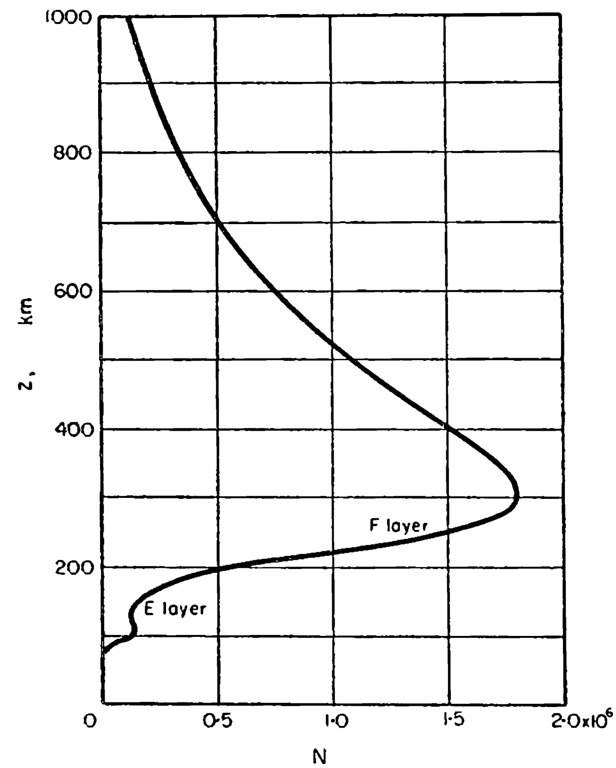

FIG. 30.1. Electron density N in the ionosphere (medium latitudes, noon, years of maximum solar activity; the values given are averages suitably defined).

methods of radio astronomy (examining the cosmic radio waves), the observation of radar reflections from the Moon, and measurements by means of rockets and artificial satellites. The averaged dependence of the electron density N on height above the Earth's surface thus obtained is shown in Fig. 30.1, where the data pertain to medium latitudes at noon in years of maximum solar activity; see [23], whence Figs. 30.1–30.3 are taken. The density of molecules (meaning all neutral particles) is shown in Fig. 30.2, while Fig. 30.3 gives the temperature in the upper parts of the atmosphere.

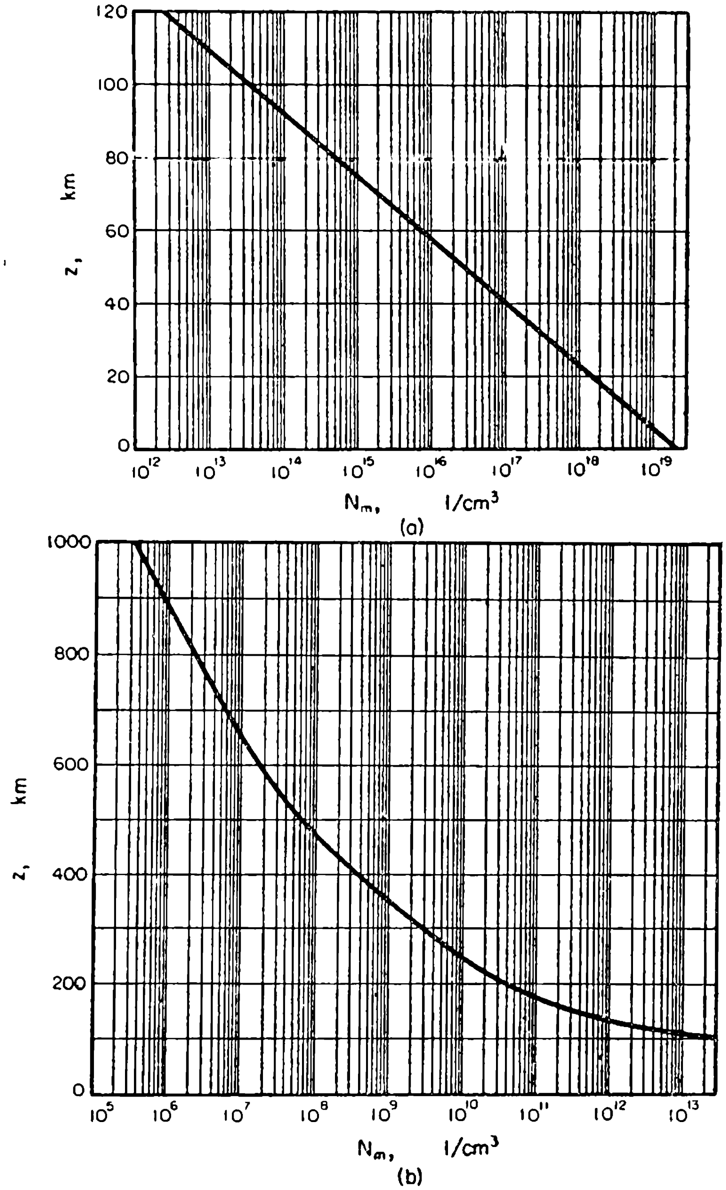

FIG. 30.2. Density N_m of neutral particles (molecules) in the atmosphere.
(a) Heights below 120 km (b) Heights above 100 km

In these diagrams the properties of the ionosphere are shown only in broad outline. In reality, the distribution of ionisation varies appreciably, depending on a number of factors (latitude, time of year, time of day, etc.). As an example, we may mention the appearance of an F_1 layer in the lower part of the F layer in summer, the appearance of a sporadic E layer, etc.

The actual layers of the ionosphere, even apart from local variations of electron density, have no simple geometrical form. Admittedly the lower part of the F layer, for example, can very often be represented to a good approximation by a parabolic layer:

$$\left.\begin{aligned} N &= N_{\max}(1 - z^2/z_m^2), \\ \varepsilon &= n^2 = 1 - 4\pi e^2 N(z)/m\,\omega^2 \\ &= 1 - (f_{\rm cr}^2/f^2)(1 - z^2/z_m^2), \\ \omega &= 2\pi f, \\ f_{\rm cr} &= \sqrt{(e^2 N_{\max}/\pi m)} \approx 9 \times 10^3 \sqrt{N_{\max}}, \\ N_{\max} &= 1{\cdot}24 \times 10^4 f_{\rm cr}(\text{Mc/s}). \end{aligned}\right\} \tag{30.1}$$

At the boundary of the layer, however, the deviations from the parabolic form are considerable, and there is no reason to suppose that the layer is

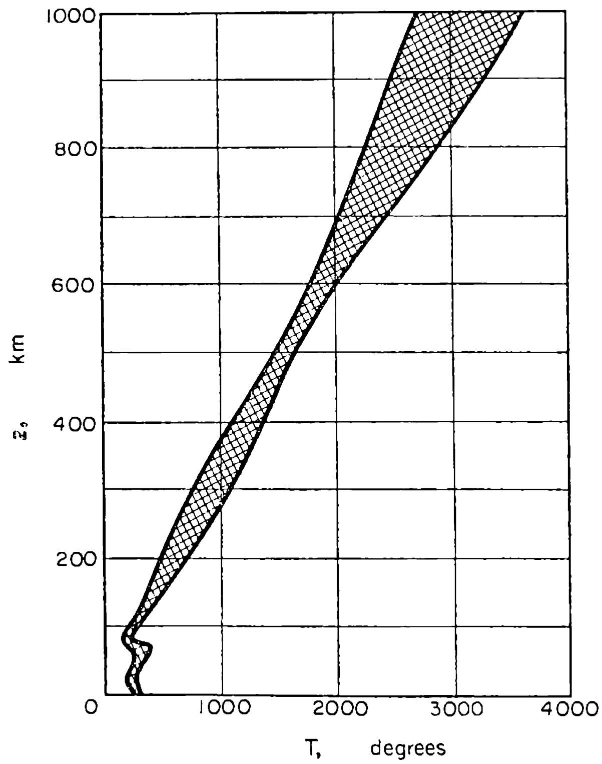

FIG. 30.3. Temperature in the upper layers of the atmosphere (the temperature values lie in the hatched area).

strictly parabolic; in particular, the "simple layer" often used in ionosphere calculations [20, 22, 23] is not parabolic.

Reflection of waves from an arbitrary layer

It is therefore very important to note that the propagation and reflection of radio waves can be treated for an ionospheric layer which is (to a considerable extent) arbitrary, viz. any smooth and sufficiently thick layer with one maximum, of the type shown in Fig. 30.4.

The reason is that, far from the reflection point for waves of a given frequency ω (the point A in Fig. 30.4a)† we can use the approximation of geometrical optics, which is valid (on certain assumptions) for any form of $\varepsilon'(z)$. In the neighbourhood of the reflection point A, the layer may usually be regarded as linear or, if the point A is near the maximum, as parabolic. In the former case, when absorption is absent, there is total reflection of the waves from the layer, the behaviour of which for values of z much exceeding $z(\varepsilon = 0)$ is unimportant. In the second case, which occurs at frequencies near the critical value, there may be penetration of waves through the layer. The solutions for both linear and parabolic layers are known (§ 17), and so, by joining these

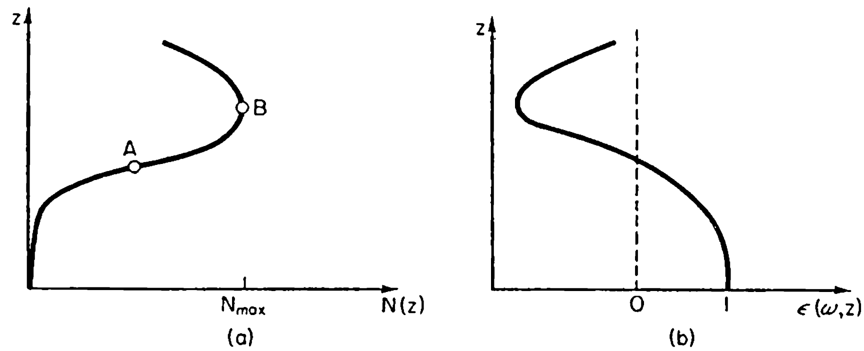

FIG. 30.4. Arbitrary smooth layer with one maximum.

solutions to that given by geometrical optics, we have the solution for an arbitrary layer. The condition for this solution to be valid (when the layer is regarded as linear near A) is that the deviation from linearity near A is small, i.e.

$$|d^2\varepsilon'/dz^2|_0 \Delta z \ll |d\varepsilon'/dz|_0; \tag{30.2}$$

here the derivatives are taken at the point $z(\varepsilon = 0)$, and Δz is the distance from the point A at which the difference between the exact solution and that given by geometrical optics is sufficiently small.

It will be clear from what follows that the above procedure has a wide range of applicability, and the more interesting case is the first, where the layer may be regarded as linear near A. The replacement of the layer by a parabolic one is necessary only in the immediate neighbourhood of the critical frequency. This case is of very slight practical importance, and will be discussed in § 33.

In the study of various irregular phenomena it is also desirable to consider the transmission and reflection of radio waves by thin layers, where the approximation of geometrical optics is in general invalid. The reflection and transmission of waves by layers of arbitrary thickness, including thin layers, have already been treated in § 18.

† The reflection point is somewhat arbitrarily taken to be in every case the point where $\varepsilon(\omega) = 0$, i.e. the point $z(\varepsilon = 0)$.

We shall therefore regard the layer as linear near the reflection point. The solution for this case is given by formulae (17.2), (17.4), (17.5), (17.6), etc.

If absorption is absent, and the distance Δz from the point $z(\varepsilon = 0)$ is sufficiently large to satisfy the inequality [cf. (17.7)]

$$\Delta z = z(\varepsilon = 0) - z \gg \left(\frac{c^2}{\omega^2 |d\varepsilon/dz|_0}\right)^{\frac{1}{3}} = \left(\frac{\lambda_0^2}{4\pi^2 |d\varepsilon/dz|_0}\right)^{\frac{1}{3}}, \tag{30.3}$$

then the field E for a linear layer is [cf. (17.6), (17.2)]

$$\left.\begin{aligned} E &= \frac{3A}{\sqrt{\pi}} \zeta^{-\frac{1}{4}} \cos\left(\frac{2}{3}\zeta^{\frac{3}{2}} - \frac{1}{4}\pi\right), \\ \zeta &= \left(\frac{\omega^2}{c^2}\left|\frac{d\varepsilon}{dz}\right|_0\right)^{\frac{1}{3}} \Delta z \\ &= \left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{2}{3}} \left|\frac{d\varepsilon}{dz}\right|_0 [z(\varepsilon = 0) - z] \\ &= \left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{2}{3}} \varepsilon(z). \end{aligned}\right\} \tag{30.4}$$

The only change from the formulae of § 17 is that there $\varepsilon = 1 - z/z_1 = 1 - |d\varepsilon/dz|_0 z$, but here $\varepsilon = |d\varepsilon/dz|_0 [z(\varepsilon = 0) - z]$, and therefore in the formulae of § 17 we must put $z_1 = (|d\varepsilon/dz|_0)^{-1} = -[(d\varepsilon/dz)_0]^{-1}$ and $z_1 - z = \Delta z = z(\varepsilon = 0) - z$.

The condition (30.3) signifies that $\zeta \gg 1$, and it is for this reason that we can use for the field E the asymptotic representation of the Bessel functions as in (30.4), which corresponds to the approximation of geometrical optics. This is clear, in particular, from the fact that the condition (30.3) is essentially the condition (16.22) for geometrical optics to be valid:

$$\frac{\lambda_0 |dn/dz|}{2\pi n^2} \ll 1,$$

since

$$\frac{\lambda_0 |dn/dz|}{2\pi n^2} = \frac{\lambda_0 |d\varepsilon/dz|}{4\pi \varepsilon^{\frac{3}{2}}} = \frac{\lambda_0}{4\pi (|d\varepsilon/dz|_0)^{\frac{1}{2}} (\Delta z)^{\frac{3}{2}}}.$$

The expression (30.4) may be written

$$E = \frac{3A}{\sqrt{\pi}\left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{1}{6}} [\varepsilon(z)]^{\frac{1}{4}}} \cos\left(\frac{\omega}{c} \int\limits_z^{z(\varepsilon = 0)} \sqrt{\varepsilon(z)}\, dz - \frac{1}{4}\pi\right). \tag{30.5}$$

Comparing (30.5) with the general expression (16.12) for the field in the approximation of geometrical optics (in our case $\varkappa = 0$, $n = \sqrt{\varepsilon}$ and we must take the solution corresponding to a standing wave), we see that the expressions are the same. Hence, if the layer is not linear further away from the reflection point, but the approximation of geometrical optics is valid, the only difference is that in (30.5) the function $\varepsilon(z)$ must be taken to be not linear but that corresponding to the layer under consideration. Thus the expression (30.5) is the desired solution for an arbitrary layer.

If at the boundary of the layer ($z = 0$) the amplitude of the incident wave is unity, its phase is zero and $\varepsilon = 1$, then the field at $z = 0$ has the form $E = E_+ + E_- = 1 + e^{-i\varphi}$, where, by (30.5),

$$\varphi(\omega) = \frac{2\omega}{c} \int\limits_0^{z(\varepsilon(\omega)=0)} n(\omega, z)\, dz - \frac{1}{2}\pi. \tag{30.6}$$

In this case the quantity A in (30.5) is [cf. also (17.9)]

$$A = \frac{2\sqrt{\pi}}{3} \left(\frac{\omega}{c|dn^2/dz|_0}\right)^{\frac{1}{6}} \exp\left(-i\frac{\omega}{c} \int\limits_0^{z(\varepsilon=0)} n(z)\, dz + \frac{1}{4} i\pi\right). \tag{30.7}$$

The field near the point $z(\varepsilon = 0)$ is given by formulae (17.14) and (17.5), where the constant A is equal to (30.7); in this case the field at the boundary of the layer is, as already stated,

$$E = 1 + e^{-i\varphi} \quad \text{for} \quad z = 0. \tag{30.8}$$

The solution derived here for an arbitrary layer is valid under the following conditions; firstly, geometrical optics must be valid for the whole layer where Δz satisfies the inequality (30.3); secondly, the inequality (30.2) must hold for values of Δz which still satisfy (30.3) (i.e. the layer must be approximately linear in the reflection region).

The expression (30.4) is valid to within terms of order $5/72.\ \frac{2}{3}\zeta^{3/2}$ (see, for instance, [126]), and so its accuracy is better than 1 per cent for $\zeta \geqq 5$. According to (30.3) and (30.4), the value $\zeta = 5$ corresponds to

$$\Delta z = \Delta z_0 = 5\left(\frac{c^2}{\omega^2 |d\varepsilon/dz|_0}\right)^{\frac{1}{3}}.$$

When $\lambda_0 = 2\pi c/\omega \sim 60\,\text{m}$ and $|d\varepsilon/dz|_0 \sim 10^{-7}$, we have $\Delta z_0 \sim 2 \times 10^4 = 200\,\text{m}$. If $\Delta z = \Delta z_0$, the condition (30.2) becomes

$$|d^2\varepsilon/dz^2|_0 \ll \frac{1}{5}\left(\left|\frac{d\varepsilon}{dz}\right|_0\right)^{4/3} \left(\frac{2\pi}{\lambda_0}\right)^{\frac{2}{3}} \approx \left(\left|\frac{d\varepsilon}{dz}\right|_0\right)^{4/3} \lambda_0^{-\frac{2}{3}}. \tag{30.9}$$

This condition must hold if the neighbourhood of the reflection point is to be regarded as linear.

Formula (30.6), which shows the phase shift between the reflected and incident waves, is of particular importance, since, if the phase is known, the group delay time $\Delta t_{gr} = \varphi'(\omega_0)$ may also be found.

The expression (30.6), apart from the term $-\frac{1}{2}\pi$, is obtained from geometrical optics by assuming that the latter is valid up to the reflection point $z(\varepsilon = 0)$. In other words, if we augment geometrical optics by the condition for reflection at the point $z(\varepsilon = 0)$, formula (30.6) without the term $-\frac{1}{2}\pi$ is obtained. However, this "supplementing" of geometrical optics by the reflection condition can, of course, be made a rigorous procedure only on the basis of the preceding treatment. The phase $-\frac{1}{2}\pi$ is much less than the principal term $(2\omega/c)\int n(z)\,dz$; this follows from the fact that the applicability of geometrical optics to the greater part of the layer [far from the point $z(\varepsilon = 0)$] presupposes that the thickness of the layer is much greater than $\lambda_0 = 2\pi c/\omega$.

The group delay time is, by (21.12) and (30.6),

$$\begin{aligned}\Delta t_{\mathrm{gr}} = \varphi'(\omega) &= \frac{2}{c}\int\limits_0^{z(\varepsilon(\omega)=0)} n(\omega, z)\,dz + \frac{2\omega}{c}\int\limits_0^{z(\varepsilon(\omega)=0)} \frac{dn(\omega, z)}{d\omega}\,dz \\ &= 2\int\limits_0^{z(\varepsilon(\omega)=0)} \frac{dz}{v_{\mathrm{gr}}(\omega, z)};\end{aligned} \tag{30.10}$$

here $v_{\mathrm{gr}}(\omega, z)$ is the group velocity (21.17), and the suffix zero to ω is omitted, since only one frequency is involved, namely the carrier frequency ω of the signal. Moreover, in differentiating the integral (30.6) with respect to the upper limit we have used the fact that, by definition, $n(z(\varepsilon = 0)) = 0$.

By (21.13), (21.14) and (30.10) we have

$$\left.\begin{aligned}\Delta t_{\mathrm{ph}} &= \frac{2}{c}\int\limits_0^{z(\varepsilon(\omega)=0)} n(\omega, z)\,dz = 2\int\limits_0^{z(\varepsilon(\omega)=0)} \frac{dz}{v_{\mathrm{ph}}(\omega, z)} = \frac{L_o}{c}, \\ L_{\mathrm{gr}} &= c\,\Delta t_{\mathrm{gr}} = 2\int\limits_0^{z(\varepsilon(\omega)=0)} \frac{c\,dz}{v_{\mathrm{gr}}(\omega, z)},\end{aligned}\right\} \tag{30.11}$$

where v_{ph} is the phase velocity (21.16) and the small term $-\pi/2\omega$ has been omitted from Δt_{ph}.

The effective height of reflection z_a. Height-frequency characteristics

In normal (i.e. vertical) probing of the ionosphere, the signal reflected from the layer returns after a time Δt_{gr}, and the point of reflection $z(\varepsilon = 0)$ is not directly determined. Hence, instead of the true height $z_t = z(\varepsilon = 0)$, it is usual to define the apparent or effective height of the reflection point z_a as the height of reflection of a signal moving with the velocity of light in vacuum for a time $\frac{1}{2}\Delta t_{\mathrm{gr}}$ (the factor $\frac{1}{2}$ entering because of the two paths up and down). Thus

$$z_a(\omega) = \frac{1}{2} c \,\Delta t_{gr} = \frac{1}{2} L_{gr} = \int\limits_0^{z_t} \frac{c\,dz}{v_{gr}(\omega, z)}. \tag{30.12}$$

The effective height z_a is always greater than the true height $z_t = z(\varepsilon = 0)$, because $v_{gr} < c$. Neglecting the effect of the magnetic field, we have $v_{gr} = cn = c\sqrt{[1 - 4\pi e^2 N(z)/m\omega^2]}$ [see (21.18)], and

$$\left.\begin{aligned} z_a(\omega) &= \int\limits_0^{z_t(\omega)} \frac{dz}{n} = \int\limits_0^{z_t(\omega)} \frac{dz}{\sqrt{[1 - 4\pi e^2 N(z)/m\,\omega^2]}}; \\ &1 - 4\pi e^2 N(z_t)/m\,\omega^2 = 0. \end{aligned}\right\} \tag{30.13}$$

Equation (30.13) may be solved for z_t. The result is (see [114; 22, § 94])

$$z_t(\omega) = \frac{2}{\pi} \int\limits_0^{\pi/2} z_a(\omega \sin\chi)\, d\chi, \tag{30.14}$$

where $z_a(\omega \sin\chi)$ is the value of the function z_a (30.13) when ω is replaced by $\omega \sin\chi$.

Thus, if we know the effective height z_a as a function of ω (the carrier frequency of the signal), in this case (with $n = \sqrt{[1 - 4\pi e^2 N(z)/m\omega^2]}$) we can use (30.14) to find the true height $z_t(\omega)$. The function $z_a(\omega)$, which gives the effective or apparent height of the ionosphere as a function of frequency, is called the height-frequency characteristic of the ionosphere. This quantity is found directly at ionospheric stations by measuring the group delay time $\Delta t_{gr} = 2z_a(\omega)/c$ for signals with various carrier frequencies ω. At these stations the whole curve $z_a(\omega)$ is now recorded automatically in less than a minute [22, 23]. Fig. 30.5 shows typical height-frequency characteristics of the ionosphere at medium latitudes. As is customary, the abscissa is not the circular frequency ω but the ordinary frequency $f = \omega/2\pi$ in megacycles per second.

Experimentally, in consequence of the effect of the Earth's magnetic field, the height-frequency characteristics split into two neighbouring curves; this splitting is particularly marked near the critical frequencies, and is shown by the broken line in Fig. 30.5b. It is not shown in Fig. 30.5a, and is usually only very slight in the E layer. The effect of the magnetic field on the reflection of radio waves from the ionosphere will be further discussed in § 35.

The experimental data concerning the ionosphere and, in particular, a detailed analysis of the height-frequency characteristics are given in [22, 23]. Here we shall mention only some qualitative features of the characteristics, which are evident from Fig. 30.5. Near the critical frequency of the E layer, the height-frequency characteristic for that layer rises fairly steeply, and the same happens even more distinctly in the F_1 and F_2 layers. Moreover, near the critical frequency of the E layer $f_{cr}(E)$ the height-frequency characteristic of the F_1 layer in the summer and the F_2 layer in the winter has a bend as if the F_1 and F_2 layers were higher at these frequencies than at slightly greater frequencies. This is, of course, impossible, and shows the considerable difference between the effective and true heights.

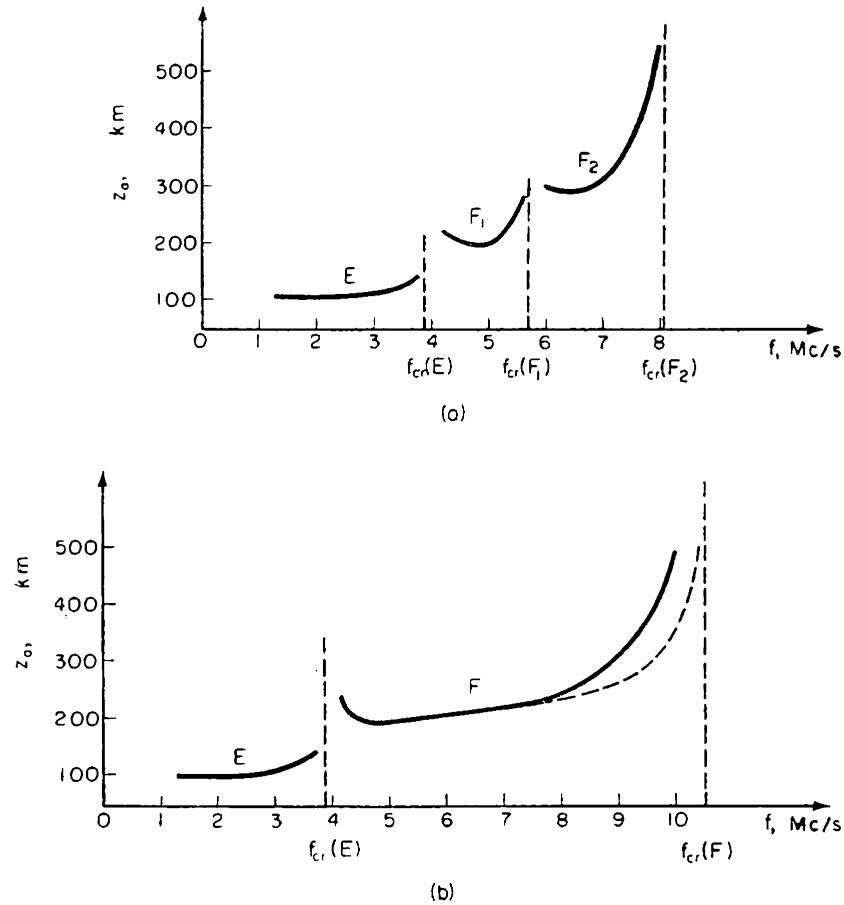

FIG. 30.5. Height-frequency characteristics of the ionosphere (diagrammatic). (a) in summer (b) in winter

The steep rise of the height-frequency characteristics near the critical frequency is explained by the fact that in this case the signal is propagated in the region near the maximum of the layer and so traverses a relatively long path in the region where the refractive index n is small. The integral (30.13) which gives the effective height z_a has n in the denominator (the group velocity $v_{gr} = cn$), and so the contribution of the region near the maximum of the layer to the value of z_a is especially large. Quantitatively this is clear from the example of a parabolic layer given below. The bend in the characteristic for the F layer at frequencies just above the critical frequency $f_{cr}(E)$ of the E layer (Fig. 30.5b) is explained by the additional delay of the signal when it passes through the E layer. Since $f > f_{cr}(E)$, the signal passes freely through the E layer, with almost no reflection, even if the difference in frequencies is only a few per cent (see § 33), but so long as f is close to f_{cr} the refractive index for the frequency f in the E layer is still considerably less than unity, and the E layer also contributes to the expression (30.13) which gives the effective height z_a for the F layer.

Near the critical frequency the difference between the effective height z_a and the true height of reflection $z_t = z(\varepsilon = 0)$ may be very great. This is seen from Fig. 30.6 [127], which shows for one particular case both the experimental curve $z_a(\omega)$ (i.e. the height-frequency characteristic) and the true

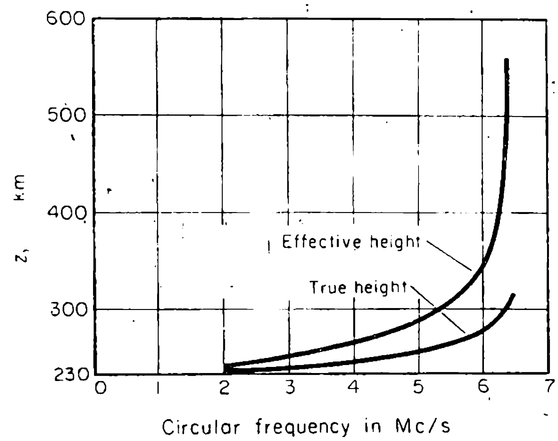

FIG. 30.6. Effective (or apparent) and true height of reflection for the F layer.

height z_t calculated from formula (30.14). It may be noted that for a linear layer the true height is half the effective height, i.e. $z_t(\omega) = \frac{1}{2} z_a(\omega)$, where of course both heights are measured from the boundary of the layer.

If we know the true height of reflection $z_t(\omega)$ we can immediately determine the distribution of electron density in the layer, since by definition we have $\varepsilon(\omega) = 0$ at $z = z_t(\omega)$, and the electron density is $N(z_t) = m\omega^2/4\pi e^2 = 3{\cdot}14 \times 10^{-10}\,\omega^2 = 1{\cdot}24 \times 10^{-8} f^2$.

A parabolic layer

It follows from the experimental data that the F_2 layer, as well as the other layers (or more precisely their lower parts), can often be well approximated by a parabolic layer [22, 23]. Moreover, the parabolic layer is a simple and yet qualitatively good approximation to an arbitrary thick smooth layer (Fig. 30.4). We shall therefore apply to the parabolic layer the general formulae derived above.

According to (30.1), the coordinates of the points of reflection for frequency f are

$$z(\varepsilon = 0) = \pm \sqrt{\frac{f_{cr}^2 - f^2}{f_{cr}^2}}\, z_m, \tag{30.15}$$

where the origin is at the maximum of the layer and the $\pm$ signs correspond to two points where $\varepsilon = 0$, lying symmetrically about this origin. If the z-axis is drawn as in Fig. 17.2 (with $z = -z_m$ at the lower boundary of the layer), the lower point $z(\varepsilon = 0)$, which is the only one of interest here, corresponds to the minus sign in (30.15). For the phase φ and the optical path length L_0 we have from (30.6), (30.11) and (30.1)

$$\left.\begin{aligned} \varphi &= 2\frac{\omega}{c}\int\limits_0^{z(\varepsilon=0)} n(z)\,dz - \frac{1}{2}\pi = \omega L_0/c - \frac{1}{2}\pi, \\ L_0 &= z_m\left[1 - \frac{f_{\mathrm{cr}}^2 - f^2}{2f f_{\mathrm{cr}}}\ln\frac{f_{\mathrm{cr}} + f}{f_{\mathrm{cr}} - f}\right]. \end{aligned}\right\} \tag{30.16}$$

The group delay time Δt_{gr}, the group path length L_{gr} and the effective (or apparent) height z_a are, from (30.10), (30.11), (30.12) and (30.1),

$$\left.\begin{aligned} \Delta t_{\mathrm{gr}} &= \varphi'(\omega) = L_{\mathrm{gr}}/c = 2z_a/c, \\ z_a &= \frac{1}{2} z_m \frac{f}{f_{\mathrm{cr}}}\ln\frac{f_{\mathrm{cr}} + f}{f_{\mathrm{cr}} - f}. \end{aligned}\right\} \tag{30.17}$$

Let us now consider what is the condition for formulae (30.16) and (30.17) to be valid in the case of a parabolic layer. This condition is evidently the same as the condition (30.9) for the validity of the fundamental formula (30.6) for the phase. Using (30.9) and (30.1), we easily find that the required condition is

$$\Delta f = f_{\mathrm{cr}} - f \gg c/3z_m, \tag{30.18}$$

or

$$\Delta f/f_{\mathrm{cr}} \sim \Delta f/f \gg \lambda_{\mathrm{cr}}/3z_m, \tag{30.19}$$

where $\lambda_{\mathrm{cr}} = c/f_{\mathrm{cr}}$ and it is assumed that $\Delta f \ll f_{\mathrm{cr}}$ [the condition (17.27)].

It will be shown in § 33 that the difference between formula (30.17) and the exact formula obtained by solving the wave equation for a parabolic layer is entirely negligible even when $\Delta f/f_{\mathrm{cr}} = \lambda_{\mathrm{cr}}/3z_m$. Thus we can essentially replace the conditions (30.18) and (30.19) by

$$\Delta f > c/3z_m, \quad \Delta f/f > \lambda_{\mathrm{cr}}/3z_m. \tag{30.20}$$

This possibility of relaxing the conditions (30.18) and (30.19) is due, in particular, to the fact that the original equation (30.9) was derived on the basis of a requirement of better than 1 per cent accuracy; but the main argument for replacing formulae (30.18) and (30.19) by (30.20) is, of course, obtained by the comparison with the exact solution for a parabolic layer.

For the F layer the half-thickness $z_m \sim 100$ km, and the condition (30.18) becomes $\Delta f \gg 10^3$ c/s. In this case, therefore, with $f_{\mathrm{cr}} \sim 10$ Mc/s $= 10^7$ c/s ($\lambda_{\mathrm{cr}} \sim 30$ m) the above formulae are valid if $\Delta f/f_{\mathrm{cr}} \gg 10^{-4}$, and in practice even for $\Delta f/f_{\mathrm{cr}} \gtrsim 10^{-4}$.

For the E layer $z_m \sim 20$ km, $f_{cr} \sim 3$ Mc/s, $\lambda_{cr} \sim 100$ m, and the conditions (30.18) and (30.19) become $\Delta f \gg 5 \times 10^3$ and $\Delta f/f_{cr} \gg 2 \times 10^{-3}$. Here formulae (30.16) and (30.17) are entirely valid even for $\Delta f/f_{cr} \gtrsim 10^{-3}$, i.e. practically always.

An arbitrary smooth layer of the type shown in Fig. 30.4 can always, except in some special cases, be approximated near the maximum by the parabola (30.1) with the appropriate value of z_m and the critical frequency f_{cr} equal to that of the actual layer. Hence the above estimate of the range of validity of the fundamental formula (30.6) and its consequences are fully applicable

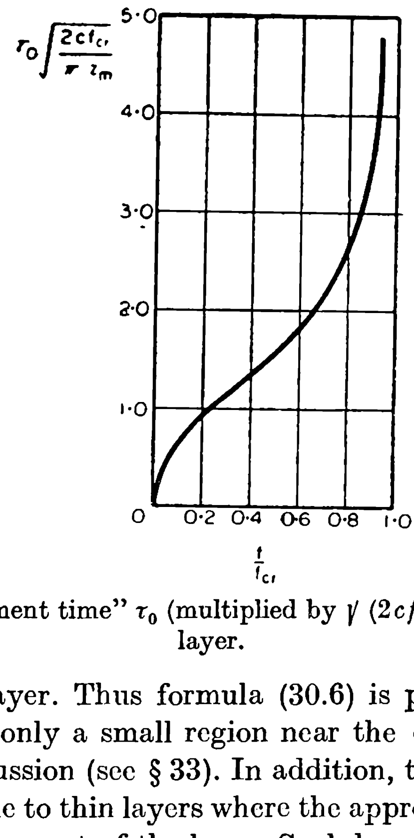

FIG. 30.7. "Establishment time" τ_0 (multiplied by $\sqrt{(2cf_{cr}/\pi z_m)}$) for a parabolic layer.

even to an arbitrary layer. Thus formula (30.6) is practically always valid. As already mentioned, only a small region near the critical frequency of the layer needs special discussion (see § 33). In addition, the results of the present section are not applicable to thin layers where the approximation of geometrical optics is invalid in all or most of the layer. Such layers may, it seems, appear sporadically in the ionosphere, and the study of them is therefore of some interest. The layer being thin, the principal theoretical problem here is to find the reflection coefficient R as a function of the frequency and the parameters of the layer. This problem has been discussed in § 18.

The spreading of the signal is characterised in accordance with (21.33) by the "establishment time" $\tau_0 = \sqrt{[\pi\varphi''(\omega_0)]}$; for a parabolic layer we have

$$\tau_0 = \sqrt{[\pi\,\varphi''(\omega_0)]} = \sqrt{\left\{\frac{\pi z_m}{2cf_{cr}}\left[\ln\frac{1+f/f_{cr}}{1-f/f_{cr}} + \frac{2f/f_{cr}}{1-(f/f_{cr})^2}\right]\right\}}. \quad (30.21)$$

Fig. 30.7 shows $\tau_0\sqrt{(2cf_{cr}/\pi z_m)}$ as a function of f/f_{cr}. As a typical example we may mention that, for $f/f_{cr} = 0{\cdot}8$ with $z_m \sim 100$ km and $f_{cr} \sim 10$ Mc/s (the

F layer) or with $z_m \sim 20$ km and $f_{cr} \sim 3$ Mc/s (the E layer) we have $\tau_0 = 10^{-5}$ s. The time τ_0 is especially large near the critical frequency; with the condition (17.27) $\Delta f \ll f_{cr}$, we have

$$\left.\begin{aligned} \tau_0 &\approx \sqrt{(z_m/2c\,\Delta f)}, \\ \Delta f &= f_{cr} - f \ll f_{cr}. \end{aligned}\right\} \tag{30.22}$$

With $z_m \sim 100$ km and $\Delta f \sim 10^4$ we have $\tau_0 \sim 10^{-4}$ sec, and so τ_0 is of the same order as the duration T of the signals customarily used in ionosphere probing ($T \sim 10^{-4}$ sec). Thus the spreading of the signal may be quite perceptible near the critical frequency. For a truncated sinusoid, i.e. the rectangular signal (21.4) and (21.5), the shape of the quasimonochromatic reflected signal is given by (21.8). If $T \gg \tau_0$ [the condition (21.30)] the shape of the front and back of the signal is as shown in Fig. 21.1 [see also formula (21.31)]. If, however, the condition (21.30) is not satisfied, as may happen near the critical frequency, then by (21.28) the shape of the signal is determined by the parameter T/τ_0, and for values of this parameter of 1, 3 and 5 is shown by Fig. 21.2 (the penetration of the wave through the layer is neglected).

Allowance for the variation of the layer with time

Until now we have always assumed that the reflecting layer undergoes no variation with time. The experimental data, however, indicate that the time variations which occur in the reflecting medium are often considerable and lead to a marked Doppler effect. Let us therefore ascertain what is the change in the frequency of the reflected wave as a function of the time variation of the refractive index $n(t) = \sqrt{\varepsilon(t)}$.

If the properties of the medium vary sufficiently slowly with time, the expression for the phase φ of the reflected wave remains unchanged in form. The dependence of φ on the time t appears only in that the function n, which determines φ, now depends on t. In such quasisteady conditions we have for reflection from a smooth layer, by (30.6),

$$\varphi(\omega_0, t) = \frac{2\omega_0}{c} \int^{z(\varepsilon(\omega_0)=0)} n(\omega_0, z, t)\, dz - \frac{1}{2}\pi, \tag{30.23}$$

where ω_0 is the frequency when the Doppler effect is neglected. The frequency ω of the oscillations is, by definition, the time derivative of the total phase of the wave, i.e. in our case, where the field of the reflected wave has the form $E = \text{constant} \times \exp\{i\omega_0 t - i\varphi(\omega_0, t)\}$, it is $\omega = \omega_0 - \partial\varphi/\partial t = \omega_0 + \Delta\omega$. When (30.23) holds,

$$\Delta\omega = -\partial\varphi/\partial t = -\frac{2\omega_0}{c} \int_0^{z(\varepsilon(\omega_0)=0)} \frac{\partial n}{\partial t}\, dz; \tag{30.24}$$

the expression (30.23) need not be differentiated with respect to the upper limit, since by definition $n = 0$ at $z(\varepsilon(\omega_0) = n^2(\omega_0) = 0)$. If reflection takes place not from the region $n = 0$ but from some boundary which is the same for all frequencies (e.g. from an ionospheric cloud), then

$$\varphi(\omega_0, t) = \frac{2\omega_0}{c}\int_0^{z_0} n\,dz,$$

where z_0 is the position of the boundary, at which $n = n(\omega_0, z_0) = n_0$. In this case

$$\Delta\omega = -\frac{2\omega_0}{c}\int_0^{z_0}\frac{\partial n}{\partial t}dz - \frac{2\omega_0}{c}n_0\frac{dz_0}{dt}. \tag{30.25}$$

The simplest pattern, which is apparently a typical one, occurs when the change in frequency is entirely due to the movement of the boundary, i.e. when the first term in (30.25) may be neglected and $z_0 = v_0 t$. Then

$$\Delta\omega = -2\omega_0 n_0 v_0/c. \tag{30.26}$$

This is the usual formula for the Doppler effect in the motion of a reflecting mirror in a homogeneous medium; it may be noted that $v_0 > 0$ if the boundary is moving upwards, while if it is moving downwards $v_0 < 0$ and $\Delta\omega > 0$, as it should be. Here it should be remembered that only the case where the boundary is perpendicular to the z-axis (the direction of the incident wave vector $\boldsymbol{k}$) is considered; in a homogeneous medium we have

$$\Delta\omega = -\frac{2\omega_0}{c}n_0\frac{\boldsymbol{v}_0\cdot\boldsymbol{k}}{|\boldsymbol{k}|} = -\frac{2\omega_0}{c}n_0 v_0\cos\theta$$

for any value of the angle θ between $\boldsymbol{k}$ and $\boldsymbol{v}_0$.

The condition for n to vary slowly with time (quasisteady-state condition) mentioned earlier is

$$\frac{1}{\omega_0}\frac{\partial\varepsilon(\omega_0, t)}{\partial t} \ll \varepsilon(\omega_0, t), \tag{30.27}$$

i.e. the change in ε during the period $T_0 = 2\pi/\omega_0$ must be much smaller than ε itself. The condition (30.27), which is fairly obvious, may also be derived by considering the propagation of waves in a homogeneous medium with $\varepsilon = \varepsilon(t)$. When (30.27) holds, the term of the form $(\partial\varepsilon/\partial t)(\partial E/\partial t)$ may be neglected in comparison with that of the form $\varepsilon\,\partial^2 E/\partial t^2$ in the relevant equation. Besides using the condition (30.27), we have everywhere used the undisplaced frequency ω_0 as the argument of n. This is, of course, possible only if

$$\Delta\omega\cdot\partial n(\omega_0)/\partial\omega_0 \ll n(\omega_0). \tag{30.28}$$

In the ionosphere, except for the scattering of radio waves by high-frequency "plasma waves", the inequalities (30.27) and (30.28) must undoubtedly hold.

§ 31. ALLOWANCE FOR ABSORPTION

The effect of absorption on reflection of waves

There is no difficulty in generalising the results of § 30 to the case where absorption is present. We could, in fact, have dealt with this case immediately, had it not been for reasons of convenience. Moreover, the absorption in the ionosphere (except in the D layer) is usually small, and in many cases it may be ignored.

The derivation of the formulae for the phase φ and the reflection coefficient R for a wave reflected by an arbitrary layer (Fig. 30.4) is similar to that given in § 30 for a layer where absorption is absent. Joining the solution of geometrical optics (16.12) near the reflection point $z(\varepsilon = 0)$ to the solution for a linear layer (17.4), (17.12)–(17.14), (17.18)–(17.20), we easily find the following expressions for an arbitrary layer with absorption:†

$$\varphi = \frac{2\omega}{c}\int_0^{z(\varepsilon=0)} n(\omega, z)\,dz - \frac{1}{2}\pi - \frac{2\sqrt{2}\,\omega[4\pi\,\sigma(0)/\omega]^{3/2}}{3c\,|d\varepsilon/dz|_0}, \tag{31.1}$$

$$-\ln R = \frac{2\omega}{c}\int_0^{z(\varepsilon=0)} \varkappa(\omega, z)\,dz + \frac{2\sqrt{2}\,\omega[4\pi\,\sigma(0)/\omega]^{3/2}}{3c\,|d\varepsilon/dz|_0}, \tag{31.2}$$

where the values of $\sigma(0)$ and $|d\varepsilon/dz|_0$ relate to the point $z(\varepsilon(\omega) = 0)$. For a layer which is linear for all z, formulae (31.1) and (31.2), of course, become (17.18) and (17.19), since then $z(\varepsilon = 0) = z_1$ and $|d\varepsilon/dz| = 1/z_1$.

Formulae (31.1) and (31.2) differ from those frequently used by the correction terms

$$\Delta(-\ln R) = -\Delta\varphi = \frac{2\sqrt{2}\,\omega[4\pi\,\sigma(0)/\omega]^{3/2}}{3c\,|d\varepsilon/dz|_0} = \frac{2\sqrt{2}[\nu_{\text{eff}}(0)]^{3/2}}{3c\sqrt{\omega}\,|d\varepsilon/dz|_0}; \tag{31.3}$$

here ν_{eff} is the effective collision frequency at the point $z(\varepsilon = 0)$, and we have used the fact that for $\varepsilon = 0$, by (3.7), $4\pi\sigma/\omega = \nu_{\text{eff}}/\omega$. The correction $\Delta(-\ln R) = -\Delta\varphi$ is usually small; for example, when $|d\varepsilon/dz|_0 = 10^{-7}\,\text{cm}^{-1}$, $\omega = 2\pi c/\lambda_0 = 1{\cdot}9 \times 10^6\,\text{sec}^{-1}$ ($\lambda_0 = 1000$ m) and $\nu(0) = 10^4\,\text{sec}^{-1}$, we have $\Delta(-\ln R) = -\Delta\varphi = 0{\cdot}23$. Thus the correction to the phase $\Delta\varphi$ is usually even less than the term $-\frac{1}{2}\pi$, which is itself small in comparison with φ (see § 30), and therefore the presence of absorption has almost no effect on the form of the expression for φ, which continues to be given by (30.6). Here, of course, allowance must be made for absorption when substituting the refractive index in (31.1), i.e. strictly speaking, we should use (7.12).

† R here denotes throughout the modulus of the amplitude coefficient of reflection, also denoted previously by $|R|$.

The correction $\Delta(-\ln R)$ to the reflection coefficient is in general small compared with $-\ln R$, but it cannot be immediately ignored. Let us therefore estimate the quantity $\Delta(-\ln R)$ for a parabolic layer (30.1), where

$$\left|\frac{d\varepsilon}{dz}\right|_0 = \frac{2 f_{cr}^2 z(\varepsilon = 0)}{f^2 z_m^2} = \frac{2\sqrt{(f_{cr}^2 - f^2)}}{f^2 z_m} f_{cr}. \tag{31.4}$$

Here we have used the fact that, if $\omega^2 \gg \nu_{eff}^2$, which holds in the F layer, the expression for ε in the presence of absorption remains the same as in the absence of absorption. Near the critical frequency,

$$\left|\frac{d\varepsilon}{dz}\right|_0 \approx \frac{2}{z_m}\sqrt{\frac{2\Delta f}{f_{cr}}}. \tag{31.5}$$

When $z_m = 100$ km, $f_{cr} = 10$ Mc/s and $\Delta f = 10^5$ c/s we have

$$|d\varepsilon/dz|_0 = 2{\cdot}8 \times 10^{-8}\,\mathrm{cm}^{-1}.$$

In the F layer $\nu \lesssim 10^4$, and so $\Delta(-\ln R) \lesssim 0{\cdot}15$ to $0{\cdot}20$, whereas in these conditions $-\ln R$ is of the order of several units, so that the correction $\Delta(-\ln R)$ is at most 10 per cent of $-\ln R$.

The range of applicability of formulae (31.1) and (31.2) is restricted by the requirement that the layer should be smooth and sufficiently thick (see § 30), and by the condition (30.2), which for relatively weak absorption reduces to (30.9) or in practice to (30.20): $\Delta f > c/3z_m$. It may be noted that the modulus signs are used in (30.2) only in order that the inequality should remain meaningful when $(d\varepsilon/dz)_0$ and $(d^2\varepsilon/dz^2)_0$ have opposite signs. When absorption is present the condition analogous to (30.2) must be applied to the real and imaginary parts of ε' separately. Hence, in addition to (30.9), we should strictly require that

$$|d^2\sigma/dz^2|_0 \Delta z \ll |d\sigma/dz|_0. \tag{31.6}$$

If σ varies with height not more rapidly than ε, as is usually the case in the ionosphere, the condition (31.6) is not more stringent than (30.9). Moreover, as already mentioned, the inequality (30.9) is derived from the condition $|d^2\varepsilon/dz^2|_0 \Delta z \ll |d\varepsilon/dz|_0$ on the assumption that the absorption is relatively weak; this is clear on account of the use of the expressions (30.3) and (30.4), which are valid in the absence of absorption.

The assumption that absorption is weak is usually correct for the ionosphere. For example, the inequality

$$\omega^2 \gg \nu_{eff}^2 \tag{31.7}$$

always holds for the F layer and usually for the E layer, and so for an isotropic plasma [see (3.9)]

$$\left.\begin{aligned} \varepsilon &\approx 1 - 4\pi e^2 N/m\omega^2, \\ \sigma &= (1-\varepsilon)\nu_{eff}/4\pi \approx e^2 N \nu_{eff}/m\omega^2. \end{aligned}\right\} \tag{31.8}$$

We always have

$$\left.\begin{aligned} n &= \sqrt{\{\tfrac{1}{2}\varepsilon + \sqrt{[(\tfrac{1}{2}\varepsilon)^2 + (2\pi\,\sigma/\omega)^2]}\}}, \\ \varkappa &= 2\pi\,\sigma/\omega\, n = \sqrt{\{-\tfrac{1}{2}\varepsilon + \sqrt{[(\tfrac{1}{2}\varepsilon)^2 + (2\pi\,\sigma/\omega)^2]}\}}. \end{aligned}\right\} \tag{31.9}$$

When $\varepsilon = 0$, i.e. at the point $z(\varepsilon = 0)$, the equations

$$n(0) = \varkappa(0) = \sqrt{[2\pi\,\sigma(0)/\omega]} = \sqrt{[\nu_{\text{eff}}(0)/2\omega]} \tag{31.10}$$

are always valid. If the condition (31.7) holds, the value of n differs only slightly from $\sqrt{\varepsilon}$ and from the value of n in the absence of absorption. For example, when $\nu(0) \sim 3 \times 10^3\,\text{sec}^{-1}$ and $f \sim 10^7$ c/s we have $n(0) \sim 10^{-2}$. For a parabolic layer (30.1) near the point $z(\varepsilon = 0)$, $\varepsilon = |d\varepsilon/dz|_0\, \Delta z$, where $\Delta z = z(\varepsilon = 0) - z$ and $|d\varepsilon/dz|_0$ is given by formulae (31.4) and (31.5). When $f_{\text{cr}} = 10^7$ and $z_m = 100$ km, even if $\Delta f = 10^5$ (the worst practical case), $\sqrt{\varepsilon} = 10^{-2}$ for $\Delta z \approx 40$ m, or $\Delta z/z_m \approx 4 \times 10^{-4}$. For larger Δz (e.g. 100 m) it is clear from the above formulae and estimates that we can in practice take $n = \sqrt{\varepsilon}$. In the F layer, therefore, except for the range of frequencies directly adjoining the critical frequency (see § 33), we can use for the wave phase φ and the effective height $z_a = \frac{1}{2}c\varphi'(\omega)$ the expressions obtained in the absence of absorption [see (30.16) and (30.17)].

The reflection coefficient when absorption is small. Determination of ν_{eff} from measurements of absorption

The smallness of the absorption also leads to a considerable simplification of the expression for $-\ln R$. Since, by (31.8) and (31.9),

$$\varkappa = 2\pi\,\sigma/\omega\, n = (1-\varepsilon)\,\nu_{\text{eff}}/2\omega\, n = (1 - n^2 + \varkappa^2)\,\nu_{\text{eff}}/2\omega\, n, \tag{31.11}$$

we have, omitting the correction $\Delta(-\ln R)$ in (31.2),

$$-\ln R = \frac{1}{c}\int\limits_0^{z(\varepsilon=0)} \nu_{\text{eff}} \frac{1 - n^2 + \varkappa^2}{n}\, dz. \tag{31.12}$$

In the F layer, where the condition (31.7) holds, equations (31.9), (31.10) and (31.11) and the above discussion of the accuracy of the formula $n = \sqrt{\varepsilon}$ give†

$$-\ln R \approx \frac{1}{c}\int\limits_0^{z(\varepsilon=0)} \frac{1-n^2}{n}\,\nu_{\text{eff}}\, dz = \frac{\bar{\nu}_{\text{eff}}}{2c}(L_{\text{gr}} - L_o). \tag{31.13}$$

---

† Strictly speaking, it is obvious that the expression (31.13) is valid if $1 - n^2 \gg \varkappa^2$. It may therefore be used only in the region where n^2 is not too close to unity. This is usually called the "deviating" region.

Another expression for $\ln R$, valid if $4\pi\sigma/\omega \ll 1$, is obtained from (17.20a): generalising the latter to the case of a smooth layer (see § 30 and [263]), we have

$$-\ln R = \frac{2\omega}{c}\int \frac{2\pi\,\sigma}{\omega\sqrt{\varepsilon}}\, dz, \quad 4\pi\,\sigma/\omega \ll 1. \tag{31.13a}$$

For $|\varepsilon| \lesssim 1$ the inequality $4\pi\sigma/\omega \ll 1$ is certainly implied by (31.7).

Here L_{gr} and L_0 are the group and optical paths (30.11):

$$L_{gr} = 2 \int\limits_0^{z(\varepsilon=0)} \frac{dz}{n}, \quad L_0 = 2 \int\limits_0^{z(\varepsilon=0)} n\, dz, \tag{31.14}$$

where it is assumed that $v_{gr} = cn$, as is true if $n = \sqrt{(1 - 4\pi e^2 N/m\omega^2)}$. The quantity $\bar{\nu}_{eff}$ is some mean value of the effective collision frequency. For the F layer the inaccuracy of formula (31.13) due to using it in place of (31.12) and neglecting the absorption in the expression for $n = \sqrt{\varepsilon}$ does not exceed a few per cent [113].

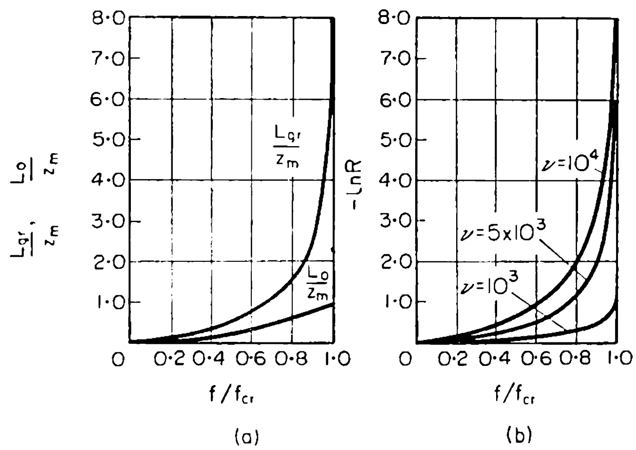

FIG. 31.1. (a) Values of L_{gr}/z_m and L_0/z_m for a parabolic layer as functions of f/f_{cr}. (b) Logarithm of the reflection coefficient for a parabolic layer with $z_m = 100$ km as a function of f/f_{cr} for a mean effective collision frequency $\bar{\nu}_{eff} = 10^3$, 5×10^3 and 10^4.

The quantity $L_{gr}(\omega) = 2z_a(\omega)$ is directly measured experimentally; near the critical frequency it increases rapidly with frequency [see (30.17) and Figs. 30.5 and 30.6], whereas L_0 increases very slowly [see (30.16)]. Hence, using the difference δL_{gr} for two frequencies f_1 and f_2 in the region where L_{gr} increases rapidly, we can put

$$\delta(-\ln R) \approx \bar{\nu}_{eff}\, \delta L_{gr}/2c, \tag{31.15}$$

where $\bar{\nu}_{eff}$ is now the average over the range of heights between the reflection points for the frequencies f_1 and f_2. If the quantity L_0 is not neglected, then, multiplying (31.13) by f, differentiating this expression with respect to frequency and using the equation $d(L_0 f) = L_{gr}\, df$ [see (21.19)], we have

$$-d(f \ln R) = \nu_{eff}\, f\, dL_{gr}/2c. \tag{31.16}$$

For a parabolic layer (30.1) the values of L_{gr} and L_0 in formula (31.13) are determined by the expressions (30.16) and (30.17); the ratios L_{gr}/z_m and L_0/z_m calculated from these formulae are shown in Fig. 31.1a. Fig. 31.1b gives the values of $-\ln R$ from formula (31.13) for $\bar{\nu}_{eff} = 10^3$, 5×10^3 and 10^4 with $z_m = 100$ km.

The measurement of the reflection coefficient R and the determination of $\bar{\nu}_{\mathrm{eff}}$ therefrom are discussed in [22, § 102; 23]; the complications which arise when the Earth's magnetic field is taken into account are considered in § 35.

In conclusion, it may be noted that in the general approximation of geometrical optics the reduction in the wave amplitude R in traversing some path l is seen from (16.11) and (19.10) to be given by

$$-\ln R = \frac{\omega}{c} \int\limits_{l} \varkappa(s)\, ds, \tag{31.17}$$

where ds is an element of the ray path and the index of absorption $\varkappa$ is determined by (31.9), or in particular cases by (7.17) or (7.20). The intensity S of the wave decreases in accordance with the formula

$$S = S_0 \exp\left(-\frac{2\omega}{c} \int\limits_{l} \varkappa(s)\, ds\right) = S_0 \exp\left(-\int\limits_{l} \mu(s)\, ds\right), \tag{31.18}$$

where μ is the absorption coefficient (7.10).

§ 32. THE FIELD STRUCTURE NEAR THE REFLECTION POINT

The field structure

In most cases only the phase and amplitude of the wave reflected from an ionospheric layer are of interest. However, in the study of a number of problems such as the non-linear interactions of radio waves in the ionosphere, it is also of importance to know the structure of the field near the reflection point. In this region, if the ratio $\Delta f/f_{\mathrm{cr}}$ is not too small (see §§ 30, 31), the layer may be regarded as linear, and so the complete solution of the problem is given by formulae (17.2), (17.4) and (17.5). (In the presence of absorption, ζ is given by formula (17.12) in place of (17.2).) Here it is more convenient, with a view to calculations for an arbitrary layer, to write ζ not in the form (17.2) but as in (30.4):

$$\zeta = \left[\frac{\omega^2}{c^2} \left|\frac{d\varepsilon}{dz}\right|_0\right]^{\frac{1}{3}} \Delta z = \left[\frac{4\pi^2}{\lambda_0^2} \left|\frac{d\varepsilon}{dz}\right|_0\right]^{\frac{1}{3}} \Delta z, \tag{32.1}$$

where Δz is the distance from the point $z(\varepsilon = 0)$, and $|d\varepsilon/dz|_0$ is the absolute value of $d\varepsilon/dz$ at that point $(|d\varepsilon/dz|_0 \equiv |(d\varepsilon/dz)_0|)$; λ_0 is of course the wavelength in vacuum.

The constant A in (17.4) and (17.5) depends on the value of the field at the boundary of the layer; for an arbitrary smooth layer and an incident wave of amplitude unity, A is given by (30.7). The form of the field near the reflection point may be made more precise by considering the properties of Bessel functions or, more simply, of the Airy integral (see, for example, [125, 126]).

In the absence of absorption the ratio $|E/A|^2$ according to (17.4) and (17.5) is represented by the continuous line in Fig. 32.1, where the abscissa is the parameter ζ (32.1). The field E is zero at the points at distances from $z(\varepsilon = 0)$ given by

$$\Delta z_{0m} = \beta_m \left(\frac{4\pi^2}{\lambda_0^2}\left|\frac{d\varepsilon}{dz}\right|_0\right)^{-\frac{1}{3}} = \beta_m \left(\frac{\omega^2}{c^2}\left|\frac{d\varepsilon}{dz}\right|_0\right)^{-\frac{1}{3}}, \tag{32.2}$$

where m is the number of the zero; $\beta_1 = 2{\cdot}338$, $\beta_2 = 4{\cdot}088$, $\beta_3 = 5{\cdot}521$, $\beta_4 = 6{\cdot}787$, $\beta_5 = 7{\cdot}944$, $\beta_{10} = 12{\cdot}8$, etc. The values of β_m can be conveniently represented as $\beta_m = (3\eta_m/2)^{\frac{2}{3}}$, where $\eta_1 = 2{\cdot}38$, $\eta_2 = 5{\cdot}61$, $\eta_3 = 8{\cdot}64$, $\eta_{10} = 30{\cdot}63$ and in general

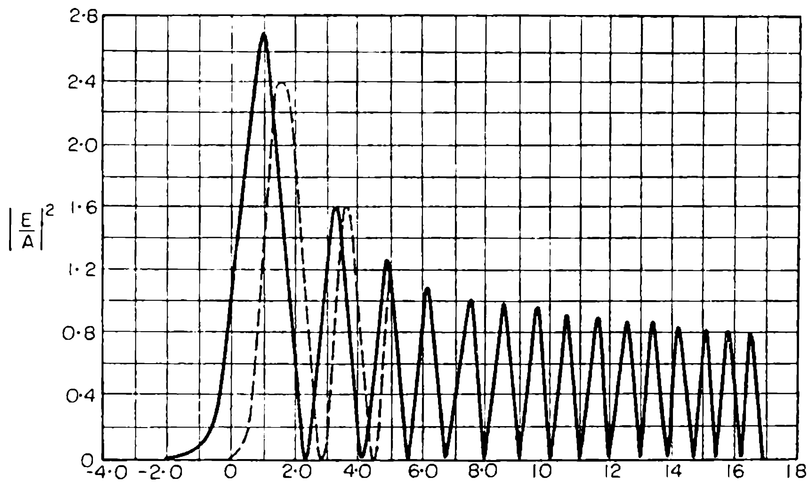

FIG. 32.1. Field structure near the reflection point (continuous line). The broken line shows the ratio $|E/A|^2$ in the approximation of geometrical optics (32.7). The parameter $\zeta = [(\omega^2/c^2)\,|d\varepsilon/dz|_0]^{1/3}\,\Delta z$.

$$\eta_m = \left(m - \frac{1}{4}\right)\pi + \frac{0{\cdot}0884194}{4m-1} - \frac{0{\cdot}08328}{(4m-1)^3} + \dots .$$

The maxima of the field E are at the points z_{Mm} given by (32.2) with β_m replaced by γ_m, where $\gamma_1 = 1{\cdot}019$, $\gamma_2 = 3{\cdot}248$, $\gamma_3 = 4{\cdot}820$, $\gamma_4 = 6{\cdot}163$, $\gamma_5 = 7{\cdot}372$.

As an example, for $\lambda_0 = 60$ m ($f = 5 \times 10^6$ c/s) and $|d\varepsilon/dz| = 10^{-7}\text{cm}^{-1}$ we have $\Delta z_{01} \approx 520$ m and $z_{M1} \approx 230$ m.

If the layer were absent and an ideal mirror were placed at the point $z(\varepsilon = 0)$, the first node of the standing wave formed by reflection would be at a distance $\frac{1}{2}\lambda_0 = 30$ m, and the first maximum at $\frac{1}{4}\lambda_0 = 15$ m from the mirror. Thus in this example the smallness of n in the region of reflection has the result that the field is "stretched out" by a factor of more than 15.

The value of $|E/A|^2$ at the first maximum is 2·68, and, since by (30.7)

$$|A|^2 = \frac{4\pi}{9}\left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{1}{3}},$$

the square of the field at the first maximum is

$$|E_{M1}|^2 = 3{\cdot}6\left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{1}{3}}. \tag{32.3}$$

Far from the reflection point we have by (30.4)

$$|E/A|^2 = (9/\pi)\,\zeta^{-\frac{1}{2}}\cos^2\left(\frac{2}{3}\zeta^{3/2} - \frac{1}{4}\pi\right), \quad \zeta \gg 1. \tag{32.4}$$

In [120] the field structure near the reflection point has been considered on the basis of the exact solution of the wave equation for a layer where

$$\left.\begin{aligned} &\text{for } z < 0 \qquad \varepsilon = \varepsilon_1 - \frac{4p^2 - 1}{4\omega^2(a - z^2)/c^2}, \\ &\text{for } z > 0 \qquad \varepsilon = \varepsilon_2 - \frac{4p^2 - 1}{4\omega^2(b + z^2)/c^2}, \end{aligned}\right\} \tag{32.5}$$

p, a and b being constants such that for $z = 0$

$$\varepsilon_1 - \frac{4p^2 - 1}{4\omega^2 a/c^2} = \varepsilon_2 - \frac{4p^2 - 1}{4\omega^2 b/c^2} = \varepsilon_0.$$

Actually, only the particular case of (32.5) where $\varepsilon_1 = \varepsilon_2 = 1$, $a = b$ was treated in [120].

The solution of the wave equation for a layer where (32.5) holds can be expressed in terms of Bessel functions of order p. If there is total reflection, then in the neighbourhood of the reflection point the layer (32.5) can be replaced by a linear layer with the same value of $(d\varepsilon/dz)_0$. The value of $|E_{M1}|^2$ then differs from (32.3) only in the third significant figure.

The approximation of geometrical optics

As has been said in § 30, despite the fact that when $\sqrt{\varepsilon} = n \to 0$ geometrical optics is not valid, the phase of the reflected wave (30.6) can be very accurately obtained from that approximation together with the condition of reflection at the point $\varepsilon = 0$. In this connection it is of some interest to consider the corresponding form of the wave field. For a standing wave with a node at the point $z(\varepsilon = 0)$ formed from the solutions (16.11) in the absence of absorption, we have (with $n = \sqrt{\varepsilon}$)

$$E = \frac{2}{\sqrt{n}}\sin\left(\frac{\omega}{c}\int_0^z n(z)\,dz\right), \tag{32.6}$$

where z is measured from the point $z(\varepsilon = 0)$ and the coefficient 2 is taken because the amplitude of the incident wave outside the layer is assumed to be unity, as in the solution (17.4) with

$$|A|^2 = \frac{4\pi}{9}\left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{\frac{1}{3}}.$$

For a linear layer, where $\varepsilon = |d\varepsilon/dz|_0\, z$, the square of (32.6) is (with Δz in place of z)

$$|E|^2 = \frac{4}{\sqrt{(|d\varepsilon/dz|_0 \Delta z)}} \sin^2 \left\{\frac{2\omega}{3c\,|d\varepsilon/dz|_0}\left(\left|\frac{d\varepsilon}{dz}\right|_0 \Delta z\right)^{3/2}\right\}$$

$$= \frac{4}{\sqrt{\zeta}}\left(\frac{\omega}{c\,|d\varepsilon/dz|_0}\right)^{1/3} \sin^2\left(\frac{2}{3}\zeta^{3/2}\right) = \frac{9}{\pi\sqrt{\zeta}}\,|A|^2 \sin^2\left(\frac{2}{3}\zeta^{3/2}\right). \quad (32.7)$$

A comparison of (32.7) and (32.4) shows, as we should expect, that for $\zeta \gg 1$ the asymptotic approximation to the exact solution differs from the solution (32.6) only by the phase $\frac{1}{4}\pi$ (see §§ 17, 30). The zeros of the function (32.7) are given by (32.2) with β_m replaced by $(3n\pi/2)^{\frac{2}{3}}$, i.e. for example, $\beta_1 \approx 2{\cdot}8$, $\beta_2 \approx 4{\cdot}5$, $\beta_{10} \approx 13{\cdot}0$. For the first zero the difference is about 20 per cent of β_m, for the second zero about 10 per cent and for the tenth zero about 2 per cent. The position and value of the first maximum of the field are now determined by formulae (32.2) and (32.3) with $\gamma_1 = 1{\cdot}019$ replaced by $\gamma_1' \approx 1{\cdot}75$ and the coefficient 3·6 replaced by 3. Thus even in the neighbourhood of the first maximum the functions (32.6) and (32.7) may furnish reasonable approximations to the exact solution in a number of problems. The ratio $|E/A|^2$ as given by (32.7) is shown by the broken line in Fig. 32.1.†

If the layer were replaced by an ideal mirror the value of $|E_M|^2$ would be 4 (the amplitude of the incident wave being unity). When the layer is present there is some increase in the field, which may be characterised by the quantity

$$\delta^2 = |E_{M1}|^2/4 = 0{\cdot}9\left(\frac{2\pi}{\lambda_0\,|d\varepsilon/dz|_0}\right)^{\frac{1}{3}}. \quad (32.8)$$

For $\lambda_0 = 60$ m and $|d\varepsilon/dz|_0 = 10^{-7}$ we have $\delta^2 \approx 20$. For the E layer, where the increase in the field may be of some interest in connection with non-linear effects, it seems that $|d\varepsilon/dz|_0 \gtrsim 10^{-7}$, and with $\lambda_0 = 1000$ m we have $\delta^2 \lesssim 8$.

Allowance for absorption

When absorption is present, the structure of the field may again be investigated by means of formula (17.4), but with ζ given by (17.12). The main

† It may be noted that, whereas the electric field is "peaked" in the neighbourhood of small values of n, the magnetic field decreases there. This is immediately clear from the fact that, for example, in a travelling wave in the absence of absorption the energy flux is constant, and therefore EH = constant. The same conclusion is reached from (32.6): if the field E is in the x-direction, then H is in the y-direction, and

$$H_y = i\frac{c}{\omega}\,\mathrm{curl}_y \boldsymbol{E} = i \cdot 2\sqrt{n}\cos\left(\frac{\omega}{c}\int\limits_0^z n(z)\,dz\right).$$

difference in the resulting expressions for $|E_{M1}|^2$ and δ^2 is that the amplitude coefficient of reflection R appears as a factor. In addition, the numerical coefficients in (32.2) and (32.3) are somewhat changed and depend on the amount of absorption. For example, if

$$\frac{4\pi\sigma(0)}{\omega} = \frac{\nu_{\text{eff}}(0)}{\omega} = \frac{1}{2}\left(\frac{|d\varepsilon/dz|_0}{\omega/c}\right)^{\frac{2}{3}}, \tag{32.9}$$

then

$$\left.\begin{aligned} z_{M1} &\approx 1{\cdot}22\left(\frac{4\pi^2}{\lambda_0^2}\left|\frac{d\varepsilon}{dz}\right|_0\right)^{-\frac{1}{3}}, \\ |E_{M1}|^2 &\approx 6R\left(\frac{2\pi}{\lambda_0\,|d\varepsilon/dz|_0}\right)^{\frac{1}{3}}. \end{aligned}\right\} \tag{32.10}$$

Putting $|d\varepsilon/dz|_0 = 10^{-7}$, $\lambda_0 = 1000$ m and $R = \frac{1}{8}(-\ln R \approx 2)$, we obtain $\delta^2 = 1{\cdot}6$. In this example, according to (32.9), $\nu_{\text{eff}} \approx 1{\cdot}3 \times 10^4$, i.e. even less than usually occurs in the E layer. Thus this example shows that the effect of the "peaking" of the field is in general simply that the field in the ionosphere can be estimated from the formulae derived for the case where the ionospheric layer is replaced by an ideal mirror (i.e. a totally reflecting boundary surface). The "peaking" approximately compensates the attenuation of the field by absorption. It need be "explicitly" taken into account only when the absorption and the derivative $|d\varepsilon/dz|_0$ are both very small. The effect of inhomogeneities near the reflection point on the "peaking" of the field in that region is discussed in [209].

§ 33. REFLECTION AND PENETRATION OF WAVES WITH NEARLY THE CRITICAL FREQUENCY IN A LAYER

A parabolic layer

It has already been mentioned in § 30 that the reflection of waves from a layer in the frequency range near the critical frequency needs special consideration. For example, in a parabolic layer the formulae of § 30, which were derived by linearising the layer near the reflection point, are valid only if

$$\Delta f = f_{\text{cr}} - f > c/3z_m. \tag{33.1}$$

For frequencies which do not satisfy this condition, the layer cannot be regarded as linear in the reflection region, but it may usually be approximated by a parabolic layer. For a layer with a maximum, however, and in particular for a parabolic layer, there is some penetration of waves through the layer near that maximum (i.e. for small Δf). It has been shown in § 17 that the coefficient of reflection of a wave from a parabolic layer without absorption is, for $\Delta f \ll f_{\text{cr}}$, given by

$$\left.\begin{aligned} R^2/(1 - R^2) &= \exp\,(4\pi^2 z_m \Delta f/c), \\ D^2 &= 1 - R^2; \end{aligned}\right\} \tag{33.2}$$

see (17.28). Here and below $R \equiv |R|$. It is clear from (33.2) that the condition (33.1) is also the condition for penetration to be small (i.e. for the transmission coefficient D^2 to be small), since

$$D^2 \leqq 3 \times 10^{-6} \quad \text{for} \quad \Delta f \geqq c/3z_m. \tag{33.3}$$

Thus the requirements for penetration to be small and for the layer to be replaceable by a linear one in the reflection region are the same. The estimates made in § 30 show that for ionospheric layers the region of penetration is negligibly small; this is clear also directly from (33.3), since for the F layer with $z_m = 100$ km we have $D^2 = 3 \times 10^{-6}$ for $\Delta f = 10^3$.

The function $R(\Delta f)$ from formula (33.2) is shown in Fig. 33.1 for layers with $z_m = 20$ km and $z_m = 100$ km.

In a strictly wave treatment of reflection from a layer, the critical frequency f_{cr} is in no way distinctive as regards the coefficient R, since reflection occurs both for $f < f_{cr}$ and for $f > f_{cr}$. However, the steep increase of the function $R(f)$ has the result that in practice we may usually assume that $R = 1$ for $f < f_{cr}$ and $R = 0$ for $f > f_{cr}$. In this case the critical frequency $f_{cr} = \sqrt{(e^2 N_{max}/\pi m)}$, i.e. the frequency at which the point $z(\varepsilon = 0)$ reaches the maximum of the layer, evidently has the physical significance of being the limiting frequency for reflection of waves from the layer.

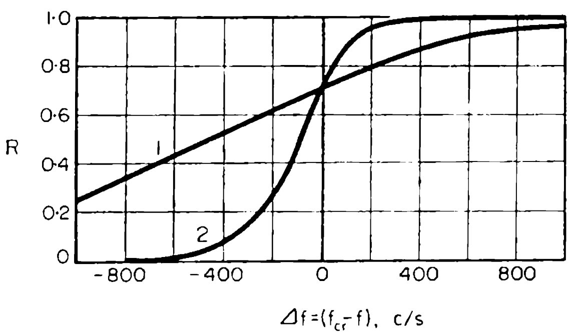

FIG. 33.1. Reflection coefficient $|R| \equiv R$ for parabolic layers with half-thickness $z_m = 20$ km (curve 1) and 100 km (curve 2) in the absence of absorption.

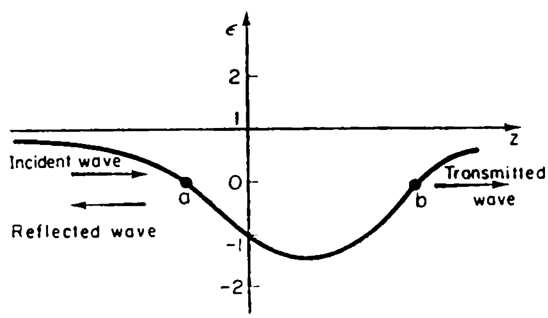

FIG. 33.2. Permittivity ε for a smooth layer with one maximum.

An arbitrary layer

In connection with the possibility of the penetration of waves through a layer we may mention the possibility of deriving an expression for the transmission coefficient $D^2 = 1 - R^2$ for a fairly arbitrary layer, provided that it is such that

$$|D| \ll 1. \tag{33.3a}$$

Let us imagine a layer of the type shown in Fig. 33.2, for which the permittivity $\varepsilon(\omega) = 0$ at $z = a$ and $z = b$. The region $a \leqq z \leqq b$ is assumed to be so large that the wave is strongly damped in traversing it, so that the condition (33.3a) is in fact fulfilled. We can suppose that $R \approx 1$, and so a standing wave will occur on the downward side, to a first approximation the same as in the case of total reflection. Hence, according to (30.5), far from the layer, where

$$a - z \gg \left(\frac{c^2}{\omega^2 |d\varepsilon/dz|_a}\right)^{\frac{1}{3}},$$

we have

$$E_1 = \frac{2C_1}{\varepsilon^{\frac{1}{4}}} \cos\left(\frac{\omega}{c}\int\limits_z^a \sqrt{\varepsilon}\, dz - \frac{1}{4}\pi\right). \tag{33.4}$$

In the region $a \leqq z \leqq b$ the wave is damped and, when z is considerably less than b but

$$z - a \gg \left(\frac{c^2}{\omega^2 |d\varepsilon/dz|_a}\right)^{\frac{1}{3}},$$

we find (to the same approximation)

$$\begin{aligned} E_2 &= \frac{C_1}{|\varepsilon|^{\frac{1}{4}}} \exp\left(-\frac{\omega}{c}\int\limits_a^z \sqrt{|\varepsilon|}\, dz\right) \\ &= \frac{C_1}{|\varepsilon|^{\frac{1}{4}}} \exp\left(-\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\, dz + \frac{\omega}{c}\int\limits_z^b \sqrt{|\varepsilon|}\, dz\right). \end{aligned} \tag{33.5}$$

The expression (33.5) is the asymptotic form (in the region $\varepsilon < 0$) of the exact solution (17.4) and (17.5), generalised to a layer which is arbitrary far from the point $z(\varepsilon = 0)$. It is important to emphasise that (33.4) and (33.5) involve the same constant C_1, for which purpose the factor 2 is included in (33.4); see [126]. The form (33.5) for the solution in the region $a < z < b$ is correct when the wave reflected from the region $z \approx b$ (or, as we usually say, from the point $z = b$) is neglected. If the region of negative ε is large, as it is if (33.3) holds, the wave reflected from the point $z = b$ would in fact play no part away from that point even if the reflection coefficient were fairly large (which it is not), simply because the factor

$$\exp\left(-\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\, dz\right)$$

is small.

For $z > b$ there is only a transmitted wave of low intensity, and when

$$z - b \gg \left(\frac{c^2}{\omega^2 |d\varepsilon/dz|_b}\right)^{\frac{1}{3}}$$

the field is

$$E_3 = \frac{C_2}{\varepsilon^{\frac{1}{4}}} \exp\left(-i\frac{\omega}{c}\int\limits_b^z \sqrt{\varepsilon}\, dz + \frac{1}{4}\pi i\right). \tag{33.6}$$

To find the transmission coefficient $|D|^2 \equiv D^2 \equiv |C_2/C_1|^2$ (we assume that $\varepsilon = 1$ for $z \to \pm\infty$), it is evidently necessary to find a relation between C_2 and C_1. Although it is not permissible to continue the solutions (33.5) and (33.6) to the point b, if we do so and put there $E_3 = E_2$, we have

$$C_2 = C_1 \exp\left(-\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\, dz\right).$$

The same result in terms of moduli, i.e.

$$|C_2|^2 = |C_1|^2 \exp\left(-2\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\, dz\right),$$

is evident from considerations of energy. The wave reflected into the layer from the point $z = b$ may be neglected (see above). This means that the region $z > b$ receives practically all the energy which reaches the point $z = b$ from below ($z < b$). The diminution of the energy flux at $z = b$ relative to that at $z = a$ is given to a close approximation by the factor

$$\exp\left(-2\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\, dz\right)$$

if the exponent is much less than -1.

Finally, a relation between C_2 and C_1 which differs from the above only by a factor $-i$ can be derived from the following rigorous arguments. Any solution of the wave equation is equal to the sum of two linearly independent solutions u and v: $E = \alpha v(z) + \beta u(z)$. Making the layer linear near the point b, we can write the two solutions u and v explicitly: one is (17.4)–(17.5) and the other is the same with the opposite sign of $J_{\frac{1}{3}}$ and $I_{\frac{1}{3}}$ in (17.4) (see [126]). At the same time (33.5) and (33.6) are the desired solutions of the problem on either side of the point b and at a sufficient distance from that point. Hence

$$E_2 = \alpha v(z \ll b) + \beta u(z \ll b),$$
$$E_3 = \alpha v(z \gg b) + \beta u(z \gg b),$$

where, as is clear from the preceding discussion, the inequalities signify that

$$|z - b| \gg \left(\frac{c^2}{\omega^2\, |d\varepsilon/dz|_b}\right)^{\frac{1}{3}}.$$

When these conditions hold, the functions u and v may be replaced by the well-known asymptotic forms (see (17.6) and [126]). Thus it is easily shown from (33.5) and (33.6) that

$$\alpha = C_2, \quad \beta = i\,C_2 = C_1 \exp\left(-\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\,dz\right),$$

and so

$$C_2 = -\,i\,C_1 \exp\left(-\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon|}\,dz\right) \tag{33.7}$$

and

$$D^2 = \exp\left(-\,2\frac{\omega}{c}\int\limits_a^b \sqrt{|\varepsilon(z,\omega)|}\,dz\right). \tag{33.8}$$

For the parabolic layer (30.1) we must evidently take a and b to be $z(\varepsilon = 0) = \pm\sqrt{[(f_{\rm cr}^2 - f^2)/f^2]}\,z_m$ [see (30.15)]. Evaluating for this case the integral in (33.8), we obtain

$$D^2 = \exp\left(-\frac{4\pi^2}{\lambda_{\rm cr}}\,\frac{f_{\rm cr}^2 - f^2}{2f_{\rm cr}^2}\,z_m\right) = \exp(-\,4\pi^2 z_m\,\Delta f/c)\,, \tag{33.9}$$

where $f_{\rm cr}^2 - f^2 = 2f_{\rm cr}\Delta f$ if $\Delta f = f_{\rm cr} - f \ll f_{\rm cr}$.

The expression (33.9) with the condition (33.3) is, as we should expect, identical with formulae (17.25) and (17.28) or (33.2), derived by solving the wave equation for the parabolic layer (30.1). Using (33.8), we can determine the extent of penetration for a layer which is (within wide limits) arbitrary.

Quantitatively, formula (33.8), which was derived in [6], is valid only if the condition $D^2 \ll 1$ used in obtaining it holds good. The method of phase integrals gives [124] the more general formula†

$$\left.\begin{aligned} |D|^2 &\equiv D^2 = e^{-2\delta_0}/(1 + e^{-2\delta_0})\,,\\ |R|^2 &\equiv R^2 = 1/(1 + e^{-2\delta_0})\,,\\ 2\delta_0 &= i\frac{\omega}{c}\oint \sqrt{\varepsilon(z)}\,dz\\ &= 2\frac{\omega}{c}\int\limits_a^b \sqrt{\varepsilon}\,dz. \end{aligned}\right\} \tag{33.10}$$

Here the path of integration in the contour integral encloses both zeros of the function $\varepsilon(z)$, and in transforming to an integral along the z-axis we assume

† The purpose of the identities $|D|^2 \equiv D^2$ and $|R|^2 \equiv R^2$ is merely to emphasise that here the symbols D^2 and R^2 always denote the squared moduli of the amplitude coefficients D and R.

that the zeros of ε lie on the real axis, i.e. that $f < f_{cr}$. When $D^2 \ll 1$, formula (33.10) for D^2 becomes (33.8). Formulae (33.10) hold also for $f > f_{cr}$ if we put $2\delta_0 = -i(\omega/c)\oint\sqrt{\varepsilon(z)}dz$ with the integration taken along a contour enclosing the zeros of the function $\varepsilon(z)$, which in this case lie on the imaginary axis. For a parabolic layer, the expressions (33.10) are the same as (17.25), which was obtained by an exact solution of the problem.

Thus the method of phase integrals in this case leads to exact results for $|D|^2$ and $|R|^2$ even for thin layers, whereas for thin parabolic layers it does not lead to the correct values for the phases of the reflected and transmitted waves. For an arbitrary smooth layer with one maximum, formulae (33.10) cannot in general be regarded as exact for all layer thicknesses, but their range of validity is wider than that of (33.8). The calculation of the small reflection coefficient for a smooth layer and wave frequencies considerably above the critical value may be treated by using the results of [215]; for a parabolic layer, the solution is again given by (17.25).

Allowance for absorption

Hitherto we have neglected absorption. Its effect when the condition (33.1) holds has been examined in § 31. In the immediate neighbourhood of the critical frequency the condition (33.1) does not hold, and formulae (33.2)

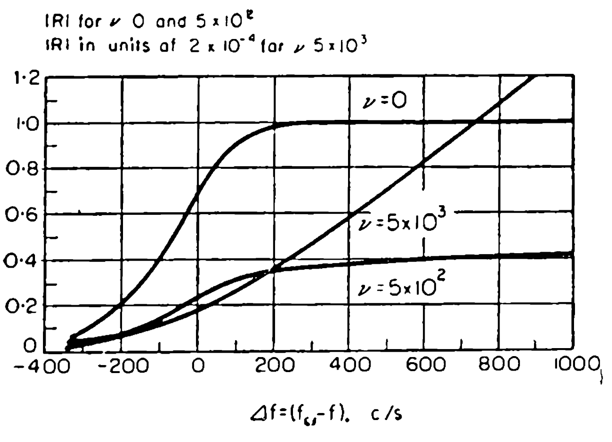

FIG. 33.3. Reflection coefficient $|R|$ for a parabolic layer with $z_m = 120$ km, $\lambda_{cr} = c/f_{cr} = 30$ m and various values of the collision frequency $\nu_{eff} \equiv \nu$.

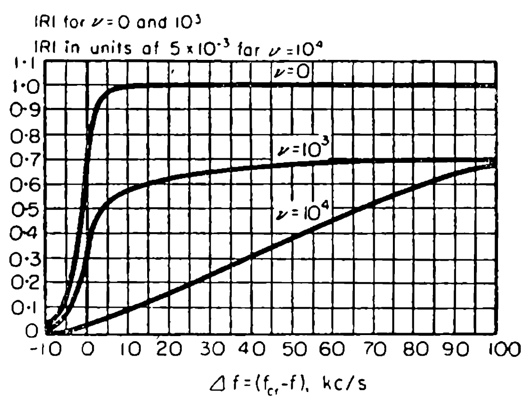

FIG. 33.4. Reflection coefficient for $z_m = 6$ km and $\lambda_{cr} = 90$ m.

and (33.8) must be generalised to the case where absorption is present. For formula (33.2) this has been done in [127]; the process is very complex. Since the penetration of waves is appreciable only in a negligible range of frequencies, we shall merely give graphs of R in various cases (Figs. 33.3 and 33.4).

An expression of the type (33.8) can also be obtained when absorption is taken into account. Without giving the details of the derivation, we may mention that in this case, as is easily seen, in order of magnitude we have

$$D^2 \approx R_0 \exp\left(-2\frac{\omega}{c}\int\limits_a^b \varkappa\, dz\right). \qquad (33.10\,\mathrm{a})$$

Here R_0 is the modulus of the amplitude coefficient of reflection (from the whole layer) in the approximation where penetration is neglected, and $\varkappa$ is the index of absorption (31.9), which for $\sigma = 0$ equals $\sqrt{|\varepsilon|}$. The field at the point $z = a$ differs from (33.4) by a factor

$$\sqrt{R_0} \approx \exp\left(-\frac{\omega}{c}\int\limits_0^a \varkappa\, dz\right),$$

since the coefficient R_0 corresponds to a twofold passage of the wave through the medium. In the absence of absorption, formula (33.10a) becomes (33.8), and the sign $\approx$ is used because a more exact calculation may lead to slight additional corrections, as happens, for example, with regard to the term $\Delta(-\ln R)$ in (31.2). In the presence of absorption the penetration is less important than in the absence of absorption, since the observed reflection coefficient R is reduced on account of absorption and, moreover, in the region of the layer where penetration begins, the wave field is already weakened by absorption [the factor R_0 in (33.10a)].

The method of phase integrals, with allowance for absorption, leads to formulae (33.10) with $2\delta_0$ replaced by $2\delta = i(\omega/c)\oint\sqrt{\varepsilon'(z)}\,dz$, the contour of integration enclosing the zeros of the function $\varepsilon'(z)$.

The effective height for a parabolic layer (exact solution)

Near the maximum of the layer, when the condition (33.1) does not hold, not only is there penetration but, as has been mentioned several times, the formulae for φ, L_o, L_{gr} and z_a given in § 30 are invalid. This is particularly clear as regards the expression (30.17) for z_a, according to which the effective (or apparent) height $z_a \to \infty$ as $f \to f_{\mathrm{cr}}$. In reality, the effective height tends to a finite limit. In § 17 an expression has been given (17.24) for the phase of the wave reflected from a parabolic layer. By means of this expression we can derive [114, 127] the following formula for z_a in the case of the parabolic layer (30.1):

$$z_a = \frac{1}{2}c\,\varphi'(\omega) = \frac{z_m f}{2f_{\mathrm{cr}}}\left[0{\cdot}5772 + \ln\left(16\pi\frac{z_m}{\lambda_{\mathrm{cr}}}\right) + \right.$$
$$\left. + \varrho\,\mathrm{re}\sum_{n=1}^{\infty}\frac{2\varrho - 3in}{n(n+i\varrho)(n+2i\varrho)}\right], \qquad (33.11)$$

where the term $-2\varrho/u^2$, which is usually very small, has been omitted, and the parameters ϱ and u are given by (17.22). For $f = f_{\mathrm{cr}}$ the height z_a has its maximum value

$$z_a(f = f_{cr}) = \frac{1}{2} z_m \left[0{\cdot}5772 + \ln\left(16\pi \frac{z_m}{\lambda_{cr}}\right)\right]. \tag{33.12}$$

When the condition (30.19) holds (which is equivalent to

$$\varrho = \pi(z_m/\lambda_{cr})(f_{cr}^2 - f^2)/f_{cr}^2 \gg 1),$$

the expression (33.11) becomes (30.17), i.e.

$$z_a = \frac{1}{2} z_m \frac{f}{f_{cr}} \ln \frac{f_{cr} + f}{f_{cr} - f} \qquad (\varrho \gg 1). \tag{33.13}$$

It may be shown that

$$\frac{(z_a)_{\text{exact}}}{(z_a)_{\text{approx}}} \to 1 \quad \text{for} \quad \frac{z_m}{\lambda_{cr}} \to \infty, \tag{33.14}$$

where $(z_a)_{\text{exact}}$ and $(z_a)_{\text{approx}}$ are given by (33.11) and (33.13) respectively.

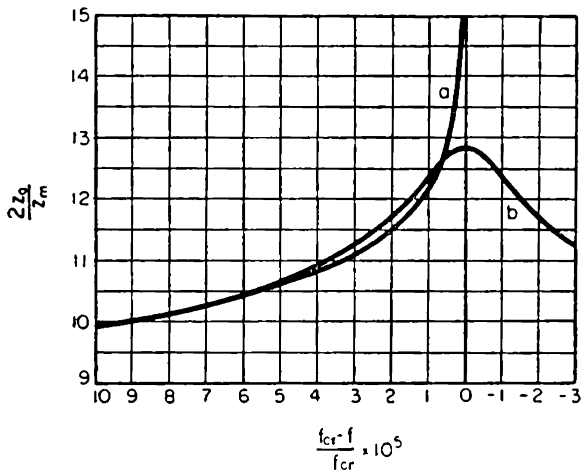

FIG. 33.5. Height-frequency characteristic for a parabolic layer (z_m = 120 km, λ_{cr} = 30 m) near the critical frequency.
(a) approximation of geometrical optics
(b) exact solution

Fig. 33.5 shows the function $2z_a/z_m$ as given by formulae (33.11) and (33.13) for z_m = 120 km and λ_{cr} = 30 m (z_m/λ_{cr} = 4000). It is seen that the two formulae give indistinguishable results even for $\Delta f/f_{cr} = 10^{-4}$, i.e. $\Delta f = 10^3$.

The function (33.11) for $f \approx f_{cr}$ depends on the frequency only through the parameter $\varrho = 2\pi z_m \Delta f/c = 2\pi z_m \Delta f/\lambda_{cr} f_{cr}$. We may therefore say, on the basis of Fig. 33.5, that the difference between formulae (33.11) and (33.13) is negligible for $\varrho > 2$ or if Δf satisfies the condition (33.1). This proves the validity of the latter condition as a criterion for the replacement of the layer by a linear one in the region of reflection, as has been done in § 30. The ratio $2z_a/z_m$ [see (33.11)] depends on z_m/λ_{cr} as well as on ϱ. The general form of the curves of $z_a(\varrho)$ for different z_m/λ_{cr} is therefore similar but not identical. The curves of $z_a(\Delta f/f_{cr})$, however, depend only on the ratio z_m/λ_{cr}. Fig. 33.6 gives the values of $2z_a/z_m$ and the reflection coefficient R for a layer with $z_m = 12$ km and $\lambda_{cr} = 30$ m, i.e. $z_m/\lambda_{cr} = 400$. Figs. 33.3–33.8 are taken from [127].

In the presence of absorption, as has been mentioned in § 31, the value of z_a is usually only very slightly different, since in the F layer $\nu_{eff} \ll \omega$. The most marked change in z_a owing to absorption is found near the critical

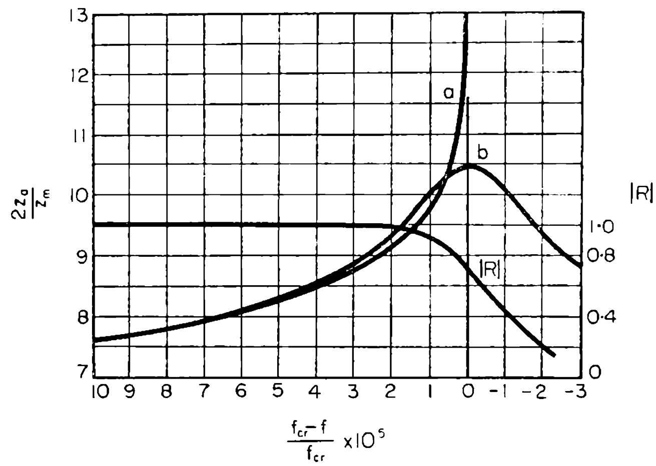

FIG. 33.6. Height-frequency characteristic and reflection coefficient $|R|$ for a parabolic layer ($z_m = 12$ km, $\lambda_{cr} = 30$ m).
(a) approximation of geometrical optics
(b) exact solution

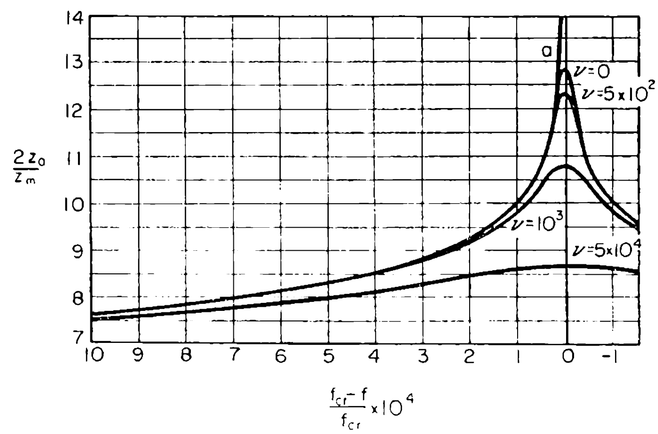

FIG. 33.7. Height-frequency characteristics for a parabolic layer ($z_m = 120$ km, $\lambda_{cr} = 30$ m) with various values of the collision frequency $\nu_{eff} \equiv \nu$.
(a) approximation of geometrical optics for $\nu = 0$
(other curves) exact solutions

frequency, since in this case the region where $n = \sqrt{\varepsilon}$ is small is especially wide [see (30.13)]. The values of $2z_a/z_m$ for a parabolic layer with various constant values of ν_{eff} are shown in Figs. 33.7 and 33.8. It is clear from

Figs. 33.5–33.8 that the ratio z_a/z_m takes large values near the critical frequency.

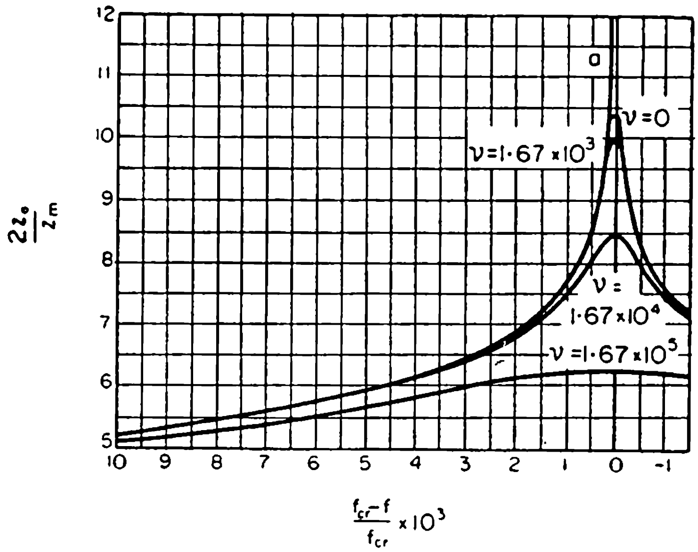

FIG. 33.8. Height-frequency characteristics for a parabolic layer (z_m = 36 km, λ_{cr} = 90 m) with various values of $\nu_{eff} \equiv \nu$.
(a) approximation of geometrical optics for $\nu = 0$
(other curves) exact solutions

The time to establish the signal amplitude

The curves in Figs. 33.5–33.8 give essentially the group delay time $\Delta t_{gr} = 2z_a/c$ as a function of the carrier frequency of the signal. It should now be noted that near the critical frequency the time τ_0 needed to establish the signal amplitude is particularly large [see (30.22)]. For example, when $z_m = 120$ km and $\Delta f = 10^2$, the difference between $(z_a)_{exact}$ and $(z_a)_{approx}$ is large (Fig. 33.5), and $\tau_0 = 10^{-3}$ sec, which is ten times the duration of the signals generally employed. Moreover, the formulae of § 21 have been derived on the assumption that the coefficient $R(\omega)$ is constant or at least varies only slightly over the spectral width of the signal. Near the critical frequency this condition is obviously not fulfilled, since when penetration occurs R depends greatly on the frequency [see (33.2)]. Hence, even if we neglect the spreading of the signal, its shape varies on account of the different penetration of different frequencies appearing in the Fourier expansion of the signal, and the concept of the group velocity is not immediately applicable. It is clear from Fig. 33.3 that in the absence of absorption, for a layer with $z_m = 120$ km and $\lambda_{cr} = 30$ m, the coefficient R depends greatly on frequency over a range of $\sim$200 c/s near f_{cr}. In our example, therefore, with $\Delta f = f_{cr} - f = 10^2$, f being the carrier frequency of the signal, in order that the formulae of § 21 and the usual concept of group velocity should be valid it is certainly necessary that the spectral width δf of the signal should be much less than 200 c/s, i.e. the duration of the signal should be $T \sim 2\pi/\delta f \gg 3 \times 10^{-2}$ sec. In this case $\tau_0 \ll T$ also.

Thus in the present example the height z_a corresponding to (33.11) and the curves in Figs. 33.5–33.8 can be determined experimentally only by using very long signals with $T \gtrsim 0{\cdot}1$ sec and making the measurement in the middle of the signal; the establishment time is $\gtrsim 10^{-3}$ sec, giving $\Delta z_a \geqq 300$ km, so that the establishment time cannot be neglected even for long signals.

§ 34. REFLECTION OF OBLIQUELY INCIDENT WAVES

The reflection point. The critical frequency

On the basis of the results of § 19 it is not difficult to deal with the reflection of radio waves from an ionospheric layer in the case of oblique incidence by analogy with the discussion in § 30 for normal incidence. For example, in the absence of absorption the phase of the reflected wave for oblique incidence is given by formula (19.15), where the function $n(z)$ may be regarded as arbitrary with the same reservations as in § 30. The reflection of the wave takes place at $z = z_{\text{refl}}$, where [see (19.12)]

$$n(z_{\text{refl}}) = \sin\theta_0, \quad \varepsilon(z_{\text{refl}}) = n^2(z_{\text{refl}}) = \sin^2\theta_0; \tag{34.1}$$

here it is assumed that at the boundary of the layer $n(0) = 1$, and θ_0 is the angle of incidence of the wave on the layer, as shown in Fig. 19.1. For $z > z_{\text{refl}}$ the wave field is exponentially damped, and the point $z = z_{\text{refl}}$ is distinguished by the fact that the coefficient of the function F in the wave equation (19.6) is zero there. (For a wave in which the vector $\boldsymbol{E}$ lies in the plane of incidence, according to (19.22) the corresponding coefficient is zero for $z < z_{\text{refl}}$, but the difference is usually negligible in ionospheric conditions.) For normal incidence $\sin\theta_0 = 0$ and reflection occurs when $n(z_{\text{refl}}) = 0$.

If $n^2 = 1 - 4\pi e^2 N(z)/m\omega^2$, it is clear from (34.1) that, at a given height z, where $N = N(z)$, reflection occurs for waves of higher frequencies at oblique incidence than at normal incidence, and

$$f_{\text{obl}} = f_{\text{nor}}/\cos\theta_0, \quad n(f_{\text{nor}}) = 0, \quad n(f_{\text{obl}}) = \sin\theta_0. \tag{34.2}$$

Accordingly, if the critical frequency of the layer is f_{cr} (i.e. if for normal incidence the point where $n(f_{\text{cr}}) = 0$ is at the maximum of the layer), then the critical frequency $f_{\text{cr,obl}}$ for oblique incidence is

$$f_{\text{cr,obl}} = f_{\text{cr}}/\cos\theta_0. \tag{34.3}$$

The ray treatment

The ray treatment of the propagation and reflection of radio waves obliquely incident on ionospheric layers is of very great importance, especially in practice. The most general concept of a ray is related to the consideration of the propagation of signals (i.e. pulses, wave groups or wave packets) bounded in space and time. The path of the "centre of gravity" of such a signal, provided that it is not greatly spread and distorted in passing through the medium, is the ray path. In any homogeneous medium, and in an inhomogeneous medium in the approximation of geometrical optics, the tangent to the signal path coincides with the group-velocity vector, as shown in § 24. Moreover, in a homogeneous or quasihomogeneous medium the direction of the group-velocity vector is the same as that of the time average energy flux vector $\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{H}/4\pi$ (§ 24). In an isotropic medium, the direction of the group-velocity vector is in turn the same as that of the wave vector $\boldsymbol{k}$, i.e. a vector normal to the wave front, which in the present case is (see § 19)

$$\left.\begin{aligned} &k_x = 0, \quad k_y = (\omega/c)\, n(0) \sin\theta_0 = (\omega/c)\, n(z) \sin\theta, \\ &k_z = (\omega/c) \sqrt{[n^2(z) - n^2(0) \sin^2\theta_0]} = (\omega/c)\, n(z) \cos\theta, \\ &k^2 = (\omega^2/c^2)\, n^2(z). \end{aligned}\right\} \tag{34.4}$$

Thus, in the case of oblique incidence of a signal on a plane isotropic ionospheric layer with $\varepsilon = \varepsilon(z)$, when the plane of incidence is taken to be the yz-plane, the direction of the ray at every point is given by (34.4). This means that we have for the angle θ between the tangent to the ray and the z-axis

$$\left.\begin{aligned} \sin\theta &= n(0) \sin\theta_0/n(z), \\ \cos\theta &= \sqrt{[n^2(z) - n^2(0) \sin^2\theta_0]}/n(z). \end{aligned}\right\} \tag{34.5}$$

The equation of the ray path is given by (19.17).

Near the reflection point, where $\sin\theta \approx 1$ (for $z = z_{\text{refl}}$, according to (34.1) and (34.5), $\sin\theta = 1$), geometrical optics is invalid, and so the expressions (34.4) and (34.5) cannot be used to determine the direction of the ray. To find the ray path in the region where reflection occurs we must investigate the motion in that region of a wave packet consisting of two adjacent solutions of the wave equation (19.2)–(19.3); for a linear layer, the corresponding solutions can be expressed in terms of Bessel functions or Airy functions. In the general case the wave packet may be markedly spread and distorted, and so a purely ray treatment is not in general suitable in the region where geometrical optics is invalid.

In the ionosphere, however, the region near the "reflection point" where the exact solution of the wave equation must be used is very small, as has several times been mentioned. A still more important circumstance is that we are usually interested only in the direction of the ray when the pulse leaves the layer. In this case, as will be seen shortly, the direction of the ray is correctly given by the approximation of geometrical optics, provided only that this approximation is applicable, as it usually is, to determine the phase of the wave reflected from the layer.

To prove this statement, and to find the point of arrival on the ground of the ray reflected from the ionosphere, let us consider the following problem. At some point $Q(0, y_1, 0)$ on the ground ($z = 0$) let a narrow beam of rays, i.e. a wave packet with wave vectors $\boldsymbol{k}$ lying close to some "carrier" wave vector $\boldsymbol{k}_0$, be emitted into the ionosphere. For simplicity, we shall assume that the duration of this pulse is infinite, i.e. that for all directions $\boldsymbol{k}/k$ the emission is monochromatic with frequency ω. The field of the incident signal (at $z = 0$) can then be written as $\boldsymbol{E}_1 = \int \boldsymbol{g}(\boldsymbol{k}) \exp(i\omega t - i k_y y_1) d\boldsymbol{k}$, where $\boldsymbol{g}(\boldsymbol{k})$ is a function having a sharp maximum at $\boldsymbol{k} = \boldsymbol{k}_0$. The field of the reflected signal at the point P, with the coordinates $(0, y, 0)$, is

$$\boldsymbol{E}_2 = \int \boldsymbol{g}(\boldsymbol{k}) \exp(i\omega t - i k_y y_1 - i\varphi(\boldsymbol{k}, y_1, y))\, d\boldsymbol{k},$$

where φ is the phase shift of the reflected monochromatic plane wave relative to the incident wave (absorption is assumed to be absent; the plane of propagation of the signal is taken as the yz-plane, so that we can put $x = 0$ as well as $z = 0$ at the points Q and P). The field E_2 will evidently be large not for all y but only near some point $y = y_2$ corresponding to the point of arrival of the reflected ray. To find this point, we expand the phase φ in series in powers of $\Delta\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}_0$:

$$\varphi(\boldsymbol{k}, y_1, y) = \varphi(\boldsymbol{k}_0, y_1, y) + \Delta\boldsymbol{k} \cdot (\partial\varphi/\partial\boldsymbol{k})_{k_0} + \cdots,$$

$$\partial\varphi/\partial\boldsymbol{k} \equiv \boldsymbol{i}\,\partial\varphi/\partial k_x + \boldsymbol{j}\,\partial\varphi/\partial k_y + \boldsymbol{k}'\,\partial\varphi/\partial k_z,$$

with $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}'$ unit vectors along the axes of x, y and z. From this expansion it is clear that, if $\partial\varphi/\partial\boldsymbol{k} = 0$, we have $\boldsymbol{E}_2 = \boldsymbol{E}_1 \exp(-i\varphi(\boldsymbol{k}_0))$ to within terms of a higher order of smallness, i.e. the field strength in the reflected signal is the same as in the incident signal. Hence it follows that the position of the point y_2 is given by that condition:

$$[\partial\varphi(\boldsymbol{k}, y_1, y_2)/\partial\boldsymbol{k}]_{k_0, z=0} = 0. \tag{34.6}$$

Thus, if the phase φ is determined with sufficient accuracy from the approximation of geometrical optics, then the distance l between the points Q and P (Fig. 34.1) can be determined by that approximation also, which gives (§ 19)

$$\varphi = \frac{2\omega}{c} \int\limits_0^{z_{\mathrm{refl}}} \sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}\, dz + \\ + \frac{\omega}{c} n(0)(y_2 - y_1)\sin\theta_0 - \frac{1}{2}\pi, \tag{34.7}$$

and for $z = 0$ we have $k_y = (\omega/c)\, n(0) \sin\theta_0$, $k_z = (\omega/c)\, n(0) \cos\theta_0$. Hence, from (34.4),

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial k_y} &= \frac{1}{(\omega/c)\, n(0)} \frac{\partial\varphi}{\partial \sin\theta_0} \\
&= y_2 - y_1 - 2\int\limits_0^{z_{\text{refl}}} \frac{n(0)\sin\theta_0\, dz}{\sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}} = 0, \\
\frac{\partial\varphi}{\partial k_z} &= \frac{1}{(\omega/c)\, n(0)} \frac{\partial\varphi}{\partial \cos\theta_0} \\
&= -(y_2 - y_1)\cot\theta_0 + 2\int\limits_0^{z_{\text{refl}}} \frac{n(0)\cos\theta_0\, dz}{\sqrt{[n^2(z) - n^2(0)\sin^2\theta_0]}} = 0.
\end{aligned}\right\} \tag{34.8}$$

Putting $n(0) = 1$, we find

$$\begin{aligned}
l = y_2 - y_1 &= 2\int\limits_0^{z_{\text{refl}}} \frac{\sin\theta_0\, dz}{\sqrt{[n^2(z) - \sin^2\theta_0]}} = 2\int\limits_0^{z_{\text{refl}}} \frac{\sin\theta_0\, dz}{n(z)\cos\theta} \\
&= 2\sin\theta_0 \int\limits_0^{z_{\text{refl}}} \frac{ds}{n(z)},
\end{aligned} \tag{34.9}$$

where $ds = dz/\cos\theta$ is an element of length along the ray path. In this case the two equations $\partial\varphi/\partial k_y = 0$ and $\partial\varphi/\partial k_z = 0$ evidently give the same result.

If geometrical optics is not applicable for the calculation of the phase φ on the ground, as may happen for very long waves or for angles θ_0 close to $\frac{1}{2}\pi$ (in each case the wave is reflected from the very boundary of the layer), then to find the distance l we must again use formula (34.6), but with the phase φ determined on the basis of the exact solution of the wave equation [121, 216].

In the cases of principal interest here, where geometrical optics is valid, there is no need to use the relation (34.6) in order to find the distance l between the corresponding points Q and P and to resolve other problems. Instead, it is more convenient to use throughout the concept of the ray path and simply to ignore the fact that the ray treatment is not valid near the reflection point. The justification of this procedure follows from the agreement between formula (34.9) and the results obtained below by a purely ray treatment. Proceeding thus, we may write down the change in phase of the reflected wave relative to the incident wave as

$$\begin{aligned}
\varphi &= \frac{2\omega h}{c\cos\theta_0} + \frac{2\omega}{c}\int\limits_{AB} n(z)\, ds \\
&= \frac{2\omega h}{c\cos\theta_0} + 2\omega\int\limits_{AB} \frac{ds}{v_{\text{ph}}(\omega, z)},
\end{aligned} \tag{34.10}$$

where $v_{ph}(\omega, z) = c/n$ is the phase velocity (21.16) and ds an element of the ray path; the point A is on the boundary of the layer and B is at the vertex of the path (at $z = z_{refl}$). The integration is along the ray path shown in Fig. 34.1; the small term $-\frac{1}{2}\pi$ in the phase is neglected. The group delay time is

$$\Delta t_{gr,obl} = \varphi'(\omega) = \frac{2h}{c\cos\theta_0} + 2\int\limits_{AB} \frac{ds}{v_{gr}(\omega, z)}, \tag{34.11}$$

where v_{gr} is the group velocity (21.17), which in our case is cn.

Theorems giving relations between the group paths for oblique and normal incidence

The optical and group path lengths for oblique incidence are evidently

$$\left.\begin{aligned} L_{o,obl} &= c\,\varphi/\omega = \frac{2h}{\cos\theta_0} + 2\int\limits_{AB} n(\omega, z)\, ds, \\ L_{gr,obl} &= c\,\Delta t_{gr,obl} = \frac{2h}{\cos\theta_0} + 2\int\limits_{AB} \frac{ds}{n(\omega, z)} \\ &= 2\int\limits_0^{z_{refl}} \frac{ds}{n(\omega, z)}. \end{aligned}\right\} \tag{34.12}$$

Fig. 34.1 shows that $ds = dz/\cos\theta = dy/\sin\theta$, and since $n(z)\sin\theta = \sin\theta_0$ we have

$$\begin{aligned} L_{gr,obl} &= \frac{2h}{\cos\theta_0} + 2\int\limits_{y_A}^{y_B} \frac{dy}{\sin\theta_0} = \frac{2(h + z_C - z_A)}{\cos\theta_0} \\ &= 2z_{a,obl}/\cos\theta_0 = 2QC = QCP. \end{aligned} \tag{34.13}$$

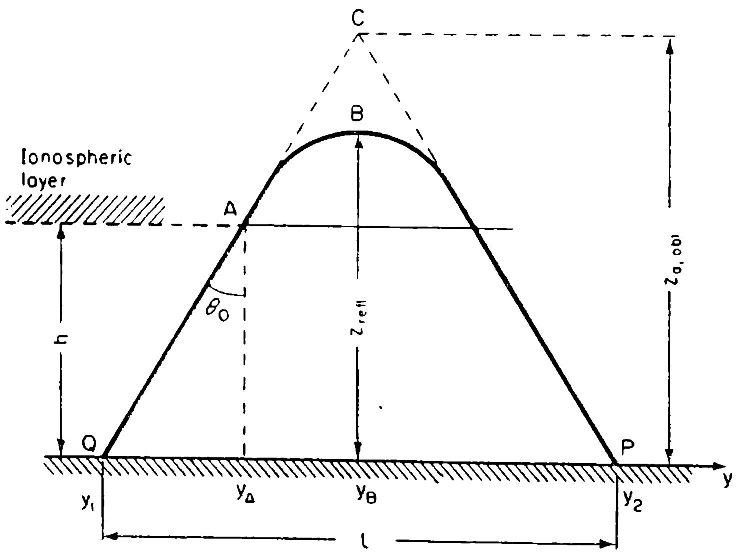

FIG. 34.1. Ray path for oblique incidence on a plane isotropic layer.

Thus the group path is equal to the sum of the sides of the isosceles triangle described about the true path as shown in Fig. 34.1. Accordingly, the group delay time $\Delta t_{\mathrm{gr,\,obl}} = (1/c) L_{\mathrm{gr,\,obl}}$ is equal to the time taken by the wave to traverse the path QCP (i.e. the two sides of the triangle) with the velocity of light in vacuum; this is sometimes called Breit and Tuve's theorem. The height $z_{a,\mathrm{obl}}(\theta_0, f)$ is called the effective or apparent height for oblique incidence at an angle θ_0.

The distance between the points QP, i.e. the distance between corresponding points, is (see Fig. 34.1)

$$l = 2 z_{a,\mathrm{obl}} \tan\theta_0 = L_{\mathrm{gr,\,obl}} \sin\theta_0 = \Delta t_{\mathrm{gr,\,obl}}\, c \sin\theta_0 . \tag{34.14}$$

This is the same as the expression obtained from formula (34.9).

The reflection occurs at greater values of $n(z_{\mathrm{refl}}) = \sin\theta_0$ as the angle of incidence θ_0 of the wave on the layer increases, and so waves of a certain frequency $f > f_{\mathrm{cr}}$ will be reflected from the layer when $\theta_0 > \theta_{0,\mathrm{min}}$, where, from (34.3),

$$\cos\theta_{0,\,\mathrm{min}} = f_{\mathrm{cr}}/f \tag{34.15}$$

(in other words, by definition, the frequency $f = f_{\mathrm{cr,\,obl}}$ corresponds to the angle $\theta_{0,\,\mathrm{min}}$). As the angle increases ($\theta_0 > \theta_{0,\mathrm{min}}$), the distance $l = QP$ at first decreases, until for some angle $\theta_0 = \theta_{0,\mathrm{min}(l)}$ this distance becomes a minimum l_{min}. The region $l < l_{\mathrm{min}}$ is called the dead zone. The value of l_{min} is given by the condition

$$dl/d\theta_0 = 0, \tag{34.16}$$

further discussion of which may be found in [22, 23].

Let us now find the relation between the effective heights for oblique incidence ($z_{a,\,\mathrm{obl}}$) and normal incidence (z_a). From (34.12), (34.13) and (34.14),

$$z_{a,\mathrm{obl}}(f, \theta_0) = h + \cos\theta_0 \int\limits_{AB} \frac{ds}{n(f, z)} = h + \int\limits_{h}^{z_{\mathrm{refl}}} \frac{dz \cos\theta_0}{n(f, z)\cos\theta}$$

$$= h + \int\limits_{h}^{z_{\mathrm{refl}}} \frac{dz}{n(f\cos\theta_0, z)} = z_a(f\cos\theta_0), \tag{34.17}$$

since

$$n^2(f\cos\theta_0, z)\cos^2\theta_0 = \left(1 - \frac{e^2 N(z)}{\pi m f^2 \cos^2\theta_0}\right)\cos^2\theta_0$$
$$= n^2(f, z) - \sin^2\theta_0 = n^2(f, z)\cos^2\theta$$

and $n(f\cos\theta_0, z_{\mathrm{refl}}) = 0$; here we have used the facts that $n(z_{\mathrm{refl}}) = \sin\theta_0$ and $n(z)\sin\theta = n(0)\sin\theta_0$.

It is also clear from (34.17) that, if $f_1\cos\theta_{01} = f_2\cos\theta_{02}$, then

$$z_{a,\mathrm{obl}}(f_1, \theta_{01}) = z_{a,\mathrm{obl}}(f_2, \theta_{02})$$
$$= z_a(f_1\cos\theta_{01}) = z_a(f_2\cos\theta_{02}),$$

where, of course, θ_{01} and θ_{02} are the angles of incidence on the layer of waves with frequencies f_1 and f_2. In this case the true heights z_{refl} of reflection are also equal, since

$$
\begin{aligned}
n(f_1, z^{(1)}_{\text{refl}}) &= \sin\theta_{01},\\
n(f_2, z^{(2)}_{\text{refl}}) &= \sin\theta_{02},\\
f_1\cos\theta_{01} &= f_2\cos\theta_{02}\\
&= f_1\sqrt{[1-n^2(f_1, z^{(1)}_{\text{refl}})]}\\
&= \sqrt{[(4\pi e^2/m)\,N(z^{(1)}_{\text{refl}})]}\\
&= f_2\sqrt{[1-n^2(f_2, z^{(2)}_{\text{refl}})]}\\
&= \sqrt{[(4\pi e^2/m)\,N(z^{(2)}_{\text{refl}})]},
\end{aligned}
$$

i.e. $z^{(1)}_{\text{refl}} = z^{(2)}_{\text{refl}} = z_{\text{refl}}$.

Next, from (34.13) and (34.17)

$$L_{\text{gr,obl}}(f, \theta_0)\cos\theta_0 = L_{\text{gr}}(f\cos\theta_0), \tag{34.18}$$

where $L_{\text{gr}} = 2z_a$ is the group path for normal incidence.

The relations (34.17) and (34.18), sometimes called Martyn's theorem, make it possible to find $z_{a,\text{obl}}$, $L_{\text{gr,obl}}$, $\Delta t_{\text{gr,obl}}$ and the distance l from the height-frequency characteristic recorded at normal incidence under conditions where the effect of the Earth's magnetic field may be neglected. For example, (34.14) and (34.18) give

$$l(f, \theta_0) = 2z_a(f\cos\theta_0)\tan\theta_0. \tag{34.19}$$

The absorption of the wave as it traverses the path $QABP$ causes a reduction in its amplitude by a factor R_{obl}, where

$$R_{\text{obl}} = \exp\left(-2\frac{\omega}{c}\int\limits_{AB}\varkappa\,ds\right);$$

$$
\begin{aligned}
-\ln R_{\text{obl}} &= 2\frac{\omega}{c}\int\limits_{AB}\varkappa\,ds = \int\limits_{AB}\mu\,ds\\
&\approx \int\limits_{AB}\frac{\nu_{\text{eff}}(1-n^2)}{cn}\,ds = \frac{\bar{\nu}_{\text{eff}}}{c}\left(L_{\text{gr,obl}} - L_{o,\text{obl}}\right),
\end{aligned}
\tag{34.20}
$$

where the same approximation as in (31.13) has been used.

By a similar derivation to that of (34.17) we easily find

$$\ln R_{\text{obl}}(f, \theta_0, \bar{\nu}_{\text{eff}}) = \cos\theta_0\cdot\ln R(f\cos\theta_0, \bar{\nu}_{\text{eff}}), \tag{34.21}$$

where R is the reflection coefficient (31.13) for normal incidence.

Reflection from a spherical layer

Up to this point the ionosphere has been regarded as plane, with properties depending only on the coordinate z. For oblique incidence at medium and large angles, however, the sphericity of the Earth must be taken into account, and as a first approximation we may assume that $\varepsilon = \varepsilon(r)$, where r is the distance from the centre of the Earth. We shall not discuss this case in detail; the procedure is exactly the same as in § 19 except that the wave equation in spherical or cylindrical coordinates is used. We shall simply mention that for a spherical Earth (with $\varepsilon = \varepsilon(r)$) the ray path lies in a plane passing through the centre of the Earth and the arc of a great circle between the corresponding points Q and P (Fig. 34.2). The ray path is given by the generalised law of refraction†

$$n(r) r \sin\theta = \text{constant}, \tag{34.22}$$

where θ is the angle between the direction of the ray and the radial direction at the point concerned.

On the Earth's surface $r = \varrho \approx 6360$ km (where ϱ is the radius of the Earth), $n(r) = \sqrt{\varepsilon(r)} = 1$, and $\theta = \theta_0$, the angle between the ray and the radius (towards the centre of the Earth) at the point Q. Hence

$$n(r) r \sin\theta = \varrho \sin\theta_0. \tag{34.23}$$

At the vertex of the path $\sin\theta = 1$ and

$$n(r_{\text{refl}}) = \varrho \sin\theta_0 / r_{\text{refl}}. \tag{34.24}$$

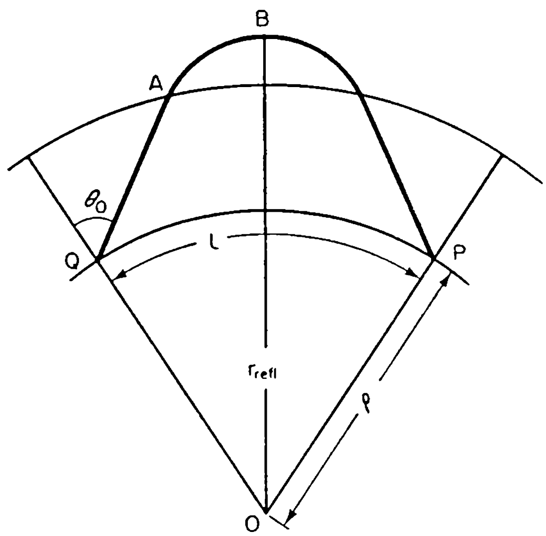

FIG. 34.2. Ray path for oblique incidence on a spherical isotropic layer.

For a given angle θ_0, but with allowance for the Earth's sphericity, the reflection occurs higher than if the Earth's surface and the ionosphere were plane, since then $n(z_{\text{refl}}) = \sin\theta_0$. In practice the difference is slight, since $r_{\text{refl}} - \varrho \sim 100$ to 400 km and $(r_{\text{refl}} - \varrho)/r_{\text{refl}} < 1/15$.

† Formula (34.22) is readily obtained by considering the path of a ray in a medium consisting of concentric spherical layers with the refractive index varying from one layer to the next.

The refraction of radio waves in the ionosphere has to be taken into account not only in communications between ground stations but also in radio astronomy, in obtaining radar echoes from the Moon, and in radar tracking of rockets and artificial satellites. The calculations are based on the law of refraction (34.22); the equation of the path and specific examples are given in [150, 217, 219, 23].

The field strength in signals reflected from the ionosphere

In actual conditions of radio-wave propagation in the ionosphere, various types of multiple reflection of waves from the Earth and the ionospheric layers occur (see the diagrammatic representation in Fig. 34.3a, b). Here it must be borne in mind that over distances comparable with the Earth's radius ϱ we cannot regard the ionosphere as having uniform properties, even if sporadic phenomena are neglected. This is evident from the fact that over such distances the Sun has different altitudes at the points of emission, reflection and reception.

As a result of multiple reflections, the wave field at the point of reception is composed of waves arriving by various paths. Multiple reflections are important even for normal or almost normal incidence, since the coefficient of reflection of waves from the Earth's surface is usually large even when the

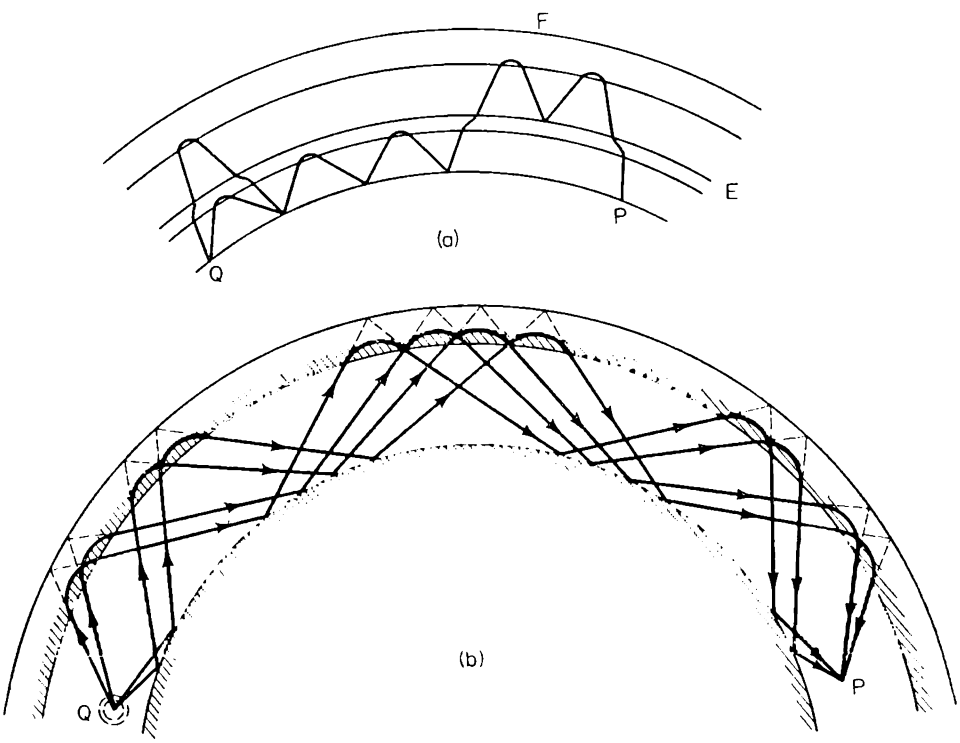

Fig. 34.3. Possible ray paths in reflection from the ionosphere (diagrammatic).
(a) ray in only one direction shown at P
(b) reflection from only one layer shown

angle of incidence is small. Accordingly, when pulses reflected from the ionosphere at normal incidence are observed, signals due to multiple reflection often appear; they can easily be distinguished, since (e.g.) for a twice reflected signal the effective height is twice that for a once reflected signal, and so on. The presence of signals due to multiple reflection can be utilised to determine the coefficient R for reflection of radio waves from the ionosphere [22, § 102].

On account of the presence of multiple reflections and many possible ray paths (see Fig. 34.3) it is very difficult to calculate the field strength at the point of reception. For an isotropic ionosphere (i.e. neglecting the effect of the Earth's magnetic field) a general solution has indeed been obtained, but it is very complex [127, 150, 220]. The cumbersome nature of the exact general solutions, together with the complexity of the actual conditions of radio wave propagation due to the inhomogeneity of the ionosphere and of the electrical properties of the Earth's surface, the presence of absorption, etc., have the result that the calculation of the field strength in reflected waves is usually made only approximately. Moreover, especially for short waves, the use of approximate methods is necessitated by the very nature of the problem. The possibility of a considerable simplification of the calculations is due to the fact that the distance of the ionosphere is of the order of 100 km, i.e. much greater than radio wavelengths, especially short ones. In other words, the ionosphere is in the wave zone of ground and airborne transmitters, and so the whole process of radio-wave propagation in the ionosphere may be considered independently of the situation of the transmitter and receiver. Their situation is important only in finding the wave field of the transmitter at a great height (at the base of the ionosphere) and in determining the field strength in the reflected signal at the position of the receiver (taking account of reflection from the Earth, etc.).

Thus the problem can be, so to speak, divided, and the problem of propagation in the ionosphere is seen to be independent; if it is necessary to find the field strength, the solutions "below" (at the Earth) and "above" (in the ionosphere) can be combined. The convenience of this separation is still more evident if we take into account the fact that the calculation of the field strength from a complex emitter (a transmitting aerial and surrounding objects) is itself a very complex matter. It should further be emphasised that, in radar probing of the ionosphere, the field strength in the reflected wave or signal is not usually of especial interest, since in this method we measure only the phase shift of the reflected wave or the delay time of the reflected signal relative to the incident one. It should also be borne in mind that for sufficiently short signals the interference of different waves several times reflected from the ionosphere and from the Earth's surface is of no importance, and the ray treatment is generally valid; in this treatment the lack of interdependence between propagation in the ionosphere and conditions on the ground becomes particularly clear. Approximate methods of calculating the field strength are given in [22, 221]. Here we shall make only a few simple remarks to give an idea of the range of problems involved and to enable some simple estimates to be made.

It is well known that the electric field of a vertical dipole (Hertzian oscillator) on the surface of a perfectly conducting Earth is

$$E_\theta = 120(\pi h_a I/\lambda r)\sin\theta \text{ mV/m}$$
$$\approx 300(\sqrt{P}/r)\sin\theta \text{ mV/m}, \tag{34.25}$$

where E_θ is the field observed in a direction at an angle θ to the axis of the oscillator (Fig. 34.4), P is the total emitted power, h_a the effective height of the oscillator, λ the wavelength in vacuum, r the distance QP from the oscillator to the point of observation, and I the current at the base of the oscillator (at an antinode of the current). In (34.25) the coefficients are so chosen as to give the field in millivolts per metre, P in kilowatts, λ and h_a in metres, r in kilometres and I in ampères.† For an arbitrary emitter the field may conveniently be represented in the form

$$E_\theta(\text{mV/m}) = \frac{60\pi h_a(\text{m})\, I(\text{A})}{\lambda(\text{m})\, r(\text{km})}\psi(\theta), \tag{34.26}$$

where the function $\psi(\theta)$ characterises the directional properties of the aerial. For an oscillator in free space, $\psi(\theta) = \sin\theta$; for one placed above a perfectly conducting Earth, $\psi(\theta) = 2\sin\theta$ for $0 \leqq \theta \leqq \frac{1}{2}\pi$ and $\psi = 0$ for $\theta < 0$.

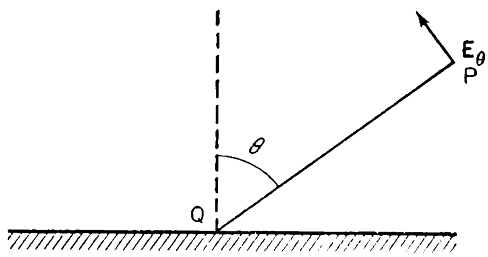

FIG. 34.4. Field of a vertical dipole at a point Q on the Earth's surface.

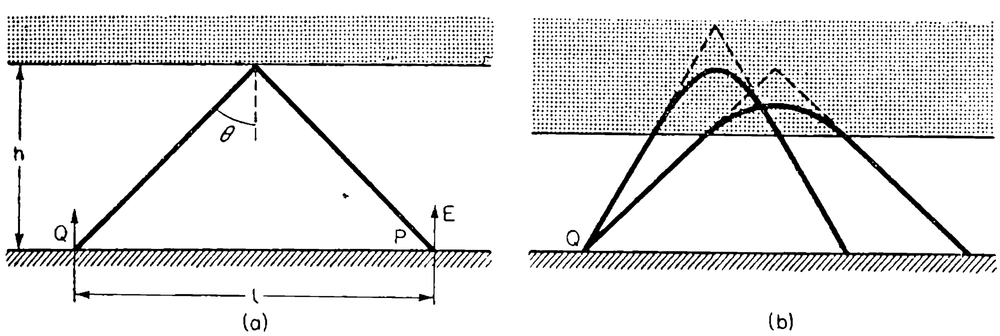

FIG. 34.5. Reflection of waves
(a) when the layer is replaced by a mirror
(b) for an isotropic extended layer

† If the absolute Gaussian sytem of units is used, the coefficients 300 and 120 in (34.25) must be replaced by 10^{-5} and $2/c = \frac{2}{3} \times 10^{-10}$ respectively.

The simplest assumption, which permits an easy calculation of the field in the wave reflected from an ionospheric layer, is to replace the layer by a mirror with reflection coefficient R. Then, for example, for a plane layer (Fig. 34.5a) the field in the reflected wave at the point P is

$$E \approx R \frac{300 \sqrt{P} \sin^2\theta}{\sqrt{[(\frac{1}{2}l)^2 + h^2]}} = 300 R \sqrt{P} \frac{(\frac{1}{2}l)^2}{[(\frac{1}{2}l)^2 + h^2]^{3/2}} \text{ mV/m}, \tag{34.27}$$

where for simplicity the emitter is taken to be a vertical oscillator on a perfectly conducting Earth. This also has the result that there is only a vertical field component at the point P, and owing to reflection this component is twice the field in free space at the same distance. This fact has been used in (34.27). The generalisation to a spherical Earth with finite conductivity and to any emitter offers no difficulty in principle. The reflection coefficient R must be calculated from ionosphere data. If several rays reach the point P, the field can be found as the resultant of the individual fields.

If the ray treatment of propagation is valid, as it almost always is in the ionosphere, the above method of calculating the field suffers only from the following defect. In reflection of waves from an inhomogeneous layer, unlike reflection from a mirror (with a sharp boundary), the field does not decrease with distance in strict proportion to $1/r$. In mirror reflection the $1/r$ law holds because the reflected wave is the same as if it were emitted by a fictitious source at a point which is the mirror image of the true emitter Q. In an inhomogeneous medium, the effective and true heights of reflection are different for rays in different directions (Fig. 34.5b), and so the reflected beam does not diverge in the same way as in reflection from a single mirror.

Within the limits of the ray treatment the divergence of the beam reflected from an isotropic inhomogeneous layer can be calculated without especial difficulty (see, for example, [222]). For the relatively thin D and E layers, however, the divergence is small and therefore unimportant, while for the F layer it is usually necessary to take account of the anisotropy of the medium due to the Earth's magnetic field. In the anisotropic case the ray path is very complex (§ 29) and the divergence of the beam is not easily calculated. We shall not pause to discuss this problem further, and merely note that in the rough estimates of field strength usually made in practice it is not reasonable to take into account the difference between reflection from the ionosphere and mirror reflection, because ordinarily this difference cannot affect the results as regards order of magnitude.

§ 35. WAVE REFLECTION WITH ALLOWANCE FOR THE EFFECT OF A MAGNETIC FIELD

The effect of a magnetic field. Critical frequencies

Both in the Earth's ionosphere and in the solar corona (and probably in the ionospheres of the other planets) the constant magnetic field exerts a considerable influence on the propagation and reflection of radio waves. This influence is generally determined by the values of the parameters

$$\sqrt{u} = \omega_H/\omega = |e|\, H^{(0)}/m\, c\, \omega$$

and

$$\frac{\omega_H}{\omega_0} = \frac{\sqrt{u}}{\sqrt{v}} = \frac{|e|\, H^{(0)}}{m\, c\, \sqrt{(4\pi e^2 N/m)}}.$$

In the Earth's ionosphere at high and medium latitudes $H^{(0)} \sim 0{\cdot}5$ oersted, for which $\omega_H = 8{\cdot}82 \times 10^6$, $f_H = \omega_H/2\pi = 1{\cdot}4 \times 10^6$ and $\lambda_H = 2\pi c/\omega_H = 214$ m. At the maximum of the F layer, $\omega_0 \lesssim 8 \times 10^7 (N_{\max} \lesssim 2 \times 10^6)$, and so $\omega_H/\omega_0 \gtrsim 0{\cdot}1$; in the metre-wavelength range, $\omega_H/\omega \sim 10^{-2}$, and for the longest waves used and for whistlers $\omega_H/\omega \sim 10^2$ to 10^3. Thus in considering the propagation of radio waves in the ionosphere we encounter various conditions, and the problem of the effect of the Earth's magnetic field has no single answer. Below we shall discuss mainly the short-wavelength range, for which $\sqrt{u} = \omega_H/\omega \ll 1$. Where the case $\sqrt{u} > 1$ is considered, it will be assumed that the wavelength is still not very large, and so the approximation of geometrical optics will be valid in the greater part of the layer.

The general picture of the propagation and reflection of monochromatic waves and signals by an inhomogeneous magnetoactive layer has in essence been described in Chapter V. When a wave (or signal) is incident on the layer it divides into an ordinary and an extraordinary wave. In a smooth layer, when the angle α is not too small, the two waves are propagated entirely independently. If $u < 1$, wave 1 (extraordinary) undergoes reflection at $v = v_{10}^{(-)} = 1 - \sqrt{u}$ (normal incidence will be assumed throughout). When absorption is absent and the maximum density in the layer is considerably higher than the point $v_{10}^{(-)}$, the reflection is total. Hence there is no wave 1 for larger values of v, even though it could be propagated in the medium for $v > v_{1\infty}$. Wave 2 is reflected from the point $v = v_{20} = 1$. If $u > 1$, the reflection of wave 1 takes place at the point $v_{10}^{(+)}$, as is clear, for instance, from Fig. 11.3b.

Thus in probing the ionosphere with a signal of carrier frequency ω the ordinary wave is reflected from the level where the electron density N is

$$N_2 = m\omega^2/4\pi e^2 = 1{\cdot}24 \times 10^{-8} f^2, \tag{35.1}$$

since $v_{20} = 4\pi e^2 N/m\omega^2$ and $f = \omega/2\pi$.

When $u = \omega_H^2/\omega^2 < 1$ the extraordinary wave is reflected from the level where N is

$$\begin{aligned} N_{1-} &= m\omega(\omega - \omega_H)/4\pi e^2 \\ &= 1{\cdot}24 \times 10^{-8} f(f - f_H), \end{aligned} \tag{35.2}$$

and when $u > 1$ it is reflected where N is

$$N_{1+} = m\omega(\omega + \omega_H)/4\pi e^2 = 1{\cdot}24 \times 10^{-8} f(f + f_H), \tag{35.3}$$

with $f_H = \omega_H/2\pi = |e|\, H^{(0)}/2\pi mc$.

The critical frequency f_{cr} is defined as that for which the reflection point corresponds to the maximum of the layer, where $N = N_{max}$. From (35.1), (35.2) and (35.3) we evidently have

$$f_{cr,2} \equiv f_{cr,o} = \sqrt{(e^2 N_{max}/\pi m)} = 0{\cdot}9 \times 10^4 \sqrt{N_{max}}, \tag{35.4}$$

$$f_{cr,1-} \equiv f_{cr,x} = \frac{1}{2} f_H + \sqrt{\left(\frac{1}{4} f_H^2 + 0{\cdot}81 \times 10^8 N_{max}\right)} = \frac{1}{2} f_H + \sqrt{\left(\frac{1}{4} f_H^2 + f_{cr,o}^2\right)}, \tag{35.5}$$

$$f_{cr,1+} \equiv f_{cr,z} = -\frac{1}{2} f_H + \sqrt{\left(\frac{1}{4} f_H^2 + 0{\cdot}81 \times 10^8 N_{max}\right)} = -\frac{1}{2} f_H + \sqrt{\left(\frac{1}{4} f_H^2 + f_{cr,o}^2\right)}. \tag{35.6}$$

From these formulae we have the relation

$$f_{cr,x} - f_{cr,z} = f_H = |e|\, H^{(0)}/2\pi m c = 2{\cdot}8 \times 10^6 H^{(0)}; \tag{35.7}$$

if $f_{cr,o}^2 \gg \frac{1}{4} f_H^2$, then also

$$f_{cr,x} - f_{cr,o} \approx \tfrac{1}{2} f_H = 1{\cdot}4 \times 10^6 H^{(0)}, \tag{35.8}$$

or, in the next approximation,

$$f_{cr,x} - f_{cr,o} \approx \frac{1}{2} f_H + \frac{1}{8} f_H^2 / f_{cr,o}. \tag{35.9}$$

The wave phase and the reflection coefficient. The course of the rays

In the region near the reflection points, where $n_{1,2} = 0$, geometrical optics is not valid, and to obtain an exact expression for the phase of the reflected wave we must use equations (23.2). The solution of these equations is more complex than in the absence of a magnetic field, but, as shown in § 25, the reflection of radio waves in the presence of a magnetic field is equivalent (except in special cases such as that of small angles α) to their reflection from a certain isotropic layer: if the reflection point $z(n_{1,2} = 0)$ is sufficiently far from the maximum of the layer, then the phase of the reflected wave is

$$\varphi_{1,2} = \frac{2\omega}{c} \int_0^{z(n_{1,2}=0)} n_{1,2}(\omega, z)\, dz - \frac{1}{2}\pi. \tag{35.10}$$

This formula differs from (30.6) only in that n is replaced by $n_{1,2}$.

When absorption is present, the results are again similar to those for an isotropic layer. Omitting the small terms mentioned in § 31, we obtain for $\varphi_{1,2}$ formula (35.10), and for the reflection coefficient

$$-\ln R_{1,2} = \frac{2\omega}{c} \int\limits_{0}^{z(n_{1,2}^2 - \varkappa_{1,2}^2 = 0)} \varkappa_{1,2}(\omega, z)\, dz, \tag{35.11}$$

where the point $z(n_{1,2}^2 - \varkappa_{1,2}^2 = 0)$ evidently corresponds to the point $z(\varepsilon = n^2 - \varkappa^2 = 0)$ in the isotropic case. When absorption is present the integration in (35.10) must be taken up to that point also; in this case $n_{1,2}$ is nowhere zero.

The other results of §§ 30–33 in which no specific form of the functions n and $\varkappa$ was used, likewise remain valid when n and $\varkappa$ are replaced by $n_{1,2}$ and $\varkappa_{1,2}$. From (24.15) this applies also to the group delay time Δt_{gr}, and therefore to the group path $L_{gr} = c\Delta t_{gr}$ and the effective height $z_a = \frac{1}{2}L_{gr}$, despite the fact that the direction of the ray does not coincide with that of the wave normal in an anisotropic medium. The latter fact, however, has the result that in normal probing of the ionosphere the point where the signal is reflected is not exactly above the point where it enters the layer, but to one side; the ordinary and extraordinary signals are reflected from different regions of the layer in a horizontal plane, quite apart from the fact that these regions are at different heights (Fig. 35.1).

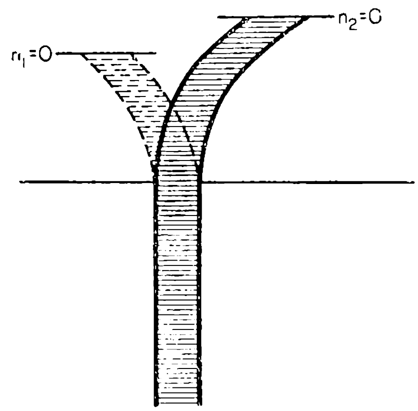

FIG. 35.1. Reflection of ordinary and extraordinary signals.

This deflection of signals may be important if we take into account the inhomogeneity of the ionosphere in the horizontal direction, which does exist to some extent, although it does not correspond to the customary idealised regular pattern. The distance by which the regions of reflection of the ordinary and extraordinary signals are displaced depends on the angle α between the field $\boldsymbol{H}^{(0)}$ and the vertical, and also on the characteristics of the corresponding ionospheric layer.

As an example, Fig. 35.2 shows some results of calculations given in [58] for a parabolic layer $N = N_{max}(1 - z^2/z_m^2)$. At the boundary of the layer ($z = -z_m$) a radio signal is normally incident, which then splits into an ordinary and an extraordinary signal. In the northern hemisphere the ordinary signal deviates north of the vertical and the extraordinary signal deviates south; in the southern hemisphere the reverse is true.

Fig. 35.2 gives the path lying in the yz-plane, which also contains the magnetic vector $\boldsymbol{H}^{(0)}$ (i. e. the yz-plane is the plane of the magnetic meridian). The ordinate is the distance $z_m - |z|$ from the boundary of the layer in units of z_m; the abscissa is the deviation Δy of the signal from the vertical in the same unit z_m, the half-thickness of the layer. Here it is assumed that reflection takes place from the maximum of the layer, i.e. that the carrier frequency f of the signal is equal to the critical frequency $f_{cr,o} \equiv f_{cr,2}$ for the ordinary ray and $f_{cr,x} \equiv f_{cr,1}$ for the extraordinary ray. The frequency $f_{cr,o}$ is taken to be 9 Mc/s, so that for the ordinary ray $f = f_{cr,o} = 9$ Mc/s and for the extraordinary ray $f = f_{cr,x} = \frac{1}{2}f_H + \sqrt{(\frac{1}{4}f_H^2 + f_{cr,o}^2)} \approx 9{\cdot}75$ Mc/s, since the

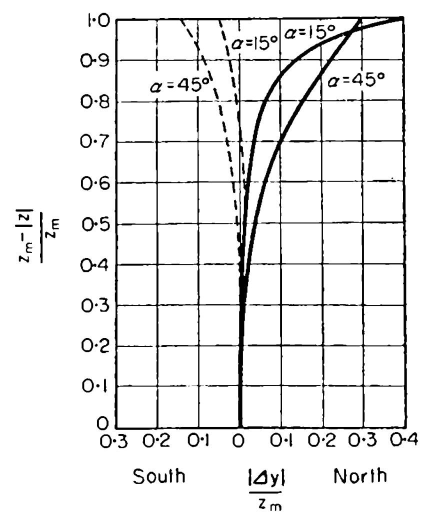

Fig. 35.2. Deviation of rays from the vertical for normal incidence with $\alpha = 15°$ and $45°$ and $f_H = 1{\cdot}4$ Mc/s. The ordinary ray (continuous lines) has $f = f_{cr,o} = 9$ Mc/s, and the extraordinary ray (broken lines) has $f = f_{cr,x} \approx 9{\cdot}75$ Mc/s.

gyration frequency f_H is taken as $1{\cdot}4$ Mc/s ($H^{(0)} = 0{\cdot}5$ oersted). Moreover, in drawing the curves in Fig. 35.2 absorption has been assumed absent, and the angle α is taken to be 45° and 15°.

A significant feature of the calculations is that they are based entirely on the approximation of geometrical optics, and so are inexact near the reflection point itself (i.e. as $|z| \to 0$). In ionospheric conditions, with normal probing and when the angle α is not too small, the resulting inaccuracy seems to be of little importance.

From Fig. 35.2 we see that, for example, when $\alpha = 15°$ the region of reflection of the ordinary ray is displaced from the point where the signal enters the layer by a distance of about $\frac{1}{2}z_m$, which for the F layer is 50 to 100 km, i.e. quite considerable. This deviation of the ray from the direction of the normal is particularly clear in the case of waves obliquely incident on the ionosphere, discussed in § 29.

The displacement of the reflection point from the vertical is shown in Fig. 35.3 as a function of α; here it is assumed that $f_{cr,o} = f_{cr,x} = 10$ Mc/s, $f = 10$ Mc/s, $f_H = 1{\cdot}44$ Mc/s and $\nu = 0$. For the ordinary ray, the curve

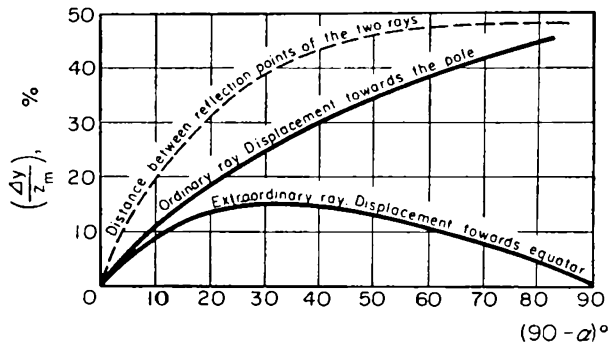

Fig. 35.3. Displacement of reflection point from the vertical in the northern hemisphere as a function of the angle α between the vertical (the direction of the wave vector) and the direction of the Earth's magnetic field.

shown in the diagram is incorrect for small α, where the effect of "tripling" of the signals begins (§ 28). For $\alpha = 0$ the reflection points of both the ordinary and the extraordinary ray lie on the vertical.

The group delay time of the signal is, by (24.15),

$$\Delta t_{gr} = 2\int_0^{z_t} \frac{dz}{v_{gr,z1,2}}; \tag{35.12}$$

here $z_t = z_{t1,2}$ is the reflection point, at which $n_{1,2} = 0$ (i.e. the "true height" of reflection). The effective height is, by definition,

$$\begin{aligned} z_{a1,2} = \frac{1}{2}c\,\Delta t_{gr} &= \int_0^{z_{t1,2}} \frac{c\,dz}{v_{gr,z1,2}} \\ &= \int_0^{z_{t1,2}} \frac{\partial(n_{1,2}\,\omega)}{\partial\omega}\,dz. \end{aligned} \tag{35.13}$$

In view of the complexity of the function $n_{1,2}(v, u, \alpha)$ the analysis of the dependence of $z_{a1,2}$ on the various parameters usually has to be made graphically [223]. In [224] a method is given for determining the electron density from height-frequency characteristics, taking into account the effect of the magnetic field. Here we shall simply give (Fig. 35.4) graphs of the function $c/v_{gr,z1,2} = \partial(\omega n_{1,2})/\partial\omega$ for $\lambda = 250$ m ($\omega = 0{\cdot}756 \times 10^7$, $u = 1{\cdot}36$), $H^{(0)}\cos\alpha = 0{\cdot}447$ and $H^{(0)}\sin\alpha = 0{\cdot}218$ ($\alpha = 25°\,50'$, $H^{(0)} = 0{\cdot}497$ oersted).

Fig. 35.5 gives height-frequency characteristics of the ionosphere for the same values of α and $H^{(0)}$ calculated for a linear layer (z_a is the effective and z_t the true height of reflection). Fig. 35.5 shows clearly the singularity of the

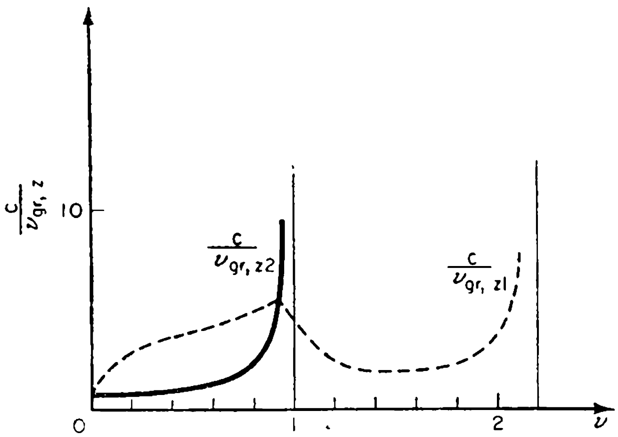

FIG. 35.4. Ratio of the velocity of light to the z-component of the group velocity (the component along the normal to the layer) as a function of $v = \omega_0^2/\omega^2$. The values of α and u are given in the text.

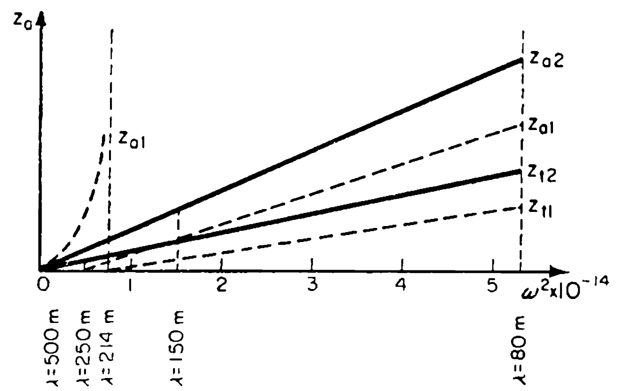

FIG. 35.5. Effective height of reflection z_a and true height of reflection z_t for a linear ionospheric layer as a function of frequency ω. The broken lines relate to a wave of type 1 and the continuous lines to a wave of type 2 (ordinary wave).

curve $z_a(\omega)$ for the extraordinary wave near the frequency ω_H ($\lambda_H \approx 214$ m). As $\omega \to \omega_H$, if absorption is neglected, z_a becomes very large in the approximation of geometrical optics (see [223, 225, 226]).†

† The behaviour of the effective height z_a for the ordinary ray near the gyration frequency depends considerably on the relation between the effective field $\boldsymbol{E}_a$ and the mean macroscopic field $\boldsymbol{E}$ [225]. Attempts have been made to utilise this fact to resolve experimentally the question whether a "polarisation term" is necessary. As shown in § 3, this term is not necessary, and in a plasma we have very nearly $\boldsymbol{E} = \boldsymbol{E}_a$, as we always assume. According to [156, p. 278] the results of the study of whistlers [53] form a very strong experimental argument against the use of a "polarisation term".

Quasilongitudinal and quasitransverse propagation

The complexity of the expression for $n_{1,2}$ and $\varkappa_{1,2}$ makes it necessary to use extensively in practice the approximate formulae for "quasilongitudinal" and "quasitransverse" propagation (§ 11).

In what is called the "deviating" region of the ionosphere, where $n_{1,2}$ differs considerably from unity, the approximation of quasitransverse propagation is frequently valid for the ordinary wave, and that of quasilongitudinal propagation for the extraordinary wave (where $u < 1$). This may be seen from Figs. 11.2, 11.3 and 11.6 and from estimates based on the inequalities (11.36) and (11.39). For example, we have

$$-\ln R_2 = \bar{\nu}_{\text{eff}}(L_{\text{gr},2} - L_{o2})/2c, \tag{35.14}$$

since for quasitransverse propagation wave 2 is the same as that propagated in an isotropic medium, and in the latter case formula (31.13), which is identical with (35.14), is valid under the conditions given in § 31. For wave 1 in the quasilongitudinal case we usually have

$$-\ln R_1 \approx \frac{\bar{\nu}_{\text{eff}}}{2c}(L_{\text{gr},1} - L_{o1})\frac{\omega}{\omega - \omega_L}; \tag{35.15}$$

for, according to (11.37), (24.11) and (35.13), when $v_{\text{gr},z1} \approx cn_1$ (for which the conditions of validity are evident from (24.11)),

$$\begin{aligned}
-\ln R_1 &= \frac{2\omega}{c}\int\limits_0^{z(n_1=0)} \varkappa_1\, dz \\
&= \frac{1}{c}\int \nu_{\text{eff}}\frac{1 - n_1^2 + \varkappa_1^2}{n_1}\frac{\omega}{\omega - \omega_L}\, dz \\
&\approx \frac{\bar{\nu}_{\text{eff}}}{2c}\left\{2\int\frac{dz}{n_1} - 2\int n_1\, dz\right\}\frac{\omega}{\omega - \omega_L} \\
&\approx \frac{\bar{\nu}_{\text{eff}}}{2c}(L_{\text{gr},1} - L_{o1})\frac{\omega}{\omega - \omega_L}.
\end{aligned}$$

In addition, in deriving the above formulae for $\ln R_{1,2}$ assumptions are made similar to those discussed in detail in § 31, and it must be remembered that in (35.14) and (35.15) also $R_{1,2}$ signifies only the part of the total reflection coefficient which is due to the passage of the wave through the "deviating" region.

In the "non-deviating" region, where $n_{1,2} \approx 1$ and v is small (as happens in the D and E layers for waves reflected from the F layer), we usually have quasilongitudinal propagation for both waves (cf. the condition (11.36)). Hence (11.37) gives

$$\varkappa_{1,2} = \frac{\nu_{\text{eff}}}{\omega}\frac{2\pi e^2 N(z)}{(\omega \pm \omega_L)^2 + \nu_{\text{eff}}^2}, \quad n_{1,2} \approx 1, \tag{35.16}$$

where the plus sign, of course, corresponds to the ordinary wave 2 and the minus sign to the extraordinary wave 1.

We shall not consider the particular cases where $\nu_{\text{eff}}^2 \gg$ or $\ll (\omega \pm \omega_L)^2$ or $\nu_{\text{eff}} \approx \omega \pm \omega_L$, since they merit discussion only in connection with the experimental data.

The preceding discussion, together with Figs. 30.5 and 35.5, demonstrates the qualitative effect of the magnetic field on the height-frequency characteristics. In the case of Fig. 35.5 the typical upward inflection of the characteristics near the critical frequencies is absent, simply because the layer is assumed to be linear (i.e. Fig. 35.5 really shows the form of $z_a(\omega)$ and $z_t(\omega)$ only for frequencies considerably below the critical value). In normal probing with $u < 1$ and not too small values of α the regular reflection from the layer leads, as already often mentioned, to the appearance of only two signals. The corresponding critical frequencies for the ordinary and extraordinary waves, $f_{\text{cr},o}$ and $f_{\text{cr},x}$, are given by formulae (35.4) and (35.5). As α decreases, the "tripling" effect appears, and the third signal, reflected from the point $v_{10}^{(+)} = 1 + \sqrt{u}$, is the first to disappear with increasing frequency (Fig. 35.6). The critical frequency for the third signal is $f_{\text{cr},z} = f_{\text{cr},x} - f_H$ (see (35.7)), and so, by measuring the frequencies $f_{\text{cr},x}$ and $f_{\text{cr},z}$, we can determine the field

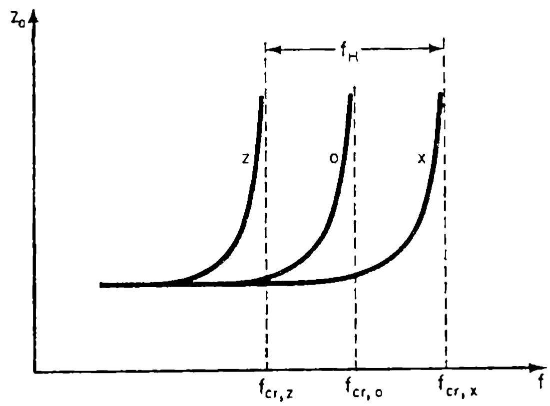

FIG. 35.6. Height-frequency characteristics (diagrammatic) for normal probing at small angles α.

$H^{(0)}$ in the layer. The same result may be achieved by measuring the frequency $f_{\text{cr},o}$ and the frequency difference $f_{\text{cr},x} - f_{\text{cr},o} = \frac{1}{2} f_H + \sqrt{(\frac{1}{4} f_H^2 + f_{\text{cr},o}^2)} - f_{\text{cr},o}$ (see (35.5), (35.8) and (35.9)).

Oblique incidence

The effect of the Earth's magnetic field on the propagation and reflection of radio waves obliquely incident on the ionosphere has been discussed in § 29. Here we shall merely remark that, in observations of the "tripling" effect due to oblique incidence and scattering of the waves (see § 29), the height-frequency characteristics differ in two respects from those shown in Fig. 35.6.

Firstly, the intensity of the z-reflection (the third signal) is always noticeably less than that of the o and x signals. Secondly, the difference $f_{\mathrm{cr},x} - f_{\mathrm{cr},z}$ is somewhat less than for normal incidence, and is therefore less than the gyration frequency f_H. This is because when the "tripling" effect is observed with oblique incidence, it is mainly the waves reflected normally which are responsible for the x-reflection. The z-reflection, on the other hand, is essentially a phenomenon occurring at oblique incidence, and so the wave is reflected from a level somewhat below the point $v_{10}^{(+)} = 1 + \sqrt{u}$. Accordingly $f_{\mathrm{cr},z,\mathrm{obl}} > f_{\mathrm{cr},z,\mathrm{nor}}$ and $f_{\mathrm{cr},x,\mathrm{nor}} - f_{\mathrm{cr},z,\mathrm{obl}} < f_H$.

Allowance for the non-uniformity of the Earth's magnetic field

In all the problems and examples discussed hitherto the external field $\boldsymbol{H}^{(0)}$ has been regarded as uniform in space. This assumption is, however, unjustified not only when considering the propagation of radio waves in the solar corona (see § 36), but sometimes also in the Earth's ionosphere.

The Earth's magnetic field is, to a first approximation, a dipole field, and at a height z above the Earth's surface it is

$$\left.\begin{aligned} H^{(0)}(\varrho + z) &= H^{(0)}(\varrho)\, \varrho^3/(\varrho + z)^3, \\ f_H(\varrho + z) &= f_H(\varrho)\, \varrho^3/(\varrho + z)^3, \end{aligned}\right\} \tag{35.17}$$

where $\varrho \approx 6360$ km is the radius of the Earth and $H^{(0)}$ is the field at the urface. These formulae are strictly valid only for the poles and the equator,

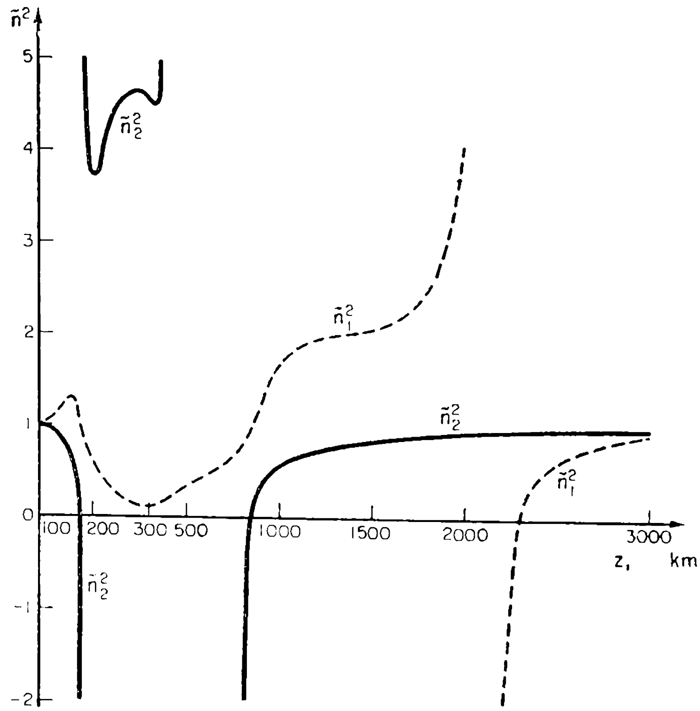

FIG. 35.7. The quantities $\tilde{n}_{1,2}^2$ as functions of height above the Earth's surface. The abscissa scales are different for $z < 300$ km and $z > 300$ km.

since they do not take the dependence on angle into account. It is clear from (35.17) that, when z is (e.g.) 200 km, the difference $f_H(\varrho) - f_H(\varrho + z) \approx \approx 1{\cdot}5 \times 10^{-5} f(\varrho)$. Hence, in probing the ionosphere, unless a measurement of f_H as a function of z is desired, the magnetic field may in fact be regarded as uniform. The situation is different when the ionosphere and the cosmic radio emission are being investigated by means of artificial satellites and high-altitude rockets, and also in the study of whistlers and some other phenomena. For example, when long-wave cosmic radio emission is received on the ground or by a satellite [227] the effect of the inhomogeneity of the Earth's magnetic field on the propagation of such waves must be taken into account. It will suffice to mention that, for a frequency $f < f_H(\varrho)$, a transition from the condition $u = f_H^2(\varrho + z)/f^2 < 1$ to $u > 1$ occurs at a certain height as the Earth is approached.

As an example [228], Fig. 35.7 shows the functions $\tilde{n}_{1,2}^2(z)$ for $f_H(\varrho) = = 1{\cdot}4 \times 10^6$, $f = 0{\cdot}6 \times 10^6$ and $\alpha = 20°$ for a layer of electron density $N = N_{\max}[1 - (z - 300)^2/4 \times 10^4]$ when $100 < z < 300$ km and $N = N_{\max} \times \times \exp[-1{\cdot}8 \times 10^{-3}(z - 300)]$ when $z > 300$ km. The value of $N_{\max}$ is taken as $1{\cdot}25 \times 10^4$, corresponding to night-time conditions. The dependence of f_H on z is taken from (35.17). Fig. 35.7 shows that for normal incidence and the selected frequency, neither an ordinary nor an extraordinary wave of extra-terrestrial origin can reach the Earth's surface (the penetration effect at normal incidence is entirely negligible in the conditions of Fig. 35.7). For oblique incidence under certain conditions the ordinary wave may become an extraordinary wave and reach the Earth by the penetration effect described in § 29.

CHAPTER VII

RADIO WAVE PROPAGATION IN COSMIC CONDITIONS

§ 36. PROPAGATION OF RADIO WAVES IN THE SUN'S ATMOSPHERE

Introduction

WITH the development of radio astronomy, the Earth's ionosphere has ceased to be the principal field of application of the theory of electromagnetic wave propagation in plasmas. Moreover, the centre of interest in this subject is increasingly shifting from the ionosphere towards radio astronomy, cosmic (magnetic) fluid dynamics and laboratory studies of plasmas.

Despite its recent origins, radio astronomy already embraces a wide range of topics. These include the radio emission of the Galaxy and the Metagalaxy (both general and from discrete sources), of the Sun and of the planets (particularly Jupiter), radar echo tracking of meteors and of the Moon, and the monochromatic radio emission from interstellar neutral hydrogen; in addition, mention should be made of theoretical problems concerning the elucidation of the nature of the sporadic radio emission from the Sun and of the non-thermal cosmic radio emission, together with theories of the origin of cosmic rays which bear on radio astronomy. There have also been developed, as part of radio astronomy, methods of studying the Earth's ionosphere, and moreover radio astronomy is of course closely interlinked with many other branches of astronomy.

Here we shall not attempt to discuss even a few of these topics; see [205, 229–240], and also [85, 89, 136, 204, 206, 207, 227, 228]. The following treatment merely describes some properties and characteristic features of the propagation of radio waves in the Sun's atmosphere (§ 36) and in the ionised interstellar medium (§ 37).

In addition to the propagation of radio waves, that of plasma, acoustic and especially hydromagnetic waves and discontinuities in cosmic conditions has received considerable attention. When the plasma may be regarded as homogeneous, the propagation of these waves is as described in Chapters II and III. The propagation of plasma and low-frequency waves in an inhomogeneous medium has been considered in Chapters IV–VI. In general, however, the propagation of such waves under various conditions has been much less thoroughly studied than the propagation of radio waves. Some results are given in [241, 242], for example. Only waves of radio frequencies will be considered here.

The solar corona

The solar corona is a kind of very extensive ionosphere. Apart from the scale and the quantitative differences of electron density and temperature, the conditions in the solar chromosphere and corona differ from those in the ionosphere in two respects.

Firstly, the solar corona and the upper part of the chromosphere are almost completely ionised, and consist practically entirely of electrons and protons. Hence it follows from the quasineutrality condition that the electron density N is equal to the proton density N_+. Secondly, in the propagation of radio waves in the chromosphere and corona we must take into account the presence of non-uniform magnetic fields. Since the corona extends over several solar radii, even the general magnetic field of the Sun cannot be regarded as constant along the path of the wave. In recent years the strength of this field at the level of the photosphere has been of the order of one oersted; at some epochs it may be considerably greater. The spot fields reach thousands of oersteds, but decrease rapidly away from the photosphere and are no longer uniform even at distances much less than the Sun's radius. In many cases the field is weak ($\omega_H \ll \omega$) in the region of importance in wave propagation, and the coronal plasma may be regarded as isotropic.

The electron density in the corona is determined from optical observations: the continuous-spectrum emission of the corona is due to the scattering of photospheric light by electrons in the corona. Some values derived from [243] and from the frequently used empirical formula [229, 244]

$$\left.\begin{aligned} N(\eta) &= 10^8(1{\cdot}55\,\eta^{-6} + 2{\cdot}99\,\eta^{-16})\ \text{electrons/cm}^3, \\ \eta &= r/r_\odot, \end{aligned}\right\} \tag{36.1}$$

are given in Table 36.1. Here r is the distance from the centre of the Sun, and $r_\odot = 6{\cdot}96 \times 10^{10}$ cm is the radius of the photosphere.†

† It must be remembered that the corona itself and the density $N(\eta)$ vary during the cycle of solar activity, and also as a result of various sporadic processes. Moreover, the corona is not in reality spherically symmetrical. Consequently the values in Table 36.1 can give only a rough idea, and the difference between the second and third columns is not large. The two series of values of $N(\eta)$ are both given because both have been used in calculations. It may also be noted that in the plane of the Sun's equator, according to [230], $N(\eta = 6) = 4 \times 10^4$, $N(8) = 1{\cdot}8 \times 10^4$, $N(10) = 1{\cdot}0 \times 10^4$, $N(12) = 6{\cdot}6 \times 10^3$, $N(14) = 4{\cdot}8 \times 10^3$, $N(16) = 3{\cdot}7 \times 10^3$, $N(18) = 3{\cdot}0 \times 10^3$ and $N(20) = 2{\cdot}6 \times 10^3$. These values were measured at the time of minimum solar activity.

TABLE 36.1

Electron density in the solar corona

$\eta = r/r_\odot$	N from [243]	N from (36.1)	$\eta = r/r_\odot$	N from [243]	N from (36.1)
1·03	$2{\cdot}9 \times 10^8$	$3{\cdot}2 \times 10^8$	2·2	$1{\cdot}2 \times 10^6$	$1{\cdot}4 \times 10^6$
1·06	$2{\cdot}1 \times 10^8$	$2{\cdot}3 \times 10^8$	2·4	$7{\cdot}0 \times 10^5$	$8{\cdot}1 \times 10^5$
1·10	$1{\cdot}4 \times 10^8$	$1{\cdot}5 \times 10^8$	2·6	$4{\cdot}2 \times 10^5$	$5{\cdot}0 \times 10^5$
1·2	$5{\cdot}8 \times 10^7$	$6{\cdot}8 \times 10^7$	2·8	$2{\cdot}9 \times 10^5$	$3{\cdot}2 \times 10^5$
1·3	$3{\cdot}0 \times 10^7$	$3{\cdot}7 \times 10^7$	3·0	$1{\cdot}9 \times 10^5$	$2{\cdot}1 \times 10^5$
1·4	$1{\cdot}8 \times 10^7$	$2{\cdot}2 \times 10^7$	3·5	8×10^4	$8{\cdot}4 \times 10^4$
1·6	$7{\cdot}5 \times 10^6$	$9{\cdot}4 \times 10^6$	4·0	4×10^4	$3{\cdot}8 \times 10^4$
1·8	$3{\cdot}8 \times 10^6$	$4{\cdot}6 \times 10^6$	5·0	1×10^4	1×10^4
2·0	$2{\cdot}0 \times 10^6$	$2{\cdot}4 \times 10^6$			

The base of the corona corresponds approximately to $\eta = 1{\cdot}03$ ($r - r_\odot \approx \approx 20{,}000$ km). However, the value $r - r_\odot = 10{,}000$ km ($\eta = 1{\cdot}014$) is frequently taken as the boundary between the chromosphere and the corona. For the chromosphere we can use the empirical formula (see, for example, [244])

$$N = 5{\cdot}7 \times 10^{11} \exp[-7{\cdot}7 \times 10^{-4}(h - 500)] \text{ for } 500 < h < 10{,}000, \quad (36.2)$$

where h is the height in kilometres above the photosphere; see [245] for further details.

The temperature in the chromosphere increases from $T \approx 5000°$ at its base to $T \approx 3$ to 5×10^5 at $h \approx 10{,}000$ km ($\eta = 1{\cdot}014$); in the corona we have $T_c = 10^6$ for $\eta \gtrsim 1{\cdot}05$, and as η increases further the temperature remains fairly constant.†

Table 36.1 shows that for $\eta < 2$ the electron density in the corona exceeds the maximum density $N \lesssim 2 \times 10^6$ in the F layer of the ionosphere. The difference in temperature is even more marked. The corona has such a high temperature that it is a source of intense thermal radiation at radio frequencies. The high intensity of this radio emission is due to the fact that at radio frequencies the optical thickness of the corona not only is not small, but may be very large (see [246–248] and below). The Sun's atmosphere also generates, in addition to this thermal radiation, an even more powerful sporadic radio emission.

Propagation of radio waves in the corona

To interpret the experimental data on the thermal and sporadic radio emission of the Sun, it is necessary to consider the nature of radio wave pro-

† Here we are considering the temperature of the coronal electrons and ions, which in the absence of streaming have almost a Maxwellian velocity distribution. The thermal radiation in the optical part of the spectrum is not in equilibrium with the particles, on account of the low optical thickness of the corona, and has a temperature of the order of the photospheric temperature $T_\odot \approx 6000°$.

pagation in the corona. For this purpose we can use the same formulae as have been established for the ionosphere. For the sake of convenience we may repeat the formulae here, using immediately the fact that, in the range of experimental interest, we always have

$$\omega^2 \gg \nu_{\text{eff}}^2. \tag{36.3}$$

In this case, in the absence of a magnetic field,

$$\left.\begin{aligned} \varepsilon &= 1 - 4\pi e^2 N/m\,\omega^2, \\ \sigma &= (1-\varepsilon)\,\nu_{\text{eff}}/4\pi = e^2 N\,\nu_{\text{eff}}/m\,\omega^2, \end{aligned}\right\} \tag{36.4}$$

where N is the electron density, and the effect of the ions (protons) may always be neglected, since $N_+ = N$ and $m/M = 1/1836$.

For ν_{eff} we must use formula (6.14):

$$\begin{aligned} \nu_{\text{eff}} &= \pi\frac{e^4}{(\varkappa T)^2} N\bar{v}\ln\left(0{\cdot}37\frac{\varkappa T}{e^2 N^{\frac{1}{3}}}\right) \\ &= \frac{5{\cdot}5}{T^{3/2}} N \ln\left(220\frac{T}{N^{\frac{1}{3}}}\right), \end{aligned} \tag{36.5}$$

where it is assumed that $N_i = N_+ = N$, and T is the electron temperature.

As has been mentioned in §§ 4 and 6, the accuracy of formula (36.5) is at best 5 per cent. Moreover, the argument of the logarithm in (36.5) is correct only when $T \ll 3 \times 10^5$ (see (4.28)). When $T \gg 3 \times 10^5$, instead of $\ln(0{\cdot}37\varkappa T/e^2 N^{\frac{1}{3}}) = \ln(220\,T/N^{\frac{1}{3}})$ we must use $\ln[\gamma_1(me^4/\hbar^2\varkappa T)^{\frac{1}{3}}\varkappa T/e^2 N^{\frac{1}{3}}] = \ln[\gamma_2 \times 10^4 T^{\frac{2}{3}}/N^{\frac{1}{3}}]$m where γ_1 and γ_2 are factors of the order of unity. The need for this change is evident from a comparison of formulae (4.19) and (4.19a) and from the calculations in § 6; the values of the numerical factors γ_1 and γ_2 in the argument of the logarithm have not been given precisely, since (e.g.) when $T \sim 10^6$ and $N \approx 10^6$ we have $10^4 T^{\frac{2}{3}}/N^{\frac{2}{3}} \approx 10^6$ and the factor $\gamma_2 \sim 1$ may be neglected. In the region $T \sim 3 \times 10^5$ the two formulae for the limiting cases give almost the same result and are fairly accurate. In (36.5) it is assumed that the electrons collide with protons; if they collide not with protons but with ions of charge Ze and density N_i, a factor Z^2 appears in (36.5) and N is replaced by N_i. In consequence, the highly ionised atoms of iron, nickel and other elements in the corona are Z times more effective than protons (an ion of change Ze is Z^2 times more effective than a proton, but on account of the quasineutrality of the corona, one ion replaces Z protons for a given electron density N). In the corona $Z \lesssim 20$ and apparently $N_i Z$ is always much less than N, so that the effect of the ions may be neglected.

In general

$$\left.\begin{aligned} n &= \sqrt{\{\tfrac{1}{2}\varepsilon + \sqrt{[(\tfrac{1}{2}\varepsilon)^2 + (2\pi\sigma/\omega)^2]}\}}, \\ \varkappa &= 2\pi\sigma/n\,\omega, \quad \mu = 2\omega\varkappa/c, \end{aligned}\right\} \tag{36.6}$$

but in practice we can always use the formulae valid for the case

$$|\varepsilon| \gg 4\pi\sigma/\omega, \tag{36.7}$$

namely

$$\left.\begin{aligned} n = \sqrt{\varepsilon} &= \sqrt{(1 - 4\pi e^2 N/m\,\omega^2)} \\ &= \sqrt{(1 - 3{\cdot}18 \times 10^9 N/\omega^2)}, \\ \varkappa &= 2\pi\,\sigma/n\,\omega = (1 - n^2)\,\nu_{\text{eff}}/2n\,\omega, \\ \mu = 2\omega\,\varkappa/c &= \nu_{\text{eff}}(1 - n^2)/c\,n \\ &= \frac{4\pi e^2 N \nu_{\text{eff}}}{m\,c\,\omega^2 \sqrt{(1 - 4\pi e^2 N/m\,\omega^2)}}. \end{aligned}\right\} \tag{36.8}$$

Let us consider a radio wave propagated radially from the surrounding space towards the Sun. The energy flux of radiation for a plane wave is evidently attenuated according to the relation $S = S_0 e^{-\tau(\eta)}$, where S_0 is the flux outside the corona and τ is called the optical thickness (a term which is not very appropriate in connection with radio waves):

$$\begin{aligned} \tau &= \int_r^\infty \mu\,dr = r_\odot \int_\eta^\infty \mu(\eta)\,d\eta \\ &= \frac{r_\odot}{c} \int_\eta^\infty \frac{\nu_{\text{eff}}(\eta)[1 - n^2(\eta)]}{n(\eta)}\,d\eta. \end{aligned} \tag{36.9}$$

This formula is valid in the approximation of geometrical optics and when the condition (36.7) holds. If the wave is not completely damped in the corona and reaches the point where $\varepsilon = 0$, the two above assumptions are both invalid over some range of values of ε. The reflection coefficient $|R|^2$ is then [see (31.2)]

$$\left.\begin{aligned} |R|^2 &= e^{-2\tau_0}; \\ \tau_0 &= \int_{r(0)}^\infty \mu\,dr + \frac{2\sqrt{2}\,[\nu_{\text{eff}}(0)]^{3/2}}{3c\,\omega^{\frac{1}{2}}\,|d\varepsilon/dr|_0} \\ &= \int_{r(0)}^\infty \mu\,dr + \Delta\tau, \end{aligned}\right\} \tag{36.10}$$

where $r(0)$, $|d\varepsilon/dr|_0$ and $\nu_{\text{eff}}(0)$ are the values of the corresponding quantities at the point where $\varepsilon = 0$; μ must be taken as given by (36.6) and not (36.8). In practice, however, as we shall show below, the value of τ_0 is given by formula (36.9), where the integration must be taken up to a point very close to $r(0)$ but still such that the condition (36.7) holds.

We may give here some calculated results [249], derived from the above formulae and the values of N in the second column of Table 36.1. The values of n^2 for various wavelengths from formula (36.8) are given in Table 36.2. These figures are, of course, not very exact, because the values of the density N in Table 36.1 are only approximate.

TABLE 36.2

The squared refractive index n^2 in the corona for various wavelengths

η	λ (m) = 50	25	15	5	3·76	1·5	1·0	0·6
	$\omega \times 10^{-7}$ = 3·76	7·52	12·5	37·6	50	125	188	313
1·03	–	–	–	–	–	0·40	0·74	0·906
1·06	–	–	–	–	–	0·57	0·81	0·932
1·10	–	–	–	–	–	0·72	0·88	0·956
1·155	–	–	–	–	≈ 0	–	–	–
1·157	–	–	–	–	0·02	–	–	–
1·200	–	–	–	–	0·26	0·882	0·948	0·981
1·245	–	–	–	≈ 0	–	–	–	–
1·247	–	–	–	0·01	–	–	–	–
1·30	–	–	–	0·32	0·62	0·939	0·973	0·990
1·40	–	–	–	0·59	0·77	0·963	0·984	0·994
1·60	–	–	–	0·83	0·904	0·985	0·993	0·9986
1·723	–	–	≈ 0	–	–	–	–	–
1·726	–	–	0·01	–	–	–	–	–
1·80	–	–	0·22	0·915	0·952	0·9922	0·997	0·9988
2·040	–	≈ 0	–	–	–	–	–	–
2·043	–	0·01	–	–	–	–	–	–
2·20	–	0·33	0·76	0·973	0·985	0·9976	0·9989	0·9996
2·40	–	0·61	0·86	0·984	0·991	0·9986	0·9994	0·9998
2·580	≈ 0	–	–	–	–	–	–	–
2·584	0·01	–	–	–	–	–	–	–
2·60	0·06	–	–	–	–	–	–	–
2·80	0·35	0·84	0·941	0·993	0·996	0·9994	0·9997	0·9999
3·00	0·57	0·89	0·961	0·996	0·9976	0·9996	0·9998	–
3·20	0·71	0·928	0·973	0·997	0·9984	0·9997	0·9999	–
3·40	0·81	0·952	0·983	0·998	0·9989	0·9998	0·9999	–
3·60	0·85	0·963	0·987	0·9985	0·9992	0·9999	0·9999	–
3·80	0·89	0·972	0·990	0·9989	0·9994	0·9999	–	–
4·0	0·91	0·977	0·992	0·9991	0·9995	–	–	–

Table 36.3 gives the values of ν_{eff} from formula (36.5) for various temperatures (the values $T = 6 \times 10^3$ and $T = 6 \times 10^4$ do not apply to the corona and are given only by way of illustration). We have roughly $\nu_{\text{eff}} \sim T^{-3/2}$, but the logarithmic term is still appreciable.

TABLE 36.3

Values of ν_{eff} for various temperatures

η	$N \times 10^{-6}$	$T = 6 \times 10^3$	6×10^4	3×10^5	6×10^5	10^6
1·03	290	$2{\cdot}71 \times 10^4$	1103	115	43	21
1·06	210	$1{\cdot}99 \times 10^4$	811	84	31	15
1·1	137	$1{\cdot}32 \times 10^4$	537	55	20	10
1·2	58	$5{\cdot}80 \times 10^3$	233	24	9·0	4·3
1·3	30	$3{\cdot}10 \times 10^3$	124	12·7	4·7	2·3
1·4	18	$1{\cdot}88 \times 10^3$	75	7·7	2·9	1·4
1·6	7·5	809	32	3·3	1·20	0·59
1·8	3·8	420	16	1·7	0·6	0·30
2·0	2·0	226	8·9	0·9	0·3	0·16
2·2	1·2	138	5·4	0·55	0·2	0·10
2·4	0·7	82	3·2	0·32	0·12	0·06
2·6	0·42	50	1·9	0·20	0·07	0·04
2·8	0·29	35	1·4	0·14	0·05	0·024
3·0	0·19	23	0·9	0·09	0·03	0·016
3·4	0·085	11	0·4	0·04	0·015	0·007
3·8	0·050	6·4	0·25	0·025	0·007	0·004
4·0	0·040	5·2	0·20	0·020	0·006	0·0035
4·4	0·025	3·3	0·12	0·012	0·0035	0·0022
4·8	0·014	1·8	0·07	0·007	0·0026	0·0013
5·0	0·010	1·3	0·05	0·005	0·0019	0·0009

It is clear from Table 36.3 that the condition (36.3) is always far from being violated in the corona. For example, even in the worst case when $\lambda = 50$ m and $T = 6 \times 10^3$ deg K we have $\nu_{\text{eff}}^2/\omega^2 < 10^{-6}$, and moreover such long waves cannot penetrate deep into the corona (since $\varepsilon < 0$ there) and the important values of $\nu_{\text{eff}}^2/\omega^2$ are of the order of 10^{-11} or less; for $\lambda = 1{\cdot}5$ m and $T = 3 \times 10^5$ we have $\nu_{\text{eff}}^2/\omega^2 \lesssim 10^{-14}$.

The smallness of the ratio $\nu_{\text{eff}}^2/\omega^2$ leads also to the possibility, already mentioned, of using formula (36.9) even when reflection occurs, and neglecting the small region where formulae (36.6) must replace (36.8). For example, when $T = 10^6$ and $\lambda = 50$ m, at the point $r(0)$ where $\varepsilon = 0$ we have $n^2 = n\varkappa = 2\pi\sigma/\omega = \nu_{\text{eff}}/2\omega = 5{\cdot}1 \times 10^{-10}$, and so $\varepsilon \gg 4\pi\sigma/\omega$ at a very small distance from $r(0)$ (e.g. $\varepsilon = 5{\cdot}1 \times 10^{-10}$ for $(r - r_\odot)/r_\odot = \Delta\eta = 2{\cdot}0 \times 10^{-10}$), and the change in τ when the difference between n^2 and ε is taken into account is of the order of 3×10^{-6}, whereas τ itself is 0·0553. For $\lambda = 5$ m and $T = 3 \times 10^5$ we have $n^2(0) = 8{\cdot}56 \times 10^{-9}$ at $r(0)$ and the corresponding change in τ is of the order of 10^{-4} with τ itself equal to 2·037.

For the same reason, the smallness of the ratio $\nu_{\text{eff}}^2/\omega^2$ has the result that the correction $\Delta\tau \sim \nu_{\text{eff}}{}^{3/2}$ in formula (36.10) is small. For instance, when $\lambda = 50$ m we have $\Delta\tau \approx 10^{-7}$ and 5×10^{-4} for $T = 10^6$ and 6×10^4 respectively.

TABLE 36.4

Optical thickness $\tau(\eta)$ of the corona for various wavelengths and $T = 6 \times 10^5$

η	λ(m) = 50	25	15	5	3·76	1·5	1·0	0·6
4·0	0·0004	0·0001	0·00000	—	—	—	—	—
3·5	0·0016	0·0004	0·0001	—	—	—	—	—
3·0	0·010	0·002	0·002	—	0·00000	—	—	—
2·8	0·025	0·004	0·003	0·00000	—	—	—	—
2·6	0·087	0·010	0·005	0·0003	0·0003	—	—	—
2·580	0·116	—	—	—	—	—	—	—
2·4	—	0·026	0·01	0·0008	—	—	—	—
2·2	—	0·086	0·03	—	0·0007	0·00000	0·00000	—
2·043	—	0·245	—	—	—	—	—	—
2·040	—	0·335	—	—	—	—	—	—
2·0	—	—	0·07	0·018	0·003	0·0004	0·0001	0·00000
1·8	—	—	0·14	0·046	0·01	0·0018	0·0007	—
1·726	—	—	0·53	—	—	—	—	—
1·723	—	—	0·63	—	—	—	—	—
1·7	—	—	—	0·076	0·02	—	—	—
1·6	—	—	—	0·12	0·04	0·006	0·002	0·0008
1·5	—	—	—	0·19	0·08	—	—	—
1·4	—	—	—	0·40	0·20	0·029	0·012	0·004
1·3	—	—	—	1·04	0·50	0·068	0·028	0·01
1·247	—	—	—	4·18	—	—	—	—
1·20	—	—	—	—	1·85	0·20	0·07	0·03
1·17	—	—	—	—	4·17	—	—	—
1·10	—	—	—	—	—	0·89	0·37	0·14
1·05	—	—	—	—	—	2·22	0·89	0·31
1·02	—	—	—	—	—	4·72	1·77	0·56
1·00	—	—	—	—	—	9·18	2·87	0·90
	$\|R\|^2 = 0·79$	$\|R\|^2 = 0·51$	$\|R\|^2 = 0·28$	$\|R\|^2 = 0·00005$	—	—	—	—

As an example, Table 36.4 gives the results of calculating $\tau(\eta)$ for various wavelength with $T = 6 \times 10^5$. This table also gives the values of $|R|^2$ from formula (36.10) for cases where the reflection coefficient is not too small.†

Table 36.3 shows that the ratio $\nu_{\text{eff}}(\eta, T_1)/\nu_{\text{eff}}(\eta, T_2)$ is almost independent of η. Hence formula (36.9) gives $\tau(\eta, T_1)/\tau(\eta, T_2) = \nu_{\text{eff}}(T_1)/\nu_{\text{eff}}(T_2)$ and so, if the optical thickness τ is known for any one temperature, it can easily be found for any other. It may also be noted that the accuracy of the values in the table is much smaller than might appear from the number of significant figures shown, on account of the extrapolation of the values of N, the graphical method of calculation, etc.

It is evident from Table 36.4 that, even for a uniform temperature $T = 6 \times 10^5$ throughout the corona, the waves longer than 1 metre are almost completely absorbed. The shorter waves, especially those in the centimetre range, are appreciably absorbed only in the chromosphere. We shall not give the relevant calculations, which are in essence entirely analogous to those for the corona or the ionosphere.

Emission of radio waves. Allowance for refraction

Since waves in the metre range are absorbed by the corona, it is evident that, whatever the mechanism by which they may be generated in the Sun's atmosphere, they must be emitted from the corona also. Thus the corona is the source of the Sun's radio emission in the metre wavelength range. The emission at a wavelength λ comes from the region $\eta = \eta(\lambda)$ where $\tau(\lambda)$ is of the order of unity (only $\frac{1}{10}$ of the radiation flux emerges from the region where $\tau = 2{\cdot}3$).

The intensity of the radiation can be given immediately only if it is thermal radiation for some temperature T. In that case an amount of energy

$$\begin{aligned} S\Delta f &= (2\pi f^2/c^2)\,\varkappa T\Delta f \\ &= (2\pi\varkappa T/\lambda^2)\,\Delta f \end{aligned} \tag{36.11}$$

is emitted from unit area (of a black body) per unit time in a frequency range Δf. Here $\varkappa = 1{\cdot}38 \times 10^{-16}$ erg/deg is Boltzmann's constant, and we have used the Rayleigh-Jeans law, since for the Sun at radio frequencies $\varkappa T \gg hf = \hbar\omega$.

If the corona has spherical symmetry and completely absorbs waves of length λ, the flux of thermal radiation at the Earth is $S\Delta f$, where now

$$\begin{aligned} S &= \frac{2\pi\varkappa T}{\lambda^2}\left(\frac{r_\odot}{R}\right)^2 \eta^2(\lambda) = \frac{1{\cdot}86 \times 10^{-21}}{\lambda_m^2}\, T\,\eta^2(\lambda)\,\frac{\text{W}}{\text{m}^2\,\text{M c/s}} \\ &= \frac{1{\cdot}11 \times 10^{-17}}{\lambda_m^2}\,\frac{T}{T_\odot}\,\eta^2(\lambda)\,\frac{\text{W}}{\text{m}^2\,\text{M c/s}}; \end{aligned} \tag{36.12}$$

† The table also gives the values of τ for $\eta = 1{\cdot}0$, obtained by arbitrarily extrapolating the density N in the second column of Table 36.1 to the photosphere itself. The value of N thus found is $4{\cdot}3 \times 10^8$ for $\eta = 1{\cdot}0$.

here λ_m is the wavelength in metres, $r_\odot = 6{\cdot}965 \times 10^{10}$ cm is the radius of the photosphere, $R = 1{\cdot}495 \times 10^{13}$ cm is the distance from the Earth to the Sun, $T_\odot = 6000$ is an arbitrarily chosen temperature of the photosphere, and $\eta(\lambda)$ is the value of η for which $\tau(\lambda) \approx 1$ and the temperature is T.

In (36.12) it is assumed that the value of $\eta(\lambda)$ is the same for both radial and non-radial propagation of waves, i.e. that the source of radio emission is a spherical surface of radius $\eta(\lambda) r_\odot$. This is, of course, not strictly true. Rays which do not travel along a radius of the Sun must undergo refraction, and moreover for these rays $\tau \approx 1$ at a distance from the photosphere which is not the same as for radially propagated rays. Hence the radio brightness of the disc is different at the centre and at the periphery, as in the optical wavelength range.

For a quantitative analysis of these problems it is necessary to consider the propagation in the Sun's atmosphere of radio waves which then reach the Earth. Here we may certainly use a ray treatment of the problem (i.e. use geometrical optics) [229, 244, 250, 251].

For rays propagated in a spherically symmetrical refracting medium the law of refraction is

$$n(r)\, r \sin\varphi(r) = r_\infty \sin\varphi(r_\infty) = p, \tag{36.13}$$

where $n(r)$ is the refractive index at a point at a distance r from the centre of the Sun, φ the angle between the direction of the ray and the radius vector, and p the distance between a ray which enters the Sun's atmosphere (before

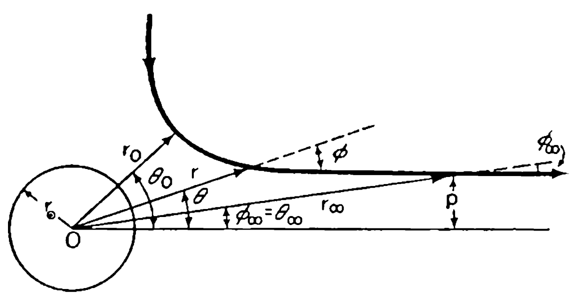

FIG. 36.1. Ray path in the solar corona.

refraction) and a radial ray (Fig. 36.1). In (36.13) it is assumed, of course, that $n(r_\infty) = 1$. Using (36.13) together with simple geometrical arguments, it may be shown that the element of length of the ray is

$$ds = \frac{dr}{\cos\varphi} = \frac{dr}{\sqrt{(1 - p^2/n^2 r^2)}}$$

and the optical thickness along the ray is

$$\tau(r) = \int_r^\infty \mu\, ds$$
$$= r_\odot \int_\eta^\infty \frac{\mu(\eta)\, d\eta}{\sqrt{[1 - \zeta^2/n^2(\eta)\,\eta^2]}}, \tag{36.14}$$

where $\zeta = p/r_\odot$ and $\mu(\eta)$ is the absorption coefficient (36.8).

The path of the ray in polar coordinates r, θ is

$$\theta = \theta_\infty + \int_r^{r_\infty} \frac{p\, dr}{r\sqrt{(r^2 n^2 - p^2)}}, \tag{36.15}$$

where we consider only rays incident on the Sun in a parallel beam (so that $\theta_\infty = \varphi_\infty$ and $r_\infty \sin\varphi_\infty = p$).

At the reflection point (or, better, the "turning point") $\varphi = \frac{1}{2}\pi$, $r = r_0$ and $\theta = \theta_0$, with

$$r_0 n(r_0) = p. \tag{36.16}$$

The ray path is symmetrical about a line through the centre of the Sun and the turning point. At this point the denominator in the integrals (36.14) and (36.15) is zero, but this is not important either in the graphical construction

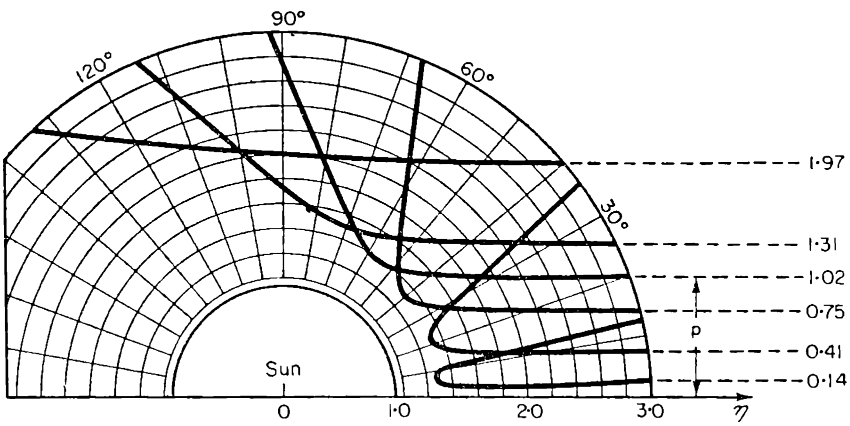

FIG. 36.2. Ray paths in the solar corona for radio waves with $\lambda_0 = 5$ m. The diagram shows the values of the "impact parameter" p in terms of the radius $r_\odot$ of the photosphere.

or in the calculations which use the linear approximation to the function $n^2(r)$ near the turning point. In order to avoid trouble with the signs, the expressions (36.14) and (36.15) may be used only for $\theta < \theta_0$, since by symmetry the values of τ for $\theta > \theta_0$ need not be calculated separately.

The ray paths in the corona are shown in Fig. 36.2 for $f = 60$ Mc/s ($\lambda = 5$ m). All the rays enter from the right along parallel paths and therefore differ from one another only in the value of the "impact parameter" p (see Fig. 36.1).

The intensity of the radiation along the various rays is given by the equation of transfer, which is widely used in astrophysics. The only difference between radio and optical frequencies is that for the latter we may usually disregard the difference between n and unity. We shall not pause to discuss the use of the equation of transfer, especially because in the case of greatest practical importance the result may be obtained directly. The reason is that, with our present knowledge concerning the temperature and electron density in the corona, it is usually appropriate to regard the temperature of the corona as a constant (i.e. independent of r). The sharp fall in temperature at the chromosphere may likewise be conveniently taken into account by assuming that at some boundary $r = r_{\mathrm{ch}}$ the temperature changes suddenly from T_c to T_{ch} (where T_c and T_{ch} are the temperatures of the corona and the chromosphere). Even with this highly idealised model of the Sun's atmosphere [244] the parameters T_c, T_{ch} and r_{ch} remain unknown, nor do we have more than a rough form for the function $N(r)$.

These unknown parameters must be determined from a comparison of the theoretically calculated distribution of the intensity of radio emission over the Sun's disc with the results obtained experimentally. If a more complex distribution of temperature in the Sun's atmosphere is used, then no unique interpretation of the limited experimental data is in general possible.

Assuming the temperature of the corona to be constant, we may determine the intensity along the ray directly by using Kirchhoff's law. For rays which do not enter the chromosphere the specific intensity I of thermal radiation along a ray at distance p from a radial ray (see Fig. 36.1) is

$$I(p) = \frac{2\varkappa f^2}{c^2} \cdot T_c[1 - \exp(-2\tau_c(r_0))], \tag{36.17}$$

where $2\varkappa f^2 T_c/c^2$ is the specific intensity† of black-body radiation of temperature T_c, and $2\tau_c(r_0)$ is the optical thickness of the corona along the ray; $\tau_c(r_0)$ is the optical thickness up to the turning point $r_0 = p/n(r_0)$, and its value is given by formula (36.14) with $\eta = \eta_0 = r_0/r_\odot$. For rays which penetrate into the chromosphere we have

$$I(p) = \frac{2\varkappa f^2}{c^2} T_c[1 - \exp(-\tau_c(r_{\mathrm{ch}}))] + \frac{2\varkappa f^2}{c^2} T_{\mathrm{ch}} \exp(-\tau_c(r_{\mathrm{ch}})), \tag{36.18}$$

where $\tau_c(r_{\mathrm{ch}})$ is the optical thickness of the corona along the ray up to the point r_{ch} where the chromosphere begins, and we have used the fact that in the chromosphere itself the ray is almost completely absorbed (i.e. $\tau_{\mathrm{ch}} \gg 1$).

† It may be recalled that, by definition, $I\Delta f\Delta\Omega$ is the quantity of energy passing in unit time through unit area normal to the direction of the ray and belonging to a frequency range Δf and solid angle $\Delta\Omega$. The above expression for I is valid for $n = 1$. In the general case we have in our problem $I = 2\varkappa f^2 n^2 T_e/c^2$, i.e. c in the expression valid for a vacuum must be replaced by c/n, where n is the refractive index for frequency f. The derivation of the expression (36.17) is given at the end of the present section in connection with its generalisation to the case of a magnetoactive plasma.

The intensity (36.18) corresponds to that of the radiation from a black body of effective temperature

$$T_{\text{eff}} = T_c[1 - \exp(-\tau_c(r_{\text{ch}}))] + T_{\text{ch}} \exp(-\tau_c(r_{\text{ch}})). \tag{36.19}$$

With (36.17) we evidently have

$$T_{\text{eff}} = T_c[1 - \exp(-2\tau_c(r_0))]. \tag{36.20}$$

The flux of radiation at the Earth is $\Delta S\, \Delta f$ from an annulus of the Sun of area $2\pi p\, \Delta p$, and the total flux is $S\Delta f$, where

$$\left.\begin{aligned} \Delta S &= 2\pi p\, \Delta p I(p)/R^2, \\ S\Delta f &= (2\pi\, \Delta f/R^2)\int\limits_0^\infty I(p)\, p\, dp. \end{aligned}\right\} \tag{36.21}$$

If the emitting surface is that of a black sphere of radius $r = r_{\odot}\eta(\lambda)$, then $I = 2\varkappa f^2 T_c/c^2$ for $p \leqq r$ and $I = 0$ for $p > r$, so that formula (36.21) for S becomes (36.12).

The intensity of radio emission from the Sun as a whole may conveniently be characterised by the effective temperature $T_{\odot,\text{eff}}$, defined as that photospheric temperature which would give the observed radio emission. Evidently

$$\begin{aligned} T_{\odot,\text{eff}}(\lambda) &= S(R/r_{\odot})^2\, \lambda^2/2\pi\varkappa \\ &= 5{\cdot}4 \times 10^{20} S\, \lambda^2, \end{aligned} \tag{36.22}$$

where $S\Delta f$ is the flux of radio emission from the entire Sun as observed at the Earth for wavelength λ (expressed in metres, with S in watts per square metre per Mc/s).

By means of the above formulae, the problem of the thermal radio emission from the Sun may be analysed, on the basis of the model used, without any difficulty of principle. We shall not give the calculations here (see [205, 229, 234, 244, 250]), but merely note that one obvious consequence of this model is a brightening towards the limb of the Sun's disc. This occurs for waves which in the central part of the disc (i.e. for $r < 0{\cdot}8$ to $0{\cdot}9\, r_{\odot}$) are strongly absorbed only in the chromosphere, where $T = T_{\text{ch}} \ll T_c$.

The effect of the magnetic field

The sporadic radiation from the Sun in the metre wavelength range is non-equilibrium (non-thermal) radiation, and its intensity sometimes reaches very large values ($T_{\odot,\text{eff}} \sim 10^{12}$ to 10^{13} deg). Whatever the mechanism responsible for the sporadic radio emission (see [136, 252]), the above discussion remains valid as regards the propagation of this radiation in the corona and its emergence therefrom. We cannot, however, consider only an isotropic coronal plasma. The effect of the magnetic field may, indeed, be important even for the thermal radio emission, but this effect is relatively small, because the Sun's general magnetic field is weak, and it is the general magnetic field which is significant as regards the thermal radio emission from the Sun as a whole. When the sporadic radio emission is not polarised or only slightly polarised, the coronal plasma also may in general be regarded as isotropic, but the sporadic emission is often strongly polarised, and then the coronal plasma must certainly be considered to be magnetoactive.

The necessary general formulae have been derived in Chapters III and V. The magnetic field in the corona, unlike that in the ionosphere, cannot in general be regarded as uniform. Moreover, the much higher temperature in the corona has the result that the effeet of the thermal motion (spatial dispersion) is greater.

By way of examples which afford some idea of the importance of the magnetic field, we may give the results of calculations [89] of the refractive index $n_{1,2}$ above a sunspot (absorption is neglected; in the corona it usually has very little effect on the form of the function $n_{1,2}(\eta)$). The field source is taken to be an extended magnetic pole at the level of the photosphere.† Then the field on the axis is

$$\left.\begin{aligned} H^{(0)}(r) &= H_b\left(1 - \frac{h}{\sqrt{(h^2+b^2)}}\right), \\ h &= r - r_\odot \\ &= r_\odot(\eta - 1). \end{aligned}\right\} \qquad (36.23)$$

Here b is the radius of the magnetic pole and H_b the field at the level of the photosphere (Fig. 36.3). The electron density is given by (36.1). Figs. 36.4 to

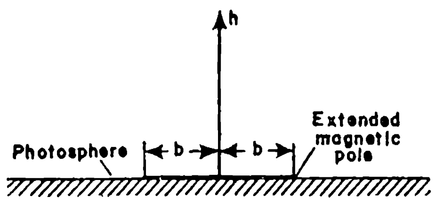

FIG. 36.3. Extended magnetic pole forming a circle of radius b at the level of the photosphere.

36.6 give graphs of the function $\tilde{n}^2_{1,2}(\eta)$ for $\alpha = 0°$, $15°$ and $90°$ and $H_b = 250$ and 2500 oersted, with $\omega = 2\pi \times 10^8 (\lambda_0 = 2\pi c/\omega = 3$ m) and $b^2 = 10^{19}$ cm^2, i.e. of the order of the area of a fairly large spot.

The difference between the cases shown in Fig. 36.5a and b is that for $H_b = 2{\cdot}5 \times 10^3$ the level $\omega_H = \omega$ (i.e. $u = 1$) is higher in the corona than the level $\omega_0 = \omega$ (i.e. $v = 1$), but for $H_b = 2{\cdot}5 \times 10^2$ it is lower. Hence, in particular, it follows that in Fig. 36.5a the interaction of the normal waves occurs

† This model gives a good approximation to the fields of unipolar spots, which form about 35 per cent of all spots [253]. For bipolar spot groups this approximation still applies, within certain limits, in the region above a spot of a given polarity.

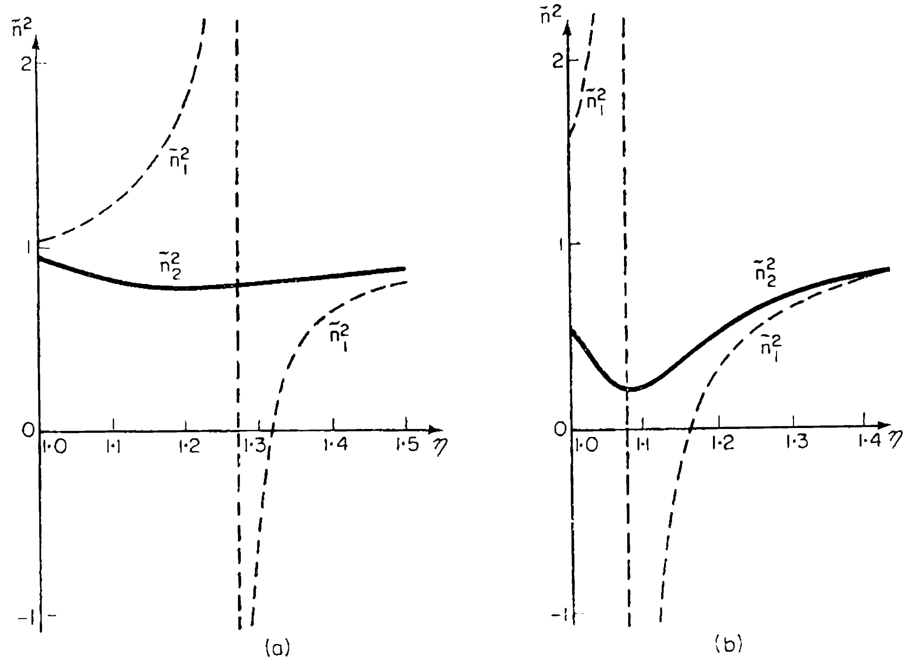

FIG. 36.4. The functions $\tilde{n}^2_{1,2}(\eta)$ in the coronal plasma for $\alpha = 0°$; the vertical broken lines correspond to the point $v_{1\infty}$.
(a) $H_b = 2{\cdot}5 \times 10^3$
(b) $H_b = 2{\cdot}5 \times 10^2$

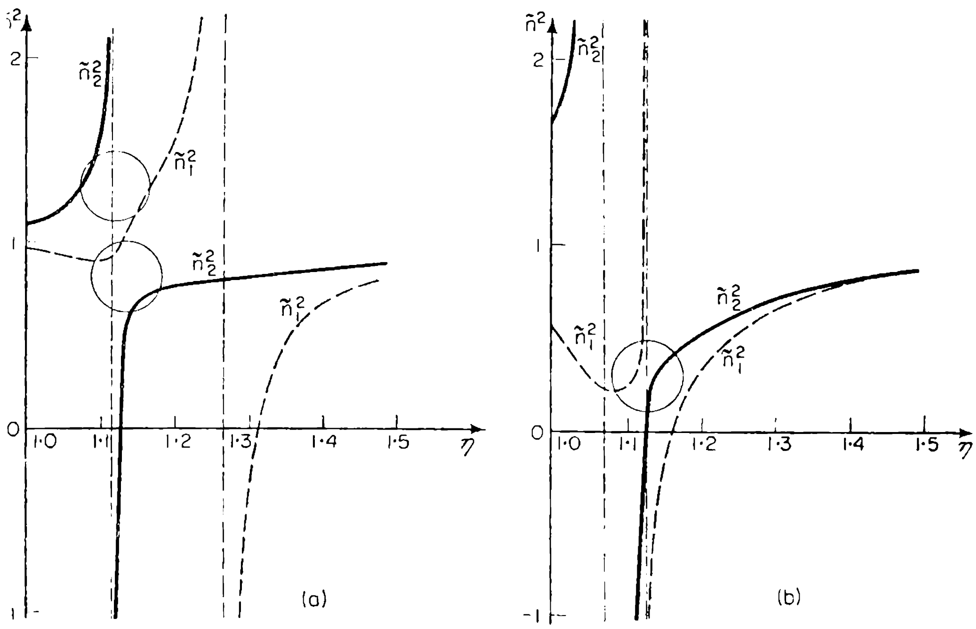

FIG. 36.5. The functions $\tilde{n}^2_{1,2}(\eta)$ in the coronal plasma for $\alpha = 15°$; the vertical broken lines correspond to the points $v_{1\infty}$ and $v_{2\infty}$.
(a) $H_b = 2{\cdot}5 \times 10^3$
(b) $H_b = 2{\cdot}5 \times 10^2$

in a region where $\omega_H > \omega$, whereas in Fig. 36.5b the interaction regions lie in a layer where $\omega_H < \omega$; the regions concerned are encircled in Fig. 36.5. It may be noted that this difference in the forms of the curve for strong and weak fields occurs only for intermediate values of the angle α $(0 < \alpha < \frac{1}{2}\pi)$;

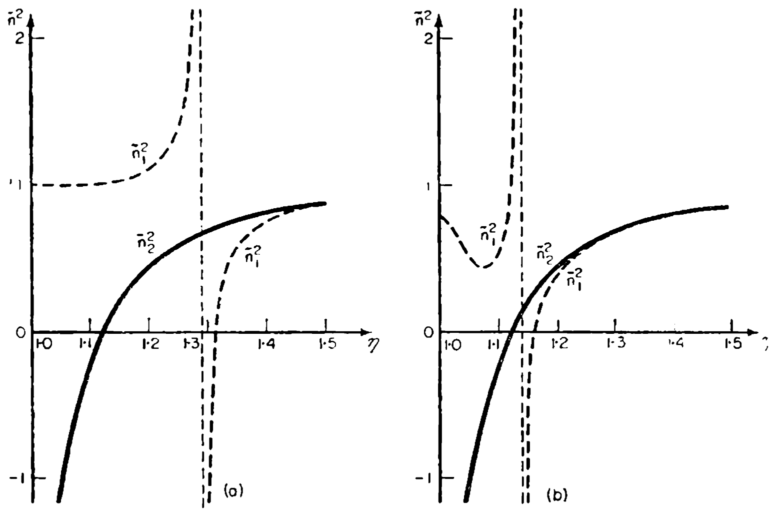

FIG. 36.6. The functions $\tilde{n}^2_{1,2}(\eta)$ in the coronal plasma for $\alpha = 90°$.
(a) $H_b = 2{\cdot}5 \times 10^3$
(b) $H_b = 2{\cdot}5 \times 10^2$

it is clear from Figs. 36.4 and 36.6 that for $\alpha = 0°$ and $\alpha = 90°$ the change in the magnetic field does not lead to any marked change in the curves $\tilde{n}^2_{1,2}(\eta)$.

Fig. 36.7a shows the curves of $\tilde{n}^2_{1,2}(\eta)$ in a weak magnetic field $H_b = 25$ oersted with $\alpha = 15°$ (in the quasihydrodynamic approximation, allowing for the

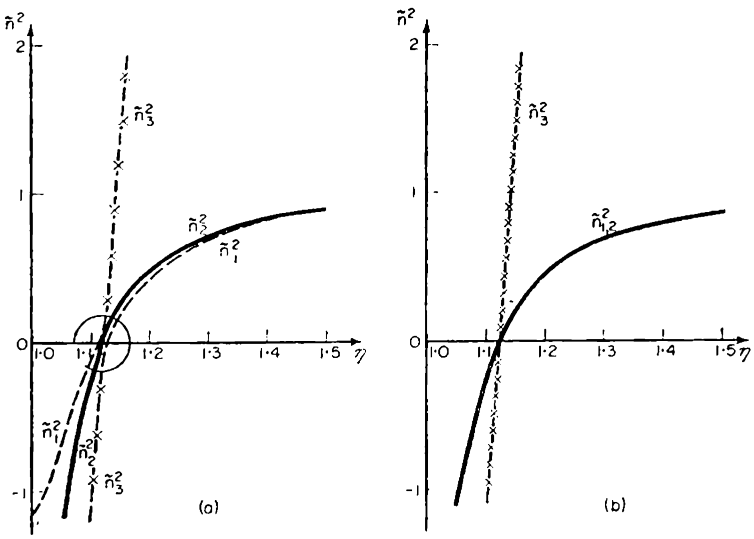

FIG. 36.7. The functions $\tilde{n}^2_{1,2,3}$ in the coronal plasma for $\alpha = 15°$.
(a) $H_b = 25$ oersted
(b) $H_b = 0$ (isotropic plasma)

thermal motion). Fig. 36.7b shows graphs of $\tilde{n}^2_{1,2}$ and $\tilde{n}^2_3$ under the same conditions, but when the magnetic field is absent:

$$\tilde{n}^2_{1,2} = 1 - \omega_0^2/\omega^2,$$

$$\tilde{n}^2_3 = \frac{1 - \omega_0^2/\omega^2}{v_T^2/c^2}.$$

It is important to note that, in a sufficiently weak field, the cross-and-dash part of the curves in Fig. 36.7a has the same properties as the similar curve of $\tilde{n}^2_3$ in Fig. 36.7b. This is in agreement with what has been said in § 12, and has the result that when $u = \omega_H^2/\omega^2 \ll 1$ we often need consider only transverse and plasma waves in an isotropic plasma, instead of having to investigate the more complex problem of the generation and propagation of waves in a magnetoactive plasma.

Transformation of plasma waves into radio waves

Fluxes of particles generate plasma waves in the corona (in the magnetoactive case this means that waves are formed in the neighbourhood of the poles of the functions $\tilde{n}^2_{1,2}$). On emergence from the corona, only radio waves are of interest in radio astronomy. Thus the problem of the transformation and emergence of waves from the corona is one of the most important in the theory of sporadic radio emission from the Sun. We shall not give a detailed discussion (sec [89, 133, 136, 252]), but merely call attention to certain points.

In a homogeneous plasma the transformation of various types of wave can occur only by scattering at inhomogencities (fluctuations) of either thermal or non-thermal origin.† For example, in an isotropic plasma there are two types of fluctuations: those of plasma density, which are unrelated to the appearance of space charge, and those where the ion density remains unchanged, i.e. fluctuations of the electron density only. The fluctuations of the latter type are actually fluctuation plasma waves which can scatter the plasma wave or transverse wave under consideration. In the scattering of plasma waves, which have been generated in any way in the corona, by fluctuations of the two types, transverse (radio) waves are formed, which then leave the corona if the conditions are favourable. A similar mechanism of transformation of various types of normal waves by scattering occurs in a magnctoactive plasma.

In an inhomogencous plasma, the transformation of waves may also occur regularly, i.e. without the intervention of scattering processes. In an isotropic plasma near the point $\varepsilon(\omega, \eta) = 0$ a plasma wave may become a transverse wave by means of the interaction discussed in § 20. In a magnetoactive plasma

† The term "homogeneous plasma" is here used in a conventional sense, of course; we mean one which is homogeneous "on the average" (in the absence of fluctuations or local inhomogeneities).

the transformation is due to an interaction of waves leading to a "tripling" effect (see §§ 28, 29; the corresponding regions of interaction are encircled in Fig. 36.5).

Collisionless absorption

In the magnetoactive coronal plasma, because of the high temperature, we cannot take account only of the absorption due to collisions. In addition, there is absorption resulting from processes inverse to the emission of magnetic bremsstrahlung and Cherenkov radiation (see § 12). The Cherenkov absorption occurs only in the region where $n_{1,2} > 1$. As regards the problem of emergence of radiation, this absorption mechanism is not of great importance (see Figs. 36.4–36.6; the Cherenkov absorption is non-zero for waves which can leave the corona only as a result of interaction). The magnetic bremsstrahlung will be observed when $\alpha \neq 0$, for both the ordinary and the extraordinary waves at the frequencies $\omega = s\omega_H$ ($s = 1, 2, 3, \ldots$). The corresponding values of the absorption coefficient $\mu = 2q = 2\omega\varkappa/c$ have already been given in § 12; see formulae (12.36)–(12.46).

For $s = 1$ and $s = 2$ the resonance absorption coefficient $\mu_{1,\text{res}}\,(s = 1) \sim \sim \mu_{1,\text{res}}\,(s = 2) \sim (\omega/c)(\omega_0^2/\omega^2)\,\beta_T$, while the collision absorption coefficient $\mu_{1,2,\text{coll}} \sim (\nu_{\text{eff}}/c)\,\omega_0^2/\omega^2$. Thus $\mu_{1,2,\text{coll}}/\mu_{1,2,\text{res}} \sim \nu_{\text{eff}}/\omega\beta_T \sim 10^{-6}$ for $\nu_{\text{eff}} \sim 10$, $\omega \sim 2\pi \times 10^8$ and $\beta_T \sim 10^{-2}$, and so the resonance absorption is very strong. Some rough estimates [85, 89] show that in the corona the optical thickness due to resonance absorption is, for the extraordinary wave,

$$\left.\begin{aligned} &\tau_1(s = 1) \sim \tau_1(s = 2) \sim 10^5,\\ &\tau_1(s = 3) \sim 60,\\ &\tau_1(s = 4) \sim 7 \times 10^{-2} \end{aligned}\right\} \tag{36.24}$$

for $\omega \sim 2\pi \times 10^8$, $\omega_0 \sim \omega$, $\beta_T = \sqrt{(\varkappa T/mc^2)} \sim 10^{-2}$, $T \sim 10^6$ and $L_H \sim 10^{10}$ cm (L_H being a characteristic distance over which the field $H^{(0)}$, and therefore the frequency ω_H, vary appreciably). For the ordinary wave the values of $\tau = \tau_2$ under the same conditions are smaller by one or two orders of magnitude. Nevertheless $\tau_2(s = 1) \sim \tau_2(s = 2) \gtrsim 10^3 \gg 1$.

Thus resonance absorption for $\alpha \sim 1$ may greatly attenuate waves passing through the levels $\omega \approx \omega_H$, $\omega \approx 2\omega_H$ and (for the extraordinary wave) $\omega \approx 3\omega_H$. For the model of the corona above a spot used to obtain Figs. 36.4 to 36.7, we have with $H_b = 2{\cdot}5 \times 10^3$ oersted, the levels $\omega = \omega_H$, $\omega = 2\omega_H$ and $\omega = 3\omega_H$ lying respectively at $\eta = 1{\cdot}27$, $1{\cdot}38$ and $1{\cdot}47$. It follows from Fig. 36.5a that in this case the resonance absorption greatly modifies the conditions for the emergence of waves from the corona. In a field $H_b = 2{\cdot}5 \times 10^2$ oersted, however, the effect of the resonance absorption is not great, since the corresponding values of η are $1{\cdot}085$, $1{\cdot}121$ and $1{\cdot}148$.

Kirchhoff's law in a magnetoactive plasma

To conclude, let us consider the question of the use of Kirchhoff's law in a magnetoactive plasma (see, in particular, [62]).

For convenience, we shall begin by repeating the argument for an isotropic medium. The intensity of radiation $I'_{0\omega}$ entering a vacuum from a bounded isotropic medium consists of the radiation I_ω due to that medium itself and the radiation $I_{0\omega}$ incident on the medium, reduced by a factor $e^{-\tau}$:

$$\left.\begin{aligned} I'_{0\omega} &= I_\omega + I_{0\omega}\, e^{-\tau}, \\ \tau &= 2\,\frac{\omega}{c}\int_a^b \mu\, ds, \end{aligned}\right\} \qquad (36.25)$$

where μ is the absorption coefficient and ds is an element of the ray shown in Fig. 36.8; it is assumed that the ray is not partly reflected and partly transmitted, i.e. that reflection is either complete or absent. The relation (36.25)

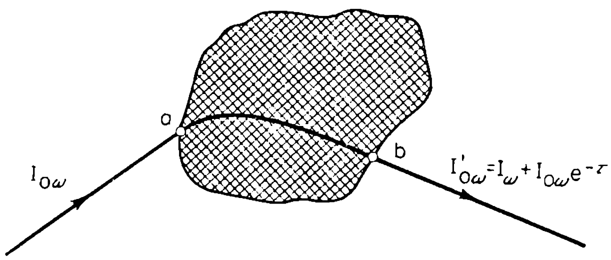

FIG. 36.8. Change in intensity of radiation when a ray passes through an absorbing medium.

expresses the conservation of energy, and moreover is based on the concept of rays, which is equivalent to using the approximation of geometrical optics.

In the case of complete thermodynamic equilibrium we have

$$\left.\begin{aligned} I_{0\omega}(T) &= \frac{1}{2\pi}\, I_{0f}(T) \\ &= \frac{\hbar\,\omega^3}{4\,\pi^3\, c^2}\, \frac{1}{e^{\hbar\,\omega/\varkappa\, T}-1}\,; \\ I_{0\omega} &= \omega^2\,\varkappa\, T/4\,\pi^3\, c^2 \quad \text{for} \quad \hbar\,\omega/\varkappa\, T \ll 1, \end{aligned}\right\} \qquad (36.26)$$

where both polarisations are taken into account and $I_{0f} \equiv I$ is the intensity used previously, referred to the frequency range $df = d\omega/2\pi$.

In a vacuum the equilibrium radiation is homogeneous and isotropic; thus in equilibrium we have $I'_{0\omega}(T) = I_{0\omega}(T)$, and the expression (36.25) becomes

$$I_\omega(T) = I_{0\omega}(T)\,(1 - e^{-\tau}). \qquad (36.27)$$

It is of interest to note that, according to its derivation, I_ω is the radiation leaving the medium to the right in Fig. 36.8, and τ is the optical thickness when the wave medium is likewise traversed from left to right. However, the ordinary reciprocity theorem, which is valid in non-magnetoactive media, shows that the optical thickness τ is the same for passage through the medium in either direction. It is only for this reason that Kirchhoff's law, i.e. the relation (36.27), can be formulated without mentioning the direction of propagation of the wave.

Considering now the radiation in the vacuum which has left the magnetoactive plasma, we may note that in thermodynamic equilibrium this radiation must be unpolarised. (The magnetic field does not destroy the thermodynamic equilibrium, and the equilibrium radiation in the vacuum is unpolarised.) Moreover, the equilibrium radiation in the vacuum can be represented as the sum of two non-coherent waves whose polarisation ellipses are mutually perpendicular and have the same axis ratio [255, § 6–7]. The intensity of each of these waves is evidently $\frac{1}{2}I_{0\omega}(T)$. The polarisation ellipses of the normal waves on leaving the magnetoactive plasma are also mutually perpendicular and have the same axis ratio (§ 11). Hence it follows that the intensity of the equilibrium radiation in the vacuum can be represented as a sum of the intensities of normal waves corresponding to the magnetoactive medium under consideration. Each of the normal waves corresponds to an intensity $\frac{1}{2}I_{0\omega}(T)$.

Assuming that the normal waves do not undergo partial reflection in the plasma and do not interact (i.e. ignoring effects such as "tripling" and the limiting polarisation; see §§ 26, 28, 29), we find that Kirchhoff's law for a magnetoactive medium is

$$I_{\omega\, 1,2}(T) = \tfrac{1}{2}\, I_{0\omega}(T)\,(1 - e^{-\tau_{1,2}}), \tag{36.28}$$

with

$$\left.\begin{aligned} \tau_{1,2} &= \frac{2\omega}{c}\int \mu_{1,2}\cos\left(\boldsymbol{k}, \frac{d\omega}{d\boldsymbol{k}}\right) ds, \\ \mu_{1,2} &= 2\omega\, \varkappa_{1,2}/c. \end{aligned}\right\} \tag{36.29}$$

Of course, for $\tau_1 \gg 1$ and $\tau_2 \gg 1$ we have $I_\omega(T) = I_{\omega 1}(T) + I_{\omega 2}(T) = I_{0\omega}(T)$, as it should be. The expression (36.29) involves the cosine of the angle between the wave vector $\boldsymbol{k}$ and the direction of the ray (the group-velocity vector) $d\omega/d\boldsymbol{k}$, because $\varkappa_{1,2}$ and $\mu_{1,2}$ characterise the absorption in the direction of $\boldsymbol{k}$ and in calculating $\tau_{1,2}$ we need to know the absorption along the ray; the path element along the normal is $ds' = \cos(\boldsymbol{k}, d\omega/d\boldsymbol{k})\, ds$, where ds is the path element; see, for instance, Fig. 36.1.

Since in a magnetoactive plasma the ordinary reciprocity theorem is in general invalid, the intensity of radiation $I_{\omega 1,2}$ in any direction (36.28) is determined by the values of $\tau_{1,2}$ for a wave in the same direction. Hence, for example, a system of the optical-valve or radio-valve type will emit strongly in any given direction only those waves which it strongly absorbs when they pass through it in the same direction.

By means of formula (36.28) we can discuss the thermal radiation of the isothermal corona with allowance for the effect of the magnetic field. Moreover, as in the isotropic case, it is easy to obtain more general expressions for a stratified medium with layers at different temperatures [see, for example, (36.18)].

§ 37. PROPAGATION OF RADIO WAVES IN THE INTERSTELLAR MEDIUM

Absorption of radio waves in the interstellar gas: general remarks

The cosmic radio emission, apart from that due to the Sun, planets and comets, is generated in interstellar space, in isolated galactic and extragalactic nebulae, and also in intergalactic space. Some part of this emission is thermal, but the greater part is non-equilibrium emission due to the acceleration, in weak magnetic fields, of relativistic electrons which occur in the cosmic radiation [205, 231–233]. Moreover, there is observed a monochromatic cosmic radio emission from neutral hydrogen ($\lambda = 21$ cm); this is due to transitions between sub-levels in the hyperfine structure of the ground state of the hydrogen atom.

Whatever the nature of the cosmic radio emission, it is important to note that the interstellar electron gas, despite its low density, absorbs a considerable amount of radiation in the range here considered (wavelengths from centimetres to hundreds of metres) [205, 227, 229–233]. We shall discuss this problem in more detail here, but shall not consider the absorption in atomic hydrogen of waves of length close to 21 cm [232] or the absorption of low-frequency (hydromagnetic) waves in the interstellar medium (§ 14).

The electron density in the interstellar gas varies over a wide range, but does not usually exceed $N = 10\ \mathrm{cm}^{-3}$, and in most regions of the Galaxy it is considerably less than this [205, 232, 233, 254]. Hence, even for the longest waves considered ($\lambda \sim 1$ km, $\omega \sim 2 \times 10^6\ \mathrm{sec}^{-1}$), we have $1 - n^2 = 3{\cdot}18 \times 10^9 N/\omega^2 \lesssim 10^{-2}$, i.e. in determining the absorption we can always suppose that $n = 1$. Thus it might seem that for the absorption coefficient μ we should take the expression (36.8) with $n = 1$, i.e.†

$$\mu = 4\pi e^2 N \nu_{\mathrm{eff}}/m c \omega^2$$
$$= \frac{4\pi e^6 N^2}{(\varkappa T)^2 m c \omega^2} \sqrt{\frac{8\varkappa T}{\pi m}} \ln\left(0{\cdot}37 \frac{\varkappa T}{e^2 N^{\frac{1}{3}}}\right)$$
$$= \frac{0{\cdot}58 N^2}{T^{3/2} \omega^2} \ln\left(220 \frac{T}{N^{\frac{1}{3}}}\right). \tag{37.1}$$

† The temperature of the interstellar electron gas, even in the hottest strongly ionised regions, is of the order of 10,000°, and so the condition (4.28) may be regarded as fulfilled.

This formula, however, is usually invalid for a gas of very low density. At first sight it may appear that this invalidity is due to the violation of the condition $\lambda^3 \gg 1/N$ (in other words, the interstellar gas is so rarefied that for short radio waves there is only a small number of electrons in a volume $\sim\lambda^3$). In deriving the formulae for n and $\varkappa$ we have assumed that $\lambda \gg N^{-\frac{1}{3}}$ (see § 4); nevertheless, it is easy to see that, even if that inequality does not hold, formula (37.1) is not invalidated, unless for other reasons.

This may be proved as follows. Even if the inequality $\lambda \gg N^{-\frac{1}{3}}$ is not satisfied, the conductivity σ and the permittivity ε may still be assigned a certain meaning, since in calculating these quantities in § 6 by the kinetic method we made no assumptions concerning the electron density N, but merely assumed that the region considered is much smaller than the wavelength λ (the field being therefore regarded as uniform). Hence, if the volume $\sim\lambda^3$ contains few particles, i.e. if $\lambda^3 N \lesssim 1$, the values of σ and ε calculated by the kinetic method are quantities averaged over a large number of such small volumes or over a time $\Delta t \gg 2\pi/\omega$.

In other words, σ and ε cannot now be regarded as the ordinary macroscopic parameters of the medium, simply because the phenomenological wave equations cannot be applied to a rarefied medium where the inequality $\lambda \gg N^{-\frac{1}{3}}$ does not hold. But if we consider the absorption of waves over a distance $L \gg \lambda$, the absorption coefficient μ calculated from the values of σ and ε thus averaged has its usual significance. Thus from this point of view formula (37.1) remains valid if μ is taken to be the mean coefficient of absorption and used to calculate the absorption of a wave traversing a sufficiently long path. The same conclusion may be reached without using the quantities σ and ε at all, but using the microscopic theory throughout and calculating the absorption coefficient as the mean energy transmitted by the wave to individual electrons. Of course this does not signify a different statement of the problem, but only a somewhat different treatment.

Thus, if formula (37.1) itself were correct, we should have no reason to carry out any further calculations. However, as has been stated in §§ 4 and 6, formula (37.1) is strictly correct only if

$$\sqrt{(4\pi e^2 N/m\omega^2)} \gg 1. \tag{37.2}$$

In the interstellar gas we have, on the contrary,

$$\sqrt{(4\pi e^2 N/m\omega^2)} \ll 1, \tag{37.3}$$

since $N \lesssim 10$ and in cases of interest to us $\omega > 2 \times 10^6$; for $N = 1$ and $\omega = 2 \times 10^6$, the parameter $\sqrt{(4\pi e^2 N/m\omega^2)} \approx 3 \times 10^{-2}$.

We shall therefore give here the derivation of a formula for μ in the case (37.3). The significance of the latter inequality is evident from the discussion in § 4: when it is satisfied, the distance traversed by an electron during one period of the high-frequency field (the field of the radio wave) is much less than the

Debye length D. Consequently the screening of the ion field by other ions and electrons is only slight and the maximum impact parameter p_m is the distance traversed by an electron in one period, viz. $p_m \sim 2\pi\bar{v}/\omega \sim (2\pi/\omega)\sqrt{(\varkappa T/m)}$. These conclusions will be confirmed below.

Calculation of the absorption coefficient in a highly rarefied plasma

Thus our problem is to calculate the coefficient of absorption of radio waves owing to the motion of electrons in the Coloumb field of a point charge. This is most simply done by using Einstein's relations between probabilities of emission and absorption of radiation. For, from the quantum viewpoint, the absorption of radio waves in collisions is a process of absorption of photons, accompanied by a transition of an electron from one state of the continuous spectrum to another of higher energy. In a transition to a state of lower energy there is spontaneous and stimulated emission of radiation. The spontaneous emission is just bremsstrahlung.

For any system the number of photons absorbed per unit time in transitions from state 1 to state 2 is $Z_a = B_{12} N_1 U_\omega$, where B_{12} is a constant coefficient, N_1 the number of atoms in state 1 and U_ω the radiation energy density per unit interval of ω. The number of photons emitted in transitions from state 2 to state 1 is $Z_e = (A_{21} + B_{21} U_\omega) N_2$, where A_{21} is the probability of spontaneous emission per unit time and $B_{21} N_2 U_\omega$ is the number of stimulated emissions. If the statistical weights of states 1 and 2 are the same, Einstein's relations are†

$$B_{12} = B_{21},\ A_{21}/B_{12} = \hbar\,\omega^3/\pi^2 c^3. \tag{37.4}$$

For bremsstrahlung of frequency ω the transition is between two states of the continuous spectrum, with energy difference $\hbar\omega$. If the inequalities

$$\left.\begin{array}{ll} \hbar\,\omega \ll \frac{1}{2} m v^2 \sim \varkappa T, & \\ e^2/\hbar v \gg 1 & (\text{i.e. } T \sim m v^2/\varkappa \ll 3 \times 10^5) \end{array}\right\} \tag{37.5}$$

hold, the energy emitted by the electron moving in a Coloumb field can be calculated by the classical theory. In this method [255, § 9–5] we first calculate the energy $d\varepsilon_\omega$ emitted by an electron moving at a certain impact parameter p, and then derive the desired quantity $2\pi\int d\varepsilon_\omega p\,dp = dq_\omega \hbar\,\omega$, which is the energy emitted in the frequency range $d\omega$ by an electron moving at any impact parameter. The general expression for $dq_\omega \hbar\,\omega$ is fairly complex, but it becomes much simpler if we consider the emission at low or high frequencies. In the present case we are interested in low frequencies, where

$$\omega \ll m v^3/e^2 \sim (\varkappa T/e^2)\sqrt{(\varkappa T/m)} \sim 10^8\, T^{3/2}. \tag{37.6}$$

† See, for instance, [9]. When the density U_f per unit interval of f is used, it must be remembered that $U_\omega = U_f/2\pi$. Moreover, the ratio A_{21}/B_{12} as given here is for isotropic radiation, and A_{21} is the total probability of a spontaneous transition, which is then independent of the direction of the photon and the state of polarisation.

Even if $T \sim 10^2$ (the lowest temperature of interest), the condition (37.6) is $\omega \ll 10^{11}$ and is therefore satisfied at radio frequencies.

With the condition (37.6) we have†

$$dq_\omega \hbar \omega = \frac{16e^6}{3v^2 c^3 m^2} \ln \frac{2v^3 m}{\gamma e^2 \omega} d\omega, \tag{37.7}$$

where v is the velocity of the electron at infinity, $\gamma = 1{\cdot}781 = e^C = e^{0{\cdot}577}$. We use the notation $dq_\omega \hbar\omega$ because here dq_ω is the bremsstrahlung cross-section (i.e. by definition dq_ω is the number of photons emitted in the frequency range ω to $\omega + d\omega$ for all values of p and for unit flux of incident particles). The number of bremsstrahlung photons emitted per unit time is $dq_\omega N_2 v$, where $N_2 v$ is the flux of incident particles. If we also use the fact that there are N ions per unit volume, the total number of spontaneous emissions per unit volume of gas is

$$Z_{e,\mathrm{sp}} = \frac{16\, e^6 N N_2}{3v\, c^3 m^2 \hbar \omega} \ln \frac{2v^3 m}{\gamma e^2 \omega} d\omega. \tag{37.8}$$

Assuming the electrons to have a Maxwellian velocity distribution $f_{00} dv = 4\pi (m/2\pi\varkappa T)^{3/2} \exp(-mv^2/2\varkappa T) v^2 dv$, the mean value is

$$\bar{Z}_{e,\mathrm{sp}} = \int Z_{e,\mathrm{sp}} f_{00} dv = \frac{32 e^6 N N_2 \ln[2(2\varkappa T)^{3/2}/\gamma^{5/2} e^2 m^{\frac{1}{2}} \omega]}{3\sqrt{(2\pi)}\sqrt{(\varkappa T)}\, m^{3/2} c^3 \hbar \omega} d\omega. \tag{37.9}$$

Knowing $\bar{Z}_{e,\mathrm{sp}}$, we can use (37.4) to find the total number of emission processes:

$$\bar{Z}_e = \bar{Z}_{e,\mathrm{sp}} + \bar{Z}_{e,\mathrm{st}} = \overline{(A_{21} + B_{21} U_\omega) N_2}.$$

The experimentally measured absorption is evidently the difference between the "true" absorption and the stimulated emission. If the number of electrons of energies 1 and 2 were the same, it is clear from the above discussion that there would be no absorption at all, since $B_{12} = B_{21}$ and $N_1 = N_2$, so that $B_{12} N_1 U_\omega = B_{21} N_2 U_\omega$. In thermal equilibrium, or if the electrons at least have a Maxwellian velocity distribution, $N_2 - N_1 = N\hbar\omega/\varkappa T$; we assume that the first condition (37.5) holds. N is the mean number of particles in states 1 and 2, which in our case is equal to the ion or electron density N, on account of the integration over velocities. Using the above results and (37.4), we find that the measured number of absorption processes is

$$\bar{Z}_{a,\mathrm{eff}} = \bar{Z}_a - \bar{Z}_{e,\mathrm{st}} = \bar{Z}_{e,\mathrm{sp}} (\pi^2 c^3 \hbar \omega / \hbar \omega^3 \varkappa T)\, U_\omega.$$

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† It may be noted that, if the inequality opposite to (37.6) holds, we have

$$dq_\omega \hbar\omega = (16\pi e^6 / 3\sqrt{3} v^2 c^3 m^2)\, d\omega; \tag{37.7a}$$

here, as in (37.7), it is assumed that the electron collides with a singly charged ion, whose motion is neglected. If the ion has a charge Ze, a factor Z^2 appears in (37.7) and (37.7a), and in (37.7) the argument of the logarithm is divided by Z. Moreover, whereas (37.7) is valid for ions of either sign, (37.7a) holds only for positive ions. For collisions with negative ions a factor $\exp(-2\pi e^2 \omega / m v^3)$ is needed; see [255, §9–5].

The energy absorbed in unit volume is (for a wave propagated in the z-direction) $dS/dz = -Z_{a,\,\mathrm{eff}}\hbar\omega$, and the flux of incident radiation is $S = cU_\omega d\omega$, since the velocity of the radiation is assumed equal to that of light (i.e. the refractive index $n = 1$). Hence we find the following final formula for the absorption coefficient $\mu = -(1/S)dS/dz$;

$$\begin{aligned}\mu &= \frac{32\pi^2 e^6 N^2 \ln[(2\varkappa T)^{3/2}/2{\cdot}115\, e^2 m^{\frac{1}{2}}\omega]}{3\sqrt{(2\pi)}\,(\varkappa T m)^{3/2} c\,\omega^2}\\ &= \frac{0{\cdot}58\, N^2}{T^{3/2}\omega^2}\ln\left(\frac{4{\cdot}6\times 10^5\, T}{\omega^{2/3}}\right)\\ &\approx \frac{10^{-2}N^2}{T^{3/2} f^2}\left[17{\cdot}7 + \ln\frac{T^{3/2}}{f}\right].\end{aligned} \tag{37.10}$$

This formula can also be obtained by the same method as in § 6 if we substitute in (6.12) a maximum impact parameter p_m given by

$$p_{m0} = \frac{1}{15}\,\frac{2\pi}{\omega}\sqrt{\frac{8\varkappa T}{\pi m}}, \tag{37.11}$$

i.e. $\frac{1}{15}$ of the path traversed by an electron moving with the mean thermal velocity during one period of variation of the field. Of course the value of the coefficient in this result could not be obtained except as above. The ratio $p_{m0}/D \sim 4\pi e^2 N/m\omega^2$, and thus, in accordance with (37.2) and (37.3), the value of the parameter $4\pi e^2 N/m\omega^2$ determines the ranges of validity of formulae (37.1) and (37.10). These formulae lead to the same result if

$$4\pi e^2 N/m\omega^2 \sim 1. \tag{37.12}$$

However, it must not be forgotten that formulae (37.1) and (37.10) pertain to the limiting cases $4\pi e^2 N/m\omega^2 \gg$ and $\ll 1$, and so a comparison of them for $4\pi e^2 N/m\omega^2 \sim 1$ is an extrapolation showing only that they agree in this range. The occurrence of the parameter p_{m0} or D only in a logarithm has the result that for $4\pi e^2 N/m\omega^2 \sim 0{\cdot}1$ to 10 we can with sufficient accuracy use either (37.1) or (37.10), since then the quantity

$$\Delta = \frac{\mu(37.10) - \mu(37.1)}{\mu(37.1)} \approx \frac{\ln(3\times 4\pi e^2 N/m\omega^2)}{3\ln(220\, T/N^{\frac{1}{3}})} \tag{37.13}$$

is very small. For example, even if $4\pi e^2 N/m\omega^2 = 1 - \varepsilon \sim 10^{-2}$, we have $|\Delta| \approx 15$ per cent only for $T = 10^4$ and $N = 10^9$; since in the corona $T \sim 10^6$ and $N < 10^9$, the actual value of Δ is less than this. In the ionosphere, with $4\pi e^2 N/m\omega^2 \sim 10^{-2}$ and $T \sim 300$, we have $|\Delta| \approx 15$ per cent when $N \sim 3\times 10^4$ (in this example $\omega \sim 10^8$, and so it is again "pessimistic", since when ω is smaller and $4\pi e^2 N/m\omega^2$ the same the value of $|\Delta|$ is smaller). Thus the use of μ with $p_m = D$ [i.e. formula (37.1) and its analogues] everywhere except in the present section cannot lead to an error greater than 5–10 per cent. Formula (37.1) is, as already noted, inapplicable only for the interstellar gas.

Knowing the absorption coefficient μ given by expression (37.10), we can calculate the optical thickness of the gas in any direction in the Galaxy. If N and T are constant along the whole path, then, of course,

$$\tau = \mu L = \frac{10^{-2} N^2}{T^{3/2} f^2} \left[17{\cdot}7 + \ln \frac{T^{3/2}}{f}\right] L, \tag{37.14}$$

where L is the length of the path.

The electron temperature in highly ionised regions of the interstellar gas is $T \sim 10^4$. Hence, if $\tau \geqq 1$, the Galaxy must be a fairly strong source of thermal radio emission. The effective temperature of this radiation in any direction is, by Kirchhoff's law,

$$T_{\text{eff}} = T[1 - e^{-\tau(\lambda)}], \tag{37.15}$$

where $\tau(\lambda)$ is the optical thickness in the direction considered for the wavelength λ in question. The specific intensity of this radiation is the intensity of black-body radiation of temperature T_{eff}, i.e.

$$\begin{aligned} I &= 2f^2 \varkappa T_{\text{eff}}/c^2 \\ &= 2\varkappa T_{\text{eff}}/\lambda^2 \\ &= 2{\cdot}76 \times 10^{-16} T_{\text{eff}}/\lambda^2 \,\text{erg/cm}^2.\,\text{sterad.}\,(\text{c/s}).\,\text{sec} \\ &= 2{\cdot}76 \times 10^{-17} T_{\text{eff}}/\lambda_m^2 \,\text{W/m}^2.\,\text{sterad.}\,(\text{Mc/s}). \end{aligned} \tag{37.16}$$

Evidently for $\tau \gg 1$ we have $T_{\text{eff}} = T$, i.e. the effective temperature cannot exceed T. We find experimentally for long waves ($\lambda \gtrsim 10$ m) $T_{\text{eff}} \gtrsim 10^5$, and moreover in most cases the optical thickness of the Galaxy $\tau < 1$. Hence the thermal radiation of the interstellar electron gas certainly cannot be responsible for the whole of the galactic radio emission. As already mentioned, the non-thermal component of the cosmic radio emission is magnetic bremsstrahlung (synchrotron radiation), i.e. is due to the acceleration of relativistic electrons in interstellar fields.

Rotation of the plane of polarisation of radio waves in the interstellar medium

The interstellar magnetic fields $H^{(0)} \lesssim 10^{-5}$ oersted are so weak that it seems quite unnecessary to take account of their effects on the propagation of radio waves; for $H^{(0)} \sim 10^{-5}$, the gyration frequency $\omega_H \sim 200$, and for wavelengths $\lambda \sim 1$ to 10 m we have $\sqrt{u} = \omega_H/\omega \sim 10^{-6}$ to 10^{-7}. This conclusion is in fact correct as regards the calculation of the optical thickness or refractive index of the interstellar plasma, but in one respect even a weak field $H^{(0)} \lesssim 10^{-5}$ is important: it causes a rotation of the plane of polarisation, and a depolarisation, of the cosmic radio emission.

The magnetic bremsstrahlung radio emission generated in a uniform magnetic field is strongly polarised [231, 233, 256]. We might therefore expect a considerable polarisation of the cosmic radio waves received at the surface of the Earth. In reality the polarisation, though it apparently exists, is very slight [257]. This is due to two causes: firstly, the magnetic field has different directions in different regions of interstellar space, and so only the radiation from one "cloud" (a region with a quasiuniform field) should be polarised; secondly, the rotation of the plane of polarisation in the interstellar medium leads to a depolarisation of the radiation within each cloud. (For non-monochromatic radio waves, such as are involved here, there is also observed a depolarisation due to the dispersion of the rotation of the plane of polarisation, i.e. the wavelength dependence of the angle of rotation.)†

For radio waves propagated in the interstellar medium we have

$$\sqrt{u} = \omega_H/\omega \ll 1, \quad v = \omega_0^2/\omega^2 \ll 1, \quad s = \nu_{\text{eff}}/\omega \ll 1, \tag{37.17}$$

and so the conditions (11.36) for "quasilongitudinal" propagation become

$$u \sin^4\alpha/4 \cos^2\alpha \ll 1, \quad u \sin^2\alpha \ll 1. \tag{37.18}$$

It has been noted above that, when $\lambda = 10$ m and $H^{(0)} \sim 10^{-5}$, the parameter $u \sim 10^{-12}$; even for $\lambda = 1$ km, $u \sim 10^{-8}$. Thus the condition for "quasilongitudinal" propagation in the interstellar medium is satisfied for practically any angle α between $\boldsymbol{H}^{(0)}$ and the wave vector $\boldsymbol{k}$; using therefore (11.37) with $u \ll 1$, $s \ll 1$ and $n \approx 1$, we find

$$\begin{aligned} n_2 - n_1 &= n_- - n_+ \\ &\approx \sqrt{u}\,(v \cos\alpha) \\ &= (\omega_H\, \omega_0^2/\omega^3) \cos\alpha \\ &= \frac{4\pi\, e^3\, H^{(0)}\, N}{m^2\, c\, \omega^3} \cos\alpha \\ &= 5{\cdot}6 \times 10^{16} \frac{H^{(0)}\, N \cos\alpha}{\omega^3}. \end{aligned} \tag{37.19}$$

Hence it follows that, after a path L has been traversed, the plane of polarisation of the wave is rotated through an angle

$$\begin{aligned} \Psi &= \tfrac{1}{2}(\omega/c)(n_2 - n_1)L \\ &\approx 0{\cdot}93 \times 10^6 (H^{(0)} N L/\omega^2) \cos\alpha; \end{aligned} \tag{37.20}$$

this formula is easily derived from (11.10) and (11.12) by taking the real expressions for the fields E_x and E_y, which are the sum of waves 1 and 2 with equal amplitudes. For $H^{(0)} \sim 10^{-5}$, $N \sim 1$, $\cos\alpha \sim 1$ and $\omega \sim 6 \times 10^8$

† The plane of polarisation of the cosmic radio emission also undergoes rotation in the Earth's ionosphere. We shall not discuss here this effect or the effect of the ionosphere on the cosmic radio emission in general (see [238, 240]).

($\lambda \sim 3$ m) we have $\Psi \sim 3 \times 10^{-17} L$, i.e. $\Psi \sim 1$ for $L \sim 3 \times 10^{16}$ cm; if $N \sim 10^{-2}$ and $H^{(0)} \sim 3 \times 10^{-6}$, then $\Psi \sim 1$ for $L \sim 10^{19} \approx 3$ pc. The regions of the Galaxy having a quasiuniform magnetic field seem usually to be larger than 10^{19} cm, and so even with $N \sim 10^{-2}$ a depolarisation of the radiation from a single "cloud" will be observed.† Thus the observed polarisation of the cosmic emission, owing to the rotation of the plane of polarisation in the interstellar medium, depends on the electron density N. This shows, in particular, a possibility of estimating N from observations of polarisation.

In June the radiation from the Crab Nebula passes through the solar corona. At this time we may attempt to estimate the magnetic field in the outer corona, as well as making a study of coronal inhomogeneities [202–205]. For this purpose it is necessary to observe the rotation of the plane of polarisation of centimetre-wavelength radiation from the nebula [288].

The above effect of rotation is a typical example of the fact that the effect of the magnetic field on the properties of a plasma is sometimes not negligible even if the field is very weak.

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† The situation is exactly similar as regards the discrete sources (galactic and extragalactic nebulae). For example, the polarisation of the long radio waves from the Crab Nebula is practically zero, but the magnetic bremsstrahlung from this nebula in the optical range and at centimetre wavelengths is partly polarised. The difference is due to the fact that the angle Ψ is inversely proportional to ω^2 [see (37.20)].

CHAPTER VIII

NON-LINEAR PHENOMENA IN A PLASMA IN A VARIABLE ELECTROMAGNETIC FIELD

§ 38. INTRODUCTION. A PLASMA IN A STRONG UNIFORM ELECTRIC FIELD

The condition for the field in the plasma to be weak. Some examples

One of the characteristic properties of a plasma enumerated in § 1 is the occurrence of non-linear effects even in relatively small and easily obtainable electric fields. This is due to the slowness of energy transfer from electrons to heavy particles (atoms, molecules and ions) resulting from the smallness of the ratio m/M, whereas electrons in a plasma may acquire considerable energy from the field, because the free path is very long. Consequently, the plasma electrons in an electric field become heated, and the complex permittivity ε' (or ε'_{ik}) depends on the field strength. In other words, the polarisation $\boldsymbol{P}$ and the conduction current $\boldsymbol{j}$ are no longer proportional to the field $\boldsymbol{E}$, so that the electrodynamic processes in the plasma (in particular, wave propagation) become non-linear (the superposition principle does not hold, and so on).

The effect of the field on the properties of the plasma is discussed below (see also § 4). It is useful to point out immediately, however, that this effect may be neglected in a first approximation if the field $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$ satisfies the condition

$$\begin{aligned} E_0 \ll E_p &= \sqrt{[3 m \varkappa T \delta (\omega^2 + \nu^2_{\text{eff},0})/e^2]} \\ &= 4{\cdot}2 \times 10^{-10} \sqrt{[\delta T (\omega^2 + \nu^2_{\text{eff},0})]} \text{ V/cm}, \end{aligned} \tag{38.1}$$

where $\nu_{\text{eff},0}(T)$ is the effective collision frequency in an equilibrium plasma, used everywhere in the preceding chapters, and δ the effective (mean) relative fraction of energy transferred by the electron in a collision with a heavy particle (see § 5; in elastic collisions $\delta = \delta_{el} = 2m/M$). In (38.1) the external magnetic field $\boldsymbol{H}^{(0)}$ is, for simplicity, assumed absent.

The characteristic field E_p is sometimes called the plasma field. In this field the mean energy of the electrons varies by an amount of the order of $\varkappa T$ (see below and § 4).

A field which satisfies the condition (38.1) is said to be weak. In a strong field ($E_0 \gtrsim E_p$), and especially in a very strong field ($E_0 \gg E_p$), the properties of the plasma change considerably.

At low frequencies (where $\omega^2 \ll \nu_{\mathrm{eff},0}^2$) in the ionosphere $E_p \sim 10^{-5}$ to 10^{-7} V/cm, since $\nu_{\mathrm{eff},0} \sim 10^5$, $T \sim 300$, $\delta \sim 10^{-3}$ in the E layer, and $\nu_{\mathrm{eff},0} \sim 10^3$, $T \sim 10^3$, $\delta \sim 10^{-4}$ in the F layer. In the solar corona, as in any fully ionised hydrogen plasma, $\delta = \delta_{\mathrm{el}} = 1/918$. Hence for low frequencies in the corona with $\nu \sim 10$ and $T \sim 10^6$ we have $E_p \sim 10^{-7}$ V/cm.

For a plasma of higher density or at high frequencies such that $\omega^2 \gg \nu_{\mathrm{eff},0}^2$ the plasma field E_p is considerably greater. For example, in the ionosphere for $\omega = 2 \times 10^6$ ($\lambda_0 \sim 1$ km) we have $E_p \sim 10^{-3}$ V/cm, and for $\omega \sim 2 \times 10^7$ ($\lambda_0 \sim 100$ m), $E_p \sim 10^{-2}$ V/cm. In the corona at metre wavelengths $E_p \sim 10$ V/cm, and for $\lambda_0 \sim 1$ cm we have $E_p \sim 10^4$ V/cm. Finally, in laboratory conditions ($\nu_{\mathrm{eff},0} \sim 10^6$ to 10^9, $T \sim 10^4$, $\delta \sim 10^{-1}$ to 10^{-3}) the values are $E_p \sim 10^{-3}$ to 10 V/cm in a field of low frequency and $E_p \sim 10^{-11}$ to $10^{-10} \sqrt{T}\omega$ V/cm at high frequencies.

Thus in a plasma, non-linearity can in fact occur in fields which are not particularly large in comparison with the values usual in the laboratory or in the wave zone of powerful radio transmitters. In non-conducting pure liquids and solids (except ferroelectrics) the situation is different: here the effect of the field on the properties of the medium may be neglected up to fields of the order of 10^5 to 10^7 V/cm, which are already approaching the fields of the atomic scale $E_a \sim e/d^2 \sim 10^8$ V/cm, where d is the dimension of the atoms or the lattice constant. In metals and semiconductors the conduction electrons may to a certain extent be compared to those in a gaseous plasma of the type under consideration (as already mentioned in § 8). However, the region of non-linearity in metals is almost inaccessible in practice, since the attainment of a sufficiently strong field in a metal is prevented by the high electrical conductivity and moreover the non-linearity is reduced by the degeneracy of the electrons, when the temperature T is replaced by the temperature of degeneracy $T_0 \sim \hbar^2 N^{2/3} m \varkappa$). In semiconductors, non-linearity is quite easily observed, and here many of the conclusions derived in the discussion of non-linear phenomena in a gaseous plasma are qualitatively applicable. However, we shall not pause to consider the case of semiconductors.†

Statement of the problem for a strong field

The equations of plasma dynamics are themselves non-linear [see, for instance, equations (13.1)–(13.4) or (13.15)–(13.19)], and so the theory of non-linear phenomena embraces in general a very extensive portion of plasma physics. In what follows we shall be concerned to describe a much more restricted but quite well defined range of problems. In this section we shall

† References to work on non-linear effects in metals and semiconductors are given in [258], which also refers to numerous papers concerned with non-linear effects in plasmas. In consequence few references other than [258] will be given below.

discuss the effect of a uniform electric field $\boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$, with arbitrary $\boldsymbol{E}_0$ and ω, on a non-relativistic and non-degenerate plasma. The plasma may also be in a constant external magnetic field $\boldsymbol{H}^{(0)}$. Macroscopic (hydrodynamic) motions in the plasma will be assumed absent.

The effect of the field on the plasma in this formulation of the problem amounts to a change in the velocity distribution of the plasma electrons, which must be found as a function of $\boldsymbol{E}_0$, ω, $\boldsymbol{H}^{(0)}$ and the plasma parameters. The distribution function for the heavy particles will be assumed to be Maxwellian with temperature T; in a steady state, which condition we shall presuppose, this assumption is usually valid.

Knowing the electron velocity distribution function, we can find the mean kinetic energy of the electrons (or, in the case of a Maxwellian distribution, their temperature T_e) and the total current density $\boldsymbol{j}_t$. In the particular case of a weak field, $T_e = T$ and the current $\boldsymbol{j}_t$ is proportional to the field $\boldsymbol{E}$.

The determination of the properties of a plasma in a uniform field of any strength is of interest in the analysis of various problems of gas-discharge physics, plasma heating, etc. The calculation of the current $\boldsymbol{j}_t$ is also necessary as a first step in the solution of problems of electrodynamics. These include, in particular, problems of electromagnetic wave propagation in plasmas. The non-linear effects occurring in wave propagation are discussed in § 39 mainly with reference to the ionosphere. We shall not here consider gas discharges, including those at high and very high frequencies [259]; plasma heating in a non-uniform field; the theory of non-steady processes, including the problem of runaway electrons [83, 88]; or certain other topics.

The elementary theory

To solve the above problem in the general case, we must use the Boltzmann equation for the electron distribution function. This will be done later, but it is convenient to begin with the "elementary theory", which is often sufficient not only to understand the fundamental results but also to obtain quantitative formulae of practical use, even in strong fields.

In the elementary theory the state of the plasma is characterised by two quantities: the mean velocity $\dot{\boldsymbol{r}}$ of the directed motion of the electrons and the effective electron temperature T_e. By definition, $\dot{\boldsymbol{r}}$ is related to the total current density $\boldsymbol{j}_t$ by

$$\boldsymbol{j}_t = \boldsymbol{j} + \partial \boldsymbol{P}/\partial t = eN\dot{\boldsymbol{r}}; \tag{38.2}$$

see (3.1). In the elementary theory the electron temperature T_e is given by

$$3\varkappa T_e/2 = \overline{K} = \tfrac{1}{2} m \overline{v^2}, \tag{38.3}$$

where the averaging is over all electrons. Since the electron velocity distribution is by no means always Maxwellian, the temperature T_e is in general an effective electron temperature.

The equation for $\dot{\boldsymbol{r}}$ has already been derived in § 3, and has the form

$$m d\dot{\boldsymbol{r}}/dt = e\boldsymbol{E} + e\dot{\boldsymbol{r}} \times \boldsymbol{H}^{(0)}/c - m\nu_{\text{eff}}(T_e)\dot{\boldsymbol{r}}. \tag{38.4}$$

This equation can be used for any field, but in a strong field ν_{eff} depends on the field, since $\nu_{\text{eff}} = q_{\text{eff}}(\bar{v})N_{i,m}\bar{v}$ and so depends on the mean electron velocity $\bar{v}$ or, alternatively, the temperature T_e. For example, in collisions with molecules we have [see (6.10)]

$$\nu_{\text{eff}} = \nu_{m0}\sqrt{(T_e/T)}, \tag{38.5}$$

and in collisions with ions, to a first approximation [see (6.14)],

$$\nu_{\text{eff}} = \nu_{i0}(T/T_e)^{3/2}, \tag{38.6}$$

where $\nu_{m,i0} \equiv \nu_{\text{eff},m,i0}$ is the effective collision frequency for collisions with molecules (m) or ions (i) when $T_e = T$.

In a weak field we in fact have $T_e = T$ and the quantity ν_{eff} in (38.4) is an independent parameter. In a strong field the plasma is heated, and T_e evidently depends on the field strength. Equation (38.4) must therefore be solved together with the equation for T_e. The latter equation in the elementary theory is obtained simply from considerations of the energy balance.

The electric field does an amount of work $\boldsymbol{j}_t \cdot \boldsymbol{E} = eN\dot{\boldsymbol{r}} \cdot \boldsymbol{E}$ on the plasma per unit time. On the other hand, in collisions with heavy particles an electron loses per unit time an average energy $\delta\nu_{\text{eff}}(\bar{K} - 3\varkappa T/2) = 3\varkappa\delta\nu_{\text{eff}}(T_e - T)/2$; the choice of this expression has been justified in § 4 [see (4.5)], and in general we have $\delta = \delta_{\text{eff}}(T_e) \equiv \delta(T_e)$ and $\nu_{\text{eff}} = \nu_{\text{eff}}(T_e)$. The energy balance can therefore be written in the form

$$\frac{d}{dt}\left(\frac{3}{2}\varkappa N T_e\right) = \boldsymbol{j}_t \cdot \boldsymbol{E} - \frac{3}{2}\varkappa\,\delta\,\nu_{\text{eff}} N(T_e - T)$$

or

$$\frac{dT_e}{dt} = \frac{2e}{3\varkappa}\dot{\boldsymbol{r}} \cdot \boldsymbol{E} - \delta(T_e)\,\nu_{\text{eff}}(T_e)(T_e - T). \tag{38.7}$$

The simultaneous solution of equations (38.4) and (38.7) will give $\dot{\boldsymbol{r}}$ and T_e as functions of the field $\boldsymbol{E}$.

The successful use of these equations depends greatly on the fact that in a steady state $\delta \ll 1$. Consequently, even in a strong electric field, the random velocity $\bar{v}$ of an electron is much greater than its directed velocity $|\dot{\boldsymbol{r}}|$ (see below and § 4). For this reason we may assume that δ and ν_{eff} in (38.4) depend only on $T_e = \frac{1}{3}m\overline{v^2}$ and not on $\dot{\boldsymbol{r}}$.†

† This applies to a steady or quasisteady state. Otherwise the velocity $|\dot{\boldsymbol{r}}|$ may be comparable with or greater than $\bar{v}$, as for example during a certain time after the application of a strong uniform electric field.

In the absence of the field $\boldsymbol{E}$, if δ = constant and $\nu_{\rm eff}$ = constant (i.e. if δ and $\nu_{\rm eff}$ are independent of T_e), we have

$$T_e - T = (T_e - T)_{t=0} \exp(-\delta \nu_{\rm eff} t) \tag{38.8}$$

and

$$\dot{\boldsymbol{r}}(t) = \dot{\boldsymbol{r}}(0) \exp(-\nu_{\rm eff} t), \tag{38.9}$$

where in the latter case we assume for simplicity that $\boldsymbol{H}^{(0)} = 0$ also (but in (38.9) the condition $\nu_{\rm eff}$ = constant is not necessary, and we may have $\nu_{\rm eff} = \nu_{\rm eff}(T_e)$ if T_e is independent of time).

It is clear from the above relations that the temperature (or energy) relaxation time $\tau_k = 1/\delta\nu_{\rm eff}$ is $1/\delta \gg 1$ times greater than the relaxation time $\tau = 1/\nu_{\rm eff}$ for the directed velocity (or momentum). On account of the restriction to steady or quasisteady processes, we shall not use the relaxation solutions (38.8) and (38.9) below. Hence, in a field $\boldsymbol{E} = \boldsymbol{E}_0 \cos\omega t$ with $\boldsymbol{H}^{(0)} = 0$, δ = constant and $\nu_{\rm eff}$ = constant, we have

$$\left.\begin{aligned} m\, d\dot{\boldsymbol{r}}/dt &= e\boldsymbol{E}_0 \cos\omega t - m\nu\dot{\boldsymbol{r}}, \\ \dot{\boldsymbol{r}} &= e\boldsymbol{E}_0(\nu\cos\omega t + \omega\sin\omega t)/m(\omega^2+\nu^2), \end{aligned}\right\} \tag{38.10}$$

$$\left.\begin{aligned} \frac{dT_e}{dt} &= \frac{e^2E_0^2}{3m\varkappa(\omega^2+\nu^2)}(\nu + \nu\cos 2\omega t + \omega\sin 2\omega t) - \delta\nu(T_e - T), \\ T_e - T &= \frac{e^2E_0^2}{3m\varkappa\delta(\omega^2+\nu^2)}\left[1 + \frac{(\delta\nu^2 - 2\omega^2)\delta}{4\omega^2+\delta^2\nu^2}\cos 2\omega t + \right. \\ &\qquad \left. + \frac{\omega\nu(2+\delta)\delta}{4\omega^2+\delta^2\nu^2}\sin 2\omega t\right]. \end{aligned}\right\} \tag{38.11}$$

Here, and often below, the suffix eff is omitted, i.e. $\nu \equiv \nu_{\rm eff}$ and $\nu_0 \equiv \nu_{\rm eff,0}$.

For very low frequencies, when

$$\omega \ll \delta\nu, \tag{38.12}$$

we have to within small terms of order $\omega/\delta\nu$

$$\begin{aligned} T_e - T &= (2e^2E_0^2/3m\varkappa\delta\nu^2)\cos^2\omega t \\ &= 2e^2E^2(t)/3m\varkappa\delta\nu^2, \end{aligned} \tag{38.13}$$

where we have used the fact that the condition (38.12) certainly gives $\omega \ll \nu$, since $\delta \ll 1$.

In the opposite limiting case

$$\omega \gg \delta\nu \tag{38.14}$$

we have to within terms of order $\delta\nu/\omega$ and δ

$$\begin{aligned} T_e - T &= e^2E_0^2/3m\varkappa\delta(\omega^2+\nu^2) \\ &= 2\overline{E^2}/3m\varkappa\delta(\omega^2+\nu^2), \end{aligned} \tag{38.15}$$

where $\overline{E^2}$ is the time average value of $E^2 = E_0^2\cos^2\omega t$.

Thus in the case (38.14) the temperature T_e is constant to a first approximation; the variable component of T_e has a frequency 2ω and a small amplitude of order $\delta\nu/\omega$ or δ. The fact that the electron temperature (or mean energy) is approximately constant in a variable electric field of frequency $\omega \gg \delta\nu$ is entirely reasonable, simply because the temperature relaxation time $\tau_k = 1/\delta\nu \gg 2\pi/\omega$, and so the temperature cannot vary appreciably during the period of oscillation $2\pi/\omega$ of the field. Consequently the temperature takes some mean value (38.15) and deviates only slightly therefrom.

It is very important to note that the situation is unchanged when the dependence of δ and $\nu \equiv \nu_{\rm eff}$ on T_e is taken into account. Equations (38.4) and (38.7) can then be solved, with the condition (38.14) and $\delta \ll 1$, by expanding in powers of $\delta\nu/\omega$ and δ. It is, however, immediately clear that in a first approximation the temperature T_e has the constant value given by

$$T_e - T = e^2E_0^2/3m\varkappa\delta(T_e)\left(\omega^2 + \nu^2(T_e)\right); \qquad (38.16)$$

this is evidently obtained from (38.7) and (38.10) by neglecting the term dT_e/dt, which is small if $\delta \ll 1$ and $\delta\nu/\omega \ll 1$ [see (38.14)]. The result is easily verified by calculating the time-dependent part of T_e in a first approximation, with (38.16) serving as the zero-order approximation.

In a constant field $\boldsymbol{E}$ and in a steady state we have

$$\left.\begin{aligned} \dot{\boldsymbol{r}} &= e\boldsymbol{E}/m\,\nu(T_e), \\ T_e &= T + 2e^2E^2/3m\varkappa\,\delta(T_e)\,\nu^2(T_e), \end{aligned}\right\} \qquad (38.17)$$

i.e. the temperature T_e is the same as the mean temperature (38.16) for $\omega = 0$ with E_0 replaced by $\sqrt{2}E$. This is reasonable, since when $\omega \ll \nu_{\rm eff}$ the variable field has the same average effect as a constant field $E = E_{\rm eff} = E_0/\sqrt{2}$.

It is immediately seen from (38.16) and (38.17) that in a steady field of any frequency the mean velocity of random motion $v \sim \sqrt{(\varkappa T_e/m)}$ considerably exceeds $|\dot{\boldsymbol{r}}|$. For, according to (38.16), even for $T_e \gg T$

$$\begin{aligned} v &\sim \sqrt{(\varkappa T_e/m)} \\ &\sim eE_0/m\sqrt{\delta}\sqrt{(\omega^2 + \nu^2)}, \end{aligned} \qquad (38.18)$$

while

$$|\dot{\boldsymbol{r}}| \lesssim e\,E_0/m\sqrt{(\omega^2 + \nu^2)} \sim \sqrt{\delta}\bar{v}; \qquad (38.19)$$

see (38.10). This estimate is, of course, equivalent to that in § 4. In a constant field, or when the approximation (38.16) (T_e constant) may be used, equation (38.4) can be solved independently of (38.7). For a field $\boldsymbol{E} = \boldsymbol{E}_0e^{i\omega t}$ we have

$$\dot{\boldsymbol{r}} = \frac{e\boldsymbol{E}}{m}\left[\frac{\nu(T_e)}{\omega^2 + \nu^2(T_e)} - i\frac{\omega}{\omega^2 + \nu^2(T_e)}\right] = \frac{\boldsymbol{j}_t}{eN}. \qquad (38.20)$$

In such conditions, as in § 3, it is convenient to use instead of $\boldsymbol{j}_t$ or $\dot{\boldsymbol{r}}$ the permittivity ε and the conductivity σ, defined by (3.1):

$$\boldsymbol{j}_t = [\sigma + i(\varepsilon - 1)\omega/4\pi]\boldsymbol{E}.$$

For ε and σ we then obtain formulae (3.7), where $\nu_{\text{eff}} = \nu_{\text{eff}}(T_e) \equiv \nu(T_e)$. In a strong field T_e depends on E_0^2, and the problem of wave propagation becomes non-linear.

The expressions (38.16) and (38.17) do not give T_e explicitly if δ and ν depend considerably on T_e. Before deriving the final formula for T_e, we may write (38.16) in the form

$$\frac{T_e}{T} = 1 + \left(\frac{E_0}{E_p}\right)^2 \frac{\omega^2 + \nu_0^2}{\omega^2 + \nu^2(T_e)}, \tag{38.21}$$

using the plasma field E_p [see (38.1)] and $\nu_0 \equiv \nu_{\text{eff},0}(T)$, and for simplicity assuming δ to be constant. The latter assumption will also be made below, since it is often entirely correct (e.g. for elastic collisions $\delta = \delta_{\text{el}} = 2m/M$). Hence from (38.21) we immediately see the significance of the condition (38.1): in a field of amplitude $E_0 \ll E_p$ the plasma is only slightly perturbed and $T_e \approx T$.

For collisions with molecules we use the expression (38.5) and substitute it in (38.21). Thus

$$T_e = T\left[1 + \frac{\omega^2 + \nu_0^2}{2\nu_0^2}\left(\sqrt{\left\{1 + \frac{4\nu_0^2}{\omega^2 + \nu_0^2}\left(\frac{E_0}{E_p}\right)^2\right\}} - 1\right)\right], \tag{38.22}$$

where now $\nu_0 \equiv \nu_{\text{eff},m0}(T)$. Fig. 38.1 shows T_e as a function of E_0/E_p for $\omega^2 \gg \nu_0^2$ and $\omega^2 \ll \nu_0^2$. The electron temperature increases monotonically with the field for collisions with molecules.

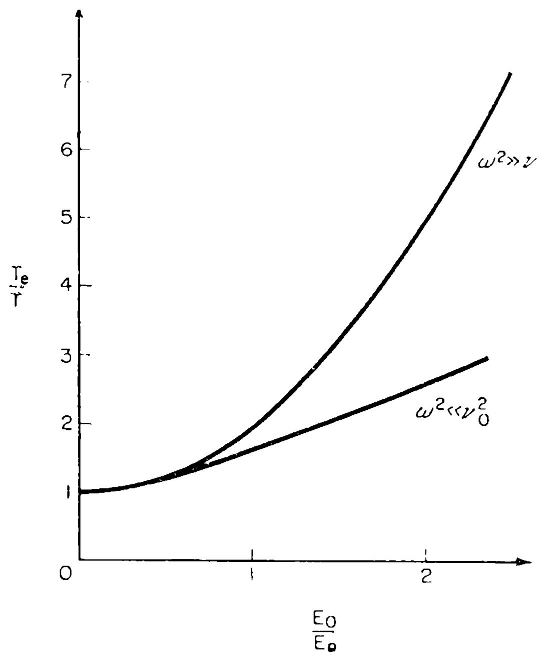

FIG. 38.1. Electron temperature as a function of electric field for a weakly ionised plasma (electrons colliding with molecules).

At high frequencies $\omega^2 \gg \nu^2(T_e)$, it is clear from (38.16) that

$$\frac{T_e}{T} = 1 + \frac{e^2 E_0^2}{3m\varkappa\delta T\omega^2}. \tag{38.23}$$

This expression is independent of ν and ν_0, and is therefore valid for collisions with ions also.

At frequencies $\omega^2 \ll \nu_{0i}^2$ there is an interesting feature for collisions with ions: the relation between T_e and E_0/E_p is no longer one-to-one [34]. Consequently, in the region $E_k^{\mathrm{II}} < E_0 < E_k^{\mathrm{I}}$ three values of the steady temperature T_e correspond to any given value of E_0 (Fig. 38.2). Only two of these values are stable, however, namely the lowest and highest. The absence of a possible steady state for $E_0 > E_k^{\mathrm{I}} = 0{\cdot}28\, E_p(\omega = 0)$ is due to the fact that the

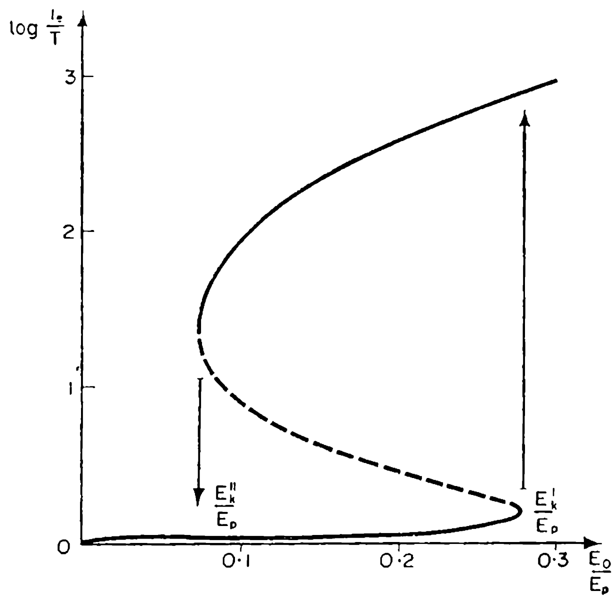

FIG. 38.2. Electron temperature as a function of electric field for collisions of electrons with ions ($\omega = 0{\cdot}01\nu_0$).

energy acquired by the electrons from the field increases rapidly with $T_e(\dot{\boldsymbol{r}}\cdot\boldsymbol{E} \sim 1/\nu \sim T_e^{3/2})$, whereas the energy acquired by the ions decreases $(\delta\nu T_e \sim T_e^{-\frac{1}{2}})$. Hence, in a sufficiently strong field, the temperature T_e must increase until the collision frequency $\nu(T_e)$ becomes less than ω and a second, "high-temperature", steady state (38.23) can exist. The transition from the low-temperature to the high-temperature state is shown by an arrow in Fig. 38.2. The reverse transition occurs in a field $E_k^{\mathrm{II}} \approx 1{\cdot}7\,(\omega/\nu_0)^{\frac{2}{3}} E_p(\omega = 0) < E_k^{\mathrm{I}}$, i.e. hysteresis must be observed in the dependence of T_e on E_0. The instability of the low-temperature state of the plasma (for collisions with ions) occurs also in a constant field $E > E_k = E_k^{\mathrm{I}}/\sqrt{2} = 0{\cdot}2\, E_p(\omega = 0)$. The second stable state is then, of course, absent. Moreover, in a certain field exceeding E_k there ceases to be a steady state even for the mean directed velocity $\dot{\boldsymbol{r}}$ of the electrons, and this gives rise to the problem of "runaway" electrons, discussed in more detail in [258].

In the presence of an external magnetic field $\boldsymbol{H}^{(0)}$ the temperature T_e in the region (38.14) is also constant in a first approximation, and

$$\frac{T_e}{T} = 1 + \left(\frac{E_0}{E_p}\right)^2 (\omega^2 + \nu^2) \times$$
$$\times \left\{\frac{\cos^2\beta}{\omega^2 + \nu^2(T_e)} + \frac{\sin^2\beta}{2[(\omega - \omega_H)^2 + \nu^2(T_e)]} + \frac{\sin^2\beta}{2[(\omega + \omega_H)^2 + \nu^2(T_e)]}\right\}. \quad (38.24)$$

The same result is reached from (38.7) when $dT_e/dt = 0$ by substituting for $\dot{\boldsymbol{r}} = \boldsymbol{j}_t/eN$ the expression (10.9), which when T_e = constant is valid even if $\nu_{\text{eff}} = \nu(T_e)$. In (38.24) β is the angle between $\boldsymbol{E}$ and $\boldsymbol{H}^{(0)}$; when $\omega_H = |e|H^{(0)}/mc \to 0$ formula (38.24), of course, becomes (38.21). When $\omega \to \omega_H$ the electron temperature has a resonance (for $\beta \neq 0$), which is simply a result of the corresponding increase in the conductivity [see (10.12) or (10.32)].

In the elementary theory, equations (38.4) and (38.7) are the starting point in an analysis of the behaviour of the plasma in a longitudinal field, for any frequency ω or for a more complex dependence on time (for example, if the amplitude of the variable field is modulated at a low frequency Ω). These equations are in fact those generally used in the theory of non-linear effects in the ionosphere (see § 39) and in certain other cases.

The accuracy of the results of the elementary theory

The elementary theory is strictly valid only if δ and ν have the same values for all electrons, i.e. are independent of their velocity. In a plasma we actually have $\nu = \nu(v)$ and $\delta = \delta(v)$. The replacement of ν and δ by their effective values $\delta(T_e)$ and $\nu_{\text{eff}}(T_e)$ and the use of these quantities in equations (38.4) and (38.7) is not logical. It is therefore clear that the accuracy of the results obtained must be tested by means of the kinetic theory. For a weak field this has been done in §§ 6 and 10. It is seen from Tables 6.2 and 6.3 and Figs. 6.1 and 6.2 that the inaccuracy of the elementary formulae is greatest for a constant field ($\omega = 0$) and disappears when $\omega^2 \gg \nu_{\text{eff}}^2$ (because in these formulae the collision frequency is just that given by the kinetic theory for $\omega^2 \gg \nu_{\text{eff}}^2$). For $\omega = 0$ and collisions with molecules (hard spheres) the elementary theory gives σ with an error of 13 per cent and ε with one of 51 per cent. For collisions with ions, when collisions between electrons are also taken into account, the kinetic calculation gives factors of 1·95 and 4·59 in the expressions for σ and ε respectively.

In a strong field, if T_e = constant, the accuracy of the elementary theory in the majority of cases is the same as in a weak field (for the same value of $\nu_{\text{eff}}(T_e)$, of course). This is due mainly to the possibility of taking the symmetrical part of the distribution function of a strongly ionised plasma to be

Maxwellian, even in a strong field. Here a strongly ionised plasma is one in which

$$\nu_{ee} \sim \nu_i \gg \delta\, \nu_m, \tag{38.25}$$

where ν_m and ν_{ee} are the electron–molecule and electron–electron collision frequencies, the latter being of the same order of magnitude as the electron–ion collision frequency ν_i (the ions are assumed positive and singly charged). In collisions between electrons the transfer of energy and momentum takes place at the same rate, and so the condition (38.25) has a simple significance: the relaxation time $\tau_{ee} \sim 1/\nu_{ee}$ for redistribution of energy between electrons is less than the relaxation time $\tau_k \sim 1/\delta\nu_m$ for transfer of energy from electrons to molecules. Thus, if (38.25) holds, the value of T_e is not some effective quantity equal to $2\bar{K}/3\varkappa$, but is in fact the kinetic temperature of the electrons.

In a weakly ionised plasma, where $\nu_{ee} \ll \delta\nu_m$, and also in the intermediate case, the electron distribution function is not Maxwellian. Here, however, $\nu_i \ll \nu_m$ and the collisions are mainly with molecules, i.e. the cross-section usually depends only slightly on the velocity. Thus a change in the distribution function does not greatly affect the effective collision frequency ν_{eff} (for the same mean energy $\bar{K} = 3\varkappa T_e/2$ as that of the Maxwellian velocity distribution).

The kinetic theory

Apart from the quantitative refinements explained above, the kinetic theory gives no further new results in steady-state conditions if we consider such mean quantities as the current $\boldsymbol{j}_t = eN\dot{\boldsymbol{r}}$ or the temperature $T_e = 2\bar{K}/3\varkappa$. The situation is in general different for non-steady processes. Moreover, not only the mean quantities but also the distribution function itself is of interest. Finally, the kinetic theory is usually necessary if account has to be taken of the spatial dispersion, or more generally if the state of the plasma depends significantly on the coordinates.

The Boltzmann equation which serves to determine the electron distribution function $f(t, \boldsymbol{r}, \boldsymbol{v})$ has the form (4.2)

$$\left.\begin{aligned} &\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\mathbf{grad}_r f + \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H}/c)\cdot\mathbf{grad}_v f + S = 0,\\ &N = \int f\, d\boldsymbol{v}, \quad \boldsymbol{j}_t = e\int \boldsymbol{v} f\, d\boldsymbol{v},\\ &\bar{K} = (1/N)\int \tfrac{1}{2} m v^2 f\, d\boldsymbol{v}. \end{aligned}\right\} \tag{38.26}$$

The Boltzmann equation may usually be considerably simplified. To do so, let us first consider an isotropic plasma ($\boldsymbol{H}^{(0)} = 0$), with the spatial gradient parallel to the field $\boldsymbol{E}$. Then there is only one preferred direction, that of $\boldsymbol{E}$ (the z-axis), and the distribution function may be expanded in Legendre polynomials $P_k(\cos\alpha)$, where α is the angle between $\boldsymbol{E}$ and $\boldsymbol{v}$:

$$f(t, \boldsymbol{r}, \boldsymbol{v}) = \sum_{k=0}^{\infty} P_k(\cos\alpha)\, f_k(t, \boldsymbol{r}, v). \tag{38.27}$$

Substituting this expansion in (38.26), multiplying by $P_{k'}(\cos\alpha)$ and integrating over angles, using the relation $\boldsymbol{E}\cdot\mathbf{grad}_v f = E\cos\alpha\,\partial f/\partial v + (E\sin^2\alpha/v)\,\partial f/\partial\cos\alpha$, we obtain a system of coupled equations for the functions $f_0, f_1, f_2, \ldots$:

$$\left.\begin{aligned}
&\frac{\partial f_0}{\partial t} + \frac{1}{3} v \frac{\partial f_1}{\partial z} + \frac{eE}{3mv^2}\frac{\partial}{\partial v}(v^2 f_1) + S_0 = 0,\\
&\frac{\partial f_1}{\partial t} + v\left(\frac{\partial f_0}{\partial z} + \frac{2}{5}\frac{\partial f_2}{\partial z}\right) + \\
&\qquad + \frac{eE}{m}\left\{\frac{\partial f_0}{\partial v} + \frac{2}{5v^3}\frac{\partial}{\partial v}(v^3 f_2)\right\} + S_1 = 0,\\
&\frac{\partial f_2}{\partial t} + v\left(\frac{2}{3}\frac{\partial f_1}{\partial z} + \frac{3}{7}\frac{\partial f_3}{\partial z}\right) + \\
&\qquad + \frac{eE}{m}\left\{\frac{2}{3} v \frac{\partial}{\partial v}\left(\frac{f_1}{v}\right) + \frac{3}{7v^4}\frac{\partial}{\partial v}(v^4 f_3)\right\} + S_2 = 0,\\
&\qquad\qquad \ldots,
\end{aligned}\right\} \tag{38.28}$$

where $S_k = (2k+1)\int P_k(\cos\alpha)\, S\, d\Omega/4\pi$.

The use of the functions f_k affords a simplification, of course, only where a few such functions are sufficient. In particular, two functions f_0 and f_1 are sufficient if $|\partial f_0/\partial z| \gg |\partial f_2/\partial z|$ and $|\partial f_0/\partial v| \gg (1/v^3)|\partial(v^3 f_2)/\partial v|$. In a spatially homogeneous plasma ($\partial f/\partial z = 0$) we have in a steady state $\partial f_1/\partial t = i\omega f_1$, $\partial f_2/\partial t \sim i\omega f_2$ and, since f_2 is assumed small,

$$f_1 = \frac{eE}{m(i\omega + \nu)}\frac{\partial f_0}{\partial v}$$

and

$$\begin{aligned}
|f_2| &\sim \left|\frac{eE}{m(i\omega+\nu)} v \frac{\partial}{\partial v}\left(\frac{f_1}{v}\right)\right|\\
&\sim \left|\frac{e^2E^2}{m^2(i\omega+\nu)^2} v \frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial f_0}{\partial v}\right)\right|,
\end{aligned}$$

where we have also used the facts that $S_1 = \nu f_1$ and $S_2 \sim \nu f_2$ (see below). Thus the necessary condition $|\partial f_0/\partial v| \gg (1/v^3)|\partial(v^3 f_2)/\partial v|$ becomes

$$\left|\frac{e^2E^2}{m^2(\omega^2+\nu^2)}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial f_0}{\partial v}\right)\right| \ll f_0.$$

If we take only this condition for the mean velocity $\bar{v} \sim \sqrt{(\varkappa T_e/m)}$, and put $\partial f_0/\partial v \sim f_0/\bar{v}$, we arrive at the requirement

$$e^2E_0^2/m\varkappa T_e[\omega^2 + \nu^2(T_e)] \ll 1. \tag{38.29}$$

The condition (38.1) for the field to be weak differs from (38.29) in that the quantity δ appears in the denominator. Thus the field may be strong and yet the condition (38.29) may hold. Expressing T_e in (38.29) by means of (38.16), we have essentially the condition $\delta \ll 1$. This inequality may be regarded as always valid in cases of interest to us. When spatial inhomogeneities are present, the above-mentioned condition $|\partial f_0/\partial z| \gg |\partial f_2/\partial z|$ is also of importance; assuming for simplicity that the field is absent or very weak, we have from equation (38.28) for f_2 the result $(i\omega + \nu)f_2 \sim v\,\partial f_1/\partial z$, whence we obtain the necessary condition for the function f_2 to be small:

$$\frac{\bar{v}}{\sqrt{[\omega^2 + \nu^2(T_e)]}} \left| \frac{\partial^2 f_1}{\partial z^2} \right| \ll \left| \frac{\partial f_0}{\partial z} \right|. \tag{38.30}$$

This condition has already been given in § 4. It gives rise to another condition which is important in non-steady processes: the first equation (38.28) shows that in some cases $\partial f_0/\partial t \sim v\,\partial f_1/\partial z$ (as, for instance, in the absence of a field when $\partial f_0/\partial t$ is large, since the integral $S_0 \sim \delta \nu f_0$ is small). In such conditions (38.30) leads to

$$|\partial f_0/\partial t| \ll \sqrt{[\omega^2 + \nu^2(T_e)]} f_0. \tag{38.31}$$

Here and in (38.30) the frequency ω is essentially given by $\omega \sim |\partial f_2/\partial t| \div f_2$. In order that the function f_2 should be negligible, the condition (38.31) must hold also in a spatially homogeneous but non-steady case in the presence of an electric field. This may be seen by retaining in (38.28) only the terms such as $\partial f_i/\partial t$, S_1 and S_2, and those involving the field E.

These conditions (38.29)–(38.31) have not been derived with great rigour, since quantities have been replaced by their mean values, and so on. Nevertheless it is clear that over a very wide range we may take only the first two equations (38.28), omitting the terms in f_2.†

When a magnetic field is present, whatever the direction of the spatial gradient, these equations become [13, 258]

$$\frac{\partial f_0}{\partial t} + \frac{1}{3} v \operatorname{div}_r \boldsymbol{f}_1 + \frac{e}{3mv^2} \frac{\partial}{\partial v} (v^2 \boldsymbol{E} \cdot \boldsymbol{f}_1) + S_0 = 0, \tag{38.32}$$

$$\frac{\partial \boldsymbol{f}_1}{\partial t} + v \,\mathbf{grad}_r f_0 + \frac{e\boldsymbol{E}}{m} \frac{\partial f_0}{\partial v} + \frac{e}{mc} \boldsymbol{H} \times \boldsymbol{f}_1 + \boldsymbol{S}_1 = 0. \tag{38.33}$$

The distribution function f is in this case of the form

$$f(t, \boldsymbol{r}, \boldsymbol{v}) = f_0(t, \boldsymbol{r}, v) + \boldsymbol{v} \cdot \boldsymbol{f}_1(t, \boldsymbol{r}, v)/v. \tag{38.34}$$

In a weak field we may suppose that the symmetrical part of f, i.e. the function f_0, is not affected by the field, and in the homogeneous case is a Max-

† This omission of f_2 relates to the calculation of the principal ("large") terms. In calculating small corrections, of the order of $\delta\nu/\omega$, we can omit f_2 only if $\delta\nu/\omega \gg \delta$ (see [258]).

wellian function f_{00} (4.16). Equation (38.33) then becomes independent of (38.32), and in the form (4.17) or (6.1) it has been used in §§ 4, 6, 10 etc.

The form of the collision integral S and its "moments" S_k has not yet been made explicit. This subject is discussed in detail in [258], and here we shall give only the result. Since in any collision between an electron and a heavy particle $\delta \ll 1$, the modulus of the velocity is changed only very slightly, and to a good approximation we have

$$\left.\begin{aligned} S_1 &= \nu(v) f_1, \\ \nu_{i,m}(v) &= N_{i,m} v \int q(v,\theta)(1-\cos\theta)\, d\Omega; \end{aligned}\right\} \tag{38.35}$$

this result has already been given and discussed in § 4, where the notation also is explained. It may be noted that to the same accuracy

$$\begin{aligned} S_2 &= \nu_2(v) f_2, \\ \nu_2(v) &= N_{i,m} v \int q(v,\theta)\,[1 - P_2(\cos\theta)]\, d\Omega; \end{aligned}$$

evidently $\nu_2(v) \sim \nu(v)$, a result already used in deriving the condition (38.29).

When only elastic collisions occur ($\delta = \delta_{\mathrm{el}} = 2m/M$),

$$S_0 = -\frac{1}{2v^2}\frac{\partial}{\partial v}\left\{v^2 \delta_{\mathrm{el}}\, \nu(v)\left[\frac{\varkappa T}{m}\frac{\partial f_0}{\partial v} + v f_0\right]\right\}. \tag{38.36}$$

Here $\nu(v)$ is the same as in (38.35), and T is the temperature of the heavy particles, which are assumed to have a Maxwellian velocity distribution. Collisions between electrons are ignored in both (38.35) and (38.36); this is legitimate if the number of neutral particles is sufficiently large. The expression (38.36) has an evident physical significance. Firstly, when the exchange of energy with heavy particles is neglected, collisions with such particles cannot affect the velocity modulus distribution of the electrons. Hence it is reasonable that S_0 should be proportional to δ. Secondly, if we have mean quantities and replace $\partial/\partial v$ by $1/\bar{v}$, we have $S_0 \sim \delta_{\mathrm{el}}\nu(1 + T/T_e) f_0 \sim \delta_{\mathrm{el}}\nu f_0 = f_0/\tau_k$, where τ_k is the relaxation time for energy. The quantity $\tau_k = 1/\delta\nu$ must determine the rate of variation of $f_0(v)$ owing to collisions with heavy particles A more detailed discussion of the expression for S_0 and its component terms is given in [258].

A strongly ionised plasma

In a strongly ionised plasma, where the condition (38.25) holds, and in the intermediate case, we cannot in general neglect collisions between electrons, and in (38.33) and (38.32) we must include the respective contributions $S_{1,ee}$ and $S_{0,ee}$ due to these collisions [258] as well as the terms (38.35) and (38.36). We have already discussed in § 6 the importance of such collisions in the determination of ε and σ, i.e. the effect of the term $S_{1,ee}$. The effect of collisions between electrons on the function f_0 is very much greater. The reason is that $S_{0,ee} \sim f_0/\tau_{ee} \sim \nu_{ee} f_0$, since in collisions between electrons, energy is transferred as efficiently as momentum. It is thus clear that, in a strongly ionised plasma [see the condition (38.25)] collisions between electrons ensure that the function f_0 is close to a Maxwellian with electron temperature T_e. This temperature itself is given by the same equation (38.32), which leads to the relation

$$\left.\begin{aligned} \frac{dT_e}{dt} &= \frac{2}{3\varkappa N} \boldsymbol{j}_t \cdot \boldsymbol{E} - \delta(T_e)\, \nu_{\text{eff}}(T_e)\,(T_e - T), \\ \nu_{\text{eff}}(T_e) &= \frac{2}{3\sqrt{(2\pi)}} \left(\frac{m}{\varkappa T_e}\right)^{5/2} \int\limits_0^\infty \nu(v)\, v^4 \exp\left(-\frac{m v^2}{2\varkappa T_e}\right) dv, \\ \delta(T_e) &\equiv \delta_{\text{eff}}(T_e) \\ &= \delta_{\text{el}} + \delta_{\text{inel}}, \\ \delta_{\text{el}} &= 2m/M. \end{aligned}\right\} \tag{38.37}$$

Here δ_{inel} is the contribution to δ from inelastic collisions. The corresponding expression and the derivation of equation (38.37) may be found in [258]; we shall not repeat that derivation here, since the equation is the same as (38.7) and evidently represents the law of conservation of energy. The only refinement concerns $\nu_{\text{eff}}(T_e)$, which is now entirely definite and exactly the same as (6.9), with T of course replaced by T_e.

The current density $\boldsymbol{j}_t$ is determined by the asymmetrical part of the distribution function:

$$\begin{aligned} \boldsymbol{j}_t &= e \int \boldsymbol{v} f\, d\boldsymbol{v} \\ &= e \int \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{f}_1)\, d\boldsymbol{v}/v \\ &= (4\pi e/3) \int v^3 \boldsymbol{f}_1(v)\, dv. \end{aligned} \tag{38.38}$$

In a strongly ionised homogeneous plasma, equation (38.33) becomes

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{f}_1}{\partial t} + \frac{e\boldsymbol{E}}{m} \frac{\partial f_{00}(v)}{\partial v} + \frac{e}{mc} \boldsymbol{H} \times \boldsymbol{f}_1 + \nu(v) \boldsymbol{f}_1 = 0, \\ &f_{00} = N\left(\frac{m}{2\pi\varkappa T_e}\right)^{3/2} \exp\left(-\frac{m v^2}{2\varkappa T_e}\right), \end{aligned}\right\} \tag{38.39}$$

where collisions between electrons are ignored; this is legitimate if $\omega^2 \gg \nu^2$, while if $\omega^2 \lesssim \nu^2$ an error of the order of unity is committed; see § 6.

In discussing equation (38.7) it has already been shown that in the range of greatest interest the temperature T_e is constant apart from small terms. In the approximation where T_e = constant we evidently have $\partial f_{00}/\partial t = 0$, and it is easily seen that the solution of equation (38.39) may be written

$$\left.\begin{aligned} f_1 &= -\dot{\boldsymbol{r}}\,\partial f_{00}/\partial v, \\ m\,d\dot{\boldsymbol{r}}/d\,t &= e\,\boldsymbol{E} + e\,\dot{\boldsymbol{r}} \times \boldsymbol{H}/c - m\,\nu(v)\,\dot{\boldsymbol{r}}, \\ \boldsymbol{j}_t &= e\,N\,\dot{\boldsymbol{r}}. \end{aligned}\right\} \tag{38.40}$$

The difference between this equation for $\dot{\boldsymbol{r}}$ so defined and equation (38.4) is that $\nu_{\text{eff}}(T_e)$ is replaced by $\nu(v)$. If ν is constant the two equations are the same, and this proves the statement made earlier that the elementary theory gives the correct result when $\nu(v) = \text{constant}$.

For $T_e = \text{constant}$ the calculation of the current $\boldsymbol{j}_t$ in a strong field is formally the same as in a weak field: for ε'_{ik} we obtain the expressions (10.32) but with $\nu_{\text{eff}} = \nu_{\text{eff}}(T_e)$, and T_e depending on E_0^2. The steady-state value of T_e is, according to (38.37), proportional to $\boldsymbol{j}_t$, and so an allowance for the velocity distribution leads to the appearance of the factor $K_\sigma[\omega/\nu_{\text{eff}}(T_e)]$ used in § 6 [see (6.25)]. In consequence the expression (38.21), for example, becomes

$$\frac{T_e}{T} = 1 + \left(\frac{E_0}{E_p}\right)^2 \frac{\delta(T)}{\delta(T_e)}\,\frac{\omega^2+\nu_0^2}{\omega^2+\nu^2(T_e)}\,K_\sigma\left(\frac{\omega}{\nu}\right). \tag{38.41}$$

Here E_p is the plasma field [see (38.1)], $\nu_0 \equiv \nu_{\text{eff}}(T)$ and $\nu(T_e) \equiv \nu_{\text{eff}}(T_e)$ defined by (38.37). To obtain (38.21) we must put $K_\sigma = 1$ and $\delta = \text{constant}$ [the latter having been assumed in going from (38.16) to (38.21)].

A weakly ionised plasma

In a weakly ionised plasma, where

$$\nu_{ee} \ll \delta\,\nu \approx \delta\,\nu_m \tag{38.42}$$

($\nu_m \equiv \nu_{\text{eff},m}(T_e)$ being the effective frequency of collisions between electrons and molecules), collisions between electrons may be neglected even in equation (38.32) for f_0. When $T_e \approx \text{constant}$ (i.e. $\partial f_0/\partial t \approx 0$) the solution of equation (38.33) has the form (38.40) with $\partial f_0/\partial v$ in place of $\partial f_{00}/\partial v$. Substituting this solution in (38.32), we obtain†

$$\frac{\partial f_0}{\partial t} - \frac{1}{2v^2}\frac{\partial}{\partial v}\left\{v^2\left[\left(\delta_{\text{el}}\,\nu(v)\,\frac{\varkappa T}{m} + \frac{2e\dot{\boldsymbol{r}}\cdot\boldsymbol{E}}{3m}\right)\frac{\partial f_0}{\partial v} + \delta_{\text{el}}\,v\,\nu(v)\,f_0\right]\right\} = 0\,, \tag{38.43}$$

where only elastic collisions are taken into account [the expression (38.36) being used].

Both in a constant or quasisteady electric field, where $\omega \ll \delta\,\nu$, and in a rapidly varying field, where $\omega \gg \delta\,\nu$, we have in a first approximation

† When $T_e \approx \text{constant}$ we can, as stated, neglect the derivative $\partial f_0/\partial t$ in going from (38.39) to (38.40), but in the equation for the function f_0, which varies slowly with time, the derivative $\partial f_0/\partial t$ must in general be retained. It may also be noted that for a non-Maxwellian distribution T_e signifies, of course, a quantity proportional to the mean energy $\overline{K}$.

$\partial f_0/\partial t = 0$, and moreover in (38.43) we can put $\dot{\boldsymbol{r}}\cdot\boldsymbol{E} = eE_0^2\nu/2m(\omega^2+\nu^2)$. Consequently we have in this approximation

$$\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\left[\left(\delta_{\rm el}\,\nu\frac{\varkappa T}{m}+\frac{e^2E_0^2\,\nu}{3m^2(\omega^2+\nu^2)}\right)\frac{\partial f_0}{\partial v}+\delta_{\rm el}\,v\,\nu f_0\right]\right\}$$

$$\equiv\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 j_v)\equiv 0. \tag{38.43a}$$

Multiplying this equation by $v^2\,dv$ and integrating from zero to v, we see that $v^2 j_v(v) = 0$, since in the absence of sources $v^2 j_v(v) = 0$ for $v = 0$. Integrating over velocity the equation $j_v(v) = 0$, we now find

$$f_0 = C\exp\left\{-\int\limits_0^v\frac{m\,v\,dv}{\varkappa T+e^2E_0^2/3m\,\delta_{\rm el}(\omega^2+\nu^2(v))}\right\}. \tag{38.44}$$

Here C is a constant given by the normalisation condition $\int f_0\,d\boldsymbol{v} = N$. In a very strong constant field, (38.44) gives the distribution

$$f_0 = C\exp\left\{-\int\limits_0^v\frac{3m^2\,\delta_{\rm el}\,\nu^2(v)\,v\,dv}{2e^2E^2}\right\}$$

$$= C\exp\{-3m^3v^4/4e^2Ml^2E^2\}, \tag{38.45}$$

where in addition E_0^2 has been replaced by $2E^2$, and to derive the last expression we have put $\delta_{\rm el} = 2m/M$ and $\nu = \nu_m = \pi a^2 N_m v = v/l$ (collisions with molecules). The Druyvesteyn distribution (38.45) differs greatly from the Maxwellian for high electron velocities. When inelastic collisions are taken into account, $\delta_{\rm el}$ in (38.44) must be replaced by $\delta(v)$ (see [258]).

Knowing the function f_0, it is easy to find $\boldsymbol{f}_1$ by means of equations (38.39) and (38.40), with f_0 instead of f_{00}. The quantities $\bar{K}$, ε and σ which are needed in the solution of problems of electrodynamics are then

$$\left.\begin{aligned}
\bar{K} &= \frac{2\pi m}{N}\int\limits_0^\infty v^4 f_0\,dv,\\
\varepsilon &= 1+\frac{(4\pi e)^2}{3}\int\limits_0^\infty\frac{v^3}{\omega^2+\nu^2(v)}\frac{\partial f_0}{\partial v}\,dv,\\
\sigma &= -\frac{4\pi e^2}{3}\int\limits_0^\infty\frac{\nu(v)\,v^3}{\omega^2+\nu^2(v)}\frac{\partial f_0}{\partial v}\,dv.
\end{aligned}\right\} \tag{38.46}$$

These are all integral quantities and therefore depend relatively little on the form of the function f_0, and in particular on the velocity dependence of f_0

in the "tail" of the distribution. It is therefore reasonable that in a weakly ionised plasma $\bar{K}$, ε and σ usually differ only slightly from the values calculated by means of the Maxwellian function $f_{00}(T_e)$ with $T_e = 2\bar{K}/3\varkappa$. For example, in a strong constant field with elastic collisions with molecules we have for a weakly ionised plasma $\bar{K} = 0{\cdot}604\, elE/\sqrt{\delta_{\text{el}}}$, and for a strongly ionised plasma $\bar{K} = 0{\cdot}613\, elE/\sqrt{\delta_{\text{el}}}$. In such conditions it is natural that in the intermediate case $\nu_{ee} \sim \delta\nu$ also we obtain results close to those which hold for a strongly ionised plasma (where $\nu_{ee} \gg \delta\nu$). Thus we reach the same conclusions as we did earlier in discussing the limits of validity of the elementary theory.

To conclude, we may note that, if the condition (38.1) for the field to be weak is satisfied, the perturbation of the distribution function may certainly be regarded as small,† but of course we cannot be sure that non-linear effects will be completely absent. The reason is that these effects are such that they may be significant even when the function f_0 differs only slightly from the Maxwellian; in other words, the condition (38.1) ensures that the non-linear effects are small, but the possibility of entirely neglecting them depends on the formulation of the problem and on the accuracy of the measurements. In such conditions (a weak field but non-linear effects beginning to appear in a way which is not negligible) the perturbation method (method of successive approximations) is sometimes very convenient and effective. In this method we put $f_0 = f_{00} + f_{01}$, where $|f_{01}| \ll f_{00}$ and f_{00} is the Maxwellian distribution. This treatment is described in detail in [22, § 64]; see also [260, 261].

§ 39. NON-LINEAR EFFECTS IN RADIO WAVE PROPAGATION IN THE IONOSPHERE

Introduction

The non-linearity of electromagnetic processes in a plasma is clearly seen, in particular, in the propagation of intense radio waves. For example, when a wave is propagated, its effect on the plasma leads to a non-linear "self-interaction", which changes the absorption and phase of the wave, and also produces harmonics of the fundamental frequency. When several waves are propagated, the principle of superposition no longer holds good: the incident and reflected, the ordinary and extraordinary, and indeed any two waves are no longer independent; they interact in a non-linear manner because the properties of the medium themselves vary. This non-linear interaction, which

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† That is, the perturbation of the main part of the distribution function (velocities $v \lesssim \sqrt{(\varkappa T/m)}$).

TABLE 39.1

Electron temperature T_e in the field of stations of various powers

Ionospheric layer	ω	E_p(mV/m)	T_e/T for power (kW)						
			10	100	500	1000	5000	10^5	10^6
D (daytime, height 60 km)	$\lesssim 3 \times 10^6$	325	1·002	1·02	1·1	1·2	1·7	4	15
$\nu_0 = 10^7$ sec^{-1}	10^7	470	1·001	1·01	1·05	1·1	1·4		
$T = 300$	10^8	3200	1·00002	1·0002	1·001	1·002	1·01		
Lower E	$\lesssim 10^5$	19	1·3	2·4	4·6	6·7	13	55	170
(night-time,	10^6	32	1·1	1·8	3·7	5·3	12		
height 90 km)	2×10^6	56	1·04	1·3	2·5	3·7	10		
$\nu_0 = 7 \times 10^5$	5×10^6	130	1·006	1·06	1·3	1·6	3·8		
$T = 200$	10^7	270	1·002	1·02	1·08	1·2	1·8		
	10^8	2700	1·00002	1·0002	1·0008	1·002	1·008		
F (height 300 km)	$\lesssim 3 \times 10^2$	0·02							
$\nu_0 = 10^3$	10^8	1900	1·000004	1·00004	1·0002	1·0004	1·002		
$T = 2000$									

occurs even in a homogeneous medium, is of course quite distinct from the "interaction" of normal waves discussed in § 20 and elsewhere.†

In a weak field [when the condition (38.1) holds], the effect of the field on the plasma is usually neglected, as it has been throughout the preceding chapters. It should, however, be again emphasised that even in a weak field some small non-linear effects may be observed on account of their particular nature. To analyse such weak non-linear effects we may naturally use the method of successive approximations (see [22, § 64]). On the other hand, non-linear effects in a weak field may also, of course, be discussed on the basis of the general expressions which are valid for any value of the ratio E_0/E_p. For strong fields ($E_0/E_p \gtrsim 1$) and especially for very strong fields ($E_0/E_p \gg 1$) the non-linear phenomena are very marked, and the customary linear theory of radio wave propagation cannot in general be used even as a first approximation.

The principal results of the non-linear theory of radio wave propagation in plasmas are briefly described below in terms of their application to the Earth's ionosphere. For this reason Table 39.1 shows, as a supplement to what has been said at the beginning of § 38, the values of the "plasma field" E_p for the ionosphere, together with the maximum values of the electron temperature in the field of stations of various powers; only electron-molecule collisions are taken into account, the field E_0 at the boundary of the ionosphere is determined by formula (34.22) with $\sin\theta = 1$, and $\delta = 2 \times 10^{-3}$ for the D and E layers and 10^{-4} for the F layer. It is evident from the table that intense waves of medium and long wavelength may considerably affect the energy (temperature) of the electrons in the lower part of the E layer. On the other hand, the effect on the ionosphere of short waves and of medium and long waves of low power is only slight.

Basic relations

Considering an arbitrary non-magnetic medium, we write the field equations in the form

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† V. B. Gil'denburg and I. G. Kondrat'ev have pointed out to the author the existence in the ionosphere of a non-linear effect due to the action on the plasma medium of a force which occurs in an inhomogeneous field. The value of this force per electron, averaged over the period $2\pi/\omega$, is

$$\boldsymbol{F} = -\frac{\varepsilon - 1}{16\pi N}\,\mathrm{grad}\,|\boldsymbol{E}|^2 = -\frac{e^2}{4m\omega^2}\,\mathrm{grad}\,|\boldsymbol{E}|^2, \quad \text{where } \boldsymbol{E} = \boldsymbol{E}_0 e^{i\omega t}$$

(collisions are neglected). This non-linear effect is in general much stronger than the one discussed in § 20.

$$\left.\begin{aligned}
\mathbf{curl}\,\boldsymbol{H} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} \\
&= \frac{4\pi}{c}\boldsymbol{j}_t + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \\
\operatorname{div}\boldsymbol{D} &= 4\pi\varrho, \\
\mathbf{curl}\,\boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}, \\
\operatorname{div}\boldsymbol{H} &= 0;
\end{aligned}\right\} \tag{39.1}$$

here $\boldsymbol{D} = \hat{\varepsilon}\boldsymbol{E}$, $\boldsymbol{j} = \hat{\sigma}\boldsymbol{E}$, and $\hat{\varepsilon}$ and $\hat{\sigma}$ are some operators which depend on the properties of the medium, being linear operators only in a sufficiently weak field. In a plasma we have

$$\boldsymbol{j}_t = \hat{\sigma}\boldsymbol{E} + \frac{\partial}{\partial t}\left(\frac{\hat{\varepsilon} - 1}{4\pi}\boldsymbol{E}\right) = e\int \boldsymbol{v} f(t, \boldsymbol{r}, \boldsymbol{v})\, d\boldsymbol{v}. \tag{39.2}$$

Despite the fact that the wave field is not uniform in space, this non-uniformity may usually be ignored in determining the function f, and therefore $\hat{\varepsilon}$ and $\hat{\sigma}$; that is, the term $\boldsymbol{v}\cdot\mathbf{grad}_r f$ in equation (38.26) may be neglected. This assumes that $\hat{\varepsilon}$ and $\hat{\sigma}$ are local operators, i.e. that the current density $\boldsymbol{j}_t$ at a given point is determined by the field $\boldsymbol{E}$ at that point. In a weak field this assumption is equivalent to neglecting spatial dispersion.† In a strong field the local approximation is valid if the field amplitude varies only slightly both over the free path $l = \bar{v}/\nu_{\text{eff}}$ and over the longer energy relaxation length $l/\sqrt{\delta} = \bar{v}/\nu_{\text{eff}}\sqrt{\delta}$; the length $l/\sqrt{\delta} = \sqrt{(l\bar{v}/\delta\nu_{\text{eff}})} \sim \sqrt{(D\tau_k)}$ is the diffusion path traversed by an electron in time $\tau_k = 1/\delta\nu_{\text{eff}}$.

In the propagation of radio waves the field may be regarded as rapidly varying if

$$\delta\nu_{\text{eff}}/\omega \ll 1. \tag{39.3}$$

In the ionosphere this condition holds for waves with $\lambda < 100$ to 1000 km, in the solar corona for $\lambda < 10^5$ km, and in electronic systems and laboratory apparatus usually for $\lambda < 10$ to 100 m.

When the condition (39.3) holds, as we have seen in § 38, the electron temperature T_e in a field of any strength (in a steady state) is constant to a first approximation, and the current density $\boldsymbol{j}_t$ varies with the frequency of the field $\boldsymbol{E}$. Hence, when the above condition for the validity of the local approximation is satisfied, the problem of wave propagation may be divided into two parts. Firstly, as in a weak field when spatial dispersion is absent, the

† The term "spatial dispersion" relates to the possibility of using quantities $\varepsilon(\omega, \boldsymbol{k})$ and $\sigma(\omega, \boldsymbol{k})$ which depend not only on ω (time dispersion) but also on the wave vector $\boldsymbol{k}$. In the non-linear theory the use of Fourier's method is limited and $\hat{\varepsilon}$ and $\hat{\sigma}$ are in general complex operators for which the use of the term "dispersion" requires a refinement of its definition at least.

current $\boldsymbol{j}_t$ is found as a function of $\boldsymbol{E}$; secondly, the field equations are solved with this current.

This approximation will be used below to discuss the self-interaction effect and the cross-modulation of radio waves. It is only when we consider the "subsidiary" waves with combination frequencies, calculating not the fundamental quantities but small corrections to them, that the terms of order $\delta\nu_{\text{eff}}/\omega$ are no longer negligible, and the problem is thereby complicated.

The self-interaction effect

Let us now examine the non-linear self-interaction of radio waves. To do so, we shall consider the propagation in an isotropic plasma of a wave whose field at the boundary of the medium (the plane $z = 0$) is $\boldsymbol{E}_0(0)\cos\omega t$. In a steady state (i.e. at a sufficiently long time $\Delta t \gg 1/\delta\nu_{\text{eff}}$ after the field is applied) the electron temperature T_e takes some constant value (small terms of the order of $\delta\nu_{\text{eff}}/\omega$ being neglected). Accordingly, the wave is propagated with unchanged frequency ω in a medium where ε and σ are constant in time but depend on the amplitude E_0 of the wave field. From the field equations (39.1) we derive in the usual manner (see § 2) the equation

$$\triangle\boldsymbol{E} - \mathbf{grad}\,\mathrm{div}\boldsymbol{E} + (\omega^2/c^2)\,\varepsilon'\big(\omega, \boldsymbol{r}, E_0(\boldsymbol{r})\big)\,\boldsymbol{E} = 0. \tag{39.4}$$

The expression for $\varepsilon' = \varepsilon - i\cdot 4\pi\sigma/\omega$ has been derived in § 38:

$$\varepsilon' = 1 - 4\pi e^2 N(\boldsymbol{r})/m\omega[\omega - i\nu_{\text{eff}}(T_e)]; \tag{39.5}$$

here and henceforward we use the formulae of the elementary theory, the temperature T_e being a function of $E_0(\boldsymbol{r})$ given by (38.16).

On account of the complexity of the non-linear equation (39.4) we shall consider its solution only with various simplifying assumptions, namely that the plasma is not only isotropic but also plane-parallel with slowly varying properties, so that the approximation of geometrical optics is valid. Then equation (39.4) becomes

$$d^2E/dz^2 + (\omega^2/c^2)\,\varepsilon'(\omega, z, E_0)\,E = 0,$$

and its approximate solution may be written (see § 16)

$$\left.\begin{aligned} &E = C\exp\Big(i\omega t - i\omega\int n\,dz/c\Big)\exp\Big(-\omega\int \varkappa\,dz/c\Big),\\ &\varepsilon'\big(\omega, z, E_0(z)\big) = (n - i\varkappa)^2, \end{aligned}\right\} \tag{39.6}$$

where in the zero-order approximation C = constant and only the wave travelling in one direction (that of the z-axis) is considered. Since n and $\varkappa$ depend on E_0, the formal solution (39.6) is an integral equation, which for the amplitude may be written

$$\left.\begin{aligned} &E_0 = C\exp\left[-\omega\int_0^z \varkappa(\omega, z, E_0)\,dz/c\right],\\ &dE_0/dz + \omega\varkappa(\omega, z, E_0)\,E_0/c = 0. \end{aligned}\right\} \tag{39.7}$$

If $|\varepsilon| \gg 4\pi\sigma/\omega$ the index of absorption is $\varkappa = 2\pi\sigma/\omega n \approx 2\pi\sigma/\omega\sqrt{\varepsilon}$, and in a weak field we have [see (7.17)]

$$\varkappa = \varkappa_0(\omega, z) = \frac{2\pi e^2 N \nu_0}{m\,\omega(\omega^2 + \nu_0^2)\sqrt{[1 - 4\pi e^2 N/m(\omega^2 + \nu_0^2)]}}. \tag{39.8}$$

In a strong field under these conditions the expressions for ε and σ are the same as in a weak field, but with $\nu_0 \equiv \nu_{\text{eff}}(T)$ replaced by $\nu_{\text{eff}}(T_e) \equiv \nu(T_e)$. Hence

$$\varkappa(\omega, z, E_0) = \varkappa_0(\omega, z)\,\frac{[\nu(T_e)/\nu_0]\,[(\omega^2/\nu_0^2) + 1]}{(\omega^2/\nu_0^2) + [\nu^2(T_e)/\nu_0^2]}, \tag{39.9}$$

where for simplicity we have neglected the dependence of $n \approx \sqrt{\varepsilon}$ on T_e, as is correct if $n \approx 1$, and also always when $\omega^2 \gg \nu^2$.

To find the field $E_0(z)$ we must evidently solve equation (39.7), using the expression (39.9) and the relation between T_e and E_0.

For collisions with molecules $\nu(T_e)/\nu_0 = \sqrt{[T_e(E_0)/T]}$ and, using the variable $\tau = \sqrt{[T_e(E_0)/T]}$, we can rewrite the relation (38.21) and equation (39.7) as

$$\left(\frac{E_0}{E_p}\right)^2 = (\tau^2 - 1)\,\frac{\omega^2 + \nu_0^2\tau^2}{\omega^2 + \nu_0^2}, \quad \tau = \sqrt{\frac{T_e(E_0)}{T}} = \frac{\nu(T_e)}{\nu_0}, \tag{39.10}$$

$$\frac{d\tau}{dz}\left(\frac{1}{\tau^2 - 1} + \frac{2\nu_0^2}{\omega^2 + \nu_0^2}\right) + \frac{\omega}{c}\,\varkappa_0(z) = 0; \tag{39.11}$$

in deriving (39.11) from (39.7) and (39.9) we have expressed the amplitude E_0 in terms of τ by means of (39.10).

From (39.11) we derive by integration the following expression for τ:

$$\left.\begin{aligned}
&\frac{\tau - 1}{\tau + 1}\exp\left(\frac{4\nu_0^2}{\omega^2 + \nu_0^2}\,\tau\right) = \frac{\tau_0 - 1}{\tau_0 + 1}\exp\left(\frac{4\nu_0^2}{\omega^2 + \nu_0^2}\,\tau_0\right)\exp(-2\mathscr{K}(z)),\\
&\qquad \mathscr{K}(z) = \frac{\omega}{c}\int_0^z \varkappa_0(z)\,dz,\\
&\qquad \tau_0 \equiv \tau(0) = \sqrt{\frac{T_e(E_0(0))}{T}},\\
&\qquad \tau = \tau(z) = \sqrt{\frac{T_e(E_0(z))}{T}},
\end{aligned}\right\} \tag{39.12}$$

where it is everywhere assumed that T and ν_0 are independent of z. It may also be recalled that $E_0(0)$ is the field amplitude at $z = 0$ and $\mathscr{K}$ a quantity which determines the absorption of a weak wave over a distance z.

Using the expressions (39.10) and (39.12), we can determine the amplitude of the field $E_0(z)$, which may conveniently be written

$$E_0(z) = E_0(0)\exp[-\mathscr{K}(z)]\,P(E_0(0)/E_p,\ \omega/\nu_0,\ \mathscr{K}(z)). \tag{39.13}$$

The factor P, which is evidently equal to unity in a weak field, represents the self-interaction of the wave during its propagation.

Within the plasma the wave always becomes weak if $\mathscr{K}(z) \gg 1$, and $\tau \to 1$. In this region we have [258]

$$P = 2\frac{E_p}{E_0(0)}\sqrt{\frac{\tau_0 - 1}{\tau_0 + 1}}\exp\left[\frac{2\nu_0^2}{\omega^2 + \nu_0^2}(\tau_0 - 1)\right]. \tag{39.14}$$

In the limiting case $\omega^2 \gg \nu_0^2 \tau_0$ and $\tau_0 \gg 1$ the factor P is independent of τ_0, and

$$E_0(z) = 2E_p \exp[-\mathscr{K}(z)]. \tag{39.15}$$

If the opposite inequality $\omega^2 \ll \nu_0^2 \tau_0$ holds, the factor P increases with τ_0 (i.e. with $E_0(0)$), since the absorption coefficient at low frequencies decreases with increasing T_e. Thus we have

$$E_0(z) = 2E_p \exp\left\{\sqrt{\left[4\left(\frac{\nu_0^2}{\omega^2 + \nu_0^2}\right)^{3/2}\frac{E_0(0)}{E_p}\right]}\right\}\exp[-\mathscr{K}(z)]. \tag{39.16}$$

The dependence of $E_0(z)\exp[\mathscr{K}(z)]/E_p$ on $E_0(0)/E_p$ in the interior of the plasma in both limiting cases is shown in Fig. 39.1.

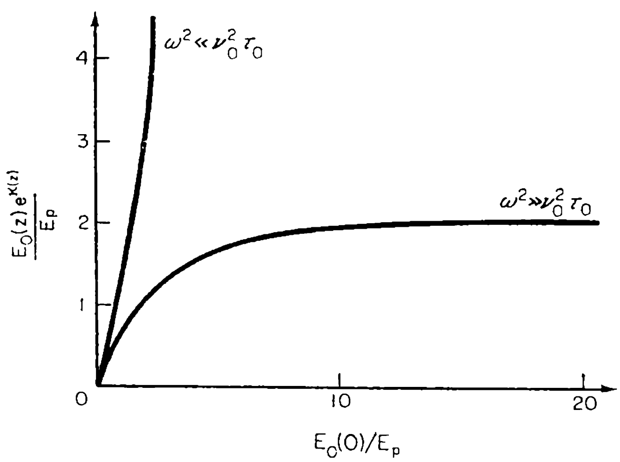

FIG. 39.1. $E_0(z)\exp(\mathscr{K}(z))/E_p$ as a function of $E_0(0)/E_p$ for collisions with molecules.

For arbitrary $\mathscr{K}(z)$, i.e. for any distance from the boundary of the plasma layer, a simple expression for the field $E_0(z)$ is obtained only at high frequencies $\omega^2 \gg \nu_0^2 \tau$, when

$$E_0(z) = 2E_p\sqrt{\frac{\tau_0 - 1}{\tau_0 + 1}}\frac{\exp[-\mathscr{K}(z)]}{1 - [(\tau_0 - 1)/(\tau_0 + 1)]\exp[-2\mathscr{K}(z)]}. \tag{39.17}$$

The field $E_0(z)$ is evidently strong only for $\mathscr{K}(z) \lesssim 1$, whatever the value of $E_0(0) \gtrsim E_p$.

For collisions with ions it is sufficient to consider only waves of high frequency, since the condition $e^2 N^{\frac{1}{3}}/\varkappa T \ll 1$ for the gas-kinetic approximation to be valid† leads to the inequality

$$\omega_0 = \sqrt{(4\pi e^2 N/m)} \gg \nu_{0i} \sim e^4 N/(\varkappa T)^{3/2} m^{\frac{1}{2}}.$$

Hence $\varepsilon = 1 - \omega_0^2/(\omega^2 + \nu_{0i}^2) > 0$, i.e. waves can be propagated, only for $\omega \gtrsim \omega_0 \gg \nu_{0i}$ (the high-frequency case). The absorption coefficient for waves of high frequency is seen from (39.9) to decrease rapidly as the temperature T_e increases (since $\nu_i \sim T_e^{-3/2}$). In consequence the self-interaction factor P increases rapidly with $E_0(0)$, and the wave, as it were, drives its way through the plasma; when $E_0(0) > 2[\mathscr{K}(z)]^{\frac{1}{3}} E_p$, the wave is hardly absorbed at all in a layer of thickness z. $\mathscr{K}(z)$ is given by (39.12); the dependence of $E_0(z)/E_p$ on $E_0(0)/E_p$ for two values of $\mathscr{K}(z)$ is shown in Fig. 39.2.

The action of the wave on the plasma causes not only the above-mentioned change of amplitude but also a change in the phase of the wave (i.e. a deviation of the phase from the value $(\omega/c)\int n_0(z)\,dz$ which corresponds to a weak wave). The phase change concerned is not large. The modulation of the wave as a

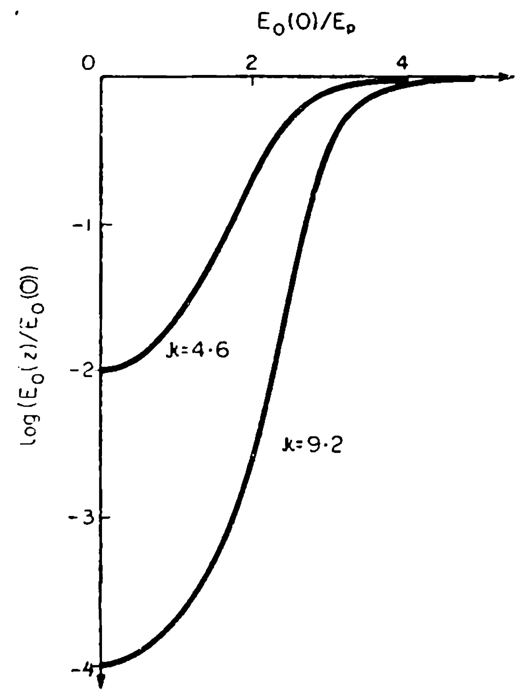

FIG. 39.2. $E_0(z)/E_p$ as a function of $E_0(0)/E_p$ for collisions with ions.

result of its self-interaction is of greater interest. If the modulation frequency $\Omega \ll \delta\nu_0$, the process may be regarded as quasisteady. This means that the

† This condition signifies that the kinetic energy $\varkappa T$ of the electron is large compared with the potential $e^2/\bar{r} \sim e^2 N^{\frac{1}{3}}$.

problem is solved in the same way as for a field of constant amplitude, but in the final formulae $E_0(0)$ is replaced by the modulated field $E_0(0, t) = E_0(0)(1 + \mu_0 \cos \Omega t)$. The expression (39.13) then becomes

$$\left.\begin{aligned} E_0(z,t) &= E_0(0,t)\exp[-\mathscr{K}(z)]\,P\big(E_0(0,t)/E_p,\ \omega/\nu_0,\ \mathscr{K}(z)\big),\\ E_0(0,t) &= E_0(0)\,(1+\mu_0\cos\Omega t). \end{aligned}\right\} \tag{39.18}$$

Since the self-interaction factor P depends non-linearly on $E_0(0, t)/E_p$ it is clear from (39.18) that the self-interaction leads to a change in the depth of modulation and to the appearance of harmonics of frequency 2Ω, 3Ω, etc.

If $\Omega \gtrsim \delta\nu_0$, however, the problem cannot be regarded as quasisteady and equations (39.7) for the field and (38.7) for the temperature T_e must be solved simultaneously.

We shall not discuss here any of the above effects of self-interaction (the change in the phase, the depth of modulation and the frequency spectrum) or the significance of self-interaction in the propagation of radio waves in the ionosphere. For these, reference may be made to [258].

Non-linear interaction of waves. Cross-modulation

The perturbations caused in a plasma by an intense wave affect not only its own propagation but also other waves which pass through the perturbed region. Here we may mention three types of effect.

If an intense wave is modulated in amplitude with a low frequency Ω, the perturbations which it causes in the plasma are also modulated, and therefore so are waves which pass through the perturbed region. This is called cross-modulation or the Luxembourg–Gor'kiĭ effect. It can quite easily be observed in the propagation of radio waves in the ionosphere and is of practical importance in radio communications in the medium wavelength range.

In the propagation of intense unmodulated waves they perturb the plasma and, first of all, cause constant changes in the electron temperature, and therefore also in the conductivity and permittivity of the medium. Thus the conditions of propagation (absorption, refraction) in the perturbed region are changed for all other waves. Secondly, in addition to the constant perturbations of ε and σ, there are weak variable perturbations with frequencies which are multiples of that of the perturbing wave. These perturbations cause the appearance of waves with combination frequencies if other waves are propagated in the plasma.†

In order to calculate the depth and phase of the cross-modulation, we must first determine the magnitude of the low-frequency perturbations caused in the

† The waves with combination frequencies also appear, of course, in the interaction of modulated waves, but the presence of modulation is not of fundamental importance here. It is therefore natural to discuss the subject of combination frequencies for unmodulated waves only.

plasma by an intense wave (with field E_1) and then find how these perturbations affect another wave with field E_2. If either of waves 1 and 2 is strong, self-interaction must be taken into account. We shall, however, treat only the case where not only is wave 2 weak but the perturbing wave may also be regarded as weak in terms of the condition (38.1). The perturbations of the plasma due to the propagation of wave 1 will, of course, be taken into account and may be considerable. The reason is, as already mentioned, that specifically non-linear effects (in particular, cross-modulation) can easily be observed even when $\Delta T_e/T \ll 1$. Moreover, this case occurs in the Earth's ionosphere for waves shorter than 1 km and stations of power less than 500 kW (see Table 39.1).

The amplitude of the electric field of the modulated weak wave 1 (of frequency ω_1) is, in the approximation of geometrical optics,

$$E_{01}(z,t) = [\varepsilon_1(0)/\varepsilon_1(z)]^{1/4}\, E_{01}(0,0)\,(1+\mu_0 \cos\Omega\, t)\exp[-\mathscr{K}_1(z)]; \tag{39.19}$$

here $\varepsilon_1(0)$ and $\varepsilon_1(z)$ are the values of $\varepsilon(z,\omega_1) \equiv \varepsilon_1(z)$ at the points $z=0$ and z,

$$\mathscr{K}_1(z) = \frac{\omega_1}{c}\int\limits_0^z \varkappa_1\, dz,$$

$\varkappa_1 \equiv \varkappa_0(\omega_1)$, and the absorption is assumed to be weak [and so in the amplitude (39.19) ε' has been replaced by ε (see § 16)].

The field of wave 1, taking account of the time factor, is $E_1 = E_{01}(z,t)\cos(\omega_1 t - \varphi_1)$, where φ_1 is some phase and E_{01} is given by (39.19). We usually have

$$\omega \gg \Omega, \tag{39.20}$$

and we may suppose in determining the directed velocity $\dot{\boldsymbol{r}}$ that the process is quasisteady. In other words, we can use the expression (38.10) for $\dot{\boldsymbol{r}}$, replacing $\boldsymbol{E}_0$ by $\boldsymbol{E}_{01}$ and ωt by $\omega_1 t - \varphi_1$. We can not do exactly the same for the temperature T_e, which varies slowly, but in equation (38.11) we can, as stated in § 38, omit the terms in $\cos\omega t$ and $\cos 2\omega t$. Consequently this equation becomes

$$\frac{dT_e}{dt} = \frac{e^2 E_{01}^2(z,t)\,\nu_0}{3m\varkappa(\omega_1^2+\nu_0^2)} - \delta\nu_0(T_e - T). \tag{39.21}$$

Substituting (39.19), we find for the part of the temperature perturbation $\Delta T_e = T_e - T$ which varies with frequencies Ω and 2Ω

$$\left.\begin{aligned} \frac{\Delta_\Omega T_e}{T} &= \frac{2\mu_0 e^2 E_{01}^2(0,0)}{3\varkappa T m\delta(\omega_1^2+\nu_0^2)}\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(z)}}\times \\ &\times \exp[-2\mathscr{K}_1(z)]\left\{\frac{\delta\nu_0\cos(\Omega t-\varphi_\Omega)}{\sqrt{(\delta^2\nu_0^2+\Omega^2)}} + \frac{\mu_0\delta\nu_0\cos(2\Omega t-\varphi_{2\Omega})}{4\sqrt{(\delta^2\nu_0^2+4\Omega^2)}}\right\}, \\ \varphi_\Omega &= \tan^{-1}(\Omega/\delta\nu_0), \\ \varphi_{2\Omega} &= \tan^{-1}(2\Omega/\delta\nu_0). \end{aligned}\right\} \tag{39.22}$$

The change in temperature causes a corresponding change in the collision frequency, so that $\nu = \nu_0 + \Delta\nu$. Considering only collisions with molecules, we have†

$$\left.\begin{aligned} \nu &= \nu_0 \sqrt{(T_e/T)}\,, \\ \Delta_\Omega \nu &= \nu_0 \Delta_\Omega T_e/2T. \end{aligned}\right\} \tag{39.23}$$

The amplitude of any other weak wave 2 (which we assume unmodulated) also has the form (39.19), of course:

$$\left.\begin{aligned} E_{02}(s) &= [\varepsilon_2(0)/\varepsilon_2(s)]^{\frac{1}{4}} E_{02}(0) \exp[-\mathscr{K}_2(s)]\,, \\ \mathscr{K}_2(s) &= \frac{\omega_2}{c}\int \varkappa_2(s)\,ds, \end{aligned}\right\} \tag{39.24}$$

where the integration is along the path of the ray.

The index of absorption $\varkappa$ depends on the collision frequency, and so the perturbing action of wave 1, which changes ν, causes an amplitude modulation of wave 2: $\varkappa_2(\nu) = \varkappa_2(\nu_0) + (\partial\varkappa_2/\partial\nu_0)\Delta_\Omega\nu$ and the amplitude (39.24) of wave 2 after passing through the perturbed region may be written

$$\begin{aligned} E_{02} = E_{02}(0) \exp\left(-\frac{\omega_2}{c}\int\limits_s \varkappa_2(\nu_0)\,ds\right) \times \\ \times \left(1 - \frac{\omega_2}{2c}\int\limits_s \nu_0 \frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{\Delta_\Omega T_e}{T}\,ds\right), \end{aligned} \tag{39.25}$$

where we have used the relations (39.23) and (39.24), omitted the factor $[\varepsilon_2(0)/\varepsilon_2(z)]$, and$^{1/4}$ assumed the depth of cross-modulation to be small, so that

$$\exp\left(-\frac{\omega_2}{2c}\int\limits_s \nu_0 \frac{\partial\varkappa_2(\nu_0)}{\partial\nu_0}\frac{\Delta_\Omega T_e}{T}\,ds\right)$$

can be replaced by the first two terms of its series expansion.

A comparison of the expressions (39.22) and (39.25) shows that the field is

$$E_{02} = \text{constant}\,[1 - \mu_\Omega\cos(\Omega t - \varphi_\Omega) - \mu_{2\Omega}\cos(2\Omega t - \varphi_{2\Omega})],$$

---

† In collisions with ions we have to a fairly good approximation $\nu = \nu_0(T/T_e)^{3/2}$ and $\Delta_\Omega\nu = -\frac{3}{2}\nu_0(T/T_e)^{3/2}\Delta_\Omega T_e/T_e$, i.e. the changes $\Delta_\Omega\nu$ and $\Delta_\Omega T$ have opposite signs. Consequently the coefficients μ_Ω and $\mu_{2\Omega}$ which characterise the depth of cross-modulation (see below) have opposite signs for collisions with molecules and with ions.

and the depth of cross-modulation is

$$\left.\begin{aligned}\mu_\Omega &= \frac{\mu_0 e^2 E_{01}^2(0,0)}{3\varkappa T m \delta}\,\frac{\omega_2}{c}\int\limits_s \frac{\nu_0\,\partial\varkappa_2(\nu_0)/\partial\nu_0}{\omega_1^2+\nu_0^2}\,\frac{\delta\,\nu_0}{\sqrt{(\delta^2\nu_0^2+\Omega^2)}}\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(s)}}\times\\ &\quad\times\exp[-2\mathscr{K}_1(s)]\,ds,\\ \mu_{2\Omega} &= \frac{\mu_0^2 e^2 E_{01}^2(0,0)}{12\varkappa T m \delta}\,\frac{\omega_2}{c}\int\limits_s \frac{\nu_0\,\partial\varkappa_2(\nu_0)/\partial\nu_0}{\omega_1^2+\nu_0^2}\,\frac{\delta\,\nu_0}{\sqrt{(\delta^2\nu_0^2+4\Omega^2)}}\sqrt{\frac{\varepsilon_1(0)}{\varepsilon_1(s)}}\times\\ &\quad\times\exp[-2\mathscr{K}_1(s)]\,ds.\end{aligned}\right\}\qquad(39.26)$$

In order to derive the final expressions for the depth of cross-modulation we must effect the integration along the path s and use the expression (39.8) for $\varkappa_2\,[=\varkappa_0(\omega_2)]$. The resulting formulae [258] depend, of course, on the geometry of the problem: the relative position of the reflection points of waves 1 and 2, the angles of incidence of the two waves on the layer, etc. Here we shall merely note the presence of the characteristic factor $\delta\nu_0/\sqrt{(\delta^2\nu_0^2+\Omega^2)}$, which indicates that the depth of cross-modulation decreases as the frequency Ω increases (in the range $\Omega \gtrsim \delta\nu_0$). Moreover, in this approximation, if $\mu_0 \ll 1$ then evidently $\mu_{2\Omega} \sim \frac{1}{4}\mu_\Omega\mu_0 \ll \mu_\Omega$.

When the effect of the constant magnetic field $\boldsymbol{H}^{(0)}$ is taken into account we obtain results similar to those given above; the only essential difference is that in (39.22) and (39.26) the expression $E_{01}^2(0,0)/(\omega_1^2+\nu_0^2)$ is replaced by

$$\frac{E_{01}^2(0,0)\cos^2\beta}{\omega_1^2+\nu_0^2}+\frac{E_{01}^2(0,0)\sin^2\beta}{2[(\omega_1-\omega_H)^2+\nu_0^2]}+\frac{E_{01}^2(0,0)\sin^2\beta}{2[(\omega_1+\omega_H)^2+\nu_0^2]},\qquad(39.27)$$

where β is the angle between $\boldsymbol{E}_{01}$ and $\boldsymbol{H}^{(0)}$. If the angle $\beta \neq 0$, the perturbations in the plasma may increase greatly near gyromagnetic resonance, when $\omega_1 \to \omega_H$. From this, however, we can not in general draw any conclusion regarding resonance behaviour of the cross-modulation (i.e. an increase in μ_Ω and $\mu_{2\Omega}$ when $\omega_1 \to \omega_H$). The reason is that the depth of cross-modulation obviously depends on the size of the perturbed region as well as the magnitude of the perturbations $\Delta_\Omega T$ and $\Delta_\Omega \nu$. Near resonance, this region becomes smaller, because the waves are strongly damped, and simply do not penetrate into the ionosphere (for instance). The cross-modulation in the ionosphere therefore does not in general exhibit any clearly marked resonance behaviour, although in some cases such behaviour may occur quite clearly. In this connection it may be noted that expressions of the type (39.27) must be used with some caution because of the need to take account of the change in polarisation of waves propagated in a magnetoactive plasma. Consequently, as shown in detail in § 11, the resonance at $\omega = \omega_H$ occurs only when $\alpha = 0$ (longitudinal propagation), whereas for other angles α the resonance frequency differs from ω_H and is given by formula (12.2a). The shift in resonance frequency in relation to non-linear effects is considered in [262]; in the Earth's ionosphere this shift is not large [263], since waves of frequency near to resonance penetrate only the lower part of the layer, where the plasma frequency ω_0 is small and so $\omega_{\text{res}} \approx \omega_H$.

The nature of the cross-modulation in the ionosphere under various conditions and for various values of the parameters μ_0, Ω, $\delta\nu_0$, ω_1, ω_2, etc., has been considered by various authors and the results are compared in [258].

Non-linear interaction of unmodulated waves. Combination frequencies

The non-linear interaction of unmodulated radio waves, as already mentioned, leads first of all to a change in the absorption and refraction of a weak wave passing through a perturbed region of the plasma. It is in general fairly difficult to observe this effect for monochromatic waves. If, however, some kind of "label" is applied to the wave, we return essentially to the problem of modulation and cross-modulation. Here particular attention need be paid only to the non-linear interaction of short pulses [264, 265].

An intense non-modulated wave E_1 of frequency ω_1 causes in the plasma not only a constant perturbation but also perturbations with frequencies which are multiples of ω_1. If wave 1 may nevertheless be supposed weak (i.e. $E_{01} \ll E_p$), then the variations of the temperature T_e with frequency $2\omega_1$ [see (38.11)] are almost the only ones to appear.† Similar harmonics occur for modulated waves, but when the condition (39.20) holds, the presence of modulation leads to no essentially new features, and so we shall for simplicity regard the intense wave as unmodulated. Moreover, we shall take only the simplest case, where the solution (38.11) for T_e is valid, and so the changes in T_e and σ may be written

$$\left.\begin{aligned} \Delta T = T_e - T = T_{e0} + \Delta T_{e+} e^{2i\omega_1 t} + \Delta T_{e-} e^{-2i\omega_1 t}, \\ \sigma = \sigma_0 + \Delta\sigma_+ e^{2i\omega_1 t} + \Delta\sigma_- e^{-2i\omega_1 t}; \end{aligned}\right\} \tag{39.28}$$

we neglect the change in ε, as is correct for a homogeneous isotropic plasma when $\omega^2 \gg \nu_0^2$. Thus the properties of the medium are no longer constant in time, and the propagation in this medium of other waves (field E_2, frequency ω_2) will be accompanied by the appearance of "subsidiary" waves with the combination frequencies $\omega_2 \pm 2\omega_1$. For, when the fields E_1 and E_2 are present, the current density due to a sufficiently weak field $E_2 = E_{02} e^{i\omega_2 t}$ is

$$j_2(E_1, E_2) = \sigma_0 E_{02} e^{i\omega_2 t} + \Delta\sigma_+ E_{02} e^{i(\omega_2 + 2\omega_1)t} + \Delta\sigma_- E_{02} e^{i(\omega_2 - 2\omega_1)t}. \tag{39.29}$$

Calculations using the Boltzmann equation, of which we shall not give details [258], lead for high frequencies ($\omega_1^2 \gg \nu_0^2$, $\omega_2^2 \gg \nu_0^2$, $(\omega_2 - 2\omega_1)^2 \gg \nu_0^2$) to the expressions

$$\Delta\sigma_+ = \Delta\sigma_- = -\frac{3e^4 N E_{01}^2(z)\,\nu_0}{80 m^2 \varkappa T \omega_1^2}\left[\frac{1}{(\omega_2 + 2\omega_1)^2} + \frac{1}{(\omega_2 - 2\omega_1)^2}\right], \tag{39.30}$$

† When wave 1 is strong, higher harmonics also appear, but even then the field strength decreases with successive harmonics owing to the presence of the parameter $\delta\nu/\omega \ll 1$.

where field 1, of amplitude $E_{01}(z) = E_{01}(0)\exp[-\mathscr{K}_1(z)]$, is for simplicity supposed weak.

When the frequency ω_2 approaches $2\omega_1$, we have a resonance increase in $\Delta\sigma_\pm$, which reach at $\omega_2 = 2\omega_1$ their maximum values

$$\Delta\sigma_+ = \Delta\sigma_- = 2e^4 N E_{01}^2(z)/15\pi m^2 \varkappa T \nu_0 \omega_1^2. \tag{39.31}$$

It is important to note that the amplitudes $\Delta\sigma_\pm$ are small in comparison with σ_0 in strong as well as weak fields, if the condition $\delta\nu_0/\omega_1 \ll$ holds (see § 38).

Unlike the case of cross-modulation, where the condition $\omega \gg \Omega$ (39.20) was sufficient to justify the use of the quasisteady approximation, the problem of the propagation of "subsidiary" waves requires a more detailed consideration; see [22, § 64; 258, § 3.5b]. The procedure is to solve the field equations or, in this particular case, the equation

$$\frac{\partial^2 E_2}{\partial z^2} - \frac{\varepsilon}{c^2}\frac{\partial^2 E_2}{\partial t^2} = \frac{4\pi}{c}\frac{\partial j_2}{\partial t}$$

which results from them, the current j_2 being given by (39.29). Since the right-hand side of this equation involves terms of frequencies ω_2 and $\omega_2 \pm 2\omega_1$, it is evident that the field E_2 likewise cannot have only the original frequency ω_2. Substitution of the solution

$$E_2 = E_{02}\, e^{i\omega_2 t} + E_{02+}\, e^{i(\omega_2 + 2\omega_1)t} + E_{02-}\, e^{i(\omega_2 - 2\omega_1)t}, \tag{39.32}$$

where $|E_{02\pm}| \ll |E_{02}|$, gives the result [22, § 64]

$$\left.\begin{aligned} E_{02}(z) &= E_{02}(0)\exp\left(-i\omega_2 \sqrt{[\varepsilon'(\omega_2)]}\, z/c\right) \\ E_{02\pm} &= E_{02}(z) \times \frac{\begin{aligned}&4\pi i(\omega_2 \pm 2\omega_1)\,\Delta\sigma_\pm \{1 - \exp[-(iz/c)\, \times \\ &\times \{(\omega_2 \pm 2\omega_1)\sqrt{[\varepsilon'(\omega_2 \pm 2\omega_1)]} - \omega_2\sqrt{[\varepsilon'(\omega_2)]}\}]\}\end{aligned}}{(\omega_2 \pm 2\omega_1)^2\,\varepsilon'(\omega_2 \pm 2\omega_1) - \omega_2^2\,\varepsilon'(\omega_2)}. \end{aligned}\right\} \tag{39.33}$$

Here the plasma has been regarded as homogeneous, and it is assumed that at $z = 0$ (the boundary of the layer) the amplitudes of the "subsidiary" waves $E_{02\pm} = 0$, i.e. the field E_2 has the form $E_{02}(0)\, e^{i\omega_2 t}$. The field 1 is assumed independent of z (like the field in a condenser). If, however, the field E_1 depends on z, as in the propagation of radio waves in the ionosphere, then the size of the interaction region has a considerable effect on the result [258].

As wave 2 passes through a layer of plasma perturbed by a fairly strong field 1, the amplitudes $E_{02\pm}$ at first increase, and then decrease only as the whole field 2 is absorbed. At a sufficient distance from the boundary $z = 0$ we have

$$\eta = \left|\frac{E_{02\pm}}{E_{02}}\right| = \left|\frac{4\pi(\omega_2 \pm 2\omega_1)\,\Delta\sigma_\pm}{(\omega_2 \pm 2\omega_1)^2\,\varepsilon(\omega_2 \pm 2\omega_1) - \omega_2^2\,\varepsilon(\omega_2)}\right|, \tag{39.34}$$

where ε' has also been replaced by ε in the denominator, as is legitimate if $|\varepsilon| \gg 4\pi\sigma/\omega$. In ionospheric conditions, taking account of the non-uniformity of the field $E_1(z)$, estimates [258] show that $\eta \sim 10^{-5}$ to 10^{-8} when $\omega_1 \approx \omega_2 \sim 10^6$ to 10^7 and the station power is 100 kW. For the same power at resonance $\eta \approx 3 \times 10^{-5}$ (when $\omega_2 = 2\omega_1 = 4 \times 10^6$), whereas when $\omega_1 = \omega_2 = 2 \times 10^6$ we have $\eta \approx 10^{-6}$. The smallness of the ratio η in the ionosphere explains why the appearance of combination frequencies is of no practical importance in radio communications and has never been observed. This effect is, nevertheless, quite distinctive and could certainly be detected if a special investigation were made. Most important, the appearance of combination frequencies may be of significance in conditions differing from those which usually hold in the propagation of radio waves in the ionosphere.

Non-linearity due to changes in electron density

All the non-linear phenomena discussed above are ultimately due to a change in electron velocity brought about by the field, which in turn causes changes in such quantities as T_e, $\nu_{\text{eff}} \equiv \nu$, ε and σ. The non-linear interaction of waves in a plasma (and, generally, the non-linear dependence of the current $\boldsymbol{j}_t$ on the field $\boldsymbol{E}$) may, however, be of a quite different nature: it may be due to a change, not of the velocity, but of the electron density [266, 258]. The tensor ε'_{ik} depends on N and, for example, in an isotropic plasma $\varepsilon'_{ik} = \varepsilon' \delta_{ik} = \delta_{ik}[1 - 4\pi e^2 N/m\omega(\omega - i\nu_{\text{eff}})]$. If, therefore, the electric field causes a change ΔN in the electron density, the properties of the plasma will depend on the field, and so the medium will become non-linear.

Neglecting the motion of the ions, which merely compensate the mean charge eN of the electrons, we evidently have

$$\Delta N = \bar{\varrho}/e = \operatorname{div} \boldsymbol{E}/4\pi e, \tag{39.35}$$

and the non-linear effect exists if $\operatorname{div} \boldsymbol{E} \neq 0$.

In the propagation of transverse (electromagnetic) waves in a homogeneous isotropic plasma, $\operatorname{div} \boldsymbol{E} = 0$, and this is also true, of course, in a uniform field; but for longitudinal plasma waves, even in a homogeneous isotropic plasma, $\operatorname{div} \boldsymbol{E} \neq 0$. In consequence we have, for example, scattering of electromagnetic waves by plasma waves, and in general a non-linear interaction of waves of these two types (see, for instance, [136]). In an inhomogeneous isotropic plasma without "external" currents, $\operatorname{curl} \boldsymbol{H} = i\omega\varepsilon' \boldsymbol{E}/c$, and so $\operatorname{div} (\varepsilon' \boldsymbol{E}) = 0$. Thus (39.35) gives

$$\Delta N = -(1/4\pi e\varepsilon')\, \boldsymbol{E} \cdot \operatorname{grad} \varepsilon'. \tag{39.36}$$

In a magnetoactive plasma $\operatorname{div} \boldsymbol{E} \neq 0$ in general, even for a homogeneous medium (see Chapter III; the condition $\operatorname{div}(\boldsymbol{D} - i \cdot 4\pi \boldsymbol{j}/\omega) = \partial(\varepsilon'_{ik} E_k)/\partial x_i = 0$ signifies in the general case that $\operatorname{div} \boldsymbol{E} = \partial E_i/\partial x_i \neq 0$ even if $\varepsilon'_{ik} = \text{constant}$).

For a monochromatic plane wave in a homogeneous magnetoactive medium the elliptically polarised wave has the form $\boldsymbol{E} = \boldsymbol{E}_{0a}\cos\varphi + \boldsymbol{E}_{0b}\sin\varphi$, and

$$\begin{aligned} \Delta N &= (1/4\pi e)\,\mathrm{div}\boldsymbol{E} \\ &= -(\omega n/4\pi e c N)(E_{0a}\cos\theta_a \sin\varphi + E_{0b}\cos\theta_b\cos\varphi). \end{aligned} \tag{39.37}$$

Here $\varphi = \omega - \boldsymbol{k}\cdot\boldsymbol{r}$, $n = n_{1,2} = ck/\omega$ is the refractive index for the ordinary or extraordinary wave (for simplicity, absorption is neglected) and θ_a, θ_b are the angles between $\boldsymbol{E}_{0a}$, $\boldsymbol{E}_{0b}$ and $\boldsymbol{k}$. If the external magnetic field $\boldsymbol{H}^{(0)}$ is not too weak and exceptional directions (i.e. angles α between $\boldsymbol{k}$ and $\boldsymbol{H}^{(0)}$ which are close to 0 or $\frac{1}{2}\pi$) are not considered, $\cos\theta_a$ and $\cos\theta_b$ are of the order of unity; the angles θ_a and θ_b are easily calculated from the formulae of § 10. In such conditions the effect (39.37) is greater than (39.36) if $\lambda\!\!\!^{-} = cn/\omega = \lambda/2\pi$ is less than the characteristic length L over which the properties of the plasma change significantly (i.e. $L \sim |\varepsilon'/\mathbf{grad}\,\varepsilon'|$).

Thus the field E_1 of frequency ω_1 causes in the plasma changes in ε and σ of the order given by $\Delta\sigma/\sigma \sim \Delta\varepsilon/\varepsilon \sim \Delta N/N \sim E_1/4\pi e N L$, or changes $\Delta\sigma_{ik}/\sigma_{ik} \sim \Delta\varepsilon_{ik}/\varepsilon_{ik} \sim \omega n E_1/4\pi e c N$; for simplicity we assume that $\varepsilon \sim \varepsilon_{ik} \sim 1$. The frequency of these variations is, of course, the same as that of the field E_1. When another wave E_2 of frequency ω_2 is propagated in a medium perturbed by the wave E_1, "subsidiary" waves of frequencies $\omega_2 \pm \omega_1$ are produced. The frequency of the "subsidiary" waves discussed previously, which are due to the effect of the field on the electron velocity, is $\omega_2 \pm 2\omega_1$. The difference arises because the change ΔN is proportional to the field E_1 [see (39.36) and (39.37)], whereas the changes in the random velocity, T_e, $\nu_{\text{eff}}(T_e)$, ε and σ are proportional to E_1^2 and more generally to various even powers of E_1.

For the effect (39.30) with $\omega_2 \sim \omega_1$ we have roughly $\Delta\sigma/\sigma \sim e^2E_1^2/m\varkappa T\omega_1^2$ and $\sigma \approx \sigma_0 \sim e^2N\nu_0/m\omega_1^2$. Hence the ratio of amplitudes of the "subsidiary" waves due to the change in the electron density (where, as already mentioned, $\Delta\sigma/\sigma \sim \omega_1 n E_1/4\pi e c N$) and those of the former type is

$$\xi \sim \frac{\varkappa T m \omega_1^3 n}{e^3 c N E_{01}(z)} \sim \frac{\omega_1^2 n}{\omega_0^2}\sqrt{\frac{\varkappa T_e}{mc^2\delta}}\,\frac{E_{p1}}{E_{01}(z)}, \tag{39.38}$$

where a numerical factor close to unity has been omitted, n is the index of refraction for a wave with field E_1, $\omega_0^2 = 4\pi e^2 N/m$, and E_{p1} is the "plasma field" (38.1) for wave 1. When $\sqrt{(\varkappa T_e/mc^2\delta)} \sim 10^{-2}$ $(T_e \sim 500,\ \delta \sim 10^{-3})$, $\omega_1^2/\omega_0^2 \sim 1$ to 10 and $E_{p1}/E_{01}(z) \sim 1$, we have $\xi \sim 10^{-2}$ to 10^{-1}. Thus in this case the amplitude of "subsidiary" waves in the ionosphere with frequencies $\omega_2 \pm \omega_1$ is 10 to 100 times less than that of waves with frequencies $\omega_2 \pm 2\omega_1$. When $\xi > 1$, the former waves are the stronger. The condition $\xi \approx 1$ corresponds to a field $E_{p1}/E_{01}(z) \sim 10^{-1}$ to 10^{-2} for $\omega_1^2/\omega_0^2 \sim 1$ to 10 and $\sqrt{(\varkappa T_e/mc^2\delta)} \sim 10^{-2}$. If the wave E_1 is modulated in amplitude, the additional cross-modulation of the wave E_2 due to the effect (39.37) is very small, its depth being of the same order as the ratio of the amplitude of the "subsidiary" wave of frequency $\omega_2 \pm \omega_1$ to that of the wave E_2 itself. This result is entirely reasonable, since the change in density due to the field, unlike the change in the electron temperature, has no component which is constant or only slowly varying.

APPENDIX A

ELECTROMAGNETIC WAVE PROPAGATION IN AN ANISOTROPIC DISPERSIVE MEDIUM†

IN THE theory of electromagnetic wave propagation in homogeneous media, particular attention has recently been given to the allowance for the effects of spatial dispersion, especially in magnetoactive plasmas and in crystals. Such media are anisotropic, and the expressions obtained for the refractive index n, the index of absorption $\varkappa$ and the group-velocity vector $\boldsymbol{v}_{gr}$ as functions of the frequency ω and the wave vector $\boldsymbol{k}$ are usually very cumbersome. It is therefore useful to bear in mind some theorems and relations of very general validity. We have not met some of these relations in the literature, and of the others we have obtained simpler or more general forms than those given in the papers we have seen.

§ A 1. FUNDAMENTAL EQUATIONS; RELATIONS BETWEEN EXPRESSIONS QUADRATIC IN THE AMPLITUDES OF PLANE WAVES PROPAGATED IN A DISPERSIVE MEDIUM

We begin from the following equations of the electromagnetic field:

$$\operatorname{curl} \boldsymbol{B} = (1/c)\,\partial \boldsymbol{D}'/\partial t; \quad \operatorname{curl} \boldsymbol{E} = -(1/c)\,\partial \boldsymbol{B}/\partial t. \tag{A 1.1}$$

Here the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are defined so that the force acting on a charge e moving with velocity $\boldsymbol{v}$ is

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c).$$

Taking the medium to be homogeneous in space and uniform in time, we shall consider monochromatic plane waves. For the electric field $\boldsymbol{E}$ we have

$$\boldsymbol{E}(\boldsymbol{r}, t) = \tfrac{1}{2}\left[\boldsymbol{E}_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} + \boldsymbol{E}_0^* e^{-i(\omega t - \boldsymbol{k}^*\cdot\boldsymbol{r})}\right]. \tag{A 1.2}$$

Similar expressions are used for $\boldsymbol{D}'$ and $\boldsymbol{B}$. In (A 1.2) the frequency ω is assumed real, in accordance with the problem under consideration here.

† By B. N. Gershman and V. L. Ginzburg. Published in *Izvestiya vysshikh uchebnykh zavedenii: Radiofizika* **5**, 31–46, 1962.

The vectors $\boldsymbol{D}_0'$ and $\boldsymbol{E}_0$ are related by

$$\left.\begin{aligned} D_{0i}' &= \varepsilon_{ij}'(\omega, \boldsymbol{k})\,E_{0j}; \quad \varepsilon_{ij}'(\omega, \boldsymbol{k}) = \varepsilon_{ij}(\omega, \boldsymbol{k}) - i\cdot\frac{4\pi\sigma_{ij}(\omega, \boldsymbol{k})}{\omega};\\ D_{0i}'^{*} &= \varepsilon_{ij}'^{*}\,E_{0j}^{*}; \quad \varepsilon_{ij}'^{*}(\omega, \boldsymbol{k}) = \varepsilon_{ij}(-\omega, -\boldsymbol{k}^{*}), \end{aligned}\right\} \tag{A 1.3}$$

where the tensors ε_{ij} and σ_{ij} are Hermitian, summation over repeated suffixes is understood, and the last relation follows from the requirement that $\boldsymbol{D}'$ should be real when $\boldsymbol{E}$ is real. The relations (A 1.3) are the most general following from the above assumptions. In particular, if $\boldsymbol{D}'$ depends on $\boldsymbol{B}$, the expressions (A 1.3) are still valid, since $\boldsymbol{B}$ can be related to $\boldsymbol{E}$ by the second field equation (A 1.1). Neglecting the spatial dispersion in a non-magnetic medium is equivalent to using the tensor

$$\varepsilon_{ij}'(\omega, 0) \equiv \varepsilon_{ij}'(\omega); \tag{A 1.4}$$

in an isotropic medium, $\varepsilon_{ij}'(\omega) = \varepsilon'(\omega)\delta_{ij}$.

For a magnetic medium the passage to the limit $k \to 0$ in $\varepsilon_{ij}'(\omega, \boldsymbol{k})$ is not entirely trivial [349]. In using equations (A 1.1) and (A 1.3) below for the case (A 1.4), we shall for simplicity assume the medium to be non-magnetic ($\mu_{ij}' = \delta_{ij}$). Instead of taking the limit $k \to 0$ in the absence of spatial dispersion, we can use from the beginning the equations valid for magnetic media also†:

$$\left.\begin{aligned} \mathbf{curl}\,\boldsymbol{H} &= (1/c)\,\partial\boldsymbol{D}'/\partial t; \quad \mathbf{curl}\,\boldsymbol{E} = -(1/c)\,\partial\boldsymbol{B}/\partial t;\\ D_{0i}' &= \varepsilon_{ij}'(\omega)\,E_{0j}; \quad B_{0i} = \mu_{ij}'(\omega)\,H_{0j}. \end{aligned}\right\} \tag{A 1.5}$$

For inhomogeneous waves $\boldsymbol{k} = \boldsymbol{k}_1 - i\boldsymbol{k}_2$, and the real vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are not parallel. For this reason the planes of equal phase and amplitude do not coincide in inhomogeneous waves (the field varying in space as $e^{-\boldsymbol{k}_2\cdot\boldsymbol{r}\mp i\boldsymbol{k}_1\cdot\boldsymbol{r}}$). In an isotropic medium without spatial dispersion, $k^2 = \omega^2\varepsilon'(\omega)/c^2$, and in the absence of absorption $\varepsilon'(\omega) = \varepsilon(\omega)$ is real. Hence in such a non-absorbing isotropic medium $\boldsymbol{k}_1\cdot\boldsymbol{k}_2 = 0$ and $k_1^2 - k_2^2 = \omega^2\varepsilon(\omega)/c^2$. In an anisotropic medium, even in the absence of absorption, the nature of inhomogeneous waves is less simple and needs investigation (which has not been done so far as we know). In what follows we shall consider only homogeneous plane waves, in which $\boldsymbol{k} = \omega\tilde{n}\boldsymbol{s}/c \equiv \omega(n - i\varkappa)\boldsymbol{s}/c$, the unit vector $\boldsymbol{s}$ being real.

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† The notation used in the literature varies, and we have therefore been compelled to define all quantities. In [36], for example, the form (A 1.1) is used, but with the notation $\boldsymbol{D}$ instead of $\boldsymbol{D}'$. In [349] the form (A 1.1) is used, but in (A 1.3) we denote the complex permittivity tensor by ε_{ij}' (instead of $\varepsilon_{ij} = \varepsilon_{ij}' + i\varepsilon_{ij}''$ as in [349]; moreover, in [349] and [36] the form $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ is used instead of $e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}$ as here). Finally, in the present book and in several textbooks on the theory of fields the expression $\boldsymbol{D}' = \boldsymbol{D} - i\cdot 4\pi\boldsymbol{j}/\omega$ is used, where $\boldsymbol{j}$ is the conduction current. The authors do not regard the present notation as the best, but have been concerned to use for convenience' sake a notation as close as possible to that found in the rest of the book and in [349].

For fields of the type (A 1.2), equations (A 1.1) become

$$\omega \boldsymbol{D}_0' = -c\,\boldsymbol{k} \times \boldsymbol{B}_0; \quad \omega \boldsymbol{B}_0 = c\,\boldsymbol{k} \times \boldsymbol{E}_0 \tag{A 1.6}$$

and similarly

$$\omega \boldsymbol{D}_0'^* = -c\,\boldsymbol{k}^* \times \boldsymbol{B}_0^*; \quad \omega \boldsymbol{B}_0^* = c\,\boldsymbol{k}^* \times \boldsymbol{E}_0^*. \tag{A 1.6a}$$

For a transparent medium† $\boldsymbol{k}^* = \boldsymbol{k}$ and equations (A 1.6) and (A 1.6a) are the same. It should be emphasised that so long as the complete set of field equations is not used [for instance, if only the relations (A 1.3) are used] the frequency ω and the wave vector $\boldsymbol{k}$ are independent, but for solutions of the field equations (A 1.6) we have $\omega = \omega(\boldsymbol{k})$.

Multiplying equations (A 1.6) by $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ respectively, we have (for $\omega \neq 0$)

$$\boldsymbol{D}_0' \cdot \boldsymbol{E}_0 = \boldsymbol{B}_0 \cdot \boldsymbol{B}_0 = c\,\boldsymbol{k} \cdot \boldsymbol{E}_0 \times \boldsymbol{B}_0/\omega. \tag{A 1.7}$$

In a transparent medium

$$\boldsymbol{D}_0' \cdot \boldsymbol{E}_0^* = \boldsymbol{B}_0 \cdot \boldsymbol{B}_0^*, \tag{A 1.8}$$

as follows from (A 1.6) and (A 1.6a). From (A 1.6), after multiplication by $\boldsymbol{E}_0^*$ and $\boldsymbol{B}_0^*$ respectively, we obtain

$$\omega(\boldsymbol{D}_0' \cdot \boldsymbol{E}_0^* \pm \boldsymbol{B}_0 \cdot \boldsymbol{B}_0^*) = c\,\boldsymbol{k} \cdot [\boldsymbol{E}_0 \times \boldsymbol{B}_0^* \pm \boldsymbol{E}_0^* \times \boldsymbol{B}_0]. \tag{A 1.9}$$

For a transparent medium this gives, with (A 1.8),

$$\boldsymbol{E}_0 \times \boldsymbol{B}_0^* = \boldsymbol{E}_0^* \times \boldsymbol{B}_0. \tag{A 1.10}$$

From (A 1.6) and (A 1.6a) we similarly have

$$\omega[\boldsymbol{D}_0' \times \boldsymbol{B}_0^* \pm \boldsymbol{D}_0'^* \times \boldsymbol{B}_0] = -c[(\boldsymbol{k} \times \boldsymbol{B}_0) \times \boldsymbol{B}_0^* \pm (\boldsymbol{k} \times \boldsymbol{E}_0) \times \boldsymbol{D}_0'^*]; \tag{A 1.11}$$

$$\omega[\boldsymbol{D}_0'^* \times \boldsymbol{B}_0 \pm \boldsymbol{D}_0' \times \boldsymbol{B}_0^*] = -c[(\boldsymbol{k}^* \times \boldsymbol{B}_0^*) \times \boldsymbol{B}_0 \pm (\boldsymbol{k}^* \times \boldsymbol{E}_0^* \times)\boldsymbol{D}_0']; \tag{A 1.11a}$$

$$\omega[D_{0i}' B_{0j}^* \pm B_{0i} D_{0j}'^*] = c[(\boldsymbol{k} \times \boldsymbol{B}_0)_i B_{0j}^* \pm (\boldsymbol{k} \times \boldsymbol{E}_0)_i D_{0j}'^*]. \tag{A 1.11b}$$

To complete the picture, we may mention the other obvious relations which follow from (A 1.6):

$$\boldsymbol{k} \cdot \boldsymbol{D}_0' = 0; \quad \boldsymbol{k} \cdot \boldsymbol{B}_0 = 0; \quad \boldsymbol{B}_0 \cdot \boldsymbol{D}_0' = 0; \quad \boldsymbol{B}_0 \cdot \boldsymbol{E}_0 = 0. \tag{A 1.12}$$

If we have a wave packet instead of a field of the type (A 1.2), then the relations (A 1.6)–(A 1.12) of course remain valid for each Fourier amplitude of the packet. The amplitudes $\boldsymbol{E}_0$ etc., however, depend on $\boldsymbol{k}$ and on $\omega(\boldsymbol{k})$. Hence the relations derived from (A 1.7)–(A 1.12) by differentiation with respect to k_l ($l = 1, 2, 3$) can apply to an arbitrary monochromatic packet only if the derivatives of the amplitudes with respect to k_l cancel. We shall see that such relations do exist.

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† A transparent medium is, of course, also a non-absorbing medium, but the converse is not true. A well-known example of a medium which is non-absorbing but not transparent is an isotropic collisionless plasma in the range of frequencies where $\varepsilon = 1 - 4\pi e^2 N/m\omega^2 < 0$ (see, for example, § 7).

Let us differentiate (A 1.9) with respect to k_l, using (A 1.3). The result is

$$\frac{\partial \omega}{\partial k_l}\left\{\frac{\partial[\omega\varepsilon'_{ij}(\omega,\boldsymbol{k})]}{\partial\omega}E_{0j}E^*_{0i}\pm\boldsymbol{B}_0\cdot\boldsymbol{B}^*_0\right\}$$

$$=c[(\boldsymbol{E}_0\times\boldsymbol{B}^*_0)_l\pm(\boldsymbol{E}^*_0\times\boldsymbol{B}_0)_l]-\omega\frac{\partial\varepsilon'_{ij}}{\partial k_l}E_{0j}E^*_{0i}+$$

$$+\left[(c\boldsymbol{k}\times\boldsymbol{E}_0\mp\omega\boldsymbol{B}_0)\cdot\frac{d\boldsymbol{B}^*_0}{dk_l}\pm(c\boldsymbol{k}\times\boldsymbol{E}^*_0-\omega\boldsymbol{B}^*_0)\cdot\frac{d\boldsymbol{B}_0}{dk_l}\right]-$$

$$-\left[\{c(\boldsymbol{k}\times\boldsymbol{B}^*_0)_j+\omega\varepsilon'_{ij}E^*_{0i}\}\frac{dE_{0j}}{dk_l}+\{\pm c(\boldsymbol{k}\times\boldsymbol{B}_0)_i+\omega\varepsilon'_{ij}E_{0j}\}\frac{dE^*_{0i}}{dk_l}\right]. \quad \text{(A 1.13)}$$

For a transparent medium $\boldsymbol{k}=\boldsymbol{k}^*$ and $\varepsilon_{ij}=\varepsilon^*_{ji}$. Then, using (A 1.6) and (A 1.6a), equation (A 1.13) with the upper signs becomes

$$\left.\begin{aligned}&\overline{W}\boldsymbol{v}_{\text{gr}}=\overline{\boldsymbol{S}};\quad \boldsymbol{v}_{\text{gr}}=\partial\omega/\partial\boldsymbol{k};\quad \overline{\boldsymbol{S}}=\overline{\boldsymbol{S}}_0+\overline{\boldsymbol{S}'};\\ &\quad\overline{W}=\frac{1}{16\pi}\left\{\frac{\partial}{\partial\omega}[\omega\varepsilon_{ij}(\omega,\boldsymbol{k})]E_{0j}E^*_{0i}+\boldsymbol{B}_0\cdot\boldsymbol{B}^*_0\right\};\\ &\quad\overline{\boldsymbol{S}}_0=\frac{c}{16\pi}(\boldsymbol{E}_0\times\boldsymbol{B}^*_0+\boldsymbol{E}^*_0\times\boldsymbol{B}_0);\quad \overline{S}'_l=-\frac{\omega}{16\pi}\frac{\partial\varepsilon_{ij}(\omega,\boldsymbol{k})}{\partial k_l}E^*_{0i}E_{0j}.\end{aligned}\right\} \quad \text{(A 1.14)}$$

Thus for real $\boldsymbol{k}$ we in fact obtain, from (A 1.13) with the upper signs, the relation (A 1.14), which does not depend on the derivatives dE_{0i}/dk_l etc.

For a non-transparent medium (and for a transparent medium also, if the lower sign is taken), equations (A 1.13) are evidently valueless, since there remain derivatives of the field amplitude with respect to k_l. The first formula (A 1.14) relates the group velocity vector $\boldsymbol{v}_{\text{gr}}=\partial\omega/\partial\boldsymbol{k}$ to the time-average energy density $\overline{W}$ and the time-average energy flux $\overline{\boldsymbol{S}}$ (see §§ 22 and B 3, and also [36, 142, 349, 352]). In the absence of spatial dispersion the flux $\boldsymbol{S}=\boldsymbol{S}_0$, the Poynting vector. The significance of the flux $\boldsymbol{S}'$ will be clear from the particular case considered in § A 3.

It may be noted that, in the absence of spatial dispersion, the relation (A 1.9) with the upper signs becomes

$$\overline{W}_0=(\boldsymbol{D}'_0\cdot\boldsymbol{E}^*_0+\boldsymbol{B}_0\cdot\boldsymbol{B}^*_0)/16\pi=\boldsymbol{k}\cdot\overline{\boldsymbol{S}}_0/\omega. \quad \text{(A 1.15)}$$

In the absence of both spatial and frequency dispersion $\overline{W}=\overline{W}_0$ and equation (A 1.14) becomes $\overline{W}_0\boldsymbol{v}_{\text{gr}}=\overline{\boldsymbol{S}}_0$. From this and (A 1.15) we have $\boldsymbol{k}\cdot\boldsymbol{v}_{\text{gr}}/\omega=1$ or $v_{\text{gr},k}=v_{\text{ph}}$, where $v_{\text{gr},k}$ is the component of $\boldsymbol{v}_{\text{gr}}$ in the direction of $\boldsymbol{k}$, and $v_{\text{ph}}=\omega/k$ is the phase velocity.

From (A 1.11) and (A 1.11a), by the same procedure as leads from (A 1.9) to (A 1.14), we obtain some relations which we shall write out here only for a transparent medium and with the upper signs, when the terms involving the derivatives of the field amplitudes with respect to k_l cancel:

$$\left.\begin{aligned}
&\bar{T}_{ij} = -\bar{g}_i v_{\mathrm{gr},j}; \quad v_{\mathrm{gr},j} = \partial\omega/\partial k_j;\\
&\bar{T}_{ij} = \bar{T}_{ij,0} + \bar{T}'_{ij};\\
&\bar{g}_i = \frac{1}{16\pi c}[(\boldsymbol{D}'_0 \times \boldsymbol{B}^*_0)_i + (\boldsymbol{D}'^*_0 \times \boldsymbol{B}_0)_i] + \frac{k_i}{16\pi}\frac{\partial\varepsilon_{jm}(\omega, \boldsymbol{k})}{\partial\omega} E^*_{0j} E_{0m};\\
&\bar{T}_{ij,0} = \frac{B_{0i}B^*_{0j} + B^*_{0i}B_{0j} + E_{0i}D'^*_{0j} + E^*_{0i}D'_{0j}}{16\pi} -\\
&\qquad - \frac{\delta_{ij}}{32\pi}(2\boldsymbol{B}_0 \cdot \boldsymbol{B}^*_0 + \boldsymbol{E}_0 \times \boldsymbol{D}'^*_0 + \boldsymbol{E}^*_0 \cdot \boldsymbol{D}'_0);\\
&\bar{T}'_{ij} = -\frac{k_i}{16\pi}\frac{\partial\varepsilon_{ml}(\omega, \boldsymbol{k})}{\partial k_j} E^*_{0m} E_{0l}.
\end{aligned}\right\} \quad (\text{A}\,1.16)$$

Formulae (A 1.14) and (A 1.16) have been derived in [142, 352], where the relativistically invariant form of the field equations was used, and the medium was throughout assumed transparent. The latter assumption is justified *a posteriori*, since for a non-transparent medium, equation (A 1.13) and the corresponding equations apparently yield no results of value. The form in which the equations are written and the derivation of the relations (A 1.14) and (A 1.15) are, of course, a matter of taste or habit, but the derivation given above seems to us so simple that to employ the relativistic equations, which are rarely used in macroscopic electrodynamics, is to introduce a complication greater than that of the proof itself, and is therefore not to be recommended.

It may be noted that in [142, 352] the vector $\bar{g}_i$ is taken to be the time-average momentum density, and $\bar{T}_{ij}$ the mean stress tensor. There is, however, now no doubt that this choice of the four-dimensional energy-momentum tensor is incorrect (see [36, §§ 16 and 56; 353]). In certain conditions (in wave propagation in a medium) the vector $\bar{g}_i$ has the significance of the total momentum density of the field and the medium [353, 81]. Thus equation (A 1.16) will probably be of importance in considering the momenta and forces due to electromagnetic waves in a medium. Further analysis is necessary, however, and we have not effected this.

§ A 2. INEQUALITIES WHICH FOLLOW FROM THE DISPERSION RELATIONS IN THE REGION OF TRANSPARENCY

It is known that the dispersion relations for isotropic media in the region of transparency can be used to derive certain conditions which give the manner of variation of the permittivity ε with the frequency ω. From these conditions we can show that $v_{\text{gr}} = |\partial\omega/\partial \boldsymbol{k}| \leqq c$ (see [36, § 64]). Below we shall give a similar discussion for the more complex case of an anisotropic dispersive medium whose electromagnetic properties are specified by the tensor $\varepsilon'_{ij} = \varepsilon_{ij} - i \cdot 4\pi\sigma_{ij}/\omega$. For such media we can obtain the following formulae, which are a simple generalisation of the usual dispersion relations (see [349, 351]). In (A 2.1) it is assumed for simplicity that the non-diagonal elements ε'_{ij} are either symmetric or antisymmetric, and ε_{ij} and σ_{ij} are assumed to be even functions of $\boldsymbol{k}$:

$$\left.\begin{aligned} \varepsilon_{ij}(\omega, \boldsymbol{k}) - \delta_{ij} &= 8\int_0^\infty \frac{\sigma_{ij}(\omega', \boldsymbol{k})}{\omega'^2 - \omega^2}\, d\omega'; \\ \frac{4\pi[\sigma_{ij}(\omega, \boldsymbol{k}) - \sigma_{ij}(0)]}{\omega} &= -\frac{2}{\pi}\,\omega \int_0^\infty \frac{\varepsilon_{ij}(\omega', \boldsymbol{k}) - \delta_{ij}}{\omega'^2 - \omega^2}\, d\omega'. \end{aligned}\right\} \tag{A 2.1}$$

In (A 2.1) the integrals must be taken as principal values at $\omega = \omega'$. It may be noted that, in the presence of spatial dispersion, formulae (A 2.1) are a particular case of the more general dispersion relations (see [354; 349, Appendix]). For complex wave vectors $\boldsymbol{k}$, the question of the significance and values of the tensor $\varepsilon_{ij}(\omega, \boldsymbol{k})$ in general requires further analysis. So far as we know, the problem has not yet been fully elucidated, and in [349], for example, the vector $\boldsymbol{k}$ is generally taken to be real. For a transparent medium, with which we shall be mainly concerned, this procedure may apparently be used without particular misgivings (a fact which is not quite evident, since in general we cannot regard the absorption as strictly equal to zero).

Writing the tensor $\varepsilon'_{ij}(\omega)$ in the form $\varepsilon'_{ij}(\omega) = \varepsilon_{ij}(\omega) - i \cdot 4\pi\sigma_{ij}(\omega)/\omega$, we can assume the tensors $\varepsilon_{ij}(\omega)$ and $\sigma_{ij}(\omega)$ to be Hermitian. The heat evolved is expressed in terms of σ_{ij}. For a monochromatic field (A 1.2), averaging over high frequencies, the heat per unit time and volume is (see § B 2, Appendix C and [349, 36])

$$\begin{aligned} q &= -\frac{i\omega}{16\pi}(\boldsymbol{E}_0 \cdot \boldsymbol{D}_0'^* - \boldsymbol{E}_0 \cdot \boldsymbol{D}_0') \\ &= -\frac{i\omega}{16\pi}[\varepsilon_{ij}'^*(\omega, \boldsymbol{k}) - \varepsilon'_{ji}(\omega, \boldsymbol{k})]\, E_{0i} E_{0j}^* \\ &= \tfrac{1}{2}\,\sigma_{ij}^* E_{0i} E_{0j}^* \\ &= \tfrac{1}{2}\,\sigma_{ij} E_{0j} E_{0i}^*. \end{aligned} \tag{A 2.2}$$

This formula for the time-average quantity of heat is obtained only if we neglect spatial dispersion, when $\varepsilon'_{ij}(\omega, \boldsymbol{k}) = \varepsilon'_{ij}(\omega)$, or for an almost transparent medium (the passage to the limit of real $\boldsymbol{k}$). If $\varepsilon'_{ij}(\omega, \boldsymbol{k})$ depends on $\boldsymbol{k}$ and $\boldsymbol{k}$ is complex, the quantity q is not the mean heat generated (Appendix C). To prove this, it is sufficient to show that q may be non-zero even when absorption is completely absent (§ A 3). In these cases, however, the mean fluxes $\bar{S}_0$ and $\bar{S}'$ are not zero, and their variation in space is responsible for a contribution to q. In what follows we shall use formula (A 2.2) only for real $\boldsymbol{k}$ or when spatial dispersion is absent.

If the medium is in a state of thermodynamic equilibrium, then for $\sigma_{ij} \neq 0$

$$2q = \sigma_{ij} E_{0i}^* E_{0j} > 0. \tag{A 2.3}$$

The Hermitian quadratic form (A 2.3) can be reduced to a sum of squares of the form $2q = \lambda_1 |E_1'|^2 + \lambda_2 |E_2'|^2 + \lambda_3 |E_3'|^2 > 0$. Since $\boldsymbol{E}$ is arbitrary, the coefficients λ_1, λ_2, λ_3 must all be positive. The reduction of the quadratic form (A 2.3) to a sum of squares is analogous to the representation of the matrix of the elements σ_{ij} in diagonal form. Using the determinant $D(\lambda) = |\sigma_{ij} - \lambda\delta_{ij}|$, we can formulate the inequality (A 2.3) as a requirement that all three roots λ_1, λ_2, λ_3 of the equation

$$D(\lambda) = |\sigma_{ij}(\omega, \boldsymbol{k}) - \lambda(\omega, \boldsymbol{k})\,\delta_{ij}| = 0 \tag{A 2.4}$$

should be positive. For this it is necessary and sufficient that the inequalities

$$D(\lambda = 0) = |\sigma_{ij}| > 0, \tag{A 2.5}$$

$$\Delta_{ii} > 0, \tag{A 2.5a}$$

$$\sigma_{ii} > 0 \tag{A 2.5b}$$

should hold, where Δ_{ij} are the minors of the elements σ_{ij} in the determinant $|\sigma_{ij}|$, and Δ_{ii} and σ_{ii} are the respective sums over i.

To derive the inequalities of interest to us, we differentiate the first equation (A 2.1) with respect to the frequency ω. The allowance for spatial dispersion is here unimportant, since the real vector $\boldsymbol{k}$ appears in (A 2.1) as an independent parameter (the field equations are not used here). To simplify the notation, the argument $\boldsymbol{k}$ will be omitted in the rest of this section. As in the isotropic case [36], for the region of transparency we may ignore the singularities of the integrands at $\omega = \omega'$. For this reason a simple differentiation with respect to the parameter ω is possible, giving

$$\frac{\partial \varepsilon_{ij}(\omega)}{\partial \omega} = 16\omega \int_0^\infty \frac{\sigma_{ij}(\omega')}{(\omega'^2 - \omega^2)}\, d\omega'. \tag{A 2.6}$$

Adding the left-hand and right-hand sides of the three relations (A 2.6) for the diagonal components of the tensors ε_{ij} and σ_{ij}, we have

$$\frac{\partial \varepsilon_{ii}(\omega)}{\partial \omega} = 16\omega \int_0^\infty \frac{\sigma_{ii}(\omega')}{(\omega'^2 - \omega^2)^2}\, d\omega'. \tag{A 2.7}$$

Since, according to the inequality (A 2.5b), the integrand in (A 2.7) is always positive throughout the range of integration, we obtain in the absence of absorption at the frequency ω (more precisely, when the absorption is very weak)

$$\frac{\partial \varepsilon_{ii}(\omega)}{\partial \omega} = \frac{\partial \varepsilon_{11}}{\partial \omega} + \frac{\partial \varepsilon_{22}}{\partial \omega} + \frac{\partial \varepsilon_{33}}{\partial \omega} > 0. \tag{A 2.8}$$

As in the isotropic case [36], we can obtain by using (A 2.5b) another inequality besides (A 2.8). Multiplying both sides of the first equation (A 2.1) by ω^2 and, as for (A 2.6), differentiating with respect to the frequency ω without taking account of the singularity at $\omega = \omega'$, we have

$$\frac{\partial}{\partial \omega} [\omega^2 (\varepsilon_{ij} - \delta_{ij})] = 16\omega \int\limits_0^\infty \frac{\sigma_{ij}(\omega')\, \omega'^2}{(\omega'^2 - \omega^2)^2}\, d\omega'. \tag{A 2.9}$$

Adding the relations (A 2.9) for $i = j$ and using the condition (A 2.5b), we obtain

$$\frac{\partial \varepsilon_{ii}(\omega)}{\partial \omega} > \frac{2[3 - \varepsilon_{ii}(\omega)]}{\omega}. \tag{A 2.10}$$

For the isotropic case without spatial dispersion we must put $\varepsilon_{ij} = \varepsilon \delta_{ij}$. Then the inequalities (A 2.8) and (A 2.10) become the simple conditions [36]

$$\partial\varepsilon/\partial\omega > 0; \quad \partial\varepsilon/\partial\omega > 2(1 - \varepsilon)/\omega. \tag{A 2.11}$$

Our calculations so far have been based on only one of the restrictions (A 2.5)–(A 2.5b). It is very difficult to derive in the general case definite results like (A 2.8) and (A 2.10) from the other two conditions (A 2.5) and (A 2.5a). The difficulties are due to the fact that the conditions (A 2.5) and (A 2.5a) involve combinations of products of the components of the tensor σ_{ij}. Hence the inequalities (A 2.5) and (A 2.5a) cannot be used in the same way as we have used (A 2.5b).

The situation is more favourable for anisotropic media of certain types if we use the simplest forms of the Hermitian tensor ε_{ij}, which are obtained by an appropriate choice of the coordinates.

In the absence of external magnetic fields and of spatial dispersion, the tensor ε_{ij} is symmetrical ($\varepsilon_{ij} = \varepsilon_{ji}$) and real. Such a tensor can be brought to its principal axes, and its components with $i = j$ are ε_1, ε_2, ε_3, while $\varepsilon_{ij} = 0$ for $i \neq j$. We shall also consider crystalline media in which the principal axes are fixed, for reasons of symmetry. Such crystals include all except those of the triclinic and monoclinic systems, where there is dispersion of the axes and the axes of the tensors ε_{ij} and σ_{ij} cannot be taken to coincide for all frequencies ω. In crystals of higher symmetry, the axes of these tensors coincide for all ω. Then we easily obtain from (A 2.1) and (A 2.5), after transforming to the principal axes, the inequalities

$$\partial\varepsilon_1/\partial\omega > 0; \quad \partial\varepsilon_2/\partial\omega > 0; \quad \partial\varepsilon_3/\partial\omega > 0; \tag{A 2.12}$$

$$\frac{\partial \varepsilon_1}{\partial \omega} > \frac{2(1-\varepsilon_1)}{\omega}; \quad \frac{\partial \varepsilon_2}{\partial \omega} > \frac{2(1-\varepsilon_2)}{\omega}; \quad \frac{\partial \varepsilon_3}{\partial \omega} > \frac{2(1-\varepsilon_3)}{\omega}. \qquad \text{(A 2.13)}$$

When spatial dispersion is taken into account, in general $\varepsilon_{ij}(\omega, \boldsymbol{k}) = \varepsilon_{ji}(\omega, -\boldsymbol{k})$, and only in a non-gyrotropic medium (in particular, when there is a centre of symmetry) $\varepsilon_{ij}(\omega, \boldsymbol{k}) = \varepsilon_{ji}(\omega, \boldsymbol{k})$. In the latter case the results (A 2.12) and (A 2.13) sometimes hold good, but the dependence of the principal axes on $\boldsymbol{k}$ complicates the situation in practice; see [351] for a discussion of crystal optics with allowance for spatial dispersion.

For the simplest magnetoactive medium (for instance, a plasma), taking the direction of the external magnetic field as the z-axis, we have

$$\varepsilon_{ij}(\omega, \boldsymbol{k}) = \begin{bmatrix} \varepsilon_1 & ig & 0 \\ -ig & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix}; \qquad \text{(A 2.14)}$$

where the quantities ε_1, ε_2 and g are real and depend only on ω and k, and not on $\boldsymbol{k}/k$. From (A 2.1), (A 2.5)–(A 2.5b) we have in this case

$$\frac{\partial \varepsilon_2}{\partial \omega} > 0; \quad \frac{\partial(\varepsilon_1 + g)}{\partial \omega} > 0; \quad \frac{\partial(\varepsilon_1 - g)}{\partial \omega} > 0. \qquad \text{(A 2.15)}$$

In addition, we have the inequalities

$$\frac{\partial \varepsilon_2}{\partial \omega} > \frac{2(1-\varepsilon_2)}{\omega}; \quad \frac{\partial(\varepsilon_1 + g)}{\partial \omega} > \frac{2(1-\varepsilon_1 - g)}{\omega}; \quad \frac{\partial(\varepsilon_1 - g)}{\partial \omega} > \frac{2(1-\varepsilon_1 + g)}{\omega}. \qquad \text{(A 2.16)}$$

The inequalities for $\partial \varepsilon_{ij}/\partial \omega$ impose certain restrictions on the refractive index $n(\omega, \boldsymbol{s})$ for homogeneous plane waves propagated in a transparent medium ($\boldsymbol{k} = \omega n \boldsymbol{s}/c$, $n > 0$). In the general case it is fairly difficult to derive the corresponding conditions on n, since the relation between ε_{ij} and n (i.e. the dispersion relation) which results from the field equations is very complicated; it is well known to have the form

$$\left| \varepsilon_{ij}(\omega, \boldsymbol{k}) - n^2 \delta_{ij} + n^2 s_i s_j \right| = 0. \qquad \text{(A 2.17)}$$

In an isotropic medium without spatial dispersion, $n^2 = \varepsilon(\omega)$ for transverse waves, and it follows from (A 2.11) that

$$dn/d\omega > 0; \quad d(n\omega)/d\omega > n; \quad d(n\omega)/d\omega > 1/n. \qquad \text{(A 2.18)}$$

Hence it is clear that the group velocity in this case,

$$\boldsymbol{v}_{\mathrm{gr}} = \partial \omega / \partial \boldsymbol{k} = c \left[\frac{d(n\omega)}{d\omega} \right]^{-1} \boldsymbol{s},$$

is always parallel to $\boldsymbol{s}$, and $v_{\mathrm{gr}} < c$. At first sight this proof [36] of the inequality $v_{\mathrm{gr}} < c$, which is evidently relativistic, may appear puzzling, since the dispersion relations (A 2.1) are a consequence only of the principle of causality. The same applies to the inequalities (A 2.11) apart from the use of the condition (A 2.3), which follows from the principle of increase of entropy. In this case, however, the use of the relation $n^2 = \varepsilon$ is equivalent to using the field equations, which are relativistically invariant. In combination with the principle of causality this in fact gives immediately the result that the signal cannot be propagated with a velocity exceeding c.

In the presence of spatial dispersion the use of inequalities of the type (A 2.11)–(A 2.13), and still more so of more complex ones, to derive any information about $n(\omega, \boldsymbol{s})$ is especially difficult, since n then appears in the expression for ε itself: $\varepsilon = \varepsilon(\omega, \omega n/c)$. For an anisotropic medium without spatial dispersion, on the other hand, we can probably deduce from (A 2.12), (A 2.13), (A 2.15) and (A 2.16) the inequality

$$v_{\mathrm{gr}} = |\partial\omega/\partial\boldsymbol{k}| < c. \tag{A 2.19}$$

This inequality is demonstrated in § A 4 for uniaxial crystals. A more general case has not been considered, since the proof is certainly very involved and there is no particular reason to cast doubt on the validity of (A 2.19). For, from the familiar kinematic derivation of the expression for the group velocity, and formula (A 1.14), the significance of the vector $\partial\omega/\partial\boldsymbol{k}$ as the velocity of the signal and the rate of flow of energy in a transparent medium is evident. Hence, from relativistic considerations, we may expect the inequality (A 2.19) to be valid in any transparent medium. In this respect the proof of (A 2.19) from the field equations and inequalities of the type (A 2.11)–(A 2.13) is of purely methodological interest.†

The component of the group velocity in the direction of $\boldsymbol{k}$ is (see § 24)

$$\begin{aligned} v_{\mathrm{gr},k} &= \frac{\boldsymbol{k}}{k}\cdot\frac{\partial\omega}{\partial\boldsymbol{k}} \\ &= \partial\omega/\partial k \\ &= \frac{c}{\partial(n\omega)/\partial\omega} \\ &= v_{\mathrm{gr}}\cos\Phi, \end{aligned} \tag{A 2.20}$$

where Φ is the angle between $\boldsymbol{v}_{\mathrm{gr}}$ and $\boldsymbol{k}$.

† It should be noted that the signal velocity can be identified with the vector $\boldsymbol{v}_{\mathrm{gr}} = \partial\omega/\partial\boldsymbol{k}$ from kinematic considerations only under certain conditions (for example, this cannot be done in a region of anomalous dispersion, where the signal is greatly deformed and we may have $|\partial\omega/\partial\boldsymbol{k}| > c$). In this connection the validity of inequality (A 2.19) for media of the class concerned indicates that the use of the expression $\partial\omega/\partial\boldsymbol{k}$ as the signal velocity in these media is consistent (in a region of anomalous dispersion the inequality (A 2.19) may be violated on account of the existence of absorption, which in (A 2.19) has been assumed absent). The most general, simple and fairly convincing proof of the condition (A 2.19) appears to be based on the use of the theorem (A 1.14) and the requirement that the velocity of energy transfer should be less than c. The above argument was developed in discussion with V. P. Silin, to whom the authors are also indebted for a number of other comments.

From (A 2.19) and (A 2.20) it is clear that

$$|\partial(n\omega)/\partial\omega| > 1. \tag{A 2.21}$$

As will be seen in § A 3, in the absence of spatial dispersion the angle Φ is always acute, and $\partial\omega/\partial k > 0$. Hence, in the absence of spatial dispersion,

$$\partial(n\omega)/\partial\omega > 1. \tag{A 2.22}$$

§ A 3. TWO THEOREMS CONCERNING WAVE PROPAGATION IN THE ABSENCE OF SPATIAL DISPERSION

We may mention here two theorems concerning the propagation of electromagnetic waves in anisotropic media in the absence of spatial dispersion. We shall also explain the reasons why these theorems may be invalid in media with spatial dispersion.

The first theorem states that in the absence of spatial dispersion the angle Φ between the directions of the group velocity $\boldsymbol{v}_{gr}$ and the wave vector $\boldsymbol{k}$ (the phase velocity) is acute, i.e. $|\Phi| < \frac{1}{2}\pi$.

To prove this, we use the relation (A 1.14), which is valid in the region of transparency; in the absence of spatial dispersion this is

$$\overline{W}\boldsymbol{v}_{gr} = \overline{\boldsymbol{S}}_0, \tag{A 3.1}$$

where $\overline{W}$ is the time average energy density of the electromagnetic field ($\overline{W} > 0$), and $\boldsymbol{S}_0 = (c/16\pi)(\boldsymbol{E}_0 \times \boldsymbol{B}_0^* + \boldsymbol{E}_0^* \times \boldsymbol{B}_0) = c\boldsymbol{E}_0 \times \boldsymbol{B}_0^*/8\pi$[see (A 1.10)]. Using the field equations (A 1.6) and (A 1.6a) for the time-average flux $\overline{\boldsymbol{S}}_0$, we obtain

$$\begin{aligned}\overline{\boldsymbol{S}}_0 &= (c/16\pi)(\boldsymbol{E}_0 \times \boldsymbol{B}_0^* + \boldsymbol{E}_0^* \times \boldsymbol{B}_0)\\ &= (c^2/16\pi\omega)\left[(\boldsymbol{k}^* + \boldsymbol{k})(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*) - \boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{E}_0^*) - \boldsymbol{E}_0^*(\boldsymbol{k}^*\cdot\boldsymbol{E}_0)\right].\end{aligned} \tag{A 3.2}$$

Let us first consider, for generality, the propagation of waves in the presence of absorption, when $\boldsymbol{k} = \boldsymbol{k}_1 - i\boldsymbol{k}_2$. Taking the component of equation (A 3.2) in the direction of propagation of the wave, which is given by the vector $\boldsymbol{k}_1$, we have

$$\begin{aligned}\boldsymbol{k}_1\cdot\overline{\boldsymbol{S}}_0 = (c^2/16\pi\omega)\{&2k_1^2(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*) - 2(\boldsymbol{k}_1\cdot\boldsymbol{E}_0)(\boldsymbol{k}_1\cdot\boldsymbol{E}_0^*) +\\ &+ i[(\boldsymbol{k}_1\cdot\boldsymbol{E}_0)(\boldsymbol{k}_2\cdot\boldsymbol{E}_0^*) - (\boldsymbol{k}_1\cdot\boldsymbol{E}_0^*)(\boldsymbol{k}_2\cdot\boldsymbol{E}_0)]\}.\end{aligned} \tag{A 3.3}$$

If the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are parallel or antiparallel (homogeneous plane wave), the last term is zero, and we easily find that $\boldsymbol{k}_1\cdot\overline{\boldsymbol{S}}_0 > 0$. In the region of transparency, where $\boldsymbol{k} = \boldsymbol{k}_1$, we have

$$\boldsymbol{k}\cdot\overline{\boldsymbol{S}}_0 > 0. \tag{A 3.4}$$

In the absence of absorption it is easy to show that $\boldsymbol{k}\cdot\boldsymbol{S}_0 > 0$, a more general inequality than (A 3.4) ($\boldsymbol{S}_0$ being the instantaneous value of the energy flux).

The inequality (A 3.4) signifies that the angle Φ between the directions of $\overline{\boldsymbol{S}}_0$ and $\boldsymbol{k}$ is acute (this is true of the angle between $\overline{\boldsymbol{S}}_0$ and $\boldsymbol{k}_1$ even in the presence of absorption). This result, together with (A 3.1), proves that $|\Phi| < \frac{1}{2}\pi$.

The second theorem concerns the propagation of homogeneous plane waves in a non-absorbing medium ($\sigma_{ij} \to 0$) in the absence of spatial dispersion. It states that under these conditions the square of the complex index of absorption $\tilde{n}^2 = (n - i\varkappa)^2$ always takes real values.

Various proofs are possible, but the one given below, which is mainly clear from [36], is one of the simplest. We use the properties of the inverse tensor ε_{ij}^{-1}, taking into account the facts that for a non-absorbing medium $E_i = \varepsilon_{ij}^{-1} D_j'$ and the tensor ε_{ij}^{-1} is Hermitian. Substituting the second formula (A 1.6) in the first, we obtain

$$\boldsymbol{D}' = \tilde{n}^2[\boldsymbol{E} - \boldsymbol{s}(\boldsymbol{s} \cdot \boldsymbol{E})]. \tag{A 3.5}$$

From (A 3.5) it follows that $\boldsymbol{D}' \cdot \boldsymbol{s} = 0$, i.e. the longitudinal component of the vector $\boldsymbol{D}'$ is zero. For the transverse components D_1 and D_2 we have from (A 3.5) $D_\alpha = \tilde{n}^2 E_\alpha (\alpha = 1, 2)$ Using the relation $E_\alpha = \varepsilon_{\alpha\beta}^{-1} D_\beta (\beta = 1, 2)$, we obtain from (A 3.5) $D_\alpha - \tilde{n}^2 \varepsilon_{\alpha\beta}^{-1} D_\beta = 0$. The values of $\tilde{n}^2$ are determined from the condition for there to be a non-trivial solution of this equation:

$$\begin{vmatrix} 1/\tilde{n}^2 - \varepsilon_{11}^{-1} & -\varepsilon_{12}^{-1} \\ -\varepsilon_{21}^{-1} & 1/\tilde{n}^2 - \varepsilon_{22}^{-1} \end{vmatrix} = 0. \tag{A 3.6}$$

The discriminant of the quadratic equation (A 3.6) for $1/\tilde{n}^2$ is $\Delta = (\varepsilon_{11}^{-1} - \varepsilon_{22}^{-1})^2 + 4\varepsilon_{12}^{-1}\varepsilon_{21}^{-1}$ and, since the tensor $\varepsilon_{\alpha\beta}^{-1}$ is Hermitian, $\Delta = (\varepsilon_{11}^{-1} - \varepsilon_{22}^{-1})^2 + 4|\varepsilon_{12}^{-1}|^2 \geq 0$. For $\Delta \geqq 0$ the values of $1/\tilde{n}^2$, and therefore those of $\tilde{n}^2$, are real; if $\tilde{n}^2 > 0$, then $n \neq 0$ and $\varkappa = 0$, but if $\tilde{n}^2 < 0$, then $n = 0$ and $\varkappa \neq 0$. In the former case the field oscillates in space but is not damped, while in the latter case it is damped without oscillations.

In another statement of the problem, where the vector $\boldsymbol{k}$ is real and the frequency $\omega = \omega' - i\gamma$ is complex, the statement that $\tilde{n}^2$ is real is equivalent to the conclusion that $\omega^2 = k^2\tilde{n}^2/c^2$ is real.

The two theorems just proved† are of interest also in that their validity is not certain in the presence of spatial dispersion. For example, in the latter case we can give examples of wave propagation in a hot magnetoactive plasma [268] or in crystalline media [1] where the inequality $\boldsymbol{k} \cdot \boldsymbol{v}_{\text{gr}} > 0$ is definitely not satisfied. Moreover, an analysis of particular cases shows that the angle Φ

† In the absence of spatial dispersion another theorem is also valid, concerning the propagation of plane waves in anisotropic media with absorption: it states that $n\varkappa > 0$, i.e. n and $\varkappa$ have the same sign. For an isotropic medium this is obvious, since $(n - i\varkappa)^2 = \varepsilon - i \cdot 4\pi\sigma/\omega$ and $n\varkappa > 0$ for $\sigma > 0$. The general proof makes use of the condition (A 2.3). It may be noted that for an emitting medium (e.g. in lasers and masers) $n\varkappa < 0$, since $\sigma < 0$. When spatial dispersion is taken into account, the inequality $n\varkappa > 0$ may not be valid even for an absorbing medium (see e.g. [91]).

may have any value. For example, it is possible for the vectors $\boldsymbol{k}$ and $\boldsymbol{v}_{gr}$ to be antiparallel, i.e. $\Phi = \pi$.

We may mention the reasons why the above proof of the inequality $\boldsymbol{k}\cdot\boldsymbol{v}_{gr} > 0$ becomes invalid. When spatial dispersion is taken into account, the relation (A 3.1) does not follow from the equation $\overline{W}\boldsymbol{v}_{gr} = \overline{\boldsymbol{S}}$ [see (A 1.14)], since the total energy flux of the electromagnetic field is $\boldsymbol{S} = \boldsymbol{S}_0 + \boldsymbol{S}'$. The components of the vector $\overline{\boldsymbol{S}'}$, it will be recalled, are given by

$$\overline{S'_l} = -\frac{\omega}{16\pi}\frac{\partial\varepsilon_{ij}(\omega,\boldsymbol{k})}{\partial k_l}E^*_{0i}E_{0j}.$$

If $\boldsymbol{S}' \neq 0$, we of course cannot be sure of the validity of statements derived from the approximation $\boldsymbol{S} = \boldsymbol{S}_0$.

The above expression for $\overline{\boldsymbol{S}'}$ was obtained in deriving formula (A 1.14). For media with weak spatial dispersion [1] we can make the form of the function $\varepsilon'_{ij}(\omega,\boldsymbol{k})$ somewhat more specific, and sometimes its explicit form need not be used. For instance, in an isotropic non-gyrotropic non-magnetic medium the allowance for weak spatial dispersion amounts to using the relations (with ε, α and β functions of ω)

$$\boldsymbol{D}' = \varepsilon\boldsymbol{E} + \alpha\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{E} + \beta\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E};\quad \boldsymbol{B} = \boldsymbol{H}. \tag{A 3.7}$$

We substitute these expressions in the general relation which follows from the field equation (A 1.1):

$$\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}'}{\partial t} + \frac{1}{8\pi}\frac{\partial H^2}{\partial t} = -\,\mathrm{div}\,\boldsymbol{S}_0;\quad \boldsymbol{S}_0 = c\boldsymbol{E}\times\boldsymbol{H}/4\pi. \tag{A 3.8}$$

Then, if for simplicity we neglect the dependence of ε, α and β on ω, we have

$$\left.\begin{aligned}
&\partial W/\partial t = -\,\mathrm{div}\,\boldsymbol{S};\\
&8\pi W = \varepsilon E^2 + H^2 - \alpha(\mathrm{div}\,\boldsymbol{E})^2 + \beta(\mathbf{curl}\,\boldsymbol{E})^2;\\
&\boldsymbol{S} = \boldsymbol{S}_0 + \frac{\alpha}{4\pi}\boldsymbol{E}\,\mathrm{div}\frac{\partial\boldsymbol{E}}{\partial t} - \frac{\beta}{4\pi}\boldsymbol{E}\times\mathbf{curl}\frac{\partial\boldsymbol{E}}{\partial t};\\
&16\pi\overline{W} = \varepsilon\boldsymbol{E}_0\cdot\boldsymbol{E}^*_0 + \boldsymbol{H}_0\cdot\boldsymbol{H}^*_0 - \alpha(\boldsymbol{k}\cdot\boldsymbol{E}_0)(\boldsymbol{k}\cdot\boldsymbol{E}^*_0) + \beta(\boldsymbol{k}\times\boldsymbol{E}_0)\cdot(\boldsymbol{k}\times\boldsymbol{E}^*_0);\\
&16\pi\overline{\boldsymbol{S}} = c(\boldsymbol{E}^*_0\times\boldsymbol{H}_0 + \boldsymbol{E}_0\times\boldsymbol{H}^*_0) + \alpha\omega[\boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{E}^*_0) + \boldsymbol{E}^*_0(\boldsymbol{k}\cdot\boldsymbol{E}_0)] -\\
&\qquad - \beta\omega[\boldsymbol{E}_0\times(\boldsymbol{k}\times\boldsymbol{E}^*_0) + \boldsymbol{E}^*_0\times(\boldsymbol{k}\times\boldsymbol{E}_0)],
\end{aligned}\right\} \tag{A 3.9}$$

where in taking the time-average values $\overline{W}$ and $\overline{\boldsymbol{S}}$ we have substituted a field of the type (A 1.2) with real $\boldsymbol{k}$.

The use of the relation (A 3.7) is equivalent to using the permittivity $\varepsilon_{ij} = (\varepsilon + \beta k^2)\delta_{ij} - (\alpha + \beta)k_ik_j$. If we substitute this ε_{ij} in (A 1.14), neglecting the derivatives of ε_{ij} with respect to ω, the result is (A 3.9), as it should be.

Let us now consider the case of a plasma, for which it is particularly easy to see the physical significance of the additional flux $\boldsymbol{S}'$. We shall use the Boltzmann equation for the electron distribution function:

$$\partial f/\partial t + \boldsymbol{v}\cdot\mathbf{grad}_r f + (e/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c)\cdot\mathbf{grad}_v f = 0. \qquad \text{(A 3.10)}$$

In (A 3.10) collisions have been neglected, and in addition we ignore the motion of the ions, i.e. consider only high-frequency fields (§ 10). The spatial dispersion in the plasma arises on account of the thermal motion, and is taken into account by the term $\boldsymbol{v}\cdot\mathbf{grad}_r f$ in (A 3.10). Multiplying (A 3.10) by $\frac{1}{2}mv^2$ and integrating over velocities, we obtain in the usual manner (see, e.g., § B 3, where spatial dispersion is not taken into account)

$$\frac{\partial}{\partial t}\left(\frac{E^2 + H^2}{8\pi} + \int \frac{1}{2} m v^2 f d\boldsymbol{v}\right) = -\operatorname{div}\left(c\,\boldsymbol{E} \times \boldsymbol{H}/4\pi + \int \frac{1}{2} m v^2 \boldsymbol{v} f d\boldsymbol{v}\right). \text{(A 3.11)}$$

Thus the total energy flux $\boldsymbol{S} = \boldsymbol{S}_0 + \int \frac{1}{2} mv^2 \boldsymbol{v} f \,\mathrm{d}\boldsymbol{v}$ consists of the field energy flux and the particle (electron) energy flux. In this case also, of course, the flux $\boldsymbol{S}'$ can be expressed in terms of $\varepsilon_{ij}(\omega, \boldsymbol{k})$ for the plasma.

On averaging over time (for a monochromatic field) the first term in (A 3.11) gives zero. If we furthermore consider the propagation of homogeneous plane waves in the z-direction, (A 3.11) gives for the mean values

$$\partial(\overline{\boldsymbol{S}_0 + \boldsymbol{S}'})/\partial z = \partial \overline{\boldsymbol{S}}/\partial z = 0.$$

Hence we have the obvious result that, when the thermal motion of the particles in a non-absorbing medium is taken into account, the time-average total energy flux is conserved:

$$\overline{\boldsymbol{S}} = \overline{\boldsymbol{S}}_0 + \overline{\boldsymbol{S}}' = \text{constant}. \qquad \text{(A 3.12)}$$

The relation (A 3.12) is of interest in that it clearly indicates the possibility of a variation of the time-average Poynting vector flux $\overline{\boldsymbol{S}}_0$ despite the absence of absorption. This result is closely related to the invalidity of the second of the above theorems when spatial dispersion is taken into account.

In solving particular problems of wave propagation in magnetoactive plasmas (Chapters III and V) and in crystalline media [1] with allowance for spatial dispersion we see why the second theorem, i.e. the statement that $\tilde{n}^2$ is real in a non-absorbing medium, may cease to be valid: it is found that in such a medium the propagation of waves with complex $\tilde{n}^2$ (and therefore complex $\tilde{n}$) is possible. In the above proof that $\tilde{n}^2$ is real we used the properties that the tensor $\varepsilon^{-1}_{\alpha\beta}(\omega)$ is Hermitian and independent of $\tilde{n}$. When spatial dispersion is present, however, $\varepsilon^{-1}_{\alpha\beta}(\omega, k) = \varepsilon^{-1}_{\alpha\beta}(\omega, \omega\tilde{n}/c)$, and the squares of the roots of equation (A 3.6) need not be real.

If $\tilde{n}^2$ is complex the field will oscillate with an amplitude which varies in space. At the same time, according to (A 3.12) the time-average total energy flux $\bar{S}$ is constant. If the medium occupies the half-space $z > 0$ and $\bar{S}$ tends to zero as $z \to \infty$, we must have $\bar{S} = 0$ everywhere. Consequently, for a half-space and waves with complex $\tilde{n}^2$ in the absence of absorption $\bar{S}_0 = -\bar{S}'$, i.e. the mean field energy flux $\bar{S}_0$ is balanced by the flux $\bar{S'}$ of non-electromagnetic origin in the medium. For a plasma, as we have seen, this non-electromagnetic flux is simply the kinetic energy flux. (We call it the non-electromagnetic flux S' because a corresponding flux can exist even when the field is absent; if we consider the flux S' in the wave field, it is of course determined by the field E_0, in accordance with (A 1.14).) It may be noted that the problem of the nature of waves in a non-absorbing medium with complex $\tilde{n}^2$ has recently been discussed also, on a slightly different basis, in [355].

§ A 4. THE PROOF OF THE INEQUALITY (A 2.19)

We shall prove the inequality $|\partial\omega/\partial \boldsymbol{k}| = v_{gr} < c$ for the example of propagation of a group of waves in a uniaxial crystal. In the system of principal axes of the symmetric tensor ε_{ij} the dispersion relation (A 2.17) becomes

$$k^2(\varepsilon_1 k_x^2 + \varepsilon_2 k_y^2 + \varepsilon_3 k_z^2) - \frac{\omega^2}{c^2}[k_x^2 \varepsilon_1(\varepsilon_2 + \varepsilon_3) +$$

$$+ k_y^2 \varepsilon_2(\varepsilon_1 + \varepsilon_3) + k_z^2 \varepsilon_3(\varepsilon_1 + \varepsilon_2)] + \frac{\omega^4}{c^4}\varepsilon_1 \varepsilon_2 \varepsilon_3 = 0. \tag{A 4.1}$$

For uniaxial crystals $\varepsilon_1 = \varepsilon_2 = \varepsilon_\perp$ and $\varepsilon_3 = \varepsilon_{||}$. Then (A 4.1) for the ordinary wave gives the equation

$$k^2 = k_0^2 \varepsilon_\perp = \omega^2 \varepsilon_\perp / c^2, \tag{A 4.2}$$

and for the extraordinary wave

$$\varepsilon_\perp k_x^2 + \varepsilon_{||} k_z^2 = \varepsilon_\perp \varepsilon_{||} k_0^2 = \omega^2 \varepsilon_\perp \varepsilon_{||}/c^2. \tag{A 4.3}$$

In deriving equations (A 4.2) and (A 4.3) we have, without loss of generality, put $k_y = 0$. The dispersion relation (A 4.2) for the ordinary wave has the same form as for an isotropic medium, with ε replaced by $\varepsilon_\perp$. The proof of the inequality $v_{gr} < c$ is here entirely similar to that given in [36] for an isotropic medium (see also § A 2).

For the extraordinary wave the proof of (A 2.19) is a little more complex. Differentiating equation (A 4.3) with respect to k_x and k_z, we find the components of the group velocity, $\partial\omega/\partial k_x$ and $\partial\omega/\partial k_z$, and its magnitude $v_{gr} = |\sqrt{[(\partial\omega/\partial k_x)^2 + (\partial\omega/\partial k_z)^2]}|$. Thus we have

$$v_{gr} = \frac{cn\sqrt{[(1/\varepsilon_{||}^2)\sin^2\alpha + (1/\varepsilon_\perp^2)\cos^2\alpha]}}{1+B}, \tag{A 4.4}$$

where

$$\tan\alpha = k_x/k_z; \quad B = \frac{\omega n^2}{2}\left(\frac{\cos^2\alpha}{\varepsilon_\perp^2}\frac{d\varepsilon_\perp}{d\omega} + \frac{\sin^2\alpha}{\varepsilon_{||}^2}\frac{d\varepsilon_{||}}{d\omega}\right).$$

Using the inequalitics (A 2.12), which in this case take thc form $d\varepsilon_{||}/d\omega > 0$ and $d\varepsilon_\perp/d\omega > 0$, we find that $v_{\mathrm{gr}} < c$ if

$$\frac{\sin^2\alpha}{\varepsilon_{||}^2} + \frac{\cos^2\alpha}{\varepsilon_\perp^2} < \frac{\sin^2\alpha}{\varepsilon_{||}} + \frac{\cos^2\alpha}{\varepsilon_\perp} = \frac{1}{n^2}. \tag{A 4.5}$$

In deriving (A 4.5) we have used equation (A 4.3). With the identity $d[\omega^2(\varepsilon - 1)]/d\omega - 2\omega(\varepsilon - 1) = \omega^2 d\varepsilon/d\omega$, we can represent the quantity B in (A 4.4) as

$$B = n^2\left\{\cos^2\alpha\,\frac{1-\varepsilon_\perp}{\varepsilon_\perp^2} + \sin^2\alpha\,\frac{1-\varepsilon_{||}}{\varepsilon_{||}^2} + \right.$$
$$\left. + \frac{1}{2\omega}\left[\frac{\cos^2\alpha}{\varepsilon_\perp^2}\frac{d}{d\omega}[\omega^2(\varepsilon_\perp - 1)] + \frac{\sin^2\alpha}{\varepsilon_{||}^2}\frac{d}{d\omega}[\omega^2(\varepsilon_{||} - 1)]\right]\right\}.$$

Using the inequalities (A 2.13) and (A 4.3) we see that $v_{\mathrm{gr}} < c$ if the inequality opposite to (A 4.5) holds. Thus we have proved that $v_{\mathrm{gr}} < c$ for the extraordinary wave also.

It may be noted that the condition $|\partial\omega/\partial\boldsymbol{k}| = v_{\mathrm{gr}} < c$ can also easily be demonstrated for a magnetoactive medium whose properties are given by the tensor (A 2.14), if propagation occurs along the constant magnetic field or perpendicular to it.

APPENDIX B

THE CONSERVATION LAW AND THE EXPRESSION FOR THE ENERGY DENSITY IN THE ELECTRODYNAMICS OF AN ABSORBING DISPERSIVE MEDIUM†

DESPITE the fact that the problem of the conservation law and the expression for the energy density in electrodynamics is a fundamental one, there are certain aspects of it which have not yet been elucidated, in particular for the case of an absorbing dispersive medium. For example, the familiar expression

$$W_E = \frac{1}{4\pi}\int_{-\infty}^{t} \boldsymbol{E}\cdot(\partial \boldsymbol{D}/\partial t)\,dt,$$

even when reduced to the form constant $\times\ E^2$, cannot be regarded as the total energy density when absorption is present, if only because the value of W_E may be negative (see below). Moreover, in the modern work [36] (§61) the concept of internal energy is regarded as meaningless in the presence of absorption. On the other hand, with the microscopic approach, using particular models of simple media (plasmas, or assemblies of oscillators), and also for certain equivalent circuits, we can derive simple expressions for the energy density, which at first sight appear entirely reasonable [22, § 68; 143, 144]. The relation between this model approach and the phenomenological approach has never been discussed so far as we know.†† Furthermore, some other cognate topics usually receive insufficiently precise treatment (as, for example, in the very recent paper [356]). Thus we may hope that the publication of the present discussion is justified, even though the matters discussed are essentially very simple; the author is aware that others besides himself have for long been unclear concerning these matters.

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† By V. L. Ginzburg. Published in *Izvestiya vysshikh uchebnykh zavedeniĭ: Radiofizika* 4, 74–88, 1961.

†† An exception occurs in § 22 of the present book, where the author was able to make corrections in proof to incorporate some of the arguments here given.

§ B 1. THE BASIC EQUATIONS AND ENERGY RELATIONS

We write the fundamental field equations in the usual form:

$$\mathbf{curl}\,\boldsymbol{H} = 4\pi\boldsymbol{j}/c + (1/c)\,\partial\boldsymbol{D}/\partial t; \quad \mathbf{curl}\,\boldsymbol{E} = -(1/c)\,\partial\boldsymbol{H}/\partial t, \tag{B 1.1}$$

where for simplicity the medium is assumed non-magnetic ($\boldsymbol{B} = \boldsymbol{H}$)†, and the external (extraneous) currents are assumed absent. In [36] the two terms $4\pi\boldsymbol{j}/c + (1/c)\,\partial\boldsymbol{D}/\partial t$ are written together as $(1/c)\,\partial\boldsymbol{D}/\partial t$, while here we use the customary notation. From (B 1.1) we have Poynting's theorem:

$$\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \frac{\partial}{\partial t}\left(\frac{H^2}{8\pi}\right) + \boldsymbol{j}\cdot\boldsymbol{E} = -\frac{c}{4\pi}\,\mathrm{div}\,(\boldsymbol{E}\times\boldsymbol{H}), \tag{B 1.2}$$

where real quantities are implied.

For harmonic fields, proportional to $e^{i\omega t}$, in a linear medium at rest, and neglecting spatial dispersion,

$$\left.\begin{aligned} &D_i^{(\omega)} = \varepsilon_{ik}(\omega)\,E_k^{(\omega)}; \quad j_i(\omega) = \sigma_{ik}(\omega)\,E_k^{(\omega)}; \\ &\varepsilon'_{ik} = \varepsilon_{ik} - i\cdot 4\pi\sigma_{ik}/\omega, \end{aligned}\right\} \tag{B 1.3}$$

where summation over repeated suffixes is understood. If the medium is also isotropic, then

$$\left.\begin{aligned} &\boldsymbol{D}^{(\omega)} = \varepsilon(\omega)\,\boldsymbol{E}^{(\omega)}; \quad \boldsymbol{j}^{(\omega)} = \sigma(\omega)\,\boldsymbol{E}^{(\omega)}; \\ &\varepsilon' = \varepsilon_1 - i\varepsilon_2 = \varepsilon - i\cdot 4\pi\sigma/\omega. \end{aligned}\right\} \tag{B 1.4}$$

The allowance for anisotropy merely complicates the notation slightly, and we shall therefore regard the medium as isotropic (but see § B 4).

In certain frequency ranges it is possible to neglect dispersion or absorption, and sometimes both. For instance, in a plasma

$$\left.\begin{aligned} &\varepsilon' = 1 - \omega_0^2/\omega(\omega - i\nu); \quad \varepsilon = 1 - \omega_0^2/(\omega^2 + \nu^2); \\ &\sigma = \omega_0^2\,\nu/4\pi(\omega^2 + \nu^2); \quad \omega_0^2 = 4\pi e^2 N/m. \end{aligned}\right\} \tag{B 1.5}$$

Here N is the electron density and ν the frequency of collisions between electrons and other particles. Formulae (B 1.5) are strictly valid when the collision frequency is independent of the velocity; in general, formulae (B 1.5) with $\nu = \nu_{\mathrm{eff}}$ correspond to the "elementary theory" (§ 3).

If $\omega^2 \ll \nu^2$, the dispersion (i.e. the dependence of ε and σ on ω) may be neglected in a first approximation:

$$\varepsilon = 1 - \omega_0^2/\nu_0^2; \quad \sigma = \omega_0^2/4\pi\nu; \quad \omega^2 \ll \nu^2. \tag{B 1.6}$$

In the opposite limiting case we have

$$\varepsilon = 1 - \omega_0^2/\omega^2; \quad \sigma = 0; \quad \nu \to 0, \tag{B 1.7}$$

† For a medium with complex permeability $\mu'(\omega)$ or tensor $\mu'_{ik}(\omega)$ not equal to unity the results are derived exactly as below for a medium with permittivity $\varepsilon'(\omega)$ or $\varepsilon'_{ik}(\omega)$ and $\mu' = 1$ (see also § B 4).

i.e. absorption is absent, but the dispersion cannot be neglected if ε is appreciably different from unity.

In the absence of dispersion, the theorem (B 1.2) takes the form

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon E^2 + H^2}{8\pi}\right) + \sigma E^2 = -\frac{c}{4\pi}\,\mathrm{div}\,(\boldsymbol{E} \times \boldsymbol{H}), \tag{B 1.8}$$

which is often used; the expression

$$W^{(0)} = W_E^{(0)} + \frac{H^2}{8\pi} = \frac{\varepsilon E^2}{8\pi} + \frac{H^2}{8\pi} \tag{B 1.9}$$

is sometimes interpreted as the field energy density in the medium. The general expression $(1/4\pi)\boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t$ reduces to the form $\partial\overline{W}_E/\partial t$ for an arbitrary medium, but for a quasimonochromatic field and with averaging over high frequencies (denoted by the bar). The derivation is generally known (see, e.g., § 22 and [22, 36, 143]), but in order to effect some generalisation and refinement the proof is also given in § B 4.

For an isotropic and non-magnetic medium (see § B 4)

$$\left.\begin{aligned} &\overline{\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}} = \frac{\partial\overline{W}_E}{\partial t};\\ &\overline{W}_E = \frac{1}{8\pi}\,\frac{d(\omega\varepsilon)}{d\omega}\,\overline{E^2} = \frac{1}{16\pi}\,\frac{d(\omega\varepsilon)}{d\omega}\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*;\\ &\overline{W} = \overline{W}_E + H^2/8\pi. \end{aligned}\right\} \tag{B 1.10}$$

In the absence of absorption (i.e. for $\sigma = 0$) the entropy of a system otherwise isolated is constant in a variable electromagnetic field. Hence the quantities $W^{(0)}$ and $\overline{W}$ [see (B 1.9) and (B 1.10)] can be regarded as the internal energy† of the field in the medium, in the thermodynamic sense of the term. In the presence of absorption the entropy is not constant, and (B 1.2) involves two volume terms, namely $(1/4\pi)\boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t + (\partial/\partial t)H^2/8\pi$ and $\boldsymbol{j}\cdot\boldsymbol{E}$. It is therefore incorrect to speak of the thermodynamic internal energy, and the distinction between energy changes and losses (dissipation) at least requires special analysis. This will be especially clear from the subsequent discussion. Here we may note that in the presence of absorption the quantities $W_E^{(0)}$ and $\overline{W}_E$ may be negative. For instance, in a plasma in the case (B 1.6) we have

$$W_E^{(0)} = \left(1 - \frac{\omega_0^2}{\nu^2}\right)\frac{E^2}{8\pi}, \quad \overline{W}_E = \overline{W}_E^{(0)}, \tag{B 1.11}$$

and evidently $W_E^{(0)} < 0$ for $\nu^2 < \omega_0^2 = 4\pi e^2 N/m$.

This result does not contradict the principle of increase of entropy, since $W^{(0)}$ is not the internal energy. Nevertheless, we shall consider this point

† For constant temperature (and not entropy) we have the free energy.

in a little more detail in order to preclude misunderstandings. For a closed system or for any region to which there is no inflow of energy (i.e. $\oint(\boldsymbol{E}\times\boldsymbol{H})_n\,dS = 0$), we have from (B 1.8) the relation

$$\int[W^{(0)}(t') - W^{(0)}(t)]\,dV = -\int_t^{t'}\int_V \sigma E^2\,dV\,dt.$$

If all field sources are cut off at time t, say, then as $t'\to\infty$ the field is entirely damped, $W^{(0)}(\infty) = 0$, and by the law of increase of entropy all the energy must be converted into heat, i.e.

$$\int W^{(0)}(t)\,dV = \int\frac{\varepsilon E^2 + H^2}{8\pi}\,dV = \int^{\infty}\int\sigma E^2\,dV\,dt > 0. \tag{B 1.12}$$

Hence it follows† that $\sigma > 0$ and $\varepsilon > 0$ (we assume that in the process under consideration the field H is sufficiently weak in comparison with the field E). The condition (B 1.12) does not contradict what was said above concerning plasmas, since (B 1.12) has been derived from (B 1.8) by considering the process of cutting off the field. Here it is not at all certain that dispersion can be neglected and the relation (B 1.8) used. For example, in the absence of sources and with the condition $\mathbf{curl}\,\boldsymbol{H} = 0$ we obtain from (B 1.1) the equation

$$\frac{\partial\boldsymbol{D}}{\partial t} + 4\pi\boldsymbol{j} = \varepsilon\frac{\partial\boldsymbol{E}}{\partial t} + 4\pi\sigma\boldsymbol{E} = 0, \tag{B 1.13}$$

where in the second expression we assume that dispersion is absent. According to (B 1.13), $\boldsymbol{E} = \boldsymbol{E}(0)e^{-4\pi\sigma t/\varepsilon}$, and for $\sigma > 0$ and $\varepsilon < 0$ the field would increase. In a quasiequilibrium system this is, of course, impossible. Equation (B 1.13) simply cannot, however, be used for a plasma: it is clear from this equation or its solution that the characteristic time of variation of the field is $T\sim\varepsilon/4\pi\sigma$, i.e. the frequencies ω in the spectral resolution are mainly $\sim 4\pi\sigma/\varepsilon$. It is easily seen that at such frequencies the condition $\omega^2 \ll \nu^2$ for dispersion to be absent and the condition $\varepsilon < 0$ cannot both be satisfied [see (B 1.6)].

The above discussion shows only that the validity of (B 1.8) is obviously restricted. But these restrictions on the properties of the medium and on the characteristic frequencies of variation of the field in the processes considered (the field in a condenser, the wave field in free space, etc.) do not in themselves prevent $W_E^{(0)}$ and $\overline{W}_E$ from being interpreted (even in the presence of absorption) as some density or mean density of energy in conditions where $(1/4\pi)\boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t = \partial W_E^{(0)}/\partial t$ or $(1/4\pi)\overline{\boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t} = \partial\overline{W}_E/\partial t$. For the quantity $W_E^{(0)}$ appears in (B 1.8) as some function of state, differentiated with respect to time, and the same is true of $\overline{W}_E$ in (B 1.10).† In this respect absorption

† The condition $\sigma(t) = \sigma > 0$ is necessary only for equilibrium and quasiequilibrium systems, where the field E is monotonically damped when the sources are cut off.

does not affect the situation. Furthermore, for sufficiently weak absorption the quantities $W_E^{(0)}$ and $\overline{W}_E$ are almost the same as the corresponding expressions for the internal energy in the absence of absorption. On the other hand, when there is considerable absorption $W_E^{(0)}$ and $\overline{W}_E$ are in general not equal to the total energy density in the system, but only to part of it which has a distinctive character (namely, that of the corresponding term in (B 1.8) or (B 1.10)). Apart from a possible question of terminology (whether or not to call $W_E^{(0)}$ and $\overline{W}_E$ the energy density) the phenomenological theory raises no problems here. It is, however, undoubtedly of interest to analyse the energy relations on the basis of the microscopic picture or of various models of the medium.

§ B 2. THE RESULTS FOR A MODEL PLASMA

Let us consider a model "plasma" consisting of free particles of charge e and mass m, moving with velocity $\boldsymbol{u}$ in an electric field $\boldsymbol{E}$ and also subject to a frictional force; the space charge is assumed to be compensated by a "background" of ions. Then

$$m\,\partial\boldsymbol{u}/\partial t = e\boldsymbol{E} - m\nu\boldsymbol{u}; \tag{B 2.1}$$

here $\boldsymbol{u}$ depends only on t, and the derivative is written in the form $\partial/\partial t$ only for convenience; in using (B 2.1) below, we also assume the field and the particle density independent of the coordinates. This equation is widely used in plasma physics in the approximation of the "elementary theory" (see above and § 3). From (B 2.1) we obtain the expression (B 1.5) for ε'. If we add to (B 2.1) the force $\boldsymbol{u} \times \boldsymbol{H}^{(0)}/c$, where $\boldsymbol{H}^{(0)}$ is the external magnetic field, we have a model of a magnetoactive plasma. This generalisation is not important in the present discussion, and we shall consider below only an isotropic plasma. It may also be noted that by adding to (B 2.1) some force **grad** U, we can take into account also bound "electrons" (e.g. for an oscillator $U = \frac{1}{2}m\omega_i^2 r^2$, $\boldsymbol{u} = \partial\boldsymbol{r}/\partial t$; a medium consisting of such oscillators is considered in § 3 and [143, 144]). The use of (B 2.1) for a plasma is more justified than for any other medium, in particular because the field acting in a plasma is equal to the mean macroscopic field (§ 3 and [22]). We are concerned with the behaviour of the medium in a quasimonochromatic field

$$\left.\begin{aligned} \boldsymbol{E}(t) &= \tfrac{1}{2}[\boldsymbol{E}_0(t)\,e^{i\omega_0 t} + \dot{\boldsymbol{E}}_0\,(t)\,e^{-i\omega_0 t}];\\ \boldsymbol{H}(t) &= \tfrac{1}{2}[\boldsymbol{H}_0(t)\,e^{i\omega_0 t} + \boldsymbol{H}_0^*(t)\,e^{-i\omega_0 t}], \end{aligned}\right\} \tag{B 2.2}$$

† The distinguishing feature of these cases is that the more general expression

$$W_E = \frac{1}{4\pi}\int_{-\infty}^{t} \boldsymbol{E}\cdot\partial\boldsymbol{D}/\partial t$$

depends on the value of the field not only at the time t considered but also at previous times, i.e. on the history as well as on the state of the system.

where $\boldsymbol{E}_0(t)$ and $\boldsymbol{H}_0(t)$ vary slowly with time (relative to the period $2\pi/\omega_0$).

For a steady process we can put $\boldsymbol{E}_0(t)$ = constant; then we have from (B 2.1) and (B 2.2)

$$\left.\begin{aligned} \boldsymbol{u} &= \frac{e}{2m}\left(\frac{\boldsymbol{E}_0\, e^{i\omega_0 t}}{i\omega_0+\nu} + \frac{\boldsymbol{E}_0^*\, e^{-i\omega_0 t}}{-i\omega_0+\nu}\right); \\ \overline{K} &= \tfrac{1}{2}\,\overline{m u^2} = e^2\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*/4m(\omega_0^2+\nu^2); \\ \overline{e\boldsymbol{u}\cdot\boldsymbol{E}} &= m\nu\,\overline{u^2} = e^2\,\nu\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*/2m(\omega_0^2+\nu^2). \end{aligned}\right\} \tag{B 2.3}$$

Let us now consider the sum of the kinetic energy of all the particles $N\overline{K}$ and the field energy (without the particles) $\overline{E^2}/8\pi = \boldsymbol{E}_0\cdot\boldsymbol{E}_0^*/16\pi$. Evidently

$$\begin{aligned} \overline{W}_E' &= N\overline{K} + \overline{E^2}/8\pi \\ &= \left[1+\frac{4\pi e^2 N}{m(\omega_0^2+\nu^2)}\right]\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi}. \end{aligned} \tag{B 2.4}$$

In the absence of absorption (when $\nu = 0$) it is seen from (B 1.5) and (B 1.10) that

$$\begin{aligned} \overline{W}_E' &= \left[1+\frac{4\pi e^2 N}{m\omega_0^2}\right]\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi} \\ &= \left[\frac{d(\omega\varepsilon)}{d\omega}\right]_{\omega_0}\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi} \\ &= \overline{W}_E. \end{aligned} \tag{B 2.5}$$

The quantity $\overline{W}_E'$ is always positive; for $\nu = 0$ it is equal to the internal energy $\overline{W}_E$, and in the presence of absorption it also has an evident physical significance as the sum of the kinetic energy and the field energy in the absence of the particles. It therefore seems quite natural to regard the energy $\overline{W}_E'$ as the mean energy density of the electric field in the medium, as has been done in [22, 143, 144]. There arises, however, the question of the relation between the expression (B 2.4) and the phenomenological quantities in the presence of absorption, in particular the quantity

$$\begin{aligned} \overline{W}_E &= \left[\frac{d(\omega\varepsilon)}{d\omega}\right]_{\omega_0}\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi} \\ &= \left[1+\frac{4\pi e^2(\omega_0^2-\nu^2)N}{m(\omega_0^2+\nu^2)^2}\right]\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi}, \end{aligned} \tag{B 2.6}$$

where we have used formula (B 1.5) for ε. This Appendix is mainly concerned with the resolution of the question just raised.

In the model (B 2.1) the total current density due to the motion of charges is

$$\begin{aligned} eN\boldsymbol{u} \equiv \boldsymbol{j}_t &= \boldsymbol{j} + \partial\boldsymbol{P}/\partial t \\ &= \boldsymbol{j} + (1/4\pi)\,\partial(\boldsymbol{D}-\boldsymbol{E})/\partial t, \end{aligned} \tag{B 2.7}$$

where $\boldsymbol{P} = (\boldsymbol{D} - \boldsymbol{E})/4\pi$ is the polarisation. Hence, multiplying (B 2.1) by $N\boldsymbol{u}$, we have for any field $\boldsymbol{E}$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,mNu^2\right) = eN\,\boldsymbol{u}\cdot\boldsymbol{E} - mN\,\nu u^2$$

$$= \boldsymbol{j}\cdot\boldsymbol{E} - \frac{\partial}{\partial t}\left(\frac{E^2}{8\pi}\right) + \frac{1}{4\pi}\,\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} - mN\,\nu u^2 \qquad \text{(B 2.8)}$$

or

$$\frac{\partial}{\partial t}\left(NK + \frac{E^2}{8\pi}\right) = \frac{\partial}{\partial t}\,W'_E$$

$$= \boldsymbol{j}\cdot\boldsymbol{E} + \frac{1}{4\pi}\,\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} - mN\,\nu u^2. \qquad \text{(B 2.9)}$$

Together with Poynting's theorem (B 1.2), (B 2.9) gives

$$\frac{\partial}{\partial t}\left(NK + \frac{E^2}{8\pi} + \frac{H^2}{8\pi}\right) = -\frac{c}{4\pi}\operatorname{div}(\boldsymbol{E}\times\boldsymbol{H}) - mN\,\nu u^2. \qquad \text{(B 2.10)}$$

This relation has an entirely evident significance: it expresses the law of conservation of energy, $-mN\nu u^2$ being the work done by the force of friction on the particles, which is converted into heat. For the quasimonochromatic field (B 2.2), after averaging over high frequencies, equation (B 2.10) becomes [see also (B 2.3) and § B 4]

$$\frac{\partial}{\partial t}\,\overline{\left(NK + \frac{E^2}{8\pi} + \frac{H^2}{8\pi}\right)}$$

$$= \frac{\partial}{\partial t}\left\{\left[1 + \frac{4\pi e^2 N}{m(\omega_0^2 + \nu^2)}\right]\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi} + \frac{\boldsymbol{H}_0\cdot\boldsymbol{H}_0^*}{16\pi}\right\}$$

$$= -\frac{c}{16\pi}\operatorname{div}(\boldsymbol{E}_0\times\boldsymbol{H}_0^* + \boldsymbol{E}_0^*\times\boldsymbol{H}_0) - mN\,\nu\overline{u^2}. \qquad \text{(B 2.11)}$$

From the phenomenological equation (B 4.9), for a medium with ε and σ given by (B 1.5), we obtain [see also (B 2.6)]

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left\{\left[1 + \frac{4\pi e^2 N(\omega_0^2 - \nu^2)}{m(\omega_0^2 + \nu^2)^2}\right]\frac{\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*}{16\pi} + \frac{\boldsymbol{H}_0\cdot\boldsymbol{H}_0^*}{16\pi}\right\} + \\
&\quad + \frac{e^2 N\nu}{2m(\omega_0^2 + \nu^2)}\,\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* + \frac{ie^2 N\omega_0\nu}{2m(\omega_0^2 + \nu^2)^2}\left(\frac{\partial \boldsymbol{E}_0}{\partial t}\cdot\boldsymbol{E}_0^* - \boldsymbol{E}_0\cdot\frac{\partial \boldsymbol{E}_0^*}{\partial t}\right) \\
&= -\frac{c}{16\pi}\operatorname{div}(\boldsymbol{E}_0\times\boldsymbol{H}_0^* + \boldsymbol{E}_0^*\times\boldsymbol{H}_0); \\
&\quad 1 + \frac{4\pi e^2 N(\omega_0^2 - \nu^2)}{m(\omega_0^2 + \nu^2)^2} = \left[\frac{d(\omega\varepsilon)}{d\omega}\right]_{\omega_0}; \\
&\quad \frac{e^2 N\nu}{m(\omega_0^2 + \nu^2)} = \sigma; \quad -\frac{2e^2 N\omega_0\nu}{m(\omega_0^2 + \nu^2)^2} = \left(\frac{d\sigma}{d\omega}\right)_{\omega_0}.
\end{aligned}\right\} \qquad \text{(B 2.12)}$$

Equations (B 2.11) and (B 2.12) pertain to the same model and cannot be contradictory. Hence it is clear that for a quasimonochromatic process (B 2.2), when equation (B 2.1) is used, the work done by the frictional force (to be equated to the amount of heat generated) per unit time is

$$m N \nu \overline{u^2} = \frac{e^2 N \nu}{2m(\omega_0^2 + \nu^2)} \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* - \frac{8\pi e^2 N \nu^2}{m(\omega_0^2 + \nu^2)^2} \frac{\partial}{\partial t}\left(\frac{\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*}{16\pi}\right) +$$
$$+ \frac{i e^2 N \omega_0 \nu}{2m(\omega_0^2 + \nu^2)^2}\left(\frac{\partial \boldsymbol{E}_0}{\partial t} \cdot \boldsymbol{E}_0^* - \boldsymbol{E}_0 \cdot \frac{\partial \boldsymbol{E}_0^*}{\partial t}\right). \tag{B 2.13}$$

The same result is easily obtained directly from (B 2.1). For we have from this equation, for any field $\boldsymbol{E}$ which vanishes at $t = -\infty$,

$$\boldsymbol{u} = e^{-\nu t} \int_{-\infty}^{t} \frac{e}{m} \boldsymbol{E}(t') e^{\nu t'} dt'. \tag{B 2.14}$$

Substituting the field (B 2.2), squaring and averaging, we find (with $\tau = t'' - t'$)

$$\frac{4m^2}{e^2} e^{2\nu t} \overline{u^2} = \int_{-\infty}^{t} \int_{-\infty}^{t} [\boldsymbol{E}_0(t') \cdot \boldsymbol{E}_0^*(t'') e^{i\omega_0(t'-t'')} e^{\nu(t'+t'')} +$$
$$+ \text{complex conjugate}]\, dt'\, dt''$$
$$= \int_{-\infty}^{t} dt'' \int_{t''-t}^{\infty} [\boldsymbol{E}_0(t'' - \tau) \cdot \boldsymbol{E}_0^*(t'') e^{-(i\omega_0+\nu)\tau} e^{2\nu t''} +$$
$$+ \text{complex conjugate}]\, d\tau$$
$$= \int_{-\infty}^{t} [\boldsymbol{E}_0(t'') \cdot \boldsymbol{E}_0^*(t'') e^{2\nu t''} \int_{t''-t}^{\infty} e^{-(i\omega_0+\nu)\tau} d\tau +$$
$$+ \text{complex conjugate}]\, dt'' -$$
$$- \int_{-\infty}^{t} \left[\frac{\partial \boldsymbol{E}_0(t'')}{\partial t''} \cdot \boldsymbol{E}_0^*(t'') e^{2\nu t''} \int_{t''-t}^{\infty} \tau e^{-(i\omega_0+\nu)\tau} d\tau +\right.$$
$$+ \text{complex conjugate}]\, dt''. \tag{B 2.15}$$

Here we have used the fact that the function $\boldsymbol{E}_0(t'' - \tau)$ varies so slowly that we need only integrate over a small range of values of τ near the point t''. Effecting the integrations with respect to τ in (B 2.15), putting $\xi = t - t''$ and again expanding:

$$\boldsymbol{E}_0(t'') = \boldsymbol{E}_0(t - \xi)$$
$$= \boldsymbol{E}_0(t) - \xi \partial \boldsymbol{E}_0(t)/\partial t,$$

we obtain (B 2.13); the terms such as $(\partial^2 \boldsymbol{E}_0/\partial t^2) \cdot \boldsymbol{E}_0^*$ and $(\partial \boldsymbol{E}_0/\partial t) \cdot (\partial \boldsymbol{E}_0^*/\partial t)$ are omitted, since on account of the assumed slow variation of the function $\boldsymbol{E}_0(t)$ these terms are small.

From the preceding discussion the picture is now clear. In the phenomenological equation (the averaged Poynting's theorem) (B 4.9) or (B 2.12) there can enter (and in fact do enter) as characteristics of the medium only the frequency-dependent functions ε, σ and their derivatives. The usual interpretation of the term $\boldsymbol{j} \cdot \boldsymbol{E} = \sigma E^2$ as the heat generated per unit time and volume is in general incorrect. In a harmonic (monochromatic) field, where $\boldsymbol{E}_0(t) =$ constant, we in fact have

$$\frac{1}{2}\sigma \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* = \frac{e^2 N \nu}{2m(\omega_0^2 + \nu^2)} \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*$$
$$= m N \nu \overline{u^2}$$

[see (B 2.12) and (B 2.13)]. In the more general case, and in particular in a quasimonochromatic field (B 2.2), $\frac{1}{2}\sigma \boldsymbol{E}_0 \cdot \boldsymbol{E}_0 \neq mN\nu\overline{u^2}$. In other words, in a phenomenological equation such as (B 4.9) [and, in particular, in (B 2.12)] each term on the left-hand side has in general no definite significance as regards energy (change of internal energy, dissipation, etc.). It is for this reason, as already mentioned in § B 1, that in the presence of absorption the expression

$$\frac{d(\omega\,\varepsilon)}{d\omega}\,\frac{\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*}{16\pi}$$

may be negative, and this result involves no paradox.

In the absence of absorption, on the other hand, we can phenomenologically find an expression for the mean density of internal energy in a quasimonochromatic field in the medium, simply because there is no term other than $\partial \overline{W}/\partial t$ on the left-hand side in Poynting's theorem (B 4.9). It is also not surprising, of course, that the use of a certain model permits the derivation of an expression for the energy density and the dissipation, i.e. of a more detailed picture of the course of processes in the medium. Making the model more specific obviously involves additional assumptions which take us beyond the limits of macroscopic electrodynamics. This is seen, in particular, from the fact that systems with the same $\varepsilon(\omega)$ and $\sigma(\omega) \neq 0$ may have entirely different energies, depending on the nature and mechanism of the dissipation.†

† These remarks are in accordance with the following result in the theory of electrical circuits, communicated to the author by L. A. Vaĭnshteĭn: for a given impedance $Z(\omega) = R + iX$ (where $R \neq 0$) the inductance L and the capacity C of a circuit, and therefore its energy, may have various values. An example is the circuit in Fig. B.1, whose impedance is always $Z = R$ for $L = \varkappa R$, $C = \varkappa/R$, $R = \sqrt{(L/C)}$ for all $\varkappa$. It may also be noted that the above discussion of equation (B 2.1) is evidently applicable in its entirety to the LCR circuit.

For a plasma, the model used here, which is based on equation (B 2.1), is suitable for finding ε and σ (for $\nu(v)$ = constant), and also for determining the temperature of the plasma in steady-state conditions (see, for instance,

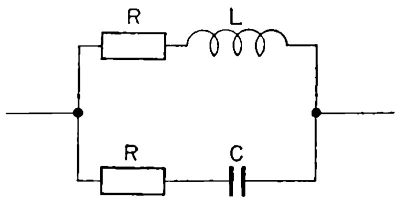

FIG. B.1.

§§ 3 and 38, and [258]). The model is not suitable, however, for considering more general cases.

§ B 3. THE APPLICATION OF THE BOLTZMANN EQUATION

In order to elucidate and develop certain points discussed above, and also on account of the great importance of plasmas, we shall here consider the energy relations when the Boltzmann equation is used.†

The electron distribution function $f(t, \boldsymbol{r}, \boldsymbol{v})$ satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c) \cdot \mathbf{grad}_v f + S = 0, \qquad \text{(B 3.1)}$$

where S is the collision integral, $\int f \, d\boldsymbol{v} = N$ and the term $\boldsymbol{v} \cdot \mathbf{grad}_{\boldsymbol{r}} f$ has been omitted; to include this term would correspond to taking account of spatial dispersion. The formulae used here and below are described in more detail in § 4 and [258]. Neglecting the motion of the ions, as we shall do henceforward, we have

$$\begin{aligned} \boldsymbol{j}_t &= \boldsymbol{j} + \frac{1}{4\pi} \frac{\partial(\boldsymbol{D} - \boldsymbol{E})}{\partial t} \\ &= e \int \boldsymbol{v} f \, d\boldsymbol{v}. \end{aligned} \qquad \text{(B 3.2)}$$

The kinetic energy of the electrons and its time derivative are respectively

$$\left.\begin{aligned} K_t &= \int \tfrac{1}{2} m v^2 f \, d\boldsymbol{v}, \\ \frac{\partial K_t}{\partial t} &= \int \frac{1}{2} m v^2 \frac{\partial f}{\partial t} d\boldsymbol{v}. \end{aligned}\right\} \qquad \text{(B 3.3)}$$

† To describe a gaseous plasma containing electrons, ions and neutral particles (molecules) the Boltzmann-equation method is applicable to a very wide extent (see § 4 and [349]).

We may note also that the following relation is easily obtained on integration by parts:

$$\int \tfrac{1}{2} e v^2 (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{H}/c) \cdot \mathbf{grad}_v f \, d\boldsymbol{v}$$
$$= -e \int \boldsymbol{v} \cdot \boldsymbol{E} f \, d\boldsymbol{v}$$
$$= -\frac{1}{4\pi} \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{j} \cdot \boldsymbol{E} + \frac{\partial}{\partial t}\left(\frac{E^2}{8\pi}\right) \qquad \text{(B 3.4)}$$

[see (B 3.2); in integrating terms of the type $\int \mathrm{div}_v \boldsymbol{F}(\boldsymbol{v})\, d\boldsymbol{v}$ we have used the fact that the respective functions $\boldsymbol{F}(\boldsymbol{v})$ decrease sufficiently rapidly as $\boldsymbol{v} \to \infty$]. Multiplying equation (B 2.1) by $\frac{1}{2} m v^2$ and integrating with respect to velocity, we find, using (B 3.3) and (B 3.4),

$$\frac{\partial}{\partial t}\left(K_t + \frac{E^2}{8\pi}\right) = \frac{1}{4\pi} \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j} \cdot \boldsymbol{E} - \int \frac{1}{2} m v^2 S \, d\boldsymbol{v}. \qquad \text{(B 3.5)}$$

Together with Poynting's theorem (B 1.2) this gives

$$\frac{\partial}{\partial t}\left(\frac{E^2 + H^2}{8\pi} + K_t\right) = -\frac{c}{4\pi} \mathrm{div}\,(\boldsymbol{E} \times \boldsymbol{H}) - \int \frac{1}{2} m v^2 S \, d\boldsymbol{v}. \qquad \text{(B 3.6)}$$

This expression for the law of conservation of energy has an evident significance. Hence, apart from the form of the last term, we can immediately write down a similar equation for any gas, replacing K_t by the sum of the kinctic and internal energies of the particles.

The last term in (B 3.6) is the energy transmitted by the electrons to the heavy particles (ions and molccules of mass $M \gg m$). This is a slow process, and for elastic collisions its rate is proportional to the ratio m/M. In certain conditions the transfer of energy to the heavy particles in a given time can be entirely neglected. This, however, does not mean that collisions can be entirely neglected and the conductivity σ equated to zero. The reason is, of course, that the transfer of momentum by an electron is a fast process (not involving the ratio m/M) and the conductivity is in fact dctermined by the effective collision frequency for momentum transfcr.

For a very wide range of problems the clectron distribution function can be put in the form

$$f(t, \boldsymbol{r}, \boldsymbol{v}) = f_0(t, \boldsymbol{r}, v) + \boldsymbol{f}_1(t, \boldsymbol{r}, v) \cdot \boldsymbol{v}/v, \qquad \text{(B 3.7)}$$

and the Boltzmann equation, omitting terms involving spatial derivatives, becomes the two equations

$$\frac{\partial f_0}{\partial t} + \frac{e}{3 m v^2} \frac{\partial}{\partial v} (v^2 \boldsymbol{E} \cdot \boldsymbol{f}_1) + S_0 = 0; \qquad \text{(B 3.8)}$$

$$\frac{\partial \boldsymbol{f}_1}{\partial t} + \frac{e\boldsymbol{E}}{m} \frac{\partial f_0}{\partial v} + \frac{e}{mc} \boldsymbol{H} \times \boldsymbol{f}_1 + \boldsymbol{S}_1 = 0. \qquad \text{(B 3.8a)}$$

Here $S_1 = \nu(v) f_1$, and roughly speaking $S_0 \sim (m/M)\,\nu(f_0 - f_{00})$, f_{00} being the equilibrium (Maxwellian) distribution function and ν the frequency of collisions which cause an appreciable change in momentum; it is this quantity, when independent of velocity, which appears in formula (B 1.5).

From (B 3.8), and qualitatively also from the problem itself, we see that in the absence of collisions the kinetic energy does not increase in a variable field of constant amplitude. When collisions are taken into account, however, and S_0 is zero, the function f_0 and the kinetic energy K_t contain terms which increase with time. This heating of the plasma by the field is limited either by the transfer of energy to the heavy particles (when $S_0 \neq 0$) or by processes neglected in (B 3.8) and (B 3.8a) (thermal conduction, radiation, etc.).

Let us now consider an isotropic plasma (external magnetic field $\boldsymbol{H}^{(0)} = 0$) in the linear approximation. In equation (B 3.8a) the function f_0 may then be taken as the unperturbed function $f_{00} = f_0(t = -\infty)$ (in a state of equilibrium, of course, f_{00} is the Maxwellian function), and we can also put $\boldsymbol{H} = 0$. The result is

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{f}_1}{\partial t} + \frac{e\boldsymbol{E}}{m}\,\frac{\partial f_{00}(v)}{\partial v} + \nu(v)\,\boldsymbol{f}_1 = 0;\\ &\boldsymbol{f}_1 = -\,e^{-\nu t}\,\frac{\partial f_{00}}{\partial v}\int\limits_{-\infty}^{t}\frac{e}{m}\,\boldsymbol{E}(t')\,e^{\nu t'}\,dt'. \end{aligned}\right\} \qquad \text{(B 3.9)}$$

We shall now assume a quasimonochromatic field [see (B 2.2)] and put $\boldsymbol{E}_0(t = -\infty) = 0$. Substituting (B 2.2) and (B 3.9) in equation (B 3.8) and immediately averaging over high frequencies, we obtain

$$\frac{\partial \overline{f_0}}{\partial t} - \frac{e^2}{3m^2 v^2}\,\frac{\partial}{\partial v}\left\{\frac{1}{4}\,v^2\,\frac{\partial f_{00}}{\partial v}\int\limits_{-\infty}^{t}[\boldsymbol{E}_0^*(t)\cdot\boldsymbol{E}_0(t')\,e^{(i\omega_0+\nu)(t'-t)} + \right.$$

$$\left. + \text{complex conjugate}]\,dt'\right\} + \overline{S_0} = 0. \qquad \text{(B 3.10)}$$

Putting $\tau = t - t'$ as in (B 2.15), we use the fact that the slowness of the variation of $\boldsymbol{E}_0(t') = \boldsymbol{E}_0(t - \tau)$ makes it possible to consider only small values of τ in (B 3.10), i.e. to put $\boldsymbol{E}_0(t - \tau) = \boldsymbol{E}_0(t) - \tau\,\partial\boldsymbol{E}_0(t)/\partial t$. This gives

$$\frac{\partial \overline{f_0}}{\partial t} - \frac{e^2}{6m^2v^2}\,\frac{\partial}{\partial v}\left\{\frac{v^2(\partial f_{00}/\partial v)\,\nu(v)}{\omega^2 + \nu^2(v)}\right\}\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* - \frac{e^2}{12m^2v^2}\,\times$$

$$\times\,\frac{\partial}{\partial v}\left\{v^2\,\frac{\partial f_{00}}{\partial v}\,\frac{(\omega_0^2 - \nu^2)\,\partial(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)/\partial t - 2i\omega_0\nu[(\partial\boldsymbol{E}_0/\partial t)\cdot\boldsymbol{E}_0^* - \boldsymbol{E}_0\cdot(\partial\boldsymbol{E}_0^*/\partial t)]}{(\omega_0^2 + \nu^2)^2}\right\} +$$

$$+\,\overline{S_0} = 0. \qquad \text{(B 3.11)}$$

Hence we have, using (B 3.3),

$$\frac{\partial \bar{K}_t}{\partial t} = \int \overline{\frac{1}{2} m v^2 \frac{\partial f}{\partial t}} \, d\boldsymbol{v}$$

$$= \int \frac{1}{2} m v^2 \frac{\partial \bar{f}_0}{\partial t} \, d\boldsymbol{v}$$

$$= \frac{e^2(\omega_0^2 - \nu^2) N}{4m(\omega_0^2 + \nu^2)^2} \frac{\partial}{\partial t} \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* + \frac{e^2 N \nu}{2m(\omega_0^2 + \nu^2)} \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* +$$

$$+ \frac{i e^2 N \omega_0 \nu}{2m(\omega_0^2 + \nu^2)^2} \left(\frac{\partial \boldsymbol{E}_0}{\partial t} \cdot \boldsymbol{E}_0^* - \boldsymbol{E}_0 \cdot \frac{\partial \boldsymbol{E}_0^*}{\partial t} \right) - \int \overline{\frac{1}{2} m v^2 S_0} \, d\boldsymbol{v}, \qquad \text{(B 3.12)}$$

where, in integrating with respect to velocity, we have put $\nu(v) = \nu =$ constant and used the normalisation condition $\int f \, d\boldsymbol{v} = 4\pi \int f_{00} v^2 \, dv = N$.

From (B 3.12) and (B 3.6) we obtain (B 2.12). This was to be expected, since for $\nu(v) = \nu =$ constant the kinetic theory leads to the same expressions (B 1.5) for ε and σ as the "elementary theory" based on equation (B 2.1).

If we compare (B 3.12) and (B 2.13), we easily see that

$$\frac{\partial \bar{K}_t}{\partial t} = \frac{\partial \overline{NK}}{\partial t} = \frac{\partial}{\partial t} \left\{ \frac{4\pi e^2 N}{m(\omega_0^2 + \nu^2)} \frac{\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^*}{16\pi} \right\} \qquad \text{(B 3.13)}$$

only if

$$m N \nu \overline{u^2} = \int \tfrac{1}{2} m v^2 \overline{S_0} \, d\boldsymbol{v}. \qquad \text{(B 3.14)}$$

In other words, the change in the total kinetic energy $\bar{K}_t$ of the electrons is equal to the change in the kinetic energy $\overline{NK}$ due to the directed velocity $\boldsymbol{u}$ only for a quite definite heat removal process: the condition (B 3.14) signifies that the work done by the frictional force (after averaging over high frequencies) is exactly equal to the energy transmitted by the electrons to the heavy particles. This requirement corresponds entirely, of course, to the mechanical significance of equation (B 2.1), in which the force of friction has been included. But in a real plasma, apart from the steady-state case $\boldsymbol{E}_0 =$ constant, the relation (B 3.14) in general does not hold.

The characteristic time for the transfer of energy by electrons to heavy particles in elastic collisions is $\tau_E = 1/\delta\nu = \tau \div 2m/M$, where $\nu = 1/\tau$ is the collision frequency in formulae (B 1.5) and elsewhere. When $\tau_E \gtrsim T$ (where T is the characteristic time of variation of the amplitude $\boldsymbol{E}_0(t)$), we cannot say that the relation (B 3.14) is exactly valid; if $\tau_E \ll T$ this relation is possible, but a special analysis is necessary: it is clear from (B 2.13) that the expression for $m N \nu \overline{u^2}$ is relatively complicated when the terms involving time derivatives are taken into account, as they must be in this case, and so the conditions for equation (B 3.14) to hold are not obvious. We have not examined this problem thoroughly, but the general situation is not dependent on whatever the answer may be in this respect. If the Boltzmann equation is solved, the law of conservation of energy need not be used, and it will automatically be satisfied; but we evidently cannot proceed conversely and find the energy $\overline{K}_t$ of the electrons in a given quasimonochromatic field without making a detailed analysis of the process of energy removal from the electrons. In other words, the expression (B 2.4) for the energy density $\overline{W'_E}$ is a particular form valid only under certain conditions.

In conclusion, we may consider the problem of calculating the "energy velocity" of a quasimonochromatic signal in an absorbing medium. This is taken as

$$v_{\mathrm{en}} = \overline{S}_p / \overline{W} \tag{B 3.15}$$

(see § 22 and [143, 144]), where $\overline{S_p}$ is the mean value of the energy flux $\boldsymbol{S}_p = c\boldsymbol{E} \times \boldsymbol{H}/4\pi$ in a plane wave and $\overline{W}$ is the mean field energy density in the medium.

Using for $\overline{W'_E} = \overline{W} - \overline{H^2}/8\pi$ the expression (B 2.4), we obtain from (B 3.15)

$$\begin{aligned} v_{\mathrm{en}} &= \frac{2c \operatorname{re} \sqrt{\varepsilon'}}{|\varepsilon'| - \varepsilon + 2} \\ &= \frac{cn}{1 + \varkappa^2}, \end{aligned} \tag{B 3.16}$$

where $\varepsilon' = \varepsilon - i \cdot 4\pi\sigma/\omega = (n - i\varkappa)^2$ with n and $\varkappa$ the indices of refraction and absorption respectively.

It is not clear, however, why we must use in (B 3.15) the expression (B 2.4), which represents a very particular case (see above), although there is some justification for taking the energy W'_E, in the hypothesis that this expression is a minimum [144].

The complete solution of the problem of the signal velocity, in our opinion, can be found only by analysing the propagation of the signal in the medium (in this case an absorbing plasma) by considering the appropriate Fourier integral. The integrand in question (see e.g. § 22) for a homogeneous isotropic medium is entirely determined by the functions $\varepsilon(\omega)$ and $\sigma(\omega)$. Hence, in principle, we can always ascertain (independently of energy considerations) whether it is meaningful to speak of any definite signal velocity in the medium concerned when appreciable absorption in present.† In conditions where this concept of velocity is used with equation (B 3.15) (here we are really using this equation as a definition) we can find some "energy" $\overline{W}_{\mathrm{en}} = \overline{S_p}/v_{\mathrm{en}}$. Thus the

† If the coefficient of absorption $2\omega\varkappa/c$ is independent of frequency, the signal velocity for any absorption is equal to the group velocity $v_{\mathrm{gr}} = c \div d(\omega n)/d\omega$ (§ 22). This is, however, a very exceptional case; as a rule, a signal propagated in a medium with strong absorption will be considerably distorted. It is then necessary to specify more precisely what is meant by the velocity of the signal, and it is not clear *a priori* whether this concept is meaningful. In [143, 144] the "centre of energy" of the signal is considered.

expression proposed in [143, 144] for v_{cn}, which for a plasma becomes (B 3.16), will be valid if $\overline{W}_{\text{cn}} = \overline{W}'_E + \overline{H^2}/8\pi$ (see above). At present the problem of the value of the signal velocity in an absorbing medium (in particular, in a plasma with $\nu \neq 0$) is still unclear.

§ B 4. THE FORM OF POYNTING'S THEOREM FOR A QUASIMONOCHROMATIC FIELD IN AN ARBITRARY LINEAR MEDIUM AT REST

Let us consider quasimonochromatic real fields:

$$\left.\begin{aligned} \boldsymbol{E}(t) &= \tfrac{1}{2}\left[\boldsymbol{E}_0(t)\, e^{i\omega_0 t} + \boldsymbol{E}_0^*(t)\, e^{-i\omega_0 t}\right] = \int_0^\infty \left[\boldsymbol{g}(\omega)\, e^{i\omega t} + \boldsymbol{g}^*(\omega)\, e^{-i\omega t}\right] d\omega; \\ \boldsymbol{H}(t) &= \tfrac{1}{2}\left[\boldsymbol{H}_0(t)\, e^{i\omega_0 t} + \boldsymbol{H}_0^*(t)\, e^{-i\omega_0 t}\right] = \int_0^\infty \left[\boldsymbol{g}_1(\omega)\, e^{i\omega t} + \boldsymbol{g}_1^*(\omega)\, e^{-i\omega t}\right] d\omega, \end{aligned}\right\} \quad \text{(B 4.1)}$$

where $\boldsymbol{E}_0(t)$ and $\boldsymbol{H}_0(t)$ are slowly varying functions of time (the characteristic frequency of variation of $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ being small compared with the "high frequency" ω_0). Any dependence of $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ on the coordinates is unimportant, because the discussion below applies entirely to any one point in space. Then we have for a linear medium at rest without spatial dispersion† and for constant $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$

$$\left.\begin{aligned} D_i &= \tfrac{1}{2}\left[\varepsilon_{ik}(\omega_0)\, E_{0k} e^{i\omega_0 t} + \varepsilon_{ik}^*(\omega_0)\, E_{0k}^* e^{-i\omega_0 t}\right]; \\ j_i &= \tfrac{1}{2}\left[\sigma_{ik}(\omega_0)\, E_{0k} e^{i\omega_0 t} + \text{complex conjugate}\right]; \\ \varepsilon'_{ik} &= \varepsilon_{1ik} - i\varepsilon_{2ik} \equiv \varepsilon_{ik} - i \cdot 4\pi\sigma_{ik}/\omega; \\ \partial \boldsymbol{B}/\partial t &= \partial \boldsymbol{B}_1/\partial t + 4\pi \boldsymbol{B}_2; \\ B_{1i} &= \tfrac{1}{2}\left[\mu_{ik}(\omega_0)\, H_{0k} e^{i\omega_0 t} + \text{complex conjugate}\right]; \\ B_{2i} &= \tfrac{1}{2}\left[\lambda_{ik}(\omega_0)\, H_{0k} e^{i\omega_0 t} + \text{complex conjugate}\right]; \\ \mu'_{ik} &= \mu_{1ik} - i\mu_{2ik} \equiv \mu_{ik} - i \cdot 4\pi\lambda_{ik}/\omega. \end{aligned}\right\} \quad \text{(B 4.2)}$$

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† The presence of spatial dispersion signifies that $\boldsymbol{D}$ and $\boldsymbol{B}$ at a given point depend on $\boldsymbol{E}$ and $\boldsymbol{H}$ not only at that point but also in some neighbourhood of it (see e.g. § 8 and [349]). The allowance for spatial dispersion is in a homogeneous medium always [349] and in crystals only under certain conditions [357] equivalent to using tensors $\varepsilon'_{ik}(\omega, \boldsymbol{k})$ and $\mu'_{ik}(\omega, \boldsymbol{k})$, where $\boldsymbol{k}$ is the wave vector (all quantities $\boldsymbol{E}, \boldsymbol{D}$, etc. are assumed proportional to $e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$; in the presence of spatial dispersion the tensors ε'_{ik} and μ'_{ik} are not independent [349, 357]). When the tensors $\varepsilon'_{ik}(\omega, \boldsymbol{k})$ and $\mu'_{ik}(\omega, \boldsymbol{k})$ are used in (B 4.2) the results obtained below are directly applicable only to fields whose dependence on the coordinates is given by a factor $e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$. It is also easy to repeat the calculations for fields of the type

$$\boldsymbol{E}(t, \boldsymbol{r}) = \tfrac{1}{2}\left[\boldsymbol{E}_0(t, \boldsymbol{r})\, e^{i(\omega_0 t - \boldsymbol{k}_0\cdot\boldsymbol{r})} + \boldsymbol{E}_0^*(t, \boldsymbol{r})\, e^{-i(\omega_0 t - \boldsymbol{k}_0\cdot\boldsymbol{r})}\right],$$

where $\boldsymbol{E}_0$ varies only slowly in time and space. This problem is discussed in Appendix C, where an equation is derived which generalises (B 4.10) to the case where spatial dispersion is present.

Here $\boldsymbol{D}$, $\boldsymbol{j}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ are real (when $\boldsymbol{E}$ and $\boldsymbol{H}$ are real). This requirement gives the conditions [already used in (B 4.2)]

$$\left.\begin{aligned} \varepsilon_{ik}(-\omega) = \varepsilon^*_{ik}(\omega), \quad \sigma_{ik}(-\omega) = \sigma^*_{ik}(\omega), \\ \mu_{1,2ik}(-\omega) = \mu^*_{1,2ik}(\omega). \end{aligned}\right\} \tag{B 4.3}$$

In a gyrotropic (magnetoactive) medium† $\varepsilon_{ik} = \varepsilon^*_{ki}$, $\sigma_{ik} = \sigma^*_{ki}$, and similarly for $\mu_{1,2ik}$. In a non-gyrotropic but anisotropic medium ε_{ik}, σ_{ik} and $\mu_{1,2ik}$ are real quantities, with $\varepsilon_{ik} = \varepsilon_{ki}$, etc. Finally, in an isotropic medium, $\varepsilon_{ik} = \varepsilon\delta_{ik}$, etc.

In what follows, in order to avoid many suffixes, we shall give the calculations for an isotropic medium with $\mu' = 1$, and finally write out the general result.

In an isotropic medium, using the relations (B 4.3) for real ε and σ we have

$$\left.\begin{aligned}
\boldsymbol{D}(t) &= \int\limits_0^\infty \varepsilon(\omega)\,[\boldsymbol{g}(\omega)\,e^{i\omega t} + \boldsymbol{g}^*(\omega)\,e^{-i\omega t}]\,d\omega;\\
\boldsymbol{j}(t) &= \int\limits_0^\infty \sigma(\omega)\,[\boldsymbol{g}(\omega)\,e^{i\omega t} + \boldsymbol{g}^*(\omega)\,e^{-i\omega t}]\,d\omega;\\
\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} &= \frac{1}{4\pi}\int\limits_0^\infty\int\limits_0^\infty [\boldsymbol{g}(\omega)\cdot\boldsymbol{g}(\omega')\,i\omega\,\varepsilon(\omega)\,e^{i(\omega+\omega')t} +\\
&+ \boldsymbol{g}(\omega)\cdot\boldsymbol{g}^*(\omega')\,i\omega\,\varepsilon(\omega)\,e^{i(\omega-\omega')t} + \text{complex conjugate}]\,d\omega\,d\omega';\\
\boldsymbol{j}\cdot\boldsymbol{E} &= \int\limits_0^\infty\int\limits_0^\infty \sigma(\omega)[\boldsymbol{g}(\omega)\cdot\boldsymbol{g}(\omega')\,e^{i(\omega+\omega')t} + \boldsymbol{g}(\omega)\cdot\boldsymbol{g}^*(\omega')\,e^{i(\omega-\omega')t} +\\
&+ \text{complex conjugate}]\,d\omega\,d\omega'.
\end{aligned}\right\} \tag{B 4.4}$$

We now average these expressions over high frequencies, i.e. over times long compared with $2\pi/\omega_0$ but short compared with the characteristic time of variation of the amplitude $\boldsymbol{E}_0(t)$. This averaging is equivalent to neglecting terms in $e^{\pm i(\omega+\omega')t}$ in comparison with those in $e^{\pm i(\omega-\omega')t}$, and is denoted by a bar††; we take into account also the fact that for a quasimonochromatic

† The separation of ε'_{ik} into ε_{ik} and σ_{ik} in a gyrotropic medium is made on the basis of the requirement that the tensor ε_{ik} should not contribute to the absorption. This gives immediately the condition $\varepsilon_{ik} = \varepsilon^*_{ki}$. The absorption is then proportional to $\sigma^*_{ik} + \sigma_{ki}$, i.e. the anti-Hermitian part of σ_{ik} makes no contribution to the absorption and can be assigned to ε_{ik}.

†† Formally, this averaging denotes, for a function $f(t)$ of the type (B 4.1),

$$\overline{f(t)} = \frac{1}{\tau_2 - \tau_1}\int\limits_{\tau_1}^{\tau_2} f(t+\tau)\,d\tau$$

$$(\tau_2 > t > \tau_1, \quad T \gg \tau_1 \sim \tau_2 \gg 2\pi/\omega_0),$$

where T is the characteristic time of variation of the amplitude. Evidently $\overline{\partial f/\partial t} = \partial \overline{f}/\partial t$.

field the function $\boldsymbol{g}(\omega)$ has a sharp maximum near the frequency ω_0. As a first approximation we can therefore put in (B 4.4) $\omega\varepsilon(\omega) = \omega_0\varepsilon(\omega_0) + [\varepsilon(\omega_0) + \omega_0(d\varepsilon/d\omega)_0]\,\Omega$ and $\sigma = \sigma(\omega)_0 + (d\sigma/d\omega)_0\Omega$, where $\omega = \omega_0 + \Omega$. The result is

$$\overline{\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}} = \frac{1}{4\pi}\left[\frac{d(\omega\varepsilon)}{d\omega}\right]_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[i\Omega\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega';$$

$$\overline{\boldsymbol{j}\cdot\boldsymbol{E}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\sigma(\omega_0) + \left(\frac{d\sigma}{d\omega}\right)_0\Omega\right][\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega', \tag{B 4.5}$$

where the lower limit $-\omega_0$ has been replaced by $-\infty$ and we have used the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[i\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega' = 0. \tag{B 4.6}$$

From (B 4.1) we have

$$\left.\begin{aligned}
\overline{E^2} &= \tfrac{1}{2}\boldsymbol{E}_0(t)\cdot\boldsymbol{E}_0^*(t)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega';\\
\frac{\partial\overline{E^2}}{\partial t} &= \overline{\frac{\partial E^2}{\partial t}} = \frac{\partial}{\partial t}\left[\frac{1}{2}\boldsymbol{E}_0(t)\cdot\boldsymbol{E}_0^*(t)\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[i(\Omega-\Omega')\,\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega'\\
&= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[i\Omega\,\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugatc}]\,d\Omega\,d\Omega';\\
\frac{\partial\boldsymbol{E}_0}{\partial t}\cdot\boldsymbol{E}_0^* - \frac{\partial\boldsymbol{E}_0^*}{\partial t}\cdot\boldsymbol{E}_0 &= 4i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega[\boldsymbol{g}(\omega_0+\Omega)\cdot\boldsymbol{g}^*(\omega_0+\Omega')\,e^{i(\Omega-\Omega')t} + \text{complex conjugate}]\,d\Omega\,d\Omega',
\end{aligned}\right\} \tag{B 4.7}$$

since

$$E_0 e^{i\omega_0 t} = 2 \int_0^\infty g(\omega)\, e^{i\omega t} d\omega;$$

$$\frac{\partial (E_0 e^{i\omega_0 t})}{\partial t} \cdot E_0^* e^{-i\omega_0 t} = i\omega_0 E_0 \cdot E_0^* + \frac{\partial E_0}{\partial t} \cdot E_0^*$$

$$= i\omega_0 E_0 \cdot E_0^* + 4i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Omega\, g(\omega_0 + \Omega) \cdot g^*(\omega_0 + \Omega') \times$$

$$\times e^{i(\Omega - \Omega')t} d\Omega\, d\Omega'.$$

It is clear from (B 4.5) and (B 4.7) that for a quasimonochromatic field

$$\left.\begin{aligned} \overline{\frac{1}{4\pi} E \cdot \frac{\partial D}{\partial t}} &= \frac{1}{16\pi} \frac{d(\omega\varepsilon)}{d\omega} \frac{\partial (E_0 \cdot E_0^*)}{\partial t}; \\ \overline{j \cdot E} &= \frac{1}{2} \sigma E_0 \cdot E_0^* - \frac{1}{4} i \frac{d\sigma}{d\omega} \left(\frac{\partial E_0}{\partial t} \cdot E_0^* - \frac{\partial E_0^*}{\partial t} \cdot E_0 \right), \end{aligned}\right\} \tag{B 4.8}$$

where the derivatives with respect to ω are taken at the frequency ω_0.

The averaged equation (B 1.2) now becomes

$$\frac{\partial}{\partial t}\left\{\frac{[d(\omega\varepsilon)/d\omega]\, E_0 \cdot E_0^* + H_0 \cdot H_0^*}{16\pi}\right\} + \frac{1}{2} \sigma E_0 \cdot E_0^* -$$

$$- \frac{1}{4} i \frac{d\sigma}{\partial\omega} \left(\frac{\partial E_0}{\partial t} \cdot E_0^* - E_0 \cdot \frac{\partial E_0^*}{\partial t} \right)$$

$$= -\frac{c}{16\pi} \operatorname{div} (E_0 \times H_0^* + E_0^* \times H_0). \tag{B 4.9}$$

Apart from the term in $d\sigma/d\omega$, which is sometimes of slight importance†, equation (B 4.9) is essentially that which is commonly used.

It may be noted that in [356] an equation of the type (B 1.2) has erroneously been written for the total field $E(t)$. This incorrect result was obtained because in [356] the functions $g(\omega)$ and $g(-\omega)$ are assumed to have sharp maxima only at $\omega = \omega_0$. For $g(\omega)$ this is true, but $g(-\omega)$ has a maximum at $\omega = -\omega_0$ which leads to the retention in the expression $(1/4\pi)\, E \cdot \partial D/\partial t$ of high-frequency terms which give zero on averaging.

We shall now write out equation (B 1.2) averaged over high frequencies for the general case (B 4.2)††:

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† Nevertheless, this term cannot justifiably be omitted. The author's thanks are due to V. P. Silin for pointing this out and for discussion of the whole problem.

†† Here, of course, the term $(1/8\pi)\, \partial H^2/\partial t$ in (B 1.2) is replaced by $(1/4\pi)\, H \cdot \partial B/\partial t$.

$$\frac{1}{32\pi}\frac{\partial}{\partial t}\left\{\frac{d[\omega(\varepsilon_{ik}^*+\varepsilon_{ki})]}{d\omega}E_{0i}E_{0k}^*+\frac{d[\omega(\mu_{ik}^*+\mu_{ki})]}{d\omega}H_{0i}H_{0k}^*\right\}+$$
$$+\frac{1}{4}(\sigma_{ik}^*+\sigma_{ki})\,E_{0i}E_{0k}^*+\frac{1}{4}(\lambda_{ik}^*+\lambda_{ki})\,H_{0i}H_{0k}^*-$$
$$-\frac{1}{4}i\left(\frac{d\sigma_{ik}^*}{d\omega}\frac{\partial E_{0i}}{\partial t}E_{0k}^*-\frac{d\sigma_{ik}}{d\omega}\frac{\partial E_{0i}^*}{\partial t}E_{0k}\right)-$$
$$-\frac{1}{4}i\left(\frac{d\lambda_{ik}^*}{d\omega}\frac{\partial H_{0i}}{\partial t}H_{0k}^*-\frac{d\lambda_{ik}}{d\omega}\frac{\partial H_{0i}^*}{\partial t}H_{0k}\right)$$
$$=-\frac{c}{16\pi}\operatorname{div}(\boldsymbol{E}_0^*\times\boldsymbol{H}_0+\boldsymbol{E}_0\times\boldsymbol{H}_0^*). \tag{B 4.10}$$

Since all the tensors are Hermitian we can here, of course, replace $\varepsilon_{ik}^*+\varepsilon_{ki}$ by $2\varepsilon_{ki}$ (or, interchanging suffixes, by $2\varepsilon_{ik}$), and likewise for the other tensors.

APPENDIX C

THE LAW OF CONSERVATION OF ENERGY IN THE ELECTRODYNAMICS OF MEDIA WITH SPATIAL DISPERSION†

So FAR as the author is aware, the problem of the law of conservation of energy in the electrodynamics of media with spatial dispersion has not yet been considered with sufficient completeness in the literature. For example, in [349] only an expression for the heat evolved in all space per unit time was derived. In this case, for fields of the type $e^{i(\omega_0 t - \boldsymbol{k}_0 \cdot \boldsymbol{r})}$ with real ω_0 and $\boldsymbol{k}_0$, the heat per unit volume is (see [349, formula (4.10)])

$$q = -\frac{i\omega_0}{16\pi}\left[\varepsilon_{ij}'^{*}(\omega_0, \boldsymbol{k}_0) - \varepsilon_{ji}'(\omega_0, \boldsymbol{k}_0)\right] E_{0i} E_{0j}^{*}$$
$$= \tfrac{1}{2}\sigma_{ij}(\omega_0, \boldsymbol{k}_0) E_{0j} E_{0i}^{*}. \tag{C.1}$$

In the present discussion, as in [349], the field equations are used in the form

$$\operatorname{curl} \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{D}'}{\partial t}, \quad \operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}. \tag{C.2}$$

For a non-magnetic medium without spatial dispersion we have $\boldsymbol{B} = \boldsymbol{H}$ and $\partial \boldsymbol{D}'/\partial t = \partial \boldsymbol{D}/\partial t + 4\pi \boldsymbol{j}$, where $\boldsymbol{j}$ is the conduction current density. We shall also, however, use a notation differing from that in [349]: the real field is written in the form (C.4) (see below), and we put $D_i'(\boldsymbol{k}, \omega) = \varepsilon_{ij}' E_j(\boldsymbol{k}, \omega)$, $\varepsilon_{ij}'(\omega, \boldsymbol{k}) = \varepsilon_{ij}(\omega, \boldsymbol{k}) - i \cdot 4\pi\sigma_{ij}(\omega, \boldsymbol{k})/\omega$, with ε_{ij} and σ_{ij} Hermitian tensors.

Since (C.1) is an averaged quantity, the question arises of the local expression for the heat and other quantities. Moreover, when there is absorption which does not tend to zero or the medium is non-absorbing but not transparent, the vector $\boldsymbol{k}_0$ is complex. If, therefore, we relate the expression (C.1) to a small volume element, we must consider its significance and the possibility of using it for complex $\boldsymbol{k}_0$; for example, in the absence of absorption the quantity q certainly does not represent the heat when $\boldsymbol{k}_0$ is complex.

The problem here is to take account of spatial dispersion in deriving the expression for the law of conservation of energy in electrodynamics or, more precisely, in averaging over high frequencies Poynting's relation

$$\frac{1}{4\pi}\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}'}{\partial t} + \frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\frac{c}{4\pi}\operatorname{div}(\boldsymbol{E}\times\boldsymbol{B}), \tag{C.3}$$

† By V. L. Ginzburg. Published in *Izvestiya vysshikh uchebnykh zavedeniĭ: Radiofizika* 5, 473–477, 1962.

which follows from the field equations (C. 2).

The calculations below are similar to those which neglect spatial dispersion and which are described in Appendix B, and so some details of the procedure may be omitted.

Let us consider a quasimonochromatic field

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},\ t) &= \tfrac{1}{2}\left[\boldsymbol{E}_0(\boldsymbol{r},\ t)\, e^{i(\omega_0 t - \boldsymbol{k}_1\cdot\boldsymbol{r})} + \boldsymbol{E}_0^*(\boldsymbol{r},\ t)\, e^{-i(\omega_0 t - \boldsymbol{k}_1\cdot\boldsymbol{r})}\right] \\ &= \int\left[\boldsymbol{g}(\omega,\ \boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} + \boldsymbol{g}^*(\omega,\ \boldsymbol{k})\, e^{-i(\omega t - \boldsymbol{k}^*\cdot\boldsymbol{r})}\right] d\omega\, d\boldsymbol{k}. \end{aligned} \tag{C.4}$$

Here the amplitude $\boldsymbol{E}_0$ varies only slightly in a time of the order of $1/\omega_0$. The spatial variation of $\boldsymbol{E}_0$, however, is subject to a less stringent condition: we shall suppose that $\boldsymbol{E}_0(\boldsymbol{r}, t) = \boldsymbol{E}_{00}(\boldsymbol{r}, t)\, e^{-\boldsymbol{k}_2\cdot\boldsymbol{r}}$, where the amplitude and phase of $\boldsymbol{E}_{00}$ vary only slightly over distances of the order of $1/|\boldsymbol{k}_2|$ and $1/|\boldsymbol{k}_1|$ respectively (in other words, the field $\boldsymbol{E}$ depends on $\boldsymbol{r}$ almost as $e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} = e^{-\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}}$, where $\boldsymbol{k}_0 = \boldsymbol{k}_1 - i\boldsymbol{k}_2$, with the vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ real). The frequency ω_0 is assumed real, which corresponds to a particular formulation of the physical problem. Calculations similar to the following may be made for a complex frequency ω_0 and real $\boldsymbol{k}_0$, with space averaging, but we shall not pause to discuss this problem. It may also be noted that the expressions given below are meaningful only if the corresponding integrals exist and the function $\varepsilon'_{ij}(\omega, \boldsymbol{k})$ can be represented by the first two terms of an expansion in series about the point $\boldsymbol{k}_0$.

On the above assumptions, the function $\boldsymbol{g}(\omega, \boldsymbol{k})$ has a sharp maximum at $\omega = \omega_0$ and $\boldsymbol{k} = \boldsymbol{k}_0 = \boldsymbol{k}_1 - i\boldsymbol{k}_2$. The integration with respect to ω in (C. 4) must be taken along the real axis from 0 to ∞, and in integrating with respect to $\boldsymbol{k}$ the contour or region of integration must include the point $\boldsymbol{k}_0$. We shall also put

$$\omega\varepsilon'_{ij}(\omega,\ \boldsymbol{k}) = a_{ij} = \omega_0(\varepsilon'_{ij})_0 + \left(\frac{\partial(\omega\varepsilon'_{ij})}{\partial\omega}\right)_0 \Omega + \omega_0\left(\frac{\partial\varepsilon'_{ij}}{\partial k_l}\right) q_l,$$

where $\omega = \omega_0 + \Omega$, $\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{q}$, and the suffix 0 denotes values for $\omega = \omega_0$ and $\boldsymbol{k} = \boldsymbol{k}_0$. Evidently

$$\begin{aligned} \frac{\partial D'_i}{\partial t} = \int &\left[i\omega\varepsilon'_{ij}(\omega,\ \boldsymbol{k})\, g_j(\omega,\ \boldsymbol{k})\, e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} - \right. \\ &\left. - i\omega\varepsilon'^*_{ij}(\omega,\ \boldsymbol{k})\, g_j^*(\omega,\ \boldsymbol{k})\, e^{-i(\omega t - \boldsymbol{k}^*\cdot\boldsymbol{r})}\right] d\omega\, d\boldsymbol{k}. \end{aligned} \tag{C.5}$$

We substitute (C. 4) and (C. 5) in (C. 3) and effect the averaging over a time interval large compared with $2\pi/\omega_0$ but small compared with the characteristic time of variation of the function $\boldsymbol{E}_0(\boldsymbol{r}, t)$. The result is

$$\begin{aligned} \frac{1}{4\pi}\overline{\boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}'}{\partial t}} = \frac{1}{4\pi}\int &\left\{i a_{ij}\, g_j(\Omega,\ \boldsymbol{q})\, g_i^*(\Omega',\ \boldsymbol{q}')\, e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{i[(\Omega-\Omega')t - (\boldsymbol{q}-\boldsymbol{q}'^*)\cdot\boldsymbol{r}]} - \right. \\ - i a_{ij}^* g_j^*(\Omega, \boldsymbol{q})\, g_i(\Omega',\ \boldsymbol{q}')\, &\left. e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{-i[(\Omega-\Omega')t - (\boldsymbol{q}^*-\boldsymbol{q}')\cdot\boldsymbol{r}]}\right\} d\Omega\, d\Omega'\, d\boldsymbol{q}\, d\boldsymbol{q}'. \end{aligned} \tag{C.6}$$

It is easy to see that

$$\left.\begin{aligned}
E_{0i}E_{0j}^* &= 4\int g_i(\Omega,\, \boldsymbol{q})\, g_j^*(\Omega',\, \boldsymbol{q}') \times \\
&\qquad \times e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{i[(\Omega-\Omega')t-(\boldsymbol{q}-\boldsymbol{q}'^*)\cdot\boldsymbol{r}]}\, d\Omega\, d\Omega'\, d\boldsymbol{q}\, d\boldsymbol{q}';\\
E_{0i}\frac{\partial E_{0j}^*}{\partial x_l} &= 4\int (iq_l^* - k_{2,l})\, g_i(\Omega',\, \boldsymbol{q}')\, g_j^*(\Omega,\, \boldsymbol{q}) \times \\
&\qquad \times e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{-i[(\Omega-\Omega')t-(\boldsymbol{q}^*-\boldsymbol{q}')\cdot\boldsymbol{r}]}\, d\Omega\, d\Omega'\, d\boldsymbol{q}\, d\boldsymbol{q}';\\
\frac{\partial}{\partial x_l}(E_{0i}E_{0j}^*) &= 4\int \{iq_l^*\, g_i(\Omega',\, \boldsymbol{q}')\, g_j^*(\Omega,\, \boldsymbol{q}) \times \\
&\qquad \times e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{-i[(\Omega-\Omega')t-(\boldsymbol{q}^*-\boldsymbol{q}')\cdot\boldsymbol{r}]} - (iq_l + 2k_{2,l}) \times \\
&\times g_i(\Omega,\boldsymbol{q})\, g_j^*(\Omega',\, \boldsymbol{q}')\, e^{-2\boldsymbol{k}_2\cdot\boldsymbol{r}}\, e^{i[(\Omega-\Omega')t-(\boldsymbol{q}-\boldsymbol{q}'^*)\cdot\boldsymbol{r}]}\}\, d\Omega\, d\Omega'\, d\boldsymbol{q}\, d\boldsymbol{q}'.
\end{aligned}\right\} \tag{C.7}$$

The corresponding expressions for $E_{0i}\,\partial E_{0j}^*/\partial t$ and $\partial(E_{0i}E_{0j}^*)/\partial t$ will not be given here (see Appendix B).

It now remains to separate the tensor $a_{ij} = b_{ij} - ic_{ij}$ in (C. 6) into Hermitian and anti-Hermitian parts in accordance with the formula $\varepsilon_{ij}'(\omega, \boldsymbol{k}) = \varepsilon_{ij} - i\cdot 4\pi\sigma_{ij}/\omega$, where ε_{ij} and σ_{ij} are Hermitian. For this purpose we put

$$\left(\frac{\partial \varepsilon_{ij}'}{\partial k_l}\right)_0 = \left(\frac{\partial \varepsilon_{ij}}{\partial k_l}\right)_0 - i\cdot\frac{4\pi}{\omega_0}\left(\frac{\partial \sigma_{ij}}{\partial k_l}\right)_0;$$

$$\left(\frac{\partial \sigma_{ij}}{\partial k_l}\right)_0^* = \left(\frac{\partial \sigma_{ij}^*}{\partial k_l}\right)_0, \quad \text{etc.,}$$

although for complex $\boldsymbol{k}_0$ we need only take the Hermitian and anti-Hermitian parts of $(\partial\varepsilon_{ij}'/\partial k_l)_0$ and their complex conjugates, and not derivatives such as $(\partial\varepsilon_{ij}/\partial k_l)_0$. Our notation is in some ways convenient, since for real $\boldsymbol{k}_0$ we do in fact have the derivatives of ε_{ij} and σ_{ij} with respect to k_l (the same is true, of course, regarding derivatives with respect to ω when the frequency ω_0 is real). Then, using (C. 7) we have the final result

$$\begin{aligned}
&\frac{1}{16\pi}\frac{\partial}{\partial t}\left[\left(\frac{\partial(\omega\varepsilon_{ij})}{\partial\omega}\right)_0 E_{0j}E_{0i}^* + \boldsymbol{B}_0\cdot\boldsymbol{B}_0^*\right] + \frac{1}{2}(\sigma_{ij})_0 E_{0j}E_{0i}^* - \\
&\quad - \frac{1}{4}i\left[\left(\frac{\partial\sigma_{ij}^*}{\partial\omega}\right)_0 \frac{\partial E_{0i}}{\partial t}E_{0j}^* - \left(\frac{\partial\sigma_{ij}}{\partial\omega}\right)_0 \frac{\partial E_{0i}^*}{\partial t}E_{0j}\right] + \\
&\quad + \frac{1}{4}i\left[\left(\frac{\partial\sigma_{ij}^*}{\partial k_l}\right)_0 \frac{\partial E_{0i}}{\partial x_l}E_{0j}^* - \left(\frac{\partial\sigma_{ij}}{\partial k_l}\right)_0 \frac{\partial E_{0i}^*}{\partial x_l}E_{0j}\right] - \\
&\quad - \frac{\omega_0}{16\pi}\left(\frac{\partial\varepsilon_{ij}}{\partial k_l}\right)_0\left[\frac{\partial}{\partial x_l}(E_{0j}E_{0i}^*) + 2k_{2,l}E_{0j}E_{0i}^*\right] \\
&= -\frac{c}{16\pi}\operatorname{div}\left[\boldsymbol{E}_0^*\times\boldsymbol{B}_0 + \boldsymbol{E}_0\times\boldsymbol{B}_0^*\right].
\end{aligned} \tag{C. 8}$$

This equation can, of course, be written in a somewhat different form, using the equations

$$\left(\frac{\partial \sigma_{ij}^*}{\partial \omega}\right)_0 \frac{\partial E_{0i}}{\partial t} E_{0j}^* = \left(\frac{\partial \sigma_{ij}}{\partial \omega}\right)_0 \frac{\partial E_{0i}}{\partial t} E_{0j}^*$$
$$= \left(\frac{\partial \sigma_{ij}}{\partial \omega}\right)_0 \frac{\partial E_{0j}}{\partial t} E_{0i}^*, \quad \text{etc.}$$

In the absence of spatial dispersion, for a non-magnetic medium, $\partial\sigma_{ij}/\partial k_l = 0$, $\partial \varepsilon_{ij}/\partial k_l = 0$; and the expression (C. 8) of course takes the familiar form, e.g. formula (B 4.10) with $\boldsymbol{B} = \boldsymbol{H}$.

The significance of (C. 8) when $\boldsymbol{k}_2 = 0$, i.e. when $\boldsymbol{k}_0 = \boldsymbol{k}_1$ is real, is also entirely evident. In that case $(\varepsilon_{ij})_0$, $(\sigma_{ij})_0$ and the corresponding derivatives are taken for real† ω_0 and $\boldsymbol{k}_0$. The field $\boldsymbol{E}$ may, of course, be slightly damped or amplified in time and space; in particular, formula (C. 8) with real ω_0 and $\boldsymbol{k}_0$ is valid for fields of the type $\boldsymbol{E}_{00} e^{i(\omega_0 t - \boldsymbol{k}_0 \cdot \boldsymbol{r}_0)} e^{-\boldsymbol{k}_2 \cdot \boldsymbol{r}} e^{\pm \gamma t}$ with $|\gamma| \ll \omega_0$ and $k_2 \ll k_0$.

The expression $\overline{S'_l} = -(\omega_0/16\pi)(\partial \varepsilon_{ij}/\partial k_l)_{\omega_0, \boldsymbol{k}_0} E_{0j} E_{0i}^*$ for a transparent medium is the time-average energy flux due to the motion in the medium. The total flux is $\overline{\boldsymbol{S}'} + \overline{\boldsymbol{S}}_p$, where $\overline{\boldsymbol{S}}_p = (c/16\pi)(\boldsymbol{E}_0^* \times \boldsymbol{B}_0 + \boldsymbol{E}_0 \times \boldsymbol{B}_0^*)$ is the averaged Poynting vector; see also Appendix A and [352].

It follows from (C. 8) and Appendix B that the expression (C. 1) may be regarded as the heat evolved only for a monochromatic field (i.e. $\partial E_{0i}/\partial t = 0$) and when terms involving derivatives with respect to x_l are neglected or, in certain conditions, after integration over the volume (as in [349]).

The invalidity of generally interpreting (C. 1) as the heat is especially clear, apart from the above discussion and Appendix B, from formula (C. 8) with $\boldsymbol{k}_2 \neq 0$, when $(\sigma_{ij})_0 = \sigma_{ij}(\omega_0, \boldsymbol{k}_1 - i\,\boldsymbol{k}_2)$. For example, let $\varepsilon'_{ij}(\omega, \boldsymbol{k}) = \varepsilon_{ij}^{(0)}(\omega) + \alpha_{ijlm}(\omega) k_l k_m$ with real $\varepsilon_{ij}^{(0)}$ and α_{ijlm} (and real ω) [1]. Then there is no dissipation, and for real $\boldsymbol{k}$ we have $\varepsilon'_{ij}(\omega, \boldsymbol{k}) = \varepsilon_{ij}(\omega, \boldsymbol{k})$, but when $\boldsymbol{k}$ is complex the tensor $\sigma_{ij}(\omega, \boldsymbol{k})$ is not zero for such a medium. For a field

$$\boldsymbol{E}(\boldsymbol{r}, t) = \tfrac{1}{2}[\boldsymbol{E}_{00} e^{-\boldsymbol{k}_2 \cdot \boldsymbol{r}} e^{i(\omega_0 t - \boldsymbol{k}_1 \cdot \boldsymbol{r})} + \text{complex conjugate}]$$

where $\boldsymbol{E}_{00} = \text{constant}$ and $\boldsymbol{E}_0 = \boldsymbol{E}_{00} e^{-\boldsymbol{k}_2 \cdot \boldsymbol{r}}$, equation (C. 8) becomes

$$\frac{1}{2}(\sigma_{ij})_0 E_{0j} E_{0i}^* = -\frac{c}{16\pi} \operatorname{div}(\boldsymbol{E}_0^* \times \boldsymbol{B}_0 + \boldsymbol{E}_0 \times \boldsymbol{B}_0^*).$$

Thus when dissipation is absent the expression $\frac{1}{2}(\sigma_{ij})_0 E_{0j} E_{0i}^*$ here represents the divergence of energy flux; in the absence of both spatial dispersion and dissipation, $(\sigma_{ij})_0 = 0$ and in this example we must suppose that $\boldsymbol{k}_2 = 0$ or $\overline{\boldsymbol{S}}_p = 0$; this also follows automatically, of course, on using the field equations.

† The value of σ_{ij} may not be zero when ω_0 and $\boldsymbol{k}_0$ are real, since in the absence of spatial dispersion σ_{ij} is independent of $\boldsymbol{k}$.

It may be mentioned, to avoid misunderstanding, that we have assumed external sources to be absent, although it is sometimes necessary to take them into account.

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