

# Trapping analysis of a magnetized electron by a circularly polarized electromagnetic wave in static electric field

## Abstract

An pseudo-potential model is presented to illustrate the trapping effect of electrons in a static electric field by an electromagnetic wave through Anomalous Doppler Resonance (ADR) under a uniform background magnetic field. When the electromagnetic wave intensity surpasses a threshold, the electron's parallel velocity becomes trapped and oscillates within a pseudo-potential well. In this trapping region, energy from the static electric field is continuously converted into gyrokinetic energy. The energy transfer ratio from the static electric field to the gyrokinetic is calculated and compared with predictions from quantum theory, showing good agreement.

## I. Introduction

The manipulation of particles via electromagnetic (E.M) waves is a powerful and versatile technique, primarily employed for two key objectives: particle acceleration and trapping. In the context of acceleration, one prominent mechanism is autoresonance, wherein a charged particle sustains phase synchronization with an E.M wave through nonlinear effects. A well-known example is the Gyro-Resonant Accelerator[1-3], in which electrons resonate within a time-varying magnetic field while interacting with a fixed-frequency E.M wave, the smooth augmentation of the magnetic field is automatically accompanied by a correspondent growth of the relativistic electron mass to match the cyclotron frequency with E.M wave. Additional acceleration mechanisms include betatron resonance[4-6] and ponderomotive-force-driven acceleration[7, 8]. On the other hand, Velocity-accelerating can also be achieved through Landau resonant, wherein electrons with velocities near the phase velocity of a longitudinal wave become confined within the associated potential well[9], and then through increasing the phase velocity along the electron trajectory, the electron will also be accelerated due to the trapping effect. For particle trapping, spatial confinement can be achieved using methods such as optical tweezers or ponderomotive potential wells generated by the spatial interference patterns of laser beats [10].

Besides the physics application, Ream of papers describe the phenomenon of the trapping effect between E.M wave and electron in universal space and plasma device such as tokamak. For example, the trapped electron in chorus wave nonlinearity or time domain structures (TDS)[11], the relativistic electron precipitation by EMIC waves (electromagnetic ion cyclotron mode[12]), and the resonance with whistle mode wave[13].

Although the interaction between test particles and E.M waves has been extensively studied[14-22], like the resonant between electron and electrostatic field or electromagnetic wave under magnetic field [23, 24]. Few studies have incorporated the static electric field into the interactions between electron and E.M wave under magnetic field, despite its relevance in phenomena such as runaway electron generation in tokamaks[25] and pitch angle scattering in earth space[11] [26], where there exist both the electrostatic field, background magnetic field and E.M wave likes whistle wave[11];

Based on pseudopotential approaches used to study electron pitch-angle scattering in E.M waves[14], we extend the model developed by Bellan P. M. In Bellan's framework, the interaction between a circularly polarized wave and an energetic gyrating particle is described by a relativistic pseudopotential that depends on the frequency mismatch. By introducing a static electric field, we reveal a previously unexplored form of resonant trapping in electromagnetic wave-particle interactions involving static electric acceleration. In this study we first numerically investigate trapping phenomenon in which electron's velocity within a static electric field become confined at Anomalous Doppler Resonant in a uniform magnetic field. During trapping, the work performed by the static electric field is continuously converted into gyrokinetic energy or E.M wave, thereby sustaining resonance condition with the E.M wave, and the energy transfer from static electric field also agree with the quantum theory prediction and satisfied with angular conservation model[27].

The structure of this paper is as follows: In Section II, we develop a mathematical framework describing the interaction between an electromagnetic wave and an electron in the presence of a static electric field collinear with the background magnetic field. Section III presents a numerical investigation of the resulting trapping dynamics, including a detailed analysis of the underlying physical mechanism and the derivation of critical threshold conditions necessary for trapping. In Section IV, the theoretical predictions are benchmarked against results from quantum theory to assess consistency and validate the model. Section V offers a comprehensive discussion of the findings, and Section VI concludes the paper with a summary of the principal results and their potential implications.

## II. electron-electromagnetic wave interaction analysis

### 2.1 Field equations

To analyze interaction between electron and E.M wave, we consider an E.M wave propagating along uniform magnetic field whose phase velocity  $v_T = \omega/k$ , where  $\omega$  is the angular frequency and  $k$  is the wavenumber. The uniform background magnetic field is  $B_0 = B_0 \hat{z}$ . The wave's magnetic field perturbation  $\tilde{B}$  is characterized by the dimensionless parameter  $\kappa \equiv |\tilde{B}|/B_0$ , such that the total magnetic field becomes  $\mathbf{B} = B_0 \hat{z} + \tilde{B}$ . The system includes a static electric field  $E_0 = E_0 \hat{z}$ , with the total electric field given by  $\mathbf{E} = E_0 \hat{z} + \tilde{E}$ , where  $\tilde{E}$  represents the electric field components of E.M wave:

$$\tilde{\mathbf{E}} = E_w [\hat{x} \cos(kz - \omega t) + g \hat{y} \sin(kz - \omega t)] \quad (1)$$

Here,  $E_w$  is the electric field amplitude of E.M wave, and the polarization satisfies the left-hand circular polarization (LCP) condition when  $g = 1$ , and right-hand circular polarization (RCP) condition when  $g = -1$ .

Faraday's law requires the associated magnetic field to be

$$\tilde{\mathbf{B}} = \frac{k}{\omega} \hat{z} \times \tilde{\mathbf{E}} = \kappa B_0 [-g \hat{x} \sin(kz - \omega t) + \hat{y} \cos(kz - \omega t)] \quad (2)$$

Here  $\kappa B_0 = B_w = \frac{k}{\omega} E_w$ .

### 2.2 Transformation to the wave frame

In wave frame, which denotes as prime and moves at constant velocity  $v_T = \omega/k$  with respect to the lab frame, the fields are

$$E'_{\parallel} = E_{\parallel} \quad (3)$$

$$B'_{\parallel} = B_{\parallel} \quad (4)$$

$$E'_{\perp} = \gamma_T (\mathbf{E} + \mathbf{v}_T \times \mathbf{B})_{\perp} \quad (5)$$

$$B'_{\perp} = \gamma_T \left( \mathbf{B} - \frac{1}{c^2} \mathbf{v}_T \times \mathbf{E} \right)_{\perp} \quad (6)$$

Where  $\gamma_T = 1/\sqrt{1 - \left(\frac{v_T}{c}\right)^2}$ . Substituting the wave fields Eq. (1), Eq. (2) into Eq.(5), Eq. (6) gives  $E'_{\perp} = 0$  and  $B'_{\perp} = \frac{B_{\perp}}{\gamma_T}$ .

Since  $\{\mathbf{k}, \frac{i\omega}{c}\}$  and  $\{\mathbf{x}, \text{ict}\}$  are relativistic four-vectors, we have

$$kz - \omega t = k\gamma_T(z' + v_T t') - \omega\gamma_T\left(t' + \frac{v_T z'}{c^2}\right) = k'z' \quad (7)$$

Here  $k' = k/\gamma_T$  and  $z'$  are the wavenumber and position in the wave frame. The magnetic field of E.M wave is

$$\mathbf{B}'_{\perp} = B'_{\perp} (-\hat{g} \hat{x} \sin(k'z') + \hat{y} \cos(k'z')) \quad (8)$$

The motion equation of the charge particle in the prime frame is

$$\frac{d}{dt'}(\gamma' \beta') = \frac{q}{m} \left( \frac{E_0}{c} \hat{z} + \beta' \times B' \right) \quad (9)$$

Where  $\gamma' = 1/\sqrt{1 - \beta'^2}$ ,  $m$  is the rest mass of electron and  $q$  is the electron charge with  $q = -e$ . Note that  $\gamma, \gamma_T$ , and  $\gamma'$  differ and should not be confused with each other. The derivation of energy  $\gamma'$  to  $t'$  and the motion equation in each direction should be

$$\frac{d\gamma'}{dt'} = \frac{qE_0 \hat{z} \cdot \beta'_z}{mc} \quad (10)$$

$$\frac{d}{dt'}(\gamma' \beta'_z) = \Omega \left( \frac{E_0}{cB_0} \hat{z} + \beta'_{\perp} \times \frac{B'_{\perp}}{B_0} \right) \quad (11)$$

$$\frac{d}{dt'}(\gamma' \beta'_{\perp}) = \Omega \left( \beta'_{\perp} \times \hat{z} + \beta'_z \hat{z} \times \frac{B'_{\perp}}{B_0} \right) \quad (12)$$

Here  $\Omega$  is the nonrelativistic electron cyclotron frequency in the lab frame with  $\Omega < 0$ . Introduce

$$\xi_z = 1 + \alpha \gamma' \beta'_z \quad (13)$$

Here  $\alpha = \frac{g\omega}{\Omega\gamma_T} n$ . And

$$\xi_{\perp} = 1 + \alpha \gamma' \beta'_{\perp} \quad (14)$$

where  $n = ck/\omega$ , according to Eq. (11), Eq. (12), we have

$$\frac{d\xi_z}{dt'} = \alpha\Omega \left( \frac{E_0}{cB_0} \hat{z} + \frac{\xi_{\perp} - 1}{\alpha\gamma'} \times \frac{B'_{\perp}}{B_0} \right) \cdot \hat{z} \quad (15)$$

$$\frac{d\xi_{\perp}}{dt'} = \alpha\Omega \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \times \hat{z} + \frac{\xi_z - 1}{\alpha\gamma'} \hat{z} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (16)$$

### 2.3 Construction of pseudo-potential equation

Taking the derivative of Eq. (15) with respect to  $t'$  gives

$$\frac{d^2\xi_z}{dt'^2} = \alpha\Omega \left( \frac{d}{dt'} \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \right) \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \cdot \hat{z} + \alpha\Omega \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \times \frac{d}{dt'} \left( \frac{\mathbf{B}'_{\perp}}{B_0} \right) \right) \cdot \hat{z} \quad (17)$$

And

$$\frac{d}{dt'} \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \right) = \frac{1}{\alpha\gamma'} \cdot \frac{d\xi_{\perp}}{dt'} - \frac{\xi_{\perp} - 1}{\alpha\gamma'^2} \frac{d\gamma'}{dt} \quad (18)$$

According to Eq. (10) and  $\xi_z$ , we have

$$\frac{d\gamma'}{dt'} = \Omega \frac{E_0}{B_0 c} \frac{\xi_z - 1}{\alpha\gamma'} \quad (19)$$

While in Eq. (29), assuming  $\beta'_{\perp} \ll \beta'_z$  and  $B'_{\perp} < B_0$ , we obtain

$$\left| \frac{d\xi_{\perp}}{dt'} \right| < \left| \Omega \frac{\xi_z - 1}{\gamma'} \right| \quad (20)$$

Finally, in the right side of Eq. (18)

$$\frac{\left| \frac{\xi_{\perp} - 1}{\alpha\gamma'^2} \cdot \frac{d\gamma'}{dt'} \right|}{\left| \frac{1}{\alpha\gamma'} \cdot \frac{d\xi_{\perp}}{dt'} \right|} < \frac{E_0 \beta'_{\perp}}{B_0 c} \quad (21)$$

While for  $\beta'_{\perp} \ll 1$  and  $\left| \frac{E_0}{B_0 c} \right| \ll 1$ , we can confidently ignore the second term in the right side of Eq. (18), then

$$\frac{d}{dt'} \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \right) = \frac{1}{\alpha\gamma'} \cdot \frac{d\xi_{\perp}}{dt'} \quad (22)$$

Combing Eq. (17) with Eq. (22), we have

$$\frac{d^2\xi_z}{dt'^2} = \alpha\Omega \left( \frac{1}{\alpha\gamma'} \cdot \frac{d\xi_{\perp}}{dt'} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \cdot \hat{z} + \alpha\Omega \left( \frac{\xi_{\perp} - 1}{\alpha\gamma'} \times \frac{d}{dt'} \left( \frac{\mathbf{B}'_{\perp}}{B_0} \right) \right) \cdot \hat{z} \quad (23)$$

Substituting Eq. (29) into Eq. (23) gives

$$\frac{d^2\xi_z}{dt'^2} = \alpha\Omega \left[ \frac{\Omega}{\gamma'} \left( \beta'_{\perp} \times \hat{z} + \beta'_z \hat{z} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \times \frac{\mathbf{B}'_{\perp}}{B_0} + \beta'_{\perp} \times \frac{d}{dt'} \left( \frac{\mathbf{B}'_{\perp}}{B_0} \right) \right] \cdot \hat{z} \quad (24)$$

The time derivation of wave magnetic field in wave frame is

$$\frac{d\mathbf{B}'_{\perp}}{dt'} = -k' \frac{dz'}{dt'} B'_{\perp} (g \hat{x} \cos(k' z') + \hat{y} \sin(k' z')) = g\omega n' \beta'_z \hat{z} \times \mathbf{B}'_{\perp} \quad (25)$$

where  $n' = \frac{ck'}{\omega} = n/\gamma_T$ . By substituting Eq. (25) into Eq. (24)(23) and simplify Eq. (24) gives

$$\frac{d^2 \xi_z}{dt'^2} = \frac{\alpha \Omega^2}{\gamma'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \left( 1 + \frac{g\omega n}{\Omega \gamma_T} \gamma' \beta'_z \right) - \beta'_z \frac{\mathbf{B}'_{\perp}}{B_0} \cdot \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (26)$$

Since  $\xi_z = 1 + \frac{g\omega n}{\Omega \gamma_T} \gamma' \beta'_z$ , finally

$$\frac{d^2 \xi_z}{dt'^2} = \frac{\alpha \Omega^2}{\gamma'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \xi_z - \beta'_z \frac{\mathbf{B}'_{\perp}}{B_0} \cdot \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (27)$$

### 2.3.1 connect $\frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0}$ with $\xi_z$

To obtain the relationship between  $\frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0}$  and  $\xi_z$ , taking the time derivation of  $\frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0}$  gives

$$\frac{d}{dt'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \right) = \frac{d\beta'_{\perp}}{dt'} \cdot \frac{\mathbf{B}'_{\perp}}{B_0} + \beta'_{\perp} \cdot \frac{d}{dt'} \left( \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (28)$$

According to Eq. (10) and Eq. (12), we have

$$\frac{d\beta'_{\perp}}{dt'} = \Omega' \left( \beta'_{\perp} \times \hat{z} + \beta'_z \left( \hat{z} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \right) - \frac{\Omega' E_0 \hat{z} \cdot \beta'_z}{c B_0} \beta'_{\perp} \quad (29)$$

Here  $\Omega' = \Omega/\lambda'$ . Substituting Eq. (29) into Eq. (28) gives

$$\frac{d\beta'_{\perp}}{dt'} \cdot \frac{\mathbf{B}'_{\perp}}{B_0} = -\hat{z} \cdot \left( \beta'_{\perp} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \Omega' - \Omega' \frac{E_0 \beta'_z}{c B_0} \left( \beta'_{\perp} \cdot \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (30)$$

and

$$\beta'_{\perp} \cdot \frac{d}{dt'} \left( \frac{\mathbf{B}'_{\perp}}{B_0} \right) = \beta'_{\perp} \cdot g\omega n' \beta'_z \left( \hat{z} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) = -g\omega n' \beta'_z \left( \beta'_{\perp} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \cdot \hat{z} \quad (31)$$

Finally, we have

$$\frac{d}{dt'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \right) = -\xi_z \Omega' \hat{z} \cdot \left( \beta'_{\perp} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) - \Omega' \frac{E_0 \beta'_z}{c B_0} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \right) \quad (32)$$

The equation is first-order linear differential equation with the form:

$$T'(\tau) + PT(\tau) = Q(\tau) \quad (33)$$

Where  $P = \frac{E_0 \beta'_z}{c B_0}$ ,  $Q = -\xi_z \hat{z} \cdot \left( \beta'_{\perp} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right)$  and  $\tau = t' \Omega'$ . The exact solution of z is

$$T = e^{-\int P d\tau} * \left[ \int e^{\int P d\tau'} Q d\tau + C_0 \right] \quad (34)$$

Since  $|P| \ll 1$  in most case of Tokamak environment, we have

$$T = \int Q d\tau \quad (35)$$

Here we choose the initial condition  $|\beta'_{\perp 0}| \ll 1$ , which implies  $C_0 = 0$ . As a result, the expression Eq. (32) can simplify to:

$$\frac{d}{dt'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \right) = -\xi_z \Omega' \hat{z} \cdot \left( \beta'_{\perp} \times \frac{\mathbf{B}'_{\perp}}{B_0} \right) \quad (36)$$

The substitution of Eq. (15) into Eq. (36) gives

$$\frac{d}{dt'} \left( \frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} \right) = -\xi_z \Omega' \left( \frac{1}{\alpha \Omega} \frac{d\xi_z}{dt'} - \frac{E_0}{c B_0} \right) \quad (37)$$

Integrating Eq. (37) with  $t'$  gives:

$$\frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} = -\frac{\xi_z^2 - \xi_{z0}^2}{2\alpha\gamma'} + \frac{E_0}{c B_0} \int \xi_z \Omega' dt' + \frac{\beta'_{\perp 0} \cdot \mathbf{B}'_{\perp 0}}{B_0} \quad (38)$$

Here  $\xi_{z0}$ ,  $\beta'_{\perp 0}$  and  $\mathbf{B}'_{\perp 0}$  represent the initial condition of  $\xi_z$ ,  $\beta'_{\perp}$  and  $\mathbf{B}'_{\perp}$ .

Noting that  $t = 0$  corresponds to  $z' = 0$ , since  $z = 0$  at  $t = 0$ , and recalling the four vectors  $\{\gamma\beta, \gamma\}$ , we have the relation:

$$\gamma'_0 \beta'_{\perp 0} = \gamma_0 \beta_{\perp 0} \quad (39)$$

Since  $\mathbf{B}'_{\perp 0}$  is in the  $y$  direction when  $z' = 0$  as shown in Eq. (8), it follows that

$$\frac{\beta'_{\perp 0} \cdot \mathbf{B}'_{\perp 0}}{B_0} = \frac{\beta'_{y0} B'_{\perp 0}}{B_0} = \frac{\gamma_0}{\gamma'_0} \kappa' \beta_{\perp 0} \sin \phi_0 \quad (40)$$

Where  $\phi_0$  is defined by  $\beta_{x0} = \beta_{\perp 0} \cos \phi_0$  and  $\beta_{y0} = \beta_{\perp 0} \sin \phi_0$ ,  $\kappa' = \frac{\kappa}{\gamma_T} = \frac{B'_{\perp}}{B_0}$ . With these definitions, Eq. (38) becomes

$$\frac{\beta'_{\perp} \cdot \mathbf{B}'_{\perp}}{B_0} = -\frac{\xi_z^2 - \xi_{z0}^2}{2\alpha\gamma'} + \frac{E_0}{c B_0} \int \xi_z \Omega' dt' + \frac{\gamma_0}{\gamma'_0} \kappa' \beta_{\perp 0} \sin \phi_0 \quad (41)$$

The substitution of Eq. (41) into Eq. (27) gives:

$$\frac{d^2 \xi_z}{dt'^2} = \frac{\alpha \Omega^2}{\gamma'} \left( \left( -\frac{\xi_z^2 - \xi_{z0}^2}{2\alpha\gamma'} + \frac{E_0}{c B_0} \int \xi_z \Omega' dt' + \frac{\gamma_0}{\gamma'_0} \kappa' \beta_{\perp 0} \sin \phi_0 \right) \xi_z - \beta'_z \kappa'^2 \right) \quad (42)$$

Simplifying Eq. (42) and substituting  $\xi_z$  for  $\beta'_z$  gives:

$$\frac{d^2 \xi_z}{dt'^2} = \Omega'^2 \left( \xi_z \left( -\frac{\xi_z^2 - \xi_{z0}^2}{2} + \frac{\zeta E_0}{c B_0} \int \Omega' \xi_z dt' + \alpha \gamma_0 \kappa' \beta_{\perp 0} \sin \phi_0 \right) - (\xi_z - 1) \kappa'^2 \right) \quad (43)$$

Here

$$\varsigma = g \frac{\omega n}{\Omega \gamma_T} \gamma' \quad (44)$$

Normalization of time  $t'$  with  $\tau = t' \Omega'$  we have

$$\frac{d^2 \xi_z}{d\tau^2} = \left( \xi_z \left( -\frac{\xi_z^2 - \xi_{z0}^2}{2} + \frac{\varsigma E_0}{c B_0} \int \xi_z d\tau + \alpha \gamma_0 \kappa' \beta_{\perp 0} \sin \phi_0 \right) - (\xi_z - 1) \kappa'^2 \right) \quad (45)$$

The Eq. (45) could also be written as

$$\frac{d^2 \xi_z}{d\tau^2} = -\frac{\partial \psi}{\partial \xi_z} \quad (46)$$

Where

$$-\frac{\partial \psi}{\partial \xi_z} = \left( \xi_z \left( -\frac{\xi_z^2 - \xi_{z0}^2}{2} + \frac{\varsigma E_0}{c B_0} \int \xi_z d\tau + \alpha \gamma_0 \kappa' \beta_{\perp 0} \sin \phi_0 \right) - (\xi_z - 1) \kappa'^2 \right) \quad (47)$$

Multiplying Eq. (46) by  $d\xi_z/d\tau$  and integrating gives a pseudo-energy equation

$$\frac{1}{2} \left( \frac{d\xi_z}{d\tau} \right)^2 + \psi(\xi_z) = W_0 \quad (48)$$

Where pseudo-potential equation is

$$\psi = \frac{1}{8} \xi_z^4 + \left( \kappa'^2 - \frac{\xi_{z0}^2}{2} - g \kappa' \sin \phi_0 \right) \frac{\xi_z^2}{2} - \kappa'^2 \xi_z - \int \xi_z \frac{\varsigma E_0}{c B_0} \int \xi_z d\tau d\xi \quad (49)$$

And

$$W_0 = \frac{1}{2} \left( \frac{d\xi_z}{d\tau} \right)^2 \Big|_{\tau=0} + \psi(\xi_{z0}) \quad (50)$$

Here  $s = \frac{\omega n \beta_{\perp 0} \gamma_0}{\Omega \gamma_T}$ . The pseudo potential  $\psi$  can only be solved numerically, as it does not have a regular form involving only the parameter  $\xi$ , here  $\psi$  also depends on  $t'$ , which is related to  $\beta'_z$  and  $\beta'_\perp$ .

When  $E_0 = 0$ , the pseudo-potential equation reduces to

$$\psi = \frac{1}{8} \xi_z^4 + \left( \kappa'^2 - \frac{\xi_{z0}^2}{2} - g \kappa' \sin \phi_0 \right) \frac{\xi_z^2}{2} - \kappa'^2 \xi_z \quad (51)$$

which has the same expression as Eq. (41) in Ref. [14], as derived by P. M. Bellan.

## 2.4 Initial condition

From Eq. (15), we see that

$$\frac{d\xi_z}{d\tau} = g \left( \frac{\omega n}{\Omega \gamma_T} \kappa' \gamma (\beta_x \cos(kz - \omega t) + g \beta_y \sin(kz - \omega t)) \right) + \frac{\varsigma E_0}{c B_0} \quad (52)$$

Here, we use the relation  $\gamma \beta_\perp = \gamma' \beta'_\perp$  and  $\kappa' = B_\perp / B_0$ . At the initial time  $t = 0$  and position  $z = 0$ , we have

$$\frac{d\xi_z}{d\tau}|_{\tau=0} = g\kappa' \cos\phi_0 + \frac{\zeta E_0}{cB_0} \quad (53)$$

### 2.6 Solve $\beta_{\perp}$

According to Eq.(12), multiplying  $\gamma'\beta'_{\perp}$  on both sides gives

$$\frac{1}{2} \frac{d(\gamma'\beta'_{\perp})^2}{dt'} = \Omega\gamma'\beta'_{\perp} \cdot \left( \beta'_z \times \frac{B'_{\perp}}{B_0} \right) \quad (54)$$

Reorganize the equation, we have

$$\frac{1}{2} \frac{d(\gamma'\beta'_{\perp})^2}{dt'} = -\Omega\gamma'\beta'_z \cdot \left( \beta'_{\perp} \times \frac{B'_{\perp}}{B_0} \right) \quad (55)$$

According to Eq.(11), multiplying  $\gamma'\beta'_z$  on both sides gives

$$\frac{1}{2} \frac{d(\gamma'\beta'_z)^2}{dt'} = \Omega\gamma'\beta'_z \cdot \left( \frac{E_0}{cB_0} \hat{z} + \beta'_{\perp} \times \frac{B'_{\perp}}{B_0} \right) \quad (56)$$

Add Eq. (55) and Eq. (56), we have

$$\frac{1}{2} \frac{d(\gamma'\beta'_{\perp})^2}{dt'} + \frac{1}{2} \frac{d(\gamma'\beta'_z)^2}{dt'} = \frac{\Omega\gamma'E_0\beta'_z}{cB_0} \quad (57)$$

Which means energy change ratio equal to work done by static electric field in moving frame.

Using the normalized time  $\tau = t'\Omega'$ , we have

$$\frac{1}{2} \frac{d(\gamma'\beta'_{\perp})^2}{d\tau} + \frac{1}{2} \frac{d(\gamma'\beta'_z)^2}{d\tau} = \frac{\gamma'^2 E_0 \beta'_z}{cB_0} \quad (58)$$

Considering that  $\gamma'$  is mainly determined by  $\beta'_z$ , integrating both sides with  $t'$  gives

$$\beta'^2_{\perp} = \frac{2}{\gamma'^2|_t} \cdot \int_0^t \frac{\gamma'^2 E_0 \beta'_z}{cB_0} d\tau + (\beta'^2_{z0} + \beta'^2_{\perp 0}) - \beta'^2_z \quad (59)$$

### III. Numerical study of the trapping effect

According to Eq. (46), Eq. (47) and Eq.(53), the velocity  $\beta'_z$  could be numerically solved by ode45, and  $\beta'_{\perp}$  can be determined from Eq.(59). Then, using the four-vector  $\{\gamma\beta, \gamma\}$ , we have:

$$\gamma = \beta_T \gamma_T \gamma' \beta'_z + \gamma_T \gamma' \quad (60)$$

$$\beta_{\perp} = \frac{\gamma' \beta'_{\perp}}{\gamma} \quad (61)$$

$$\beta_z = \frac{\gamma_T \gamma'}{\gamma} (\beta'_z + \beta_T) \quad (62)$$

Finally, all the velocities in lab frame can be solved numerically



Quantum analysis indicates that, for anomalous Doppler resonance with electrons, the wave polarization is primarily dictated by the LCP component[27], whereas normal Doppler resonance corresponds to the RCP component. We begin by considering two cases in which a LCP electromagnetic wave interacts with electron in the presence of a uniform magnetic field and a static electric field.

In case I, considering a scenario where the uniform magnetic field  $B_0 = 2$  T and the static electric field  $E_0 = -0.2$  V/m, both along the z-axis, which are close to typical Tokamak plasma conditions. A plane LCP E.M wave is assumed to propagate along z direction with refractive index  $n = 4$  where  $\omega = 1.1 \Omega$  and  $k = 1290 \text{ m}^{-1}$ [28]. The electric field of E.M wave is set to  $E_\omega = 20 \text{ V/m}$ . The numerical results are shown in Figure 1, For the static electric field  $E_0 < 0$ , the direction of the electric field is opposite to that of the background magnetic field. As a result, the parallel velocity increases over time, as shown in Figure.1(a). Since here  $\alpha < 0$ , which is given in Eq. (13), the value of  $\xi_z$  decreases as  $\beta_z$  increase. When the parallel velocity satisfies the anomalous doppler resonance condition where  $\xi_z = 0$ , the perpendicular velocity  $\beta_\perp$  increase abruptly, as shown in Figure.1 (b-d). After the parallel velocity exceeds the resonance condition, the  $\beta_\perp$  will no longer increase. The phase evolution of  $\xi_z$  and  $\frac{1}{\Omega'} \frac{d\xi_z}{dt'}$  is shown in fig.(e). It can be observed that the fluctuation of  $\frac{1}{\Omega'} \frac{d\xi_z}{dt'}$  is stronger for  $\xi_z < 0$  than for  $\xi_z > 0$ . This is because the change of  $\frac{1}{\Omega'} \frac{d\xi_z}{dt'}$  is proportional to  $\beta_\perp$  as shown in Eq. (52), An increase in  $\beta_\perp$  therefore leads to stronger fluctuations in  $\frac{1}{\Omega'} \frac{d\xi_z}{dt'}$ . Fig 1.(f) illustrates the pseudo-potential  $\Delta\psi(\xi)$ , defined as  $\Delta\psi(\xi) = \psi(\xi) - \psi(\xi_0)$ . The initial pseudo-kinetic energy  $W_0 = \frac{1}{2} \left( \frac{d\xi_z}{dt'} \right)^2 \big|_{t'=0}$ , which is indicated as red dash line in Fig.1 (f). The  $\Delta\psi(\xi)$  cannot exceed  $W_0$  due to the conservation of pseudo energy, as given in Eq. (48), where the  $\frac{1}{2} \left( \frac{d\xi_z}{d\tau} \right)^2 \geq 0$  must always be satisfied.

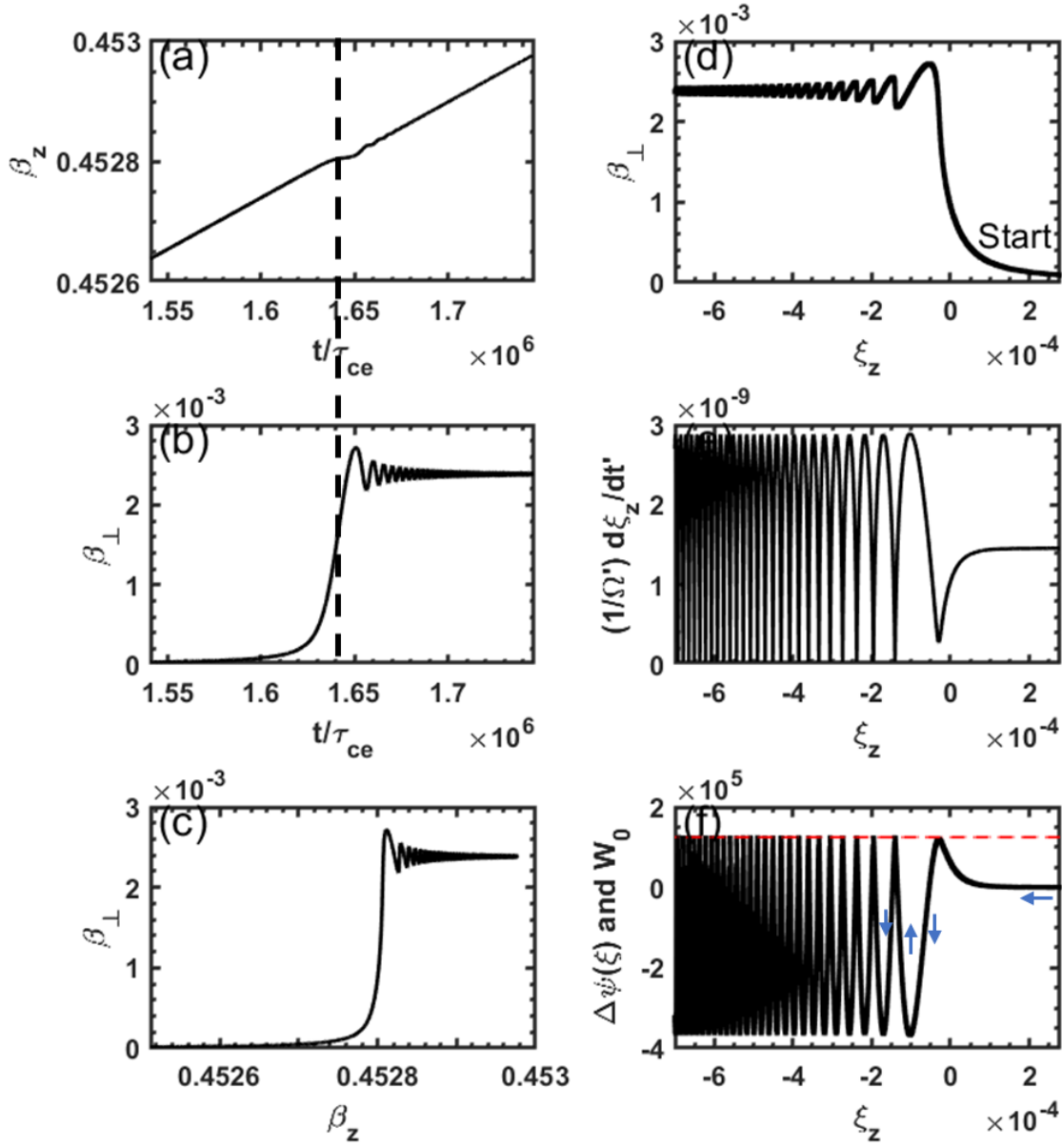


Figure 1. Numerical integration of Eq. (45) with initial equation Eq.(53). Input parameters are  $E_0 = -0.2$  V/m,  $E_w = 20$  V/m,  $B_0 = 2$  T,  $\omega/\Omega = -1.1$ ,  $g = 1$ , initial  $\beta_z = 0.45$  and  $\beta_\perp = 0$ ,  $\phi_0 = 0$ ,  $n = 4$ . This give  $\kappa' \approx 1.29 \times 10^{-7}$ ,  $\gamma_T = 1.0308$ ,  $\alpha = -4.373$ ,  $\gamma_0 = 1.1198$  and  $\gamma' = 1.0264$ ,  $\frac{cE_0}{cB_0} = 1.458 \times 10^{-9}$ . (a) The time evolution of  $\beta_z$ , here  $\tau_{ce}$  refers to the cyclotron period of electron. (b) The time evolution of  $\beta_\perp$ . (c) The velocity phase in  $(\beta_z, \beta_\perp)$ . (d) The evolution of  $\beta_\perp$  with  $\xi_z$ . (e) The evolution of  $d\xi_z/dt$  with  $\xi_z$ . (f) The pseudo-potential  $\Delta\psi(\xi)$  (black line) and the initial pseudo-kinetic energy  $W_0$  (red dash line)

In case II, the electric field of the LCP E.M wave is increased to  $E_w = 22$  V/m. As the electron's parallel velocity approaches the resonant velocity, it no longer increases continuously but begins to oscillate around the resonant velocity, as shown in Fig. 2(a). While on the other hand, the perpendicular velocity increases continuously when  $\beta_z$  trapping in resonant region, as shown in Fig.2 (b-d). The phase

trajectory of  $(\xi_z, \frac{1}{\Omega'} \frac{d\xi_z}{dt'})$  is shown in Fig.2 (e). The closed-loop structure indicates periodic motion around the resonant point, and the direction of motion is labeled with arrow. The electron can only propagate within the region where the pseudo-potential  $\Delta\psi(\xi)$  is lower than the initial pseudo-kinetic energy  $W_0$ . When the pseudo-potential tends to surpass the  $W_0$ , the electron velocity rebounds upon reaching the boundary of the pseudo-potential well. Consequently, the electron becomes confined within the well, the width of the pseudo-potential well also increases, since it is influenced by the parameter  $\beta_{\perp}$ . This bounce effect, shown in Fig.2 (f), illustrates the trapping phenomenon.

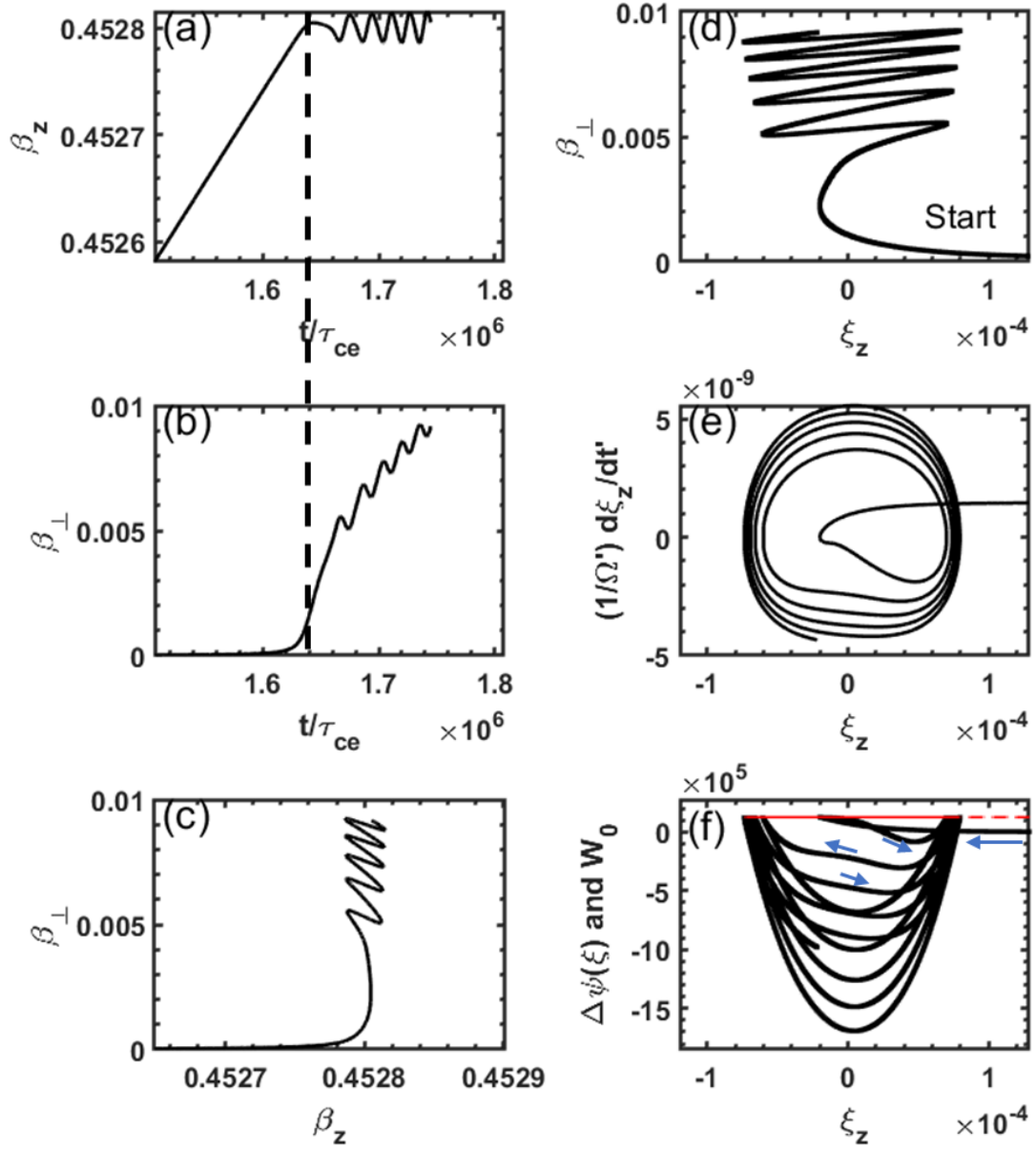


Figure 2. Same as Fig. (1) except with  $E_W = 22$  V/m. This gives  $\kappa' = 1.38 \times 10^{-7}$  but same  $\frac{\xi E_0}{c B_0} = 1.458 \times 10^{-9}$ . Panels (e) and (f) are zoom in near the resonant condition  $\xi_z = 0$ .

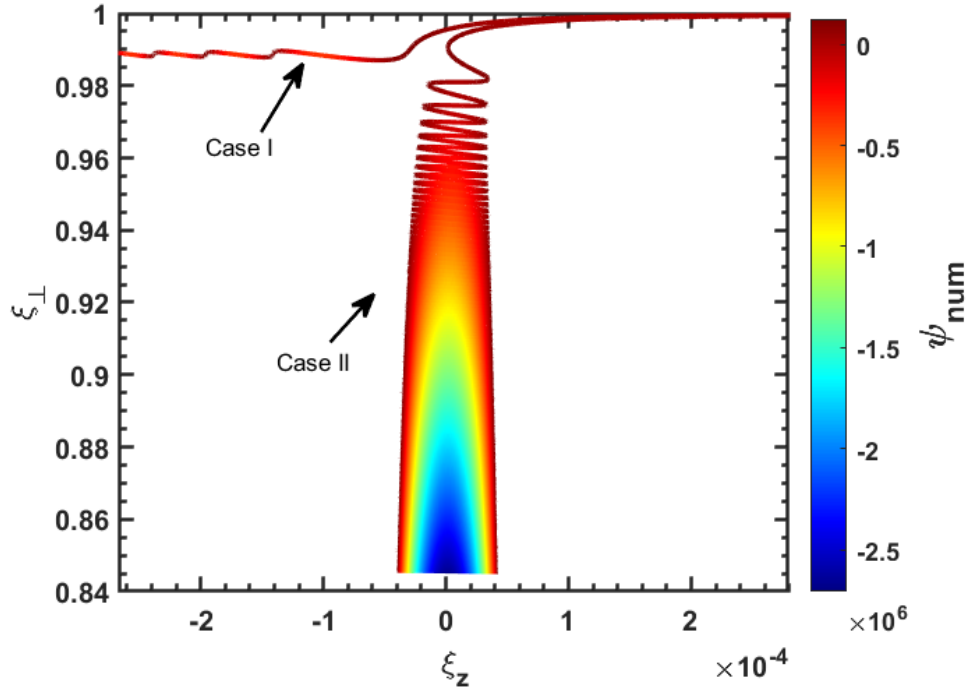


Figure 3. Pseudo-potential  $\Delta\psi$  mapped along the electron trajectory in the  $(\xi_z, \xi_\perp)$  plane.

Since the pseudo-potential  $\Delta\psi(\xi)$  is a function of both  $\beta_z$  and  $\beta_\perp$ , we traced  $\Delta\psi$  along the particle trajectory in the  $(\xi_z, \xi_\perp)$  phase space. This approach highlights the underlying structure of the pseudo-potential and enables clearer physical interpretation. As shown in Fig. 3, which compares the two scenarios described in Case I and Case II. It can be seen that when the electron becomes trapped under the resonant condition, it slips into a “deep potential valley” that extends further along the  $\xi_\perp$  direction. In contrast, if the electron passes through the resonant region without being trapped, it continues on a “highway”-like trajectory without further obstruction.

To determine the critical boundary of the trapping region, we refer to Eq. (45) and Eq. (52). Assuming the initial perpendicular velocity is approximately zero ( $\beta_{\perp 0} = 0$ ), then  $s = 0$ . Under this condition, the dynamics are governed solely by two coefficients:  $\kappa'$  and  $\frac{\zeta E_0}{cB_0}$ ,  $\xi_{z0}$  is inconsequential provided it lies outside of resonant region, as demonstrated in Fig. 2 (e) and (f), where the  $\xi_z$  is mainly shifted by static electric field before the resonance at  $\xi_z = 0$ . As shown in Fig. 4, the trapping region is indicated in yellow, while the blue region corresponds to the passing regime. Case I and Case II are marked with star symbols in the figure, located in the passing and trapping regions, respectively. For special case as given in Fig. 1, where  $\frac{\zeta E_0}{cB_0} = 1.458 \times 10^{-9}$ , the critical threshold is  $\kappa'_c = 1.367 \times 10^{-7}$ , and  $\frac{E_{\perp c}}{E_0} = \frac{\kappa'_c \gamma_T c B_0}{n E_0} \approx 102$ . Consequently, effective electron trapping in the electromagnetic wave requires the LCP electric field intensity to exceed the background static electric field by a factor of more than 102 in typical tokamak plasma with frequencies near the upper-hybrid mode.

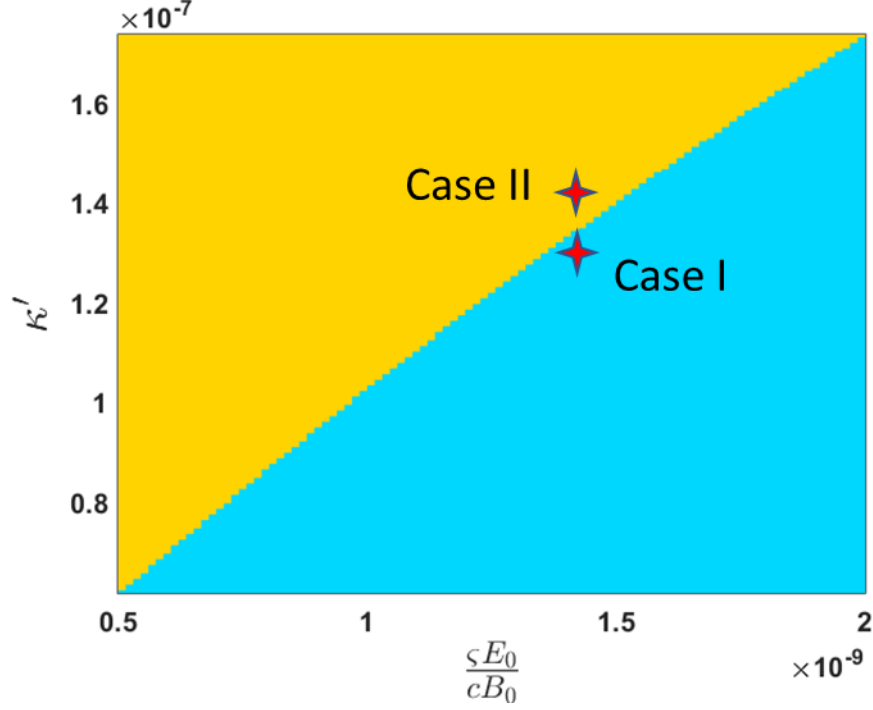


Figure 4. Parameter space of  $(\frac{\zeta E_0}{c B_0}, \kappa')$  showing electron trapping (yellow) and passing (blue) regimes.

#### IV. Benchmark with quantum theory

One characteristic worth pointing out is that when the electron is trapped in the electromagnetic wave, the energy transfer from the static electric field to the gyrokinetic energy is governed by quantum theory (QE theory) [29]

$$\eta_{\perp} = \frac{n|\Omega|/\gamma_c}{\mathbf{k} \cdot \mathbf{v}} \quad (63)$$

Here  $\gamma_c$  refers to the Lorentz factor during the resonance. For anomalous Doppler resonance,  $n = 1$  and  $\mathbf{k} \cdot \mathbf{v} = \omega + |\Omega|/\gamma_c$ ,  $\eta_p$  can be written as

$$\eta_p = \frac{1}{1 + |\omega/\Omega|\gamma_c} \quad (64)$$

For Case II,  $\frac{\omega}{\Omega} = -1.1$  and  $\gamma_c = 1.1215$  we have  $\eta_{\perp} = 0.448$ . To numerically calculate the energy transfer ratio, we evaluate the work done by the static electric field during resonance:

$$W_E = \int_0^t E_0 q v_z dt - \int_0^{t_c} E_0 q v_z dt \quad (65)$$

Here  $\tau_c$  refers to the beginning of trapping condition. Before the trapping condition where  $t < \tau_c$ , The work done by the static electric field is expressed as:

$$W_E = \int_0^t E_0 q v_z dt \quad (66)$$

The increase of perpendicular energy is given by

$$W_{\perp} = \frac{1}{2} m v_{\perp}^2 \quad (67)$$

Since the  $W_{\perp} \approx 0$  before the resonance happen, we can use the  $W_{\perp}$  directly as the increasement from the resonant condition. Finally, the energy transfer ratio is calculated as  $\eta_{\perp} = W_{\perp}/W_E$ .

The energy transfer ratio from the static electric field to parallel kinetic energy is :

$$\eta_{\parallel} = \frac{W_{\parallel}}{W_E} \quad (68)$$

Here  $W_{\parallel} = \frac{1}{2} m v_{\parallel}^2 - \frac{1}{2} m v_0^2$  before  $t < \tau_c$  where  $v_0$  refers to the initial velocity and  $W_{\parallel} = \frac{1}{2} m v_{\parallel}^2 - \frac{1}{2} m v_c^2$  after  $t > \tau_c$  after where  $v_c$  refers to the resonant velocity .

The  $\eta_{\perp}$  values obtained from the two methods are illustrated in Fig. 7. As the electron is trapped by electromagnetic wave (  $t > t_c$  with  $t_c \approx 1.67\tau_{ce}$ ), the energy transfer ratio from the numerical results tends to approach the theoretical prediction. Beside this, the parallel energy transfer ratio is 100% before resonant condition and then is 0 during the resonant condition. The other 55% energy will transfer to the stimulated wave emission as described in quantum description[29]. The agreement between the numerical and theoretical results confirms both the accuracy of the simulation and the consistency between quantum theory and classical dynamics.

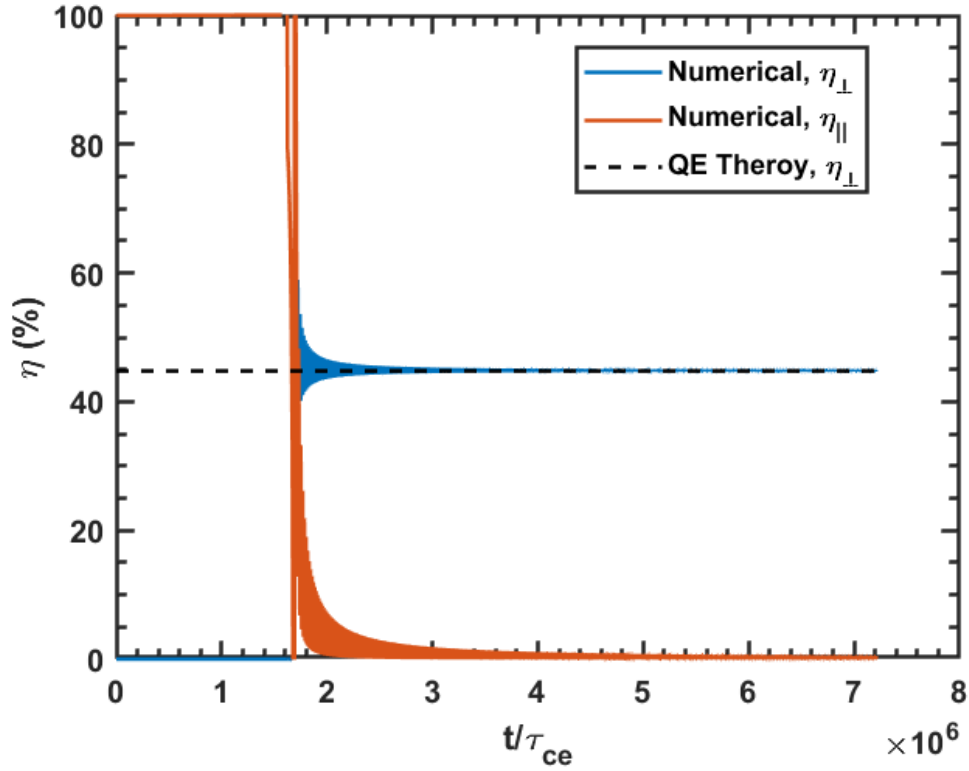


Figure 5. Energy transfer ratio from the static electric field to the gyrokinetic energy  $\eta_{\perp}$  and parallel kinetic energy  $\eta_{\parallel}$ . The parameters used here are the same as in Figure 2 (QE theory means theory based on quantum equation).

## V. Discussion

This trapping effect can be qualitatively understood through the conservation of angular momentum and linear momentum[27]. We can draw an analogy by treating the cyclotron electron as a system that contains both internal and kinetic energy, where the internal energy refers to gyrokinetic energy and kinetic energy refers to its translational motion along the magnetic field. When this system is stimulated by an external E.M wave, it undergoes stimulated emission, radiating E.M wave identical to the external one.

For anomalous doppler resonance, Since the emitted E.M wave propagates in the same direction as the electron, conservation of linear momentum requires the electron to lose some of its parallel momentum. At the same time, because the electron possesses right-hand circularly rotation (associated with positive angular momentum), while the emitted wave has left-hand circularly polarization (associated with negative angular momentum), conservation of total angular momentum requires the electron to gain angular momentum after emission. As a result, the electron loses kinetic energy and gains gyrokinetic energy. However, a static electric field continues to replenish the lost kinetic energy. When the rate of energy loss to the E.M wave balances the energy input from the electric field, the electron's parallel velocity ceases to increase, and the electron becomes trapped in the electromagnetic wave. This manifests as a continuous transfer of energy from the static electric field to the gyrokinetic energy of the system.



The redistribution of electron energy between parallel motion and gyrokinetic energy suggests a possible route for mitigating runaway electrons. Tailored wave injections (e.g., whistler-mode or electron cyclotron waves) could induce resonant trapping, helping to suppress runaway electrons. At the same time, the continuous energy transfer from static electric fields to gyrokinetic energy may provide a novel pathway for plasma heating, complementing established methods such as electron cyclotron resonance heating (ECRH). These effects merit further investigation within the framework of wave–plasma interactions.

## VI. Summary

In conclusion, trapping under anomalous Doppler resonances is analyzed via the pseudo-potential approach. The parallel velocity oscillates within a potential well, while the perpendicular velocity grows continuously. Critical trapping energy is obtained numerically, with energy conservation ratios from simulations and quantum theory showing strong agreement. The underlying mechanism is explained in terms of angular and linear momentum conservation, and potential applications for runaway electron suppression and plasma heating are discussed.

## Appendix:

### a. Prove the relationship between resonant condition and $\xi_z$

The parameter  $\xi$  characterizes the frequency mismatch relative to the resonance condition given by

$$\omega = \mathbf{k} \cdot \mathbf{v} + g \frac{\Omega}{\gamma} \quad (69)$$

This relationship can be derived as follows: Starting from the definition of  $\varsigma$  as shown in Eq.(44), we have:

$$\varsigma \beta'_z = \frac{\omega}{\Omega} \frac{n}{\gamma_T} \gamma' \beta'_z \quad (70)$$

Since  $\{\gamma\beta, \gamma\}$  are four-vector, we have

$$\gamma' \beta'_z = \gamma_T (\gamma \beta_z - \beta_T \gamma) = \gamma_T \gamma \left( \beta_z - \frac{1}{n} \right) \quad (71)$$

Substituting Eq. (71) and Eq. (70) into Eq. (13) gives

$$\xi_z = 1 + g \frac{\omega}{\Omega} \frac{n}{\gamma_T} \gamma_T \gamma \left( \beta_z - \frac{1}{n} \right) = g \frac{g \frac{\Omega}{\gamma} + \mathbf{k} \cdot \mathbf{v} - \omega}{\frac{\Omega}{\gamma}} \quad (72)$$

Here  $g \frac{\Omega}{\gamma} + \mathbf{k} \cdot \mathbf{v} - \omega = 0$  represents the resonant condition.

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