JOURNAL OF GEOPHYSICAL RESEARCH, SPACE PHYSICS VOL. 75, NO. 31, NOVEMBER 1, 1970

Particle Acceleration by Electrostatic Waves

DANIEL W. SWIFT

Geophysical Institute, University of Alaska, College 99701

The work of Gary, Montgomery, and Swift (1968) is extended to show that a coherent electrostatic wave whose wave number is a slowly varying function of position will produce a nearly monoenergetic beam of particles. The effect is demonstrated by numerical examples and by analysis of a phase plane diagram. Numerical calculations for the motion of a particle in a wave whose phase is interrupted at regular intervals indicate that an incoherent wave can produce a burst of particles with a broad energy spectrum.

Swift [1968] noted that electrostatic waves associated with the lower hybrid resonance could be excited by a proton loss-cone instability. This instability was likely to occur when the ring current protons penetrated into the tenuous upper ionosphere. The electrostatic waves propagate with a phase speed comparable to the speed of the ring current protons, but the wave normal is nearly perpendicular to the magnetic field lines. Electrons in the wave field can only be accelerated in directions parallel to the field lines. Since the component of the phase velocity parallel to the field lines is so high, resonant energy transfer between energetic electrons and the wave can take place. It was noted that, as the wave propagates into regions of increasing density of the ionospheric plasma, the wave normal becomes more nearly perpendicular to the magnetic field lines, thus resulting in an increased parallel phase velocity. As a consequence, an electron trapped in the wave may experience a *substantial* increase in velocity as it accelerates to keep up with the parallel component of the phase velocity. The significance of this proposed mechanism is that it may result in a rather substantial energy transfer from the ring current protons to the parallel motion of energetic electrons.

This suggested acceleration mechanism was further investigated by *Gary et al.* [1968], who numerically integrated the one-dimensional equation of motion of a particle in an electrostatic wave, namely

$$d^2x/dt^2 = \epsilon \cos (k_{\parallel}x - \omega t + \phi)$$

where ϵ is proportional to the charge-to-mass ratio of the particle times the component of the electric field parallel to the magnetic field, and ω and ϕ are the frequency and phase of the wave, assumed to be constant. For definiteness, $k_{\parallel}$ was chosen to be of the form $k_{\parallel} = k_0(1 + \alpha x)^{-1}$; the calculations clearly showed that particles could be accelerated up to several times their initial velocity by this mechanism, and that the smaller the value of α, the greater the final velocity of the particles that do get trapped.

Later *Laval and Pellat* [1970] derived an adiabatic invariant for a particle trapped in an accelerating wave. By demonstrating the existence of the adiabatic invariant, J, they were able to show that, once a particle starts to oscillate in the trough of an electrostatic wave, it will remain trapped in the wave until the inertial force on the particle in the accelerating reference frame of the wave exceeds the restoring force of the electric field. If the phase velocity of the wave increases substantially during the time J is conserved, the result of a coherent wave propagating through a plasma would be the production of a nearly monoenergetic beam of particles.

In this paper, I wish to demonstrate by means of numerical examples that, if particles become trapped, they will stay trapped until the conditions for J conservation break down, and that the conditions under which an electron will escape trapping are insensitive to the conditions under which the electron became trapped. The effect of wave incoherence will also be shown.

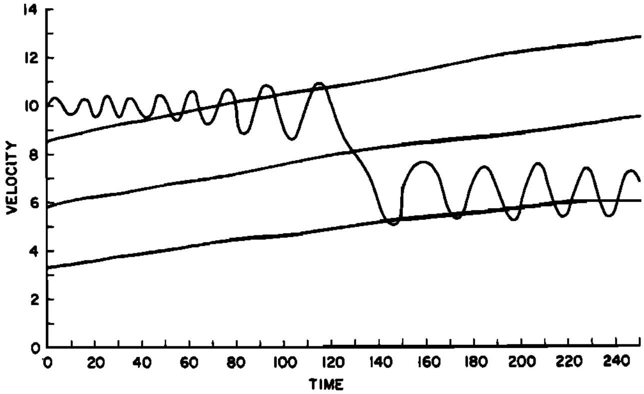

Fig. 1. Orbit of a particle, in a coherent wave, that was not trapped. The nearly straight lines show the wave phase velocity and the trapping limits on either side.

Particle Acceleration by a Coherent Wave

To illustrate the behavior of a particle in a wave field in which the wave number is a decreasing function of distance, the calculations of *Gary et al.* [1968] were repeated by integrating the equation

$$d^2x/dt^2 = -\epsilon \sin \left[\int k\, dx - \omega t + \phi \right] \quad (1)$$

where

$$k = k_0(1 + \alpha x)^{-1}$$

and with $\alpha = 0.0003$, $\omega = 1.052$, $k_0 = 0.18$, and $\epsilon = 0.3162$. The phase, ϕ, was varied from calculation to calculation to represent varying initial positions of the particle. At the beginning of the calculation, the particle was started off with an initial velocity greater than the phase velocity of the wave. Figure 1 shows the velocity as a function of time of a particle that did not get trapped. Also shown for reference is the phase velocity and the trapping width of the wave, $\Delta V = \pm 2(\epsilon/k)^{1/2}$. Notice that the particle passed through the phase velocity of the wave just once, and that the particle had its velocity decreased by about $2(\epsilon/k)^{1/2}$. This is true for all particles that passed through the wave and did not get trapped.

Figure 2 shows the orbit of a particle that became trapped. A particle that makes a second crossing of the phase velocity will continue to oscillate in the potential well of the wave until the inertial force on the particle in the accelerating reference frame of the wave exceeds the restoring force of the wave electric field. Figure

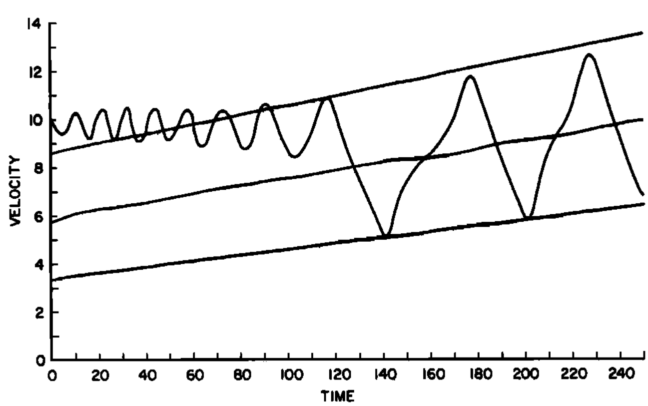

Fig. 2. Orbit of a particle that was trapped (see Figure 1).

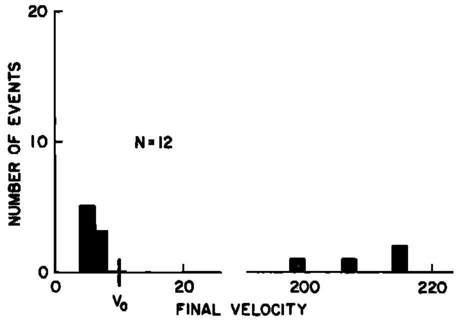

Fig. 3. Histogram of final velocities for particles in a coherent wave. V_0 is the initial velocity of all the particles.

3 shows a histogram of the final velocities for an ensemble of 12 particles, each corresponding to a different value of ϕ at regularly spaced intervals between 0 and 2π. Here we notice that particles that did not get trapped suffered a reduction in velocity by approximately one-half of the trapping width of the wave, whereas the particles that did get trapped were accelerated to 40 times their initial velocity.

The condition for escape from trapping is given by the condition that the inertial acceleration exceed the electric field acceleration, namely,

$$\frac{d}{dt}(\omega/k) \geq \epsilon$$

which in the model used for the numerical calculations gives

$$\omega/k = \frac{\epsilon k_0}{\omega\alpha} = 180$$

This value is within the trapping width of the final velocities of the trapped particles shown in Figure 3.

Phase Plane Analysis

A phase plane analysis of equation 1 in a reference frame at the wave phase velocity will be made as an aid to understanding the trapping process of a particle in a wave and the effect of an incoherent wave. Equation 1 may be transformed to a coordinate ξ, such that the point $\xi = 0$ moves at the wave phase velocity, by the substitution

$$\xi = \int_0^x k(x')\, dx' - \omega t \qquad (2)$$

Equation 1 in the ξ coordinate system takes the form

$$d^2\xi/dt^2$$

$$= \frac{1}{k^2}\frac{dk}{dx}[(d\xi/dt) + \omega]^2 - \epsilon k \sin(\xi + \phi) \qquad (3)$$

By writing $(d\xi/dt) = \eta$, we obtain

$$\eta\frac{d\eta}{d\xi} = -a(\xi, t)(\eta + \omega)^2 - \epsilon k(\xi, t)\sin(\xi + \phi) \qquad (4)$$

where a is a slowly varying function of ξ and time.

The phase plane analysis of (4) is not strictly valid when the coefficients depend on time. However, it is useful to proceed on the basis that the coefficients are frozen in time. If we set $k = k_0 (1 + \alpha x)^{-1}$, we find that the coefficient, a, in (4) is independent of time, and (4) becomes

$$\eta\frac{d\eta}{d\xi} = -\frac{\alpha}{k_0}(\eta + \omega)^2 - \epsilon k(\xi, t)\sin(\xi + \phi) \qquad (5)$$

where

$$k = k_0 \exp[(-\alpha/k_0)(\xi + \omega t)]$$

In the limit of $\alpha = 0$, equation 5 becomes that of a simple pendulum, and the trapped particle

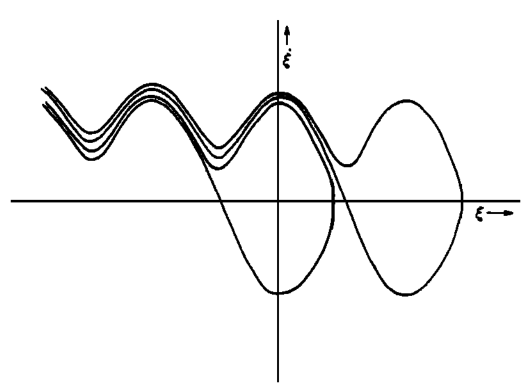

Fig. 4. A phase plane diagram in which the point $\xi = 0$ moves with the phase velocity of the wave. Two regions in which particles are trapped are indicated by the shading. This pattern is actually quasiperiodic along the ξ axis.

orbits correspond to oscillating solutions of the pendulum equation. The effect of the first term on the right-hand side of (5) is to cause a damping of particle motion such that a particle with an energy greater than the trapping energy will decay into the energy range where it can be trapped. In the x coordinate system, this result corresponds to a particle starting with a velocity greater than the phase velocity of the wave, and with the wave phase velocity gradually overtaking the particle velocity. Figure 4 shows a phase plane diagram of two regions corresponding to trapped orbits shown by the shaded areas. Although only two shaded regions are shown, the pattern indicated in Figure 4 repeats itself along the ξ axis, so there is a separate region corresponding to trapped orbits for each wave trough. A particle starting off in the shaded region at large values of $d\xi/dt$ gradually moves toward the ξ axis, and when it crosses the axis it goes into a decaying spiral. Those particles that start off in the unshaded areas fall through the ξ axis and remain below the ξ axis, moving in the negative ξ direction.

Now if we consider the effect of advancing time on the diagram in Figure 4, it can be seen that the area occupied by the shaded region will gradually shrink, so that the probability of finding a particle in a shaded area above the ξ axis will diminish. A particle becomes untrapped when the size of the shaded areas near the ξ axis shrinks to the point where a particle finds itself in an unshaded region.

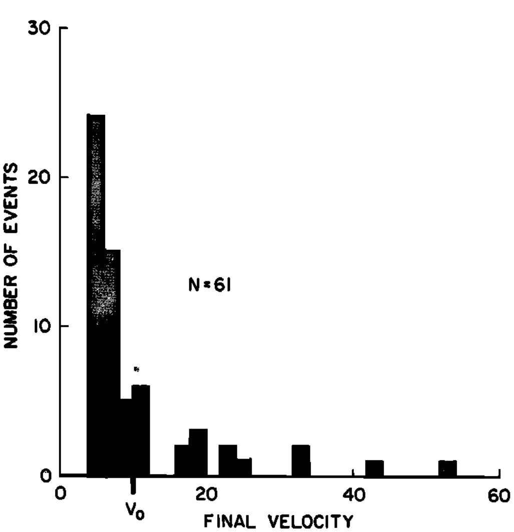

Fig. 5. A histogram of final velocities when the phase of the wave was randomized at regular intervals longer than an oscillation period of a particle in the potential well of a wave. V_0 is the initial velocity used for all calculations.

Effect of Wave Incoherence

The effect of wave incoherence can be simulated by making random changes in the phase of the wave. We introduce new random values of the variable, ϕ, in (1) or (5). The effect of a sudden change in ϕ can be viewed as a sudden shift along the ξ axis of the pattern shown in Figure 4, with the particle remaining stationary during this shift. It is difficult to give an analytic expression for the probability that a particle will be trapped or untrapped as a result of a phase change, but it can be seen that, as the size of the shaded areas decreases, the probability that a particle will find itself in an untrapped orbit as a result of a phase change increases. It can also be seen that there may be qualitatively different effects, depending on whether the coherence time of the wave is longer or shorter than the period of oscillation of a particle in the potential well of a wave. If the coherence time is longer than an oscillation period, then the probability that a particle becomes trapped as a result of a phase change is small.

Figure 5 shows a histogram of final velocities when (1) was integrated numerically with the same parameters used to make the computations for the coherent wave, except that the phase, ϕ, was changed to a new random value at regular intervals on a time scale long compared with the period of oscillation of a particle in the wave. Figure 6 shows a histogram of final velocities when the dephasing time is short compared with the oscillation time. It can be seen from Figures 5 and 6 that the final velocities for the incoherent wave are all less than the final velocities for trapped particles in a coherent wave. The final distribution of velocities is considerably broader than the Poisson distribution. The only qualitative difference between Figures 5 and 6 that seems significant is that the distribution in Figure 6 is peaked at the initial velocity, whereas the distribution in Figure 5 indicates that most of the particles lost energy. This difference probably results from the fact that a particle in an untrapped trajectory is not likely to become trapped as a result of a phase

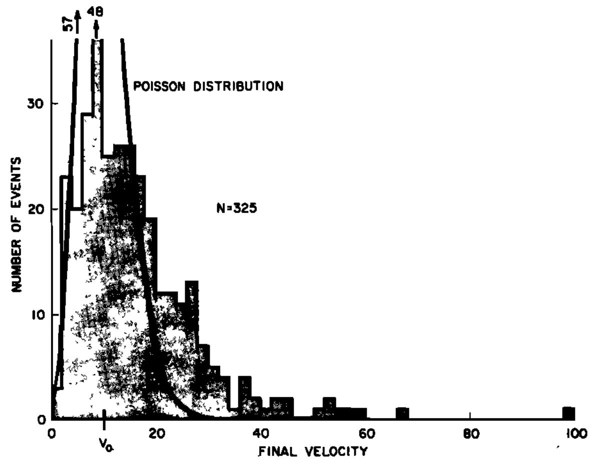

Fig. 6. A histogram of final velocities when the phase of the wave was interrupted at intervals shorter than an oscillation period (see Figure 5).

change when moving across the ξ axis of Figure 4. On the other hand, most of the particles in Figure 6 probably experienced trapping at some point in their history.

Summary

The results of the analysis and the numerical computations indicate that, if a coherent wave (with wave number being a slowly varying function of position) passes through a plasma, the result will be the production of a nearly monoenergetic beam of particles. The fraction of particles trapped by the wave may experience a very significant increase in energy as a result of the passage of a single wave train. The particles whose velocity passes through the phase velocity of the wave will have their speed reduced by approximately the trapping width of the wave, $2(\epsilon/k)^{1/2}$. If the wave is at all incoherent on a time scale of particle trapping, the result will be the production of a beam with a broad energy spectrum.

Acknowledgments. I am indebted to S. Peter Gary for the use of his computer program, which was used to calculate the particle motions. I would also like to thank Sharon Dean for her assistance in doing the calculations in preparation of Figure 4 and in running the program for calculating the particle trajectories.

This research was supported by the Atmospheric Sciences Section, National Science Foundation, NSF grants GA-771 and GA-4597.

References

Gary, S. Peter, David Montgomery, and Daniel W. Swift, Particle acceleration by electrostatic waves with spatially varying phase velocities, *J. Geophys. Res., 73*, 7524–7525, 1968.

Laval, G., and R. Pellat, Particle acceleration by electrostatic waves propagating in an inhomogeneous plasma, *J. Geophys. Res., 75*(16), 3255–3256, 1970.

Swift, Daniel W., A new interpretation for VLF chorus, *J. Geophys. Res., 73*, 7447–7456, 1968.

(Received February 6, 1970;
revised May 29, 1970.)