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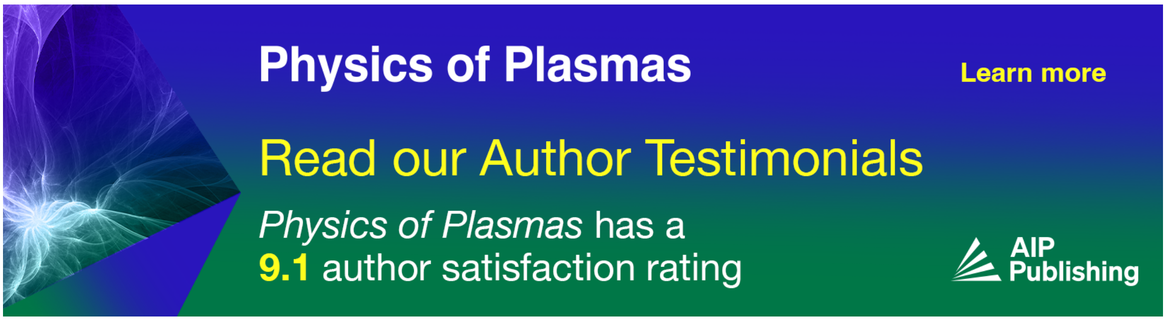

Nonlinear analysis of electron cyclotron maser based on anomalous Doppler effect

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The nonlinear dynamics of an initially rectilinear electron beam interacting with a retarded transverse circularly polarized electromagnetic wave under the condition of anomalous Doppler effect has been studied. The physical mechanism of the interaction is presented, and nonlinear ordinary equations for electron energy evolutions have been derived and discussed. The numerical calculations show that the interaction efficiency can be very high. The results are of great importance for the design of practical devices. © *2007 American Institute of Physics*.
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I. INTRODUCTION

Since the instability of electron cyclotron resonance maser (CRM) based on normal Doppler effect was proposed independently by Twiss *et al.* in the late 1950s,[1–3] the high-power microwave sources based upon the normal Doppler CRM interaction have received extensive investigations both theoretically and experimentally.[4–9]

The cyclotron resonance condition of the CRM is

$$\omega = k_z v_z + s\Omega, \tag{1}$$

where ω, k_z, v_z, s, and Ω are the wave frequency, axial wave number, electron axial velocity, cyclotron harmonic numbers, and relativistic cyclotron frequency, respectively. From the tuning condition (1), one can see that the resonance occurs at positive cyclotron harmonics ($s>0$) in the normal Doppler CRM, and it occurs only at negative cyclotron harmonics ($s<0$) in the anomalous Doppler CRM.

The anomalous Doppler effect occurs when the electron axial velocity exceeds the wave phase velocity.[10] Some theoretical and experimental studies for anomalous Doppler CRM can be found in Refs. 11–20.

The anomalous Doppler CRM has three distinct advantages. The first is that the radiation can be produced by initially rectilinear beam (i.e., with axial velocity only), which has been confirmed by experiments.[16–18] This is very important because the quality of rectilinear electron beams formed by Pierce electron guns can be much better than that of the beams of gyrating electrons produced by magnetron injection guns. The second is the potential for large frequency upshift with moderate magnetic field and beam energy. From Eq. (1), one can obtain the resonance frequency of the anomalous Doppler CRM of the fundamental harmonics as

$$\omega = \frac{\Omega}{v_z/v_{\text{ph}} - 1}, \tag{2}$$

where $v_{\text{ph}}=\omega/k$ is the wave phase velocity. If choosing $v_z \approx v_{\text{ph}}$ (a very energetic electron beam is not necessary because $v_{\text{ph}}<c$, c is the speed of light in vacuum), one can get a substantial frequency upshift $\omega \gg \Omega$. The third is that anomalous Doppler CRM can be more tolerable to the electron velocity spread than gyrotron and cyclotron autoresonance maser (CRM) due to the interaction mechanism.[13]

The study of normal Doppler CRM has been carried out in great detail and the study of anomalous Doppler CRM is rather insufficient. In some previous papers, as in Refs. 11, 13, 14, and 20, analytical estimates were given to the interaction efficiency of anomalous Doppler CRM. Most recently, in Ref. 20, Nusinovich *et al.* gave a potential design of strip line maser amplifier based on anomalous Doppler effect.

However, previous papers only studied the first stage of the electron-wave interaction, i.e., the electron axial velocity greater than the wave phase velocity, and then gave the analytical or numerical estimates of the interaction efficiency. We will expound the interaction mechanism in detail for anomalous Doppler CRM when the electron axial velocity bellows the wave phase velocity at resonance condition. In this case, the electron loses both its axial and transverse momentums, so the interaction efficiency can be very high in theory. Furthermore, for the first time, we will deduce a nonlinear equation for electron energy evolution, and by use of the equation we will study the interaction efficiency of anomalous Doppler CRM more straightforward. The paper is structured as follows. In Sec. II, we present the physical mechanism of the interaction in detail. In Sec. III, a nonlinear ordinary differential equation of electron energy is obtained by use of integral equations. The analytical solutions of these equations can be obtained with the elliptic functions. Numerical simulations are performed in Sec. IV, and in Sec. V, we provide a summary together with some discussions.

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II. THE PHYSICAL PICTURE OF THE ANOMALOUS DOPPLER ELECTRON CRM

Suppose that a transverse circularly polarized electromagnetic wave propagates in a homogeneous nondispersive medium with the refractive index $n(n>1)$, and that along the propagation direction (say, the z axis) a uniform magnetic field $\vec{B}_0$ is applied,

$$\vec{E} = E\cos(\omega t - kz)\vec{e}_x + gE\sin(\omega t - kz)\vec{e}_y, \tag{3a}$$

$$B = -gnE\sin(\omega t - kz)\vec{e}_x + nE\cos(\omega t - kz)\vec{e}_y + B_0\vec{e}_z. \tag{3b}$$

Here $\vec{E}$ and $\vec{B}$ are electric and magnetic fields of the electromagnetic wave (E the amplitude of the electric field), t is the laboratory time, $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ represent the unit vector along the x, y, and z axes, respectively, and g is the polarization parameter. In the case of $g=1$, we say the wave is left circularly polarized, and when $g=-1$ the wave is right circularly polarized.[21]

For an electron beam with an initially axial velocity along the z axis, in order to obtain the resonance interaction, the polarization parameter $g=-1$ must be chosen, and we suppose the beam is tenuous enough and can be treated as a single electron.[13] The equations for the motion and energy change of electrons can be expressed as

$$\frac{d\vec{p}}{dt} = -e\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right], \tag{4a}$$

$$\frac{dW}{dt} = -e\vec{v}\cdot\vec{E}, \tag{4b}$$

where e is the charge of a positron, $\vec{p}$ and $\vec{v}$ are, respectively, the momentum and velocity of the electron, and W is its total energy (here $\vec{p}=m\gamma\vec{v}$ and $W=m\gamma c^2$, m is the rest mass of electron, and γ is the relativistic factor, i.e., the normalized electron energy).

Let us suppose that initially the electron only has an axial velocity v_z along the z axis, then according to Eq. (4a), the total transverse force $F_\perp$ acting upon the electron is

$$F_\perp = eE\left(\frac{v_z}{c/n} - 1\right). \tag{5}$$

If the inequality $v_z > v_{\mathrm{ph}} = c/n$ is satisfied, then $F_\perp$ takes the same direction as that of $\vec{E}$. Consequently, a velocity $v_\perp$ along $\vec{E}$ is developed; therefore, the electron gives energy to the wave, giving rise to an axial force F_z acting upon the electron,

$$F_z = -\frac{neEv_\perp}{c}. \tag{6}$$

Here F_z has the opposite direction of the z axis, as is shown in Fig. 1. Then, the transverse velocity increases gradually and the axial velocity decreases due to these two forces.

When the axial velocity v_z is smaller than the wave phase velocity v_{ph}, the transverse force $F_\perp$ changes its direction while the axial force F_z does not. As a result, both the axial and transverse velocities are decelerated, which is indicated in Fig. 2. In this case, the energy is transferred more efficiently from the electron to the electromagnetic wave.

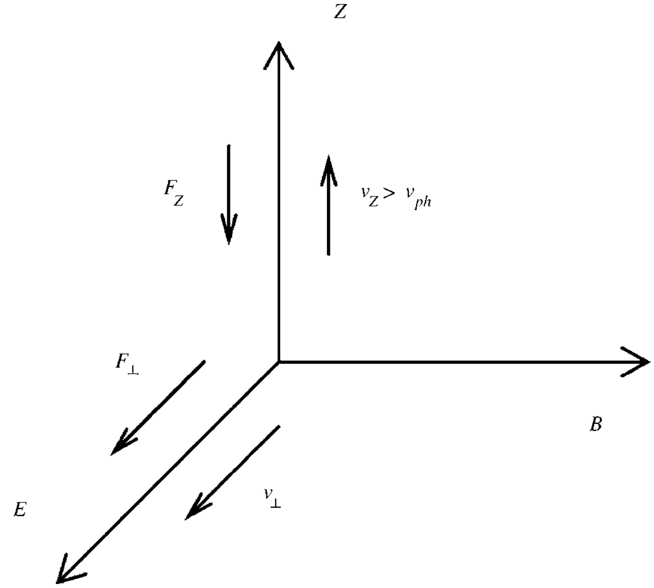

FIG. 1. The first stage of the electron wave interaction ($v_Z > v_{ph}$). The electron loses its axial momentum and builds up a transverse momentum.

III. THE NONLINEAR EVOLUTION EQUATION OF THE NORMALIZED ELECTRON ENERGY

For the interaction between electron and a left circularly polarized wave, a nonlinear ordinary equation of the electron energy has been obtained by the use of integral equations.[22] In our case, in order to generate the anomalous Doppler CRM, the interaction of electron to right circularly polarized

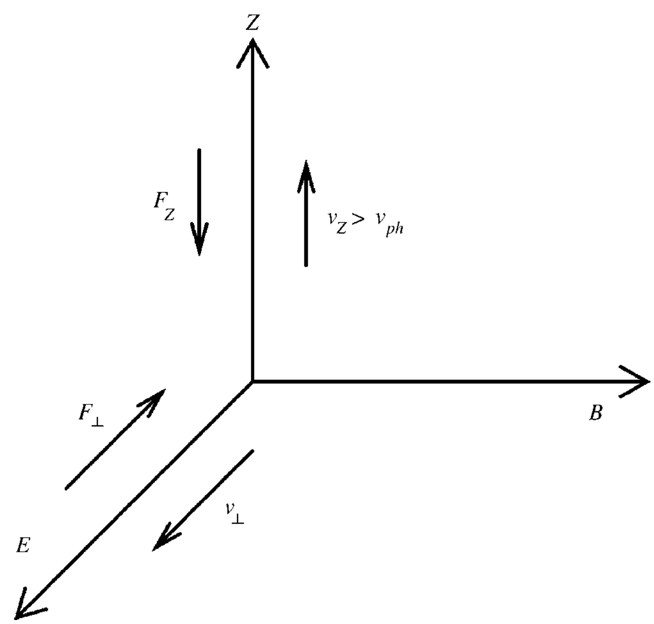

FIG. 2. The second stage of the electron wave interaction ($v_Z < v_{\mathrm{ph}}$). The electron loses both its axial and transverse momentum, and the interaction is more effective than in the first stage.

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wave will be considered. Following Ref. 22, a similar nonlinear equation has been derived and presented in the Appendix.

The components of Eq. (4) can be written as follows:

$$\frac{dp_x}{dt} + \Omega p_y = -\frac{eE}{\omega}\left[\omega - k\frac{dz}{dt}\right]\cos(\omega t - kz), \tag{7a}$$

$$\frac{dp_y}{dt} - \Omega p_x = \frac{eE}{\omega}\left[\omega - k\frac{dz}{dt}\right]\sin(\omega t - kz), \tag{7b}$$

$$\frac{dp_z}{dt} = -\Omega\frac{nE}{B_0}[p_x\cos(\omega t - kz) - p_y\sin(\omega t - kz)], \tag{7c}$$

$$\frac{dW}{dt} = -\Omega\frac{cE}{B_0}[p_x\cos(\omega t - kz) - p_y\sin(\omega t - kz)], \tag{7d}$$

where $\Omega = \frac{eB_0}{m\gamma c}$ is the relativistic cyclotron frequency. By using Eqs. (7c) and (7d), immediately we can obtain one of the constants of motion,

$$\gamma(nc - v_z) = \gamma_0(nc - v_{z0}). \tag{8}$$

Here the subscript 0 denotes the initial conditions, and with Eq. (8), it is easy to obtain the identity,

$$\omega - kv_z + \Omega = \Omega d_1 + \omega d_z, \tag{9}$$

with $d_1 = (n^2\omega - kv_{z0} + \Omega_0)/\Omega_0$, $d_2 = 1 - n^2$. The integration of Eq. (9) from 0 to t gives

$$\omega t - kz(t) + kz_0 = (d_1 - 1)\sigma(t) + \omega t d_2, \tag{10}$$

where $\sigma(t) = \int_0^t \Omega(t)dt$ is the cyclotron phase angle. Finally, we obtain the nonlinear equation of electron energy as

$$\begin{aligned}\left(\gamma\frac{d\gamma}{dt}\right)^2 &+ \frac{d_2^2\omega^2}{4}\Bigg\{(\gamma-\gamma_0)^4 + \frac{4(d_1\Omega_0 + d_2\omega)\gamma_0}{\omega d_2}(\gamma-\gamma_0)^3 \\ &+ 4\left[\left(\frac{d_1\Omega_0 + d_2\omega}{\omega}\cdot\frac{\gamma_0}{d_2}\right)^2 - \frac{eE}{mc\omega}\cdot\frac{v_{\perp 0}}{c}\cdot\frac{\gamma_0}{d_2} \right. \\ &\left. - \frac{1}{d_2}\left(\frac{eE}{mc\omega}\right)^2\right](\gamma-\gamma_0)^2 \\ &- \frac{8\gamma_0}{d_2^2}\left[\frac{d_1\Omega_0 + d_2\omega - \Omega_0}{\omega}\left(\frac{eE}{mc\omega}\right)^2\right. \\ &\left. + \frac{d_1\Omega_0 + d_2\omega}{\omega}\cdot\frac{v_{\perp 0}\sin\theta_0}{c}\cdot\frac{eE}{mc\omega}\right](\gamma-\gamma_0) \\ &- 4\left(\frac{\gamma_0}{d_2}\cdot\frac{v_{\perp 0}\cos\theta_0}{c}\cdot\frac{eE}{mc\omega}\right) = 0.\end{aligned} \tag{11}$$

The analytic solutions of Eq. (11) can be reduced to the integrals of the form $\int R[\gamma, \sqrt{V(\gamma)}]d\gamma$, where $V(\gamma)$ is a polynomial of third or fourth degree and R is a rational function. Such integrals can further be reduced to linear combinations of elementary functions and elliptic functions.[23–25]

For an electron beam, if the initial transverse velocity $v_{\perp 0}=0$ and the resonance condition $\omega - kv_{z0} + \Omega_0 = d_1\Omega_0 + d_2\omega = 0$ are satisfied, then Eq. (11) can be written as

TABLE I. Typical values of anomalous Doppler ECM: n is the refractive index, E is the amplitude of the electric field, f is the wave frequency, B_0 is the static magnetic field, γ_0 is the initial relativistic factor, γ_{ph} is the effective relativistic factor.

Case	Figure 3	Figure 5	Figure 7
n	3	1.5	3
E (Statvolt)	10	10	10
f (GHz)	15.09	23.39	15
B_0 (Gauss)	10 000	5000	0
γ_0	1.5	2	1.25
γ_{ph}	1.06	1.34	1.06

$$\left(\frac{d\gamma}{dt}\right)^2 + \frac{d_2^2\omega^2}{4\gamma^2}V(\gamma) = 0. \tag{12}$$

Here $V(\gamma) = (\gamma-\gamma_0)^4 + a(\gamma-\gamma_0)^2 + b(\gamma-\gamma_0)$ is a fourth-order polynomial, and $a = -\frac{4}{d_2}\left(\frac{eE}{mc\omega}\right)^2$, $b = \frac{8\gamma_0}{d_2^2}\cdot\Omega_0/\omega\left(eE/mc\omega\right)$. It is obvious that the polynomial $V(\gamma)$ always has two real roots and a pair of conjugate complex roots,

$$\gamma_1 = \gamma_0, \tag{13a}$$

$$\gamma_2 = \gamma_0 + \sqrt[3]{-b/2 + \sqrt{R}} + \sqrt[3]{-b/2 - \sqrt{R}}, \tag{13b}$$

$$\gamma_3 = \gamma_0 + \Lambda\cdot\sqrt[3]{-b/2 + \sqrt{R}} + \Lambda^2\cdot\sqrt[3]{-b/2 - \sqrt{R}}, \tag{13c}$$

$$\gamma_4 = \gamma_0 + \Lambda^2\cdot\sqrt[3]{-b/2 + \sqrt{R}} + \Lambda\cdot\sqrt[3]{-b/2 - \sqrt{R}}, \tag{13d}$$

where $R = (b/2)^2 + (a/3)^3$, $\Lambda = -1 + i\sqrt{3}/2$, and $i^2 = -1$.

It is found with Ref. 23 that we can express t as elliptic functions of γ, but the final expression is rather complicated (we cannot inverse γ into any known functions of t). For further study, numerical simulations will be presented in the next section.

If the static magnetic field is absent and the beam were rectilinear, i.e., $\Omega_0 = 0$ and $v_{\perp 0} = 0$, then Eq. (11) takes the following form:

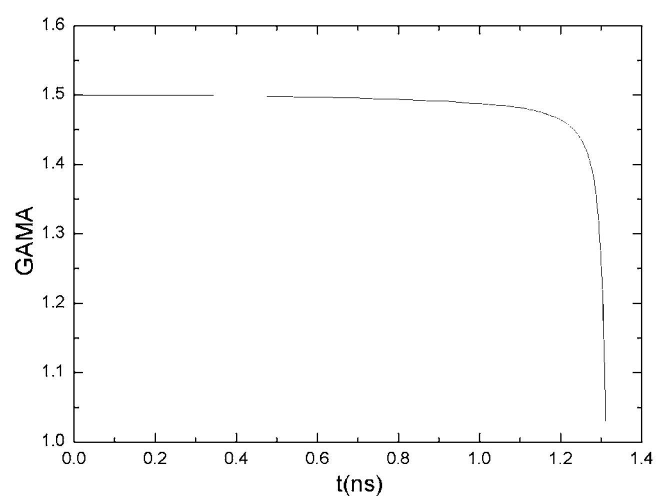

FIG. 3. Relativistic factor vs laboratory time, for initial parameters listed in Table I.

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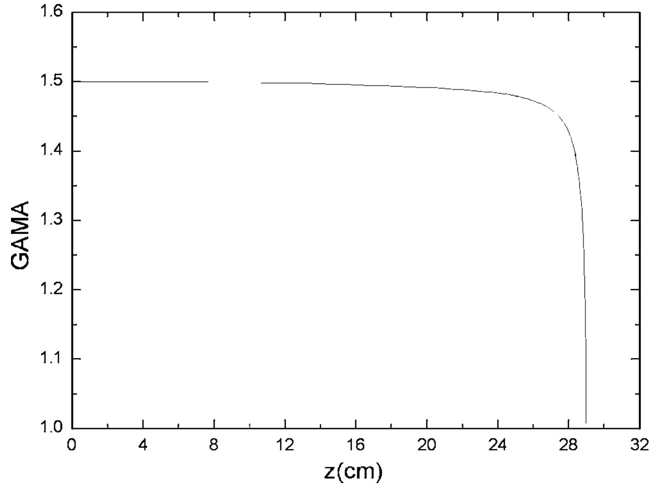

FIG. 4. Relativistic factor vs interaction length (for initial parameters as in Fig. 3).

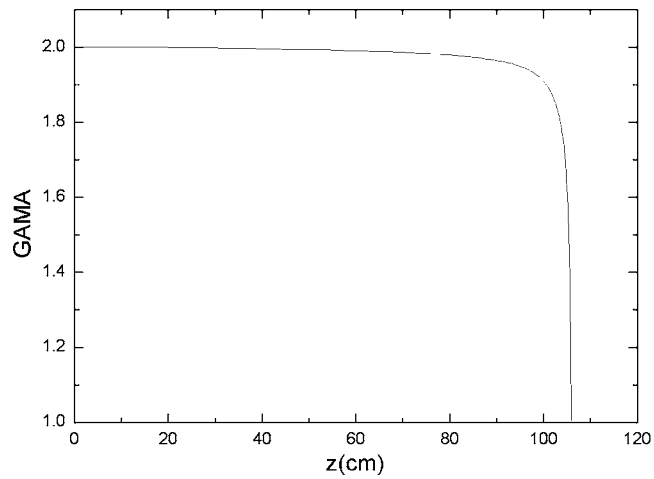

FIG. 6. Relativistic factor vs interaction length (for initial parameters as in Fig. 3).

$$\left(\gamma\frac{d\gamma}{dt}\right)^2+\frac{d_2^2\omega^2}{4}\times\left\{(\gamma-\gamma_0)^4+4\gamma_0(\gamma-\gamma_0)^3+4\left[\gamma_0^2-\frac{1}{d_2}\left(\frac{eE}{mc\omega}\right)^2\right]\times(\gamma-\gamma_0)^2-8\gamma_0\left(\frac{eE}{mc\omega}\right)^2(\gamma-\gamma_0)\right\}=0. \tag{14}$$

Equation (14) can be analyzed similarly as Eq. (12).

It is worth noticing that by use of invariant (8), we have the following relation: $d/dt=[nc-\gamma_0/\gamma(nc-v_{z0})]d/dz$. Then, the variable t in Eq. (11) can be transformed easily to the interaction length z. An analysis of these equations is of fundamental importance for the design of practical devices.

IV. NUMERICAL SIMULATIONS

In this section, with the use of nonlinear Eq. (11), we give some numerical examples of CRM based on anomalous Doppler effect, and the initial parameters are listed in Table I. The parameter γ_{ph} is defined as the relativistic factor of an electron has the velocity of the wave phase velocity, i.e., $\gamma_{ph}=1/\sqrt{(1-v_{ph}^2/c^2)^2}$. We plot the picture of relativistic factor versus laboratory time and interaction length in Figs. 3–8, and as for Figs. 3–6, initially the anomalous resonant condition (1) is satisfied. From these numerical examples we can see clearly that the beam wave interaction has some special features. The first is that the interaction efficiency can approach 100% and the very energetic electron beam is not needed. The second is that the interaction length spans about 20–100 cm when the confine magnetic field is presented. Thus, it is optimum for the design of a practical device. The third is that the curves become much steeper when γ is less than γ_{ph}. This occurs because in the first stage of the interaction the axial force acting upon the electron is small, the axial velocity decreases and the transverse velocity increases, and when the electron axial velocity is below the wave phase velocity, the interaction becomes still more efficient because the electron loses both its transverse and axial momentum in this stage.

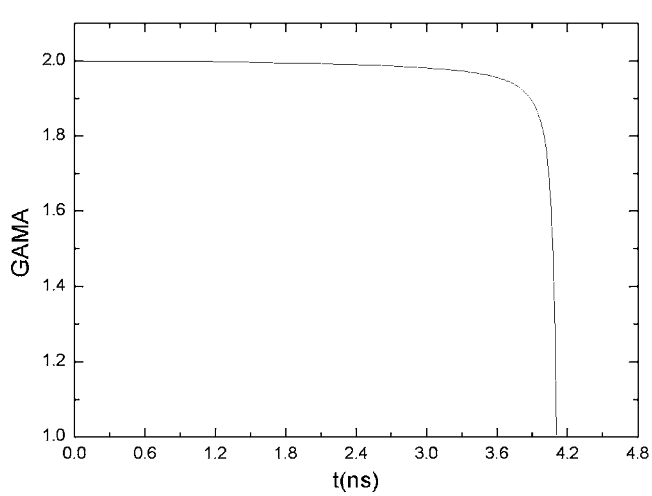

FIG. 5. Relativistic factor vs laboratory time, for initial parameters listed in Table I.

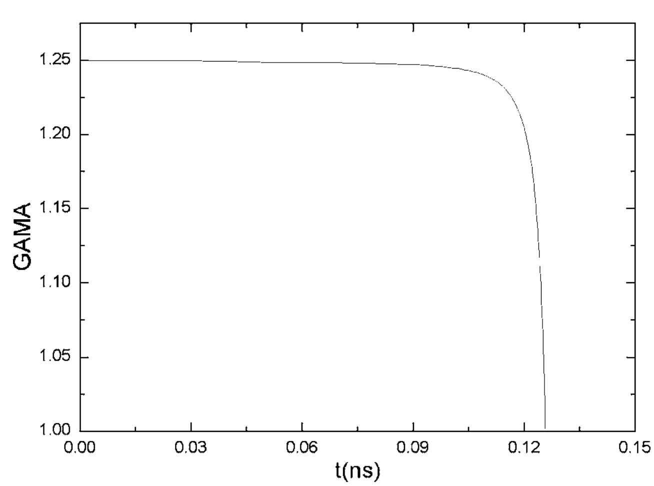

FIG. 7. Relativistic factor vs laboratory time, for initial parameters listed in Table I.

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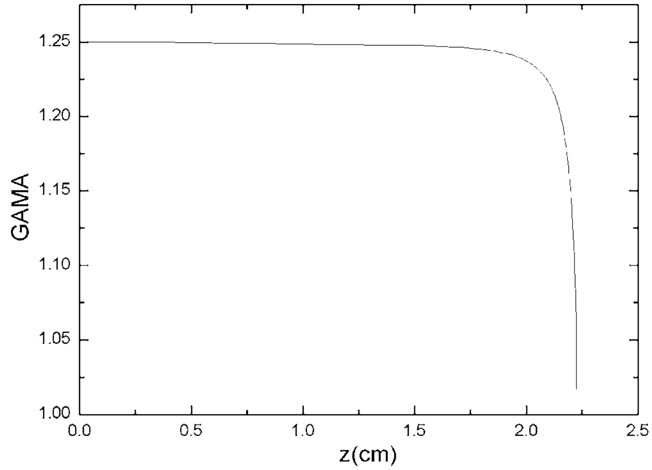

FIG. 8. Relativistic factor vs interaction length (for initial parameters as in Fig. 3).

V. CONCLUSIONS

In summary, we have studied the interaction between a rectilinear electron beam and a slow right circularly polarized electromagnetic wave under the condition of anomalous Doppler effect. The interaction mechanism is presented, and the nonlinear equation for the electron energy has been derived and analyzed. The numerical examples show that the conversion efficiency could be very high and the interaction length spans about 20–100 cm. However, in this study the electron beam is very tenuous and the space charge effect has not been taken into account. These deserve further studies.

ACKNOWLEDGMENTS

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APPENDIX: THE EVOLUTION EQUATION OF THE NORMALIZED ELECTRON ENERGY

The solutions of Eqs. (7a) and (7b) can be expressed as

$$p_x = p_{\perp 0}\cos[\sigma(t)+\alpha] - \frac{eE}{\omega}\int_0^t\left[\omega - k\frac{dz(\tau)}{d\tau}\right]\cos[-\sigma(t) + \sigma(\tau) + \omega\tau - kz(\tau)]d\tau, \tag{A1}$$

$$p_y = p_{\perp 0}\sin[\sigma(t)+\alpha] + \frac{eE}{\omega}\int_0^t\left[\omega - k\frac{dz(\tau)}{d\tau}\right]\sin[-\sigma(t) + \sigma(\tau) + \omega\tau - kz(\tau)]d\tau. \tag{A2}$$

Here $p_{\perp 0}=(p_{x0}^2+p_{y0}^2)^{1/2}$ is the initial transverse momentum, and $\alpha=\arctan(p_{y0}/p_{x0})$. Substituting Eqs. (A1) and (A2) into (7c) and with the use of Eq. (10), we have

$$\frac{1}{\Omega^3}\frac{d\Omega(t)}{dt} = \frac{E}{eB_0^2}\cdot\left\{p_{\perp 0}\cos[d_1\sigma(t)+d_2\omega t+\theta_0] - \frac{eE}{\omega}\int_0^t[(d_1-1)\Omega(\tau)+d_2\omega]\cdot\cos[d_1\sigma(t) + d_2\omega t - d_1\sigma(\tau) - d_2\omega\tau]d\tau\right\}, \tag{A3}$$

where $\theta_0=-kz_0+\alpha$ is the initial angle between $\vec{v}_\perp$ and $\vec{E}$. We multiply Eq. (A3) by $d_1\Omega(t)+d_2\omega$, and then integrate the result from 0 to t, yielding

$$-\frac{d_1}{\Omega} - \frac{d_2\omega}{2\Omega^2} = \frac{E}{eB_0^2}p_{\perp 0}\sin[d_1\sigma(t)+d_2\omega t+\theta_0] - \frac{E^2}{\omega B_0^2}\int_0^t[(d_1-1)\Omega(\tau)+d_2\omega]\cdot\sin[d_1\sigma(t) + d_2\omega t - d_1\sigma(\tau) + d_2\omega\tau]d\tau - \frac{d_1}{\Omega_0} - \frac{d_2\omega}{2\Omega_0^2} - \frac{Ep_{\perp 0}\sin\theta_0}{eB_0^2}. \tag{A4}$$

Differentiating Eq. (A3) with respect to t, we get

$$\frac{d}{dt}\left[\frac{1}{\Omega^3}\frac{d\Omega(t)}{dt}\right] = \frac{E}{eB_0^2}\cdot\left\{-p_{\perp 0}\sin[d_1\sigma(t)+d_2\omega t + \theta_0]\cdot[d_1\Omega(t)+d_2\omega] - \frac{eE}{\omega}[(d_1 - 1)\Omega(t)+d_2\omega] + \frac{eE}{\omega}\int_0^t[(d_1-1)\Omega(\tau) + d_2\omega]\cdot[d_1\Omega(t)+d_2\omega]\cdot\sin[d_1\sigma(t) + d_2\omega t - d_1\sigma(\tau) - d_2\omega\tau]\right\}. \tag{A5}$$

By multiplying Eq. (A4) with $d_1\Omega(t)+d_2\omega$, and adding the result with (A5), noticing that $\Omega=eB_0/m\gamma c$, $\Omega_0 = eB_0/m\gamma_0 c$, we have

$$\frac{1}{\Omega_0^2\gamma_0^2}\cdot\frac{d}{dt}\left(\gamma\frac{d\gamma}{dt}\right) + \frac{\omega^2 d_2^2}{2\Omega_0^2\gamma_0^2}\gamma^2 + \frac{3\omega d_1 d_2}{2\Omega_0\gamma_0}\gamma + \left(d_1^2 - \frac{\omega d_1 d_2}{\Omega_0} - \frac{\omega^2 d_2^2}{2\Omega_0^2} - \frac{\omega d_2 E p_{\perp 0}\sin\theta_0}{eB_0^2}\right) - \frac{1}{\gamma}\left[d_1^2\gamma_0 + \frac{\omega d_1 d_2\gamma_0}{2\Omega_0} + \frac{\Omega_0\gamma_0 d_1 E p_{\perp 0}\sin\theta_0}{eB_0^2} + \frac{\Omega_0\gamma_0 E^2(d_1-1)}{\omega B_0^2}\right] = 0. \tag{A6}$$

We multiply Eq. (A6) with $\gamma d\gamma/dt$, then integrate the result from 0 to t. By noticing that $d\gamma/dt|_{t=0}= -eEv_{\perp 0}\cos\theta_0/mc^2$, the general nonlinear equation of γ (11) can thus be obtained.

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