An Exact Solution of the Relativistic Equation of Motion of a Charged Particle Driven by a Circularly Polarized Electromagnetic Wave and a Constant Magnetic Field

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***Abstract*—An exact solution is found for the relativistic equation of motion of a charged particle driven by a circularly polarized electromagnetic wave and a constant magnetic field. The explicit expressions of particle position and velocity are obtained for certain initial conditions. The results are of interest to the interaction of the high-power laser with the magnetized plasma, electromagnetically pumped free-electron laser with a guide magnetic field, propagation of electromagnetic wave signals through a re-entry plasma sheath in the presence of a strong magnetic field, and magnetic confinement plasmas.**

***Index Terms*— Electromagnetic wave, exact solution, magnetized plasma, relativistic equation of motion.**

I. Introduction

THE solutions of the relativistic equation of motion (REM) of a charged particle driven by a linearly or circularly polarized electromagnetic wave have been investigated by several scientists [1]–[3]. Recently, Acharya and Saxena [4] obtained an exact solution of the REM of a charged particle in the presence of an elliptically polarized electromagnetic wave. These results have applications in many different areas such as the particle acceleration, laser-plasma interaction, free-electron laser, and magnetic confinement plasmas.

Adding a constant magnetic field in the above-mentioned problem results in difficulties in solving the equations of motion. Roberts and Buchsbaum [5] have studied the motion of a charged particle in the presence of a constant magnetic field and a circularly polarized electromagnetic wave. They obtained a single ordinary differential equation of energy using a method of integrating the equations of motion, however, they did not work out the explicit expressions of particle position and velocity. Later, Villalon and Burke [6] studied this problem in a different situation using a numerical method and obtained some valuable results.

In this paper, we are motivated to derive the exact solution of REM of a charged particle in the presence of a constant magnetic field and a circularly polarized electromagnetic wave. The explicit expressions of particle position and velocity are obtained for certain initial conditions.

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II. Solution of the Equation of Motion

Consider a particle with charge q, rest mass $\boldsymbol{m}$, position $\boldsymbol{r} = (x, y, z)$, and velocity $\boldsymbol{v} = (v_x, v_y, v_z)$, moving in the presence of a constant magnetic field

$$\boldsymbol{B}_0 = B_0\hat{z} \tag{1}$$

and an electromagnetic wave with an electric field

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{2}$$

and a magnetic field

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{3}$$

where

$$\boldsymbol{A} = \hat{A}[\cos(kz - \omega t + \phi_0)\hat{x} - \sin(kz - \omega t + \phi_0)\hat{y}] \tag{4}$$

with ω, $\boldsymbol{k}$, and ϕ_0 being the angular frequency, wave number, and initial phase of the electromagnetic wave, respectively. Here $\boldsymbol{A}$ is the vector potential of the electromagnetic wave and $\hat{A}$ is the amplitude of the vector potential.

The equation of motion of the charged particle is given by

$$\frac{d\boldsymbol{p}}{dt} = q[\boldsymbol{E} + \boldsymbol{v} \times (\boldsymbol{B}_0 + \boldsymbol{B})] \tag{5}$$

where $\boldsymbol{p} = m\gamma\boldsymbol{v}$ is the momentum of the particle, $\gamma = (1 - v^2/c^2)^{-1/2}$ represents the relativistic factor, and c denotes the speed of light in vacuum. The components of the equation of motion can be written as

$$\frac{d}{dt}(p_x + q\hat{A}\cos\phi) = qv_yB_0 \tag{6}$$

$$\frac{d}{dt}(p_y - q\hat{A}\sin\phi) = -qv_xB_0 \tag{7}$$

and

$$\frac{dp_z}{dt} = -qk\hat{A}(v_x\sin\phi + v_y\cos\phi) \tag{8}$$

where

$$\phi = kz - \omega t + \phi_0. \tag{9}$$

The expression for the rate of change of energy of the charged particle is given by

$$\frac{d}{dt}(mc^2\gamma) = -q\hat{A}\omega(v_x \sin\phi + v_y \cos\phi). \tag{10}$$

Substituting (8) into (10) results in

$$\frac{d}{dt}\left(mc^2\gamma - v_p p_z\right) = 0 \tag{11}$$

giving

$$mc^2(\gamma - \gamma_0) = v_p(p_z - p_{z0}) \tag{12}$$

where $v_p = \omega/k$ is the phase velocity of the electromagnetic wave, γ_0 is the initial relativistic factor of the particle, and p_{z0} is the initial momentum of $\hat{z}$ component of the particle.

Solving (6), (7) and (8), and assuming $\boldsymbol{r} = 0$ at $t = 0$, one can obtain

$$\gamma v_x = -\frac{\Omega_w}{k}\cos\phi + \Omega_0 y + \gamma_0 v_{\perp 0}\cos\theta_0 \tag{13}$$

$$\gamma v_y = \frac{\Omega_w}{k}\sin\phi - \Omega_0 x - \gamma_0 v_{\perp 0}\sin\theta_0 \tag{14}$$

and

$$\begin{aligned}\gamma v_z =& \gamma_0 v_{z0} - k \\ &\cdot \left[\frac{1}{2}\Omega_0\left(x^2 + y^2\right) + \gamma_0 v_{\perp 0}(y\cos\theta_0 + x\sin\theta_0)\right]\end{aligned} \tag{15}$$

with

$$\begin{aligned}\gamma =& \gamma_0 - \frac{kv_p}{c^2} \\ &\cdot \left[\frac{1}{2}\Omega_0\left(x^2 + y^2\right) + \gamma_0 v_{\perp 0}(y\cos\theta_0 + x\sin\theta_0)\right]\end{aligned} \tag{16}$$

where $\Omega_w = kq\hat{A}/m$, $\Omega_0 = qB_0/m$, $v_{\perp 0}$ and θ_0 are constants that can be determined by the initial conditions, and v_{z0} is the initial velocity of $\hat{z}$ component of the particle.

Now it is useful to define variables ρ and θ satisfying

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta. \end{cases} \tag{17}$$

Substituting (17) into (13)–(15) and (16), we have

$$\frac{d\rho}{dt} = \frac{-\frac{\Omega_w}{k}\cos(\phi+\theta) + \gamma_0 v_{\perp 0}\cos(\theta+\theta_0)}{\gamma_0 - \frac{kv_p}{c^2}\left[\frac{1}{2}\Omega_0\rho^2 + \gamma_0\rho v_{\perp 0}\sin(\theta+\theta_0)\right]} \tag{18}$$

$$\frac{d\theta}{dt} = \frac{\frac{\Omega_w}{k\rho}\sin(\phi+\theta) - \Omega_0 - \frac{\gamma_0 v_{\perp 0}}{\rho}\sin(\theta+\theta_0)}{\gamma_0 - \frac{kv_p}{c^2}\left[\frac{1}{2}\Omega_0\rho^2 + \gamma_0 p v_{\perp 0}\sin(\theta+\theta_0)\right]} \tag{19}$$

and

$$\frac{dz}{dt} = \frac{\gamma_0 v_{z0} - k\left[\frac{1}{2}\Omega_0\rho^2 + \gamma_0\rho v_{\perp 0}\sin(\theta+\theta_0)\right]}{\gamma_0 - \frac{kv_p}{c^2}\left[\frac{1}{2}\Omega_0\rho^2 + \gamma_0\rho v_{\perp 0}\sin(\theta+\theta_0)\right]} \tag{20}$$

with

$$\gamma = \gamma_0 - \frac{kv_p}{c^2}\left[\frac{1}{2}\Omega_0\rho^2 + \gamma_0\rho v_{\perp 0}\sin(\theta+\theta_0)\right]. \tag{21}$$

Using $\gamma = (1 - v^2/c^2)^{-1/2}$, we obtain an exact solution of (18), (19), and (20), which can be expressed as

$$\begin{aligned}&\frac{1}{4}k^2\Omega_0^2\left(1 - \frac{v_p^2}{c^2}\right)\rho^4 + k^2\Omega_0 v_{\perp 0}\gamma_0\left(1 - \frac{v_p^2}{c^2}\right)\rho^3\sin(\theta+\theta_0) \\ &\quad + \Big[\gamma_0\Omega_0(\Omega_0/\gamma_0 + \omega - kv_{z0}) \\ &\qquad + k^2\gamma_0^2 v_{\perp 0}^2\left(1 - \frac{v_p^2}{c^2}\right)\sin^2(\theta+\theta_0)\Big]\rho^2 \\ &\quad + 2\Big\{\frac{\Omega_0}{k}[k\gamma_0 v_{\perp 0}\sin(\theta+\theta_0) - \Omega_w\sin(\phi+\theta)] \\ &\qquad + \gamma_0^2 k v_{\perp 0}(\omega - kv_{z0})\sin(\theta+\theta_0)\Big\}\rho \\ &\quad + \frac{2\Omega_w}{k}v_{\perp 0}\gamma_0[\cos(\phi_0 - \theta_0) - \cos(\phi - \theta_0)] = 0.\end{aligned} \tag{22}$$

We note that (22) is a quartic equation of variable ρ which may be solved analytically in some situations. For example, when the phase velocity of the electromagnetic wave is equal to the speed of light in vacuum, i.e., $v_p = c$, equation (22) reduces to a quadratic equation of variable ρ which can be solved easily. In addition, for special initial conditions, such that $v_{\perp 0} = 0$, equation (22) becomes

$$\left(1 - \frac{v_p^2}{c^2}\right)\rho^3 + \frac{4\gamma_0\Delta\Omega_0}{k^2\Omega_0}\rho - \frac{8\Omega_w}{k^3\Omega_0}\sin\psi = 0 \tag{23}$$

where $\psi = \phi + \theta$ and $\Delta\Omega_0 = \Omega_0/\gamma_0 + \omega - kv_{z0}$. The condition of $\Delta\Omega_0 = 0$ indicates that the charged particle is initially at cyclotron resonance, and in this case one may obtain an interesting result from (23).

Equation (23) is a cubic equation of variable ρ whose solution may be found in conventional hand books of mathematics. Here, we only consider three situations in which the analytical solutions of (23) could be simplified.

A. The Solution in the Case of $v_p = c$ and $\Delta\Omega_0 \neq 0$

When $v_p = c$ and $\Delta\Omega_0 \neq 0$, it is easy to rewrite (23) as

$$\rho = \frac{\sin\psi}{\Delta k} \tag{24}$$

where

$$\Delta k = \frac{\Delta\Omega_0\gamma_0 k}{2\Omega_w}. \tag{25}$$

We assume that

$$\rho = \psi = \phi_0 + \theta_0 = 0 \text{ at } t = 0. \tag{26}$$

Using (24) and

$$\frac{dz}{d\psi} = \frac{dz/dt}{d\psi/dt} = \frac{v_z}{kv_z - \omega + d\theta/dt} \tag{27}$$

one can obtain

$$z = \left(\frac{k^2\Omega_0}{4\Omega_w\Delta k^3} - \frac{k\gamma_0 v_{z0}}{\Delta k\Omega_w}\right)\psi - \frac{k^2\Omega_0}{8\Delta k^3\Omega_w}\sin(2\psi). \tag{28}$$

In this case, the expressions of velocity components can be written as

$$v_\perp = \sqrt{(d\rho/dt)^2 + \rho^2(d\theta/dt)^2} = \frac{\sqrt{\frac{\Omega_w^2}{k^2} + \frac{\Omega_0}{\Delta k}\left(\frac{\Omega_0}{\Delta k} - \frac{2\Omega_w}{k}\right)\sin^2\psi}}{\gamma_0 - \frac{\Omega_0 k}{2\Delta k^2 c}\sin^2\psi} \tag{29}$$

and

$$v_z = dz/dt = \frac{\gamma_0 v_{z0} - \frac{\Omega_0 k}{2\Delta k^2}\sin^2\psi}{\gamma_0 - \frac{\Omega_0 k}{2\Delta k^2 c}\sin^2\psi}. \tag{30}$$

Because

$$\gamma = \gamma_0 - \frac{\Omega_0 k}{2\Delta k^2 c}\sin^2\psi \geq 1 \tag{31}$$

the trajectory of the charged particle is constrained by

$$\frac{\Omega_0 k}{2\Delta k^2 c}\sin^2\psi \leq \gamma_0 - 1 \tag{32}$$

which is related to the initial conditions of the particle motion. It may be mentioned that $\Omega_0 > 0$ for $q > 0$ and $\Omega_0 < 0$ for $q < 0$, and that (32) is always satisfied for $\Omega_0 k < 0$.

B. The Solution in the Case of $v_p = c$ and $\Delta\Omega_0 = 0$

When $v_p = c$ and $\Delta\Omega_0 = 0$, equation (23) becomes

$$\sin\psi = 0 \tag{33}$$

and one can get

$$\rho = \pm\frac{\Omega_w}{k\Omega_0}(\theta - \theta_0) \tag{34}$$

and

$$z = \frac{\Omega_w^2}{6k\Omega_0^2}(\theta - \theta_0)^3 - \frac{\gamma_0 v_{z0}}{\Omega_0}(\theta - \theta_0). \tag{35}$$

The expressions of velocity components for $v_p = c$ and $\Delta\Omega_0 = 0$ are written as

$$v_\perp = \sqrt{(d\rho/dt)^2 + \rho^2(d\theta/dt)^2} = \frac{\sqrt{\frac{\Omega_w^2}{k^2}[1 + (\theta - \theta_0)^2]}}{\gamma_0 - \frac{\Omega_w^2}{2\omega\Omega_0}(\theta - \theta_0)^2} \tag{36}$$

and

$$v_z = dz/dt = \frac{\gamma_0 v_{z0} - \frac{\Omega_w^2}{2k\Omega_0}(\theta - \theta_0)^2}{\gamma_0 - \frac{\Omega_w^2}{2\omega\Omega_0}(\theta - \theta_0)^2} \tag{37}$$

with

$$\gamma = \gamma_0 - \frac{\Omega_w^2}{2\omega\Omega_0}(\theta - \theta_0)^2. \tag{38}$$

In this case, the motion of the charged particle is constrained by

$$\frac{\Omega_w^2}{2\omega\Omega_0}(\theta - \theta_0)^2 \leq \gamma_0 - 1. \tag{39}$$

We note that the inequality (39) is always satisfied for $\omega\Omega_0 < 0$.

C. The Solution in the Case of $v_p \neq c$ and $\Delta\Omega_0 = 0$

In the case of $v_p \neq c$ and $\Delta\Omega_0 = 0$, (23) can be rewritten as

$$\rho = \hat{\rho}\sin^{\frac{1}{3}}\psi \tag{40}$$

where

$$\hat{\rho} = \frac{2}{k}\left[\frac{\Omega_w}{\Omega_0(1 - v_p^2/c^2)}\right]^{\frac{1}{3}}. \tag{41}$$

Using (40), we have

$$z = \pm\frac{k}{\Omega_w}\int_0^{\hat{\rho}\sin^{\frac{1}{3}}\psi} \frac{\left(\gamma_0 v_{z0} - \frac{1}{2}k\Omega_0\rho^2\right)d\rho}{\sqrt{1 - \hat{\rho}^{-6}\rho^6}} \tag{42}$$

and the expressions of velocity components for $v_p \neq c$ and $\Delta\Omega_0 = 0$ are given by

$$v_\perp = \sqrt{(d\rho/dt)^2 + \rho^2(d\theta/dt)^2} = \frac{\left[\frac{\Omega_w^2}{k^2} + \Omega_0^2\hat{\rho}^2\sin^{\frac{2}{3}}\psi - \frac{2\Omega_w\Omega_0\hat{\rho}}{k}\sin^{\frac{4}{3}}\psi\right]^{\frac{1}{2}}}{\gamma_0 - \frac{kv_p\hat{\rho}^2\Omega_0}{2c^2}\sin^{\frac{2}{3}}\psi} \tag{43}$$

and

$$v_z = dz/dt = \frac{\gamma_0 v_{z0} - \frac{1}{2}k\Omega_0\hat{\rho}\sin^{\frac{2}{3}}\psi}{\gamma_0 - \frac{kv_p\hat{\rho}^2\Omega_0}{2c^2}\sin^{\frac{2}{3}}\psi}. \tag{44}$$

It is noticed that

$$\gamma = \gamma_0 - \frac{kv_p\hat{\rho}^2\Omega_0}{2c^2}\sin^{\frac{2}{3}}\psi \geq 1 \tag{45}$$

and the trajectory of the charged particle is constrained by

$$\frac{kv_p\hat{\rho}^2\Omega_0}{2c^2}\sin^{\frac{2}{3}}\psi \leq \gamma_0 - 1. \tag{46}$$

Similarly, when $kv_p\Omega_0 < 0$, the inequality (46) is always satisfied.

III. Conclusion

We have obtained an exact solution to the equation of motion for a charged particle traveling in a constant magnetic field and a circularly polarized electromagnetic wave. The expressions of particle position and velocity are derived for several situations. For special initial conditions of $v_{\perp 0} = 0$, the particle position and velocity are found to be the explicit functions of $\psi = \phi + \theta$ in the case of $v_p = c$ and $\Delta\Omega_0 \neq 0$ or in the case of $v_p \neq c$ and $\Delta\Omega_0 = 0$. However, when $v_p = c$ and $\Delta\Omega_0 = 0$, the particle position and velocity are explicit functions of θ. In addition, the trajectory of the charged particle for each situation is found to be constrained by a constraint condition.

The results obtained in this paper may be applicable to the areas of high-power laser-plasma interaction, free-electron laser with an electromagnetic wave wiggler and axial guide magnetic field, and magnetic confinement plasmas. Additionally, the results may also find applications in the propagation of electromagnetic wave signals through a re-entry plasma sheath in the presence of a strong magnetic field.

REFERENCES

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*. Oxford, U.K.: Pergamon, 1975, p. 118.

[2] H. R. Jory and A. W. Trivelpiece, "Charged particle motion in large-amplitude electromagnetic fields," *J. Appl. Phys.*, vol. 39, pp. 3053–3059, 1968.

[3] J. V. Shebalin, "An exact solution to the relativistic equation of motion of a charged particle driven by a linearly polarized electromagnetic wave," *IEEE Trans. Plasma Sci.*, vol. 16, pp. 390–392, 1988.

[4] S. Acharya and A. C. Saxena, "The exact solution of the relativistic equation of motion of a charged particle driven by an elliptically polarized electromagnetic wave," *IEEE Trans. Plasma Sci.*, vol. 21, pp. 257–259, 1993.

[5] C. S. Roberts and S. J. Buchsbaum, "Motion of a charged particle in a constant magnetic field and a transverse electromagnetic wave propagating along the field," *Phys. Rev. A*, vol. 135, pp. 381–389, 1964.

[6] E. Villalon and W. J. Burke, "Relativistic particle acceleration by obliquely propagating electromagnetic fields," *Phys. Fluids*, vol. 30, pp. 3695–3702, 1987.

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