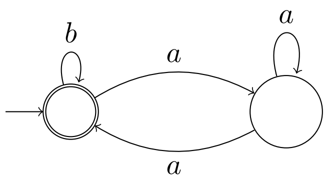
Sample solution for midterm exam

In the following, the alphabet is always $\Sigma = \{a, b\}$.

Question 1.

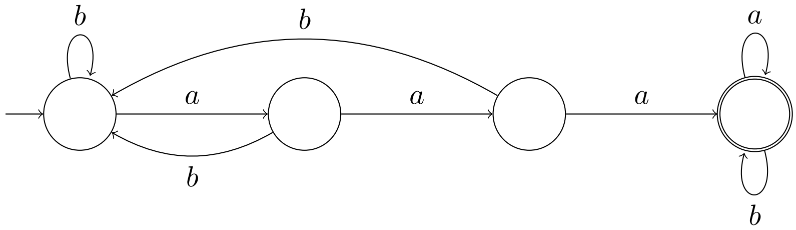
(a) $L_1 = \{a^{2n} \mid n \geqslant 1\}$ is defined by the regex: $aa(aa)^*$.

(b) $L_2 = \{w \mid \text{every } a \text{ in } w \text{ is followed immediately by } b\}$ is accepted by the following DFA:

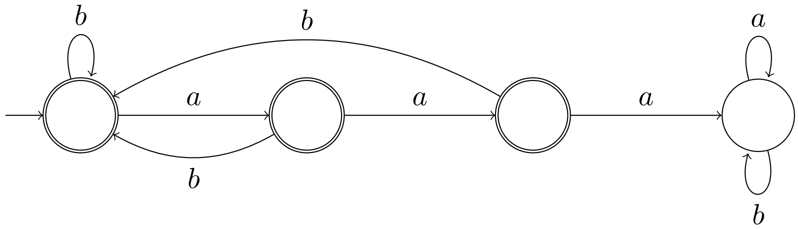


(c) $L_3 = \{w \mid w \text{ contains three consecutive } a\text{'s}\}$ is defined by the regex: $\Sigma^* aaa \Sigma^*$.

Alternatively, one can also construct the following DFA:



(d) $L_4 = \{w \mid w \text{ does not contain three consecutive } a\text{'s}\}$ is accepted by the complement of the DFA in (c):



Question 2. Abusing the notation, for two regex e_1 and e_2, we write $e_1 = e_2$ to denote $L(e_1) = L(e_2)$. Likewise, $e_1 \subseteq e_2$ denotes $L(e_1) \subseteq L(e_2)$ and $e_1 \neq e_2$ denotes $L(e_1) \neq L(e_2)$.

(a) $r^* \cup s^* \neq (r \cup s)^*$, when $r = a$ and $s = b$.

(b) $(r^* \cdot s^*)^* \neq (r \cdot s)^*$, when $r = a$ and $s = b$.

(c) $(r^* \cup s)^* = (r \cup s)^*$.

Proof: Note that $r \cup s \subseteq r^* \cup s$, hence, $(r \cup s)^* \subseteq (r^* \cup s)^*$. The other direction follows from the following.

$$(r^* \cup s)^* \quad \subseteq \quad ((r \cup s)^* \cup s)^* \quad = \quad ((r \cup s)^*)^* \quad = \quad (r \cup s)^*$$

The inclusion comes from the fact that $r \subseteq r \cup s$, and hence, $r^* \subseteq (r \cup s)^*$. The first equality comes from the fact that $s \subseteq (r \cup s)^*$, and hence, $(r \cup s)^* \cup s = s$, whereas the second equality from the fact that $(A^*)^* = A^*$, for every set A.

(d) $(r^* \cup s^*)^* = (r \cup s)^*$.

Proof: Applying the equality in (c), we have the following.

$$(r \cup s)^* \quad = \quad (r^* \cup s)^* \quad = \quad (r^* \cup s^*)^*.$$

Question 3. In the following, capital letters X, S are variables and S is always the start variable.

(a) $K_1 = \{a^n b^n \mid n \geqslant 1\}$ is generated by the following grammar:

$$S \rightarrow aSb \mid ab$$

(b) $K_2 = \{a^n x b^n \mid x \in \Sigma^* \text{ and } n \geqslant 1\}$ is generated by the following grammar:

$$\begin{array}{rcl} S & \rightarrow & aSb \mid X \\ X & \rightarrow & aX \mid bX \mid \epsilon \end{array}$$

(c) $K_3 = \{a^n b^n a^m b^m \mid n, m \geqslant 1\}$ is generated by the following grammar:

$$\begin{array}{rcl} S & \rightarrow & XX \\ X & \rightarrow & aXb \mid ab \end{array}$$

(d) $K_4 = \{a^n b^m a^m b^n \mid m, n \geqslant 1\}$ is generated by the following grammar:

$$\begin{array}{rcl} S & \rightarrow & aSb \mid X \\ X & \rightarrow & bXa \mid ba \end{array}$$

Question 4. Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ be a left-linear grammar. Construct the following NFA $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$, where the set of states Q is $V \cup \{q_f\}$, the initial state q_0 is S, the set of final states is $F = \{q_f\}$, and the set of transitions δ is as follows.

- For every rule $A \rightarrow cB$ in R, we have a transition (A, c, B) in δ.
- For every rule $A \rightarrow c$ in R, we have a transition (A, c, q_f) in δ.

To show that $L(\mathcal{G}) = L(\mathcal{A})$ holds, we can prove that for every word w, for every variable A, the following holds.

- $S \Rightarrow^* wA$ if and only if there is a run of $\mathcal{A}$ from S to A on w.
- $S \Rightarrow^* w$ if and only if there is a run of $\mathcal{A}$ from S to q_f on w.

The proof is by straightforward induction on the length of w.

Question 5. Let $\mathcal{A}_1 = \langle \Sigma, \Gamma, Q_1, q_{0,1}, F_1, \delta_1 \rangle$ be a PDA that accepts K and $\mathcal{A}_2 = \langle \Sigma, Q_2, q_{0,2}, F_2, \delta_2 \rangle$ be an NFA that accepts L.

Construct the following PDA $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ that simulates both $\mathcal{A}_1$ and $\mathcal{A}_2$ simultaneously.

- $Q = Q_1 \times Q_2$.
- $q_0 = (q_{0,1}, q_{0,2})$.
- $F = F_1 \times F_2$.

- δ is defined as follows.
  - For every $(p_1, x, \mathsf{pop}(y) \to (q_1, \mathsf{push}(z)) \in \delta_1$, where $x \neq \epsilon$ and $(p_2, x, q_2) \in \delta_2$, the following transition is in δ:

$$((p_1, p_2), x, \mathsf{pop}(y) \quad \to \quad ((q_1, q_2), \mathsf{push}(z))$$

  - For every $(p_1, x, \mathsf{pop}(y) \to (q_1, \mathsf{push}(z)) \in \delta_1$, where $x = \epsilon$, for every $p_2 \in Q_2$, the following transition is in δ:

$$((p_1, p_2), x, \mathsf{pop}(y) \quad \to \quad ((q_1, p_2), \mathsf{push}(z))$$

That $\mathcal{A}$ accepts precisely $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ can be proved in a similar manner as the fact that regular languages are closed under intersection.