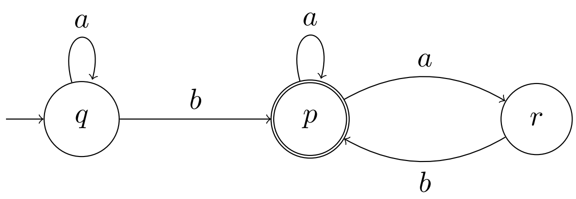
Sample solution for midterm

(1) Consider the following automaton $\mathcal{A}$.

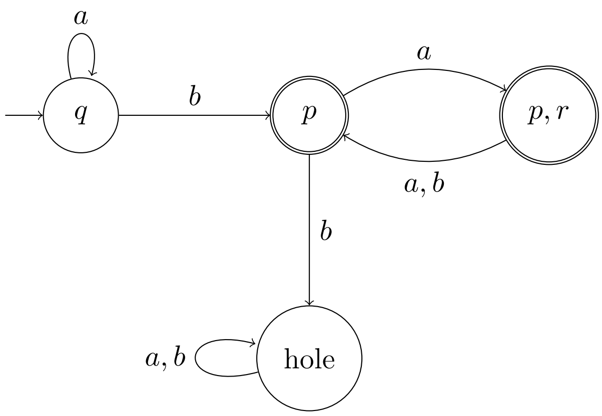


(i) Is $\mathcal{A}$ deterministic or non-deterministic?
Ans: Non-deterministic.

(ii) Is aaa accepted by $\mathcal{A}$?
Ans: No.

(iii) Is $ababababa$ accepted by $\mathcal{A}$?
Ans: Yes.

(iv) Construct the deterministic automaton for $\mathcal{A}$.
Ans:



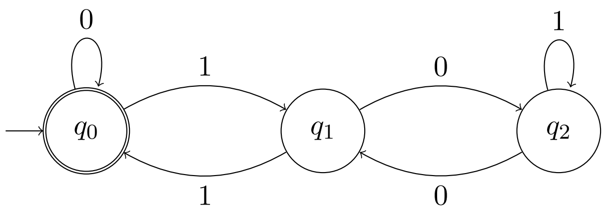
(2) Construct a DFA for the following language over the alphabet $\{0, 1\}$:

$$L_0 := \{w \mid w \text{ represents an integer divisible by } 3\}.$$

Ans: First, we calculate the following:

$$\begin{array}{ll} 2 \cdot 0 + 0 \equiv 0 \bmod 3 & 2 \cdot 0 + 1 \equiv 1 \bmod 3 \\ 2 \cdot 1 + 0 \equiv 2 \bmod 3 & 2 \cdot 1 + 1 \equiv 0 \bmod 3 \\ 2 \cdot 2 + 0 \equiv 1 \bmod 3 & 2 \cdot 2 + 1 \equiv 2 \bmod 3 \end{array}$$

Then, we can construct a DFA with three states q_0, q_1, q_2 corresponding to $0, 1, 2$, respectively.



(3) Construct the CFG for each of the following languages.

- $L_1 := \{w \# w^R \# \mid w \in \{0,1\}^*\}$.
Ans: L_1 can be generated by the following grammar with S being the start variable.

$$\begin{array}{rcl} S & \to & X\# \\ X & \to & \# \mid 0X0 \mid 1X1 \end{array}$$

- $L_2 := \{w_1 \# w_1^R \# w_2 \# w_2^R \# \cdots \# w_k \# w_k^R \# \mid \text{each } w_i \in \{0,1\}^* \text{ for some } k \geqslant 1\}$.
Ans: L_2 can be generated by the following grammar with T being the start variable.

$$\begin{array}{rcl} T & \to & ST \mid S \\ S & \to & X\# \\ X & \to & \# \mid 0X0 \mid 1X1 \end{array}$$

(4) Prove or disprove the following.

- If L is regular and K is CFL, then $L \cap K$ is regular.
Ans: The statement is wrong. Consider the following languages L and K.

$$\begin{array}{rcl} L & := & \{a^n b^n \mid n \geqslant 0\} \\ K & := & \Sigma^* \end{array}$$

We have learned that L is CFL, but not regular, while K is obviously regular. Thus, $L \cap K = L$ is not regular.
- If L is regular and K is CFL, then $L \cup K$ is regular.
Ans: The statement is wrong. Consider the following languages L and K.

$$\begin{array}{rcl} L & := & \{a^n b^n \mid n \geqslant 0\} \\ K & := & \emptyset \end{array}$$

We have learned that L is CFL, but not regular, while K is obviously regular. Thus, $L \cup K = L$ is not regular.

(5) Prove that if L is regular, then $\mathsf{half}(L)$ is also regular.

Ans: Suppose L is regular and is accepted by a DFA $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$.

Consider the following ϵ-NFA $\mathcal{A}' = \langle \Sigma, Q', q_0', F', \delta' \rangle$.

- $Q' = Q \times Q \cup \{p\}$, where $p \notin Q$.
- $q_0' = p$.
- $F = \{(q,q) \mid q \in Q\}$.
- δ' is the following set of transitions.

$$\begin{array}{rcl} \delta' & = & \{(p, \epsilon, (q_0, q_f)) \mid q_f \in F\} \\ & & \cup \ \{((q_1, q_2), a, (q_1', q_2')) \mid (q_1, a, q_2) \in \delta \text{ and } (q_2', b, q_2) \in \delta \text{ for some } b \in \Sigma\} \end{array}$$

The idea is that on input word $c_1 \cdots c_n$, $\mathcal{A}'$ simulates $\mathcal{A}$ both "going forward" from the initial state q_0 and "going backward" from one of the final states $q_f \in F$.

We will prove that $L(\mathcal{A}') = \mathsf{half}(L)$. If a word $c_1 c_2 \cdots c_n d_1 \cdots d_n$ is accepted by $\mathcal{A}$, where each $c_i, d_i \in \Sigma$, with an accepting run:

$$q_0 \; c_1 \; q_1 \; \cdots \; q_{n-1} \; c_n \; q_n \; d_n \; q_{n+1} \; \cdots \; q_{2n-1} \; d_n \; q_{2n}, \qquad \text{where } q_{2n} \in F,$$

then the following is a run of $\mathcal{A}'$ on $c_1 c_2 \cdots c_n$ by definition of δ':

$$p \; \epsilon (q_0, q_{2n}) \; c_1 \; (q_1, q_{2n-1}) \; \cdots \; (q_{n-1}, q_{n+1}) \; c_n \; (q_n, q_n).$$

Since $(q_n, q_n) \in F$, the word $c_1 \cdots c_n$ is accepted by $\mathcal{A}'$.

Vice versa, if $c_1 c_2 \cdots c_n$ is accepted by $\mathcal{A}'$, the accepting run must be of the form:

$$p \; \epsilon (q_0, q_{2n}) \; c_1 \; (q_1, q_{2n-1}) \; \cdots \; (q_{n-1}, q_{n+1}) \; c_n \; (q_n, q_n), \qquad \text{where } q_{2n} \in F.$$

By definition of δ', we have the following run on some $d_1 \cdots d_n$:

$$q_n \; d_1 \; q_{n+1}) \; \cdots \; q_{2n-1}) \; d_n \; q_{2n}$$

This means there is a run of $\mathcal{A}$ on $c_1 \cdots c_n d_1 \cdots d_n$:

$$q_0 \; c_1 \; q_1 \; \cdots \; q_{n-1} \; c_n \; q_n \; d_n \; q_{n+1} \; \cdots \; q_{2n-1} \; d_n \; q_{2n}$$

Since $q_{2n} \in F$, $c_1 \cdots c_n d_1 \cdots d_n$ is accepted by $\mathcal{A}$. Therefore, $\mathcal{A}'$ accepts $\mathsf{half}(L)$.