Solutions to Examination #1 (範圍: Counting Techniques)

$a_{n+2}-6a_{n+1}+9a_n=3\cdot 2^n+7\cdot 3^n \quad n\geq 0 \qquad a_2=$

1. Find a_n, where $a_n-6a_{n-1}+9a_{n-2}=3(2^{n-2})+7(3^{n-2})$ for $n\geq 2$ with $a_0=1$ and $a_1=4$. (10%)

$a_2-24+9=3+7 \qquad a_2=25$

2. In how many ways can we arrange the six integers $1, 2, \ldots, 6$ in a line so that there is no occurrence of patterns 12, 34, and 56 (e.g., 214563 is not permitted)? (10%)

3. Suppose that a 20-digit ternary (0, 1, 2) sequence is randomly generated. Show that the probability of having an even number of 1's is $(1/2)(3^{20}+1)/3^{20}$. (10%)

4. There are $a\times 4^{10}+b\times 3^9+c\times 2^8+d$ 10-digit telephone numbers, which use only the digits 1, 3, 5 and 7, with each appearing at least twice or not at all. Please find the integers a, b, c and d. (10%)

5. Trump tried to solve the problem of finding the number of integer solutions for $x_1+x_2+x_3+x_4=24$, where $3\leq x_i\leq 8$ for $1\leq i\leq 4$, as follows.

   The answer is the coefficient of x^{24} in

$$\begin{aligned} F(x) &= (x^3+x^4+x^5+\ldots)^4 \\ &= x^{12}(1+x+x^2+\ldots)^4 \\ &= x^{12}(1-x)^{-4}, \end{aligned}$$

   which is 455.

   Is Trump's solution correct? Give the details how to get the final answer 455 if you think it correct, and give the correct answer together with the necessary computation if you think it incorrect. (10%)

$a_1=$

6. The following solution to $a_n+2a_{n-1}=n+3$ with $a_0=3$ is not correct. Please solve it correctly. (10%)

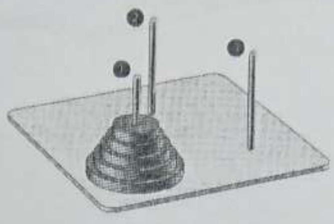
$$\begin{aligned} \text{Let } a_n^p=cn+d. \quad &\Rightarrow \quad (cn+d)+2(c(n-1)+d)=n+3 \\ &\Rightarrow \quad 3cn+(3d-2c)=n+3 \\ &\Rightarrow \quad c=1/3,\ d=11/9 \end{aligned}$$

$$a_n^h=k(-2)^n \text{ and } a_0=3 \quad \Rightarrow \quad k=3$$

$$\text{So, } a_n=a_n^h+a_n^p=3\times(-2)^n+n/3+11/9.$$

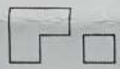
7. Consider three pegs P_1, P_2, P_3 and n disks $D_1, D_2, \ldots, D_n$ of different sizes with holes in their centers. Initially, these disks are stacked on P_1 without a larger one resting on a smaller one. Refer to the following graph for the situation of $n=5$. Moving $D_1, D_2, \ldots, D_n$

from P_1 to P_3 can be carried out by a recursive strategy as follows. First, moving D_1, $D_2, \ldots, D_{n-1}$ from P_1 to P_2; second, moving D_n from P_1 to P_3; third, moving $D_1, D_2, \ldots, D_{n-1}$ from P_2 to P_3. The procedure below is a non-recursive implementation of the strategy for $n=4$, where steps (6) and (11) are missing and "D_k: $P_i \to P_j$" denotes "move D_k from P_i to P_j". Please complete steps (6) and (11). (10%)

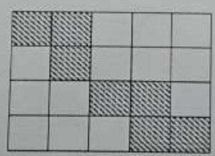


(1) D_1: $P_1 \to P_2$; (2) D_2: $P_1 \to P_3$; (3) D_1: $P_2 \to P_3$; (4) D_3: $P_1 \to P_2$;
(5) D_1: $P_3 \to P_1$; (6) omitted (7) D_1: $P_1 \to P_2$; (8) D_4: $P_1 \to P_3$;
(9) D_1: $P_2 \to P_3$; (10) D_2: $P_2 \to P_1$; (11) omitted (12) D_3: $P_2 \to P_3$;
(13) D_1: $P_1 \to P_2$; (14) D_2: $P_1 \to P_3$; (15) D_1: $P_2 \to P_3$.

8. Let a_n be the number of ways to cover a $2 \times n$ chessboard with 1-square or 3-square tiles (see below). Show that $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$. (10%)



9. Explain why there are $5! - r_1 \cdot 4! + r_2 \cdot 3! - r_3 \cdot 2! + r_4$ ways to place 4 nontaking rooks on the unshaded area of the following chessboard, where $1 + r_1x + r_2x^2 + r_3x^3 + r_4x^4$ is the rook polynomial of the shaded area of the chessboard. (10%)



10. Prove $N(\overline{c}_1\,\overline{c}_2\,\overline{c}_3\,\overline{c}_4) = N - \Sigma N(c_i) + \Sigma N(c_ic_j) - \Sigma N(c_ic_jc_k) + \Sigma N(c_ic_jc_kc_l)$. (10%)

Examination #2

(範圍: Algebra)

1. Define a binary relation R on the set of integers as follows: $x\,R\,y \Leftrightarrow x+y$ is odd. Determine whether R is (a) reflexive; (b) symmetric; (c) antisymmetric; (d) transitive. (10%) (直接給答不需解釋，每小題答錯倒扣 2.5 分，此題得分不為負數)

2. Let G be a group and S be a subgroup of G with $1 < |S| < |G|$. If $|G| = 45$, then what is the possible value for $|S|$? (10%)

3. Prove that if n is the sum of the squares of two odd integers, then n is not a perfect square. (10%)

4. Let $(K, \cdot, +)$ be a Boolean algebra. The following is a proof of $a \cdot (a+b) = a$, where $a, b \in K$. Show that $a + (a \cdot b) = a$ holds as well. (10%)

$$\begin{aligned} a \cdot (a+b) &= (a \cdot a) + (a \cdot b) = a + (a \cdot b) = (a \cdot 1) + (a \cdot b) \\ &= a \cdot (1+b) = a \cdot 1 = a. \end{aligned}$$

5. Define a commutative ring $(Z, \oplus, \odot)$ as follows: $a \oplus b = a + b - 1$ and $a \odot b = a + b - ab$ for any $a, b \in Z$ (the set of integers).

   (a) Is it an integral domain? (5%)

   (b) Is it a field? (5%)

   Explain your reason.

6. Define a binary relation R on the set of integers as follows: $x\,R\,y \Leftrightarrow x - y \geq 0$ is even.

   (a) Is R a partial ordering? (5%)

   (b) Is R a total ordering? (5%)

   Explain your reason.

7. The following is an outline of a proof for the fact that a finite integral domain $(R, +, \cdot)$ is a field.

   Let $R = \{d_1, d_2, \ldots, d_n\}$, where d_i's are all distinct.

   Then, $\{d_1, d_2, \ldots, d_n\} = \{a \cdot d_1, a \cdot d_2, \ldots, a \cdot d_n\}$ for any $a \in R$ and $a \neq z$, and so $a^{-1} = d_k$ for some $1 \leq k \leq n$.

   (a) Explain why $\{d_1, d_2, \ldots, d_n\} = \{a \cdot d_1, a \cdot d_2, \ldots, a \cdot d_n\}$. (5%)

   (b) Explain why $a^{-1} = d_k$ for some $1 \leq k \leq n$. (5%)

8. Suppose that f: $(R, +, \cdot) \to (S, \oplus, \odot)$ is a ring homomorphism and A is a subring of R. Show that $f(A)$ is a subring of S. (10%)

9. Show that $x = a_1M_1x_1 + a_2M_2x_2 + \ldots + a_kM_kx_k$ is a solution to $x \equiv a_i \pmod{m_i}$ for all $1 \le i \le k$, where a_i's, x_i's, m_i's, and M_i's are all integers satisfying

   (i) $m_i \ge 2$ and $\gcd(m_i, m_j) = 1$ for all $i \ne j$;

   (ii) $0 \le a_i \le m_i - 1$;

   (iii) $M_i = m_1 \ldots m_{i-1}m_{i+1} \ldots m_k$;

   (iv) $M_ix_i \equiv 1 \pmod{m_i}$. (10%)

10. Consider the group $(Z_k, +)$, where $k > 0$ is an integer. The following is an incomplete proof for the fact that $<[a]> = Z_k$ if and only if $\gcd(a, k) = 1$.

    (if) $\gcd(a, k) = 1 \Rightarrow as + kt = 1$ for some integers s and t

    $\vdots$

    (only if) $<[a]> = Z_k \Rightarrow [ap] = [1]$ for some integer p

    $\vdots$

    Please write down the missing parts. (10%)

Examination #3

(範圍: Graph Theory) (All graphs mentioned below are simple)

1. Draw an instance for each of the following graphs.
   (a) $K_{3,3}$. (5%)
   (b) A 5-vertex strongly connected digraph with the fewest edges. (5%)
   (c) A connected graph with six vertices having a unique maximum matching of size 3. Also show the maximum matching. (5%)
   (d) A graph with six vertices having a unique maximum independent set of size 3. Also show the maximum independent set. (5%)
   (e) A connected graph having edge connectivity greater than vertex connectivity. Also show the edge connectivity and vertex connectivity. (10%)
   (f) A digraph with six vertices a, b, c, d, e, f whose transitive closure is as follows. (10%)

$$\begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

2. Why does every planar drawing of a connected planar graph has the same number of regions? (10%)

3. The following is a correctness proof for Kruskal's MST algorithm, with the assumption that all edge costs are distinct.

   Let T be the spanning tree of G generated by Kruskal's algorithm and T^* be an MST of G.

   Supposee that T contains $e_1, e_2, \ldots, e_{n-1}$ and T^* contains $e^*_1, e^*_2, \ldots, e^*_{n-1}$, both in increasing order of costs, where n is the number of vertices.

   Assume $e_1 = e^*_1, e_2 = e^*_2, \ldots, e_{k-1} = e^*_{k-1}, e_k \neq e^*_k$, where $c(e_k) < c(e^*_k)$.

   By inserting e_k into T^*, a cycle is formed, where an edge (denoted by e^*) not in T with $c(e^*) > c(e_k)$ can be found.

   If e^* is replaced with e_k in T^*, then a spanning tree with smaller cost than T^* results, a contradiction.

   Explain why (a) $c(e_k) < c(e^*_k)$ and (b) $c(e^*) > c(e_k)$. (10%)

4. Suppose that G is a digraph and there is an arc between every two of its vertices. Show that G contains a (directed) Hamiltonian path. (10%)

5. Explain why $f(e) = c(e)$ for each $e \in E(S, \bar{S})$ and $f(e) = 0$ for each $e \in E(\bar{S}, S)$, as $F = c(S)$. (10%)

6. Suppose that G is a graph having n vertices $v_1, v_2, \ldots, v_n$, and let d_i be the degree of v_i, where $n \geq 2$ and $1 \leq i \leq n$. Show that G is connected, provided $d_i + d_j \geq n - 1$ for every two nonadjacent vertices $v_i \neq v_j$. (10%)

7. Find the maximum total flow and minimum cut for the following transport network by the Edmonds & Karp's algorithm (i.e., finding the shortest augmenting paths). Also show the augmenting paths in sequence. (10%)