Sample solution HW 1

(2.5 points) Question 1. Construct an NFA for each of the following languages, where $\Sigma = \{a, b\}$.

(a) The language L_1 that consists of all the words in which b appears at least twice.

(b) The language L_2 that consists of all the words that starts with bb and ends with ab.

(c) The language L_3 that consists of all the words that contains aba.

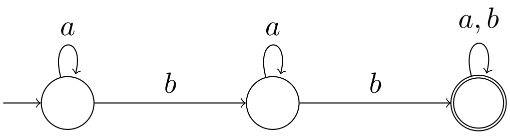
For example: aba and $ababaa$ are in L, since they contain aba. On the contrary, $aaaaa$ and $abbabbabb$ are not in L, since they do not contain aba.

(d) The language L_4 that consists of all the words that do *not* contain bb.

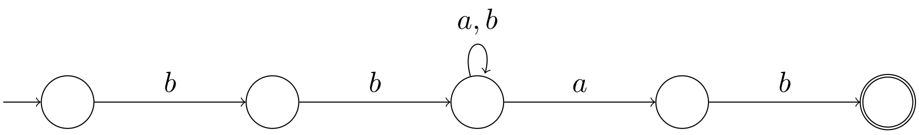
(e) The language L_5 that consists of all the words w such that if w contains bb, then w ends with ab.

Solutions for question 1.

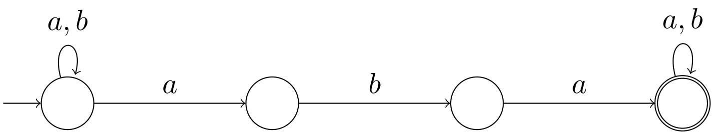
(1.a)



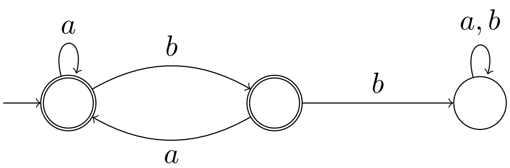
(1.b)



(1.c)



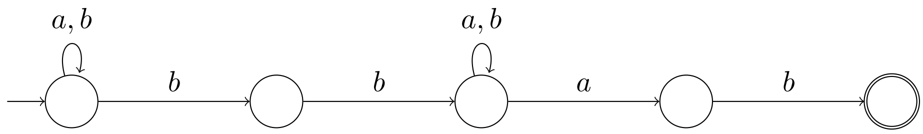
(1.d)



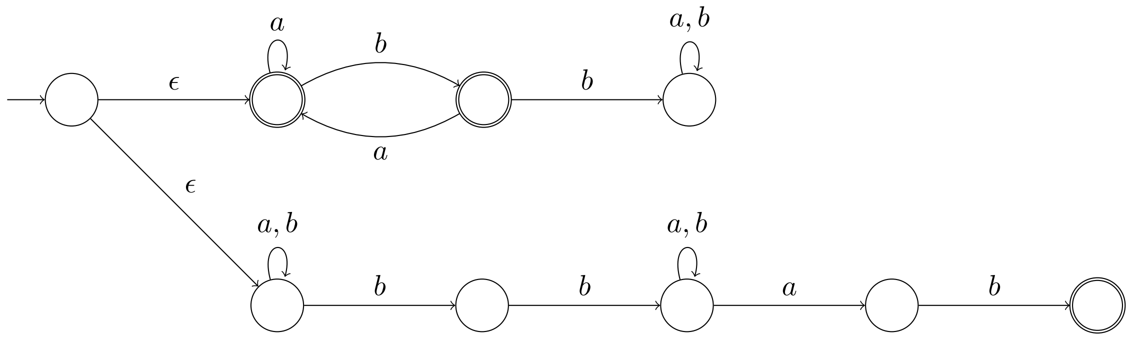
(1.e) Define the following two languages.

$$\begin{aligned} K_1 &= \{w \mid w \text{ does not contain } bb\} \\ K_2 &= \{w \mid w \text{ contains } bb \text{ and ends with } ab\} \end{aligned}$$

Note that $L_5 = K_1 \cup K_2$, and K_1 is the language in question (1.d). We can construct an NFA for K_2 as follows.



From here, we can easily construct an NFA (with ϵ-move) for the language $K_1 \cup K_2$ as follows.



(2.5 points) Question 2. Construct the regular expression for each of the languages above.

Solutions for question 2. In the following $\Sigma = \{a, b\}$.

(1.a) $a^*ba^*b\Sigma^*$.

(1.b) $bb\Sigma^*ab$.

(1.c) $\Sigma^*aba\Sigma^*$.

(1.d) $a^*(baa^*)^*(b \cup \emptyset^*)$.
Intuitively, the regex means that every b must be followed by at least one a, unless it is the last one.

(1.e) $e_1 \ \cup \ \Sigma^*bb\Sigma^*ab$, where e_1 is the regex in (1.d).

(2 points) Question 3. A string $w \in \{0,1\}^*$ represents an integer in a standard way. For example, the string 000 represents the integer 0, and so do 0 and 000000. The string 00100 and 100 both represent the integer 4.

Construct a DFA for the following language over the alphabet $\{0,1\}$:

$$L_0 \quad := \quad \{w \mid w \text{ represents an integer divisible by } 3\}$$

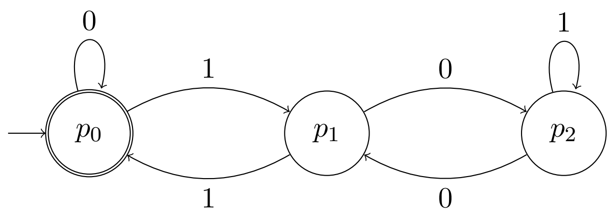
Hint: Consider $(2i+j) \mod 3$, for every $i \in \{0,1,2\}$ and $j \in \{0,1\}$.

Solution for question 3. Note that if a word $w \in \{0,1\}^*$ represents an integer N_w, then $w0$ and $w1$ represent the integer $2 \cdot N_w$ and $2 \cdot N_w + 1$, respectively.

Now, calculate the following:

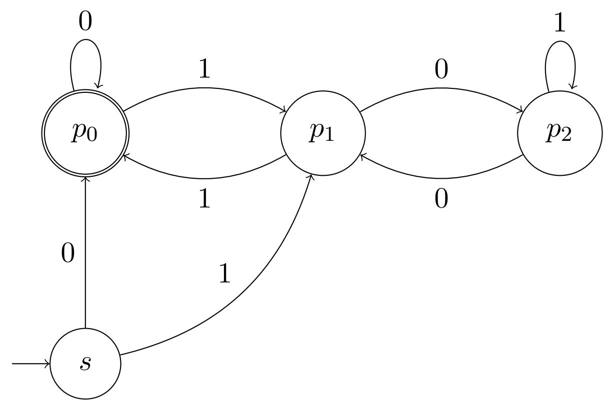
$$\begin{aligned} 2\cdot 0+0 &\equiv 0 \bmod 3 & \qquad 2\cdot 0+1 &\equiv 1 \bmod 3 \\ 2\cdot 1+0 &\equiv 2 \bmod 3 & \qquad 2\cdot 1+1 &\equiv 0 \bmod 3 \\ 2\cdot 2+0 &\equiv 1 \bmod 3 & \qquad 2\cdot 2+1 &\equiv 2 \bmod 3 \end{aligned}$$

Then, we can construct a DFA for the language L_0 with three states p_0, p_1, p_2 corresponding to $0, 1, 2$, respectively.



In the above, we assume that ϵ represents 0, which is divisible by 3.

If you insist that ϵ shouldn't be in L_0, the following DFA is also acceptable.



Question 4. For a language $L \subseteq \Sigma^*$ (not necessarily regular), we define the equivalence relation $\sim_L$ on Σ^*, where $u \sim_L v$ if for every $w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$.

Equivalently, we can say that $u \sim_L v$, if one of the following holds.

- Both uw and vw are in L.
- Both uw and vw are not in L.

(a) **(1 points)** Prove that $\sim_L$ is an equivalence relation.

(b) **(2 points)** In the following, let $\#(\sim_L)$ (read: the index of $\sim_L$) denote the number of equivalence classes of $\sim_L$.

Prove that L is a regular language if and only if $\#(\sim_L)$ is finite.

Solutions for question 4. We will show that $\sim_L$ is an equivalence relation.

- Reflexive: For every $u \in \Sigma^*$, $u \sim_L u$.

  It is rather trivial. For every $w \in \Sigma^*$, $uw \in L$ if and only if $uw \in L$, hence, $u \sim_L u$.

- Symmetric: For every $u, v \in \Sigma^*$, if $u \sim_L v$, then $v \sim_L u$.

  This is also rather trivial. Suppose $u \sim_L v$, which means for every $w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$, which is equivalent to $vw \in L$ if and only if $uw \in L$. Therefore, $v \sim_L u$.

- Transitive: For every $u, v, x \in \Sigma^*$, if $u \sim_L v$ and $v \sim_L x$, then $u \sim_L x$.

  Suppose $u \sim_L v$ and $v \sim_L x$. This means:

  - For every $w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$.
  - For every $w \in \Sigma^*$, $vw \in L$ if and only if $xw \in L$.

  Thus, for every $w \in \Sigma^*$, $uw \in L$ if and only if $xw \in L$. Therefore, $u \sim_L x$.

Now we show the second part. We start with the "only if" part. Let L be a regular language and $\mathcal{A}$ be its DFA.

For a word w, we denote by $\mathcal{A}(w)$ the state of $\mathcal{A}$ after reading w. Or, more formally, if $w = a_1 \cdots a_n$ and $q_0 a_1 q_1 \cdots a_n q_n$ is the run of $\mathcal{A}$ on w, then $\mathcal{A}(w) = q_n$.

We will first prove the following:

Claim 1 *For every words u, v, if $\mathcal{A}(u) = \mathcal{A}(v)$, then $u \sim_L v$.*

Proof. Let u and v be such that $\mathcal{A}(u) = \mathcal{A}(v)$. Let $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$.

We have to show that for every $w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$. Let $w = c_1 \cdots c_k$. Consider the run of $\mathcal{A}$ on uw:

$$p_0 \; a_1 \; p_1 \; \cdots\cdots \; a_n p_n \; c_1 r_1 \; \cdots\cdots c_k r_k$$

Likewise, consider the run of $\mathcal{A}$ on vw:

$$s_0 \; b_1 \; s_1 \; \cdots\cdots \; b_m s_m \; c_1 t_1 \; \cdots\cdots c_k t_k$$

Here both p_0, s_0 is the initial state of $\mathcal{A}$. Since $\mathcal{A}(u) = \mathcal{A}(v)$, we have $p_n = s_m$. Furthermore, $\mathcal{A}$ is deterministic. Thus, $r_1 = t_1, \ldots, r_k = t_k$, and therefore,

$$\mathcal{A}(uw) \;\; = \;\; \mathcal{A}(vw)$$

This completes proof of Claim 1. ■

Claim 1 immediately implies that $\#(\sim_L) \leqslant |Q|$, where Q is the set of states of $\mathcal{A}$. Thus, $\#(\sim_L)$ is finite.

Now, we show the "if" direction. Let L be a language over Σ, where $\sim_L$ has finitely many equivalence classes $C_1, \ldots, C_m$. Without loss of generality, we can assume that $L \neq \emptyset$.

We first prove the following claim.

Claim 2 *There is $i_1, \ldots, i_k \subseteq \{1, \ldots, m\}$ such that $L = C_{i_1} \cup \cdots \cup C_{i_k}$. In other words, L is a union of some of the equivalence classes of $\sim_L$.*

Proof. Note that if $w \sim_L v$, then either both of them belong to L, or both of them do not belong to L. Thus, Claim 2 follows immediately. ■

Now, consider the following DFA $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$.

- $Q = \{p_1, \ldots, p_m\}$, i.e., the number of states is precisely the number of equivalence classes in $\sim_L$.
- q_0 is p_j, where j is such that $\epsilon \in C_j$.
- $F = \{p_{i_1}, \ldots, p_{i_k}\}$, where $i_1, \ldots, i_k$ are the indices in (5.a).
- $\delta : Q \times \Sigma \to Q$ is defined as follows. For every $p_i \in Q$, for every $a \in \Sigma$, we pick an arbitrary $w \in C_i$, and define $\delta(p_i, a) = p_j$, where $[wa] = C_j$.

Note that δ is a well-defined function, i.e., for every $w_1, w_2 \in C_i$, $[w_1 a] = [w_2 a]$.* In other words, the end result p_j remains the same for whichever w we pick, as long as w is from C_i.

We will show that $L(\mathcal{A}) = L$. Recall that $\mathcal{A}(w)$ is the state of $\mathcal{A}$ after reading w starting from the initial state. From the construction of $\mathcal{A}$, for every word $w \in \Sigma^*$, if $[w] = C_j$, then $\mathcal{A}(w) = p_j$. Now,

$$w \in L \quad \text{if and only if} \quad w \in C_{i_1} \cup \cdots \cup C_{i_k}$$

and hence,

$$w \in C_{i_1} \cup \cdots \cup C_{i_k} \quad \text{if and only if} \quad A(w) \text{ is one of } p_{i_1}, \ldots, p_{i_k}.$$

Thus, $w \in L$ if and only if $w \in L(\mathcal{A})$, and hence, $L = L(\mathcal{A})$.

*Here, for $w \in \Sigma^*$, $[w]$ denotes the equivalence class of $\sim_L$ that contains w. See the notation in Lecture 1.