Question 1 (2 points). In the following, the alphabet is $\Sigma = \{a, b\}$. Construct a DFA/NFA for each of the following languages.

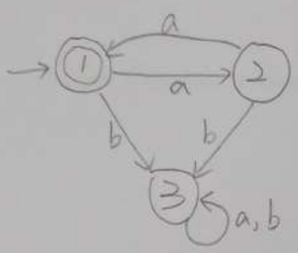
- $L_1 := \{a^{2n} \mid n \geqslant 0\}$.
- $L_2 := \{w \mid w \text{ contains } abb\}$.

  For example, the word bb and $abab$ are not in L_2, because they do not contain abb. On the other hand, $aabbab$, abb and $abbabb$ are in L_2, because they contain abb.
- $L_3 := \Sigma^* - L_1$.
- $L_4 := \Sigma^* - L_2$.

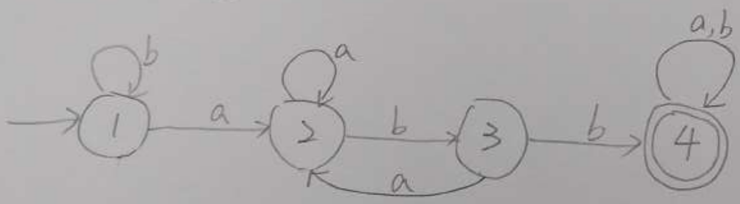
For this solution, your DFA/NFA can only have up to 4 states. You don't need to prove your DFA/NFA is correct.

Solution for question 1.

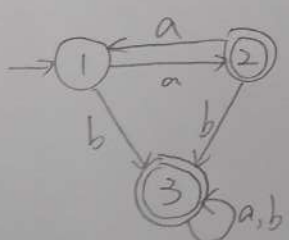
L_1:



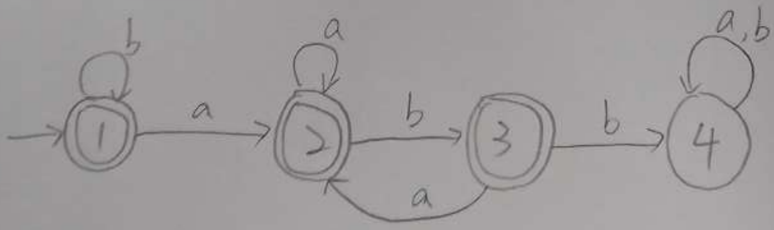
L_2:



L_3:



L_4:



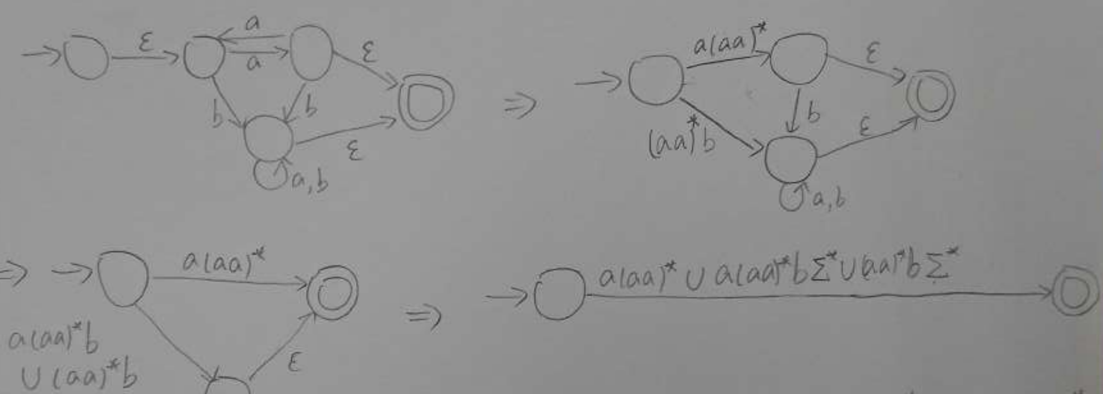
Question 2 (2 points). Construct regular expressions for each of the languages L_1–L_4 in question 1.

Solution for question 2.

L_1: $(aa)^*$ #

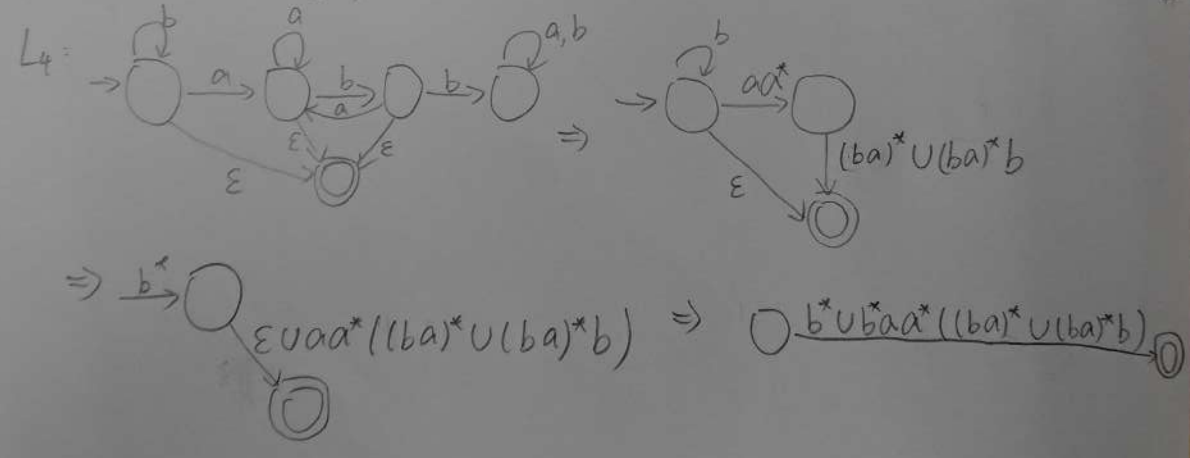
L_2: $\Sigma^* abb \Sigma^*$ #

L_3:



RE of L_3: $a(aa)^* \cup a(aa)^* b \Sigma^* \cup (aa)^* b \Sigma^*$ #

L_4:



Question 3 (2 points). Construct CFG for each of the following languages.

- $L_5 := \{a^n b^n \mid n \geqslant 1\}$.
- $L_6 := \{a^m b^n \mid m \neq n\}$.

You don't need to prove that your CFG is correct, but too complicated grammars will be considered wrong.

Solution for question 3.

L_5: $S \to aSb \mid ab$

L_6: $S \to T \mid U$

$T \to aT \mid aV$

$U \to Ub \mid Vb$

$V \to aVb \mid \varepsilon$

Question 4 (2 points). Let L_7 be the following language.

$$L_7 := \{a^n \mid n \text{ is a perfect square}\}.$$

For example, ϵ, a^4, a^9 all belong to L_7, since $0, 4, 9$ are all perfect square, i.e., $0 = 0^2$, $4 = 2^2$ and $9 = 3^2$. On the other hand, a^5 and a^8 do not belong to L_7, since the square roots of 5 and 8 are not integers.

Prove that L_7 is not CFL.

Solution for question 4.

Proof by Contradiction: Assume L_7 is a CFL, then by Pumping lemma, there exist an integer K s.t. every $w \in L_7$ with len $\geq K$ can be rewrite as:

$w = uxyzv$ and $ux^iyz^iv \in L_7$ $\forall i \geq 0$, $|xyz| \leq K$, $|xz| \geq 1$

Let $w = a^{k^2}$, $|a^{k^2}| = k^2 \geq K$, then $|ux^iyz^iv| = k^2 + (i-1) \times |xz|$

$\leq k^2 + (i-1) \times |xyz| \leq k^2 + (i-1)k$

Consider the case that $i = 2$, $|ux^2yz^2v| \leq k^2 + k$,

but next length of $|a^{k^2}| = k^2$ is $|a^{(k+1)^2}| = (k+1)^2 = k^2 + 2k + 1$,

$\because k^2 + 2k + 1 > k^2 + k$, thus $ux^2yz^2v \notin L_7$

$\rightarrow L_7$ is not a CFL #

Question 5 (2 points). For a language L, define *the square root of* L, denoted by $\mathsf{SQRT}(L)$, as follows.

$$\mathsf{SQRT}(L) := \{u \mid \text{there is } v \in L \text{ such that } |u|^2 = |v|\}.$$

Prove that if L is regular, then $\mathsf{SQRT}(L)$ is regular.

Solution for question 5.

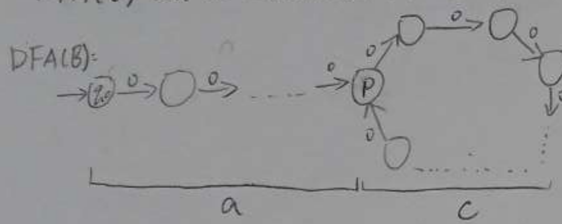
To prove the question, first we prove another following lemma:

Lemma: Let $L(A), L(B) \in RL$, we prove that $C(A,B) = \{x \in A \mid (\exists y)[|y| = |x|^2, y \in B]\}$ is also regular.

Case 1: $A = 0^*$, $B \subseteq 0^*$

DFA(B) can be constructed as below, where $a \geq 0$ & $c \geq 1$ (a, c are number of edges)

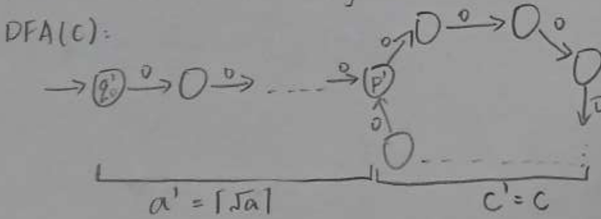
DFA(B):



There must be exactly "1 cycle" in DFA(B) because that every state has exactly "1 out edge".

Now we consider the following DFA to be DFA(C), where

DFA(C):



Now we dealing with the final states mapping between DFA(B) & DFA(C):

First, we define $len(q)$ is the number of edges of the shortest path from q_0 to q

case 1-1: For each final state f in B that $k^2 = len(f) < a$ (not in the cycle), we just map f to f', where $k = len(f') < a'$

case 1-2: For each final state f in B that $k^2 = len(f) \geq a$ (in the cycle), the following holds:

$$0^k \in C \underset{\text{This is the way we construct DFA(C)}}{\Leftrightarrow} 0^{k^2} \in B \underset{\text{obviously}}{\Leftrightarrow} 0^{k^2 + rc} \in B \Leftrightarrow 0^{k+c} \in C$$

Let $r = (2k+c)$, then $k^2 + (2k+c)c = (k+c)^2$

thus we map f to f', where $k = len(f') \geq a'$

To here, we've proved case 1 #

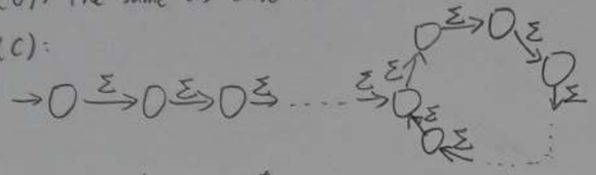
Solution for question 5.

Case 2: $A = \Sigma^*$, $B \subseteq 0^*$

To construct DFA(C), here we only need to replace all edge from 0 to Σ, cause the lemma only focus on the relationship of length between B & C.

DFA(B): The same as Case 1:

DFA(C):



Case 3: $A = \Sigma^*$, $B \subseteq \Gamma^*$

For this case, if we replace all edge in B from a ($a \in \Gamma$) to 0, we can construct B', where $B' = \{ 0^{|x|} \mid x \in B \}$ is also regular, thus Case 3 can be reduced to Case 2.

Case 4: $A \subseteq \Sigma^*$, $B \subseteq \Gamma^*$

from Case 3

because $C(A,B) = A \cap C(\Sigma^*, B)$, A & $C(\Sigma^*, B)$ are both regular, so $C(A,B) = \{ x \in A \mid (\exists y)[\,|y| = |x|^2, y \in B\,] \}$ is also regular

After proving the previous lemma, the original proposition SQRT(L) is equal to the lemma thus we've proved it #