DATA 1301
Introduction to Data Science
Logic and Probability Theory

Amir Shahmoradi

Department of Physics / College of Science
Data Science Program / College of Science
The University of Texas
Arlington, Texas

The Boolean algebra's fundamental identities

Implication

The proposition

$$A \Rightarrow B$$

to be read as 'A implies B', does not assert that either A or B is true; it means only that

$A\bar{B}$ is false,

or, the same thing,

$(\bar{A} + B)$ is true.

This can be written also as the logical equation

$$A = AB.$$

That is, if A is true then B must be true; or, if B is false then A must be false.
On the other hand,
if A is false, $A \Rightarrow B$ says nothing about B, and
if B is true, $A \Rightarrow B$ says nothing about A.

How many logical operations are needed to represent all possible logical expressions?

Suppose we have a set of **logic functions** $\{f_1(A), f_2(A), f_3(A), f_4(A)\}$

A	T	F
$f_1(A)$	T	T
$f_2(A)$	T	F
$f_3(A)$	F	T
$f_4(A)$	F	F

Using a **truth tables** show that the above logic functions are equivalent to the following logical operations,

$$f_1(A) = A + \overline{A}$$
$$f_2(A) = A$$
$$f_3(A) = \overline{A}$$
$$f_4(A) = A\ \overline{A},$$

How many logical operations are needed to represent all possible logical expressions?

We move on to claim without proof here that, the following set of logical operations

{conjunction, disjunction, negation}, i.e. {AND, OR, NOT},

is sufficient to construct all logic functions.

Now, lets consider more general cases: Suppose we have the following special functions that are TRUE only at specific points within the **logical sample space**:

A, B	TT	TF	FT	FF
$f_1(A, B)$	T	F	F	F
$f_2(A, B)$	F	T	F	F
$f_3(A, B)$	F	F	T	F
$f_4(A, B)$	F	F	F	T

We can show that the above truth table is equivalent to the following logical operations.

$$f_1(A, B) = A\ B$$
$$f_2(A, B) = A\ \overline{B}$$
$$f_3(A, B) = \overline{A}\ B$$
$$f_4(A, B) = \overline{A}\ \overline{B}$$

How many logical operations are needed to represent all possible logical expressions?

Question: Show that the following functions,

A, B	TT	TF	FT	FF
$f_5(A, B)$	F	T	F	T
$f_6(A, B)$	T	F	T	T

can be written in terms of the previous **four basis logic functions** as specified below.

$$f_5(A, B) = f_2(A, B) + f_4(A, B)$$

$$f_6(A, B) = f_1(A, B) + f_3(A, B) + f_4(A, B)$$

Which one of the above two functions is equivalent to the **logical implication** $(A \Rightarrow B)$?

The NAND and NOR operations

It turns out that we can further squeeze the minimal set of logical operations from which we can build all other operations. In fact, either NAND (NOT AND) denoted by ↑ , or equivalently, NOR (NOT OR) denoted by ↓ is sufficient to build all other logical operations.

$$A \uparrow B \equiv \overline{AB} = \overline{A} + \overline{B}$$

$$A \downarrow B \equiv \overline{A + B} = \overline{A}\;\overline{B}$$

Question:

Show that the three fundamental operations (negations, disjunction, conjunction) can be all written as a sequence of NAND or NOR operations as given below.

$$\overline{A} = A \uparrow A$$

$$AB = (A \uparrow B) \uparrow (A \uparrow B)$$

$$A + B = (A \uparrow A) \uparrow (B \uparrow B)$$

$$\overline{A} = A \downarrow A$$

$$A + B = (A \downarrow B) \downarrow (A \downarrow B)$$

$$AB = (A \downarrow A) \downarrow (B \downarrow B)$$

The desiderata of Probability Theory

There are a set of properties that we **desire** to have in a theory of probability that we wish to construct now.

(I) Degrees of plausibility are represented by real numbers.

(II) Qualitative correspondence with common sense.

(III) Consistency.

- (I) If a conclusion can be reasoned out in more than one way, then every possible way must lead to the same result.

- (II) We must always consider all the evidence relevant to a question. We should not arbitrarily ignore some of the information, basing the conclusions only on what remains. In other words, the robot is completely nonideological.

- (III) We must always represent equivalent states of knowledge by equivalent plausibility assignments. That is, if in two problems the robot's state of knowledge is the same (except perhaps for the labeling of the propositions), then it must assign the same plausibilities in both.

An example of correspondence with common sense

First, let’s learn the conditional notation: A proposition (A) whose truth is conditioned on the truth of another proposition (B) is typically denoted by,

$$A \mid B$$

Second, **by convention**, we will assume that propositions with greater degree of plausibility correspond to greater real numbers.

Therefore,

$$(A|B) > (C|B)$$

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$$A|B$$

Second, **by convention**, we will assume that propositions with greater degree of plausibility correspond to greater real numbers.

Therefore,

$$(A|B) > (C|B)$$

says that, given B, A is more plausible than C.

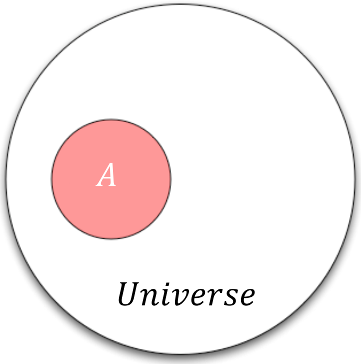
Now, what do we mean by "correspondence with common sense" ? Suppose,

$$(A|C') > (A|C);$$
$$(B|AC') = (B|AC).$$

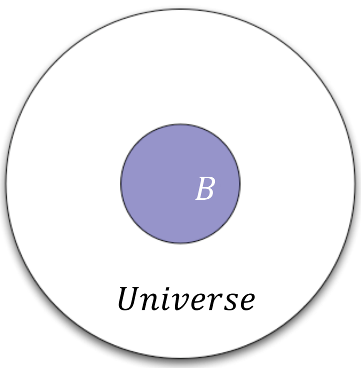
Then, the desiderata of "correspondence with commonsense" requires us to have,

$$(AB|C') \geq (AB|C);$$

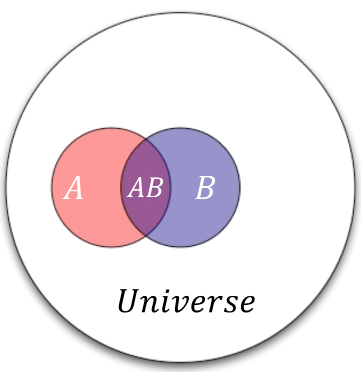
The product Rule (A prelude to the Bayesian Probability Theory)



$$P(A) = \frac{A}{U}$$



$$P(B) = \frac{B}{U}$$



$$P(AB) = \frac{AB}{U}$$

$$P(A|B) = \frac{AB}{B} = \frac{\frac{AB}{U}}{\frac{B}{U}} = \frac{P(AB)}{P(B)}$$

$$P(B|A) = \frac{AB}{A} = \frac{\frac{AB}{U}}{\frac{A}{U}} = \frac{P(AB)}{P(A)}$$

Bayes rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$