Deflection of 2D Mesh Under Applied E field

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1 Notation:

- Y: Young modulus of wire material
- g: Gap spacing at zero field
- V: Applied voltage
- A: Wire cross section
- ϵ: Dielectic permittivity of medium
- T: Wire tension with no applied field
- z(x,y): Mesh position as a function of 2D position
- σ: Stress
- s: Strain
- R: Radius of mesh

2 Calculation

The stored energy in each mesh segment has contributions from both elastic energy in the wires and from the electrostatic energy from the capacitance between planes. The equilibrium shape is determined by functionally minimizing the sum of these contributions using the Euler Lagrange equation.

Stress energy:

A wire segment of length δl_0 pre-tensioned to T stores elastic energy at zero field:

$$E^0_{elastic} = \frac{1}{2}\frac{(A\delta l_0)}{E}\sigma^2 \tag{1}$$

Or in terms of tension, T where ($\sigma = T/A$):

$$E^0_{elastic} = \frac{1}{2}\frac{\delta l_0}{EA}T^2 \tag{2}$$

The length of the pre-tensioned segment at zero field will be:

$$\delta l_{pt} = \delta l_0 \left(1 + \frac{T}{AE}\right) \tag{3}$$

With voltage applied, this segment gets stretched further. If this wire runs in the x direction and has local deformation gradient $\partial z/\partial x$, the deformed length can be related to the un-deformed length by:

$$\delta l_{deformed} = \delta l_{pt}\left[\sqrt{1 + \left(\frac{dz}{dx}\right)^2}\right]. \tag{4}$$

We will always be considering situations where the total curvature is small - that is, mesh displacement much smaller than the radius of the EL assembly. In such cases, $dz/dx \ll 1$ and so we can expand to first order:

$$\delta l_{deformed} = \delta l_{pt}\left[1 + \frac{1}{2}\left(\frac{dz}{dx}\right)^2\right]. \tag{5}$$

The total stored energy is given by:

$$E_{elastic} = \frac{1}{2}\left(A\delta l_0\right) E s^2 = \frac{AE\delta l_0}{2}\left(\frac{\delta l_{deformed} - \delta l_0}{\delta l_0}\right)^2 \tag{6}$$

$$= \frac{AE\delta l_0}{2}\left(\left(1 + \frac{T}{AE}\right)\left[1 + \frac{1}{2}\left(\frac{dz}{dx}\right)^2\right] - 1\right)^2 \tag{7}$$

$$= \frac{AE\delta l_0}{2}\left(\frac{T}{AE} + \frac{1}{2}\left(1 + \frac{T}{AE}\right)\left(\frac{dz}{dx}\right)^2\right)^2 \tag{8}$$

$$= \frac{T\delta l_0}{2}\left[\left(\frac{T}{AE}\right) + \frac{1}{2}\left(1 + \frac{T}{AE}\right)\left(\frac{dz}{dx}\right)^2\right] + \mathcal{O}\left(\frac{dz}{dx}\right)^4 \tag{9}$$

Now we have to integrate over the wires. Recall the above is the expression for the energy stored in a length that was δl when un-tensioned, but δl_{pt} when pre-tensioned. An x-wire at position y on the ring has a tensioned length that stretches from $-x_0 = -\sqrt{R^2 - y^2}$ to $x_0 = \sqrt{R^2 - y^2}$. Integrating along the pre-tensioned wire is equavalent to identifying the integration measures $dx = dl_{pt}$,

$$E^{1\,wire}_{elastic} = \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx\,\frac{T}{2}\left[\left(\frac{T}{AE}\right)\left(1 + \frac{T}{AE}\right)^{-1} + \frac{1}{2}\left(\frac{dz}{dx}\right)^2\right] \tag{10}$$

If there are N_x wires per unit length in Y, we sum to find the contributions from all X wires:

$$E^{x\,wires}_{elastic} = \sum_{j\sim -RN_x}^{j\sim RN_x} \int_{-R\sqrt{1-j^2}}^{R\sqrt{1-j^2}} dx\,\frac{T}{2}\left[\left(\frac{T}{AE}\right)\left(1 + \frac{T}{AE}\right)^{-1} + \frac{1}{2}\left(\frac{dz}{dx}\right)^2\right] \tag{11}$$

And if they are sufficiently densely packed, we can replace the sum with an integral:

$$\sum_{j\sim -RN_x}^{j\sim RN_x} \int_{-R\sqrt{1-j^2}}^{R\sqrt{1-j^2}} dx \;\rightarrow\; \int_{-RN_x}^{RN_x} dj \int_{-R\sqrt{1-j^2}}^{R\sqrt{1-j^2}} dx \tag{12}$$

Writing $j/N_x = y$:

$$= N\int_{x-R}^{R} dy \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx = N_x \int_{\circ R} dxdy \tag{13}$$

Thus the total elastic contribution from the x wires is:

$$E^{x\,wires}_{elastic} = N_x \int_{\circ R} dxdy\,\frac{T}{2}\left[\left(\frac{T}{AE}\right)\left(1 + \frac{T}{AE}\right)^{-1} + \frac{1}{2}\left(\frac{dz}{dx}\right)^2\right] \tag{14}$$

$$= E^x_0 + N_x \int_{\circ R} dxdy\,\frac{T}{4}\left(\frac{dz}{dx}\right)^2 \tag{15}$$

Adding the similar contribution from the Y wires, we find the total stored elastic energy:

$$= E^x_0 + E^y_0 + \int_{\circ R} dxdy\,\frac{T}{4}\left[N_x\left(\frac{dz}{dx}\right)^2 + N_y\left(\frac{dz}{dx}\right)^2\right] \tag{16}$$

A notable observation about this formula is that if the wire were not pre-tensioned, the only restoring force would come from the $\mathcal{O}(\frac{dz}{dx}^4)$ term, which was neglected here. For our purposes, the mesh will always be pre-tensioned, so the second order term suffices.

Electrostatic energy:

Next we have to assess the capacitve energy. We can think of each element of the surface as being a small parallel plate capacitor, all connected in parallel. Thus the total electric energy is:

$$E_{electrical} = \frac{1}{2}CV^2 = \sum_i \frac{1}{2}C_iV^2 = \sum_i \frac{1}{2}\epsilon\frac{\delta x\delta y}{g+z}V^2 \tag{17}$$

Taking the infinitessimal limit:

$$E_{electrical} \to \int_{\circ R} dxdy\,\frac{1}{2}\epsilon\frac{V^2}{(g+z)} \tag{18}$$

We will generally be able to consider small displacements of the mesh. Note that this is not quite as robust an approximation as the one above about its curvature. Nevertheless, it will provide a considerable simplification:

$$E_{electrical} \sim \int_{\circ R} dxdy\,\frac{1}{2}\epsilon\frac{V^2}{g}\left(1-\frac{z}{g}\right) \tag{19}$$

The equilibrium shape:

To find the equilibrium mesh shape z(x,y), we need to minimize the energy functional:

$$E = E_0^x + E_0^y + \int d^2x_i\mathcal{E}[z] \tag{20}$$

$$\mathcal{E}[z] \;= \frac{T}{4}\left[N_x\left(\frac{dz}{dx}\right)^2 + N_y\left(\frac{dz}{dx}\right)^2\right] + \frac{1}{2}\epsilon\frac{V^2}{g}\left(1-\frac{z}{g}\right) \tag{21}$$

Which can be solved with the Euler Lagrange equation:

$$\sum_i \partial_i\left(\frac{\partial\mathcal{E}}{\partial\left[\partial_i z\right]}\right) = \frac{d\mathcal{E}}{dz} \tag{22}$$

Applying the ELG:

$$\frac{T}{4}\left(N_x\frac{\partial^2 z}{\partial x^2} + N_y\frac{\partial^2 z}{\partial y^2}\right) = -\frac{1}{2}\epsilon\frac{V^2}{g^2} \tag{23}$$

Life will be easier if we restrict ourselves to the symmetric case, $N_x = N_y = N$, in which case this reduces to a cylindrically symmetric form:

$$\left(\partial_x^2 + \partial_y^2\right)z = -\kappa \qquad \kappa = \frac{2\epsilon V^2}{g^2TN} \tag{24}$$

This is best attacked in cylindrical polars:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)z = -\kappa \tag{25}$$

To find the general solution, we write z as a power series in ρ:

$$z = \sum_n a_n\rho^n \tag{26}$$

Which, substituted into the above differential equation above, gives:

$$\sum_n n^2a_n\rho^{n-2} = -\kappa \tag{27}$$

Since there are no ρ on the RHS, only the n=2 and n=0 terms contribute. The boundary condition $z(R) = 0$ fixes the relationship between a_0 and a_2, to yield, in the end:

$$a_2 = -\frac{\kappa}{4} \qquad a_0 = \frac{\kappa}{4R^2} \qquad a_{others} = 0 \tag{28}$$

Giving the solution:

$$z = -\frac{\kappa}{4}\left(R^2 - \rho^2\right) \qquad \kappa = \frac{2\epsilon V^2}{g^2 TN} \tag{29}$$

The extremal disortion occurs at $\rho = 0$ and is:

$$z_{max} = \frac{\epsilon V^2 R^2}{2g^2 TN} \tag{30}$$

The following points are notable:

- The Young modulus of the mesh material does not feature in this expression
- The total deflection depends only on the E field, not on the gap size and voltage independently - which I guess is obvious, actually. We can express in terms of EL field E instead of V and g:

$$z_{max} = \frac{\epsilon R^2 E^2}{2TN} \tag{31}$$

- We can also express in terms of the total mesh tension $\tau = 2TNR$ rather than tension per wire T, as:

$$z_{max} = \frac{\epsilon E^2 R^3}{\tau} \tag{32}$$

3 Putting in some real numbers

Lets put in some real units, to be followed by some real numbers.

$$z_{max} = 8.85 \times 10^{-12} * 10^{10} \frac{[kgms^{-3}A^{-1}]^2[m^3][m^{-3}kg^{-1}s^4A^2]}{[kgms^{-2}]} \frac{[E/(kV/cm)]^2[R/m]^3}{[\tau/N]} \tag{33}$$

$$z_{max} = 88.5\, mm \frac{[E/(kV/cm)]^2[R/m]^3}{[\tau/N]} \tag{34}$$

For NEXT-100 dimensions, we fix plate radius at 1.2m. The nominal operating field is 2kV/cm/bar, which means 28kV/cm at 15 bar.

To maintain a deflection less than 1mm for NEXT-100 with 1.2m diameter, a tension of around 1.6 tons is required. 2mm can be achieved with 800 kg. For NEXT-XXX, with 1.5 diameter, the required load increases to 2.4 ton equivalent for 1mm, and 1.2 ton for 2mm. Deflections at some various tensions are shown overleaf.

4 Counteracting mesh deflection by deforming the back surface

By deforming the back plane to shape $b(\rho)$ we change the capacitive stored energy term to:

$$E_{electrical} = \frac{1}{2}CV^2 = \sum_i \frac{1}{2}C_i V^2 = \sum_i \frac{1}{2}\epsilon \frac{\delta x \delta y}{g - b(\rho) + z} V^2 \tag{35}$$

If we assume $b(\rho) \ll g$ then:

$$= \int d^2x \, \frac{1}{2}\epsilon \frac{V^2}{g}\left(1 - \frac{z + b(\rho)}{g}\right) \tag{36}$$

Changing the total energy functional to:

$$\mathcal{E}[z] = \frac{T}{4}\left[N_x \left(\frac{dz}{dx}\right)^2 + N_y \left(\frac{dz}{dx}\right)^2\right] + \frac{1}{2}\epsilon \frac{V^2}{g}\left(1 - \frac{z - b(\rho)}{g}\right) \tag{37}$$

Notably the solution to the ELG equation is unaffected. The conclusion is that given small perturbations of the back surface, the first-order solution for the mesh curvature is unchanged. Thus for small perturbations relative to the gap size, adjusting the back surface to match the curvature at nominal field will achieve the desired effect. This can be understood as resulting from the fact that for small perturbations relative to gap size, the E field as a function of ρ remains approximately constant and so the distorted shape is not affected.

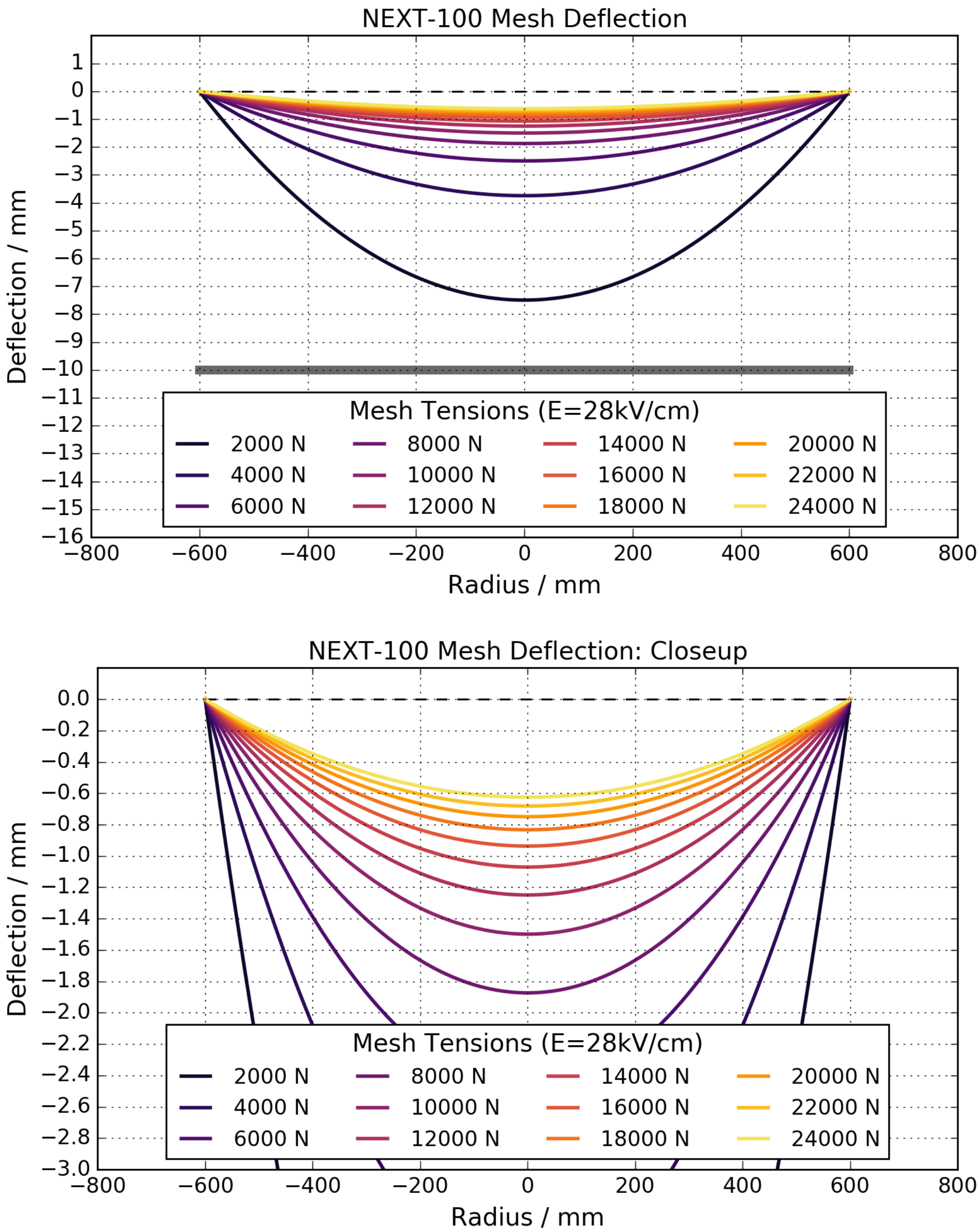

Figure 1: Diagrams showing deflected shape of meshes for NEXT-100 for E=28 kV/cm.

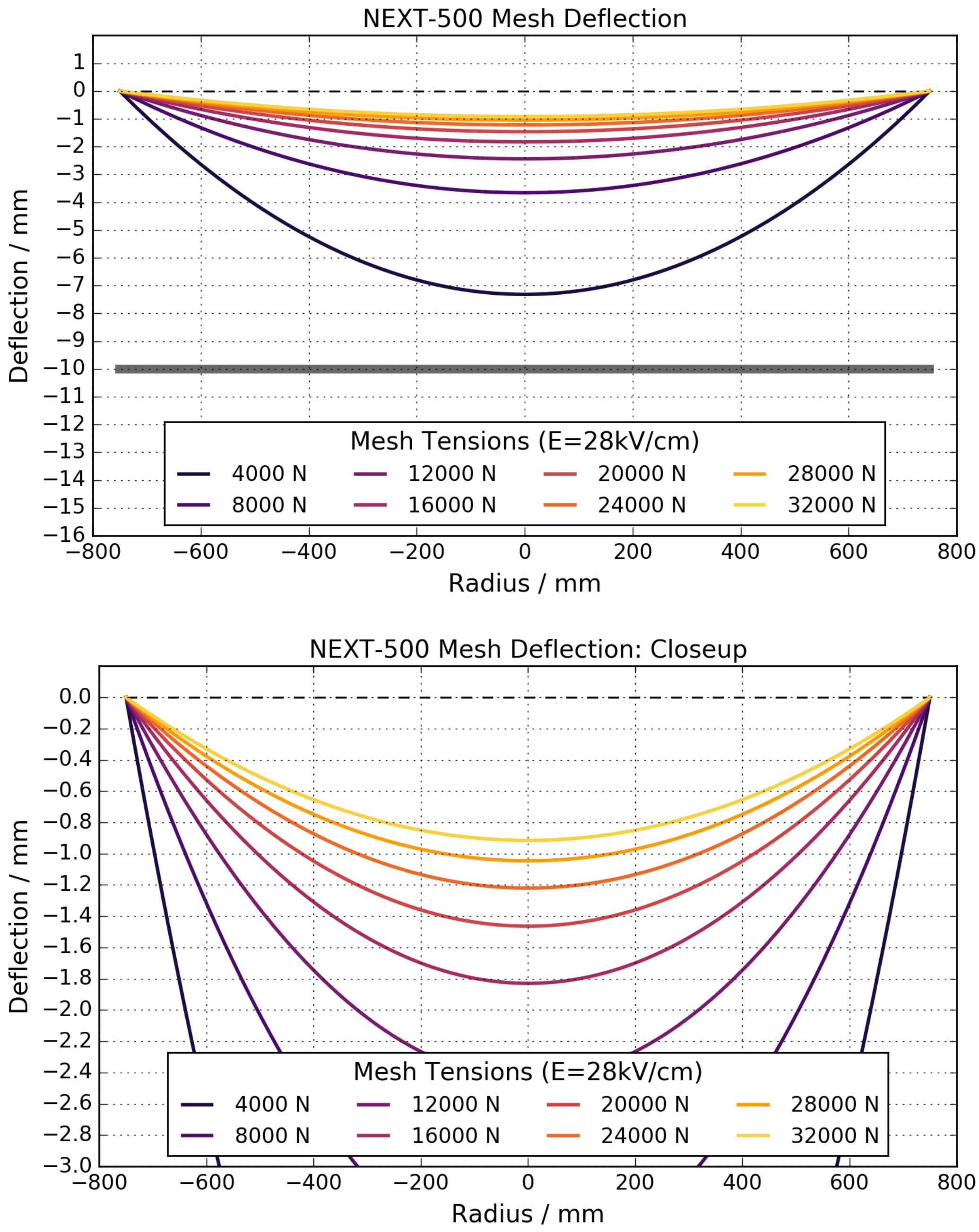

Figure 2: Diagrams showing deflected shape of meshes for NEXT-500 for E=28 kV/cm.

5 What about larger perturbations?

By virtue of the geometry in question, we can safely assume the curvature is always small, $\frac{dz}{d\rho} \ll 1$. However, in some cases we are talking about distortions relative to the gap size that are not necessarily small, $z \lesssim g$. We modify the general energy functional thus:

$$\mathcal{E}[z] = \frac{T}{4}\left[N_x\left(\frac{dz}{dx}\right)^2 + N_y\left(\frac{dz}{dx}\right)^2\right] + \frac{1}{2}\epsilon\frac{V^2}{g - b(\rho) + z} \tag{38}$$

The Euler Lagrange equation can be again solved, the only difference being in the RHS:

$$\frac{T}{4}\left(N_x\frac{\partial^2 z}{\partial x^2} + N_y\frac{\partial^2 z}{\partial y^2}\right) = -\frac{1}{2}\epsilon\frac{V^2}{(g - b(\rho) + z)^2} \tag{39}$$

A sanity check shows that this reduces to the previous form for $b + z \ll g$. Now, on to solve it. Again writing in cylindrical polars:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)z = \frac{2\epsilon V^2}{TN}\frac{1}{(g - b(\rho) + z)^2} \tag{40}$$

It is helpful to change variables to $\zeta(\rho) = g - b(\rho) + z(\rho)$, to yield:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)[\zeta(\rho) + b(\rho) - g] = \frac{2\epsilon V^2}{TN\zeta(\rho)^2} \tag{41}$$

This is not completely trivial for arbitrary $b(p)$, so let us restrict to solutions we really want - a particular choice of back-plane shape where $\zeta(\rho) = \zeta_0$, some constant independent of radius. If such a soluton exists, it would have $\zeta = g$ and $\partial\zeta/\partial\rho = 0$. The ELG would then reduce to:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial b}{\partial\rho}\right) = \frac{2\epsilon V^2}{TNg^2} \tag{42}$$

The RHS is just a constant, so this is easy to solve exactly. Going step-by-step:

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial b}{\partial\rho}\right) = \frac{2\epsilon V^2}{TNg^2}\rho \tag{43}$$

$$\frac{\partial b}{\partial\rho} = \frac{2\epsilon V^2}{TNg^2}\frac{\rho}{2} + \frac{C_1}{\rho} \tag{44}$$

$$b = \frac{2\epsilon V^2}{TNg^2}\frac{\rho^2}{4} + C_1 log(\rho) + C_2 \tag{45}$$

Where C_1 and C_2 are arbitrary constants. Because we must demand a solution regular at the origin, it must be the case that $C_1 = 0$. This leaves C_2 to be determined by boundary conditions. We defined $z = 0$ at the plate edges, so the appropriate condition is $b(R) = -g$:

$$b = \frac{\epsilon V^2 R^2}{2TNg^2} + C_2 = -g \tag{46}$$

$$b = \frac{\epsilon V^2}{2TNg^2}\left(\rho^2 - R^2\right) - g \qquad z = \frac{\epsilon V^2}{2TNg^2}\left(\rho^2 - R^2\right) \tag{47}$$

Notably, even though this was not the exact solution for the mesh streched above a flat plane - it was only the first order term - this solution is exact given a parabolic anode plane behind the mesh, with the shape arranged such that the gap size is constant.