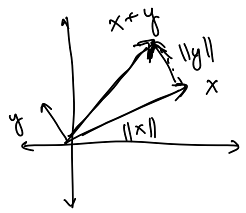
Vectors, Norms & Inner Products

$x \in \mathbb{R}^d$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$, $x^T = [x_1, x_2 \dots x_d]$



Vector Norms

$\|\cdot\| : \mathbb{R}^d \to \mathbb{R}^+$

1. $\|x\| \geq 0$ & $\|x\| = 0 \iff x = 0$ – positivity
2. $\|\alpha x\| = |\alpha| \, \|x\|$ – homogeneity
3. $\|x+y\| \leq \|x\| + \|y\|$ – triangle inequality

$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots |x_d|^p \right)^{1/p}$ – L_p norm

$\|x\|_1 = |x_1| + |x_2| + \dots |x_d|$ – L_1 norm

$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots |x_d|^2}$ – L_2 norm

$\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$ – L_∞ norm

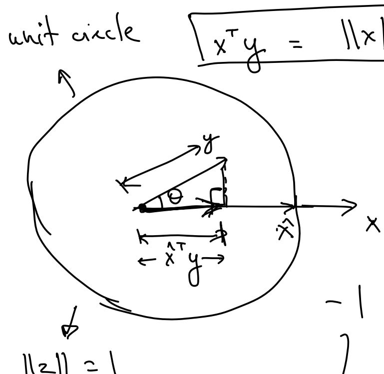
$\|x\|$ mean $\|x\|_2$

Inner Products

$x, y \in \mathbb{R}^d$

$\langle x, y \rangle = \boxed{x^T y} = x_1 y_1 + x_2 y_2 + \dots + x_d y_d = \sum_{i=1}^{d} x_i y_i$

$\boxed{x^T y = \|x\|_2 \|y\|_2 \cos\theta}$, where θ is the angle between vectors x and y.



Let $\hat{x}$ be unit vector in the direction of x

$$\hat{x} = \frac{x}{\|x\|_2}$$

$$-1 \leq \cos\theta = \frac{\hat{x}^T y}{\|y\|_2} = \frac{x^T y}{\|x\|_2 \|y\|_2} \leq 1$$

$$\Rightarrow x^T y = \|x\|_2 \|y\|_2 \cos\theta$$

$$\frac{|x^T y|}{\|x\|\,\|y\|} \le 1 \Rightarrow \boxed{|x^T y| \le \|x\|_2 \cdot \|y\|_2}$$

Cauchy Schwarz Inequality

$\theta = 90^\circ \Rightarrow x^T y = 0$

Equality holds iff x & y are collinear

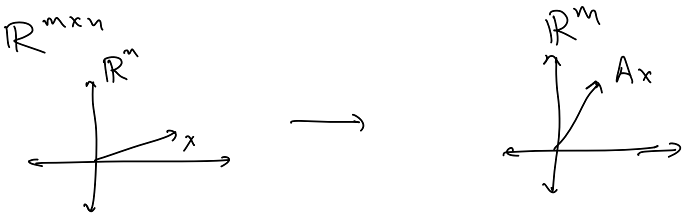
$y = \alpha x$

$|x^T y| \le \|x\|_p \|x\|_q \quad , \quad \frac{1}{p} + \frac{1}{q} = 1$?

Matrices, Matrix Norms

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longmapsto Ax$$

$A \in \mathbb{R}^{m \times n}$



$x \longmapsto Ax \quad , \quad y \longmapsto Ay \quad , \quad \alpha x + \beta y \longmapsto A(\alpha x + \beta y) = \alpha Ax + \beta . Ay$

Matrix A represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Matrix Norms , $A \in \mathbb{R}^{m \times n}$, $\|\cdot\| : \mathbb{R}^{m \times n} \to \mathbb{R}_+$

1. $\|A\| \ge 0$ & $\|A\| = 0$ iff $A = 0$ - positivity
2. $\|\alpha A\| = |\alpha| . \|A\|$ - homogeneity
3. $\|A + B\| \le \|A\| + \|B\|$ - triangle inequality

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2}$$ - Frobenius Norm of a matrix

Induced Matrix Norms :

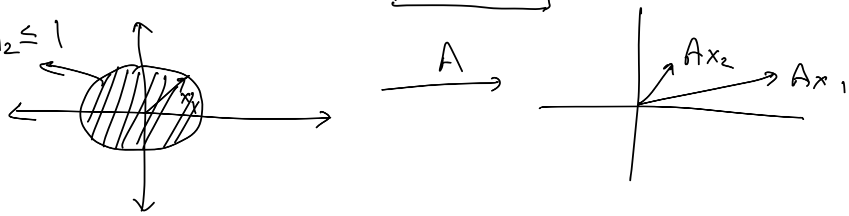
$$\|A\|_{p,q} = \max_{x \ne 0} \frac{\|Ax\|_p}{\|x\|_q}$$

$\|A\|_2 \; (= \|A\|_{2,2})$

$$\|A\|_2 = \max_{x \neq 0} \boxed{\frac{\|Ax\|_2}{\|x\|_2}} = \max_{\|x\|_2 \leq 1} \|Ax\|_2 = \sigma_1$$

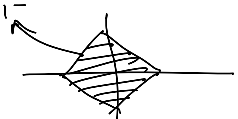
$\sigma_1 \to$ maximum singular value of A.

$\|x\|_2 \leq 1$



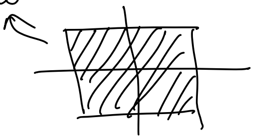
$$\|A\|_1 \; (= \|A\|_{1,1}) = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 \leq 1} \|Ax\|_1$$

$\|x\|_1 \leq 1$



$$\|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty \leq 1} \|Ax\|_\infty$$

$\|x\|_\infty \leq 1$



all rows or columns of A

take absolute value of

and take max.

$\|x\| = 1 \to \|Ax\|$

$$x \qquad \frac{\|Ax\|}{\|x\|}$$

$$\alpha x \qquad \frac{\|A\alpha x\|}{\|\alpha x\|} = \frac{|\alpha| \cdot \|Ax\|}{|\alpha| \cdot \|x\|} = \frac{\|Ax\|}{\|x\|}$$

$$\max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Matrices, Eigenvalues and Eigenvectors

$A \in \mathbb{R}^{n\times n}$

$\boxed{Ax = \lambda x}$, $x \neq 0$

λ: eigenvalue

x: eigenvector ($\|x\|_2 = 1$)

$Ax - \lambda x = 0$

$(A - \lambda I)x = 0 \Rightarrow A - \lambda I$ is singular $\equiv$ determinant is 0

$\Rightarrow \det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\right)$$ is polynomial of degree n in λ.

If A is $n\times n$, then it has n eigenvalues

If $A = A^T$ (symmetric) $\Rightarrow$ all eigenvalues are real and it has a full set (n) of mutually orthogonal eigenvectors

$A = A^T$, $Av_1 = v_1\lambda_1$, $\lambda_i \in \mathbb{R}$

$Av_2 = v_2\lambda_2$, $v_i^T v_j = 0$, $i \neq j$

$\vdots$ $\qquad = 1$, $i = j$

$Av_n = v_n\lambda_n$, $V = [v_1\ v_2 \cdots v_n] \in \mathbb{R}^{n\times n}$

$V^TV = I = VV^T$

$$A[v_1 \cdots v_n] = [v_1 \cdots v_n]\underbrace{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}_{\Lambda}$$

$AV = V\Lambda$

Since $VV^T = I \Rightarrow \boxed{A = V\Lambda V^T}$ $\longrightarrow$ Eigenvalue Decomposition

Matrix A is positive definite if $x^TAx > 0\ \forall x$ $(A \succ 0)$

A is positive semi-definite if $x^TAx \geq 0\ \forall x$ $(A \succeq 0)$

$A = A^T$, $x^TAx = x^T(V\Lambda V^T)x = (x^TV)\Lambda(V^Tx)$, $V^Tx \to z$

Let $z = V^Tx$, then $x^TAx = z^T\Lambda z = \sum_{i=1}^{n} \lambda_i z_i^2$

$$x^T A x \geqslant 0 \iff \lambda_i \geqslant 0$$

$A = GG^T$, $x^T A x = x^T G G^T x$

$$z = G^T x$$

$$x^T A x = z^T z = \sum_{i=1}^{n} z_i^2 \geqslant 0$$

$A = GG^T \iff$ Matrix A is positive semi-definite.