UTMC Sample Numerical Round

UTMC Committee

December 2019

1. There are 10 fans at a convention, each of which is either a fan of anime or videogames (nobody is a fan of both). They are split into 2 groups such that the first group has as many anime fans as videogame fans and the other group has twice as many anime fans as videogame fans. Each group has at least 1 person. How many people are in the first group?

2. If r_1, r_2, r_3 are the three distinct solutions to the equation $2x^3 + x + 9 = 0$, what is the value of $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$?

3. Let $f(n)$ denote the sum of all the distinct prime divisors of n. For example, $f(120) = 2 + 3 + 5 = 10$. What is the value of $f(f(3^{12} - 1))$?

4. The three friends Ana, Banana, and Hana are at UTMC, and want to spread their goodwill to the other contestants. What better way to do this than to hold open doors for them? A huge crowd of 903 UTMC contestants (including Ana, Banana, and Hana) are walking through a door, with one person passing through each second. The 3 friends position themselves somewhere in the line and when they reach the door, they hold it open until the next friend arrives. Each of the remaining 900 contestants give their appreciation to the friend currently holding the door for them. Their amount of appreciation is equal to the number of seconds that one of the three friends had been holding the door for (so the first person after a friend has appreciation 1, the second person has appreciation 2, and so on). What is the maximum total appreciation the three friends can get from the other 900 UTMC contestants?

5. David Tang is busy creating the last problem for UTMC. He draws, with a ruler and compass, a non-degenerate triangle with non-zero side lengths a, b, c. David then notices that, of all triangles with a perimeter of 2019, this particular triangle minimizes the value of

$$7a^2 + 7b^2 + 7c^2 + 7ab - 11bc - 13ac$$

Find $\frac{abc}{2019^3}$.