Algorithms and Data Structures III (course 1DL481)
Uppsala University – Spring 2023
Report for Assignment 1 by Team 7

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Problem 3: Boolean Satisfiability (SAT)

Task a: Ordered Resolution. Consider the following formula in conjunctive normal form (CNF):

$$\varphi \equiv (x_1 \vee x_2) \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_5) \\ \wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6)$$

We apply resolution by selecting the variables in their own given order. In blue are the new clauses from the resolution.

$$\begin{aligned}
\varphi \equiv & (x_1 \vee x_2) \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_5) \\
& \wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6) \\
\equiv & (x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_5) \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_3 \vee \neg x_5) \\
& \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6) \\
\equiv & (\neg x_3 \vee \neg x_4) \wedge (\neg x_3 \vee \neg x_6) \wedge (\neg x_5 \vee \neg x_4) \wedge (\neg x_5 \vee \neg x_6) \\
& \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_4 \vee \neg x_6) \\
\equiv & (x_4 \vee \neg x_6) \wedge (x_4 \vee \neg x_5) \wedge (\neg x_4 \vee \neg x_5) \wedge (\neg x_5 \vee \neg x_6) \\
& \wedge (x_5 \vee x_6) \wedge (\neg x_4 \vee \neg x_6) \\
\equiv & (\neg x_5 \vee \neg x_6) \wedge (\neg x_6) \wedge (\neg x_5) \wedge (x_5 \vee x_6) \\
\equiv & (\neg x_6) \wedge (x_6) \\
\equiv & ()
\end{aligned}$$

As we reach the empty clause, the formula is unsatisfiable.

Task b: DPLL. Consider again the formula φ given in Task a. On the initial formula φ we have no unit clause and no pure literal, so we select the variable x_1 and run $\mathsf{DPLL}(\varphi \wedge x_1)$: The formula we now apply DPLL is now:

$$\begin{aligned}
\varphi_1 \equiv \varphi \wedge x_1 \equiv & (x_1 \vee x_2) \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_5) \\
& \wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6) \wedge (x_1)
\end{aligned}$$

We have a unit clause now and make unit propagation with x_1

$$\begin{aligned}
\varphi_2 \equiv & (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_3) \wedge (\neg x_5) \\
& \wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6)
\end{aligned}$$

We have new unit clauses: First we apply the unit propagation on $\neg x_3$ and get:

$$\begin{aligned}\varphi_3 \equiv &(x_5 \vee x_6) \wedge (\neg x_5)\\ &\wedge (\neg x_2) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6)\end{aligned}$$

We apply unit propagation on x_4 and get:

$$\begin{aligned}\varphi_4 \equiv &(x_5 \vee x_6) \wedge (\neg x_5)\\ &\wedge (\neg x_2) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_6)\end{aligned}$$

We apply unit propagation $\neg x_5$ and get:

$$\begin{aligned}\varphi_5 \equiv &(x_6) \wedge (\neg x_2)\\ &\wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_6)\end{aligned}$$

We apply unit propagation to x_6 and get:

$$\varphi_6 \equiv (\neg x_2) \wedge (\neg x_2) \wedge ()$$

and get an empty clause. We make backtracking and run $\mathsf{DPLL}(\varphi \wedge \neg x_1)$. We make unit propagation with $\neg x_1$:

$$\begin{aligned}\varphi_7 \equiv \varphi \wedge \neg x_1 \equiv &(x_2) \wedge (x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_3 \vee \neg x_5)\\ &\wedge (\neg x_2 \vee \neg x_4) \wedge (\neg x_2 \vee \neg x_6) \wedge (\neg x_4 \vee \neg x_6)\end{aligned}$$

We make unit propagation with x_2 and get:

$$\begin{aligned}\varphi_8 \equiv &(x_3 \vee x_4) \wedge (x_5 \vee x_6) \wedge (\neg x_3 \vee \neg x_5)\\ &\wedge (\neg x_4) \wedge (\neg x_6) \wedge (\neg x_4 \vee \neg x_6)\end{aligned}$$

We make unit propagation with $\neg x_4$ and get:

$$\begin{aligned}\varphi_9 \equiv &(x_3) \wedge (x_5 \vee x_6)\\ &\wedge (\neg x_3 \vee \neg x_5) \wedge (\neg x_6)\end{aligned}$$

We we make unit propagation with x_3 and get:

$$\varphi_{10} \equiv (x_5 \vee x_6) \wedge (\neg x_5) \wedge (\neg x_6)$$

We make unit propagation with $\neg x_5$ and get:

$$\varphi_{11} \equiv (x_6) \wedge (\neg x_6)$$

We make unti propagation with x_6 and get:

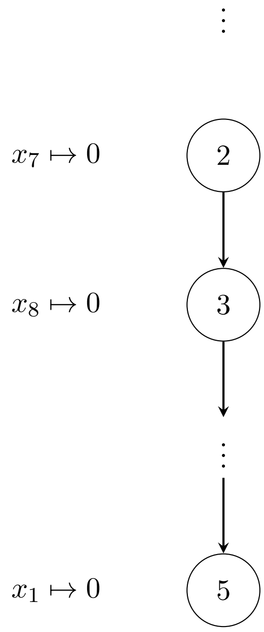
$$\varphi_{12} \equiv ()$$

Hence we get $\varphi = \mathsf{UNSAT}$.

Task c: CDCL. Consider the following CNF formula:

$$(x_1 \vee x_8 \vee \neg x_2) \wedge (x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_4 \vee \neg x_5) \wedge (x_7 \vee \neg x_4 \vee \neg x_6) \wedge (x_5 \vee x_6)$$

Assume that x_7 has been assigned 0 at decision level 2, and that x_8 has been assigned 0 at decision level 3. Moreover, assume that the current decision assignment is $x_1 = 0$ at decision level 5.



Using green text for trustified literals and gray for gray for false ones we get the formula:

$$(x_1 \vee x_8 \vee \neg x_2) \wedge (x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_4 \vee \neg x_5) \wedge (x_7 \vee \neg x_4 \vee \neg x_6) \wedge (x_5 \vee x_6)$$

Task d: Encoding. Describe your encoding, citing either [?], or [?, Section 2.2.2], or both, if you use their ideas: first explain the meaning of the Boolean variables that you use in your formula $\varphi_{d,c,e}$; then explain the encodings by the help functions of the hint that you actually use (there is no need to explain $\textsc{AtMost}(k, x_1, \ldots, x_n)$ if you use [?]); and finally explain how the constraints of the problem are encoded using those variables and help functions.

We chose the programming language $\langle \ldots \rangle$, for which a compiler or interpreter is available on the Linux computers of the IT department. All source code is uploaded with this report. The compilation and running instructions are $\langle \ldots \rangle$.

We validated the correctness of our encoding and implementation by checking its outputs on many instances via the provided polynomial-time solution checker.

Task e: Experiments. We chose the SAT solver MiniSat for our experiments. We used or did not use the provided script for running the experiments and tabling their results under Linux Ubuntu 18.04 (64 bit) on an Intel Xeon E5520 of 2.27 GHz, with 4 processors of 4 cores each, with a 70 GB RAM and an 8 MB L3 cache (a ThinLinc computer of the IT department).

The results are in Table 1. The trivially unsatisfiable instances (which are the ones that violate the inequality $e \leq \left\lfloor \frac{d-1}{c-1} \right\rfloor$) that were actually attempted in our experiments are $\langle 8, 2, 8 \rangle$, $\langle 10, 2, 10 \rangle$, $\langle 12, 2, 12 \rangle$, $\langle 14, 2, 14 \rangle$, $\langle 16, 2, 16 \rangle$, $\langle 15, 3, 8 \rangle$, $\langle \ldots \rangle$, $\langle 16, 4, 6 \rangle$, $\langle \ldots \rangle$, and $\langle \ldots \rangle$. We observe that our encoding detects their trivial unsatisfiability in $\langle \ldots \rangle$ time.

d	c	e	status	time	d	c	e	status	time	d	c	e	status	time
8	2	7	sat		12	3	4	sat		16	4	5	sat	
8	2	8	unsat		12	3	5	unsat		16	4	6	unsat	
10	2	9	sat		15	3	6	sat		20	4	4	sat	
10	2	10	unsat		15	3	7	sat		20	4	5	sat	
12	2	11	sat		15	3	8	unsat		20	4	6	?	
12	2	12	unsat		18	3	6	sat		24	4	4	sat	
14	2	12	sat		18	3	7	sat		24	4	5	sat	
14	2	13	sat		18	3	8	?		24	4	6	?	
14	2	14	unsat		21	3	6	sat		28	4	4	sat	
16	2	10	sat		21	3	7	sat		28	4	5	sat	
16	2	11	sat		21	3	8	sat		28	4	6	sat	
16	2	12	sat		21	3	9	?		28	4	7	?	
16	2	13	sat		24	3	6	sat		32	4	3	sat	
16	2	14	sat		24	3	...	sat		32	4	...	sat	
16	2	15	sat		24	3	9	sat		32	4	9	sat	
16	2	16	unsat		24	3	10	?		32	4	10	sat	

Table 1: Cruise design: satisfiability and runtime (in seconds) using MiniSat, with a timeout of 60.0 CPU seconds; a timeout is denoted by 't/o'; if no timeout occurred, then proven satisfiability is denoted by 'sat' and proven unsatisfiability by 'unsat', else trivial unsatisfiability is denoted by 'unsat' and the unknown status is denoted by '?'.

Problem 4: SAT Modulo Theories (SMT)

Task a: Encoding of instructions. We encode the transition between program states as follows for each MiniASM instruction:

- push_j:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- pop:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- dup:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- plus:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- neg:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- read:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.
- write:
  - In English: $\langle \ldots \rangle$.
  - In SMT syntax: $\langle \ldots \rangle$.

Task b: Partial-correctness checker. Describe your encoding. The final `assert` call, which ensures that the SMT solver produces `unsat` when needed, is $\langle \ldots \rangle$.

We chose the SMT solver Z3 for our experiments. We chose the programming language $\langle \ldots \rangle$, for which a compiler or interpreter is available on the Linux computers of the IT department. All source code is uploaded with this report. The compilation and running instructions are $\langle \ldots \rangle$.

Let swap be the abbreviation of push_0; write; push_1; write; push_0; read; push_1; read, that is changing the order of the two top-most numbers on the stack:

1. The program push_{10}; read is or is not reported by Z3 to be partially correct, because $\langle \ldots \rangle$.
2. The program push_1; dup; dup; write; read $\langle \ldots \rangle$

3. The program $\mathsf{push}_1; \mathsf{dup}; \mathsf{read}; \mathsf{dup}; \mathsf{neg}; \mathsf{plus}; \mathsf{plus}$ $\langle \ldots \rangle$

4. The program $\mathsf{push}_1; \mathsf{push}_0; \mathsf{read}; \mathsf{write}; \mathsf{push}_0; \mathsf{read}; \mathsf{read}$ $\langle \ldots \rangle$

5. The program $\mathsf{push}_{10}; \mathsf{push}_0; \mathsf{swap}$ $\langle \ldots \rangle$

6. The program $\mathsf{push}_{10}; \mathsf{dup}; \mathsf{read}; \mathsf{swap}; \mathsf{push}_1; \mathsf{plus}; \mathsf{read}; \mathsf{plus}$ $\langle \ldots \rangle$

7. The program $\mathsf{push}_{10}; \mathsf{dup}; \mathsf{push}_1; \mathsf{plus}; \mathsf{dup}; \mathsf{push}_1; \mathsf{plus}; \mathsf{plus}; \mathsf{plus}$ $\langle \ldots \rangle$

When Z3 produces `sat` for a partial-correctness check, the output of `get-model` means $\langle \ldots \rangle$.

Task c: Partial-equivalence checker. Describe your encoding. The final `assert` call, which ensures that the SMT solver produces `unsat` when needed, is $\langle \ldots \rangle$.

We chose the SMT solver Z3 for our experiments. We chose the programming language $\langle \ldots \rangle$, for which a compiler or interpreter is available on the Linux computers of the IT department. All source code is uploaded with this report. The compilation and running instructions are $\langle \ldots \rangle$.

Assuming that variable x is stored at heap address 0 and variable y is stored at heap address 1, the program $t \coloneqq x;\ x \coloneqq y;\ y \coloneqq t$, translated into

$$\mathsf{push}_0; \mathsf{read}; \mathsf{push}_1; \mathsf{read}; \mathsf{push}_0; \mathsf{write}; \mathsf{push}_1; \mathsf{write}$$

and the program $x \coloneqq x + y;\ y \coloneqq x - y;\ x \coloneqq x - y$, translated into

$$\mathsf{push}_0; \mathsf{read}; \mathsf{push}_1; \mathsf{read}; \mathsf{plus}; \mathsf{dup}; \mathsf{push}_1; \mathsf{read}; \mathsf{neg}; \mathsf{plus}; \mathsf{dup}; \mathsf{push}_1; \mathsf{write}; \mathsf{neg}; \mathsf{plus}; \mathsf{push}_0; \mathsf{write}$$

are or are not reported by Z3 to be partially equivalent, because $\langle \ldots \rangle$. When Z3 produces `sat` for a partial-equivalence check, the output of `get-model` means $\langle \ldots \rangle$.

Task d: Extended language. We encode the transition between program states for the $\mathsf{cmp}_\circ$ instruction as follows:

- In English: $\langle \ldots \rangle$.
- In SMT syntax: $\langle \ldots \rangle$.

For encoding the jmp_j instruction (where $j \geq 0$), we propose $\langle \ldots \rangle$. The difficulty of encoding one or more jmp_j instructions lies in $\langle \ldots \rangle$. With our approach for encoding jmp_j, the partial correctness of programs is determined by $\langle \ldots \rangle$.

Feedback to the Teachers