Deriving the Lorentz Transformation with Linear Algebra

September 7, 2020

To begin we assume the Lorentz transformation is a linear transformation. This is so there aren't any unwanted accelerations between reference frames. We'll have two reference frames, S and S'. The S' reference frame is moving with respect to the S reference frame with velocity v. We let x denote the position of the S reference frame and t denote its time. Likewise x' and t' will represent the position and time of the S' frame, respectively. Thus we begin by claiming x' is some linear combination of x and t:

$$x' = ax + bt$$

We now question about the point whose x' coordinate is zero. Since the S' reference frame is moving with a velocity v away from the S frame, we know the x coordinate of that point must be vt.

Inputting into the equation we get:

$$0 = avt + bt$$

Thus

$$-avt = bt$$

$$-av = b$$

Using this result we say

$$x' = ax - avt$$

$$x' = a(x - vt)$$

Now it is a matter of expressing t' in terms of x and t. To do this, we consider the whole linear transformation which maps the vector $\begin{bmatrix} x \\ t \end{bmatrix}$ to $\begin{bmatrix} x' \\ t' \end{bmatrix}$

We can think of this as a matrix $\begin{bmatrix} a & b \\ e & f \end{bmatrix}$ such that

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

We have already established b = -av.

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ e & f \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

Our goal is the expressed e and f in terms of a and then finally derive a.

To do so we must consider the inverse transformation which maps $\begin{bmatrix} x' \\ t' \end{bmatrix}$ to $\begin{bmatrix} x \\ t \end{bmatrix}$. We can reason that x = a(x' + vt') by stating that in the x frame of reference, x' is moving away with velocity -vt. There is no need for our scaling factor, a, to be any different. So

$$\begin{bmatrix} a & -av \\ e & f \end{bmatrix}^{-1} = \begin{bmatrix} a & av \\ u & v \end{bmatrix}$$

where, again, u and v are just arbitrary constants that we shall not concern ourselves with at the moment. From this, we can rewrite the left-hand side by using the inverse matrix formula.

$$\begin{bmatrix} \frac{f}{af+eav} & \frac{av}{af+eav} \\ \frac{-e}{af+eav} & \frac{a}{af+eav} \end{bmatrix} = \begin{bmatrix} a & av \\ u & v \end{bmatrix}$$

Now, in order for the two top right elements of each matrix to be the same, $\frac{av}{af+eav}$ must equal av. The only way this can happen is if af+eav = 1

Allowing af+eav = 1, we get

$$\begin{bmatrix} f & av \\ -e & a \end{bmatrix} = \begin{bmatrix} a & av \\ u & v \end{bmatrix}$$

In order for the two top left elements to be equal, f must equal a. Inputting this into our previous result

$$af + eav = 1$$

$$a^2 + eav = 1$$

$$e = \frac{1 - a^2}{av}$$

We have succesfully found the transformation we were looking for, namely

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ \frac{1-a^2}{av} & a \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

To only task left is to derive the value of a. To do so we must use the second postulate of Einstein's special theory of relativity: The speed of light is unchanging in all frames of reference. Thus, if we imagine both S and S' have the same position when t and t' are 0, and a light pulse is emmitted in the same direction from S as S' is moving, then we see that the position with respect to S after time t is ct, while the position with respect to S' after time t' is ct', we get

$$x' = ct'$$

$$x = ct$$

Our transformation maps any vector with position and time in one coordinate frame to another. This means that our transformation must map the vector $\begin{bmatrix} ct \\ t \end{bmatrix}$ to $\begin{bmatrix} ct' \\ t' \end{bmatrix}$. Thus

$$\begin{bmatrix} ct' \\ t' \end{bmatrix} = \begin{bmatrix} a & -av \\ \frac{1-a^2}{av} & a \end{bmatrix} \begin{bmatrix} ct \\ t \end{bmatrix}$$

If we do the multiplication, we end up with this system of equations:

$$act - avt = ct'$$

$$\frac{ct - a^2ct}{av} + at = t'$$

If we divide the first equation by c, we end up with

$$\frac{a(c-v)t}{c} = t'$$

Since both the left hand-sides are equal to t', we may set them equal to each other. Doing so results in

$$\frac{ct - a^2ct}{av} + at = \frac{a(c-v)t}{c}$$

We now proceed with the algebra

$$\frac{c - a^2c}{av} + a = \frac{a(c-v)}{c}$$

$$\frac{c - a^2c + a^2v}{av} = \frac{a(c-v)}{c}$$

$$\frac{c - a^2(c-v)}{av} = \frac{a(c-v)}{c}$$

$$c^2 - a^2c(c-v) = a^2v(c-v)$$

$$c^2 - a^2(c+v)(c-v) = 0$$

$$a^2 = \frac{c^2}{c^2 - v^2}$$

$$a^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We have successfully derived our constant. This is often representend with the greek letter γ

We now return to our equations.

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t + \frac{x - \gamma^2 x}{\gamma^2 v})$$

We can modify the last equations by rewriting the term $\frac{x-\gamma^2 x}{\gamma^2 v}$, however we must use the definition of γ

$$\frac{x(1-\gamma^2)}{\gamma^2 v}$$

$$x(\frac{1}{\gamma^2 v} - \frac{1}{v})$$

$$x(\frac{1}{\frac{v}{1-\frac{v^2}{c^2}}} - \frac{1}{v})$$

$$x(\frac{-\frac{v^2}{c^2}}{v}$$

$$\frac{-xv}{c^2}$$

This simplified things greatly, and allows us to rewrite out Lorentz Transformation in the more common and well known way

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{xv}{c^2})$$