Introduction to Nonparamteric Regression: Kernel Estimation

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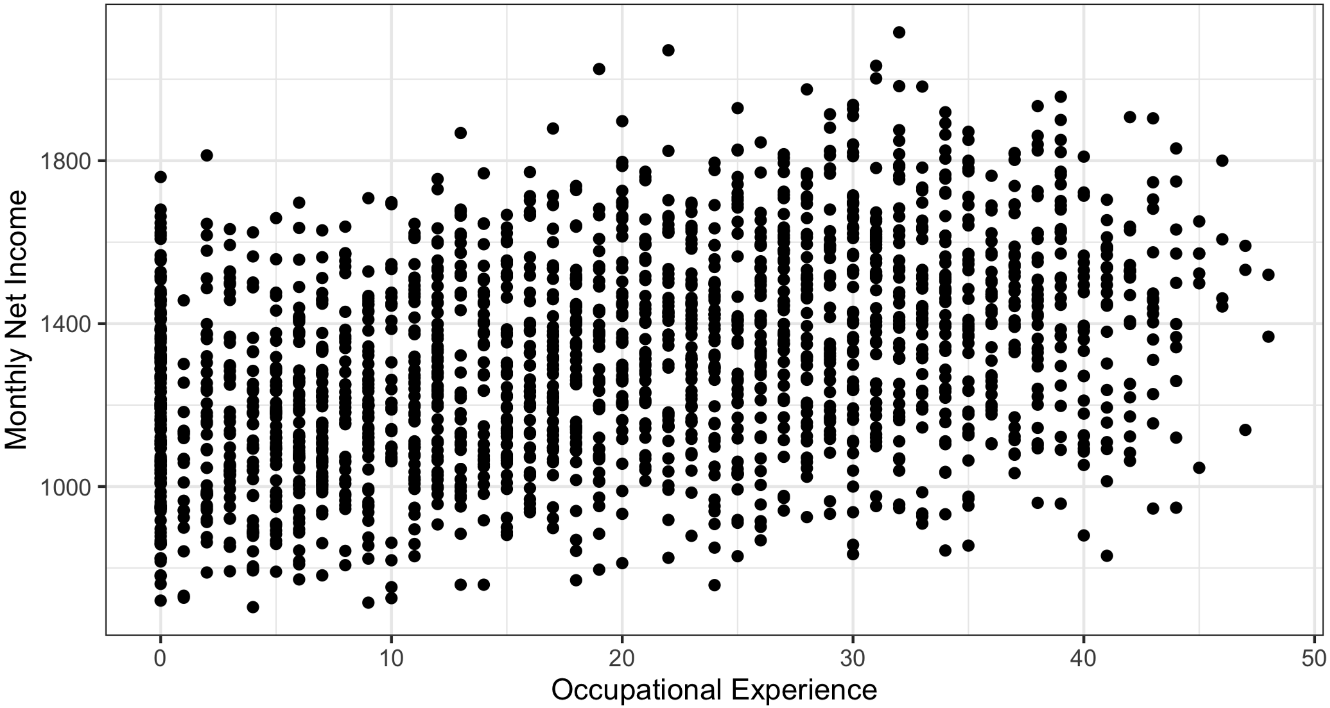
Overview

1. Motivating Example
2. Conditional Distribution and Conditional Expectation
3. Binning
4. Local Averaging and Kernel Estimation

Motivating Example

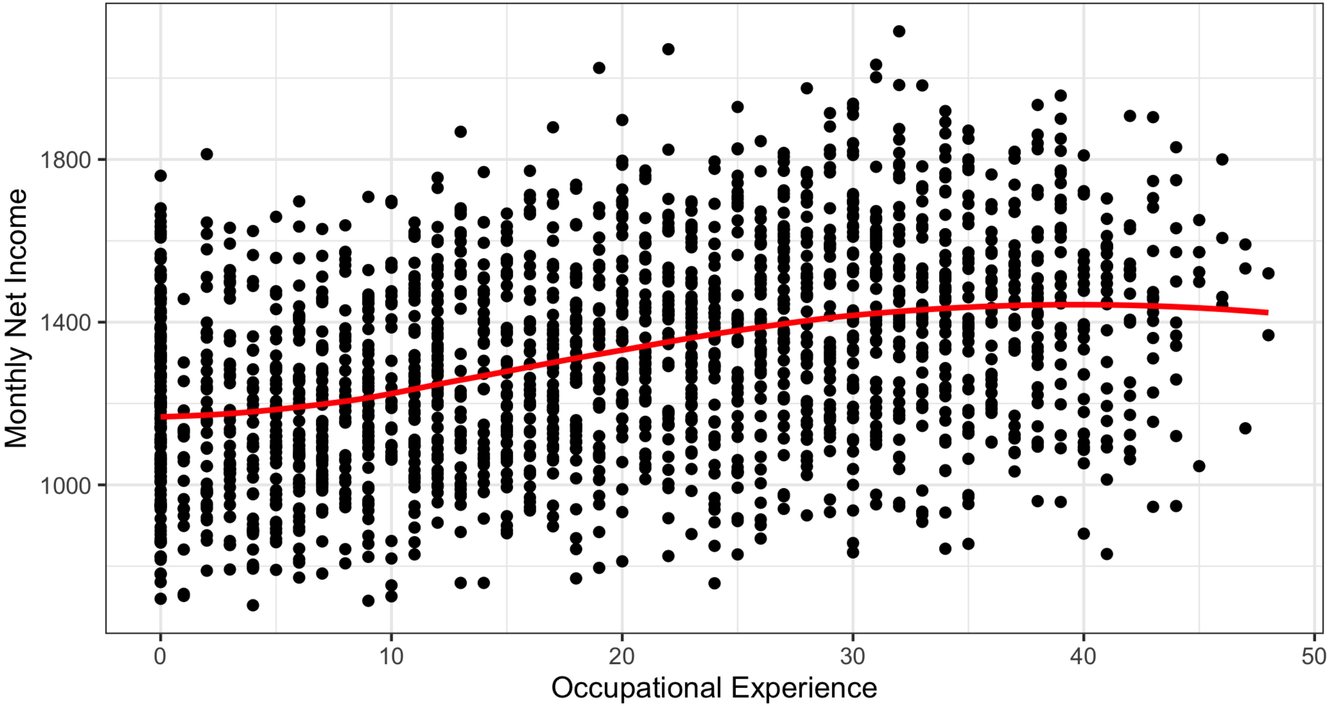
- Income and Occupational Experience:



- *Example*: simulated data of monthly net income (1922 observations)

Example: Income ~ Occup. Experience

- Scatterplot with a regression smoother:



What is Regression Analysis?

- Fox (2008):
    - Regression analysis examines the relationship between a quantitative response variable Y, and one or more quantitative explanatory variables $X_1, \ldots, X_k$.
    - Regression analysis traces the conditional distribution of Y—or some aspect of this distribution, such as its mean—as a function of explanatory variables $X_1, \ldots, X_k$, i.e., how does the distribution/mean of Y change with changes in the Xs?
- **Conditional distribution**

$$p(Y|x_1, x_2, \ldots, x_k) = g(x_1, x_2, \ldots, x_k)$$

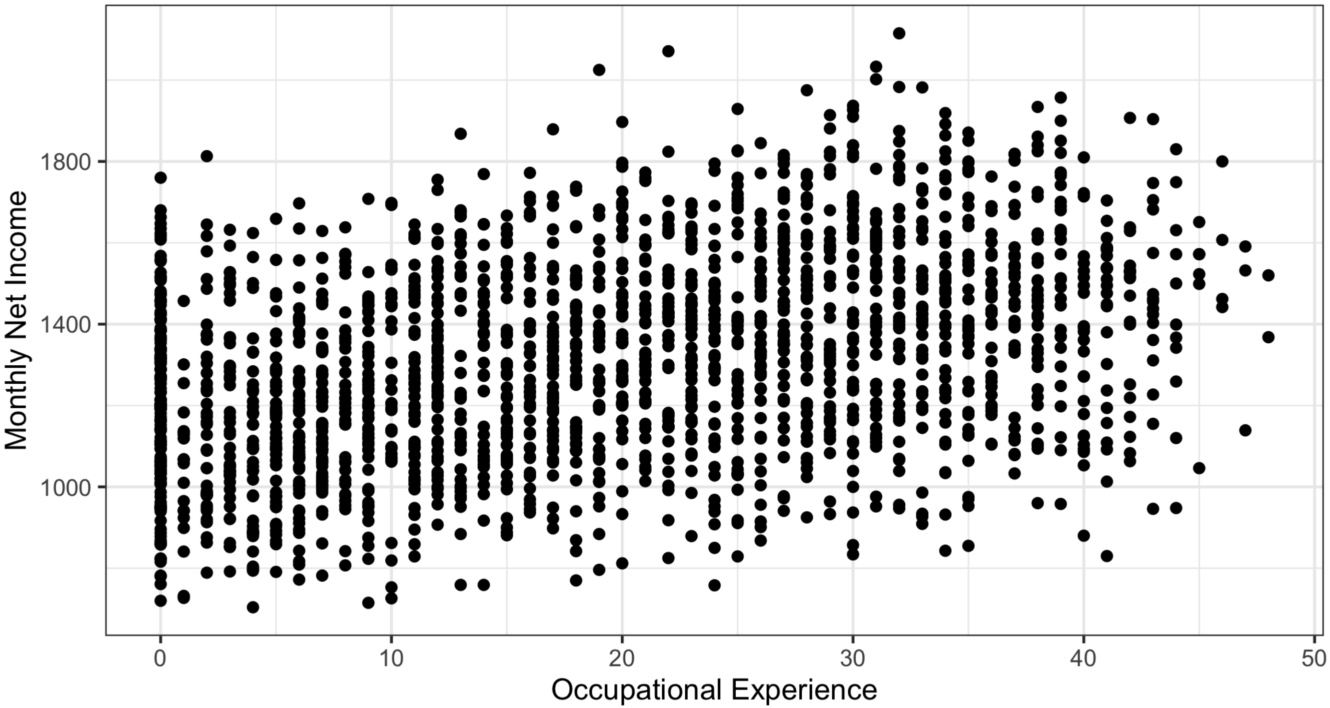
- **Conditional expectation**

$$E(Y|X_1, X_2, \ldots, X_k) = f(X_1, X_2, \ldots, X_k)$$

Regression Analysis: Example

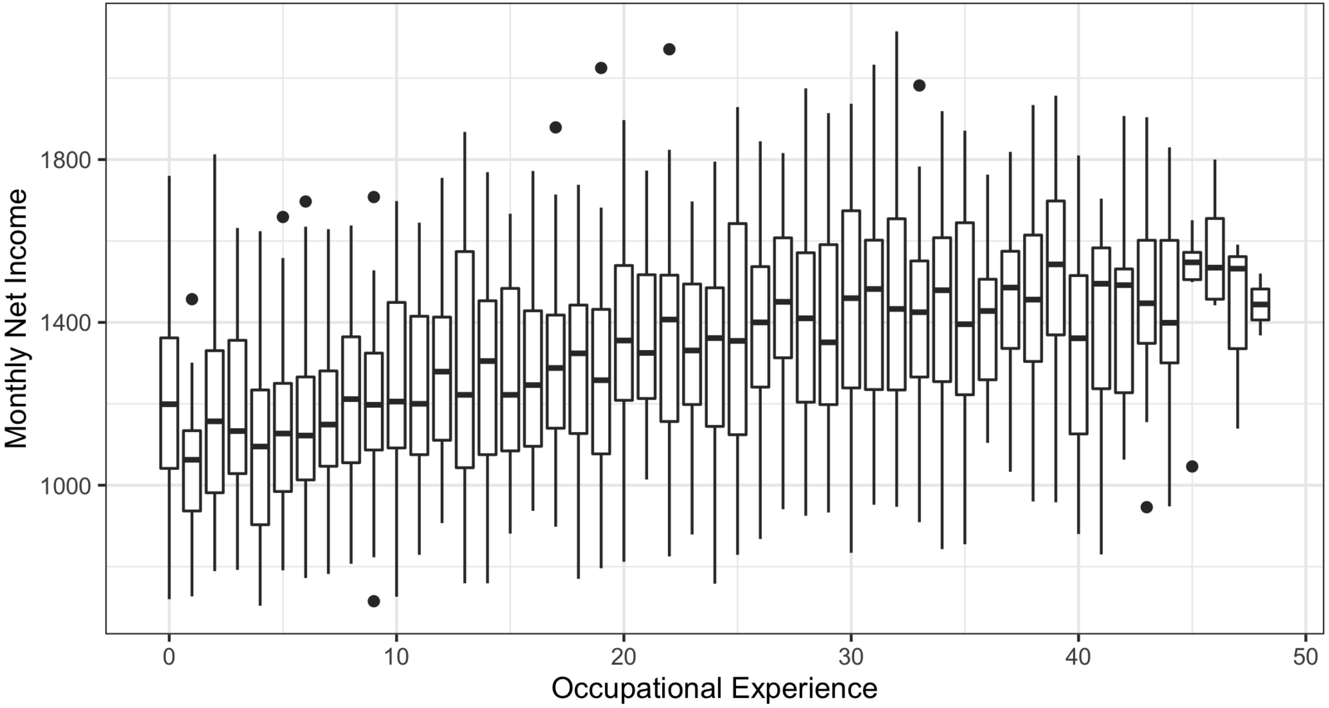
- Scatterplot: *Income* by *Occupational Experience*

```
(p <- ggplot(dat, aes(x=oexp, y=income)) + geom_point() + labs(x = '(
```



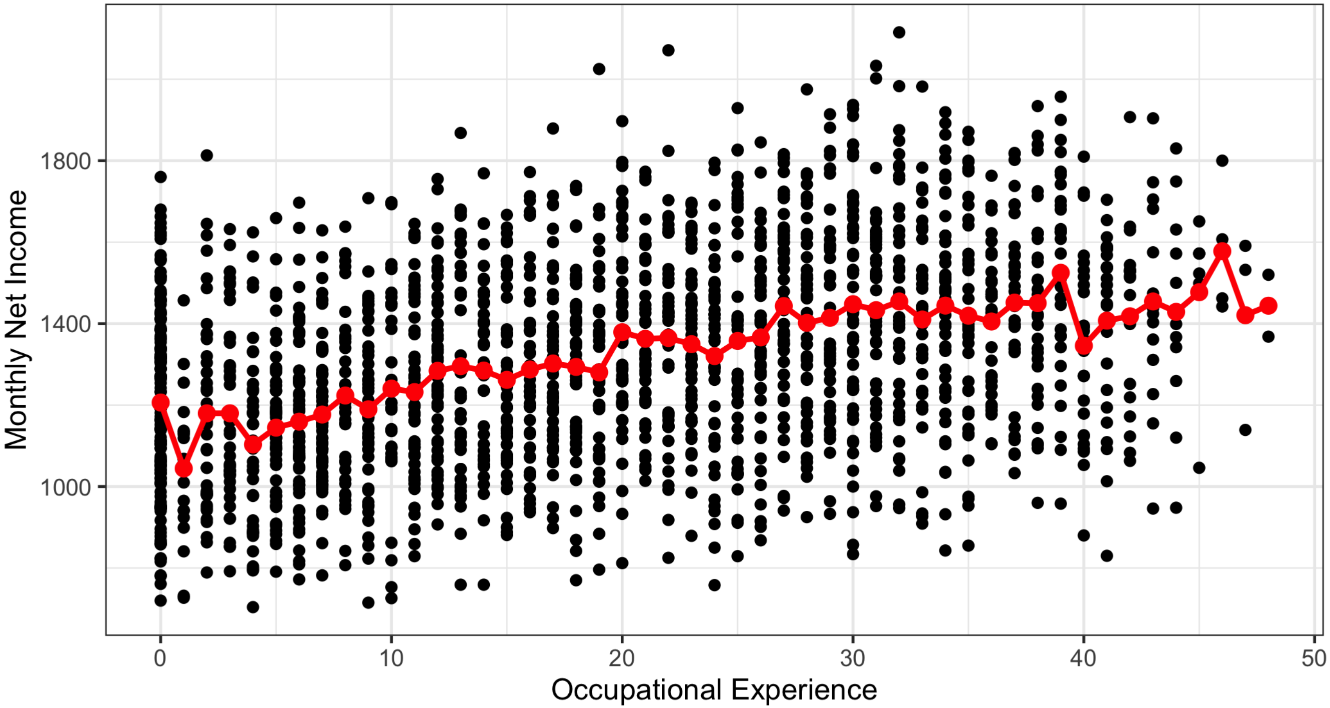
Conditional Distribution: Example

```
ggplot(dat, aes(x=oexp, y=income)) + geom_boxplot(aes(group=oexp)) +
```



Conditional Expectation: Example

```
p + stat_summary(fun = mean, geom = 'point', col = "red", size=2.5) +
    stat_summary(fun = mean, geom = 'line', col = "red", size=1)
```



Conditional Expectation Function (Path of Means)

The conditional expectation

$$E(Income|Occ.\,Experience = x)$$

tells us how the population average of *Income* changes as we move the conditioning variable (*Occ. Experience*) over all possible values. For every value of *Occupational Experience*, we might get a different average income. The collection of all such averages is called the conditional expectation function (CEF) and denoted as:

$$E(Income|Occ.\,Experience)$$

(without the specific value x for *Occ. Experience*)

More generally:

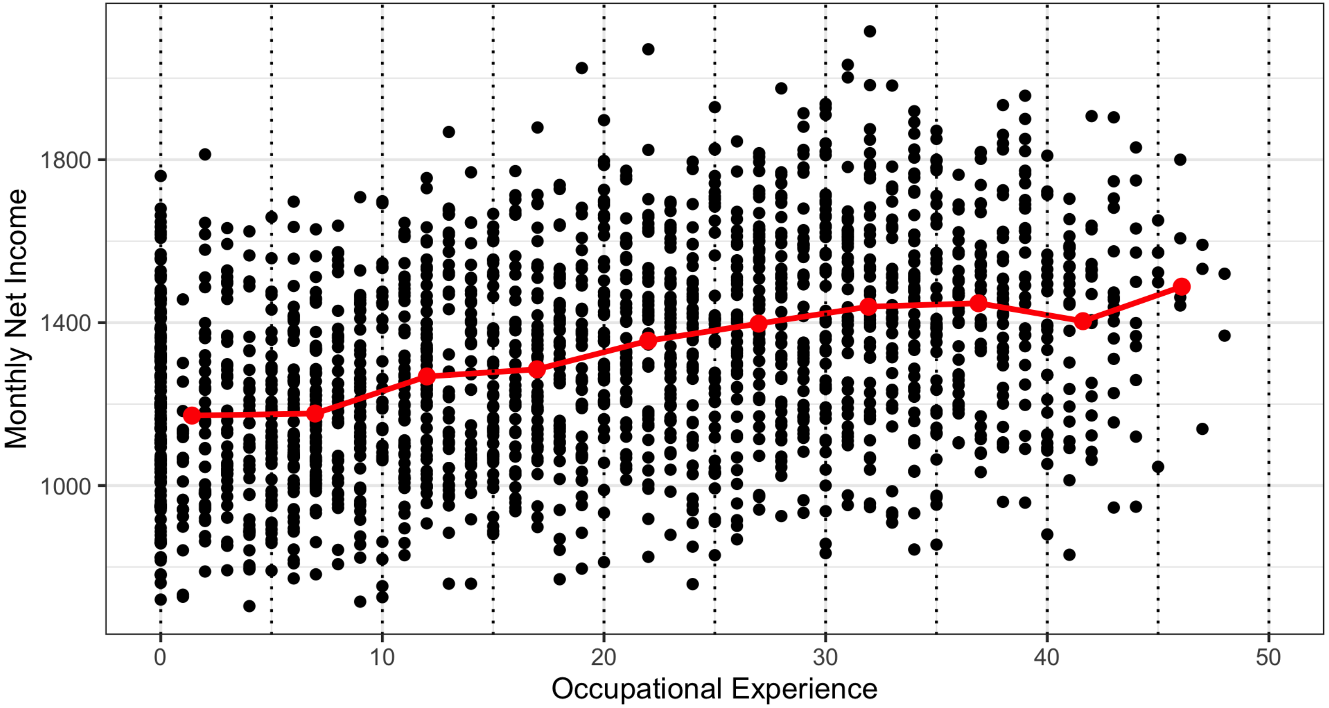
$$E(Y|X_1, X_2, \ldots, X_k)$$

Nonparametric Regression

- We investigate the conditional distribution/mean of a response variable without making any assumptions about the functional form or the shape of the distribution $\rightarrow$ nonparametric regression.
  - If an explanatory variable is discrete like age in years or educational level, we can estimate the conditional distribution/mean of Y directly.
  - If an explanatory variable is continuous (age measured in days, achievement scores) we don't have enough observations for each unique value; thus we may aggregate the explanatory variable into a large number of narrow bins and take the mean (or other statistic) value $\rightarrow$ binning

Binning

5-year bins: $C \in \{0-4, 5-9, \ldots, 45-49\}$ conditional mean given bins C



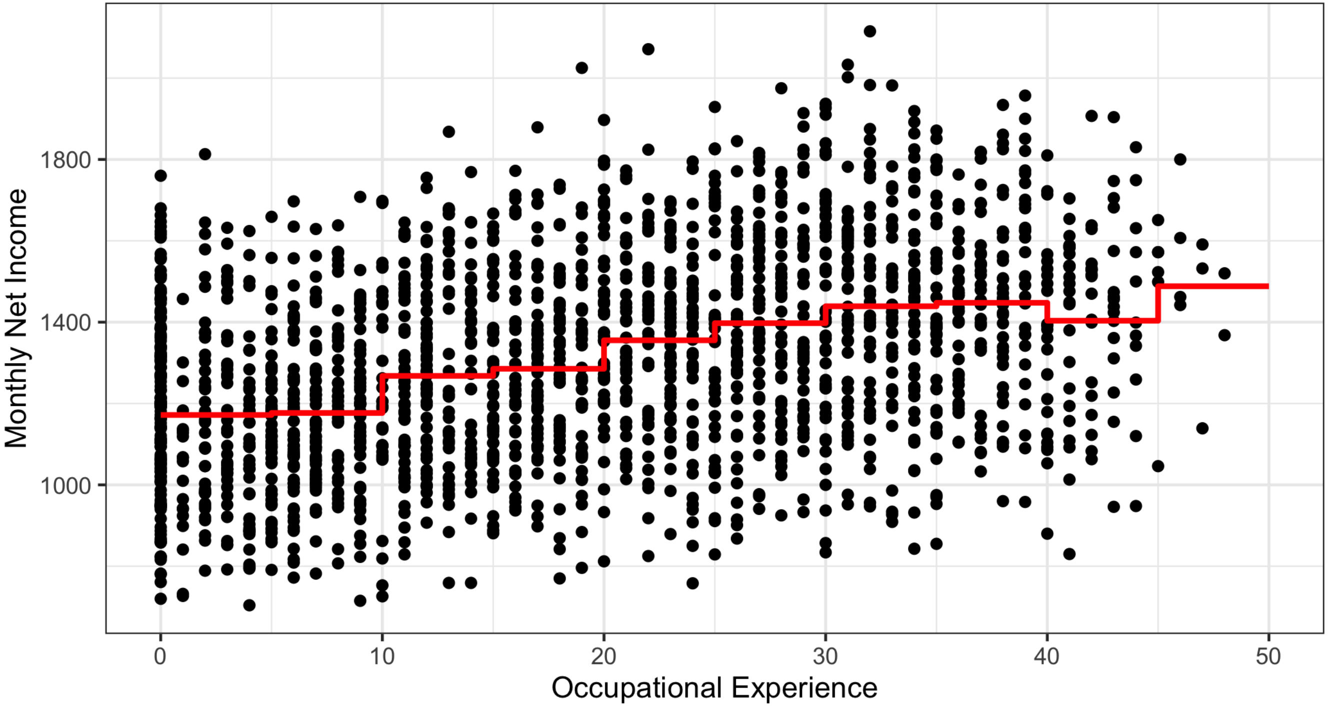
```r
# create 5-year age groups.
dat$oexpbins <- cut(dat$oexp, breaks = seq(0, 50, by = 5), right = F)

# compute the mean income and the mean oexp for each bin.
bin.stat <- dat %>% group_by(oexpbins) %>%
            summarize(m.income = mean(income), m.oexp = mean(oexp), 

# add points and lines for group means and add vertical lines at brea
p + geom_point(data=bin.stat, aes(x=m.oexp, y=m.income), col = "red",
    geom_line(data=bin.stat, aes(x=m.oexp, y=m.income), col = "red", 
    geom_vline(xintercept = seq(0, 50, by = 5), linetype="dotted", s
```

Binning

Conditional means as a step-function



Regression/Scatterplot Smoothers

- Binning is a very rough method and depends, of course, on the location and size of the bins.
- An alternative approach are so called regression or scatterplot smoothers which construct a separate bin (=window) for each unique x-value and then calculates the mean (or other statistic) from the data within the window. The window is typically defined by kernel weights.
  - Local averaging & kernel estimation
  - Local polynomial regression
  - Locally weighted regression (loess/lowess)

Local Averaging

- Local averaging calculates the mean (or other statistic) for a window centered at a specific value x_0.

- Procedure for local averaging:

1. Define value x_0 for which you want to calculate the mean

2. Construct a symmetric neighborhood around x_0 (=window); choose a window width using either

    - bandwidth (=half the window width in absolute measurement units of x) or

    - span (=a fraction of the data covered by the window, e.g., .3 = 30%).

3. For observations within the window, compute the mean of Y.

4. Do so for all other x-values

Local Averaging

- More formally, the weighted local average for a given value x_0 is given by

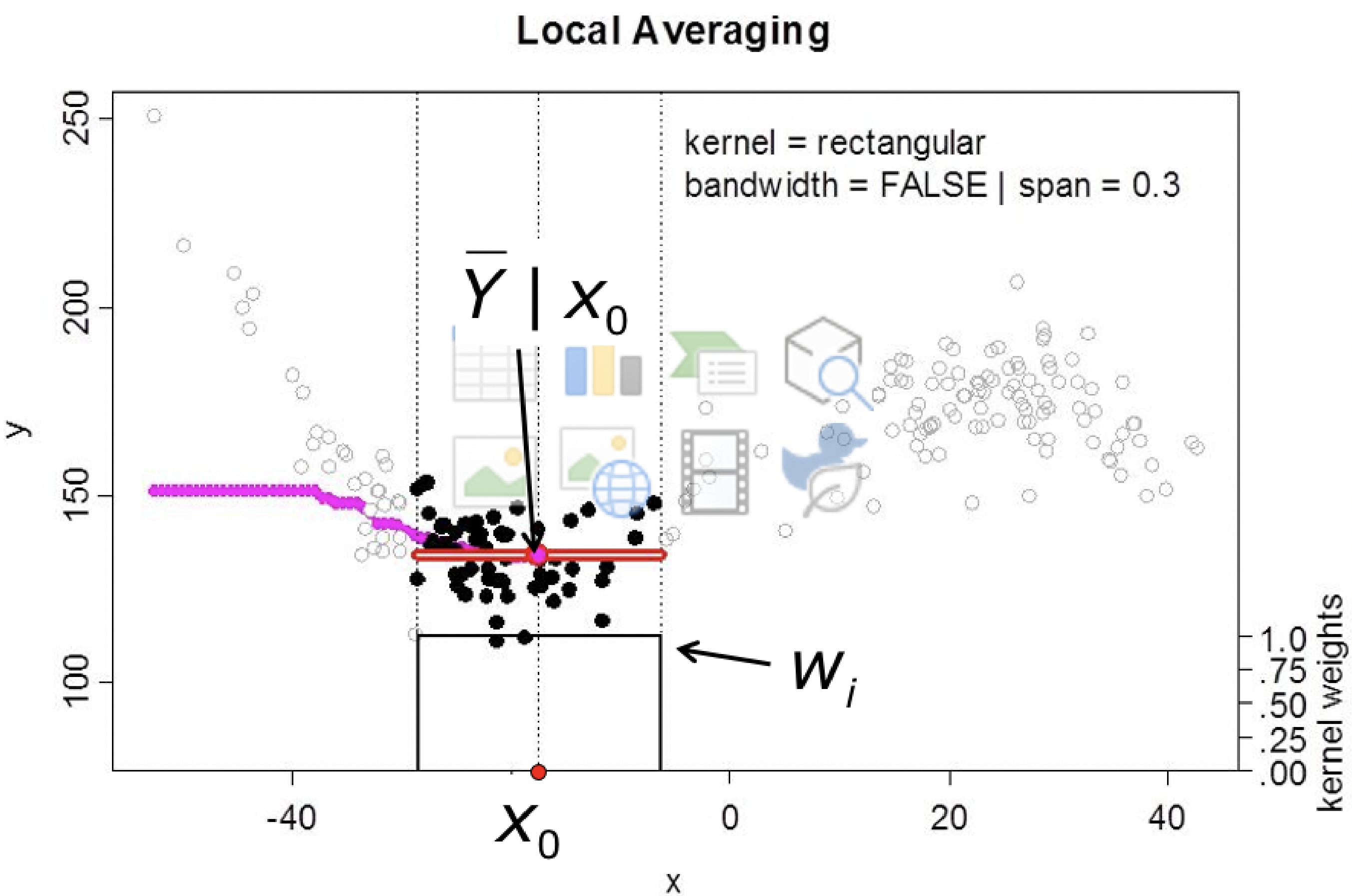
$$\overline{Y}|x_0 = \frac{\sum_{i=1}^{n} w_i Y_i}{\sum_{i=1}^{n} w_i}$$

where weight $w_i = 1$ if observation i lies within the window and $w_i = 0$ if it is outside the window:

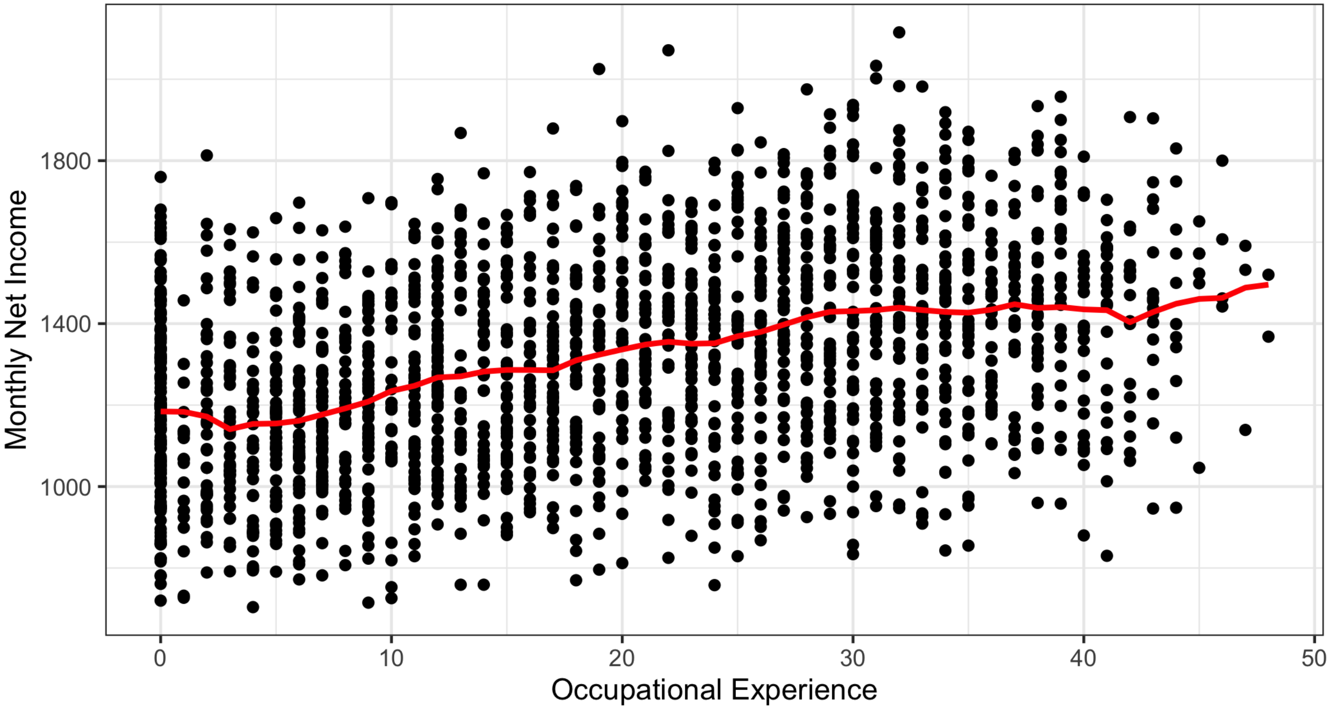
$$w_i = \begin{cases} 1 & \text{for } |(x_i - x_0)/h| < 1 \\ 0 & \text{for } |(x_i - x_0)/h| \geq 1 \end{cases}$$

h is the bandwidth of the window (half of the window width). Note that the weight function is identical to a rectangular kernel.

Local Averaging

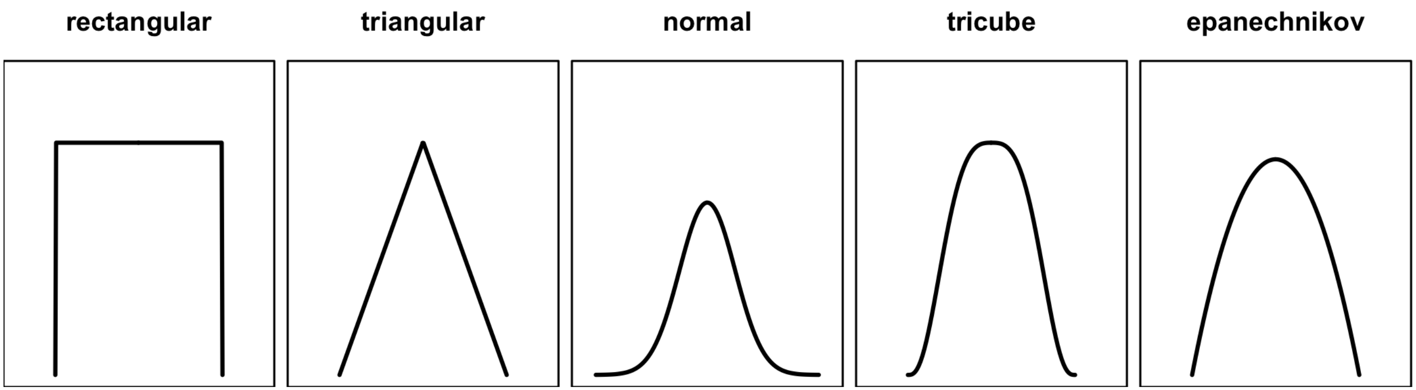


Local Averaging: Example

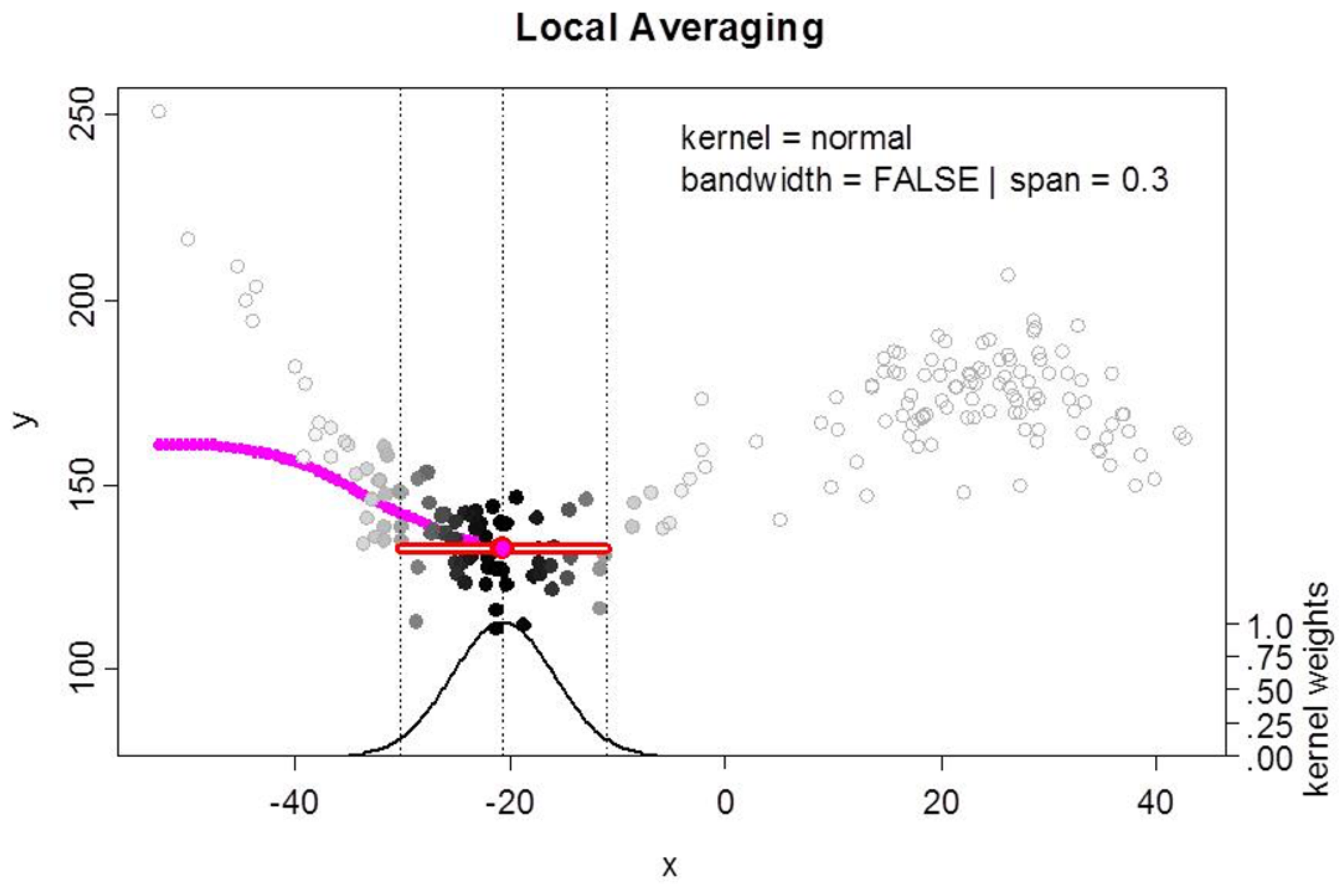


Kernel Estimation

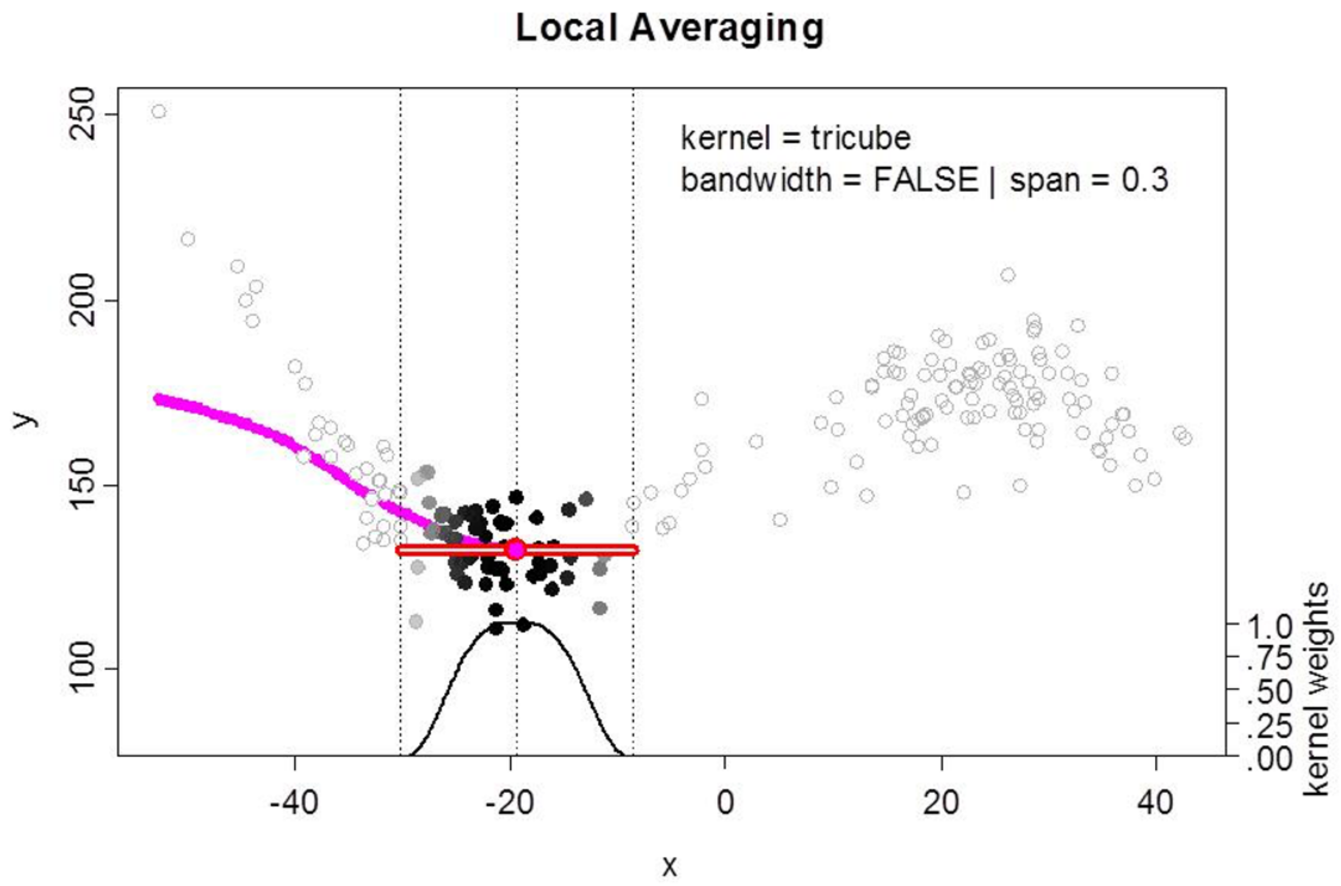
- We can do a local averaging by using different weighting functions, i.e., kernels
- Weights are defined by kernels (like in kernel density estimation), e.g.
  - Rectangular (uniform) kernel: same weight for all observations within the window = local averaging
  - Triangular, normal, tricube, Epanechnikov kernel: observations closer to the x_0 under consideration get more weight than observations further away



Kernel Estimation: Normal Kernel

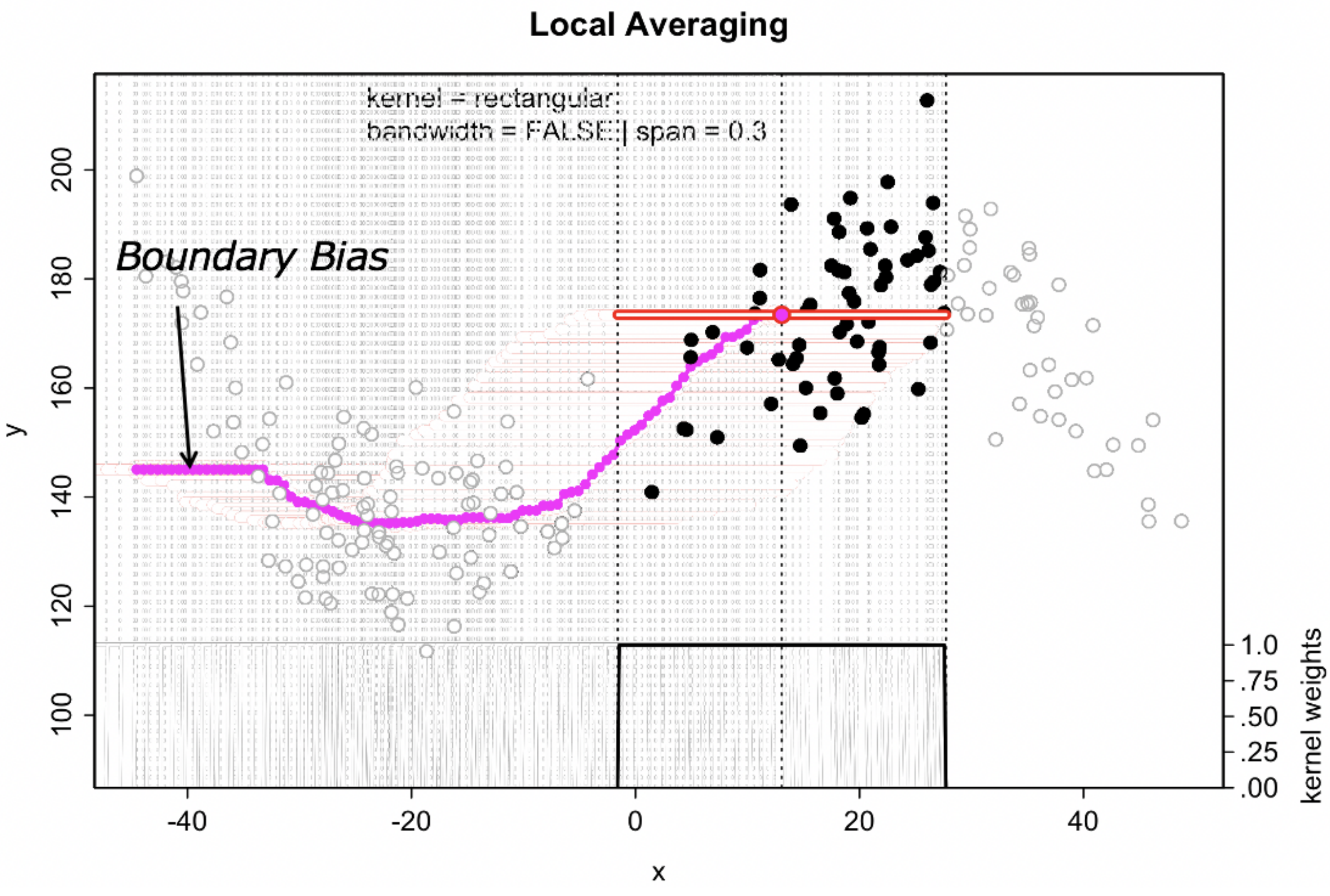


Kernel Estimation: Tricube Kernel



R Demonstration

Boundary Bias



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