Transformations and Modeling
ONEXYS Read-ahead

Introduction

Now that we have introduced the techniques of graph transformations, we will use our new tools to explore several examples of modeling involving sine and cosine functions. The main application will be to modeling the populations in a predator-prey relationship, and we will also revisit the sound and pendulum applications you have completed recently.

Instructions

After reading through the *Transformations and Modeling* context and questions below, you should complete the application reflection in Canvas. Note: *you will have a chance to talk further with your coach before answering the questions below in detail*. The point of the pre-read is to "prime the pump" for further conversations with your coaches.

Transformations and Modeling

I. Rabbits and Foxes

Predator-prey models are mathematical models that describe interactions among species. Simply put, one species eats another for survival. These models are arguably the building blocks of the biological and and ecological systems as biomasses are grown out of their resource masses. But there are many other contexts in which these mathematical models are helpful. They can be used to describe resource-consumer relationships, plant-herbivore, parasite-host, and susceptible-infectious interactions, etc. [1]

Figure 1: Predator-Prey Relationships

In the first part of this application, we're going to consider how periodic functions might be able to describe simple predator-prey relationships. In particular, we will estimate and interpret *period*, *amplitude*, and *midline* in the context of predator-prey relationships.

The data in the table below describes the populations of foxes and rabbits in a national park over a 12 month period. Each value of t corresponds to the beginning of the month, with $t = 0$ corresponding to the beginning of January.

[1] Frank Hoppensteadt (2006) Predator-prey model. Scholarpedia, 1(10):1563.

t, month	0	1	2	3	4	5	6	7	8	9	10	11	12
$r(t)$, rabbits	1000	750	567	500	567	750	1000	1250	1466	1500	1466	1250	1000
$f(t)$, foxes	150	143	125	100	75	57	50	57	75	100	125	146	150

This data can also be displayed graphically as in Figure 2 below, with r in red and f in blue.

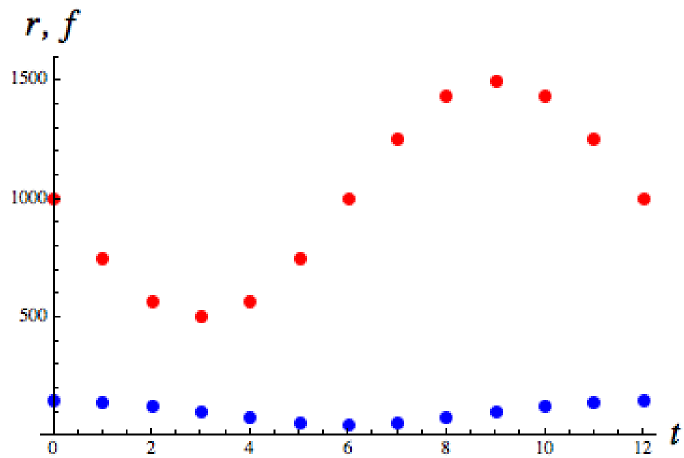

Figure 2: Figure 1: Fox and rabbit population.

1. (Graded for completeness only.) Explain why it might be appropriate to model the number of rabbits and foxes as periodic functions of time.
2. Assuming the data indicates that the rabbit population is well-modeled by a periodic function, what are the period, midline, and amplitude of this function?
3. Assuming the fox population is also modeled by a periodic function, what are the period, midline, and amplitude of this function?
4. (Graded for completeness only.) Looking at the data, it appears that one population "chases" the other, in the sense that the low values of one are quickly followed by low values of the other, and similarly for high values. Write a sentence or two explaining why we might expect this to be the case.
5. Find appropriate sinusoidal functions for $r(t)$ and $f(t)$. In both cases, write the functions so that the input for sine and/or cosine is in radians.

II. The Mathematics of Sound, Part 1[2]

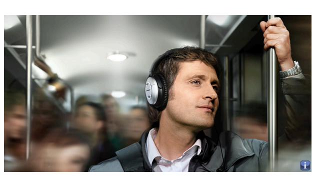

Figure 2: Ad for noise cancelling headphones.

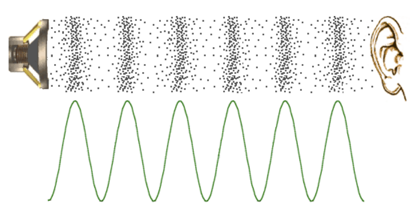

Figure 3: Compression waves.

[2] You will need to use the *Sound* simulation linked from the "Apply" section of the module to answer the next set of questions.

Sound travels through the air in compression waves (Figure 3). These waves are often represented as sinusoidal plane waves, and the sound we hear depends on the frequency and the amplitude of the wave. The standard note used for tuning is the fifth A on a piano, denoted A_4, which has a frequency of 440 Hz, i.e., 440 cycles per second. Equivalently, the period of A_4 is 1/440 seconds. If we model this sound wave with a function of the form $g(t) = A * \sin(\omega * t)$, where the input of the sine function is in radians, then we need to set $\omega = 2\pi * 440$. More generally, to produce a sound wave of frequency f, we need to set $\omega = 2\pi * f$. When you open the Sound simulation, you will see the default function written in this way.

6. (Graded for completeness only.) Using the Sound simulation, enter the function $g(t) = \sin(2\pi * 440 * t)$ by changing the sliders to $A = 1, \varphi = 0$, and $f = 440$. When you click the "Play Sound" button, you should hear the computer play the note A_4 for 10 seconds. Adjust the A slider to a wide variety of values in order to hear the impact of the amplitude on what you hear. In particular, adjust the slider to $A = 0.5$ in order to hear $g(t) = 0.5 * \sin(2\pi * 440 * t)$ and to $A = 2$ in order to hear $g(t) = 2 * \sin(2\pi * 440 * t)$. Describe in words the differences in the sound (if any) that you hear.
7. (Graded for completeness only.) Now compare the sound made by $g(t) = \sin(2\pi * 440 * t + \varphi)$, for different values of φ. (The easiest way to do this is to just use the slider.) Describe in words the differences in the sound (if any) that you hear.
8. Consider the following sound wave.

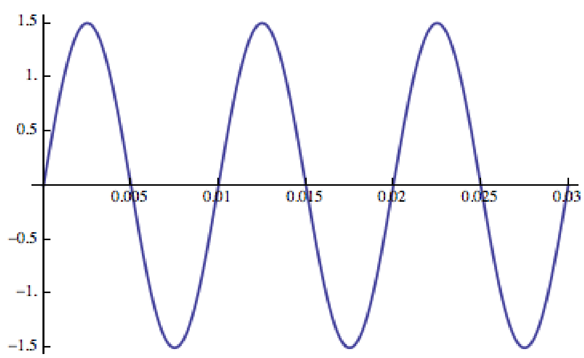

Figure 3: A sound wave.

Write a sine function $g(t)$ to model this sound wave. Enter your answer in the form $g(t) = A * \sin(2\pi * f * t)$, including the "$g(t) =$ ".
9. Noise cancellation is effected by producing a second wave $h(t)$ so that $g(t) + h(t) = 0$. This second wave has the same amplitude and frequency as the first, but has been phase-shifted so that maxima of $h(t)$ coincide with the minima of $g(t)$, and vice versa. Find the smallest positive value of k so that $h(t) = g(t+k)$ exactly cancels out the sound produced by $g(t)$, where $g(t)$ is the function you wrote in Question 8. Note: This is the principle behind noise-canceling headphones, which sample the external noise and emit sound waves at the same frequency but shifted so as to maximize the destructive interference.
10. If you want to the $h(t)$ from Question 9 in the form $h(t) = A * \sin(2\pi * f * t + \varphi)$, what should be the value of φ? Write your answer as an equation of the form "$\varphi = \ldots$".

III. The Pendulum

Recall from the last question of The Pendulum that as long as the initial angle α is relatively small and the pendulum is released from rest, the angle θ (in radians) of the pendulum, measured from vertical, will be well-approximated by the function $\theta(t) = \alpha * \cos\left(\sqrt{\frac{g}{L}} * t\right)$, where α is the initial angle, g is the acceleration due to gravity (9.8 m/s^2), L is the length of the pendulum, and t is in seconds.

11. (Graded for completeness only.) Using the language of graph transformations, describe the effect on the graph of $\theta(t) = \alpha * \cos\left(\sqrt{\frac{g}{L}} * t\right)$ of doubling the initial angle α.
12. (Graded for completeness only.) Using the language of graph transformations, describe the effect on the graph of $\theta(t) = \alpha * \cos\left(\sqrt{\frac{g}{L}} * t\right)$ of doubling the length L of the pendulum.

Instructions, part deux

After reading and reflecting on these questions, complete the pre-read assignment on Canvas. This will give your coach some insight on your thinking in order to best help you before you are required to formally answer these questions.