Cobb-Douglas Production Functions and the Solow Growth Model

CONEX Read-ahead

Introduction

The Cobb-Douglas production function represents economic output Y as a function of capital K and labor L. It is a staple of macroeconomics courses such as ECON 2020 and ECON 3020 at UVA.

Figure 1: Paul Douglas

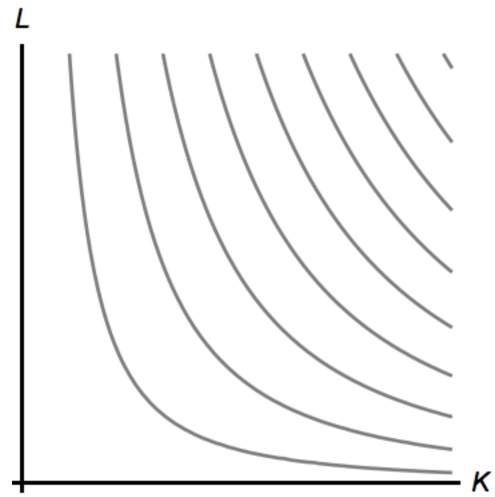

Figure 2: Isoquants of a Cobb-Douglas function

The Cobb-Douglas function is a type of function known as a "power law", in which one quantity is proportional to the product of powers of various other quantities. It takes the form

$$Y = AK^{\alpha}L^{\beta},$$

where A, α, and β are positive constants. In particular, the parameters α and β represent the "elasticities" of capital and labor respectively and are crucial in predicting the impact on production of increases or decreases in the two inputs. In this context, the greater the elasticity of capital (or labor), the more sensitive production will be to changes in capital (or labor). That is, the greater the elasticity of capital (or labor), the greater the economic production will be when capital (or labor) is changed.

In this application, we will see how logarithms can be combined with linear regression to find the Cobb-Douglas function that best fits a given data set and to estimate the elasticity of capital and labor. We will then explore an important application of Cobb-Douglass production functions called the Solow Growth Model.

Instructions

After reading through the *Cobb-Douglas Production Functions and the Solow Growth Model* context and questions below, you should complete the application reflection in Canvas. Note: *you will have* *a chance to talk further with your coach before answering the questions below in detail.* The point of the pre-read is to “prime the pump” for further conversations with your coaches.

Cobb-Douglas Production Functions and the Solow Growth Model

The Cobb-Douglass production function was introduced in 1928 in a paper by mathematician Charles Cobb and economist Paul Douglas, and is historically important as being one of the first quantitative models to be applied to the study of macroeconomics. An important application of Cobb-Douglass production functions is the Solow Growth Model, which was developed by Robert Solow in 1956.[1] In 1986 Solow was awarded the noble prize in economics “for his contributions to the theory of economic growth,” particularly his development of the growth model. Even more than Gross Domestic Product (GDP) or other economic indicators, growth is highly correlated with reducing poverty and raising living standards. In fact, if the GDP fell by as much as 30%, on par with the Great Depression, this would not be as detrimental as a slowdown of economic growth by 1% for ten years. Due to the importance of growth, economists try very hard to model it as accurately as possible. Even today Solow’s model remains one of the most used and respected models, and it is the inspiration for many other current models of economic growth.

Questions

1. In adopting the Cobb-Douglas production function, the first thing an economist would want to do is to estimate the elasticity of capital and labor (i.e., the parameters α and β). Starting with the equation $Y = AK^\alpha L^\beta$, take the base-10 logarithm of both sides, and simplify the right-hand side as much as possible. Note that we will use the notation “log” for the base-10 logarithm, rather than “$\log_{10}$”, throughout this application. Hint: Your equation should express $\log Y$ as a *linear* function of $\log K$ and $\log L$. Write your answer as an equation in the form “$\log(Y) = ...$”.

The equation you derived in Question 1 expresses $\log Y$ as a linear function of $\log K$ and $\log L$. This is a key step for many things we might want to do, such as create a graph, as linear functions are much easier to deal with than other types of functions. However, it is still a “multivariable” function, so for the moment we will simplify things by assuming the supply of labor to be constant.

2. If labor is held constant, the Cobb-Douglas function looks like $Y = CK^\alpha$, where α and $C = AL^\beta$ are positive constants. If we take the logarithm of both sides, so that we have $\log Y$ as a linear function of $\log K$, what is the slope of this line? What is its vertical intercept? (As above, use log for $\log_{10}$.)
3. (Graded for completeness only.) Consider the following table of data.

K	12	20	35	47	81
Y	235	282	347	401	490

   Is this data (roughly) linear? Explain your reasoning.
4. (Graded for completeness only.) If we compute the (base-10) logarithm of each value in the

[1] The model was developed independently by Trevor Swan, and is sometimes referred to as the “Solow-Swan” model.

table above, we get the following.

$\log K$	1.079	1.301	1.544	1.672	1.908
$\log Y$	2.371	2.450	2.540	2.603	2.690

Is this transformed data (roughly) linear? Explain your reasoning.

5. If we let $k = \log K$ and $y = \log Y$, what is the equation for the line $y = f(k)$ of best fit to the transformed data (displayed in the table in Question 4) as determined by linear regression? *See the instructions for using linear regression in the "Explore" section of this module.* Write your answer as an equation in the form "$f(k) = ...$".
6. Using your answer to Question 5, find C and α in the original Cobb-Douglas function $Y = CK^\alpha$.
7. If capital is held constant and labor varies, then the Cobb-Douglas function looks like $Y = DL^\beta$, where β and $D = AK^\alpha$ are positive constants. If we take the logarithm of both sides of this equation, we have $\log Y = \log D + \beta \log L$. Now consider the following table of data.

L	20	25	37	51	60
Y	210	224	242	269	284

Take the logarithm of each value in this table, then use linear regression to find the best-fit line $\log Y = \log D + \beta \log L$. What value do you obtain for β?

8. (Graded for completeness only.) Given the elasticities α and β you found in Questions 6 and 7, is this economy more sensitive to increases in capital, or to increases in labor? Explain your reasoning.

When talking about economic growth, the focus is generally on the movement of capital. In this context capital essentially refers to any tools or machines used in the production of goods and services. Capital is closely related to economic growth because both income and consumption per person depend on the level of capital in the economy. Capital stock per person is the fixed amount of capital per capita in a given economy at any given time. The Solow Growth Model represents the rate of change in capital stock per person, i.e., growth, as a function of the savings rate (the percentage of each person's income set aside to save), the production function (the value of goods and services produced as a function of capital stock per person) and the depreciation rate of capital (how fast capital wears out over time and subsequently needs to be replaced). More advanced versions also take into account a growing population, the increasing efficiency of labor and capital, and the amount of labor (work force) doing research versus producing goods and services.

The Solow model is built on the Cobb-Douglas production function

$$Y = K^\alpha (AL)^{1-\alpha},$$

where $K = K(t)$ and $L = L(t)$ represent capital and labor, as above, and $A = A(t)$ represents the labor-boosting effect of knowledge and technology. α is the elasticity of capital ($0 < \alpha < 1$), and the product AL is known as "effective" labor. The output per unit of effective labor (i.e. per-capita output) is given by

$$y(t) = \frac{Y(t)}{A(t)L(t)} = \frac{K(t)^\alpha (A(t)L(t))^{1-\alpha}}{A(t)L(t)} = \left(\frac{K(t)}{A(t)L(t)}\right)^\alpha = k(t)^\alpha,$$

where $k(t) = \frac{K(t)}{A(t)L(t)}$ is the capital stock per unit of effective labor.

The Solow Growth Model is expressed by the following equation, which states that the average rate of change of capital stock per unit of effective labor over a small time interval, Δt, that is $\dfrac{\Delta k}{\Delta t}$, is approximately equal to the following linear function of $k(t)$ and $y(t)$. More precisely, it states that

$$\frac{\Delta k}{\Delta t} \approx sy(t) - (n + g + \delta)k(t),$$

where s is the fraction of the output per unit of effective labor that is saved and invested, n is the growth rate of population, g is the growth rate of technology, and δ is the depreciation rate of physical capital.

9. What are the minimum and maximum possible values of the parameter s?
10. (Graded for completeness only.) $n + g + \delta$ is the total depreciation rate. Why does the total depreciation rate depend on the growth rate of population?
11. (Graded for completeness only.) Why does the total depreciation rate depend on the growth rate of technology?
12. Since $y = k^\alpha$, we can write the Solow model as

$$\frac{\Delta k}{\Delta t} \approx sk^\alpha - (n + g + \delta)k.$$

    Plot the functions sk^α and $(n + g + \delta)k$ as functions of k on the same set of axes.
13. (Graded for completeness only.) Since $0 < \alpha < 1$, the graph of sk^α should look something like a square root graph. Explain why this means the graph of $(n+g+\delta)k$ will intersect the graph of sk^α at exactly one positive value of k.
14. (Graded for completeness only.) Suppose the two graphs intersect where $k = k^* > 0$. What is happening to the value of k if $0 < k < k^*$? What is happening if $k > k^*$? What does this imply about the long-term behavior of k?

Instructions, part deux

After reading and reflecting on these questions, complete the application reflection on Canvas. This will give your coach some insight on your thinking in order to best help you before you are required to formally answer these questions.