Falling Bodies
Read-ahead

Introduction

The Greek philosopher Aristotle (384-322 BC) claimed that heavier objects fall faster than lighter ones. This theory was accepted for almost two thousand years. Around 1590 AD Galileo claimed that if two objects are dropped from the top of a tower[1], they will fall at the same speed regardless of mass. In this application we will use one-dimensional kinematics to explore these claims.

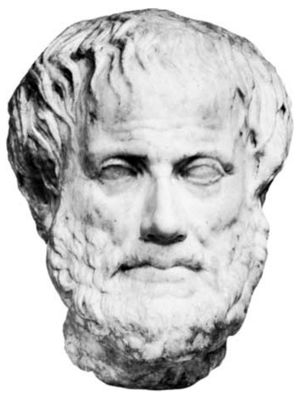

Figure 1: Aristotle

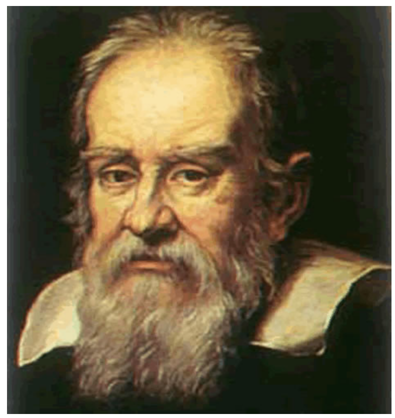

Figure 2: Galileo

Instructions

After reading through *Falling Bodies* context and questions below, you should complete the reflection assignment in Canvas. Note: *you will have a chance to talk further with your coach before answering the questions below in detail*. The point of this read-ahead and the reflection is to "prime the pump" for further conversations with your coaches.

Falling Bodies

It turns out that Galileo's claim is correct if we ignore air resistance. In this case the acceleration of the object does not depend on its mass, and is given by the constant function $a(t) = -g$, where $g = 9.8$ m/s^2. An object dropped from a height of x_0 meters will thus accelerate at a constant rate, with velocity $v(t) = -gt$, and position $x(t) = x_0 - \frac{1}{2}gt^2$. (See the video Kinematics 1D: Motion in a Straight Line for the derivation of these equations.)

Questions

1. A tourist drops a camera from the top of the Leaning Tower of Pisa, 55 meters above the ground. According to the equations given above, how long does it take the camera to reach the ground?
2. How fast is the camera going when it hits the ground?

[1]It is not clear that Galileo ever actually dropped objects from the tops of towers. In any case, given the technology of the time it is unlikely he would have been able to accurately compare the motion of objects moving at such high speeds. Instead, he attempted to verify his claim experimentally by rolling balls down incline planes.

3. How tall would the tower have to be for the camera to hit the ground at twice the speed you computed in the previous question?
4. Suppose on a different trip the tourist drops his phone down a well. He hears a splash 5 seconds later. The speed of sound at that altitude is 340 m/s. How deep is the well?

While it was certainly clear to Galileo that differently-shaped objects (such as a rock and a feather) do ***not*** all fall at the same rate, it took several more centuries for mathematicians and physicists to develop even a rough model for air resistance. In fact, the precise nature of the flow of air around a moving object is not well understood today. However, if we make some basic assumptions about the force produced, called the *drag force* we can derive equations that do a reasonably good job of predicting the motion of a falling body.

In all models of air resistance, the drag force depends on the speed of the object through the air, with faster speeds producing more resistance. If the drag force is proportional to the speed, we have

$$a(t) = -9.8 - \frac{k}{m}v,$$

where m is the mass of the object and k is a positive constant. This can be written as a differential equation,

$$\frac{dv}{dt} = -9.8 - \frac{k}{m}v,$$

whose solution (for $v(0) = 0$) is $v(t) = -\frac{9.8m}{k} + \frac{9.8m}{k}e^{-(k/m)t}$.

5. (Graded for completeness only.) What qualities of the object (or the air it is falling through) do you think will have an effect on the value of k? Explain if you can how k would depend on each qualities.
6. Compute $\lim_{t\to\infty} v(t)$, where $v(t) = -\frac{9.8m}{k} + \frac{9.8m}{k}e^{-(k/m)t}$. This quantity is called the ***terminal velocity*** of the object.
7. Suppose that k is fixed. Sketch graphs of $v(t)$ for $m = k, m = 2k$, and $m = 3k$.
8. Suppose three objects are identical on the outside, but have masses 10 kg, 20 kg, and 30 kg. If all three are dropped from the same height at the same time, which object would hit the ground first?
9. Suppose an object of mass m is dropped from a height of 55 meters. If the velocity of the object is given by $v(t) = -\frac{9.8m}{k} + \frac{9.8m}{k}e^{-(k/m)t}$, use antidifferentiation to find a formula for the height $x(t)$ as a function of t.
10. In the case $m = 2k$, use the formula you found in the previous problem to determine the time it takes for the object to hit the ground. How does your answer compare to your answer in question #1?

Another model of air resistance assumes that the drag force is proportional to the *square* of the speed, so that (when the object is moving downwards) we have

$$\frac{dv}{dt} = -9.8 + \frac{c}{m}v^2,$$

for some positive constant c. Note that for a given object the value of c in this model would be different than the value of k in the model above, though it would still depend on the qualities you listed in question #5.

11. (Graded for completeness only.) Why is the term corresponding to air resistance added in this model, where it was subtracted before?
12. At the terminal velocity, the acceleration of the object is zero. Therefore, to find the terminal velocity we can set $\frac{dv}{dt} = -9.8 + \frac{c}{m}v^2 = 0$ and solve for v. Do this, and write the terminal velocity in terms of c and m. Be careful with the sign!

Instructions, part deux

After reading and reflecting on these questions, complete the pre-read assignment on Canvas. This will give your coach some insight on your thinking in order to best help you before you are required to formally answer these questions.