Differential Equations
Read-ahead

Introduction
Differential equations are used in a wide variety of fields to model and predict changing quantities. In this application, you will solidify your understanding of the derivative by exploring the use of differential equations in various contexts, including some of the previous applications.[1]

$$\frac{dx}{dt} = a(y - x)$$
$$\frac{dy}{dt} = x(b - z) - y$$
$$\frac{dz}{dt} = xy - cz$$

Figure 1: The Lorenz system of differential equations

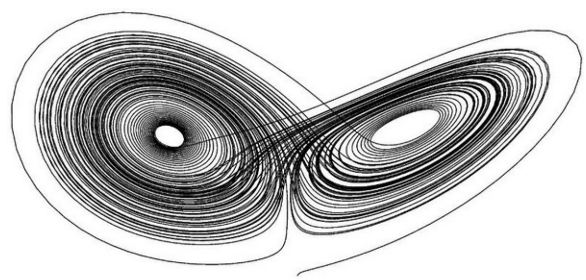

Figure 2: Solutions to the Lorenz system

Instructions
After reading through *Differential Equations* context and questions below, you should complete the reflection assignment in Canvas. Note: *you will have a chance to talk further with your coach before answering the questions below in detail*. The point of this read-ahead and the reflection is to "prime the pump" for further conversations with your coaches.

Differential Equations

A *differential equation* is an equation involving one or more derivatives. A *solution* to a differential equation is a function that satisfies the equation. For example,

$$y(t) = \sqrt{1 - t^2}$$

is a solution to the differential equation

$$\frac{dy}{dt} = -\frac{t}{y},$$

since, when we plug in the purported solution, the left-hand side

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{1-t^2}\right) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = -\frac{t}{\sqrt{1-t^2}}$$

[1] The Lorenz system shown in the figures above is more complicated than the differential equations we will be considering in this application. If you are interested in learning more about this system and its chaotic solutions, see the link in the "Explore" area for a nice introductory video on the topic.

and the right-hand side

$$-\frac{t}{y} = -\frac{t}{\sqrt{1-t^2}}$$

of the differential equation are equal.

As mentioned above, differential equations are widely used to model real-world quantities. For example, if a colony of bacteria has plenty of nutrients and ample space in which to grow, the individual bacteria reproduce and die at rates that remain fixed over time. If $P(t)$ represents the number of bacteria at time t, then **the rate of growth of the population is proportional to the size of the population**. In equation form, this bolded statement is written

$$\frac{dP}{dt} = kP,$$

where k is a positive constant[2]. This differential equation is known as the "Exponential Model", because its solutions have the form

$$P(t) = P_0 e^{kt}.$$

Looking at the graph of a solution shown below, we can see that it makes sense given our understanding of the derivative as the slope of the tangent line: the larger the value of P, the steeper the slope.

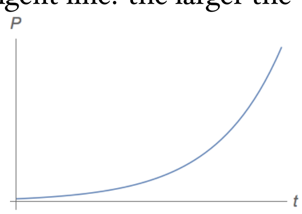

Figure 1: A solution to the exponential model.

Questions

1. *Newton's Law of Cooling* states that **an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the environment**. In differential equation form, this is written

$$\frac{dT}{dt} = -k(T - A),$$

where $T = T(t)$ represents the temperature of the object, A the temperature of the environment[3], and k is a positive constant. Suppose a turkey is taken out of the oven when the thermometer shows that it has reached 350 degrees Fahrenheit. It is placed in a room where the temperature 75 degrees Fahrenheit. According to Newton's Law of Cooling, at which of the following values of T will the turkey be cooling most quickly? (You may assume that the room is big enough that its temperate is not measurably changed by the presence of a hot turkey.)

(a) 150° F (b) 200° F (c) 250° F (d) 300° F

[2] k is the per-capita net birth rate of the population.
[3] A is also called the "ambient" temperature.

2. Suppose when the turkey has cooled to 150° F a second turkey is taken out of the freezer and placed in the same room. According to the differential equation $\frac{dT}{dt} = -k(T - A)$, where k and A are the same as in the previous problem, at what temperature would the turkey need to be so that it is warming at the same rate that the first turkey is cooling?
3. A population of aphids living in a field of corn is modeled by $\frac{dP}{dt} = 0.2(1-0.1t)P$, where the time t is measures in weeks. According to this model, the population grows to a peak, then declines as the aphids begin to run out of food. After how many weeks does the population start to decline?
4. (Graded for completeness only.) In the Keeling curve application we chose to model the baseline atmospheric CO_2 concentration over time as a linear function $C(t)$. However, a close examination of the data (shown below) indicates that the CO_2 concentration might instead be growing exponentially, i.e., according to the differential equation

$$\frac{dC}{dt} = kC,$$

for some positive constant k. Give a reason why the rate of change of CO_2 concentration might depend on the concentration of CO_2. Hint: The name of this "effect" is well-known.

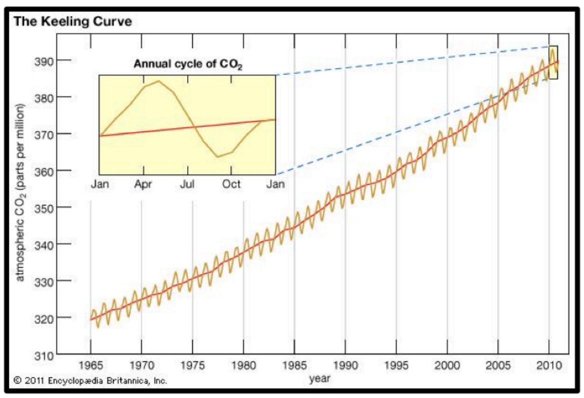

Figure 2: The Keeling Curve

5. Towards the end of the Interest Rates and the Fisher Equation application, you considered a continuous model of a bank account balance, with nominal interest rate i, inflation rate π, and real interest rate r. In fact, if $g(t)$ gives the real value of the account, modeled continuously, then

$$\frac{dg}{dt} = ig - \pi g.$$

The right-hand side of this differential equation consists of two terms, one positive and one negative. Both are rates, with the same units as $\frac{dg}{dt}$, i.e., dollars per year. Describe in words the practical meaning of these two rates.
6. Recall at the end of the Pendulum application the following expression was given as an approximation for the angle θ as a function of the time t.

$$\theta(t) = \alpha \cos\left(\sqrt{\frac{g}{L}} * t\right)$$

Here α is the initial angle, g is the acceleration due to gravity, L is the length of the pendulum, and t is in seconds. This function is the solution to the second-order differential equation

$$\frac{d^2\theta}{dt^2} = -k\theta.$$

Given that $\theta(t)$ is a solution, express k in terms of g and L.

7. (Graded for completeness only.) The differential equation in the previous problem is itself an approximation of the differential equation $\frac{d^2\theta}{dt^2} = -k \sin\theta$. Use derivatives (and the tangent line) to explain why $\sin\theta \approx \theta$ when θ is small.
8. In the Transformations and Modeling application, you used trigonometric functions to model the number of rabbits and foxes as periodic functions of time. The simplest differential equation model for this situation is the Lotka-Volterra predator-prey model, which is a system of two first-order differential equations:

$$\frac{dr}{dt} = ar - brf$$
$$\frac{df}{dt} = -cf + drf$$

The parameters a, b, c, and d are all positive. Solutions to this model are periodic, though not sinusoidal. Which of the four terms in the right-hand side of this model represents the predation of rabbits by foxes?

Instructions, part deux

After reading and reflecting on these questions, complete the pre-read assignment on Canvas. This will give your coach some insight on your thinking in order to best help you before you are required to formally answer these questions.