Vector Projection and Data Fitting
Read-ahead

Introduction

Given a vector $\vec{v}$ and a line L, the *vector projection of $\vec{v}$ onto L*, written $\text{proj}_L\vec{v}$, is found by placing the tail of $\vec{v}$ on L and dropping a perpendicular to L from the head of $\vec{v}$, as shown below in Figure 1, with the vector projection in blue.

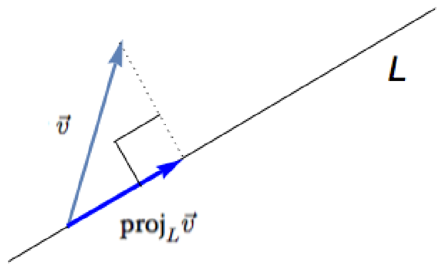

Figure 1: vector projection

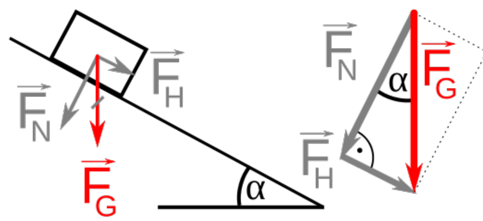

Figure 2: force decomposition diagram

Vector projections have a wide variety of applications, particularly in physics where force vectors are often "decomposed" into the sum of their projections onto the directions parallel to, and perpendicular to, the motion. An example of a force decomposition diagram is given in Figure 2. In this application you will see how to use the dot product to project a vector onto a line, and see an application of vector projection in the method of least squares.

Instructions

After reading through *Vector Projection and Data Fitting* context and questions below, you should complete the reflection assignment in Canvas. Note: *you will have a chance to talk further with your coach before answering the questions below in detail*. The point of this read-ahead and the reflection is to "prime the pump" for further conversations with your coaches.

Vector Projection and Data Fitting

The geometric description of $\text{proj}_L\vec{v}$ given above makes sense in two and three dimensions, but does not gives us a way to directly compute the projection of a given vector onto a given line, and does not generalize to higher dimensions. (How can we "drop a perpendicular" in 10-dimensional space?) It turns out there is a simple way to compute $\text{proj}_L\vec{v}$ using the dot product.

Consider the right triangle OPQ in Figure 3 below, in which $\vec{u}$ is a unit vector parallel to the line L, and the angle θ between $\vec{v}$ and $\vec{u}$ is less than 90 degrees. Then

$$\cos\theta = \frac{||\overline{OQ}||}{||\overline{OP}||} = \frac{||\text{proj}_L\vec{v}||}{||\vec{v}||},$$

and so

$$\begin{aligned}||\text{proj}_L\vec{v}|| &= ||\vec{v}||\cos\theta \\ &= ||\vec{v}||||\vec{u}||\cos\theta \quad (\text{since } ||\vec{u}|| = 1) \\ &= \vec{v}\cdot\vec{u}\end{aligned}$$

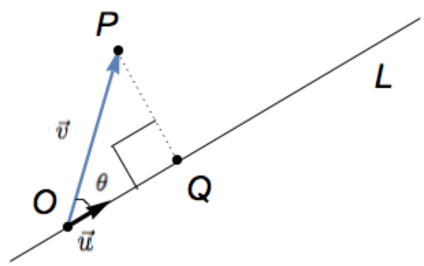

Figure 3

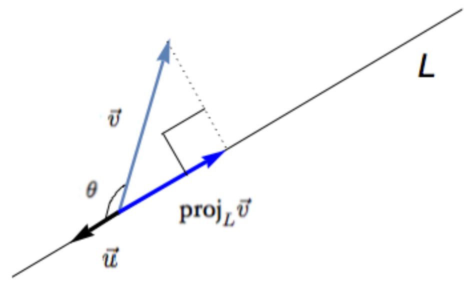

Figure 4

The vector projection $\overrightarrow{OQ} = \text{proj}_L\vec{v}$ is thus the vector in the direction of $\vec{u}$ with magnitude $\vec{v}\cdot\vec{u}$, i.e.,

$$\text{proj}_L\vec{v} = (\vec{v}\cdot\vec{u})\,\vec{u}.$$

In Figure 3 and in the work above we assumed that the angle θ is less than 90 degrees. If this angle is more than 90 degrees, as in Figure 4, we can see that the vector projection $\text{proj}_L\vec{v}$ points in the opposite direction to $\vec{u}$. In this case, the dot product $\vec{v}\cdot\vec{u}$ is negative, and the formula $\text{proj}_L\vec{v} = (\vec{v}\cdot\vec{u})\,\vec{u}$ still holds.

Note that the vector projection is sometimes written $\text{proj}_{\vec{u}}\vec{v}$, or indeed $\text{proj}_{\vec{w}}\vec{v}$ where $\vec{w}$ is any vector (not just a unit vector) parallel to L. In this way we can talk about the projection of one vector onto another vector.

Example

Suppose we want to project the vector $\vec{v} = 3\hat{\imath} + 5\hat{\jmath}$ onto the line L given by $y = -2x$. To find a vector in the direction of L, note that $A(0,0)$ and $B(1,-2)$ are on L, so the vector $\overrightarrow{AB} = \hat{\imath} - 2\hat{\jmath}$ is parallel to L. Since $\left|\left|\overrightarrow{AB}\right|\right| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$, a unit vector parallel to L is $\vec{u} = \frac{\overrightarrow{AB}}{\sqrt{5}} = \frac{\sqrt{5}}{5}\hat{\imath} - \frac{2\sqrt{5}}{5}\hat{\jmath}$. Then

$$\text{proj}_L\vec{v} = (\vec{v}\cdot\vec{u})\,\vec{u} = \left(3\left(\frac{\sqrt{5}}{5}\right) + 5\left(-\frac{2\sqrt{5}}{5}\right)\right)\left(\frac{\sqrt{5}}{5}\hat{\imath} - \frac{2\sqrt{5}}{5}\hat{\jmath}\right) = -\frac{7\sqrt{5}}{5}\left(\frac{\sqrt{5}}{5}\hat{\imath} - \frac{2\sqrt{5}}{5}\hat{\jmath}\right) = -\frac{7}{5}\hat{\imath} + \frac{14}{5}\hat{\jmath}.$$

If instead we had used the unit vector $\vec{w} = -\frac{\sqrt{5}}{5}\hat{\imath} + \frac{2\sqrt{5}}{5}\hat{\jmath}$, also parallel to L but in the opposite direction to $\vec{u}$, we would have

$$\text{proj}_L\vec{v} = (\vec{v}\cdot\vec{w})\,\vec{w} = \left(3\left(-\frac{\sqrt{5}}{5}\right) + 5\left(\frac{2\sqrt{5}}{5}\right)\right)\left(-\frac{\sqrt{5}}{5}\hat{\imath} + \frac{2\sqrt{5}}{5}\hat{\jmath}\right) = \frac{7\sqrt{5}}{5}\left(-\frac{\sqrt{5}}{5}\hat{\imath} + \frac{2\sqrt{5}}{5}\hat{\jmath}\right) = -\frac{7}{5}\hat{\imath} + \frac{14}{5}\hat{\jmath},$$

same as before. See Figure 5 below for an illustration.

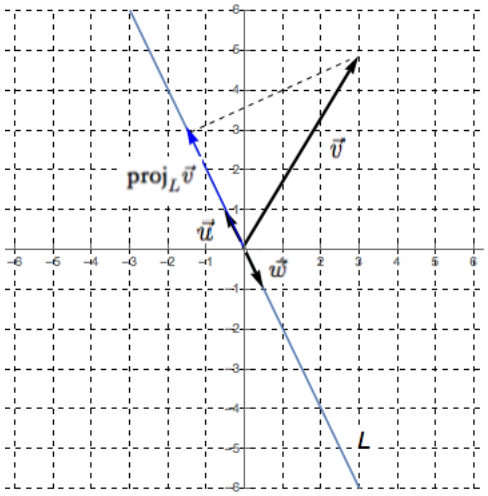

Figure 5

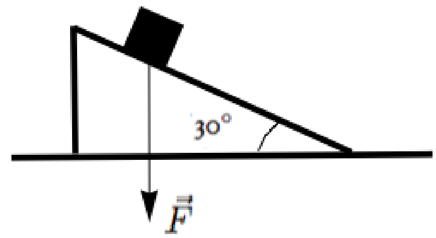

Figure 6

Questions

1. Compute the projection of the vector $\vec{v} = \hat{\imath} - \hat{\jmath}$ onto the line $y = 2x$.
2. (Graded for completeness only.) Describe in words what happens when a vector $\vec{v}$ is projected onto a line L that is parallel to $\vec{v}$, and what happens when $\vec{v}$ is perpendicular to L.
3. Given a vector $\vec{v}$ and a line L, we can also project $\vec{v}$ onto a line perpendicular to L. Compute the projection of the vector $\vec{v} = \hat{\imath} - \hat{\jmath}$ onto the line through the origin that is perpendicular to the line $y = 2x$.
4. (Graded for completeness only.) The vector you found in problem #3 is sometimes called the vector ***rejection*** of $\vec{v}$ from L, written $\text{rej}_L\vec{v}$. Explain why, for any vector $\vec{v}$ and any line L, the dot product $\text{proj}_L\vec{v} \cdot \text{rej}_L\vec{v}$ is zero and the vector sum $\text{proj}_L\vec{v} + \text{rej}_L\vec{v}$ equals the original vector $\vec{v}$. Note that this latter relationship gives us an easy way to compute $\text{rej}_L\vec{v}$ if we've already computed $\text{proj}_L\vec{v}$.
5. An object is placed on an inclined plane, as shown in Figure 6. The force due to gravity is $\vec{F} = -20\hat{\jmath}$. If the surface of the inclined plane makes an angle of 30° with the horizontal, find the projection of $\vec{F}$ onto the inclined plane and the rejection of $\vec{F}$ from the inclined plane. (These vectors are also called the ***components*** of $\vec{F}$ parallel to, and perpendicular to, the inclined plane.) Round your answers to two decimal places.

The concept of vector projection generalizes to higher dimensions. In three dimensions, we can project a vector onto a line or onto a plane. In four or more dimensions we can project onto any ***subspace***[1] W of the whole space. Although it may be difficult to visualize this projection, if the subspace W is described using mutually perpendicular unit vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_m$, then the calculation is simple:

$$\text{proj}_W\vec{v} = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2 + \ldots + (\vec{v} \cdot \vec{u}_m)\vec{u}_m.$$

[1] Subspaces are higher dimensional generalizations of lines and planes.

Least Squares and Data Fitting

Suppose you are given a bunch of data points and asked to draw a line through those points. Almost certainly, it will be impossible to hit all the points, but you can still try to draw a line that "fits" the overall trend of the data. The *line of best fit* for the data, also called the "trend line", is the line that minimizes the sum of the squares of the differences in the y values of the data points and the corresponding y values of the line. The algorithm to compute this line is called *Least Squares* or *Linear Regression*. This method is typically implemented with a program like Excel and without much thought to the underlying mathematics. In fact, the concept of vector projection plays a key role, as we shall see in the following example.

Consider the three points $(0, 3), (3, 1)$, and $(4, 1)$. If a line $y = mx + b$ passes through these three points, then we have

$$\begin{aligned} b &= 3 \\ 3m + b &= 1 \\ 4m + b &= 1. \end{aligned}$$

This is a system of three equations and two unknowns (m and b) that has no solutions. This should not be a surprise, as the points clearly do not line up. Suppose we ask which points $(0, y_1), (3, y_2)$, and $(4, y_3)$ *do* lie in a line, i.e., for which y values y_1, y_2 and y_3 does the system

$$\begin{aligned} b &= y_1 \\ 3m + b &= y_2 \\ 4m + b &= y_3 \end{aligned}$$

have a solution? The answer to this question is precisely the set vectors $y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$ such that $y_1\hat{i} + y_2\hat{j} + y_3\hat{k} = m\vec{v}_1 + b\vec{v}_2$, where $\vec{v}_1 = 3\hat{j} + 4\hat{k}$ and $\vec{v}_2 = \hat{i} + \hat{j} + \hat{k}$. This set of vectors forms a plane W in three dimensions that can be described by the mutually perpendicular unit vectors $\vec{u}_1 = \frac{3}{5}\hat{j} + \frac{4}{5}\hat{k}$ and $\vec{u}_2 = \frac{25}{5\sqrt{26}}\hat{i} + \frac{4}{5\sqrt{26}}\hat{j} - \frac{3}{5\sqrt{26}}\hat{k}$, as illustrated by Figure 7 below. The vectors $\vec{u}_1$ and $\vec{u}_2$ are found using an separate algorithm called the Gram-Schmidt Method, which also involves vector projection.

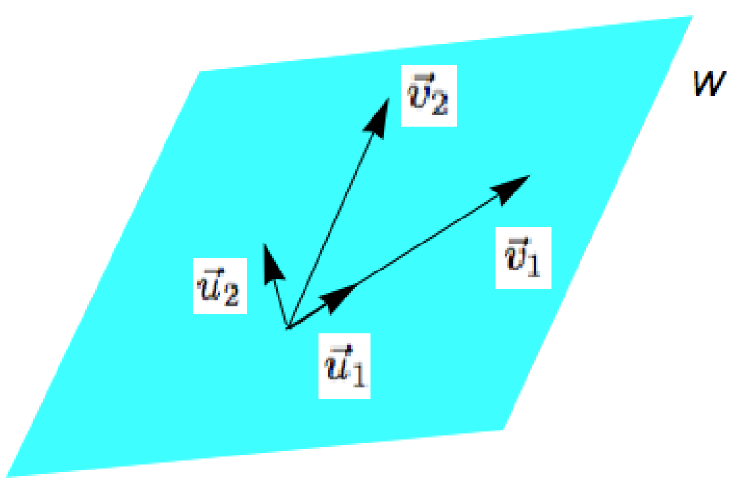

Figure 7: result of the Gram-Schmidt method

The method of Least Squares works by replacing the y values in the right-hand side of the original system with their projection onto this plane. Specifically, if we let $\vec{a} = 3\hat{\imath} + \hat{\jmath} + \hat{k}$, then we compute $\text{proj}_W \vec{a} = (\vec{a} \cdot \vec{u}_1)\vec{u}_1 + (\vec{a} \cdot \vec{u}_2)\vec{u}_2 = \frac{38}{13}\hat{\imath} + \frac{17}{13}\hat{\jmath} + \frac{10}{13}\hat{k}$. We then replace the original system

$$\begin{aligned} b &= 3 \\ 3m + b &= 1 \\ 4m + b &= 1 \end{aligned}$$

which has no solutions with the system

$$\begin{aligned} b &= 38/13 \\ 3m + b &= 17/13 \\ 4m + b &= 10/13, \end{aligned}$$

which does. In particular, setting $b = 38/13$ in the second equation gives $m = \frac{1}{3}\left(\frac{17}{13} - \frac{38}{13}\right) = -\frac{7}{13}$, and setting $b = 38/13$ in the third equation also gives $m = \frac{1}{4}\left(\frac{10}{13} - \frac{38}{13}\right) = -\frac{7}{13}$. So the line of best fit for the three points $(0, 3), (3, 1)$, and $(4, 1)$ is $y = -\frac{7}{13}x + \frac{38}{13}$. This line is shown below in Figure 8.

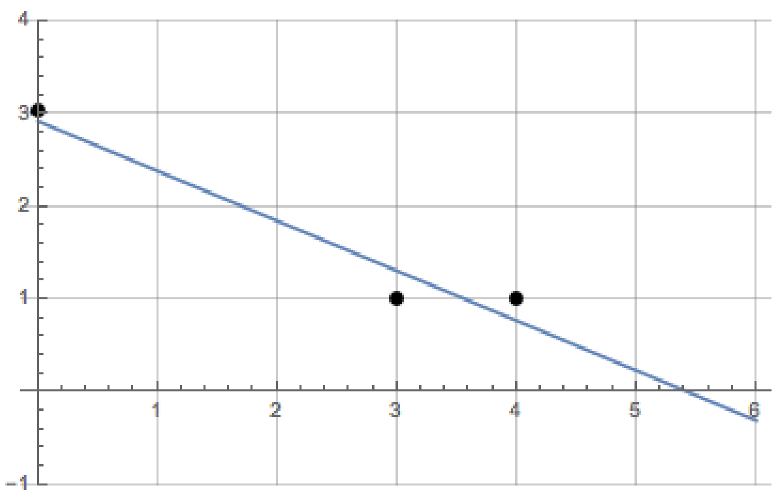

Figure 8: line of best fit

6. (Graded for completeness only.) First use software such as Excel or Google Sheets to verify the line of best fit found in the example above, then use the same software to find the quadratic function of best fit. How well do you think this quadratic function fits the data? Explain your reasoning. *See the instructions for using linear regression in the "Explore" panel of this module.*
7. A simpler problem than the one considered above is to find the constant function $y = b$ that best fits a given set of data. Consider the same three points $(0, 3), (3, 1)$, and $(4, 1)$. We now have the system

$$\begin{aligned} b &= 3 \\ b &= 1 \\ b &= 1. \end{aligned}$$

Clearly this system has no solution, since b can't simultaneously be equal to both 3 and 1. Project the vector $\vec{a} = 3\hat{\imath} + \hat{\jmath} + \hat{k}$ onto the line L parallel to the vector $\vec{v}_2 = \hat{\imath} + \hat{\jmath} + \hat{k}$. Enter the exact answers, using fraction notation, e.g., "1/2", if necessary.

8. Replace the right-hand side of the system given in problem #7 with the components of the vector projection $\text{proj}_L\vec{a}$ you computed in that problem. This new system should have a solution. What is the constant function $y = b$ of best fit for the data?
9. (Graded for completeness only.) How does the value of b you found in the previous problem relate to the original data? Is there a different way you could have computed this value?

Instructions, part deux

After reading and reflecting on these questions, complete the pre-read assignment on Canvas. This will give your coach some insight on your thinking in order to best help you before you are required to formally answer these questions.