The Z-Transform (ZT)

→ It is used to convert time domain into Z-Domain

→ complicated convolution in time domain is converted into simple multiplication in Z-domain

$$y(n) = x(n) * h(n) \longleftrightarrow Y(z) = X(z) \cdot H(z).$$

where $Y(z)$ is the ZT of $y(n)$

→ ZT is used on only DT signals

→ ZT Provide pole-zero locations of DT-LTI system to verify stability

→ ZT is used to solve Linear constant coefficient Difference Equations (LCCDE) with initial conditions

Defn:- ZT of DT signal $x(n)$ is defined as power series (infinite)

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where $z \rightarrow$ complex variable

$$z = r \cdot e^{j\omega}$$

The Region of Convergence (ROC):-

$$\text{ZT} \quad X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

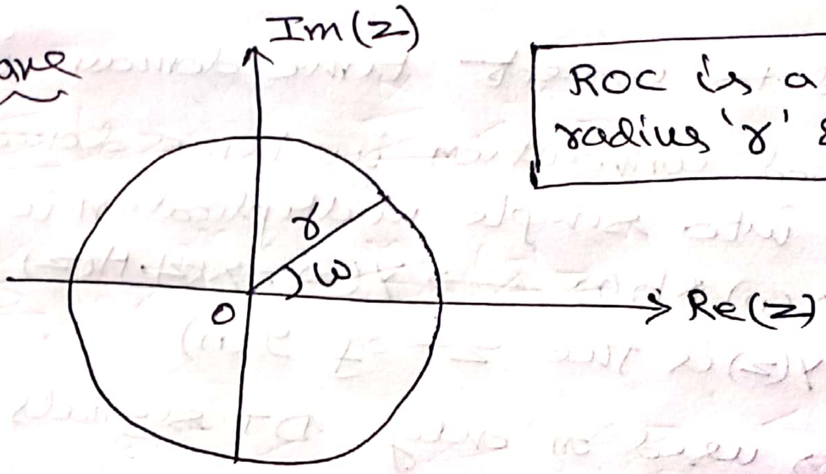
The ROC of $X(z)$ is the set of z values for which $X(z)$ attains a finite value.

$$z = r \cdot e^{j\omega}$$

where magnitude of $z = |z| = r$

Angle of $z = \angle z = \omega$

ROC Representation is as shown in Figure



ROC is a circle with radius 'γ' & angle 'ω'

Problems:-

1) Find ZT of $x(n) = a^n u(n)$ → causal sequence

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) \cdot z^{-n} =$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

only for $|a z^{-1}| < 1$ for $X(z)$ to be finite

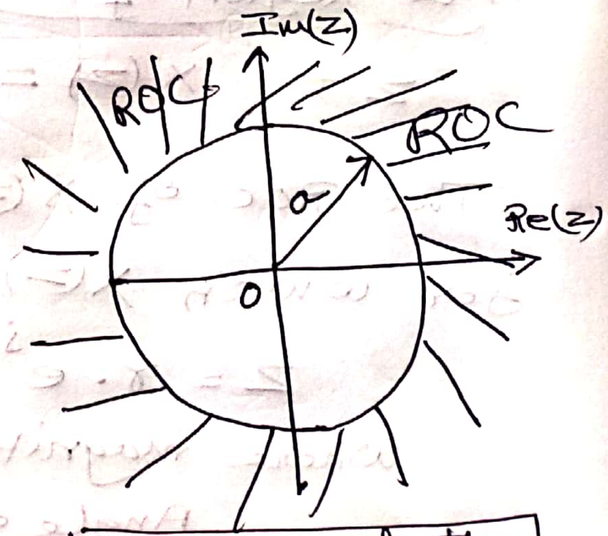
$\boxed{|a| < |z|}$ → ROC condition

$$\sum_{n=0}^{\infty} (a)^n = \begin{cases} \frac{1}{1-a}, & a<1 \\ \infty, & a \geq 1 \end{cases}$$

$$\boxed{X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}}$$

ROC: $|z| > |a|$

ROC is outside the circle for causal sequence



ROC: outside the circle

$\Rightarrow$ causal

2) Find ZT of $x(n) = a^n[-u(-n-1)]$

→ Non causal sequence.

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n(-u(-n-1))\, z^{-n} = \sum_{n=-\infty}^{-1} a^n(-1) \cdot z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

Substitute $n = -n$ to convert $-$ve limits to +ve limits

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

$$\boxed{\sum_{n=\square}^{\infty} (a)^n = \begin{cases} \dfrac{a^{\square}}{1-a}, & a<1 \\ \infty, & a \geq 1 \end{cases}}$$

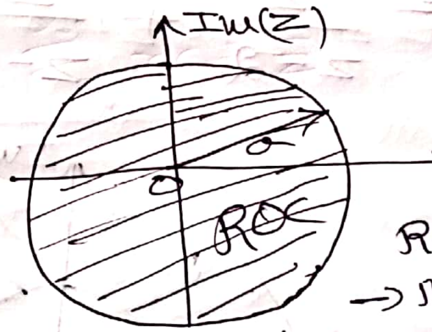
$$= -\left[\frac{a^{-1} z}{1 - a^{-1} z}\right]$$

$$\boxed{X(z) = +\left[\frac{z}{z-a}\right]}$$

ROC :- $|a^{-1} z| < 1$

$|z| < |a|$

ROC :- $|z| < |a|$



inside ~~outside~~ ROC

→ Non causal

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ROC is inside The circle for non causal sequence

3.) Find ZT of $x(n) = (2)^n u(n) - 4^n u(-n-1)$

→ Two sided sequence
→ both causal & non causal.

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [2^n u(n) - 4^n u(-n-1)]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u(n) z^{-n} - \sum_{n=-\infty}^{\infty} 4^n u(-n-1) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (2 z^{-1})^n - \sum_{n=-\infty}^{-1} 4^n \cdot z^{-n}$$

Substitute $n = -n$ in 2 term

$$= \sum_{n=0}^{\infty} (2z^{-1})^n - \sum_{n=1}^{\infty} 4^{-n} z^n$$

$$X(z) = \sum_{n=0}^{\infty} (2z^{-1})^n - \sum_{n=1}^{\infty} (4^{-1} z)^n$$

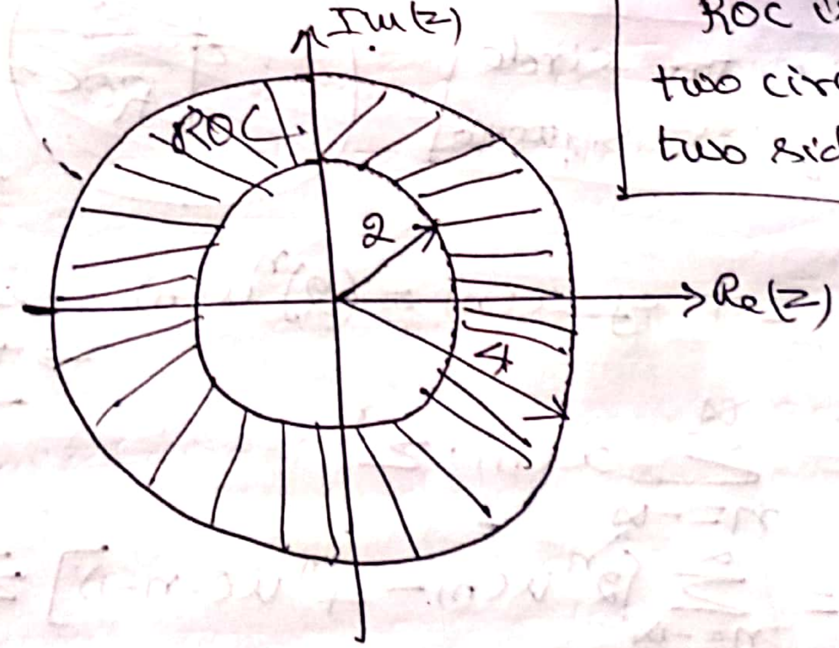
ROC: $|2z^{-1}| < 1$ & $|4^{-1} z| < 1$

$|z| > 2$ & $|z| < 4$

$$\boxed{2 < |z| < 4}$$

$$X(z) = \frac{1}{1 - 2z^{-1}} - \frac{4^{-1} z}{1 - 4^{-1} z}$$

$$\boxed{X(z) = \frac{z}{z-2} + \frac{z}{z-4}}$$ ROC: $2 < |z| < 4$



ROC is between two circles for two sided sequence

4) Find ZT of $x(n) = (2^n + 4^n)u(n)$

→ one sided sequence.

Also find pole-zero locations

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} (2^n + 4^n) u(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} 4^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n} + \sum_{n=0}^{\infty} 4^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (2z^{-1})^n + \sum_{n=0}^{\infty} (4z^{-1})^n$$

ROC: $|2z^{-1}| < 1$ & $|4z^{-1}| < 1$

$\cancel{|z| < 2}$ & $|z| > 4$ ← ?

$|z| > 2$

If $z = 3$, $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n = \infty$

$\therefore X(z) \to \infty$

$z = 5$, $X(z) \to$ Finite

$\therefore |z| > 4$

Higher

$$X(z) = \frac{z}{z-2} + \frac{z}{z-4}$$

$$= \frac{z(z-4) + z(z-2)}{(z-2)(z-4)}$$

$$= \frac{z^2 - 4z + z^2 - 2z}{(z-2)(z-4)} = \frac{2z^2 - 6z}{(z-2)(z-4)}$$

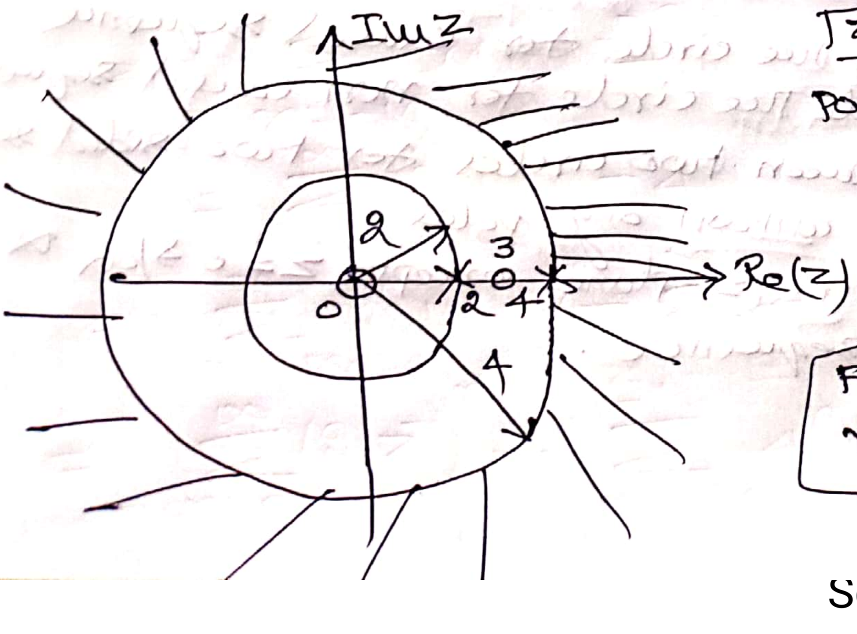
$$X(z) = \frac{z(2z-6)}{(z-2)(z-4)}, \quad \text{ROC:- } |z| > 4$$

Zeros:- $z(2z-6) = 0$

$\boxed{z = 0}$ $2z = 6$

$\boxed{z = 3}$

Poles:- $z = 2$, $z = 4$



Roc does not contain any poles

For stability no poles in ROC

5) Find ZT of $x(n) = [\underset{\uparrow}{1}, 5, 6, 7]$ Finite sequence & Right sided

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{3} x(n) z^{-n}$$

$$X(z) = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$X(z) = 1 + 5z^{-1} + 6z^{-2} + 7z^{-3}$$

ROC :- Entire z-plane except $z = 0$

6) Find ZT of $x(n) = [1, 5, 6, \underset{\uparrow}{7}]$ Finite left sided sequence.

$$= X(z) = z^{3} + 5z^{2} + 6z^{1} + 7$$

Entire z-plane, except $z = \infty$

7) Find ZT of $x(n) = [1, 5, \underset{\uparrow}{6}, 7]$ Finite. Two sided sequence

$$X(z) = z^{2} + 5z + 6 + 7z^{-1}$$

Entire z-plane except $z = 0$ & ∞

> For $x(n)$ finite duration, the ROC is entire z-plane except $z = 0$ &/OR ∞

Properties of ROC

1. ROC consists of a ring in the z-plane centred about the origin
2. ROC is outside the circle for causal sequence
3. ROC is inside the circle for non causal sequence
4. ROC is between two circles for two sided sequence
5. ROC does not contain any poles
6. ROC is entire z-plane except $z = 0$ &/OR ∞ for finite sequence.

6) Find ZT of (i) $x(n) = \delta(n)$

(ii) $x(n) = \delta(n-k)$ (iii) $x(n) = u(n)$

(iv) $x(n) = u(n-2)$ (v) $x(n) = u(-n)$

(vi) $x(n) = a^n, \quad 0 \le n \le N-1$

(vii) $x(n) = e^{j\Omega n} u(n)$

(viii) $x(n) = e^{-j\Omega n} u(n)$

(i) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$X(z) = \sum_{n=0} \delta(n) z^{-n} = \delta(0) z^{0} = 1$$

ROC: Entire z-plane.

(ii) $x(n) = \delta(n-k)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n-k) \cdot z^{-n}$$

$$= \sum_{n=k} \delta(n-k) z^{-n} = \delta(0) z^{-k}$$

$$X(z) = z^{-k}$$

ROC: Entire z-plane except $z=0$, for $k>0$
except $z=\infty$, for $k<0$

(iii) $x(n) = u(n)$

$$X(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{z}{z-1}$$

ROC: $|z^{-1}| < 1 \Rightarrow |z| > 1$

(iv) $x(n) = \cancel{a^n}\ u(n-2)$

$$X(z) = \cancel{\sum_{n=-\infty}^{\infty} a^n z^{-n}} = \sum_{n=-\infty}^{\infty} u(n-2) \cdot z^{-n}$$

$$= \sum_{n=2}^{\infty} (z^{-1})^n = \frac{(z^{-1})^2}{1-z^{-1}} = \frac{z^{-2}}{1-z^{-1}}$$

$$\boxed{X(z) = \frac{1}{z(z-1)}}$$

ROC: $|z^{-1}| < 1$
$|z| > 1$

(v) $x(n) = u(-n)$

$$X(z) = \sum_{n=0}^{-\infty} z^{-n} \qquad \text{Substitute } n=-n$$

$$= \sum_{n=0}^{\infty} z^{n} = \sum_{n=0}^{\infty} (z)^{n} = \frac{1}{1-z}$$

ROC :- $|z| < 1$

(vi) $x(n) = a^n \quad , \quad 0 \le n \le N-1$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$= \frac{1-(az^{-1})^{N-1+1}}{1-az^{-1}} \quad , \quad |az^{-1}| \ne 1$$

$\Rightarrow |az^{-1}| < 1$

$$X(z) = \frac{1-a^N z^{-N}}{1-az^{-1}}$$

$|z| > |a|$

causal

ROC: $|z| > |a|$

(vii) $x(n) = e^{j\Omega n} u(n)$

$$X(z) = \sum_{n=0}^{\infty} e^{j\Omega n} \cdot z^{-n} = \sum_{n=0}^{\infty} (e^{j\Omega} \cdot z^{-1})^n$$

$$X(z) = \frac{1}{1-e^{j\Omega} z^{-1}} \quad , \quad \text{ROC: } |e^{j\Omega} \cdot z^{-1}| < 1$$

$|z| > |e^{j\Omega}|$

$|z| > 1$

$$\therefore \boxed{X(z) = \frac{z}{z-e^{j\Omega}}} \qquad \text{ROC:- } |z| > 1$$

(viii) $x(n) = e^{-j\Omega n} u(n)$

$$X(z) = \frac{z}{z-e^{-j\Omega}} \quad , \quad \text{ROC: } |z| > 1$$

(7) Find ZT of $x(n) = (8)^{|n|}$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} (8)^{|n|} \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (8)^{(-n)} \cdot z^{-n} + \sum_{n=0}^{\infty} 8^{n} \cdot z^{-n}$$

substitute $n = -n$ in 1st term

$$= \sum_{n=1}^{\infty} 8^{n} \cdot z^{n} + \sum_{n=0}^{\infty} (8 \cdot z^{-1})^{n}$$

$$X(z) = \sum_{n=1}^{\infty} (8z)^{n} + \sum_{n=0}^{\infty} (8z^{-1})^{n}$$

ROC:

$|8z| < 1$ & $|8z^{-1}| < 1$

$|z| < \frac{1}{8}$ & $|z| > 8$

$\frac{1}{8} < |z| <$

(7) Find ZT of $x(n) = \left(\frac{1}{8}\right)^{|n|}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{8}\right)^{|n|} \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{8}\right)^{(-n)} \cdot z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^{(n)} \cdot z^{-n}$$

substitute $n = -n$ in 1st term

$$= \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^{n} z^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^{n} \cdot z^{-n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{8} z\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{8} \cdot z^{-1}\right)^{n}$$

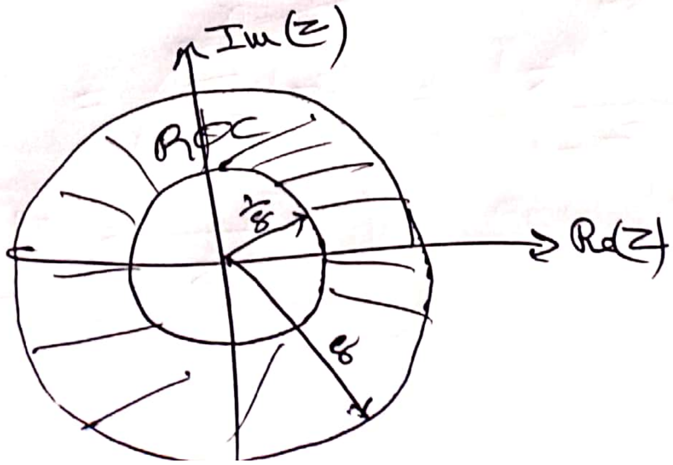
ROC:

$\left|\frac{1}{8} z\right| < 1$ & $\left|\frac{1}{8} z^{-1}\right| < 1$

$|z| < 8$ & $|z| > \frac{1}{8}$

$$X(z) = \frac{\frac{1}{8} z}{1 - \frac{1}{8} z} + \frac{1}{1 - \frac{1}{8} z^{-1}}$$

ROC $\boxed{\frac{1}{8} < |z| < 8}$



Properties of ZT :-

1. Linearity :-

If $x_1(n) \longleftrightarrow X_1(z)$ with ROC R_1

& $x_2(n) \longleftrightarrow X_2(z)$ with ROC R_2

The $[a x_1(n) + b x_2(n)] \longleftrightarrow [a X_1(z) + b X_2(z)]$

ROC is the intersection of two ROC's

Proof:- LHS $= [a x_1(n) + b x_2(n)]$

$$Z[LHS] = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z) = RHS$$

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2. Time Shift :-

If $x(n) \longleftrightarrow X(z)$ with ROC R

$x(n-n_0) \longleftrightarrow z^{-n_0} \cdot X(z)$ with ROC R

with $z \neq 0$ (or) $z \neq \infty$

Proof: $Z[LHS] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$

let $n - n_0 = p$, $n = p + n_0$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+n_0)} = \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-p} \cdot z^{-n_0}$$

$$= z^{-n_0} \sum_{p=-\infty}^{\infty} x(p) z^{-p} = z^{-n_0} X(z) = RHS$$

3. Time Reversal :-

If $x(n) \longleftrightarrow X(z)$ with ROC R

then $x(-n) \longleftrightarrow X(z^{-1})$ with ROC $\left(\frac{1}{R}\right)$

$$= Z\{LHS\} = \sum_{n=-\infty}^{\infty} x(-n) \cdot z^{-n}$$

Let $n = -n$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[z^{-1}\right]^{-n} = X(z^{-1}) = RHS$$

4. Scaling in Z-domain :-

If $x(n) \longleftrightarrow X(z)$ with ROC R

$(a)^n x(n) \longleftrightarrow X\left(\frac{z}{a}\right)$ with ROC $|a| R$

Proof :- $Z[LHS] = \sum_{n=-\infty}^{\infty} [a^n x(n)] \cdot z^{-n}$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[a^{-1} z\right]^{-n} = X(a^{-1}z) = X\left(\frac{z}{a}\right) = RHS$$

5. Differentiation in Z-domain (or) multiplication of sequence with time 'n'

If $x(n) \longleftrightarrow X(z)$ with ROC : R

Then $n \cdot x(n) \longleftrightarrow -z \frac{dX(z)}{dz}$ ROC : R

Proof :- Consider $Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

D.w.r. z

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n) (-n) \cdot z^{-n-1}$$

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} [n \cdot x(n)] \cdot z^{-n}$$

$$\boxed{-z \cdot \frac{dX(z)}{dz} = Z[n \cdot x(n)]}$$

6) Convolution :-

If $x_1(n) \longleftrightarrow X_1(z)$ ROC : R_1
& $x_2(n) \longleftrightarrow X_2(z)$ ROC : R_2

then $[x_1(n) * x_2(n)] \longleftrightarrow [X_1(z) \cdot X_2(z)]$

ROC: intersection of R_1 & R_2

= Proof: $Z[LHS] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] \cdot z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K) \cdot z^{-n}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{n=-\infty}^{\infty} x_2(n-K) \cdot z^{-n}$$

let $n-K = P$, $n = P+K$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \cdot \sum_{P=-\infty}^{\infty} x_2(P) \cdot z^{-(P+K)}$$

$$= \left[\sum_{K=-\infty}^{\infty} x_1(K) \cdot z^{-K}\right] \cdot \left[\sum_{P=-\infty}^{\infty} x_2(P) \cdot z^{-P}\right]$$

$$= X_1(z) \cdot X_2(z) = RHS$$

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7) If $x(n)$ is causal i.e., $x(n)=0$ for $n<0$

then $\underset{n \to 0}{Lt}\, x(n) = x(0) = \underset{z \to \infty}{Lt} [X(z)]$

Proof :- consider $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

For causal $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots + x(\infty)\, z^{-\infty}$$ (13)

$$\underset{z\to\infty}{Lt}\, X(z) = x(0) + 0 + 0 + \cdots + 0$$

$$\underset{n\to\infty}{Lt}\, x(n) = x(0) = \boxed{\underset{z\to\infty}{Lt}\, X(z) = x(0)}$$

1. Find ZT of ~~(i) cos(wn) . u(n)~~

(i) $x(n) = \cos(\omega n) \cdot u(n)$

(ii) $x(n) = \sin(\omega n) \cdot u(n)$

= (i) $x(n) = \cos(\omega n) \cdot u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \cos(\omega n) \cdot u(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos(\omega n) \cdot z^{-n} = \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right] z^{-n}$$

$$X(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(e^{j\omega} \cdot z^{-1}\right)^{n} + \sum_{n=0}^{\infty} \left(e^{-j\omega} \cdot z^{-1}\right)^{n}\right]$$

ROC:- $|e^{j\omega} z^{-1}| < 1$ & $|e^{-j\omega} \cdot z^{-1}| < 1$

$|z| > |e^{j\omega}|$ & $|z| > |e^{-j\omega}|$

$|z| > 1$ & $|z| > 1$

$\therefore |z| > 1$

$$X(z) = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{1 - e^{-j\omega} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{j\omega}} + \frac{z}{z - e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{(z - e^{j\omega})(z - e^{-j\omega})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{z^2 - z \cdot e^{-j\omega} - z e^{j\omega} + 1}\right]$$

$$=\frac{1}{2}\left[\frac{2z^2-z(e^{jw}+e^{-jw})}{z^2+1-z(e^{jw}+e^{-jw})}\right]$$

$$=\frac{1}{2}\left[\frac{2z^2-z\times 2\cos w}{z^2+1-z\times 2\cos w}\right]$$

$$X(z)=\frac{z^2-z\cos w}{z^2+1-2z\cos w}, \quad \text{ROC: } |z|>1$$

(ii) $x(n) = \sin(wn) \cdot u(n)$

Q. Find ZT of $x(n) = \sin\left(\frac{\pi}{8}n - \frac{\pi}{4}\right) \cdot u(n-2)$

$$= \sin\left(\tfrac{\pi}{8} - \tfrac{\pi}{4}\right) = \frac{e^{+j\left(\frac{\pi}{8}n-\frac{\pi}{4}\right)} - e^{-j\left(\frac{\pi}{8}n-\frac{\pi}{4}\right)}}{2j}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\sin\left(\frac{\pi}{8}n - \frac{\pi}{4}\right) \cdot u(n-2)\right] \cdot z^{-n}$$

$$X(z) = \sum_{n=2}^{\infty} \sin\left(\frac{\pi}{8}n - \frac{\pi}{4}\right) \cdot z^{-n}$$

Ans :- $X(z) = \dfrac{z^{-1}\sin(\pi/8)}{1-2z^{-1}\cos\left(\frac{\pi}{8}\right)+z^{-2}} \cdot \left(z^{-2}\right)$ > ROC: $|z|>1$

③ Find ZT $x(n) = e^{an}\left(\frac{1}{3}\right)^n u(n)$

$$= \quad X(z) = \sum_{n=0}^{\infty}\left[e^{a}\cdot\left(\frac{1}{3}\right)\cdot z^{-1}\right]^n \qquad \text{ROC: } \left|e^{a}\cdot\frac{1}{3}\cdot z^{-1}\right| < 1$$

$$|z| > \left|e^{a}\,\frac{1}{3}\right|.$$

$$= \frac{1}{1 - e^{a}\cdot\frac{1}{3}\cdot z^{-1}}$$

$$X(z) = \frac{z}{z - e^{a}\cdot\frac{1}{3}}, \quad \text{ROC: } |z| > \left|\frac{e^{a}}{3}\right|$$

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(4) Find ZT of $x(n) = n\cdot u(n)$

= Using Differentiation property.

If $x(n) \longleftrightarrow X(z)$

$$n\,x(n) \longleftrightarrow -z\cdot\frac{dX(z)}{dz} = \cancel{-z\cdot\frac{d}{dz}\{ZT}$$

$$= -z\cdot\frac{d}{dz}\{Z[x(n)]\}$$

$$Z[x(n)] = Z[n\cdot u(n)] = -z\cdot\frac{d}{dz}\{Z[u(n)]\}$$

$$= -z\cdot\frac{d}{dz}\left\{\frac{z}{z-1}\right\}$$

$$= -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$\boxed{n\cdot u(n) \longleftrightarrow \frac{z}{(z-1)^2}} \qquad \text{ROC: } |z| > 1$$

Find ZT of

(4) $x(n) = n \cdot a^n \cdot u(n)$

$$= Z[n \cdot a^n u(n)] = -z \cdot \frac{d}{dz}\{Z[a^n u(n)]\}$$

$$= -z \cdot \frac{d}{dz}\left\{\frac{z}{z-a}\right\}$$

$$= -z \frac{d}{dz}\{(z)(z-a)^{-1}\} = -z\left\{z(-1)(z-a)^{-2} + (z-a)^{-1}\right\}$$

$$= -z\left\{\cancel{z(z^2+a^2-2az)} \frac{-z}{(z-a)^2} + \frac{1}{(z-a)}\right\} = -z\left\{\frac{-\cancel{z} + \cancel{z} - a}{(z-a)^2}\right\}$$

$$\boxed{Z[n \cdot a^n u(n)] = \frac{az}{(z-a)^2}}$$

ROC: $|z| > |a|$

(5) $x(n) = n^2 u(n)$

$$= Z[n^2 u(n)] = Z[n(n \cdot u(n))] = -z \cdot \frac{d\{Z(n \cdot u(n))\}}{dz}$$

$$= -z \cdot \frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\} = -z \frac{d}{dz}\left\{z(z-1)^{-2}\right\}$$

$$= -z\left[z(-2)(z-1)^{-3} + (z-1)^{-2}\right] = -z\left[\frac{-2z}{(z-1)^3} + \frac{1}{(z-1)^2}\right]$$

$$= -z\left(\frac{-2z+z-1}{(z-1)^3}\right) = -z\left(\frac{-z-1}{(z-1)^3}\right)$$

$$= \frac{z(z+1)}{(z-1)^3}$$

ROC:- $|z| > 1$

6 $x(n) = n^2 a^n u(n)$

$$= Z[n.\underbrace{(n.a^n u(n))}] = -z.\frac{d}{dz}\{Z\{n.a^n u(n)\}\}$$

$$= -z.\frac{d}{dz}\left\{\frac{az}{(z-a)^2}\right\} = -z.\frac{d}{dz}\{(az)(z-a)^{-2}\}$$

$$= -z\left[az(-2)(z-a)^{-3} + (z-a)^{-2}a\right]$$

$$= -z\left[\frac{-2az}{(z-a)^3} + \frac{a}{(z-a)^2}\right] = -z\left[\frac{-2az + a(z-a)}{(z-a)^3}\right]$$

$$= -z\left[\frac{-2az + az - a^2}{(z-a)^3}\right] = -z\left[\frac{-az-a^2}{(z-a)^3}\right] = \frac{-z(-a(z+a))}{(z-a)^3}$$

$$= \frac{az(z+a)}{(z-a)^3}, \quad \text{ROC: } |z|>|a|$$

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SL NO	Signal $x(n)$	ZT	ROC
1	$n.u(n)$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
2	$n.a^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
3	$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\|>1$
4	$n^2 a^n u(n)$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\|>\|a\|$
5	$\delta(n)$	1	Entire zp

7. $x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-10)]$

$$= X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n [u(n) - u(n-10)] \cdot z^{-n}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{9} \left(\frac{1}{2} z^{-1}\right)^n$$

$$X(z) = \frac{1 - \left(\frac{1}{2} z^{-1}\right)^{10}}{1 - \frac{1}{2} z^{-1}}, \quad \left|\frac{1}{2} z^{-1}\right| \neq 1$$

$$\left|\frac{1}{2} z^{-1}\right| < 1$$

$$\boxed{|z| > \frac{1}{2}} \text{ ROC}$$

8. $x(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4} n\right) \cdot u(n)$

Ans: $$X(z) = \frac{\left(\frac{z}{3\sqrt{2}}\right)}{\left(z - \frac{1}{3} e^{j\frac{\pi}{4}}\right)\left(z - \frac{1}{3} e^{-j\frac{\pi}{4}}\right)}$$ ROC $|z| > \frac{1}{3}$

9. $x(n) = \left(\frac{1}{2}\right)^n u(n) + (2)^n u(-n-1)$

Ans:- $X(z) = \frac{z}{z - \frac{1}{2}} - \frac{z}{z - 2}$ ROC: $\frac{1}{2} < |z| < 2$

10. $x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)$

$$= X(z) = Z\left[\left(\frac{1}{2}\right)^n u(n)\right] \cdot Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$X(z) = \left(\frac{z}{z - \frac{1}{2}}\right) \cdot \left(\frac{z}{z - \frac{1}{3}}\right), \quad \text{ROC: } |z| > \frac{1}{2}$$

(11) $x(n) = n\left\{\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right\}$

$=$

$$Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right] = Z\left[\left(\frac{1}{2}\right)^n u(n)\right] \cdot Z\left[\left(\frac{1}{2}\right)^n u(n)\right]$$

$$= \left(\frac{z}{z-\frac{1}{2}}\right)\left(\frac{z}{z-\frac{1}{2}}\right) = \frac{z^2}{\left(z-\frac{1}{2}\right)^2}$$

$$Z\left[n \cdot \left(\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right)\right] = -z \cdot \frac{d}{dz}\left\{Z\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right\}$$

$$= -z \cdot \frac{d}{dz}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)^2}\right]$$

MULTIPLICATION WITH n

(12) $x(n) = u(n) * (0.1)^n u(-n)$

$\Rightarrow x(n) = x_1(n) * x_2(n) \longleftrightarrow X(z) = X_1(z) \cdot X_2(z)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \frac{z}{z-1}, \quad |z|>1$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} = \cancel{\frac{z}{z-0.1}}, \quad \cancel{|z|<}$$

$$= \sum_{n=-\infty}^{0} (0.1)^n \cdot z^{-n} = \sum_{n=-\infty}^{0} (0.1\, z^{-1})^n$$

Substitute $n = -n$

$$= \sum_{n=0}^{\infty} (0.1^{-1} z)^n = \frac{1}{1-0.1^{-1} z}$$

ROC: $|0.1^{-1} z| < 1$

$|z| < 0.1$

$$X_2(z) = \frac{0.1}{\cancel{0.1-z}}$$

(12) $x(n) = \sinh(\omega_0 n)\, u(n)$

$$\Rightarrow x(n) = \left[\frac{e^{\omega_0 n} - e^{-\omega_0 n}}{2}\right] \cdot u(n)$$

Ans:- $$X(z) = \frac{z^{-1}\sinh(\omega_0)}{1+z^{-2}-2z^{-1}\cosh(\omega_0)}$$

---

(13) $x(n) = n(n-1)\, u(n)$

$\Rightarrow$ Ans:- $$X(z) = \frac{2z}{(z-1)^3}, \quad \text{ROC } |z|>1$$

---

(14) $x(n) = n(n-1)\, a^n u(n)$

$\Rightarrow$ Ans:- $$X(z) = \frac{2za^2}{(z-a)^3}, \quad \text{ROC } |z|>|a|$$

Inverse Z-T (IZT)

It is used to recover the original DT signal $x(n)$ from its z-domain $X(z)$ using partial fraction Expansion

i.e., $x(n) = Z^{-1}[X(z)]$

→ consider $X(z)$ and decompose into sum of simple fractions $X_1(z), X_2(z) \ldots$

→ ~~The~~ $\left[\frac{X(z)}{z}\right]$ must be proper rational function

i.e., the power of Numerator < power of Denominator

1. Determine IZT of $X(z) = \frac{10z}{(z-1)(z-2)}$

for ROC's (i) $|z|>2$ (ii) $|z|<1$ (iii) $1<|z|<2$

= $x(n) = ?$ $X(z) = \frac{10z}{(z-1)(z-2)}$

$$\left[\frac{X(z)}{z}\right] = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\cancel{A = \frac{10}{(z-1)(z-2)} \times (z-1)\Big|_{\substack{(z-1)=0 \\ z=1}}}$$

$$A = \left[\frac{X(z)}{z}\right] \times (z-1)\Big|_{z=1} = \frac{10 \cancel{(z-1)}}{\cancel{(z-1)}(z-2)}\Big|_{z=1} = -10$$

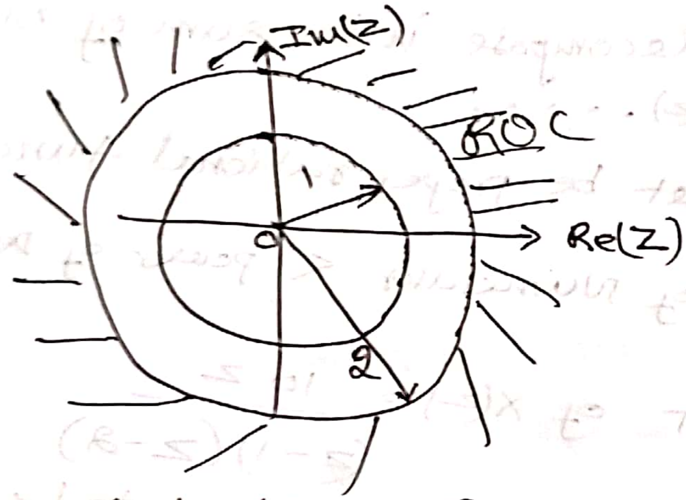
$$B = \frac{10 \cancel{(z-2)}}{(z-1)\cancel{(z-2)}}\Big|_{z=2} = 10$$

$$\frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2} = \cancel{-10\left[\frac{z}{z-1}\right] + 10\left[\frac{z}{z-2}\right]}$$

$$X(z) = -10\left(\frac{z}{z-1}\right) + 10\left(\frac{z}{z-2}\right)$$

$$X(z) = -10\left[\frac{z}{z-1}\right] + 10\left[\frac{z}{z-2}\right] \quad \text{——— (1)}$$

(i) ROC $|z| > 2$:— → ROC is outside w.r.t. both the circles



→ causal w.r.t. both the circles

→ multiply with $u(n)$

Take IZT of Eqn (1)

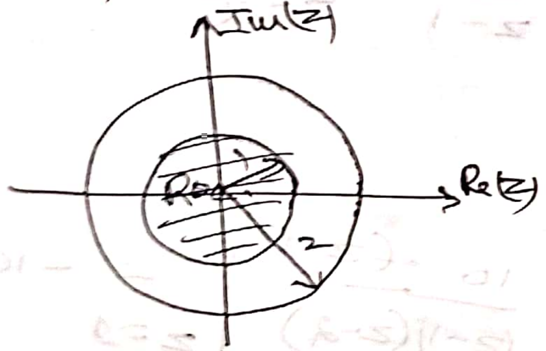
$$x(n) = -10\,(1)^n \cdot u(n) + 10 \times (2)^n \cdot u(n)$$

$$x(n) = 10\left[2^n - 1\right] u(n)$$

(ii) ROC :- $|z| < 1$:— → ROC is inside w.r.t. both the circles

→ Non causal w.r.t. both

→ multiply with $[-u(-n-1)]$

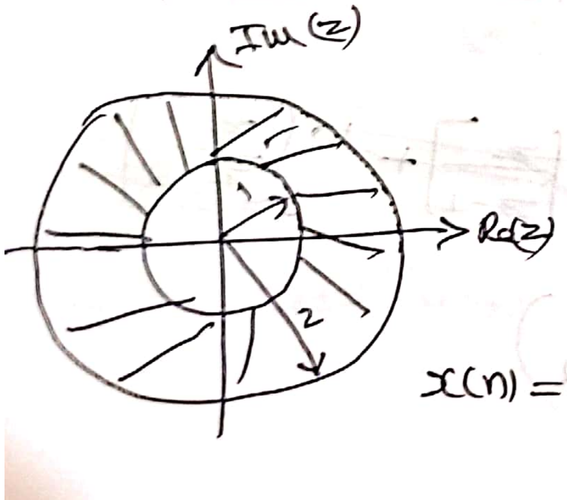


Take IZT of Eqn (1)

$$\boxed{x(n) = -10(1)^n[-u(-n-1)] + 10(2)^n[-u(-n-1)]}$$

(iii) ROC: $1 < |z| < 2$:- → ROC is outside w.r.t. inside the circle 1, indicate causal ∴ multiply with $u(n)$

→ ROC is inside w.r.t. outside the circle 2, indicate non causal ∴ multiply with $[-u(-n-1)]$



Take IZT of Eqn (1)

$$x(n) = -10(1)^n u(n) + 10(2)^n\left[-u(-n-1)\right]$$

Find IZT

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2) $X(z) = \dfrac{z}{z^2-5z+6}$, ROC: (i) $|z|<2$

(ii) $|z|>3$ (iii) $2<|z|<3$

$= \quad X(z) = \dfrac{z}{(z-3)(z-2)}$

$\left[\dfrac{X(z)}{z}\right] = \dfrac{1}{(z-3)(z-2)} = \dfrac{A}{z-3} + \dfrac{B}{z-2}$

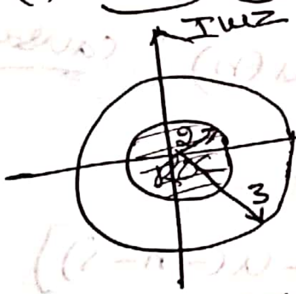
$A = \left.\dfrac{1\ \cancel{(z-3)}}{\cancel{(z-3)}(z-2)}\right|_{z=3} = \dfrac{1}{1} = 1$

$B = \left.\dfrac{1\ \cancel{(z-2)}}{(z-3)\cancel{(z-2)}}\right|_{z=2} = -1$

$\dfrac{X(z)}{z} = \dfrac{+1}{z-3} + \dfrac{(-1)}{z-2}$ ——— (1)

$X(z) = +\left[\dfrac{z}{z-3}\right] - \left[\dfrac{z}{z-2}\right]$

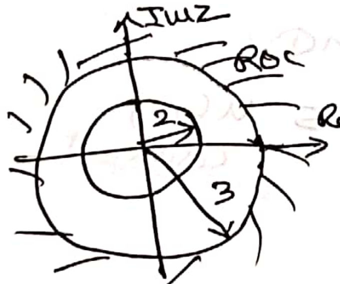
(i) ROC: $|z|<2$:— ROC inside w.r.t. both circles, indicate non-causal. ∴ multiply with $[-u(-n-1)]$



Take IZT of Eqn (1)

$x(n) = +(3)^n[-u(-n-1)] - (2)^n[-u(-n-1)]$

(ii) ROC :— $|z|>3$:— ROC out w.r.t. both poles, indicate causal. ∴ multiply with $u(n)$

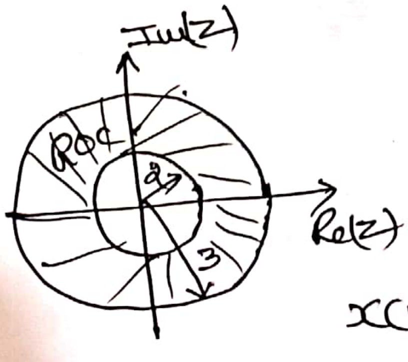


Take IZT of Eqn (1)

$x(n) = (3)^n u(n) - (2)^n u(n)$

(iii) ROC: $2<|z|<3$:— → ROC outside w.r.t circle of radius 2, indicate causal, ∴ multiply with $u(n)$

→ ROC inside w.r.t. circle of radius 3, indicate non causal, ∴ multiply with $[-u(-n-1)]$.



Take IZT of Eqn (1)

$x(n) = (3)^n[-u(-n-1)] - (2)^n u(n)$

③ Find IZT of $X(z) = \dfrac{z(z^2+4z+5)}{(z-3)(z-2)(z-1)}$

For ROC's (i) $|z|>3$ (ii) $|z|<1$ (iii) $2<|z|<3$

$$= \frac{X(z)}{z} = \frac{z^2+4z+5}{(z-3)(z-2)(z-1)} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{z-1}$$

$$A = \left.\frac{(z^2+4z+5)\cancel{(z-3)}}{\cancel{(z-3)}(z-2)(z-1)}\right|_{z=3} = 13$$

$$B = \left.\frac{(z^2+4z+5)\cancel{(z-2)}}{(z-3)\cancel{(z-2)}(z-1)}\right|_{z=2} = -17$$

$$C = \left.\frac{(z^2+4z+5)\cancel{(z-1)}}{(z-3)(z-2)\cancel{(z-1)}}\right|_{z=1} = 5$$

$$X(z) = 13\left[\frac{z}{z-3}\right] - 17\left[\frac{z}{z-2}\right] + 5\left[\frac{z}{z-1}\right] \quad \text{—— ①}$$

(i) ROC: $|z|>3$:- Take IZT of Eqn ①

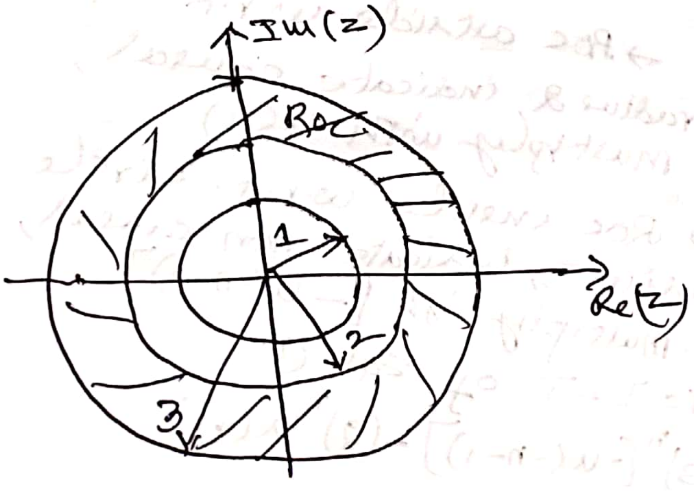
$x(n) = 13(3)^n u(n) - 17(2)^n u(n) + 5(1)^n u(n)$ causal

(ii) ROC $|z|<1$:- Take IZT of Eqn ①

$x(n) = 13(3)^n[-u(-n-1)] - 17(2)^n[-u(-n-1)] + 5[-u(-n-1)]$ Non causal

(iii) ROC $2<|z|<3$:- Take IZT of Eqn ①

$x(n) = \underbrace{13(3)^n[-u(-n-1)]}_{\text{Noncausal}} - \underbrace{17(2)^n u(n)}_{\text{causal}} + \underbrace{5\, u(n)}_{\text{causal}}$



(4) Find IZT of $X(z) = \dfrac{z^2 + z e^{-3T} - 2z e^{-2T}}{z^2 - [e^{-2T} + e^{-3T}]z + e^{-5T}}$

ROC: $|z| > e^{-2T}$

$$= \quad X(z) = \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{z^2 - z e^{-2T} - z e^{-3T} + e^{-5T}}$$

$$= \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{z(z - e^{-2T}) - e^{-3T}(z - e^{-2T})}$$

$$X(z) = \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{(z - e^{-2T})(z - e^{-3T})}$$

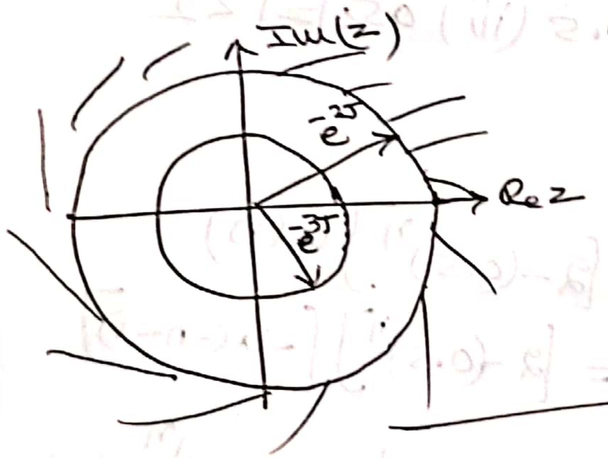
$$\frac{X(z)}{z} = \frac{z + e^{-3T} - 2e^{-2T}}{(z - e^{-2T})(z - e^{-3T})} = \frac{A}{z - e^{-2T}} + \frac{B}{z - e^{-3T}}$$

$A = -1, \quad B = 2$

$$X(z) = -1\left[\frac{z}{z - e^{-2T}}\right] + 2\left[\frac{z}{z - e^{-3T}}\right] \quad \text{——— (1)}$$

Take IZT

$$x(n) = -1 \cdot (e^{-2T})^n u(n) + 2 \cdot (e^{-3T})^n u(n)$$



(5) Find IZT of $X(z) = \dfrac{1}{1 - z^{-2}}$, ROC: $|z| > 1$

$$= \quad X(z) = \frac{z^2}{z^2 - 1} = \frac{z^2}{(z+1)(z-1)}$$

$$\frac{X(z)}{z} = \frac{z}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$A = \frac{1}{2}, \quad B = \frac{1}{2}$

$$X(z) = \frac{1}{2}\left[\frac{z}{z-(-1)}\right] + \frac{1}{2}\left[\frac{z}{z-1}\right]$$

Take IZT

$$x(n) = \frac{1}{2}(-1)^n u(n) + \frac{1}{2} u(n)$$

(6) Find IZT of $X(z) = \frac{3}{z-2}$, ROC : $|z|>2$

$\Rightarrow$

$$\frac{X(z)}{z} = \frac{3}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$A = -\frac{3}{2}, \quad B = \frac{3}{2}$$

$$\therefore X(z) = -\frac{3}{2} + \frac{3}{2}\left(\frac{z}{z-2}\right)$$

Take IZT

$$x(n) = -\frac{3}{2}\delta(n) + \frac{3}{2}(2)^n u(n)$$

(7) ~~X(z)~~ Find IZT of $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$

ROC's (i) $|z|>1$ (ii) $|z|<0.5$ (iii) $0.5<|z|<1$

$\Rightarrow$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

(i) ROC :- $|z|>1$:- $x(n) = [2-(0.5)^n]u(n)$

(ii) ROC : $|z|<0.5$:- $x(n) = [2-(0.5)^n][-u(-n-1)]$

(iii) ROC : $0.5<|z|<1$:- $x(n) = 2[-u(-n-1)] - (0.5)^n u(n)$

8 Find IZT of $x(z) = \dfrac{2z^3-5z^2+z+3}{(z-1)(z-2)}$ ROC :- $|z| < 1$

$= \dfrac{x(z)}{z} = \dfrac{2z^3-5z^2+z+3}{z(z-1)(z-2)} = \dfrac{2z^3-5z^2+z+3}{z^3-3z^2+2z}$

Not proper rational fraction

convert $\left[\frac{X(z)}{z}\right]$ into proper rational fraction by division.

$$\begin{array}{r|l} z^3-3z^2+2z & 2z^3-5z^2+z+3 \;(2 \\ & \cancel{2z^3} \mp 6z^2 \pm 4z \\ \hline & z^2-3z+3 \end{array}$$

$\therefore \dfrac{X(z)}{z} = 2 + \left(\dfrac{z^2-3z+3}{z^3-3z^2+2z}\right) = 2 + \left[\dfrac{z^2-3z+3}{z(z-1)(z-2)}\right]$ — 1

Consider $\dfrac{z^2-3z+3}{z(z-1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$

$A = \frac{3}{2}$, $B = -1$, $C = \frac{1}{2}$

$\dfrac{z^2-3z+3}{z(z-1)(z-2)} = \dfrac{3/2}{z} + \left(\dfrac{-1}{z-1}\right) + \left(\dfrac{1/2}{z-2}\right)$ — 2

Substitute 2 in 1

$\dfrac{X(z)}{z} = 2 + \dfrac{3/2}{z} + \left(\dfrac{-1}{z-1}\right) + \left(\dfrac{1/2}{z-2}\right)$

ROC $|z| < 1$
Inside
∴ Non causal.

$X(z) = 2z + \frac{3}{2} - \left(\frac{z}{z-1}\right) + \frac{1}{2}\left(\frac{z}{z-2}\right)$

$\delta(n) \xleftrightarrow{ZT} 1$
$\delta(n-1) \xleftrightarrow{ZT} z^{-1}$
$\delta(n+1) \xleftrightarrow{ZT} z^{+1}$

Take IZT

$x(n) = 2\delta(n+1) + \frac{3}{2}\delta(n) - (1)^n[-u(-n-1)]$
$+ \frac{1}{2}(2)^n[-u(-n-1)]$

⑨ Find IZT of $X(z) = \dfrac{2z^2}{(z+1)(z+2)^2}$, ROC: $|z|>2$

$$= \quad \frac{X(z)}{z} = \frac{2z}{(z+1)(z+2)^2} = \frac{A}{(z+1)} + \frac{B}{(z+2)} + \frac{C}{(z+2)^2}$$

$$A = \left.\frac{2z\cancel{(z+1)}}{\cancel{(z+1)}(z+2)^2}\right|_{z=-1} = -2$$

$$\cancel{B = \frac{2z\,(z+}{(z+1)(z+2)^2}} \quad C = \left.\frac{2z\cancel{(z+2)^2}}{(z+1)\cancel{(z+2)^2}}\right|_{z=-2} = 4.$$

$$B = \frac{1}{1!}\frac{d}{dz}\left[\left(\frac{X(z)}{z}\right)(z+2)^2\right]\Bigg|_{z=-2} = \frac{d}{dz}\{\text{Equation of 'c'}\}\Bigg|_{z=-2}$$

$$= \frac{d}{dz}\left[\frac{2z\cancel{(z+2)^2}}{(z+1)\cancel{(z+2)^2}}\right]\Bigg|_{z=-2} = \frac{d}{dz}\left[\frac{2z}{z+1}\right]\Bigg|_{z=-2}$$

$$B = \left.\frac{(z+1)2-2z}{(z+1)^2}\right|_{z=-2} = 2$$

$$\frac{X(z)}{z} = \frac{-2}{z+1} + \frac{2}{z+2} + \frac{4}{(z+2)^2}$$

$$X(z) = -2\left[\frac{z}{z-(-1)}\right] + 2\left[\frac{z}{z-(-2)}\right] + 4\left[\frac{z}{(z-(-2))^2}\right]$$

$$\boxed{n \cdot a^n u(n) \longleftrightarrow \frac{az}{(z-a)^2}}$$

$$X(z) = -2\left[\frac{z}{z-(-1)}\right] + 2\left[\frac{z}{z-(-2)}\right] + 4\left[\frac{(-2)\,z}{(-2)(z-(-2))^2}\right]$$

$$X(z) = -2\left[\frac{z}{z-(-1)}\right] + 2\left[\frac{z}{z-(-2)}\right] + \frac{4}{(-2)}\left[\frac{(-2)\,z}{(z-(-2))^2}\right]$$

ROC $|z|>2$ causal

Take IZT

$$x(n) = -2(-1)^n u(n) + 2(-2)^n u(n) - 2\,n(-2)^n u(n)$$

10. Find IZT $X(z) = \dfrac{z}{z(z-1)(z-2)^2}$, ROC $|z|>2$

$$= \frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$$A = \frac{1 \cancel{(z)}}{\cancel{z}(z-1)(z-2)^2}\Bigg|_{z=0} = -\frac{1}{4}$$

$$B = \frac{1 \cancel{(z-1)}}{z\cancel{(z-1)}(z-2)^2}\Bigg|_{z=1} = 1$$

$$D = \frac{1 \cancel{(z-2)^2}}{z(z-1)\cancel{(z-2)^2}}\Bigg|_{z=2} = \frac{1}{2}$$

$$C = \frac{1}{1!}\frac{d}{dz}\{\text{Equation of } D\} = \frac{d}{dz}\left[\frac{1}{z(z-1)}\right]\Bigg|_{z=2}$$

$$= \frac{d}{dz}\{z^{-1}(z-1)^{-1}\}\Big|_{z=2} = z^{-1}(-1)(z-1)^{-2} + (z-1)^{-1}\cdot(-1)z^{-2}\Big|_{z=2}$$

$$= -\frac{3}{4}$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left(\frac{z}{(z-2)^2}\right)$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left[\frac{2z}{2(z-2)^2}\right]$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{4}\left[\frac{2z}{(z-2)^2}\right]$$

Take IZT — ROC $|z|>2$ causal.

$$x(n) = -\frac{1}{4}\delta(n) + (1)^n u(n) - \frac{3}{4}(2)^n u(n) + \frac{1}{4}\cdot n\cdot(2)^n u(n)$$

(11) Find IZT of $X(z) = \dfrac{z^2+z}{(z-\frac{1}{4})(z-\frac{1}{2})^3}$ ROC: $|z| > \frac{1}{2}$

$$= \frac{X(z)}{z} = \frac{z+1}{(z-\frac{1}{4})(z-\frac{1}{2})^3} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}} + \frac{C}{(z-\frac{1}{2})^2} + \frac{D}{(z-\frac{1}{2})^3}$$

$$A = \left.\frac{(z+1)\cancel{(z-\frac{1}{4})}}{\cancel{(z-\frac{1}{4})}(z-\frac{1}{2})^3}\right|_{z=\frac{1}{4}} = -80$$

$$D = \left.\frac{(z+1)\cancel{(z-\frac{1}{2})^3}}{(z-\frac{1}{4})\cancel{(z-\frac{1}{2})^3}}\right|_{z=\frac{1}{2}} = 6$$

$$C = \frac{1}{1!}\frac{d}{dz}\{\text{Equation of } D\} = \left.\frac{d}{dz}\left\{\frac{(z+1)}{(z-\frac{1}{4})}\right\}\right|_{z=\frac{1}{2}}$$

$$C = \left.\frac{(z-\frac{1}{4})\cdot 1 - (z+1)\cdot 1}{(z-\frac{1}{4})^2}\right|_{z=\frac{1}{2}} = -20$$

$$B = \left.\frac{1}{2!}\frac{d}{dz}\{\text{Equation of } C\}\right|_{z=\frac{1}{2}}$$

$$= \left.\frac{1}{2!}\frac{d}{dz}\left\{\frac{(z-\frac{1}{4})-(z+1)}{(z-\frac{1}{4})^2}\right\}\right|_{z=\frac{1}{2}}$$

$$= \left.\frac{1}{2\times 1} = \frac{d}{dz}\left[\left(-\frac{5}{4}\right)\left(z-\frac{1}{4}\right)^{-2}\right]\right|_{z=\frac{1}{2}}$$

$$B = \left.\frac{1}{2}\left[\left(-\frac{5}{4}\right)(-2)\left(z-\frac{1}{4}\right)^{-3}\right]\right|_{z=\frac{1}{2}} = 80$$

$n \cdot a^n u(n) \longleftrightarrow \dfrac{az}{(z-a)^2}$	$[n(n-1)]\, a^n u(n) \longleftrightarrow \dfrac{a^2\, 2z}{(z-a)^3}$

$$X(z) = -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left(\frac{z}{z-\frac{1}{2}}\right) - 20\left(\frac{z}{(z-\frac{1}{2})^2}\right) + 6\left(\frac{z}{(z-\frac{1}{2})^3}\right),$$ ROC $|z| > \frac{1}{2}$ causal.

$$= -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left[\frac{z}{z-\frac{1}{2}}\right] - 20\left(\frac{(\frac{1}{2})z}{(\frac{1}{2})(z-\frac{1}{2})^2}\right) + 6\left(\frac{(\frac{1}{2})^2 \times 2z}{(\frac{1}{2})^2 \times 2 \times (z-\frac{1}{2})^3}\right)$$

$$X(z) = -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left(\frac{z}{z-\frac{1}{2}}\right) - 40\left[\frac{\frac{1}{2}z}{(z-\frac{1}{2})^2}\right] + 12\left[\frac{(\frac{1}{2})^2 2z}{(z-\frac{1}{2})^3}\right]$$

Take IZT

$$x(n) = \left[-80\left(\frac{1}{4}\right)^n + 80\left(\frac{1}{2}\right)^n - 40\cdot n\left(\frac{1}{2}\right)^n + 12\cdot n(n-1)\left(\frac{1}{2}\right)^n\right] u(n)$$

Inverse Z-T (IZT)

It is used to recover the original DT signal $x(n)$ from its z-domain $X(z)$ using partial fraction Expansion

i.e., $x(n) = Z^{-1}[X(z)]$

→ consider $X(z)$ and decompose into sum of simple fractions $X_1(z), X_2(z) \ldots$

→ ~~The~~ $\left[\frac{X(z)}{z}\right]$ must be proper rational function

i.e., the power of Numerator < power of Denominator

1. Determine IZT of $X(z) = \frac{10z}{(z-1)(z-2)}$

for ROC's (i) $|z|>2$ (ii) $|z|<1$ (iii) $1<|z|<2$

= $x(n) = ?$ $\quad X(z) = \frac{10z}{(z-1)(z-2)}$

$$\left[\frac{X(z)}{z}\right] = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\cancel{A = \frac{10}{(z-1)(z-2)} \times (z-1) \Big|_{\substack{(z-1)=0 \\ z=1}}}$$

$$A = \left[\frac{X(z)}{z}\right] \times (z-1)\Big|_{z=1} = \frac{10}{\cancel{(z-1)}(z-2)}\cancel{(z-1)}\Big|_{z=1} = -10$$

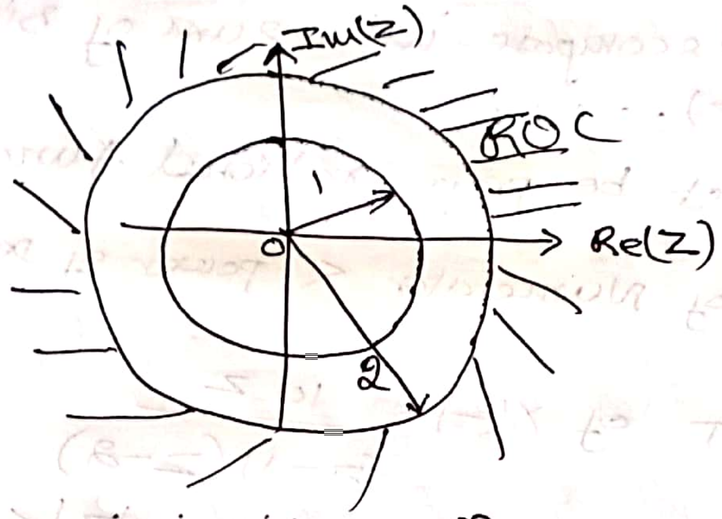
$$B = \frac{10}{(z-1)\cancel{(z-2)}}\cancel{(z-2)}\Big|_{z=2} = 10$$

$$\frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2} \cancel{= -10\left[\frac{z}{z-1}\right] + 10\left[\frac{z}{z-2}\right]}$$

$$X(z) = -10\left(\frac{z}{z-1}\right) + 10\left(\frac{z}{z-2}\right)$$

$$X(z) = -10\left[\frac{z}{z-1}\right] + 10\left[\frac{z}{z-2}\right] \quad \text{—— (1)}$$

(i) ROC $|z| > 2$:— → ROC is outside w.r.t. both the circles



→ causal w.r.t. both the circles

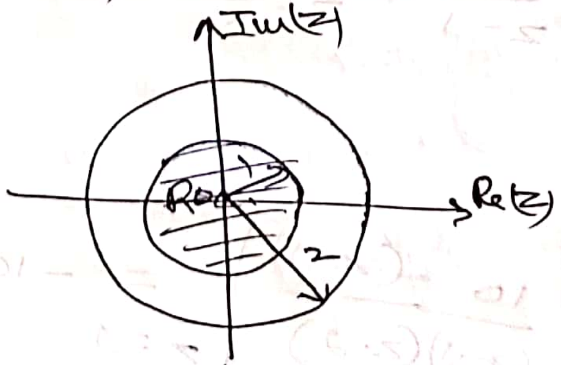
→ multiply with $u(n)$

Take IZT of Eqn (1)

$$x(n) = -10\,(1)^n \cdot u(n) + 10 \times (2)^n \cdot u(n)$$

$$x(n) = 10\left[2^n - 1\right] u(n)$$

(ii) ROC :– $|z| < 1$:— → ROC is inside w.r.t. both the circles



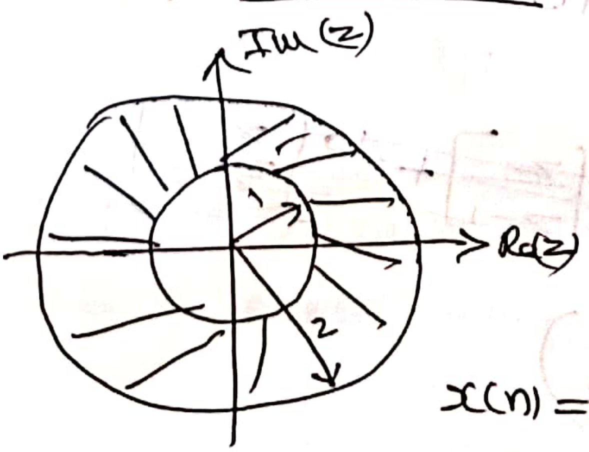
→ Non causal w.r.t. both

→ multiply with $[-u(-n-1)]$

Take IZT of Eqn (1)

$$\boxed{x(n) = -10(1)^n[-u(-n-1)] + 10(2)^n[-u(-n-1)]}$$

(iii) ROC : $1 < |z| < 2$:-



→ Roc is outside w.r.t. inside the circle 1, indicate causal ∴ multiply with $u(n)$

→ Roc is inside w.r.t. outside the circle 2, indicate non causal ∴ multiply with $[-u(-n-1)]$

Take IZT of Eqn (1)

$$x(n) = -10(1)^n u(n) + 10(2)^n\left[-u(-n-1)\right]$$

Find IZT

2) $X(z) = \dfrac{z}{z^2-5z+6}$, ROC: (i) $|z| < 2$ (ii) $|z| > 3$ (iii) $2<|z|<3$

$= X(z) = \dfrac{z}{(z-3)(z-2)}$

$\left[\dfrac{X(z)}{z}\right] = \dfrac{1}{(z-3)(z-2)} = \dfrac{A}{z-3} + \dfrac{B}{z-2}$

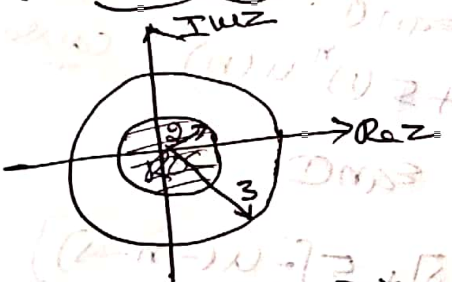
$A = \left.\dfrac{1\,\cancel{(z-3)}}{\cancel{(z-3)}(z-2)}\right|_{z=3} = \dfrac{1}{1} = 1$

$B = \left.\dfrac{1\,\cancel{(z-2)}}{(z-3)\cancel{(z-2)}}\right|_{z=2} = -1$

$\dfrac{X(z)}{z} = \dfrac{+1}{z-3} + \dfrac{(-1)}{z-2}$

$X(z) = +\left[\dfrac{z}{z-3}\right] - \left[\dfrac{z}{z-2}\right]$ —— (1)

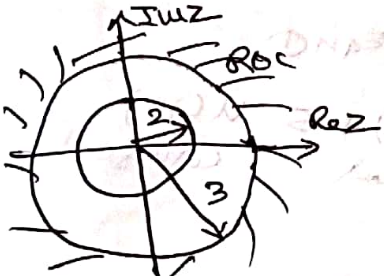
(i) ROC: $|z|<2$:— ROC inside w.r.t. both circles, indicate non-causal.
∴ multiply with $[-u(-n-1)]$



Take IZT of Eqn (1)

$x(n) = +(3)^n[-u(-n-1)] - (2)^n[-u(-n-1)]$

(ii) ROC:— $|z|>3$:— ROC out w.r.t. both poles, indicate causal. ∴ multiply with $u(n)$

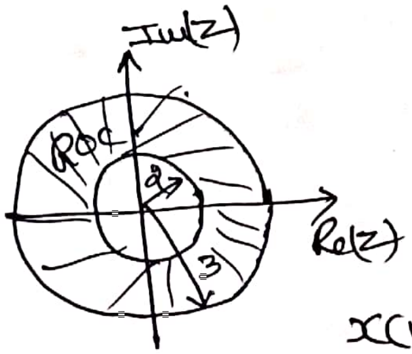


Take IZT of Eqn (1)

$x(n) = (3)^n u(n) - (2)^n u(n)$

(iii) ROC: $2<|z|<3$:— → ROC outside w.r.t circle of radius 2, indicate causal,
∴ multiply with $u(n)$

→ ROC inside w.r.t. circle of radius 3, indicate non causal,
∴ multiply with $[-u(-n-1)]$.



Take IZT of Eqn (1)

$x(n) = (3)^n[-u(-n-1)] - (2)^n u(n)$

3 Find IZT of $X(z) = \frac{z(z^2+4z+5)}{(z-3)(z-2)(z-1)}$

For ROC's (i) $|z|>3$ (ii) $|z|<1$ (iii) $2<|z|<3$

$$= \frac{X(z)}{z} = \frac{z^2+4z+5}{(z-3)(z-2)(z-1)} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{z-1}$$

$$A = \left.\frac{(z^2+4z+5)\cancel{(z-3)}}{\cancel{(z-3)}(z-2)(z-1)}\right|_{z=3} = 13$$

$$B = \left.\frac{(z^2+4z+5)\cancel{(z-2)}}{(z-3)\cancel{(z-2)}(z-1)}\right|_{z=2} = -17$$

$$C = \left.\frac{(z^2+4z+5)\cancel{(z-1)}}{(z-3)(z-2)\cancel{(z-1)}}\right|_{z=1} = 5$$

$$X(z) = 13\left[\frac{z}{z-3}\right] - 17\left[\frac{z}{z-2}\right] + 5\left[\frac{z}{z-1}\right] \quad \text{--- (1)}$$

(i) ROC: $|z|>3$:- Take IZT of Eqn (1) causal

$$x(n) = 13(3)^n u(n) - 17(2)^n u(n) + 5(1)^n u(n)$$

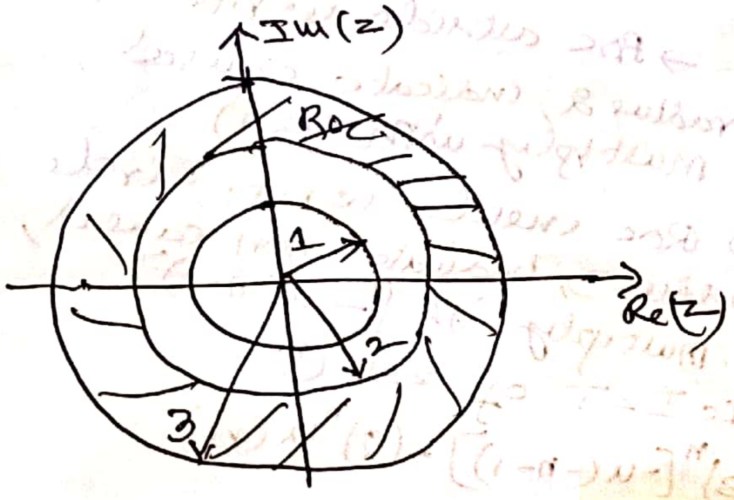
(ii) ROC $|z|<1$:- Take IZT of Eqn (1)

$$x(n) = 13(3)^n[-u(-n-1)] - 17(2)^n[-u(-n-1)] + 5[-u(-n-1)]$$

Non causal

(iii) ROC $2<|z|<3$:- Take IZT of Eqn (1)

$$x(n) = \underbrace{13(3)^n[-u(-n-1)]}_{\text{Noncausal}} - \underbrace{17(2)^n u(n)}_{\text{causal}} + \underbrace{5\, u(n)}_{\text{causal}}$$



(4) Find IZT of $X(z) = \dfrac{z^2 + z e^{-3T} - 2z e^{-2T}}{z^2 - [e^{-2T} + e^{-3T}] z + e^{-5T}}$

ROC: $|z| > e^{-2T}$

$$= \quad X(z) = \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{z^2 - z e^{-2T} - z e^{-3T} + e^{-5T}}$$

$$= \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{z(z - e^{-2T}) - e^{-3T}(z - e^{-2T})}$$

$$X(z) = \frac{z^2 + z e^{-3T} - 2z e^{-2T}}{(z - e^{-2T})(z - e^{-3T})}$$

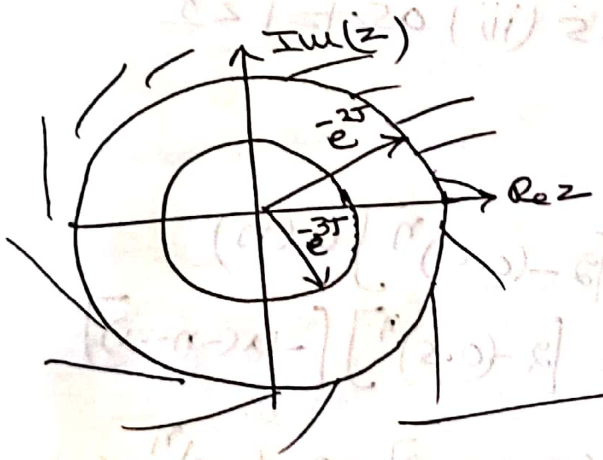
$$\frac{X(z)}{z} = \frac{z + e^{-3T} - 2e^{-2T}}{(z - e^{-2T})(z - e^{-3T})} = \frac{A}{z - e^{-2T}} + \frac{B}{z - e^{-3T}}$$

$A = -1, \quad B = 2$ ——— (1)

$$X(z) = -1\left[\frac{z}{z - e^{-2T}}\right] + 2\left[\frac{z}{z - e^{-3T}}\right]$$

Take IZT

$$x(n) = -1 \cdot (e^{-2T})^n u(n) + 2 \cdot (e^{-3T})^n u(n)$$



(5) Find IZT of $X(z) = \dfrac{1}{1 - z^{-2}}$, ROC: $|z| > 1$

$$= \quad X(z) = \frac{z^2}{z^2 - 1} = \frac{z^2}{(z+1)(z-1)}$$

$$\frac{X(z)}{z} = \frac{z}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$A = \frac{1}{2}, \quad B = \frac{1}{2}$

$$X(z) = \frac{1}{2}\left[\frac{z}{z-(-1)}\right] + \frac{1}{2}\left[\frac{z}{z-1}\right]$$

Take IZT

$$x(n) = \frac{1}{2}(-1)^n u(n) + \frac{1}{2} u(n)$$

---

(6) Find IZT of $X(z) = \frac{3}{z-2}$, ROC : $|z| > 2$

$\Rightarrow$

$$\frac{X(z)}{z} = \frac{3}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$A = -\frac{3}{2}, \quad B = \frac{3}{2}$$

$$\therefore X(z) = -\frac{3}{2} + \frac{3}{2}\left(\frac{z}{z-2}\right)$$

Take IZT

$$x(n) = -\frac{3}{2}\delta(n) + \frac{3}{2}(2)^n u(n)$$

---

(7) ~~X(z)~~ Find IZT of $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

ROC's (i) $|z| > 1$ (ii) $|z| < 0.5$ (iii) $0.5 < |z| < 1$

$\Rightarrow$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

(i) ROC :- $|z| > 1$:- $x(n) = \left[2 - (0.5)^n\right] u(n)$

(ii) ROC : $|z| < 0.5$:- $x(n) = \left[2 - (0.5)^n\right]\left[-u(-n-1)\right]$

(iii) ROC : $0.5 < |z| < 1$:- $x(n) = 2\left[-u(-n-1)\right] - (0.5)^n u(n)$

---

8) Find IZT of $X(z) = \dfrac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)}$ ROC :- $|z| < 1$

$$= \frac{X(z)}{z} = \frac{2z^3 - 5z^2 + z + 3}{z(z-1)(z-2)} = \frac{2z^3 - 5z^2 + z + 3}{z^3 - 3z^2 + 2z}$$

Not proper rational fraction.

convert $\left[\frac{X(z)}{z}\right]$ into proper rational fraction by division.

$$z^3 - 3z^2 + 2z \overline{\big)\, 2z^3 - 5z^2 + z + 3 \,} \; (2$$
$$\underline{\cancel{2z^3} - 6z^2 + 4z}$$
$$z^2 - 3z + 3$$

$$\therefore \frac{X(z)}{z} = 2 + \left(\frac{z^2 - 3z + 3}{z^3 - 3z^2 + 2z}\right) = 2 + \left[\frac{z^2 - 3z + 3}{z(z-1)(z-2)}\right] \quad \text{— (1)}$$

Consider $\dfrac{z^2 - 3z + 3}{z(z-1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$

$A = \frac{3}{2}$, $B = -1$, $C = \frac{1}{2}$

$$\frac{z^2 - 3z + 3}{z(z-1)(z-2)} = \frac{3/2}{z} + \left(\frac{-1}{z-1}\right) + \left(\frac{1/2}{z-2}\right) \quad \text{— (2)}$$

Substitute (2) in (1)

$$\frac{X(z)}{z} = 2 + \frac{3/2}{z} + \left(\frac{-1}{z-1}\right) + \left(\frac{1/2}{z-2}\right)$$

ROC $|z| < 1$ Inside
$\therefore$ Non causal.

$$X(z) = 2z + \frac{3}{2} - \left(\frac{z}{z-1}\right) + \frac{1}{2}\left(\frac{z}{z-2}\right)$$

$\delta(n) \xleftrightarrow{ZT} 1$
$\delta(n-1) \xleftrightarrow{ZT} z^{-1}$
$\delta(n+1) \xleftrightarrow{ZT} z^{+1}$

Take IZT

$$x(n) = 2\delta(n+1) + \frac{3}{2}\delta(n) - (1)^n\left[-u(-n-1)\right] + \frac{1}{2}(2)^n\left[-u(-n-1)\right]$$

(9) Find IZT of $X(z) = \dfrac{2z^2}{(z+1)(z+2)^2}$, ROC: $|z|>2$

$$= \frac{X(z)}{z} = \frac{2z}{(z+1)(z+2)^2} = \frac{A}{(z+1)} + \frac{B}{(z+2)} + \frac{C}{(z+2)^2}$$

$$A = \left.\frac{2z\cancel{(z+1)}}{\cancel{(z+1)}(z+2)^2}\right|_{z=-1} = -2$$

$$\cancel{B = \frac{2z\,(z+}{(z+1)(z+2)^2}} \quad C = \left.\frac{2z\cancel{(z+2)^2}}{(z+1)\cancel{(z+2)^2}}\right|_{z=-2} = 4.$$

$$B = \frac{1}{1!}\left.\frac{d}{dz}\left[\left(\frac{X(z)}{z}\right)(z+2)^2\right]\right|_{z=-2} = \frac{d}{dz}\left\{\text{Equation of 'c'}\right\}\Big|_{z=-2}$$

$$= \frac{d}{dz}\left.\left[\frac{2z\cancel{(z+2)^2}}{(z+1)\cancel{(z+2)^2}}\right]\right|_{z=-2} = \frac{d}{dz}\left.\left[\frac{2z}{z+1}\right]\right|_{z=-2}$$

$$B = \left.\frac{(z+1)2 - 2z}{(z+1)^2}\right|_{z=-2} = 2$$

$$\frac{X(z)}{z} = \frac{-2}{z+1} + \frac{2}{z+2} + \frac{4}{(z+2)^2}$$

$$X(z) = -2\left[\frac{z}{z-(-1)}\right] + 2\left[\frac{z}{z-(-2)}\right] + 4\left[\frac{z}{(z-(-2))^2}\right]$$

$$\boxed{n \cdot a^n u(n) \longleftrightarrow \frac{az}{(z-a)^2}}$$

$$X(z) = -2\left[\frac{z}{z-(-1)}\right] + 2\left[\frac{z}{z-(-2)}\right] + 4\left[\frac{(-2)\,z}{(-2)(z-(-2))^2}\right]$$

$$X(z) = -2\left(\frac{z}{z-(-1)}\right) + 2\left[\frac{z}{z-(-2)}\right] + \frac{4}{(-2)}\left[\frac{(-2)\,z}{(z-(-2))^2}\right]$$

ROC $|z|>2$ causal

Take IZT

$$x(n) = -2(-1)^n u(n) + 2(-2)^n u(n) - 2\,n(-2)^n u(n)$$

10) Find IZT $X(z) = \dfrac{z}{z(z-1)(z-2)^2}$, ROC $|z|>2$

$$= \frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-2)} + \frac{D}{(z-2)^2}$$

$$A = \frac{1 \cancel{(z)}}{\cancel{z}(z-1)(z-2)^2}\Bigg|_{z=0} = -\frac{1}{4}$$

$$B = \frac{1 \cancel{(z-1)}}{z\cancel{(z-1)}(z-2)^2}\Bigg|_{z=1} = 1$$

$$D = \frac{1 \cancel{(z-2)^2}}{z(z-1)\cancel{(z-2)^2}}\Bigg|_{z=2} = \frac{1}{2}$$

$$C = \frac{1}{1!}\frac{d}{dz}\{\text{Equation of } D\} = \frac{d}{dz}\left[\frac{1}{z(z-1)}\right]\Bigg|_{z=2}$$

$$= \frac{d}{dz}\{z^{-1}(z-1)^{-1}\}\Big|_{z=2} = z^{-1}(-1)(z-1)^{-2} + (z-1)^{-1}\cdot(-1)z^{-2}\Big|_{z=2}$$

$$= -\frac{3}{4}$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left(\frac{z}{(z-2)^2}\right)$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{2}\left[\frac{2z}{2(z-2)^2}\right]$$

$$X(z) = -\frac{1}{4} + \frac{z}{z-1} - \frac{3}{4}\left(\frac{z}{z-2}\right) + \frac{1}{4}\left[\frac{2z}{(z-2)^2}\right]$$

Take IZT ROC $|z|>2$ causal.

$$x(n) = -\frac{1}{4}\delta(n) + (1)^n u(n) - \frac{3}{4}(2)^n u(n) + \frac{1}{4}\cdot n\cdot(2)^n u(n)$$

(11) Find IZT of $X(z) = \dfrac{z^2+z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^3}$ (32)

ROC: $|z| > \frac{1}{2}$

$$= \frac{X(z)}{z} = \frac{z+1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)^3} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}} + \frac{C}{\left(z-\frac{1}{2}\right)^2} + \frac{D}{\left(z-\frac{1}{2}\right)^3}$$

$$A = \left.\frac{(z+1)\cancel{\left(z-\frac{1}{4}\right)}}{\cancel{\left(z-\frac{1}{4}\right)}\left(z-\frac{1}{2}\right)^3}\right|_{z=\frac{1}{4}} = -80$$

$$D = \left.\frac{(z+1)\cancel{\left(z-\frac{1}{2}\right)^3}}{\left(z-\frac{1}{4}\right)\cancel{\left(z-\frac{1}{2}\right)^3}}\right|_{z=\frac{1}{2}} = 6$$

$$C = \frac{1}{1!}\frac{d}{dz}\{\text{Equation of } D\} = \left.\frac{d}{dz}\left\{\frac{(z+1)}{\left(z-\frac{1}{4}\right)}\right\}\right|_{z=\frac{1}{2}}$$

$$C = \left.\frac{\left(z-\frac{1}{4}\right)\cdot 1 - (z+1)\cdot 1}{\left(z-\frac{1}{4}\right)^2}\right|_{z=\frac{1}{2}} = -20$$

$$B = \left.\frac{1}{2!}\frac{d}{dz}\{\text{Equation of } C\}\right|_{z=\frac{1}{2}}$$

$$= \left.\frac{1}{2!}\frac{d}{dz}\left\{\frac{\left(z-\frac{1}{4}\right)-(z+1)}{\left(z-\frac{1}{4}\right)^2}\right\}\right|_{z=\frac{1}{2}}$$

$$= \left.\frac{1}{2\times 1} = \frac{d}{dz}\left[\left(-\frac{5}{4}\right)\left(z-\frac{1}{4}\right)^{-2}\right]\right|_{z=\frac{1}{2}}$$

$$B = \left.\frac{1}{2}\left[\left(-\frac{5}{4}\right)(-2)\left(z-\frac{1}{4}\right)^{-3}\right]\right|_{z=\frac{1}{2}} = 80$$

$$X(z) = -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left(\frac{z}{z-\frac{1}{2}}\right) - 20\left(\frac{z}{\left(z-\frac{1}{2}\right)^2}\right) + 6\left(\frac{z}{\left(z-\frac{1}{2}\right)^3}\right), \quad \text{ROC } |z|>\frac{1}{2} \text{ causal}$$

$$= -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left[\frac{z}{z-\frac{1}{2}}\right] - 20\left[\frac{\left(\frac{1}{2}\right)z}{\left(\frac{1}{2}\right)\left(z-\frac{1}{2}\right)^2}\right] + 6\left[\frac{\left(\frac{1}{2}\right)^2 \times 2z}{\left(\frac{1}{2}\right)^2 \times 2 \times \left(z-\frac{1}{2}\right)^3}\right]$$

$$X(z) = -80\left[\frac{z}{z-\frac{1}{4}}\right] + 80\left(\frac{z}{z-\frac{1}{2}}\right) - 40\left[\frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}\right] + 12\left[\frac{\left(\frac{1}{2}\right)^2 2z}{\left(z-\frac{1}{2}\right)^3}\right]$$

Take IZT

$$x(n) = \left[-80\left(\frac{1}{4}\right)^n + 80\left(\frac{1}{2}\right)^n - 40\cdot n\left(\frac{1}{2}\right)^n + 12\cdot n(n-1)\left(\frac{1}{2}\right)^n\right]u(n)$$

$n\cdot a^n u(n) \longleftrightarrow \dfrac{az}{(z-a)^2}$

$[n(n-1)]\, a^n u(n) \longleftrightarrow a^2 \dfrac{2z}{(z-a)^3}$

(12) Find $x(n) = x_1(n) * x_2(n)$ using ZT & IZT

Given $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$

& $x_2(n) = \left(\frac{1}{5}\right)^n u(n)$

Also verify the result using convolution sum.

$\Rightarrow$ $X_1(z) = \frac{z}{z-\frac{1}{3}}$, ROC: $|z| > \frac{1}{3}$

$X_2 = \frac{z}{z-\frac{1}{5}}$, ROC: $|z| > \frac{1}{5}$

$x(n) = x_1(n) * x_2(n) \xleftrightarrow{ZT} X(z) = X_1(z) \cdot X_2(z)$

$X(z) = \left(\frac{z}{z-\frac{1}{3}}\right)\left(\frac{z}{z-\frac{1}{5}}\right)$

ROC: $|z| > \frac{1}{3}$

To find $x(n)$ using ZT

$$\frac{X(z)}{z} = \frac{z}{(z-\frac{1}{3})(z-\frac{1}{5})} = \frac{A}{z-\frac{1}{3}} + \frac{B}{z-\frac{1}{5}}$$

$A = \frac{5}{2}$, $B = -\frac{3}{2}$

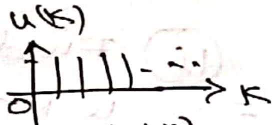
$$\frac{X(z)}{z} = \frac{5/2}{z-\frac{1}{3}} - \frac{3/2}{z-\frac{1}{5}}$$ ROC: $|z| > \frac{1}{3}$ causal

$$X(z) = \frac{5}{2}\left(\frac{z}{z-\frac{1}{3}}\right) - \frac{3}{2}\left(\frac{z}{z-\frac{1}{5}}\right)$$

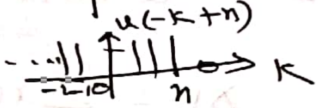
Take IZT

$$x(n) = \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)$$

Verification:- Direct convolution sum



$$x(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k) \cdot \left(\frac{1}{5}\right)^{n-k} u(n-k)$$



$$x(n) = \left(\frac{1}{5}\right)^n \sum_{k=-\infty}^{\infty} \left(\frac{5}{3}\right)^k u(k)\, u(n-k)$$

Case(i):- $n < 0$:- No overlap $\therefore x(n) = 0$

case(ii):- $n \geq 0$:- overlap $k = 0$ to n

~~$x(n) = \left(\frac{1}{5}\right)^n$~~ $x(n) = \left(\frac{1}{5}\right)^n \sum_{k=0}^{n} \left(\frac{5}{3}\right)^k = \left(\frac{1}{5}\right)^n \left[\frac{1-\left(\frac{5}{3}\right)^{n+1}}{1-\left(\frac{5}{3}\right)}\right]$

$= \left[\frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n\right]$

Ans:- $x(n) = \left[\frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n\right] u(n)$

Hence verified

(13) Using ZT find $x(n) = \alpha^n u(n) * \beta^n u(n)$

= Ans:- $x(n) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}\right] u(n)$

(14) Find IZT of $X(z) = \frac{z+1}{3z^2 - 4z + 1}$, ROC: $|z| > 1$

= Ans:- $x(n) = \delta(n) + u(n) - 2\left(\frac{1}{3}\right)^n u(n)$

(15) Find IZT of $X(z) = \log(1 + az^{-1})$, ROC: $|z| > |a|$

= W.K.T. $\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

$\therefore \log(1 + az^{-1}) = \frac{az^{-1}}{1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \cdots\cdots$

$$X(z) = \log(1 + az^{-1}) = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} (az^{-1})^n}{n}\right]$$

$$X(z) = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n} \times a^n\right] \cdot z^{-n} \quad \text{--- (1)}$$

W.K.T. $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$ --- (2)

on comparing (1) & (2), $\boxed{x(n) = \left[\frac{(-1)^{n+1}}{n} \cdot a^n\right] u(n-1)}$

(16) Find IZT of $X(z) = \sin(z)$

= W.K.T. $X(z) = \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots$ --- (1)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = x(-\infty) z^{\infty} + \cdots\, x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \quad \text{--- (2)}$$

Compare (1) & (2)

$$x(n) = \left\{ \cdots \left(\frac{-1}{7!}\right), \left(\frac{1}{5!}\right), 0, \left(\frac{-1}{3!}\right), 0, 1, \underset{\uparrow}{0} \right\}$$

Solution of Difference Equations

→ Analysis of DT-LTI systems

DT-LTI system is represented by linear constant coefficient Difference Equation (LCCDE) (or) Difference equation given by

$$\boxed{\sum_{k=0}^{N-1} a_k y(n-k) = \sum_{k=0}^{M-1} b_k x(n-k)}$$

ZT → $Z[y(n-1)] = z^{-1} Y(z)$ without initial conditions

→ $Z[y(n-1)] = z^{-1}[Y(z) + y(-1)z^{1}]$ with initial condition

→ $Z[y(n-2)] = z^{-2} Y(z)$ without initial conditions

→ $Z[y(n-2)] = z^{-2}[Y(z) + y(-1) z^{1} + y(-2) z^{2}]$ with ~~initi~~ initial condition

① Find the solution to the following difference equation

$$y(n) - \frac{3}{2} y(n-1) + \frac{1}{2} y(n-2) = \left(\frac{1}{4}\right)^n \quad \text{for } n \geq 0$$

with initial conditions $y(-1) = 4$ & $y(-2) = 10$

= ZT of Difference equation with initial conditions

$$Y(z) - \frac{3}{2} z^{-1}[Y(z) + y(-1) z] + \frac{1}{2} z^{-2}[Y(z) + y(-1) z + y(-2) z^{2}] = \frac{z}{z - \frac{1}{4}}$$

$$Y(z) - \frac{3}{2} z^{-1}[Y(z) + 4z] + \frac{1}{2} z^{-2}[Y(z) + 4z + 10z^{2}] = \frac{z}{z - \frac{1}{4}}$$

$$Y(z)\left[1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right] - 6 + 2z^{-1} + 5 = \frac{z}{z - \frac{1}{4}}$$

$$Y(z)\left[\frac{z^2 - \frac{3}{2} z + \frac{1}{2}}{z^2}\right] - 1 + \frac{2}{z} = \frac{z}{z - \frac{1}{4}}$$

$$\frac{Y(z)}{z} = \frac{2z^2 - \frac{9}{4} z + \frac{1}{2}}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)(z - 1)} = \frac{A}{z - \frac{1}{4}} + \frac{B}{z - \frac{1}{2}} + \frac{C}{z - 1}$$

$$A = \frac{1}{3}, \quad B = 1, \quad C = \frac{2}{3}$$

$$Y(z) = \frac{1}{3}\left[\frac{z}{z-\frac{1}{4}}\right] + \left(\frac{z}{z-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{z}{z-1}\right)$$

Take IZT

$$y(n) = \left[\frac{1}{3}\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n + \frac{2}{3}\right] u(n)$$

---

(2) Find the solution of the following LCCDE

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

with $y(-1) = 1$, $y(-2) = 1$, $x(n) = 3^n u(n)$

= ZT of LCCDE with initial conditions

$$Y(z) - \frac{1}{4}z^{-1}\left[Y(z) + y(-1)z\right] - \frac{1}{2}z^{-2}\left[Y(z) + y(-1)z + y(-2)z^2\right]$$

$$= X(z) + z^{-1}X(z)$$

$$Y(z) - \frac{1}{4}z^{-1}\left[Y(z) + z\right] - \frac{1}{2}z^{-2}\left[Y(z) + z + z^2\right]$$

$$= \frac{z}{z-3} + z^{-1}\cdot\frac{z}{z-3}$$

$$\frac{Y(z)}{z} = \frac{\frac{11}{8}z^2 - \frac{3}{8}}{(z-3)(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{z-3} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-\frac{1}{4}}$$

$A = \frac{96}{55}$, $B = \frac{1}{20}$, $C = -\frac{37}{88}$

$$\therefore Y(z) = \frac{96}{55}\left(\frac{z}{z-3}\right) + \frac{1}{20}\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{37}{88}\left(\frac{z}{z-\frac{1}{4}}\right)$$

Take IZT

$$y(n) = \left[\frac{96}{55}(3)^n + \frac{1}{20}\left(\frac{1}{2}\right)^n - \frac{37}{88}\left(\frac{1}{4}\right)^n\right] u(n)$$

contd

3) For the LTI system

$$y(n) - 0.5\,y(n-1) = x(n)$$

Find (i) The system function $H(z)$
(ii) The impulse response $h(n)$
(iii) The step response $s(n)$

= (i) Take ZT of LTI system.

$$Y(z) - 0.5\,z^{-1}\,Y(z) = X(z)$$

$$Y(z)\{1 - 0.5\,z^{-1}\} = X(z)$$

$$Y(z)\left(\frac{z-0.5}{z}\right) = X(z)$$

$$\boxed{\frac{Y(z)}{X(z)} = \frac{z}{z-0.5} = H(z)} \quad \text{--- } (1)$$

(ii) Impulse response $h(n) = ?$

Take IZT of eqn (1)

$$h(n) = (0.5)^n\,u(n)$$

(iii) Step Response $s(n) = ?$ $\qquad x(n) = u(n)$

$$y(n) - 0.5\,y(n-1) = u(n)$$

Take ZT

$$Y(z) - 0.5\,z^{-1}\,Y(z) = \frac{z}{z-1}$$

$$Y(z)\left[\frac{z-0.5}{z}\right] = \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$A = 2, \quad B = -1$

$$Y(z) = 2\left(\frac{z}{z-1}\right) - \left(\frac{z}{z-0.5}\right)$$

Take IZT

$$s(n) = y(n) = \left[2 - (0.5)^n\right]u(n)$$

4. For the causal system specified by

$$H(z) = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

(i) Find impulse response
(ii) Find Difference equation of the system

= (i) Take IZT of $H(z) = h(n)$

$$\frac{H(z)}{z} = \frac{(z-1)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{1}{4}}$$

$A = -\frac{2}{3}$, $B = \frac{5}{3}$

$$H(z) = -\frac{2}{3}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{5}{3}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

Take IZT

$$h(n) = \left[-\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{5}{3}\left(-\frac{1}{4}\right)^n\right]u(n)$$

(ii)
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z-1)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)} = \frac{z^2-z}{z^2-\frac{1}{4}z-\frac{1}{8}}$$

Convert positive powers of z into negative powers of z

$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

Cross multiply

$$Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

Take IZT

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Scanned with CamScanner

(5) A causal LTI system is

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

(i) Find the impulse response of the system

(ii) Find the output of the system for the input

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

= Ans: (i) $h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n)$

(ii) $y(n) = 4\left(\frac{1}{2}\right)^n - (n+3)\left(\frac{1}{4}\right)^n u(n)$

---

(6) Determine the response of the system

$$y(n) - 5y(n-1) + 6y(n-2) = x(n)$$

to an input $x(n) = [1, -2, 0]$

= Ans :- $y(n) = 3^n u(n)$

---

(7) A causal system is represented by the following difference equation

$$y(n) + \frac{1}{4}y(n-1) = x(n) + \frac{1}{2}x(n-1)$$

(i) Find the system function $H(z)$ and give the corresponding ROC

(ii) Find the unit sample response of the system in analytical form

(iii) Find the unit step response of the system in analytical form

(iv) Find the magnitude and phase of the frequency response

(v) Is the system causal? why or why not?

= (i) $H(z) = ?$ Take ZT of Difference equation.

$$Y(z) + \frac{1}{4}z^{-1}Y(z) = X(z) + \frac{1}{2}z^{-1}X(z)$$

$$Y(z)\left[1 + \frac{1}{4}z^{-1}\right] = X(z)\left[1 + \frac{1}{2}z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}$$

$$\frac{Y(z)}{X(z)} = H(z) = \left(\frac{z+\frac{1}{2}}{z+\frac{1}{4}}\right) \quad \text{--- (1)}$$

ROC :- $|z| > \frac{1}{4}$ causal

(ii) Unit sample response ? $h(n) = ?$

$$\frac{H(z)}{z} = \frac{z+\frac{1}{2}}{z(z+\frac{1}{4})} = \frac{A}{z} + \frac{B}{z+\frac{1}{4}}$$

$A = 2, \quad B = -1$

$$H(z) = 2 - \left[\frac{z}{z-(-\frac{1}{4})}\right]$$

ROC :- $|z| > \frac{1}{4}$

Take IZT

$$h(n) = 2\delta(n) - \left(\frac{-1}{4}\right)^n u(n) \quad \text{--- (2)}$$

(iii) Unit step response $s(n) = ?$ ie $x(n) = u(n)$

Take ZT

$X(z) = \frac{z}{z-1}$

W.K.T. $Y(z) = \cancel{\frac{1+\frac{1}{2}z^{-1}}{2}} \left(\frac{z+\frac{1}{2}}{z+\frac{1}{4}}\right) X(z)$

$$Y(z) = \left(\frac{z+\frac{1}{2}}{z+\frac{1}{4}}\right)\left(\frac{z}{z-1}\right)$$

$$\frac{Y(z)}{z} = \frac{(z+\frac{1}{2})}{(z+\frac{1}{4})(z-1)} = \frac{A}{z+\frac{1}{4}} + \frac{B}{z-1}$$

$A = \frac{-1}{5}, \quad B = \frac{6}{5}$

$$Y(z) = \frac{-1}{5}\left[\frac{z}{z-(-\frac{1}{4})}\right] + \frac{6}{5}\left(\frac{z}{z-1}\right),$$

ROC: $|z| > 1$ causal

Take IZT

$$y(n) = \left[-\frac{1}{5}\left(\frac{-1}{4}\right)^n + \frac{6}{5}\right] u(n)$$

(iv) Frequency Response :- consider Equation ①

$$H(z) = \frac{z+\frac{1}{2}}{z+\frac{1}{4}}$$

For Frequency response, Substitute $z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{e^{j\omega}+\frac{1}{2}}{e^{j\omega}+\frac{1}{4}} = \frac{\cos\omega + j\sin\omega + \frac{1}{2}}{\cos\omega + j\sin\omega + \frac{1}{4}}$$

$$H(e^{j\omega}) = \frac{\left[\cos(\omega)+\frac{1}{2}\right] + j\sin\omega}{\left[\cos(\omega)+\frac{1}{4}\right] + j\sin\omega}$$ Frequency Response

magnitude Response: $|H(e^{j\omega})| = \frac{\sqrt{\left(\cos\omega+\frac{1}{2}\right)^2+\sin^2\omega}}{\sqrt{\left(\cos\omega+\frac{1}{4}\right)^2+\sin^2\omega}}$

$$= \frac{\sqrt{\cos^2\omega+\frac{1}{4}+2\times\frac{1}{2}\cos\omega+\sin^2\omega}}{\sqrt{\cos^2\omega+\frac{1}{16}+2\times\frac{1}{4}\cos\omega+\sin^2\omega}}$$

$$= \frac{\sqrt{1+\frac{1}{4}+\cos\omega}}{\sqrt{1+\frac{1}{16}+\frac{1}{2}\cos\omega}} = \frac{\sqrt{\frac{5}{4}+\cos\omega}}{\sqrt{\frac{17}{16}+\frac{1}{2}\cos\omega}}$$

Phase Response $\angle H(e^{j\omega})$:-

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{\sin\omega}{\cos\omega+\frac{1}{2}}\right] - \tan^{-1}\left[\frac{\sin\omega}{\cos\omega+\frac{1}{4}}\right]$$

(v) Consider Equation ②

$$h(n) = 2\delta(n) - \left(-\frac{1}{4}\right)^n u(n)$$

$$h(n) = 0, \text{ for } n<0$$

$\therefore$ system is causal

The Unilateral ZT (UZT)

It is the one sided ZT for $n \geq 0$

i.e., for causal sequence.

It is defined as

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}}$$

Properties are same as ZT except time shift property

Time shift property of UZT

If $x(n) \xleftrightarrow{Z_{UZT}} X(z)$

$x(n-1) \longleftrightarrow$

$$Z[x(n-1)] = \sum_{n=0}^{\infty} x(n-1) z^{-n} \quad \text{--- } (1)$$

let $n-1 = p$, $n = p+1$

Lower:- $n=0$, $p+1=0$, $p=-1$

Upper:- $n=\infty$, $p+1=\infty$, $n=\infty-1=\infty$ --- (2)

Substitute (2) in (1)

$$= \sum_{p=-1}^{\infty} x(p) \cdot z^{-(p+1)} = z^{-1} \sum_{p=-1}^{\infty} x(p) \cdot z^{-p}$$

$$= z^{-1}\left[x(-1) z + \sum_{p=0}^{\infty} x(p) \cdot z^{-p}\right]$$

$$\boxed{Z[x(n-1)] = x(-1) + z^{-1} X(z)}$$

$$Z[x(n-2)] = ? = \sum_{n=0}^{\infty} x(n-2)\, z^{-n}$$

Let ~~x(n-2~~ $n-2 = p$, $n = p+2$

Lower $n=0$ Upper $p=\infty$

$p+2=0$

$p=-2$

$$\therefore Z[x(n-2)] = \sum_{p=-2}^{\infty} x(p) \cdot z^{-(p+2)} = z^{-2} \sum_{p=-2}^{\infty} x(p)\, z^{-p}$$

$$= z^{-2}\left[x(-2)\, z^{2} + x(-1)\, z^{1} + \sum_{p=0}^{\infty} x(p)\, z^{-p}\right]$$

$$= x(-2) + x(-1)\, z^{-1} + z^{-2} X(z)$$

1) Determine UZT of $x(n) = [2, 4, \underset{\uparrow}{5}, 7, 0, 1]$

$$= \quad X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= 5 + 7z^{-1} + 0 + z^{-3}$$

2) Find UZT of $y(n) = x(n-2)$

where $x(n) = \alpha^{n}$

$$= \quad X(z) = \sum_{n=0}^{\infty} \alpha^{n} z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^{n} = \frac{z}{z-\alpha}$$

$$\cancel{Y(z) =} \; Z[x(n-2)] = x(-2) + x(-1)\, z^{-1} + z^{-2} X(z)$$

$x(n) = \alpha^{n}$

For $n=-2$, $x(-2) = \alpha^{-2}$

$n=-1$, $x(-1) = \alpha^{-1}$

$$\therefore Z(x(n-2)) = \alpha^{-2} + \alpha^{-1} z^{-1} + z^{-2} \cdot \frac{z}{z-\alpha}$$

$$\therefore Y(z) = \alpha^{-2} + \alpha^{-1} z^{-1} + \frac{z}{z-\alpha}$$

Discrete-Time Fourier Transform (DTFT)

The representation of signals using complex exponentials is called Fourier representation proposed by Joseph Fourier

classification of Fourier representation based on signals

1). Discrete Time Fourier Series (DTFS):-

Fourier representation of Discrete time periodic signals

2). Fourier Series (FS): Fourier representation of continuous time periodic signals

3). Discrete Time Fourier Transform (DTFT): Fourier representation of Discrete time non periodic signals

4). Fourier Transform (FT):- Fourier representation of continuous time non-periodic signals.

DTFT

→ complicated convolution in Time domain is converted into simple multiplication using DTFT

→ Frequency analysis of DT signals (non periodic) is possible using DTFT

→ The disadvantage of DTFT is continuous one, which can't be analysed using Digital computers.

Defn: consider non periodic DT signal $x(n)$ and apply DTFT

DTFT $$\boxed{X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}}$$

~~IDTFT~~ $X(e^{j\Omega})$ is continuous and periodic ~~$x(n)$~~

IDTFT $$\boxed{x(n) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[X(e^{j\Omega})\right] \cdot e^{j\Omega n} \cdot d\Omega}$$

Assuming $x(n)$ is finite sequence, while computing DTFT.

problems:-

3

① Find DTFT of $x(n) = a^n u(n)$, Also find magnitude & phase response.

$$= X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \left[a \cdot e^{-j\omega}\right]^n$$

$$= \frac{1}{1 - a \cdot e^{-j\omega}}$$

$$\boxed{X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}}$$

Frequency Response:- $X(e^{j\omega}) = \dfrac{1}{1 - a(\cos\omega - j\sin\omega)}$

$$X(e^{j\omega}) = \frac{1}{(1 - a\cos\omega) + j(a\sin\omega)}$$

Magnitude Response $|X(e^{j\omega})|$:-

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + (a\sin\omega)^2}}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

*****
DTFT is continuous & periodic with fundamental period 2π

Phase Response $\angle X(e^{j\omega})$:-

$$\angle X(e^{j\omega}) = 0 - \tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]$$

$$\boxed{\angle a + jb = \tan^{-1}\frac{b}{a}}$$

(2) Find DTFT of $x(n) = a^n[-u(-n-1)]$

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} a^n[-u(-n-1)] \cdot e^{-j\Omega n} = -\sum_{n=-1}^{-\infty} a^n e^{-j\Omega n}$$

Replace n by $-n$

$$X(e^{j\Omega}) = -\sum_{n=1}^{\infty} a^{-n} e^{j\Omega n} = -\sum_{n=1}^{\infty} (a^{-1} \cdot e^{j\Omega})^n$$

$$= -\left(\frac{a^{-1} \cdot e^{j\Omega}}{1 - a^{-1} \cdot e^{j\Omega}}\right) = \left(\frac{a^{-1} e^{j\Omega}}{a^{-1} e^{j\Omega} - 1}\right)$$

$$\boxed{X(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - a}}$$

(3) Find DTFT of $x(n) = 5(3)^n u(-n)$

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} 5(3)^n u(-n) \cdot e^{-j\Omega n}$$

$$= 5 \sum_{n=0}^{-\infty} 3^n e^{-j\Omega n} = 5 \sum_{n=0}^{\infty} 3^{-n} \cdot e^{j\Omega n}$$

$$= 5 \sum_{n=0}^{\infty} (3^{-1} e^{j\Omega})^n = 5\left[\frac{1}{1 - 3^{-1} e^{j\Omega}}\right]$$

$$= 5\left[\frac{3}{3 - e^{j\Omega}}\right]$$

(4) Find DTFT of $x(n) = u(n) - u(n-6)$

$$= X(e^{j\Omega}) = \sum_{n=0}^{5} 1 \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{5} (e^{-j\Omega})^n = \frac{1 - (e^{-j\Omega})^{5+1}}{1 - e^{-j\Omega}}$$

$$X(e^{j\Omega}) = \frac{1-e^{-j6\Omega}}{1-e^{-j\Omega}}$$

(5) Find the DTFT of $x(n) = \delta(n)$

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta(n) \cdot e^{-j\Omega n} = \sum_{n=0} \delta(0) e^{-j\Omega n}$$

$$= \delta(0) \cdot e^{-j\Omega \times 0} = 1$$

(6) Find the DTFT of $x(n) = \delta(n-k)$

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta(n-k)\, e^{-j\Omega n} = \sum_{n=k} \delta(n-k)\, e^{-j\Omega n}$$

$$= \delta(k-k)\, e^{-j\Omega k} = e^{-j\Omega k}.$$

(7) Find the DTFT of $x(n) = a^{|n|}$, $|a|<1$

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{(-n)} \cdot e^{-j\Omega n} + \sum_{n=0}^{\infty} a^{n} \cdot e^{-j\Omega n}$$

replace n by $-n$ in 1st term.

$$= \sum_{n=1}^{\infty} a^{n}\, e^{j\Omega n} + \sum_{n=0}^{\infty} a^{n}\, e^{-j\Omega n}$$

$$= \sum_{n=1}^{\infty} (a\, e^{j\Omega})^{n} + \sum_{n=0}^{\infty} (a\, e^{-j\Omega})^{n}$$

$$= \frac{a\, e^{j\Omega}}{1-a \cdot e^{j\Omega}} + \frac{1}{1-a \cdot e^{-j\Omega}}$$

$$X(e^{j\Omega}) = \frac{a\, e^{j\Omega}}{1-a \cdot e^{j\Omega}} + \frac{e^{j\Omega}}{e^{j\Omega}-a}$$

(8) Find DTFT of $x(n) = a^n \sin(\Omega n) \cdot u(n)$

$$= X(e^{j\Omega}) = \frac{a \sin\Omega \cdot e^{-j\Omega}}{1 - 2a \cdot \cos\Omega \cdot e^{-j\Omega} + a^2 e^{-j2\Omega}}$$

(9) Find DTFT of $x(n) = \begin{cases} 1, & -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right) \\ 0, & \text{otherwise} \end{cases}$

~~Also~~ Assume N is odd.

(or)

Find the frequency response, magnitude response

$$= X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n} = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} 1 \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} (e^{-j\Omega})^n = \frac{(e^{-j\Omega})^{-\left(\frac{N-1}{2}\right)} - (e^{-j\Omega})^{\left[\left(\frac{N-1}{2}\right)+1\right]}}{1 - e^{-j\Omega}}$$

$$= \frac{e^{j\Omega\left(\frac{N-1}{2}\right)} - (e^{-j\Omega})^{\left(\frac{N+1}{2}\right)}}{1 - e^{-j\Omega}}$$

$$\frac{N-1}{2} + 1$$
$$\frac{N-1+2}{2}$$
$$\frac{N+1}{2}$$

$$= \frac{e^{j\Omega\frac{N}{2}} \cdot e^{-\frac{j\Omega}{2}} - e^{-\frac{j\Omega}{2}N} \cdot e^{-\frac{j\Omega}{2}}}{1 - e^{-\frac{j\Omega}{2}} \cdot e^{-\frac{j\Omega}{2}}}$$

$$= \frac{\cancel{e^{-\frac{j\Omega}{2}}}\left[e^{j\Omega\frac{N}{2}} - e^{-j\Omega\frac{N}{2}}\right]}{\cancel{e^{-\frac{j\Omega}{2}}}\left[e^{\frac{j\Omega}{2}} - e^{-\frac{j\Omega}{2}}\right]}$$

7

$$X(e^{j\Omega}) = \frac{\cancel{2j}\sin\left(\frac{\Omega N}{2}\right)}{\cancel{2j}\sin\left(\frac{\Omega}{2}\right)}$$

$$\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j(0)n} = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} 1$$

a, $a=1$

Frequency Response:

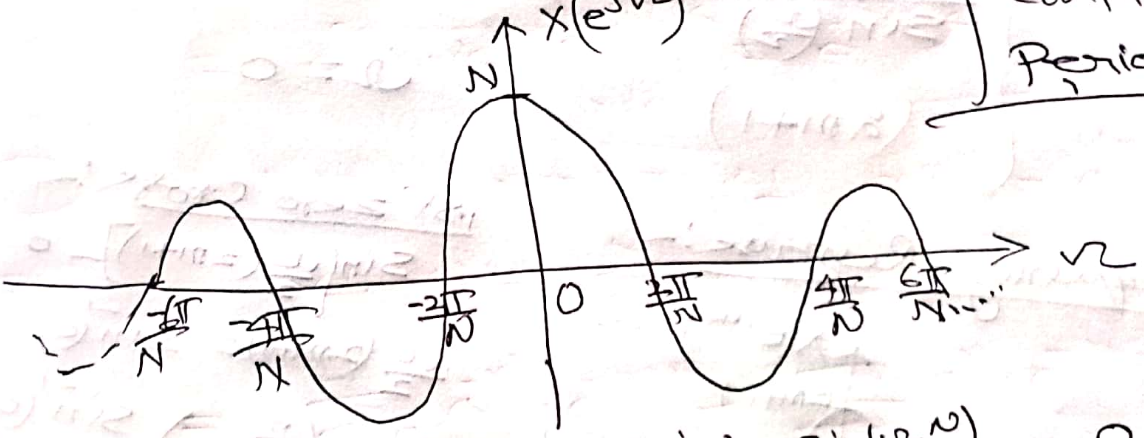
$$X(e^{j\Omega}) = \begin{cases} \dfrac{\sin\left(\frac{\Omega N}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}, & \Omega \neq 0 \\ N, & \Omega = 0 \end{cases}$$

4^{th} formula, $a=1$

So $N - \cancel{1} + \cancel{1} = N$

Frequency Response plot :—

Continuous Periodic



1^{st} Zero cross,

$$X(e^{j\Omega}) = \frac{\sin\left(\frac{\Omega N}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)} = 0$$

$$\sin\left(\frac{\Omega N}{2}\right) = 0$$

$$\frac{\Omega N}{2} = \sin^{-1}(0) = 0$$

$$\Omega = \frac{2}{N}\sin^{-1}(0) = 0 \quad \times$$

$$\Omega = \frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N}, \ldots$$

$\sin\left[\Omega \cdot \frac{N}{2}\right]$

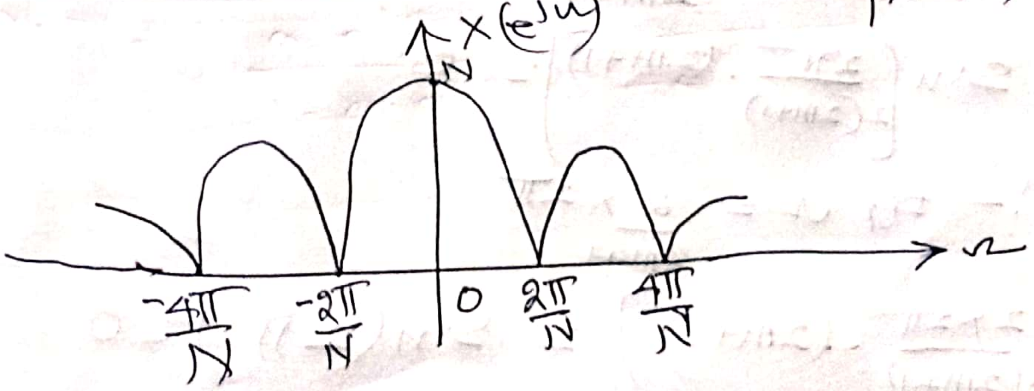
$\sin\left[\left(\frac{2\pi}{\cancel{N}}\right)\cdot\frac{\cancel{N}}{2}\right] = 0$

$\sin\left[\left(\frac{4\pi}{\cancel{N}}\right)\cdot\frac{\cancel{N}}{\cancel{2}}\right] = 0$

$\sin\left[\left(\frac{6\pi}{N}\right)\cdot\frac{N}{2}\right] = 0$

Magnitude Response $|X(e^{j\Omega})|$:—

$|X(e^{j\Omega})| =$

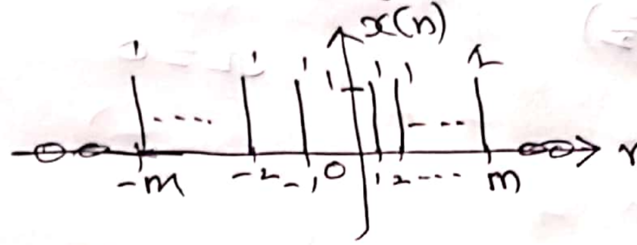


(10) Find the DTFT of rectangular pulse

$$x(n) = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$

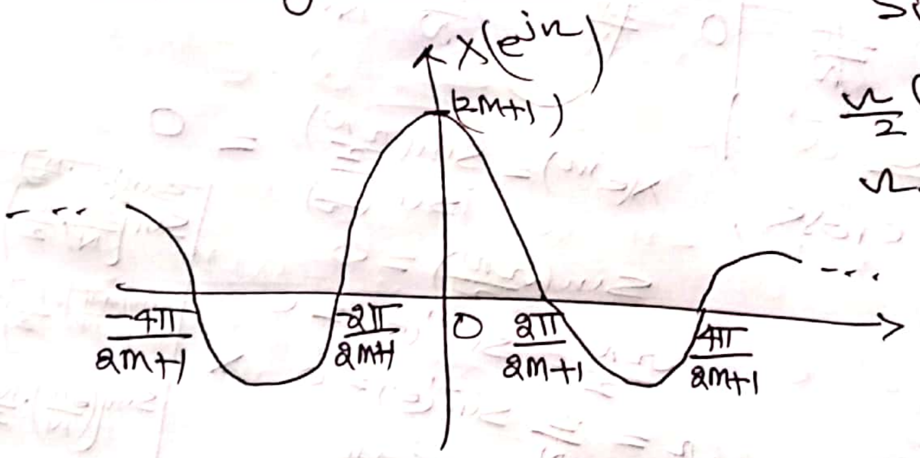
Draw its spectrum.

=



$$X(e^{j\Omega}) = \begin{cases} \dfrac{\sin\left[\frac{\Omega}{2}(2M+1)\right]}{\sin\left(\frac{\Omega}{2}\right)}, & \Omega \neq 0 \\ (2M+1) & \Omega = 0 \end{cases}$$

Frequency Response :-



1st zero cross :-

$\sin\left[\frac{\Omega}{2}(2M+1)\right] = 0$

$\frac{\Omega}{2}(2M+1) = \sin^{-1}(0)$

$\Omega = \left(\frac{2}{2M+1}\right)\sin^{-1}(0)$

$\sin^{-1}0 = \pi, 2\pi$

$\sin\left[\frac{\Omega}{2}(2M+1)\right]$

1st Zero cross :- For $\Omega = \frac{2\pi}{2M+1}$

i.e. $\sin\left[\frac{\cancel{2}\pi}{\cancel{2}\cancel{(2M+1)}}\cdot\cancel{(2M+1)}\right] = 0$

2nd Zero cross :- For $\Omega = \frac{2 \times 2\pi}{2M+1}$

i.e. $\sin\left[\frac{\cancel{2} \times 2\pi}{\cancel{2}\cancel{(2M+1)}}\cdot\cancel{(2M+1)}\right] = \sin(2\pi) = 0$

(11) Find the DTFT of

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n-4)$

Ans: $X(e^{j\Omega}) = \dfrac{\left(\frac{1}{2} e^{-j\Omega}\right)^4}{1 - \frac{1}{2} e^{-j\Omega}}$

(ii) $x(n) = \left(\frac{1}{4}\right)^n u(n+4)$

Ans:- $X(e^{j\Omega}) = \dfrac{256 \times e^{j4\Omega}}{1 - \frac{1}{4} e^{-j\Omega}}$

(12) Find DTFT of $x(n) = \delta(6-3n)$

plot spectra.

$$= \quad X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta(6-3n) \cdot e^{-j\Omega n}$$

for $\delta(\,)$ to exist

Substitute $(6-3n) = 0$

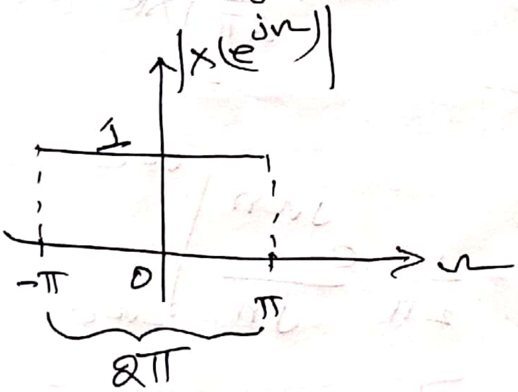
$\therefore \underline{n = 2}$

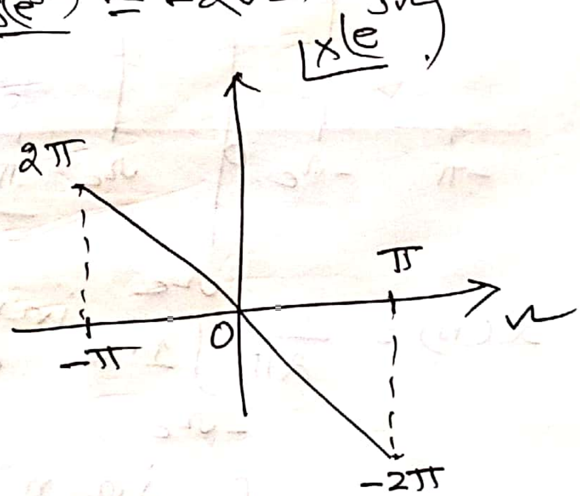
$$\therefore X(e^{j\Omega}) = \sum_{n=2} \delta(6-3n) \cdot e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \delta(0) \cdot e^{-j\Omega 2} = 1 e^{-j2\Omega}$$

magnitude spectrum $|X(e^{j\Omega})| = 1$

phase spectrum $\underline{\angle X(e^{j\Omega})} = -2\Omega$





(13) Find the inverse DTFT of

$X(e^{j\Omega}) = \delta(\Omega) \quad , \quad -\pi \le \Omega \le \pi$

=

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} \, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) \cdot e^{j\Omega n} \cdot d\Omega$$

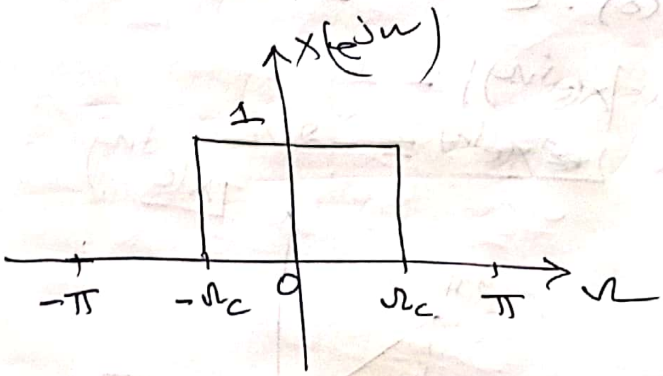
$$= \frac{1}{2\pi}\int_{0} \delta(\Omega) \, e^{j\Omega n} d\Omega = \frac{1}{2\pi}\left[\delta(0) \cdot e^{j0n}\right]$$

$$\boxed{x(n) = \frac{1}{2\pi}}$$

(14) Find IDTFT of

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & \Omega_c \le |\Omega| \le \pi \end{cases}$$

=



$x(n) = ?$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

$$x(n) = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n} d\Omega = \frac{1}{2\pi} \left. \frac{e^{j\Omega n}}{jn} \right|_{-\Omega_c}^{\Omega_c}$$

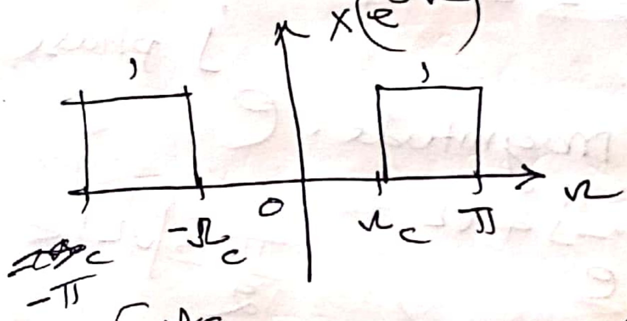
$$= \frac{1}{2\pi j n}\left[e^{j\Omega_c n} - e^{-j\Omega_c n}\right] = \frac{\cancel{2}\cancel{j}\sin(\Omega_c n)}{\cancel{2}\pi \cancel{j} n}$$

$$x(n) = \begin{cases} \dfrac{\sin(\Omega_c n)}{\pi n}, & n \ne 0 \\ (\Omega_c/\pi), & n = 0 \end{cases}$$

(15) Find IDFT of

$$X(e^{j\Omega}) = \begin{cases} 0, & |\Omega| \le \Omega_c \\ 1, & \Omega_c \le |\Omega| \le \pi \end{cases}$$

=



$x(n) = ?$

$$x(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\Omega_c} 1 \cdot e^{j\Omega n}\, d\Omega + \int_{\Omega_c}^{\pi} 1 \cdot e^{j\Omega n}\, d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\Bigg|_{-\pi}^{-\Omega_c} + \frac{e^{j\Omega n}}{jn}\Bigg|_{\Omega_c}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left\{\frac{1}{jn}\left[e^{-j\Omega_c n} - e^{-j\pi n}\right] + \frac{1}{jn}\left[e^{j\pi n} - e^{j\Omega_c n}\right]\right\}$$

$$= \frac{1}{2\pi jn}\left[\left(e^{j\pi n} - e^{-j\pi n}\right) - \left(e^{j\Omega_c n} - e^{-j\Omega_c n}\right)\right]$$

$$= \frac{1}{2\pi jn}\left[2j\sin\pi n - 2j\sin(\Omega_c n)\right]$$

$$\boxed{\sin n\pi = 0 \text{ for all } n}$$

$$= -\frac{\cancel{2}\cancel{j}}{\cancel{2}\pi \cancel{j} n}\sin\Omega_c n$$

$$x(n) = \begin{cases} -\dfrac{\sin(\Omega_c n)}{\pi n}, & n \ne 0 \\ (-\Omega_c/\pi) & n = 0 \end{cases}$$

(16) Find time domain (IDTFT) of (12)

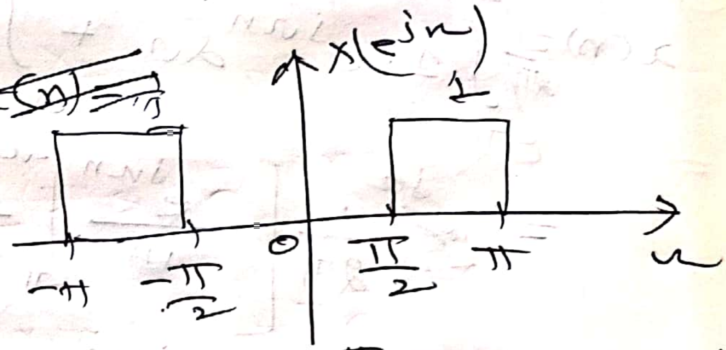
$$|X(e^{j\Omega})| = \begin{cases} 1, & \frac{\pi}{2} \le |\Omega| \le \pi \\ 0, & -\frac{\pi}{2} \le \Omega \le \frac{\pi}{2} \end{cases}$$

$$\arg X(e^{j\Omega}) = -4\Omega$$

$$= X(e^{j\Omega}) = \text{magnitude} \cdot e^{j\,\text{phase}}$$

$$X(e^{j\Omega}) = 1 \cdot e^{-j4\Omega}, \quad \frac{\pi}{2} \le |\Omega| \le \pi$$

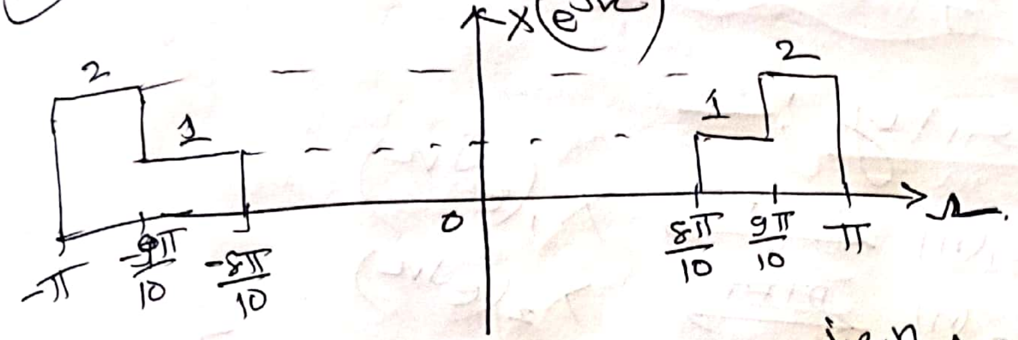
$x(n) = ?$



$$x(n) = \frac{1}{2\pi}\left\{ \int_{-\pi}^{-\pi/2} e^{-j4\Omega} \cdot e^{j\Omega n}\, d\Omega + \int_{\pi/2}^{\pi} e^{-j4\Omega} \cdot e^{j\Omega n}\, d\Omega \right\}$$

$$x(n) = \begin{cases} -\left(\dfrac{\sin\left(\frac{\pi}{2}(n-4)\right)}{\pi(n-4)}\right), & n \ne 4 \\ \left(\frac{1}{2}\right) & n = 4 \end{cases}$$

(17) Determine IDTFT of



$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} \cdot d\Omega$$

$$= \frac{1}{2\pi}\left\{ \int_{-\pi}^{-9\pi/10} 2 \cdot e^{j\Omega n} d\Omega + \int_{-\frac{9\pi}{10}}^{-8\pi/10} 1 \cdot e^{j\Omega n} d\Omega + \int_{\frac{8\pi}{10}}^{9\pi/10} 1 \cdot e^{j\Omega n} d\Omega + \int_{\frac{9\pi}{10}}^{\pi} 2 e^{j\Omega n} d\Omega \right\}$$

$$x(n) = -\frac{1}{n\pi}\left[\sin\left(\frac{8\pi n}{10}\right) + \sin\left(\frac{9\pi n}{10}\right)\right], \quad n \neq 0$$

$$, \quad n = 0$$

---

(18)

Properties of DTFT

1) linearity :-

If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\Omega})$

& $x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\Omega})$

Then $[a x_1(n) + b x_2(n)] \xleftrightarrow{DTFT} [a X_1(e^{j\Omega}) + b X_2(e^{j\Omega})]$

Superposition principle

Proof:- consider LHS, Take DTFT of LHS

$$DTFT[LHS] = \sum_{n=-\infty}^{\infty} (LHS) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] e^{-j\Omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\Omega n} + b \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\Omega n}$$

$$= a X_1(e^{j\Omega}) + b X_2(e^{j\Omega n}) = RHS$$

Hence the proof

2) Time shift :-

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

Then $x(n-n_0) \xleftrightarrow{DTFT} e^{j\Omega(-n_0)} X(e^{j\Omega})$

Proof:- consider LHS, take DTFT

$$DTFT[LHS] = \sum_{n=-\infty}^{\infty} (LHS) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) \cdot e^{-j\Omega n} \quad \text{——— (1)}$$

Let $n - n_0 = m$, $n = m + n_0$ ——— (2)

Substitute (2) in (1)

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\Omega m}$$

$$= e^{-j\Omega n_0} X(e^{j\Omega}) = RHS$$, Hence the Proof

③ Frequency Shift :-

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

$e^{j\Omega_0 n} \cdot x(n) \longleftrightarrow X(e^{j(\Omega-\Omega_0)})$.

Proof :- Consider LHS, Take DTFT

$$DTFT[LHS] = \sum_{n=-\infty}^{\infty} (LHS) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\Omega-\Omega_0)n}$$

$$= X[e^{j(\Omega-\Omega_0)}] = RHS$$, Hence the Proof

④ Convolution :-

If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\Omega})$

& $x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\Omega})$

Then $[x_1(n) * x_2(n)] \xleftrightarrow{DTFT} X_1(e^{j\Omega}) \cdot X_2(e^{j\Omega})$

Proof :- Consider LHS, Take DTFT

$$DTFT[LHS] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m) \cdot e^{-j\Omega n}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{n=-\infty}^{\infty} x_2(n-m) \cdot e^{-j\Omega n}$$

let $n-m = L$, $n = L+m$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{L=-\infty}^{\infty} x_2(L)\, e^{-j\Omega(L+m)}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m) \cdot e^{-j\Omega m} \cdot \sum_{l=-\infty}^{\infty} x_2(L) \cdot e^{-j\Omega L}$$

$$= X_1(e^{j\Omega}) \cdot X_2(e^{j\Omega}) = RHS$$, Hence Step proof

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(5) Modulation :-

If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\Omega})$

& $x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\Omega})$

Then $[x_1(n) \cdot x_2(n)] \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X_1(e^{j\Omega}) * X_2(e^{j\Omega})\right]$

= Proof :- W.K.T IDFT

$$x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\mu}) \cdot e^{j\mu n}\, d\mu$$

Take DTFT [LHS] $= \sum_{n=-\infty}^{\infty} [x_1(n) \cdot x_2(n)]\, e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\mu}) \cdot e^{j\mu n}\, d\mu \cdot x_2(n) \cdot e^{-j\Omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\mu})\, e^{j(m-\Omega)n} \cdot dm \cdot \sum_{n=-\infty}^{\infty} x_2(n)$$

$x_2(n)\, e^{-j\Omega n}$

Proof :— W.K.T. IDFT $x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{jL}) \cdot e^{jLn} dL$ (17)

$$\text{DTFT}[LHS] = \sum_{n=-\infty}^{\infty} (LHS) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x_1(n) \cdot x_2(n)] \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jL}) \cdot e^{jLn} dL \right) \cdot x_2(n) \cdot e^{-j\Omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jL}) \cdot dL \cdot \sum_{n=-\infty}^{\infty} x_2(n) \cdot e^{jLn} \cdot e^{-j\Omega n}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(e^{jL}) \cdot dL \cdot \boxed{\sum_{n=-\infty}^{\infty} x_2(n) \cdot e^{-j(\Omega - L)n}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(e^{jL}) \cdot dL \cdot X_2\left(e^{j(\Omega - L)}\right)$$

$$= \frac{1}{2\pi} \boxed{\int_{-\infty}^{\infty} X_1(e^{jL}) \cdot X_2 e^{j(\Omega - L)} \cdot dL}$$

$$= \frac{1}{2\pi} \left[X_1(e^{j\Omega}) * X_2(e^{j\Omega}) \right] = RHS$$

(6) **Scaling** :—

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $x(an) \xleftrightarrow{DTFT} X\left(e^{j\left(\frac{\Omega}{a}\right)}\right)$

Proof:

$$\text{DTFT}(LHS) = \sum_{n=-\infty}^{\infty} (LHS) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(an) \cdot e^{-j\Omega n}$$

let $an = m$, $n = \frac{m}{a}$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\Omega\left(\frac{m}{a}\right)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\left(\frac{\Omega}{a}\right) \cdot m}$$

$$= X\left(e^{j\left(\frac{\Omega}{a}\right)}\right) = RHS$$

(7) Frequency Differentiation

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

Then $-jn\, x(n) \xleftrightarrow{DTFT} \frac{d[X(e^{j\Omega})]}{d\Omega}$

(or)

$n\, x(n) \xleftrightarrow{DTFT} j \cdot \frac{d[X(e^{j\Omega})]}{d\Omega}$

= W.K.T. $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}$

D.w.r.t. Ω.

$$\frac{d[X(e^{j\Omega})]}{d\Omega} = \sum_{n=-\infty}^{\infty} x(n) \cdot (-jn) \cdot e^{-j\Omega n}$$

$$\frac{d[X(e^{j\Omega})]}{d\Omega} = \sum_{n=-\infty}^{\infty} [-jn \cdot x(n)] \cdot e^{-j\Omega n}$$

$$\frac{d[X(e^{j\Omega})]}{d\Omega} = DTFT[-jn\, x(n)]$$

Hence the proof

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(19)

(8) Perseval's Theorem

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$

then $$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 d\Omega$$

→ Energy in TD is equalent to energy Sectrum in FD

$$= \quad LHS = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) \quad \text{--- (1)}$$

w.k.T $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} d\Omega$$

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega}) e^{-j\Omega n} d\Omega \quad \text{--- (2)}$$

Substitute (2) in (1)

$$LHS = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega}) e^{-j\Omega n} d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega}) . d\Omega \boxed{\sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\Omega}) . d\Omega \cdot X(e^{j\Omega})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\Omega})|^2 . d\Omega = RHS$$

① Determine time-domain signal corresponding to DTFT $X(e^{j\Omega}) = \cos\Omega + j\sin\Omega$

$$\Rightarrow \quad X(e^{j\Omega}) = \cos(\Omega) + j\sin(\Omega) = e^{j\Omega}$$

$$\text{IDTFT} \quad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega} \cdot e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n+1)} \cdot d\Omega$$

$$x(n) = \begin{cases} \dfrac{\sin[\pi(n+1)]}{\pi(n+1)}, & n \neq -1 \\ 1, & n = -1 \end{cases}$$

$n\pi$

because sin(npi) = 0

$$x(n) = \begin{cases} 0, & n \neq -1 \\ 1, & n = -1 \end{cases}$$

$$\therefore \boxed{x(n) = \delta(n+1)}$$

(2) Find IDTFT of

$$X(e^{j\Omega}) = \frac{6e^{j2\Omega} - \frac{2}{3}e^{j\Omega} - \frac{1}{6}}{e^{j2\Omega} + \frac{1}{6}e^{j\Omega} - \frac{1}{6}}$$

$$= \frac{X(e^{j\Omega})}{e^{j\Omega}} = \frac{6 \cdot e^{j2\Omega} - \frac{2}{3}e^{j\Omega} - \frac{1}{6}}{e^{j\Omega}\left(e^{j\Omega} - \frac{1}{3}\right)\left(e^{j\Omega} + \frac{1}{2}\right)}$$

$$= \frac{A}{e^{j\Omega}} + \frac{B}{e^{j\Omega} - \frac{1}{3}} + \frac{C}{e^{j\Omega} + \frac{1}{2}}$$

$A = 1, \quad B = 1, \quad C = 4$

$$X(e^{j\Omega}) = 1 + \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{3}} + 4\,\frac{e^{j\Omega}}{e^{j\Omega} - \left(-\frac{1}{2}\right)}$$

Take IDTFT

$$x(n) = \delta(n) + \left(\frac{1}{3}\right)^n u(n) + 4\left(\frac{-1}{2}\right)^n u(n)$$

(3) Find IDTFT of $X(\Omega) = \dfrac{3 - \frac{1}{4}e^{-j\Omega}}{-\frac{1}{16}e^{-j2\Omega} + 1}$

$$= X(\Omega) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{1 - \left(\frac{1}{4}e^{-j\Omega}\right)^2} = \frac{3 - \frac{1}{4}e^{-j\Omega}}{\left(1 + \frac{1}{4}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

$$= \frac{\dfrac{3e^{j\Omega} - \frac{1}{4}}{e^{j\Omega}}}{\left(\dfrac{e^{j\Omega} + \frac{1}{4}}{e^{j\Omega}}\right)\left(\dfrac{e^{j\Omega} - \frac{1}{4}}{e^{j\Omega}}\right)}$$

$$= \frac{\left(3e^{j\Omega} - \frac{1}{4}\right) e^{j\Omega}}{\left(e^{j\Omega} - \frac{1}{4}\right)\left(e^{j\Omega} + \frac{1}{4}\right)}$$

$$\frac{X(e^{j\Omega})}{e^{j\Omega}} = \frac{A}{e^{j\Omega} - \frac{1}{4}} + \frac{B}{e^{j\Omega} + \frac{1}{4}}$$

$$A = 1, \quad B = 2$$

$$X(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - \frac{1}{4}} + 2\,\frac{e^{j\Omega}}{e^{j\Omega} - \left(-\frac{1}{4}\right)}$$

Take IDTFT

$$x(n) = \left(\frac{1}{4}\right)^n u(n) + 2\left(-\frac{1}{4}\right)^n u(n)$$

→ Frequency Response of LTI system

→ Solution of Difference equations

The LTI system i.e., the Difference equation is represented by linear constant coefficient Difference equation LCCDE

$$\sum_{k=0}^{M-1} a_k\, y(n-k) = \sum_{k=0}^{N-1} b_k\, x(n-k)$$

Solution:- $y(n) = ?$

→ Convert time domain into Frequency domain using DTFT

→ Find IDTFT to get solution

① The DT-LTI system is described by $3y(n) - 4y(n-1) + y(n-2) = 3x(n)$

Determine Frequency response and impulse response of LTI system

= Take DTFT

$$3Y(e^{j\Omega}) - 4e^{-j\Omega} \cdot Y(e^{j\Omega}) + e^{-j2\Omega} Y(e^{j\Omega}) = 3X(e^{j\Omega})$$

$$Y(e^{j\Omega})\left[3 - 4e^{-j\Omega} + e^{-j2\Omega}\right] = 3X(e^{j\Omega})$$

Frequency Response

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1}{1 - \frac{4}{3}e^{-j\Omega} + \frac{1}{3}e^{-j2\Omega}}$$

$$\boxed{H(e^{j\Omega}) = \frac{e^{+j2\Omega}}{e^{j2\Omega} - \frac{4}{3}e^{j\Omega} + \frac{1}{3}}}$$

Impulse Response

$$\frac{H(e^{j\Omega})}{e^{j\Omega}} = \frac{e^{j\Omega}}{(e^{j\Omega}-1)(e^{j\Omega}-\frac{1}{3})}$$

$$= \frac{A}{e^{j\Omega}-1} + \frac{B}{e^{j\Omega}-\frac{1}{3}}$$

$$A = 3/2, \quad B = -\frac{1}{2}$$

$$H(e^{j\Omega}) = \frac{3}{2}\left(\frac{e^{j\Omega}}{e^{j\Omega}-1}\right) - \frac{1}{2}\left(\frac{e^{j\Omega}}{e^{j\Omega}-\frac{1}{3}}\right)$$

Take I DTFT

$$h(n) = \frac{3}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

2) The impulse response of DT-LTI system is $h(n) = (0.5)^n u(n)$

(i) Find the Frequency response of ~~DTI~~ DT-LTI system

(ii) Find the magnitude and phase response

(iii) write the difference equation of the system

= (i) Take DTFT $\{h(n)\} = H(e^{j\Omega})$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n u(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (0.5 \cdot e^{-j\Omega})^n$$

FR. $$\boxed{H(e^{j\Omega}) = \frac{1}{1 - 0.5\, e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - 0.5}}$$

~~Magnitude Response~~ $|H(e^{j\Omega})|$

$$H(e^{j\Omega}) = \frac{\cos(\Omega) + j\sin(\Omega)}{\cos\Omega + j\sin\Omega - 0.5}$$

FR

$$H(e^{j\Omega}) = \frac{\cos\Omega + j\sin\Omega}{[\cos(\Omega) - 0.5] + j\sin\Omega}$$

(ii) magnitude Response $|H(e^{j\Omega})|$

$$= \frac{\sqrt{\cos^2\Omega + \sin^2\Omega}}{\sqrt{(\cos\Omega - 0.5)^2 + \sin^2\Omega}} = \cancel{\frac{1}{\sqrt{\cos^2\Omega + 0.25}}}$$

$$= \frac{1}{\sqrt{\cos^2\Omega + 0.25 - \cos\Omega + \sin^2\Omega}}$$

$$|H(e^{j\Omega})| = \frac{1}{\sqrt{1.25 - \cos\Omega}}$$

Phase Response :- $\angle H(e^{j\Omega})$

$$= \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega}\right) - \tan^{-1}\frac{\sin\Omega}{\cos\Omega - 0.5}$$

$$= -\tan^{-1}\left(\frac{0.5\sin\Omega}{1 - 0.5\cos\Omega}\right)$$

(iii) Difference Equation :

$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - 0.5} = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$$

$$e^{j\Omega}X(e^{j\Omega}) = e^{j\Omega}Y(e^{j\Omega}) - 0.5\,Y(e^{j\Omega})$$

~~Take IDTFT~~ convert $e^{j\Omega}$ into $e^{-j\Omega}$

$$X(e^{j\Omega}) = Y(e^{j\Omega}) - 0.5\,e^{-j\Omega}\,Y(e^{j\Omega})$$

Take IDTFT

$$x(n) = y(n) - 0.5\,y(n-1)$$

3) Find the solution of difference equation $y(n) - 0.5\,y(n-1) = x(n)$ using DTFT & IDTFT

where $x(n) = u(n)$

$\Rightarrow$ Take DTFT $\quad Y(e^{j\Omega}) - 0.5\, e^{-j\Omega} Y(e^{j\Omega}) = X(e^{j\Omega}) \longrightarrow (1)$

Take DTFT $[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$

$$= \sum_{n=-\infty}^{\infty} u(n)\, e^{-j\Omega n} = \sum_{n=0}^{\infty} (e^{-j\Omega})^n$$

$$X(e^{j\Omega}) = \frac{1}{1-e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}-1} \longrightarrow (2)$$

Substitute (2) in (1)

$$Y(e^{j\Omega}) - 0.5\, e^{-j\Omega} Y(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega}-1}$$

$$Y(e^{j\Omega})\left[1 - 0.5\, e^{-j\Omega}\right] = \frac{e^{j\Omega}}{e^{j\Omega}-1}$$

$$Y(e^{j\Omega})\left[\frac{e^{j\Omega}-0.5}{e^{j\Omega}}\right] = \frac{e^{j\Omega}}{e^{j\Omega}-1}$$

$$\frac{Y(e^{j\Omega})}{e^{j\Omega}} = \frac{e^{j\Omega}}{(e^{j\Omega}-1)(e^{j\Omega}-0.5)} = \frac{A}{e^{j\Omega}-1} + \frac{B}{e^{j\Omega}-0.5}$$

$A = 2,\ B = -1$

$$Y(e^{j\Omega}) = 2\left(\frac{e^{j\Omega}}{e^{j\Omega}-1}\right) - 1\left(\frac{e^{j\Omega}}{e^{j\Omega}-0.5}\right)$$

Take IDTFT

$$y(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

(4) Find the solution of difference equation

$$y(n) - \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = x(n) + x(n-1)$$

for input $x(n) = \left(\frac{-1}{2}\right)^n u(n)$

$$= \quad DTFT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left[-\frac{1}{2} \cdot e^{-j\Omega}\right]^n$$

$$X(e^{j\Omega}) = \frac{1}{1 + \frac{1}{2} e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} + \frac{1}{2}} \quad (1)$$

Take DTFT of Given Equation

$$Y(e^{j\Omega}) - \frac{1}{4} e^{-j\Omega} \cdot Y(e^{j\Omega}) - \frac{1}{8} e^{-j2\Omega} Y(e^{j\Omega})$$

$$= X(e^{j\Omega}) + e^{-j\Omega} X(e^{j\Omega})$$

$$\cancel{Y(e^{j\Omega}) - \frac{1}{4} e^{-j\Omega} Y}.$$

$$Y(e^{j\Omega})\left[1 - \frac{1}{4} e^{-j\Omega} - \frac{1}{8} e^{-j2\Omega}\right] = X(e^{j\Omega})\left[1 - e^{-j\Omega}\right]$$

$$Y(e^{j\Omega})\left[\frac{e^{j2\Omega} - \frac{1}{4} e^{j\Omega} - \frac{1}{8}}{e^{j2\Omega}}\right] = X(e^{j\Omega})\left(\frac{e^{j\Omega} - 1}{e^{j\Omega}}\right)$$

$$\frac{Y(e^{j\Omega})}{e^{j2\Omega}}\left[e^{j2\Omega}-\frac{1}{4}e^{j\Omega}-\frac{1}{8}\right]=\left(\frac{\cancel{e^{j\Omega}}}{e^{j\Omega}+\frac{1}{2}}\right)\left(\frac{e^{j\Omega}-1}{\cancel{e^{j\Omega}}}\right).$$

$$\frac{Y(e^{j\Omega})}{e^{j\Omega}}=\frac{e^{j\Omega}(e^{j\Omega}-1)}{\left(e^{j\Omega}+\frac{1}{2}\right)\left(e^{j2\Omega}-\frac{1}{4}e^{j\Omega}-\frac{1}{8}\right)}$$

$$=\frac{e^{j\Omega}(e^{j\Omega}-1)}{\left(e^{j\Omega}+\frac{1}{2}\right)\left(e^{j\Omega}-\frac{1}{2}\right)\left(e^{j\Omega}+\frac{1}{4}\right)}$$

$$=\frac{A}{e^{j\Omega}+\frac{1}{2}}+\frac{B}{e^{j\Omega}-\frac{1}{2}}+\frac{C}{e^{j\Omega}+\frac{1}{4}}$$

$A=-1$, $B=1$, $C=1$

$$\therefore Y(e^{j\Omega})=-1\left[\frac{e^{j\Omega}}{e^{j\Omega}-(-\frac{1}{2})}\right]+\left(\frac{e^{j\Omega}}{e^{j\Omega}-\frac{1}{2}}\right)+\left(\frac{e^{j\Omega}}{e^{j\Omega}-(-\frac{1}{4})}\right)$$

Take IDTFT

$$y(n)=-1\left(\frac{-1}{2}\right)^n u(n)+\left(\frac{1}{2}\right)^n u(n)+\left(\frac{-1}{4}\right)^n u(n)$$