Digital Signal Processing

Unit - I

Discrete Fourier Transform (DFT)

The Discrete Time Fourier Transform (DTFT) of discrete time signal $x(n)$ is continuous and is represented as

$$DTFT = X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$x(n) \rightarrow$ DT signal of ~~leng~~ finite length N

$X(e^{j\omega}) \rightarrow$ DTFT

$\rightarrow$ ***** continuous and periodic [Frequency domain]

The limitation of DTFT is that the digital computer can't be used for DTFT computation.

This limitation can be addressed by sampling to convert continuous into ~~dist~~ discrete.

Frequency domain Sampling and Reconstruction of DT signals

(or)

~~Consider DT signal x(n~~)

Definition of DFT and its inverse

Consider DT signal $x(n)$ of finite length 'N'

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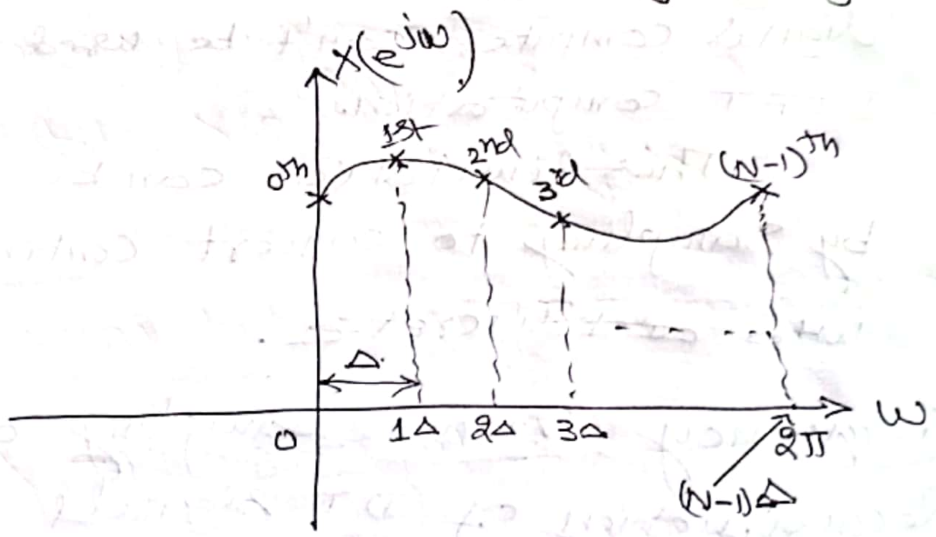
Convert Time Domain signal $x(n)$ into Frequen Domain signal $X(e^{jw})$ using DTFT

$$X(e^{jw}) = \sum_{n=0}^{N-1} x(n)\, e^{-jwn} \quad \text{—— (1)}$$

$X(e^{jw}) \rightarrow$ Frequency Domain & Continous

Frequency Domain sampling is used to convert continous frequency domain into discrete frequency domain leads to Discrete Fourier Transform (DFT)

Consider the waveform of $X(e^{jw})$



$N \rightarrow$ Number of frequency samples

$\Delta \rightarrow$ The space between successive samples

$$\Delta = \frac{2\pi}{N}$$

The discrete samples are as follows

$X(e^{j\omega}) \longrightarrow$ continous

0^{th} sample, ~~X~~ $\omega = 0$, $X(e^{j\omega}) = X(e^{j\times 0})$

1^{st} sample, $\omega = 1\Delta$, $X(e^{j\omega}) = X(e^{j\times 1\Delta})$

2^{nd} sample, $\omega = 2\Delta$, $X(e^{j\omega}) = X(e^{j\times 2\Delta})$

$\vdots$

$(N-1)^{th}$ sample, $\omega = (N-1)\Delta$, $X(e^{j\omega}) = X(e^{j\times (N-1)\Delta})$

In general $X(e^{j\omega}) = X(e^{j\times K\times\Delta})$

Where $K = 0$ to $(N-1)$

& $\Delta = \frac{2\pi}{N}$

$$\therefore X(e^{j\omega}) = X\left(e^{jK\cdot\frac{2\pi}{N}}\right)$$

$$\Rightarrow \omega = K\cdot\frac{2\pi}{N} \quad \text{——— (2)}$$

Substitute (2) in (1) to convert continous into discrete

$$X\left(e^{jK\cdot\frac{2\pi}{N}}\right) = \sum_{n=0}^{N-1} x(n)\, e^{-jK\cdot\frac{2\pi}{N}\cdot n}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}$$

$$\cancel{= \sum_{n=0}^{N-1} x(n)\, e^{-j}} \quad *****$$

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}} \longleftarrow \text{N-point DFT}$$

Where $W_N = e^{-j\frac{2\pi}{N}}$ = Twiddle Factor

= phase Factor

Inverse DFT : Reconstruction of DT signal from DFT

$$\boxed{x(n) = IDFT[X(k)] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}}$$

Synthesis equation   N-pt. IDFT

———— x ————

Relation between DTFT and DFT

DTFT: $X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$ —— ① Continuous Periodic

DFT: $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$ —— ② Discrete Periodic

on comparing ① & ②

$$\boxed{X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}}$$

Relation between ZT and DFT

ZT: $X(z) = \sum_{n=0}^{N-1} x(n)\, z^{n}$ —— ①

DFT: $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$ —— ②

Compare ① & ②

$$X(k) = X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}}$$

1. Determine DFT of the sequence $x(n) = [1, 1, 1, 1]$ for $N = 4$ & $N = 8$

Also plot magnitude and phase spectrum

$\Rightarrow$ For $N=4$, $X(K) = ?$ for $K = 0, 1, 2, 3$

$$X(K) = \sum_{n=0}^{3} x(n) W_N^{Kn} = x(0) + x(1) W_N^{K\cdot 1} + x(2) W_N^{K\cdot 2} + x(3) W_N^{K\cdot 3}$$

~~$K=0$, $X(0)=$~~

$$\boxed{X(K) = 1 + 1\cdot W_N^{K} + 1\cdot W_N^{2K} + 1\cdot W_N^{3K}}$$

$K=0$, $X(0) = 1+1+1+1 = 4$

$K=1$, $X(1) = 1 + W_N^1 + W_N^2 + W_N^3 = 1-j-1+j = 0$

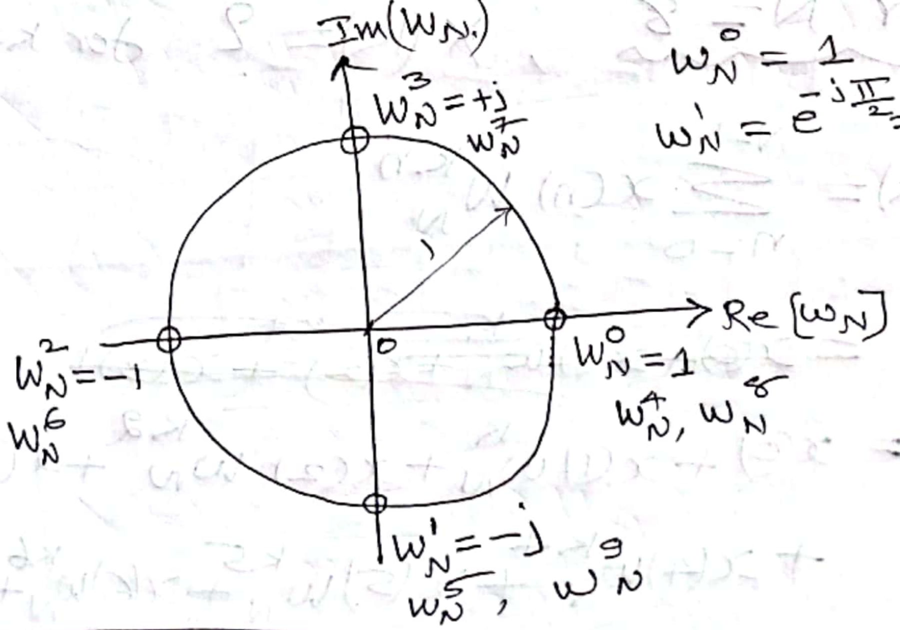
$K=2$, $X(2) = 1 + W_N^2 + W_N^4 + W_N^6 = 1-1+1-1 = 0$

$K=3$, $X(3) = 1 + W_N^3 + W_N^6 + W_N^9 = 1+j-1-j = 0$

~~$W_N = e^{-j2\pi/N} = e^{-j2\pi/4} = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$~~

Twiddle Factor Diagram

$$W_N = 1e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$



$W_N^0 = 1$

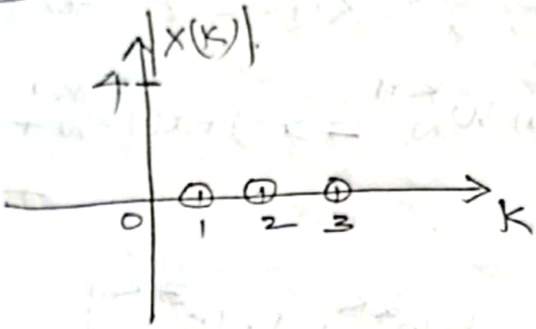
$W_N^1 = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$

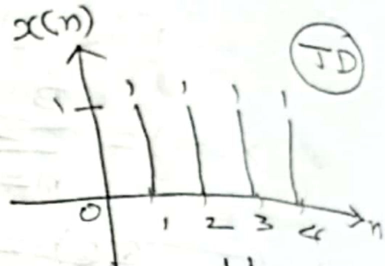
$= 0 - j$

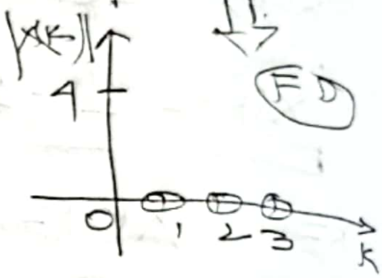
$W_N \rightarrow$ Periodic

$$X(k) = [4, 0, 0, 0]$$

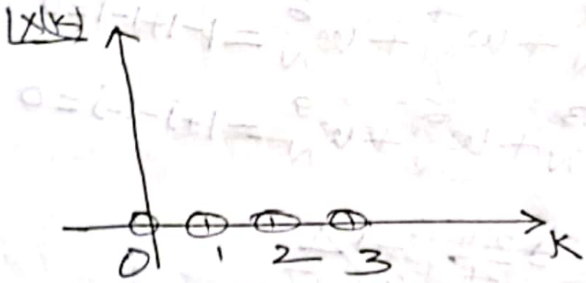
magnitude spectrum $|X(k)| = \sqrt{(\text{real})^2 + (\text{img})^2}$







Phase spectrum $\angle X(k) = \tan^{-1}\left[\frac{\text{img}}{\text{real}}\right]$



For $N = 8$, $X(k) = 2$ for $k = 0$ to 7

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$\cancel{= x(0) + x(1) W_N^{k} + x(2)^{k2} + x(3) +}$$

$$= x(0) + x(1) W_N^{k} + x(2) W_N^{k2} + x(3) W_N^{k3}$$

$$+ x(4) W_N^{k4} + x(5) W_N^{k5} + x(6) W_N^{k6} + x(7) W_N^{k7}$$

$$X(k) = x(0) + x(1)W_N^{k} + x(2)W_N^{2k} + x(3)W_N^{3k} + x(4)W_N^{4k}$$
$$+ x(5)W_N^{5k} + x(6)W_N^{6k} + x(7)W_N^{7k}$$

$$k=0,\ X(0) = x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6)$$
$$+ x(7)$$
$$= 1+1+1+1+0+0+0+0 = 4.$$

$k=1,\ X(1) = 1 + W_N^1 + W_N^2 + W_N^3 = 1 + 0.707 - j0.707 - j - 0.707 - j0.707$

$k=2,\ X(2) = 1 + W_N^2 + W_N^4 + W_N^6 = 1 - j - 0.707 + j0.707 + j$

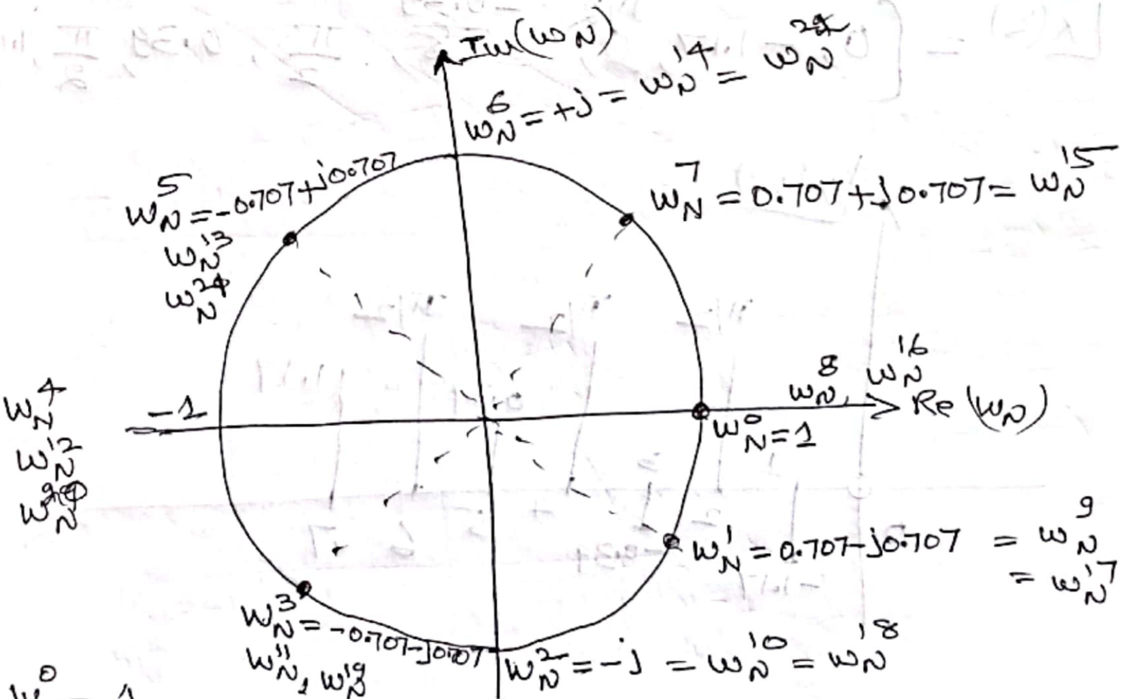
$k=3,\ X(3) = 1 + W_N^3 + W_N^6 + W_N^9 = 1 - 0.707 - j0.707 + j + 0.707 - j0.707$

$k=4,\ X(4) = 1 + W_N^4 + W_N^8 + W_N^{12} = 1 - 1 + 1 - 1 = 0$

$k=5,\ X(5) = 1 + W_N^5 + W_N^{10} + W_N^{15} = 1 - 0.707 + j0.707 - j + 0.707 + j0.707$

$k=6,\ X(6) = 1 + W_N^6 + W_N^{12} + W_N^{18} = 1 + j - 1 - j = 0$

$k=7,\ X(7) = 1 + W_N^7 + W_N^{14} + W_N^{21} = 1 + 0.707 + j0.707 + j - 0.707 + j0.707.$



$W_N^0 = 1$

$W_N^1 = e^{-j\frac{2\pi}{8}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$

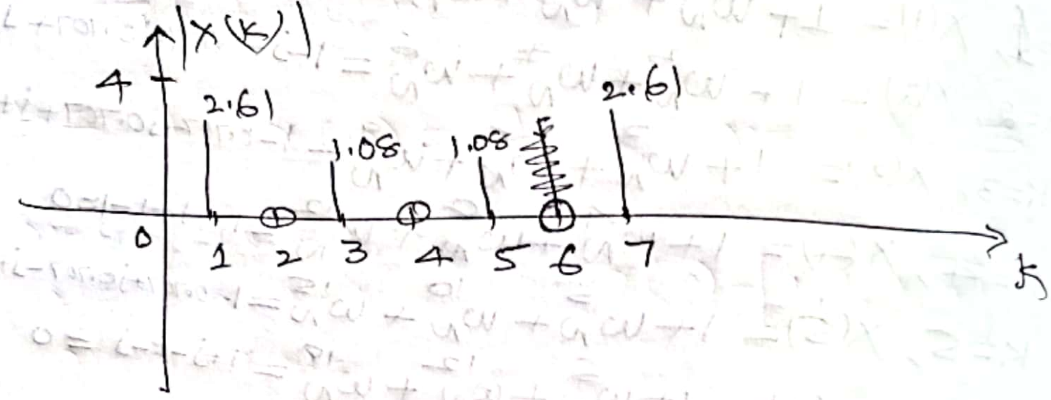
$W_N^2 = e^{-j\frac{2\times 2\pi}{8}} = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = 0 - j$

$W_N^3 = e^{-j\frac{3\times 2\pi}{8}} = \cos\frac{6\pi}{8} - j\sin\frac{6\pi}{8} = -0.707 - j0.707$

$$X(K) = [4, (1-j2.414), 0, (1-j0.414), 0, (1+j0.414), 0, (1+j2.414)]$$

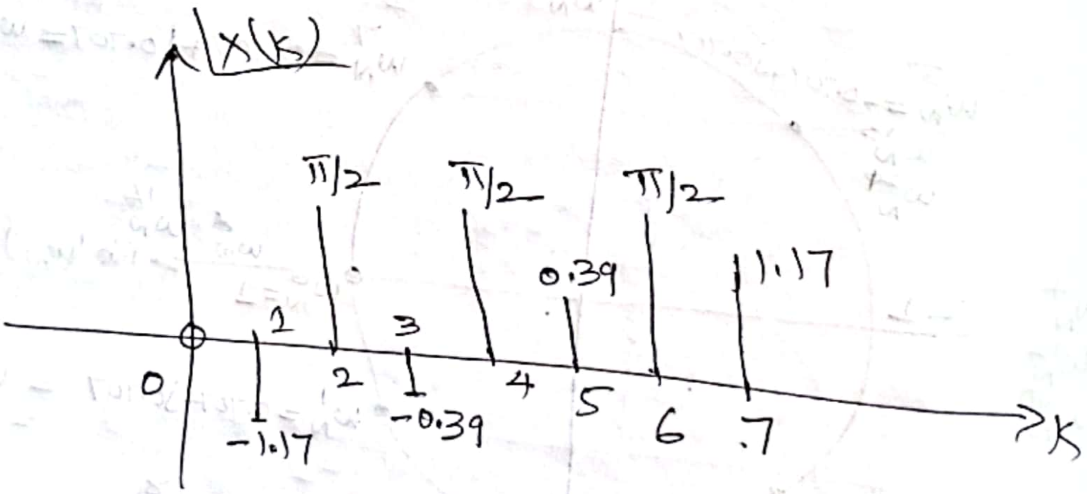
magnitude Spectrum $|X(K)| = \sqrt{r^2+i^2}$

$$|X(K)| = [4, 2.61, 0, 1.08, 0, 1.08, 0, 2.61]$$



Phase spectrum $\angle X(K) = \tan^{-1}\left(\frac{i}{r}\right)$.

$$\angle X(K) = [0, -1.17, \frac{\pi}{2}, -0.39, \frac{\pi}{2}, 0.39, \frac{\pi}{2}, 1.17]$$



2. Find DFT of the sequence

$x(n) = \frac{1}{3}$ for $0 \le n \le 2$

for $N = 4$ and $N = 8$

Also plot magnitude and phase spectrum

= For N = 4

$X(K) = \left[1, -\frac{j}{3}, \frac{1}{3}, \frac{j}{3}\right]$

= For N = 8 :-

$X(K) = [1, (0.57 - j0.57), \cancel{0.81}\ {-j/3}, (0.1 + j0.1), \frac{1}{3},$

$(0.1 - j0.1), \frac{j}{3}, (0.57 + j0.57)]$

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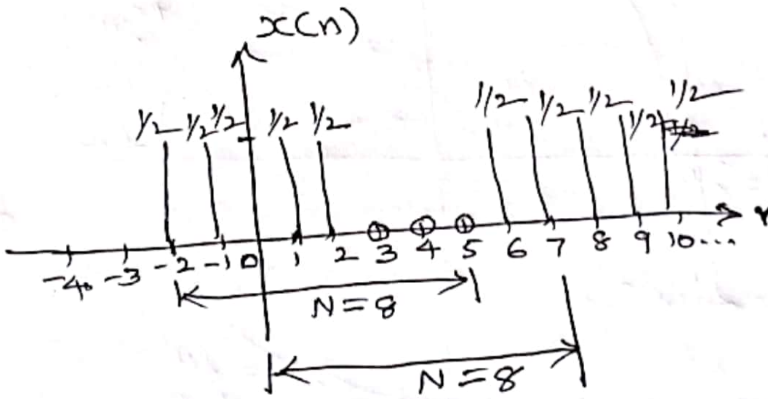
3. Determine DFT of the sequence

$$x(n) = \begin{cases} \frac{1}{2}, & -2 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

for $N = 8$

= In DFT always $x(n)$ & $X(K)$ are periodic.



$x(n) = \left[\underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2}\right]$

Ans :-

$X(K) =$

(4) Find IDFT of $X(k) = [4, 0, 0, 0]$

$$= \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$= \frac{1}{4} \sum_{k=0}^{3} X(k) W_N^{-kn}$$

$$= \frac{1}{4}\left[X(0) + X(1) W_N^{-n} + X(2) W_N^{-2n} + X(3) W_N^{-3n}\right]$$

$$= \frac{1}{4}\left[4 + 0 + 0 + 0\right]$$

$\cancel{X(k) = 1}$ $x(n) = 1$, $n = 0$ to 3

$n=0$, $x(0) = 1$,

$n=1$, $x(1) = 1$

$n=2$, $x(2) = 1$

$n=3$, $x(3) = 1$

$\therefore x(n) = [1, 1, 1, 1]$

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(5) Find DFT of $x(n) = \delta(n)$

$$= \quad X(K) = \sum_{n=0}^{N-1} \delta(n)\, W_N^{kn} = \sum_{n=0} \delta(n)\, W_N^{kn}$$

$$X(K) = \delta(0) \cdot W_N^{K \cdot 0} = 1$$

(6) Find DFT of $x(n) = \delta(n-n_0)$

$$= \quad X(K) = \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{kn}$$

$$= \sum_{n=n_0} \delta(n-n_0)\, W_N^{kn}$$

$$= \delta(0) \cdot W_N^{Kn_0} = W_N^{Kn_0}$$

$$= e^{-j\frac{2\pi}{N} \cdot Kn_0}$$

(7) Find N-point DFT of $x(n) = a^n$

$$= \quad X(K) = \sum_{n=0}^{N-1} a^n \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} (a \cdot W_N^{K})^n$$

$$= \frac{1 - (aW_N^K)^{N-1+1}}{1 - a \cdot W_N^K}$$

$$= \frac{1 - a^N \cdot W_N^{KN}}{1 - a \cdot W_N^K}$$

$$= \frac{1 - a^N \cdot e^{-j\frac{2\pi}{\cancel{N}} \cdot K\cancel{N}}}{1 - a \cdot W_N^K}$$

$$X(K) = \frac{1 - a^N}{1 - a \cdot W_N^K}$$

$$\sum_{n=0}^{N-1} (a)^n = \frac{1 - a^{N-1+1}}{1-a}, \quad a \neq 1$$
$$= N-1+1, \quad a = 1$$

$$\sum_{n=\square}^{N-1} (a)^n = \frac{a^{\square} - a^{N-1+1}}{1-a}, \quad a \neq 1$$
$$= N-1+1-\square, \quad a = 1$$

(5) Find DFT of $x(n) = \delta(n)$

$$= \quad X(K) = \sum_{n=0}^{N-1} \delta(n)\, W_N^{kn} = \sum_{n=0} \delta(n)\, W_N^{kn}$$

$$X(K) = \delta(0) \cdot W_N^{K\cdot 0} = 1$$

(6) Find DFT of $x(n) = \delta(n-n_0)$

$$= \quad X(K) = \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{kn}$$

$$= \sum_{n=n_0} \delta(n-n_0)\, W_N^{kn}$$

$$= \delta(0) \cdot W_N^{Kn_0} = W_N^{Kn_0}$$

$$= e^{-j\frac{2\pi}{N} \cdot Kn_0}$$

(7) Find N-point DFT of $x(n) = a^n$

$$= \quad X(K) = \sum_{n=0}^{N-1} a^n \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left(a \cdot W_N^{K}\right)^n$$

$$= \frac{1 - \left(a W_N^{K}\right)^{N-1+1}}{1 - a \cdot W_N^{K}}$$

$$= \frac{1 - a^N \cdot W_N^{KN}}{1 - a \cdot W_N^{K}}$$

$$= \frac{1 - a^N \cdot e^{-j\frac{2\pi}{\cancel{N}} \cdot K\cancel{N}}}{1 - a \cdot W_N^{K}}$$

$$X(K) = \frac{1 - a^N}{1 - a \cdot W_N^{K}}$$

$$\sum_{n=0}^{N-1} (a)^n = \frac{1 - a^{N-1+1}}{1-a}, \quad a \neq 1$$
$$= N-1+1, \quad a = 1$$

$$\sum_{n=\square}^{N-1} (a)^n = \frac{a^{\square} - a^{N-1+1}}{1-a}, \quad a \neq 1$$
$$= N-1+1-\square, \quad a = 1$$

(8) Find N-point DFT of

$$x(n) = e^{j\frac{2\pi}{N}n}, \quad 0 \le n \le N-1$$

$$= \quad X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} \left[e^{-j\frac{2\pi}{N}(k-1)}\right]^n$$

$$X(k) = \sum_{n=0}^{N-1} \left[W_N^{(k-1)}\right]^n$$

Case(i) :- For $W_N^{(k-1)} = 1 \Rightarrow$ for $k=1$, $W_N^{(1-1)} = 1$

$$X(k) = \sum_{n=0}^{N-1} (1)^n = N-1+1 = N$$

Case(ii) :- For $W_N^{(k-1)} \neq 1 \Rightarrow k \neq 1$

$$X(k) = \sum_{n=0}^{N-1} \left[W_N^{(k-1)}\right]^n$$

$$= \frac{1-\left(W_N^{(k-1)}\right)^{N-1+1}}{1-W_N^{(k-1)}} = \frac{1-W_N^{kN} \cdot W_N^{N}}{1-W_N^{(k-1)}}$$

$$X(k) = \frac{1-e^{-j\frac{2\pi}{\cancel{N}}k\cancel{N}} \cdot e^{-j\frac{2\pi}{\cancel{N}}\cancel{N}}}{1-W_N^{(k-1)}} = \frac{1-1}{1-W_N^{(k-1)}}$$

$$X(k) = 0$$

$$\therefore X(k) = \begin{cases} N, & \text{for } k=1 \\ 0, & \text{for } k \neq 1 \end{cases}$$

$$\boxed{X(k) = N \cdot \delta(k-1)}$$

(9) Find N-point DFT of $x(n) = e^{j\frac{2\pi}{N}mn}$

$$= X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mn} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} \left[e^{-j\frac{2\pi}{N}(k-m)}\right]^n = \sum_{n=0}^{N-1} \left[W_N^{(k-m)}\right]^n$$

$$\boxed{X(k) = N \cdot \delta(k-m)}$$

(10) Find N-point DFT of $x(n) = 1$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{N-1} 1 \cdot W_N^{kn}$$

$$X(k) = \sum_{n=0}^{N-1} \left[W_N^{k}\right]^n$$

$$\boxed{X(k) = N \cdot \delta(k)}$$

(11) Find N-point DFT of $x(n) = \cos\left(\frac{2\pi}{N} mn\right)$

$$= \quad x(n) = \frac{e^{j\frac{2\pi}{N}mn} + e^{-j\frac{2\pi}{N}mn}}{2}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} = \sum_{n=0}^{N-1} \left[\frac{e^{j\frac{2\pi}{N}mn} + e^{-j\frac{2\pi}{N}mn}}{2}\right] W_N^{kn}$$

$$X(k) = \frac{1}{2}\left\{\sum_{n=0}^{N-1} e^{+j\frac{2\pi}{N}mn} \cdot e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} \cdot e^{-j\frac{2\pi}{N}kn}\right\}$$

$$X(k) = \frac{1}{2}\left\{\sum_{n=0}^{N-1} \left[W_N^{(k-m)}\right]^n + \sum_{n=0}^{N-1} \left[W_N^{(k+m)}\right]^n\right\}$$

$$X(k) = \frac{1}{2}\left\{N \cdot \delta(k-m) + N \cdot \delta(k+m)\right\}$$

12 $x(n) = \sin\left(\frac{2\pi}{N} mn\right)$

$$= \quad X(K) = \frac{1}{2j}\left[N \cdot \delta(k-m) - N \cdot \delta(k+m)\right]$$

13 Find 8-point DFT of a sequence

$x(n) = (-1)^{n+1}, \quad 0 \leq n \leq 7$

Also plot the spectrum

$$= \quad X(k) = [0, 0, 0, 0, -8, 0, 0, 0]$$

Properties of DFT :-

1. Linearity :-

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

& $x_2(n) \xleftrightarrow{DFT} X_2(K)$

Then $[a x_1(n) + b x_2(n)] \xleftrightarrow{DFT} [a X_1(K) + b X_2(K)]$

Superposition Principle.

Proof :- $LHS = [a x_1(n) + b x_2(n)]$

Take DFT

$$= \sum_{n=0}^{N-1} (LHS) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] W_N^{kn}$$

$$= a \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$$

$$= a \cdot X_1(K) + b X_2(K) = RHS$$

2. Periodicity :-

$x(n)$ periodic $\longrightarrow$ $x(n) = x(n+N)$

$X(K)$ Periodic $\longrightarrow$ $X(K) = X(K+N)$

Proof: Consider IDFT $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-kn}$.

Replace n by $(n+N)$

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot W_N^{-K(n+N)}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-kn} \cdot \underbrace{W_N^{-KN}}_{1}$$

$$\boxed{x(n+N) = x(n)}$$

consider $X(K) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{Kn}$

Replace K by $(K+N)$,

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(K+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{Kn} \cdot W_N^{Nn} \rightarrow 1$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

$$\boxed{X(K+N) = X(K)}$$

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3. Circular Time Shift

If $x(n) \xleftrightarrow{DFT} X(K)$

Then $x((n-n_0))_N \xleftrightarrow{DFT} W_N^{n_0 K}\, X(K)$

= Circular shift

If $N=4$, $x((n-2))_4 = x(n-2, \text{modulo } 4)$ ~~Mod~~

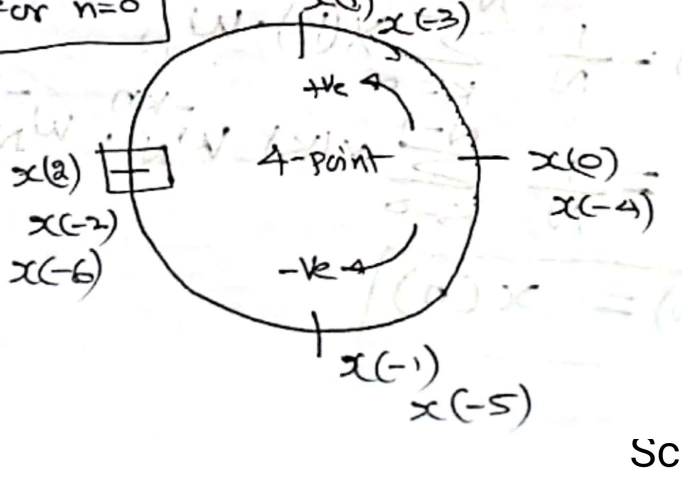
$= x(n-2) = x(n-2+4) = x(n+2)$

$= x(n-2-4) =$ ~~x(n~~ $x(n-6)$

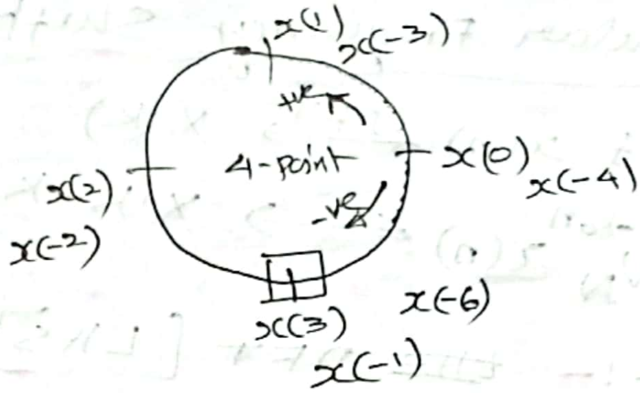
$n = 0,1,2,3$: If $n=0$, $x(-2) = x(2) = x(-6)$

If $n=1$, $x(-1) = x(3) = x(-5)$

For $n=0$



$\boxed{\text{For } n=1}$



Proof:- $LHS = x((n-n_0))_N = x(n-n_0)$

$$DFT[LHS] = \sum_{n=0}^{N-1} (LHS) \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n-n_0) \cdot W_N^{kn} \quad \text{——— (1)}$$

Let $n - n_0 = P$, $n = P + n_0$ — (2)

Lower limit:- $n=0$

$P + n_0 = 0$

$P = -n_0 + 0$

$\boxed{P=0}$

Upper limit:- $n = N-1$

$P + n_0 = N-1$

$P = -n_0 + (N-1)$

$\boxed{P = N-1}$

Substitute (2) in (1)

$$= \sum_{P=0}^{N-1} x(P)\, W_N^{k(P+n_0)}$$

$$= W_N^{kn_0} \cdot \sum_{P=0}^{N-1} x(P)\, W_N^{kP}$$

$$= W_N^{kn_0} \cdot X(k) = RHS$$

4. Circular Frequency Shift

If $x(n) \xleftrightarrow{DFT} X(k)$

$W_N^{-k_0 n} x(n) \longleftrightarrow X(k-k_0)$

= Proof!- ~~LHS~~ DFT [LHS]

$$= \sum_{n=0}^{N-1} (LHS)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} W_N^{-k_0 n} x(n) \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{(k-k_0)n} \quad \text{——— (1)}$$

W.K.T

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{——— (2)}$$

Compare (1) & (2)

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{(k-k_0)n} = X(k-k_0)$$

$= RHS$

5. Time Reversal!-

If $x(n) \xleftrightarrow{DFT} X(k)$

then $x(-n) \longleftrightarrow X(-k)$

= Proof!- DFT [LHS]

$$= \sum_{n=0}^{N-1} (LHS) \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x(-n) \cdot W_N^{kn} \quad \text{——— (1)}$$

Let $-n = n \Rightarrow n = -n$

Lower limit $n = 0$ Upper limit $n = N-1$

$-n = 0$ $-n = N-1$

$\cancel{\boxed{n=0}}$ $n = -(N-1)$

$n = -0$ $\boxed{n = N-1}$

$\boxed{n=0}$ — (2)

Substitute (2) in (1)

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{(-k)n} = X(-k)$$

$$= RHS$$

6. Circular convolution

$$x(n) = \left[x_1(n) \, (N) \, x_2(n)\right] \xleftrightarrow{DFT} X(k) = X_1(k) \cdot X_2(k)$$

Proof :- $\boxed{LHS = x(n) = \sum_{P=0}^{N-1} x_1(P)\, x_2(n-P) \quad — (1)}$

W.K.T. $X_1(k) = \sum_{\cancel{n}=0}^{N-1} x_1(\cancel{n}) \cdot W_N^{k\cancel{n}}$

$$X_2(k) = \sum_{m=0}^{N-1} x_2(m)\, W_N^{km}$$

$$RHS = X(k) = X_1(k) \cdot X_2(k) = \sum_{n=0}^{N-1} x_1(\cancel{n}) \sum_{m=0}^{N-1} x_2(m)\, W_N^{k\cancel{n}} \cdot W_N^{km}$$

$$IDFT\left[X(k)\right] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k) \cdot W_N^{-kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\cancel{v}=0}^{N-1} x_1(v) \cdot W_N^{kv} \cdot \sum_{m=0}^{N-1} x_2(m) W_N^{km} \cdot W_N^{-kn}$$

$$= \frac{1}{N} \sum_{V=0}^{N-1} x_1(V) \sum_{m=0}^{N-1} x_2(m) \sum_{k=0}^{N-1} W_N^{kV} \cdot W_N^{km} \cdot W_N^{-kn}$$

$$= \frac{1}{N} \sum_{v=0}^{N-1} x_1(v) \cdot \sum_{m=0}^{N-1} x_2(m) \cdot \sum_{k=0}^{N-1} W_N^{k(v+m-n)} \quad \text{——} (2)$$

Consider $\sum_{k=0}^{N-1} \left[W_N^{(v+m-n)}\right]^k$

If $W_N^{(v+m-n)} = 1$ ie $v+m-n=0$, $\boxed{v = n-m}$

Then $\sum_{k=0}^{N-1} (1)^k = N-1+1 = N$

If $W_N^{(v+m-n)} \neq 1$, ie $v \neq n-m$

Then $\sum_{k=0}^{N-1} \left[W_N^{(v+m-n)}\right]^k = \dfrac{1-\left[W_N^{(v+m-n)}\right]^{N-1+1}}{1-W_N^{(v+m-n)}}$

$= 0$

$$\therefore \sum_{k=0}^{N-1} \left[W_N^{(v+m-n)}\right]^k = \begin{cases} N, & v = n-m \\ 0, & v \neq n-m \end{cases} \quad \text{——} (3)$$

Substitute (3) in (2)

$$= \frac{1}{N} \sum_{v=0}^{N-1} x_1(v) \cdot \sum_{m=0}^{N-1} x_2(m) \cdot N$$

for $v = n-m$

$$\therefore = \frac{1}{N} \sum_{m=0}^{N-1} x_1(n-m) \sum_{m=0}^{N-1} x_2(m) \cdot N$$

$$= \sum_{m=0}^{N-1} x_2(m) \cdot x_1(n-m)$$

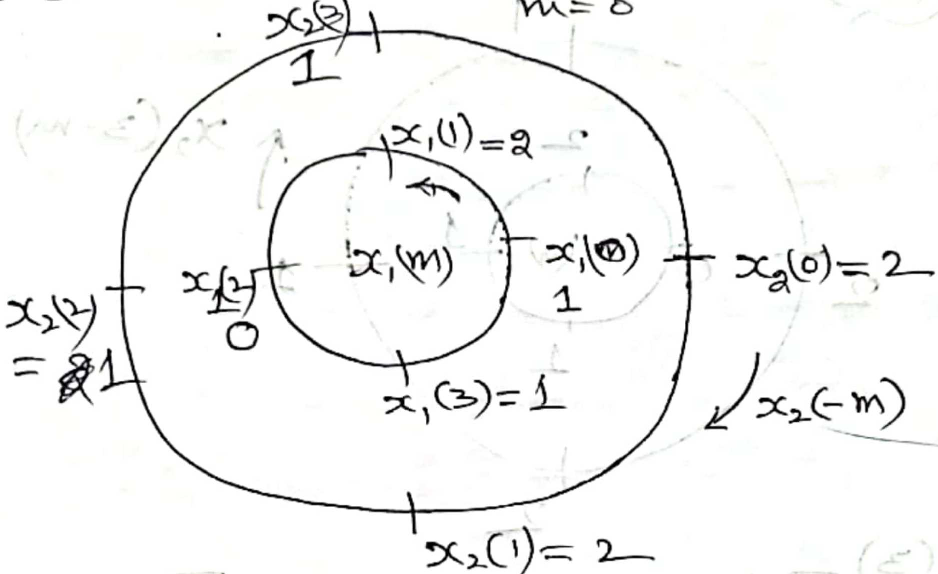
$$= x_2(n) \; (N) \; x_1(n) = LHS$$

1. Find circular convolution of two sequences $x_1(n) = [\underset{\uparrow}{1}, 2, 0, 1]$, and $x_2(n) = [\underset{\uparrow}{2}, 2, 1, 1]$ using (i) circle method (ii) Matrix method (iii) DFT and IDFT

= (i) <u>circle Method</u> :-

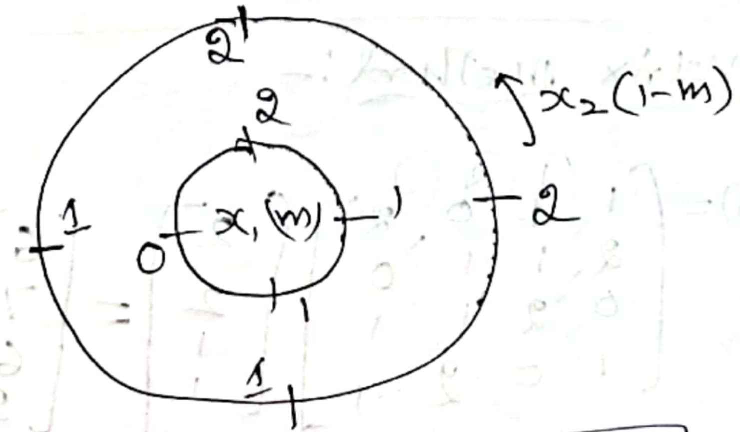
$$x(n) = x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(m) x_2(n-m)$$

For n=0, $x(0) = \sum_{m=0}^{3} x_1(m) \cdot x_2(0-m)$



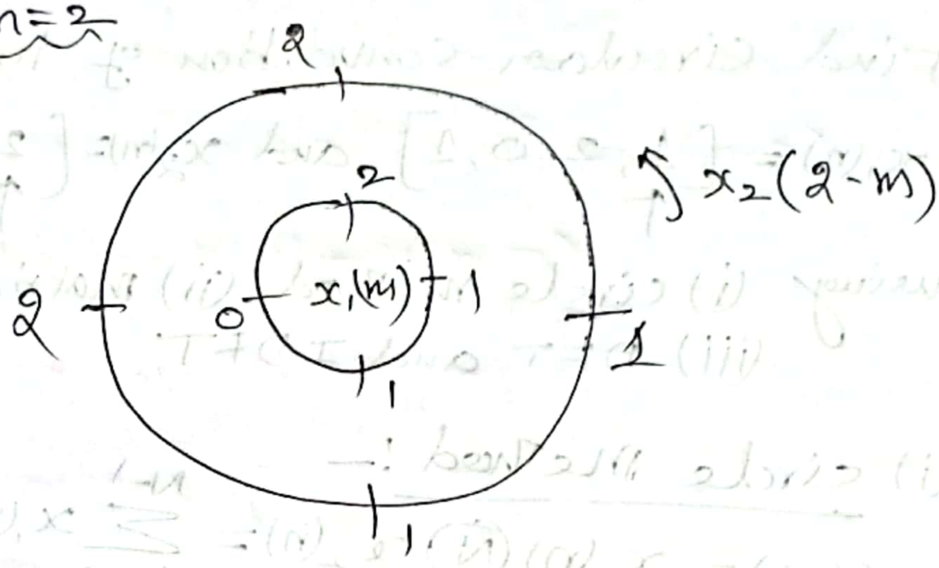
$$\boxed{x(0) = 2+2+0+2 = 6}$$

For n=1 :- $x(1) = \sum_{m=0}^{3} x_1(m) x_2(1-m)$



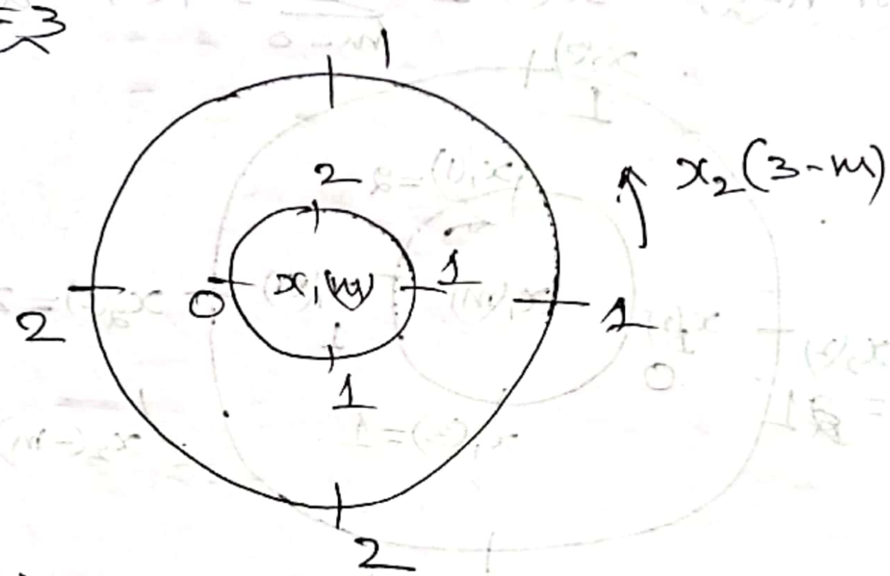
$$\boxed{x(1) = 2+4+0+1 = 7}$$

For $n=2$



$$x(2) = 1+4+0+1 = 6$$

For $n=3$



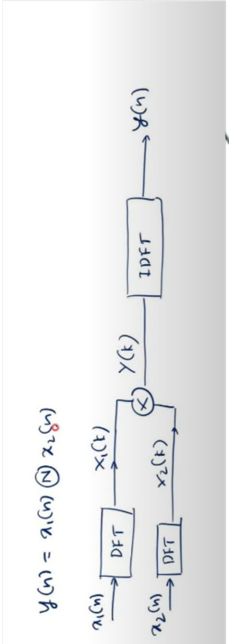
$$x(3) = 1+2+0+2 = 5$$

$$x(n) = [6, 7, 6, 5]$$

(ii) Matrix Method :-

$$x(n) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

$$\boxed{x(n) = [6, 7, 6, 5]}$$



iii) DFT and IDFT method

$$x(n) = [x_1(n) \, (N) \, x_2(n)] \longleftrightarrow X(k) = X_1(k) \cdot X_2(k)$$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn} = x_1(0) + x_1(1) W_N^{k} + x_1(2) W_N^{k2} + x_1(3) W_N^{k3}$$

$$\boxed{X_1(k) = 1 + 2 W_N^{k} + 0 + 1 W_N^{k3}}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) W_N^{kn} = x_2(0) + x_2(1) W_N^{k} + x_2(2) W_N^{k2} + x_2(3) W_N^{k3}$$

$$\boxed{X_2(k) = 2 + 2 W_N^{k} + W_N^{k2} + W_N^{k3}}$$

$$X(k) = X_1(k) \cdot X_2(k) = \left(1 + 2 W_N^{k} + W_N^{k3}\right)\left(2 + 2 W_N^{k} + W_N^{k2} + W_N^{k3}\right)$$

$$= 2 + 2 W_N^{k} + W_N^{k2} + W_N^{k3} + 4 W_N^{k} + 4 W_N^{2k} + 2 W_N^{k3} + 2 W_N^{4k} + 2 W_N^{k3} + 2 W_N^{k4} + W_N^{k5} + W_N^{k6}$$

$$= 2 + 6 W_N^{k} + 5 W_N^{k2} + 5 W_N^{k3} + 4 W_N^{k4} + W_N^{k5} + W_N^{k6}$$

$$= 2 + 6 W_N^{k} + 5 W_N^{k2} + 5 W_N^{k3} + 4 + W_N^{k1} + W_N^{k2}$$

$$X(k) = 6 + 7 W_N^{k} + 6 W_N^{k2} + 5 W_N^{k3}$$

Take IDFT

$$x(n) = 6\delta(n) + 7\delta(n-1) + 6\delta(n-2) + 5\delta(n-3)$$

$$x(n) = [6, 7, 6, 5]$$

2) Find the circular convolution of

$x_1(n) = \delta(n) + 3\delta(n-1) + 5\delta(n-3)$

$x_2(n) = u(n) - u(n-3)$

$\Rightarrow$ $x_1(n) = [1, 3, 0, 5]$

$x_2(n) = [1, 1, 1] \Longrightarrow [1, 1, 1, 0]$

$$x(n) = \begin{bmatrix} 1 & 5 & 0 & 3 \\ 3 & 1 & 5 & 0 \\ 0 & 3 & 1 & 5 \\ 5 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 4 \\ 8 \end{bmatrix}$$

$x(n) = [6, 9, 4, 8]$

3) Given $X(K) = [0, (1+j), 1, (1-j)]$ is the DFT of finite sequence $x(n)$. Find DFT of (i) $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

(ii) $x_2(n) = \cos\left(\frac{\pi}{2}n\right) x(n)$

(iii) $x_3(n) = x((n-1))_4$

$\Rightarrow$ (i) $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$ N=4

$= e^{j\frac{2\pi}{4}n} x(n)$

$x_1(n) = W_N^{-n} x(n)$

$X_1(K) = ?$

Using Circular Frequency Shift property

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$x_1(n) = W_N^{-n} x(n) \xleftrightarrow{DFT} X(k-1) = X_1(k)$$

$\therefore X_1(k) = \text{DFT of } x_1(n) = X(k-1)$

~~$X_1(k)$~~ For $k=0$, $X_1(0) = X(-1) = X(-1+4) = X(3)$

$X_1(0) = X(3) = (1-j)$

For $k=1$, $X_1(1) = X(1-1) = X(0) = 0$

For $k=2$, $X_1(2) = X(1) = (1+j)$

For $k=3$, $X_1(3) = X(2) = 1$

$\therefore X_1(k) = \{(1-j), 0, (1+j), 1]$

(ii)

4. Let $G(K)$ and $H(K)$ denote 6 point DFT's of sequences $g(n)$ and $h(n)$ respectively

(i) Determine $H(K)$ for

$$G(K) = [(1+j), (-2.1+j3.2), (-1.2-j2.4), 0, (0.9+j3.1), (-0.3+j1.1)]$$

and $h(n) = g((n-4))_6$

(ii) Determine $h(n)$ for

$$g(n) = [4.1, 3.5, 1.2, 5, 2, 3.3]$$

and $H(K) = G((K-3))_6$

= (i) Use circular time shift property

If $g(n) \xleftrightarrow{DFT} G(K)$

(N=6) $h(n) = g(n-4) \longleftrightarrow W_N^{4K} G(K) = H(K)$

$$\therefore H(K) = W_N^{4K} G(K) = e^{-j\frac{2\pi}{N}.4K} G(K)$$

$$= e^{-j\frac{2\pi}{\cancel{6}_3}4K} \cdot G(K)$$

$$\boxed{H(K) = e^{-j\frac{4\pi K}{3}} G(K)}$$

$H(K) = ?$ $K=0$ to 5

$K=0$, $\boxed{H(0) = 1 \cdot G(0) = (1+j)}$

$K=1$, $H(1) = e^{-j\frac{4\pi}{3}1} \cdot G(1) = (\cos(\frac{4\pi}{3}) - j\sin(\frac{4\pi}{3}))(G(1))$

$$= (-0.5+j0.866)(-2.1+j3.2)$$

$$\boxed{H(1) = -1.7212 - j3.41}$$

$k=2$, $H(2) = e^{-j\frac{4\pi}{3}2}\, G(2)$

$= \left[\cos\left(\frac{8\pi}{3}\right) - j\sin\left(\frac{8\pi}{3}\right)\right]\left[-1.2 - j2.4\right]$

$\boxed{H(2) = -1.4783 + j\,2.239}$

$k=3$, $H(3) = e^{-j\frac{4\pi}{3}.3}\, G(3)$

$\boxed{H(3) = \left[\cos(4\pi) - j\sin 4\pi\right]\left[0\right] = 0}$

$k=4$, $H(4) = e^{-j\frac{4\pi}{3}4}\, G(4)$

$= \left[\cos\left(\frac{16\pi}{3}\right) - j\sin\left(\frac{16\pi}{3}\right)\right]\left[0.9 + j3.1\right]$

$\boxed{H(4) = 2.23 - j\,2.3294}$

$k=5$, $H(5) = e^{-j\frac{4\pi}{3}.5}\, G(5)$

$= \left[\cos\left(\frac{20\pi}{3}\right) - j\sin\left(\frac{20\pi}{3}\right)\right]\left[-0.3 + j1.1\right]$

$\boxed{H(5) = -0.8025 - j\,0.8096}$

$\therefore H(k) = \left\{(1+j), (-1.7213 - j3.41), (-1.4783 + j\,2.239), 0, (2.23 - j\,2.3294), (-0.8025 - j0.8096)\right\}$

(ii) Use Frequency Shift Property

If $g(n) \xleftrightarrow{DFT} G(k)$

$h(n) = W_N^{-3n}\, g(n) \longleftrightarrow G((k-3))_6 = H(k)$

$\therefore \boxed{h(n) = W_N^{-3n}\, g(n)}$

$h(n) = ?$

$n = 0 \text{ to } 5$

$h(n) = e^{-j\frac{2\pi}{6}(-3n)}\, g(n) = e^{j\pi n}\, g(n)$

$\boxed{h(n) = (-1)^n\, g(n)}$

$n=0, \quad h(0) = 1 \cdot g(0) = 4.1$

$n=1, \quad h(1) = -1 \; g(1) = -3.5$

$n=2, \quad h(2) = 1 \; g(2) = 1.2$

$n=3, \quad h(3) = -1 \; g(3) = -5$

$n=4, \quad h(4) = 1 \cdot g(4) = 2$

$n=5, \quad h(5) = -1 \; g(5) = -3.3$

$\therefore h(n) = [4.1, -3.5, 1.2, -5, 2, -3.3]$

---

(5) Let $x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3)$
Find sequence $y(n)$ whose 5-point DFT is $Y(K) = [X(K)]^2$, where $X(K)$ is the 5 point DFT of $x(n)$

$X(K) = ? \implies Y(K) = ? \implies y(n) = ?$

$x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3)$

$X(K) = 2 \cdot DFT[\delta(n)] + DFT[\delta(n-1)] + DFT[\delta(n-3)]$

$$\boxed{X(K) = 2 \times 1 + W_N^K + W_N^{3K}}$$

$Y(K) = [X(K)]^2 = \left[2 + W_N^K + W_N^{3K}\right]^2$

$Y(K) = 4 + W_N^{2K} + W_N^{6K} + 2(2W_N^K + W_N^{4K} + 2W_N^{3K})$

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$y(K) = 4+4W_N^K + W_N^{2K} + 4W_N^{3K} + 2W_N^{4K} + W_N^{6K}$$

$$= 4+4W_N^K + W_N^{2K} + 4W_N^{3K} + 2W_N^{4K} + W_N^{5K} \cdot W_N^K$$

$$y(K) = 4+5W_N^K + W_N^{2K} + 4W_N^{3K} + 2W_N^{4K}$$

Take IDFT

$$y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4)$$

$$\boxed{y(n) = [4, 5, 1, 4, 2]}$$

---

(6) consider the finite length sequence $x(n) = \delta(n) + 2\delta(n-5)$, Find

(i) the 10 point DFT of $x(n)$

(ii) the ~~DFT~~ 10 point sequence $y(n)$, that has a DFT $Y(K) = X(K) \cdot W(K)$, where $X(K)$ is the 10 point DFT of $x(n)$ & $W(K)$ — " — of $w(n)$ where $w(n) = u(n) - u(n-7)$

= (i) $x(n) = \delta(n) - 2\delta(n-5)$

$\therefore x(n) = [1, 0, 0, 0, 0, +2, 0, 0, 0, 0]$

$$X(K) = \sum_{n=0}^{9} x(n) W_N^{kn} = 1 + 2 \cdot W_N^{K5}$$

$$= 1 + 2 \cdot e^{-j\frac{2\pi}{10}5K} = 1 + 2 \cdot e^{-j\pi K}$$

$$\boxed{X(K) = 1 + 2(-1)^K}$$

$k=0, \quad X(0) = 1 + 2(-1)^0 = \cancel{-1}\ 3$

$k=1, \quad X(1) = \cancel{3}\ \ -1$

$k=2, \quad X(2) = \cancel{-1}\ \ 3$

$k=3, \quad X(3) = \cancel{3}\ \ -1$

$k=4, \quad X(4) = \cancel{-1}\ \ 3$

$k=5, \quad X(5) = \cancel{3}\ \ -1$

$k=6, \quad X(6) = \cancel{-1}\ \ 3$

$k=7, \quad X(7) = \cancel{3}\ \ -1$

$k=8, \quad X(8) = \cancel{-1}\ \ 3$

$k=9, \quad X(9) = \cancel{3}\ \ -1$

$\cancel{\therefore X(k) = [-1, 3, -1, 3, -1, 3, -1, 3, -1, 3]}$

$X(k) = [3, -1, 3, -1, 3, -1, 3, -1, 3, -1]$

(ii) $y(n) = ?$

$w(n) = u(n) - u(n-7)$

$w(n) = [\underset{\uparrow}{1}, 1, 1, 1, 1, 1, 1, 0, 0, 0]$

Use circular convolution property

$y(n) = x(n) \textcircled{N} w(n) \longleftrightarrow Y(k) = X(k) \cdot W(k)$

Find $y(n)$ using circular convolution with the help of matrix method.

$y(n) = [3, 1, 1, 1, 1, 1, 3, 2, 2, 2]$

(7) Given DFT ~~X(k)~~

$X(k) = [10, (-2+j2), (-2), (-2-j2)]$

of sequence $x(n)$. Find

(i) $x_1(n) = x((n+2))_4$ (ii) $x_2(n) = x(4-n)$

= (i) $x_1(n) = x((n+2))_4$, $X_1(k) = ?$

Use circular time shift property

If $x(n) \longleftrightarrow X(k)$

$x_1(n) = x(n+2) \longleftrightarrow W_N^{-2k} X(k) = X_1(k)$

$\therefore X_1(k) = W_N^{-2k} X(k) = e^{-j\frac{2\pi}{4}(-2)k} X(k)$

$$\boxed{X_1(k) = e^{j\pi k} X(k) = (-1)^k X(k)}$$

$\therefore X_1(k) = [10, (2-j2), -2, (2+j2)]$

(ii) ~~N~~ $x_2(n) = x(4-n)$ $X_2(k) = ?$

$N = 4$

$x_2(n) = x(N-n) = x(-n)$

Use circular Time shift property.

If $x(n) \xleftrightarrow{DFT} X(k)$

$x_2(n) = x(-n) \longleftrightarrow X(-k) = X_2(k)$

$\therefore \boxed{X_2(k) = X(-k)} = X(4-k)$

$k=0$, $\therefore X_2(0) = X(4) = X(0) = 10$

$k=1$, $X_2(1) = X(3) = \cancel{(2+j2)}\ (-2-j2)$

$k=2$, $X_2(2) = X(2) = -2$

$k=3$, $X_2(3) = X(1) = \cancel{(2-j2)}\ (-2+j2)$

$\therefore X_2(k) = [10, \cancel{(2+j2)}\ (-2-j2), -2, \cancel{(2-j2)}\ (-2+j2)]$

8 Find 4 point DFT of $x(n) = \sin^2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)$

Also find DFT of $y(n) = \cos^2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)$ from $X(K)$

= $x(n) = \sin^2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)$

$$x(n) = \left[\frac{1 - \cos 2\theta}{2}\right] = \frac{1 - \cos\left[2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)\right]}{2}$$

$n = 0, \; x(0) = 0.75$

$n = 1, \; x(1) = 0.25$

$n = 2, \; x(2) = 0.75$

$n = 3, \; x(3) = 0.25$

$$\therefore x(n) = [\underset{\uparrow}{0.75}, 0.25, 0.75, 0.25] \quad \text{——} (1)$$

$$X(k) = ? = \sum_{n=0}^{3} x(n) W_N^{kn}$$

$$X(k) = [2, 0, 1, 0] \quad \text{——} (2)$$

Given $y(n) = \cos^2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)$

$$y(n) = \frac{1 + \cos\left[2\left(\frac{n\pi}{2} + \frac{\pi}{3}\right)\right]}{2}$$

$$y(n) = [\underset{\uparrow}{0.25}, 0.75, 0.25, 0.75] \quad \text{——} (3)$$

on comparing (1) & (3), we find relation between $x(n)$ & $y(n)$.

$$\boxed{y(n) = x(n-1)}$$ Shifted Right by one unit

To find $Y(K)$ from $X(K)$

Use circular Time shift property

If $x(n) \xleftrightarrow{DFT} X(K)$

$y(n) = x(n-1) \longleftrightarrow W_N^{1K} X(K) = Y(K)$

$\therefore Y(K) = W_N^{K} X(K) = e^{-j\frac{2\pi}{4}K} X(K) =$

$K=0,\ Y(0) = 1 \cdot X(0) = 2$

$K=1,\ Y(1) = e^{-j\frac{\pi}{2}} \cdot X(1) = [e^{-j\frac{\pi}{2}}][0] = 0$

$K=2,\ Y(2) = e^{-j\frac{\pi}{2}\times 2} X(2) = (-1)[1] = -1$

$K=3,\ Y(3) = e^{-j\frac{\pi}{2}3} X(3) = e^{-j\frac{\pi}{2}3}[0] = 0$

$\therefore Y(K) = [2, 0, -1, 0]$

---

(9) compute circular convolution using DFT & IDFT for

$x_1(n) = [\underset{\uparrow}{2}, 3, 1, 1]$, $x_2(n) = [\underset{\uparrow}{1}, 3, 5, 3]$

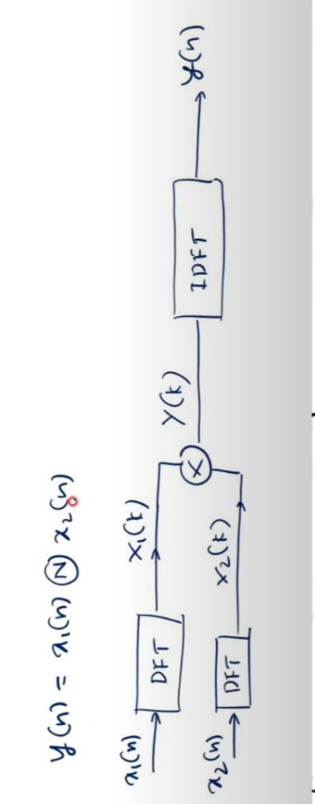
verify the result

$=$ $x(n) = x_1(n) \, \text{(N)} \, x_2(n) \longleftrightarrow X(K) = X_1(K) \cdot X_2(K)$

$X_1(K) = [7, (1-2j), 1, (1+2j)]$

$X_2(K) = [12, -4, 6, -4]$

$X(K) = X_1(K) \cdot X_2(K) = [84, (-4+j8), 6, (-4-j8)]$



Take IDFT of X(k)

$$x(n) = [\cancel{20.5}\ 19,\ \cancel{15.5}\ 17,\ \cancel{24.5}\ 23,\ \cancel{23.5}\ 25]$$

Verification :-

$$x(n) = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \\ 23 \\ 25 \end{bmatrix}$$

---

(10) Given $x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$

(i) Find 4-point DFT of x(n)

(ii) If y(n) is the circular convolution of x(n) with itself, find y(n)

(iii) Find DFT Y(k) of y(n)

= (i) $x(n) = [1, 0, 2, 1]$

$$X(k) = \sum_{n=0}^{3} x(n)\, W_N^{kn}$$

$X(k) =$

(ii) $y(n) = ? = x(n) \; (N) \; x(n)$

(iii) $Y(K) = ? = \sum_{n=0}^{3} y(n) \, W_N^{kn}$

$=$

(11) Evaluate circular convolution of $x(n)$ & $h(n)$

where $x(n) = u(n) - u(n-4)$

& $h(n) = u(n) - u(n-3)$

for $N=8$

$\Rightarrow$

$x(n) = [1, 1, 1, 1, 0, 0, 0, 0]$

$h(n) = [1, 1, 1, 0, 0, 0, 0, 0]$

$y(n) = x(n) \; Ⓝ \; h(n)$

$= [1, 2, 3, 3, 2, 1, 0, 0]$

---

(12) Let $X(K)$ is the N-point DFT of real sequence $x(n)$. S.T

(i) $X(N-K) = X(-K) = X^*(K)$

(ii) $X(0)$ is real

(iii) If N is even, $X\left(\frac{N}{2}\right)$ is real

$\Rightarrow$

(i) W.K.T. $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ — (1)

Substitute $K = N-K$

$X(N-K) = \sum_{n=0}^{N-1} x(n) W_N^{(N-K)n}$

$= \sum_{n=0}^{N-1} x(n) \underbrace{W_N^{Nn}}_{\to 1} \cdot W_N^{-kn}$

$= \sum_{n=0}^{N-1} x(n) \cdot W_N^{-kn}$

$$\boxed{X(N-k) = X(-k)}$$

$$X(N-k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}(-kn)}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{j\frac{2\pi}{N}kn}$$

$$\boxed{X(N-k) = X^*(k)} \rightarrow \text{complex conjugate}$$

(ii) Substitute $k=0$ in Equation ①

$$\boxed{X(0) = \sum_{n=0}^{N-1} x(n)} \Rightarrow \text{Real.}$$

(iii) ~~Substitute~~ $k = \frac{N}{2}$ in Equation ①

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{\frac{N}{2}n}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N} \cdot \frac{N}{2} n}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot \left(e^{-j\pi}\right)^n$$

$$\boxed{X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n) \cdot (-1)^n} \Rightarrow \text{Real}$$

---

13) Let $X(k)$ is the DFT of

$x(n) = [1, 2, 1, 0]$

Find $Y(k)$, the DFT of

$y(n) = x((-n))_N$

= $N = 4.$

$X(k) = ? = \sum_{n=0}^{3} x(n)\, W_N^{kn}$

$X(k) = [4, (-j2), 0, (j2)]$.

To find $Y(k)$, Use Time Reversal property

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) = x(-n) \longleftrightarrow X(-k) = Y(k)$

$\therefore Y(k) = X(-k) = X(N-k)$

$k=0,\quad Y(0) = X(0) = 4$

$k=1,\quad Y(1) = X(4-1) = X(3) = j2$

$k=2,\quad Y(2) = X(4-2) = X(2) = 0$

$k=3,\quad Y(3) = X(4-3) = X(1) = -j2$

$\therefore Y(k) = [4, j2, 0, -j2]$

---

(14) Find N-point circular convolution of $y(n) = x_1(n) \, (N) \, x_2(n)$

Given $x_1(n) = \cos\left(\frac{2\pi n}{N}\right)$

$x_2(n) = \sin\left(\frac{2\pi n}{N}\right)$

= $y(n) = ? = x_1(n) \, (N) \, x_2(n) \longleftrightarrow Y(k) = X_1(k) \cdot X_2(k)$

$x_1(n) = \cos\left(\frac{2\pi n}{N}\right) = \frac{e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi}{N}n}}{2}$

$X_1(k) = \sum_{n=0}^{N-1} x_1(n) \cdot W_N^{kn}$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{j\frac{2\pi}{N}n}\cdot e^{-j\frac{2\pi}{N}kn} + e^{-j\frac{2\pi}{N}n}\cdot e^{-j\frac{2\pi}{N}kn}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{-j\frac{2\pi}{N}(k-1)}\right]^n + \frac{1}{2}\sum_{n=0}^{N-1}\left[e^{-j\frac{2\pi}{N}(k+1)}\right]^n$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left[W_N^{(k-1)}\right]^n + \frac{1}{2}\sum_{n=0}^{N-1}\left[W_N^{(k+1)}\right]^n$$

$$\boxed{X_1(k) = \frac{N}{2}\delta(k-1) + \frac{N}{2}\delta(k+1)}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, W_N^{kn}$$

$$\boxed{X_2(k) = \frac{N}{2j}\delta(k-1) - \frac{N}{2j}\delta(k+1)}$$

$$Y(k) = X_1(k)\cdot X_2(k)$$

$$= \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\cdot\frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]$$

$$= \frac{N^2}{4j}\left[\delta^2(k-1) - \delta^2(k+1)\right]$$

$$\boxed{Y(k) = \frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right]}$$

$$Y(k) = \frac{N^2}{4j}\delta(k-1) - \frac{N^2}{4j}\delta(k+1)$$

~~Take IDFT~~

$$\cancel{y(n) = \frac{N^2}{4j}W_N^{(k-1)} - \frac{N^2}{4j}W_N}$$

$$\cancel{= \frac{N^2}{4j}\left\{e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}\right.}$$

$$\cancel{= \frac{N^2}{4j}\left[2j \sin\left(\frac{2\pi}{N} n\right)\right]}$$

$$\cancel{y(n) = \frac{N^2}{2} \sin\left(\frac{2\pi}{N} n\right)}$$

~~Take IDFT~~

$$Y(K) = \frac{N}{2}\left[\frac{N}{2j}\,\delta(K-1) - \frac{N}{2j}\,\delta(K+1)\right]$$

Take IDFT

$$y(n) = \frac{N}{2}\left[\frac{1}{2j} e^{j\frac{2\pi}{N} n} - \frac{1}{2j} e^{-j\frac{2\pi}{N} n}\right]$$

$$= \frac{N}{2}\left[\frac{1}{2j}\left(e^{j\frac{2\pi}{N} n} - e^{-j\frac{2\pi}{N} n}\right)\right]$$

$$= \frac{N}{2}\left[\frac{1}{\cancel{2j}} \times \cancel{2j} \sin\left(\frac{2\pi}{N} n\right)\right]$$

$$\boxed{y(n) = \frac{N}{2} \sin\left(\frac{2\pi}{N} n\right)}$$

→ Linear convolution using DFT

→ Relation between circular and linear convolution

Consider LTI system

O/p of LTI system is the linear convolution (LC)

LC: $y(n) = x(n) * h(n)$ —— ①

Assume, the length of $x(n)$ is L

—— // —— $h(n)$ is M

Then the length of $y(n) = L+M-1 = N$

Take DTFT on Eqn ① to convert Time domain into frequency domain

DTFT $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$ —— ②

The disadvantage is continous in nature, hence convert into discrete by sampling that leads to DFT

$$Y(K) = X(K)\, H(K) \quad \text{—— ③}$$

Here length of $X(K)$ is L

—— // —— $H(K)$ is M

—— // —— $Y(K)$ is $L+M-1 = N$

This is the disadvantage, since two different DFT lengths can't be multiplied

∴ Pad zeros to $X(K)$ & $H(K)$ to make equal lengths

$$Y(k) = X(k) \quad H(k)$$

$N = L+M-1$ $\quad$ $L + (M-1)$ Zeros $\quad$ M $(L-1)$ Zeros

In DFT all lengths must be same.

Pad $(M-1)$ Zeros to $X(k)$

Pad $(L-1)$ — // — $H(k)$

Now $Y(k) = X(k) \; H(k)$

$N = L+M-1$ $\quad$ $L+M-1$ $\quad$ $L+M-1$

$$\boxed{\underset{N}{Y(k)} = \underset{N}{X(k)} \cdot \underset{N}{H(k)} \longleftrightarrow y(n) = x(n) \; \textcircled{N} \; h(n)}$$

> The linear convolution is converted into circular convolution by padding Zeros and then Linear convolution can be computed using DFT

(1) Determine the response of LTI System. Given $x(n) = [\underset{\uparrow}{1}, 2, 3]$ & $h(n) = [\underset{\uparrow}{1}, 2]$ using DFT & also verify the results

= Response of LTI System $\longrightarrow$ LC

$LC :- \underset{3+2-1=4}{y(n)} = \underset{3}{x(n)} * \underset{2}{h(n)}$

Convert LC to CC by padding Zeros to use DFT

Pad one Zero to $x(n)$

& pad two Zeros to $h(n)$

$\therefore x(n) = [1, 2, 3, 0]$

$h(n) = [1, 2, 0, 0]$

$CC :- \underset{4}{y(n)} = \underset{4}{x(n)} \; (N) \; \underset{4}{h(n)}$

$y(n) = x(n) \; (N) \; h(n) \xleftrightarrow{DFT} Y(K) = X(K) \cdot H(K)$

$X(K) = DFT[x(n)] = \sum_{n=0}^{3} x(n) W_N^{Kn}$

$$\boxed{X(K) = 1 + 2W_N^{K} + 3W_N^{K2}}$$

$H(K) = DFT[h(n)] = \sum_{n=0}^{3} h(n) W_N^{Kn}$

$$\boxed{H(K) = 1 + 2W_N^{K}}$$

$$Y(k) = X(k) \cdot H(k)$$

$$= \left(1 + 2W_N^k + 3W_N^{k2}\right)\left(1 + 2W_N^k\right)$$

$$= 1 + 2W_N^k + 2W_N^k + 4W_N^{2k} + 3W_N^{k2} + 6W_N^{3k}$$

$$Y(k) = 1 + 4W_N^k + 7W_N^{2k} + 6W_N^{3k}$$

Take IDFT

$$y(n) = \delta(n) + 4\delta(n-1) + 7\delta(n-2) + 6\delta(n-3)$$

$$y(n) = [1, 4, 7, 6]$$

Verification :-

1.) Using CC :- $x(n) = [1, 2, 3, 0]$
$h(n) = [1, 2, 0, 0]$

$$y(n) = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

2.) Using LC :- $x(n) = [1, 2, 3]$
$h(n) = [1, 2]$

$h(n)$ \ $x(n)$	1	2	3
1	1	2	3
2	2	4	6

$y(n) = [1, 4, 7, 6]$

verified

2. Given $x_1(n) = [2, 1, 1, 2]$
$x_2(n) = [1, -1, -1, 1]$

(i) compute CC

(ii) compute LC

(iii) What value of N is neccessary so that LC & CC yield same result on N-point interval

= (i) CC :-

$x(n) = x_1(n) \circledN x_2(n) = [0, -2, 0, 2]$

(ii) LC :-

$x(n) = x_1(n) * x_2(n) = [2, -1, -2, 2, -2, -1, 2]$

(iii) LC :- $x(n) = x_1(n) * x_2(n)$

$4+4-1 = 7 = N$ 4   4

$\Downarrow$ padd Zeros

CC :- $x(n) = x_1(n) \circledN x_2(n)$

$7 = N$ 4 + 3 Zeros   4 + 3 Zeros

For N = 7, both CC & LC yield ~~ss~~ same result

(3) Find output of LTI system for input $x(n) = u(n) - u(n-3)$ & impulse respons $h(n) = u(n) - u(n-2)$ using (i) LC (ii) CC. Also find frequency samples using DFT

=

$$x(n) = [1, 1, 1]$$
$$h(n) = [1, 1]$$

(i) output of LTI system using LC :-

$h(n) \backslash x(n)$	1	1
1	1	1
1	1	1
1	1	1

$$y(n) = [1, 2, 2, 1]$$

(ii) output of LTI system using CC :-

$$y(n) = x(n) \; N \; h(n)$$

4   3 (+1 Zero)   2 (+2 Zeros)

$$x(n) = [1, 1, 1, 0] \qquad h(n) = [1, 1, 0, 0]$$

$$y(n) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

(iii) Frequency samples : $Y(k) = ?$

$y(n) = [1, 2, 2, 1]$

Take DFT

$$Y(k) = \sum_{n=0}^{3} y(n) W_N^{kn} =$$

Unit II

Fast Fourier Transform (FFT) Algorithms

→ Computes efficiently DFT by reducing multiplication and additions

→ It uses symmetry and periodicity properties of phase factor W_N^{kn}

- Symmetry property :- $W_N^{(k+\frac{N}{2})} = -W_N^k$

Proof: $LHS = W_N^{(k+\frac{N}{2})} = e^{-j\frac{2\pi}{N}(k+\frac{N}{2})}$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\frac{N}{2}} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j\pi}$$

$$= W_N^k(-1) = -W_N^k = RHS$$

- Periodicity property :- $W_N^{(k+N)} = W_N^k$

Proof :- $LHS = W_N^{(k+N)} = e^{-j\frac{2\pi}{N}(k+N)}$

$$= e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}N} = e^{-j\frac{2\pi}{N}k} \cdot 1 = W_N^k = RHS$$

Direct computation of DFT :-

N-point DFT $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$$\boxed{X(k) = x(0) W_N^{k\cdot 0} + x(1) W_N^{k\cdot 1} + \cdots + W_N^{k(N-1)}}$$

$k=0$, $X(0) = x(0) W_N^{0\times 0} + x(1) W_N^{0\times 1} + \cdots + W_N^{0(N-1)}$

$k=1$, $X(1) = x(0) W_N^{1(0)} + x(1) W_N^{1(1)} + \cdots + W_N^{1(N-1)}$

$\vdots$

$k=(N-1)$, $X(N-1) = x(0) W_N^{(N-1)0} + x(1) W_N^{(N-1)1} + \cdots + W_N^{(N-1)(N-1)}$

***Number of complex multiplications $= N \times N = N^2$
Number of complex Additions $= (N-1)N$

More number of complex multiplications & additions
∴ Disadvantage

Radix-2 FFT Algorithm :-

The number of samples = N

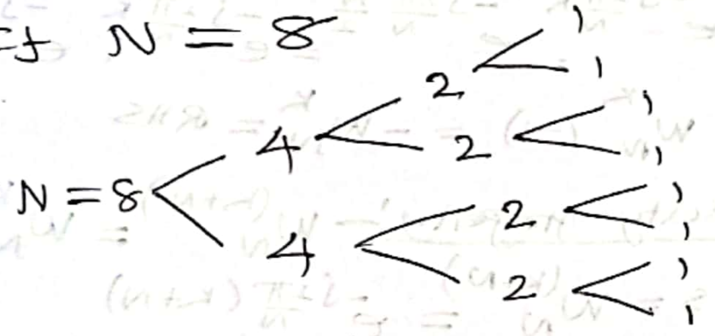
→ The value of $N = 2^{\text{integer}}$

$\therefore N = 2, 4, 8, 16, 32, 64 \ldots\ldots$

N → Even

→ The Algorithm is decompose into 2 parts

If N = 8



Divide and conquer approach

Radix-3 FFT Algorithm

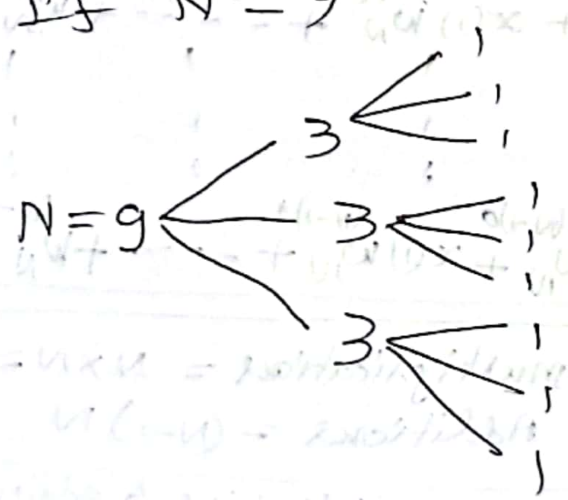
→ The value of $N = 3^{\text{integer}}$

$\therefore N = 3, 9, 27, \ldots$

N → odd

→ Decompose into 3 parts

If N = 9



FFT Algorithms :- There are two types of algorithms

(i) Decimation in Time (DIT) FFT

(ii) Decimation in Frequency (DIF) FFT

1. Radix 2 DIT-FFT :-

consider N-point DFT $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

Decompose time $x(n)$ into two parts viz., even and odd indexed.

$x(n) = x(0), x(1), x(2), x(3), \ldots \; x(N-2), x(N-1)$

Even $x(n) = x(0), x(2), x(4) \ldots x(N-2)$

odd $x(n) = x(1), x(3), x(5) \ldots x(N-1)$

$$\therefore X(k) = \underset{\text{Even 'n'}}{\sum_{n=0}^{N-2}} x(n) W_N^{kn} + \underset{\text{odd 'n'}}{\sum_{n=1}^{N-1}} x(n) W_N^{kn} \quad \text{--- (1)}$$

Substitute $n = 2n$ in 1st summation

Lower:- $n = 0$, $2n = 0$, $\boxed{n=0}$

Upper:- $n = N-2$, $2n = N-2$, $\boxed{n = \frac{N}{2} - 1}$

--- (2)

Substitute, $n = 2n+1$ in 2nd summation

Lower:- $n = 1$, $2n+1 = 1$, $\boxed{n=0}$

Upper: $n = N-1$, $2n+1 = N-1$, $\boxed{n = \frac{N}{2} - 1}$

--- (3)

Substitute (2) & (3) in (1)

$$X(k) = \underset{\text{(Even)}}{\sum_{n=0}^{\frac{N}{2}-1}} x(2n) W_N^{k \cdot 2n} + \underset{\text{(odd)}}{\sum_{n=0}^{\frac{N}{2}-1}} x(2n+1) W_N^{k(2n+1)}$$

$$\boxed{W_N^2 = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}}$$

$$\therefore X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{kn} \cdot W_N^k$$

$$\underbrace{X(K)}_{N\text{-pt. DFT}} = \underbrace{\boxed{\sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{kn}}}_{\frac{N}{2}\text{ pt. DFT Even}} + W_N^K \underbrace{\boxed{\sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{kn}}}_{\frac{N}{2}\text{-pt. DFT odd}}$$

$$\boxed{X(K) = G(K) + W_N^K H(K)}$$

Where $G(K) \rightarrow \frac{N}{2}$ pt. DFT Even

$$\xrightarrow{\text{Even DFT}} G(K) = \sum_{n=0}^{N/2-1} x(2n)\, W_{N/2}^{kn}$$

$H(K) \rightarrow \frac{N}{2}$ pt. DFT odd

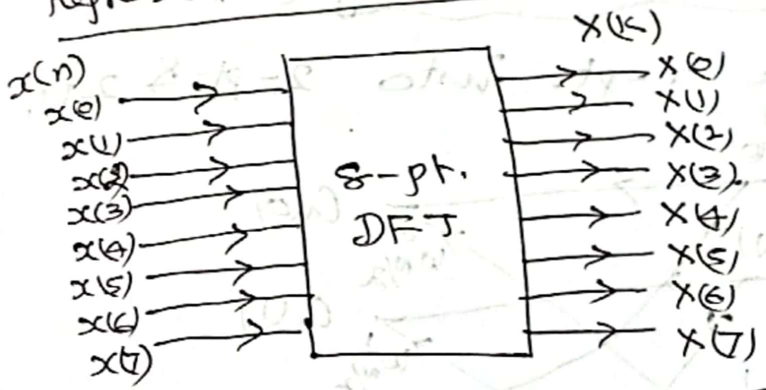
$$\xrightarrow{\text{odd DFT}} H(K) = \sum_{n=0}^{N/2-1} x(2n+1)\, W_{N/2}^{kn}$$

If $N=8$, $n=0$ to 7

$$\boxed{\underset{\substack{\uparrow \\ n=0\text{ to }7}}{X(K)} = \underset{\substack{\uparrow \\ n=0\text{ to }3 \\ \text{Even DFT}}}{G(K)} + W_N^K \underset{\substack{\uparrow \\ K=0\text{ to }3 \\ \text{odd DFT}}}{H(K)}}$$

— 1

Representation of 8-pt. DFT

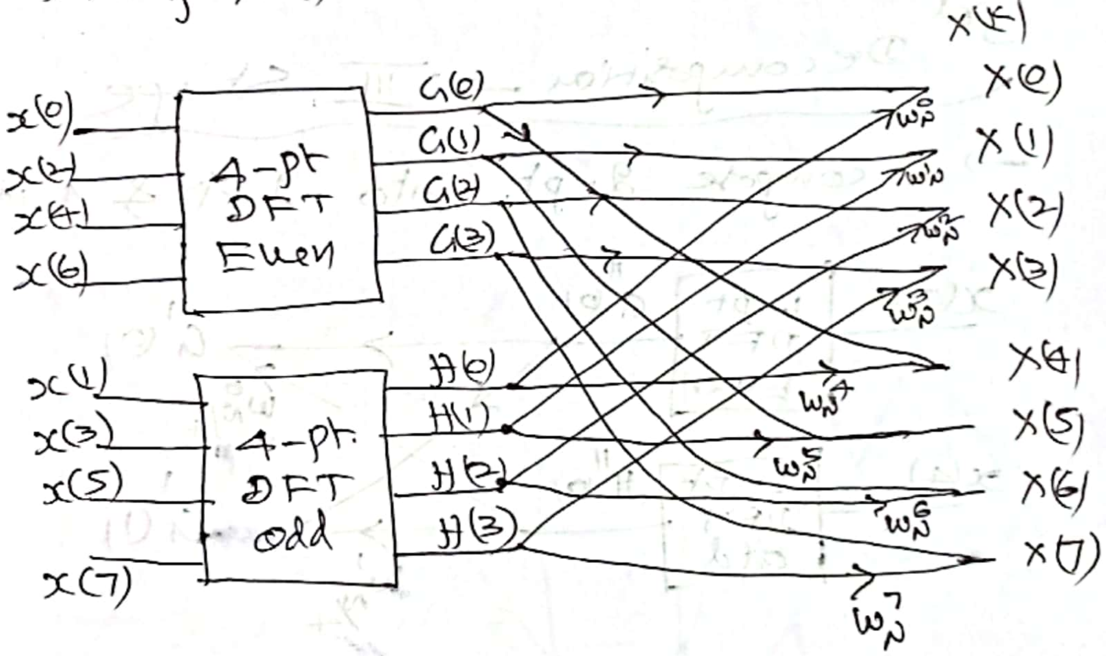


1st Decomposition :- $\boxed{\text{1st stage}}$

→ Decompose 8-pt. into 4-pt & 4-pt.

→ Separate x(n) into even x(n) & odd x(n)

→ Keep X(K) in natural order



Using Equation ①

$K=0, \quad X(0) = G(0) + W_N^0 H(0)$

$K=1, \quad X(1) = G(1) + W_N^1 H(1)$

$K=2, \quad X(2) = G(2) + W_N^2 H(2)$

$K=3, \quad X(3) = G(3) + W_N^3 H(3)$

$K=4, \quad X(4) = G(0) + W_N^4 H(4)$ ← *

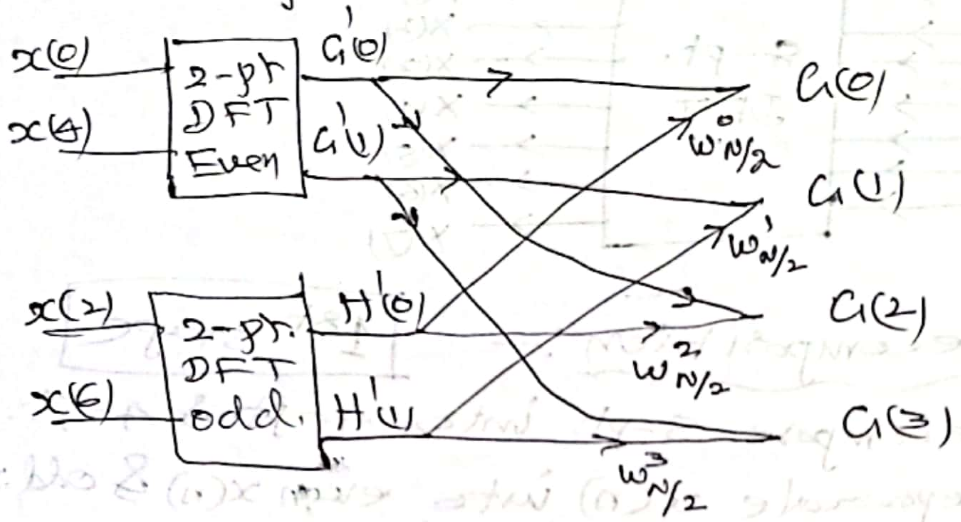
$K=5, \quad X(5) = G(1) + W_N^5 H(1)$

$K=6, \quad X(6) = G(2) + W_N^6 H(2)$

$K=7, \quad X(7) = G(3) + W_N^7 H(3)$

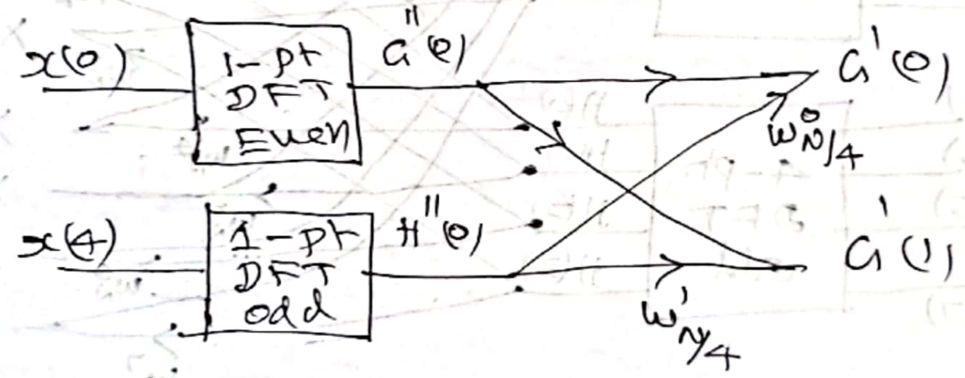
2nd Decomposition → II Stage

→ Decompose 4-pt. into 2-pt. & 2 pt.

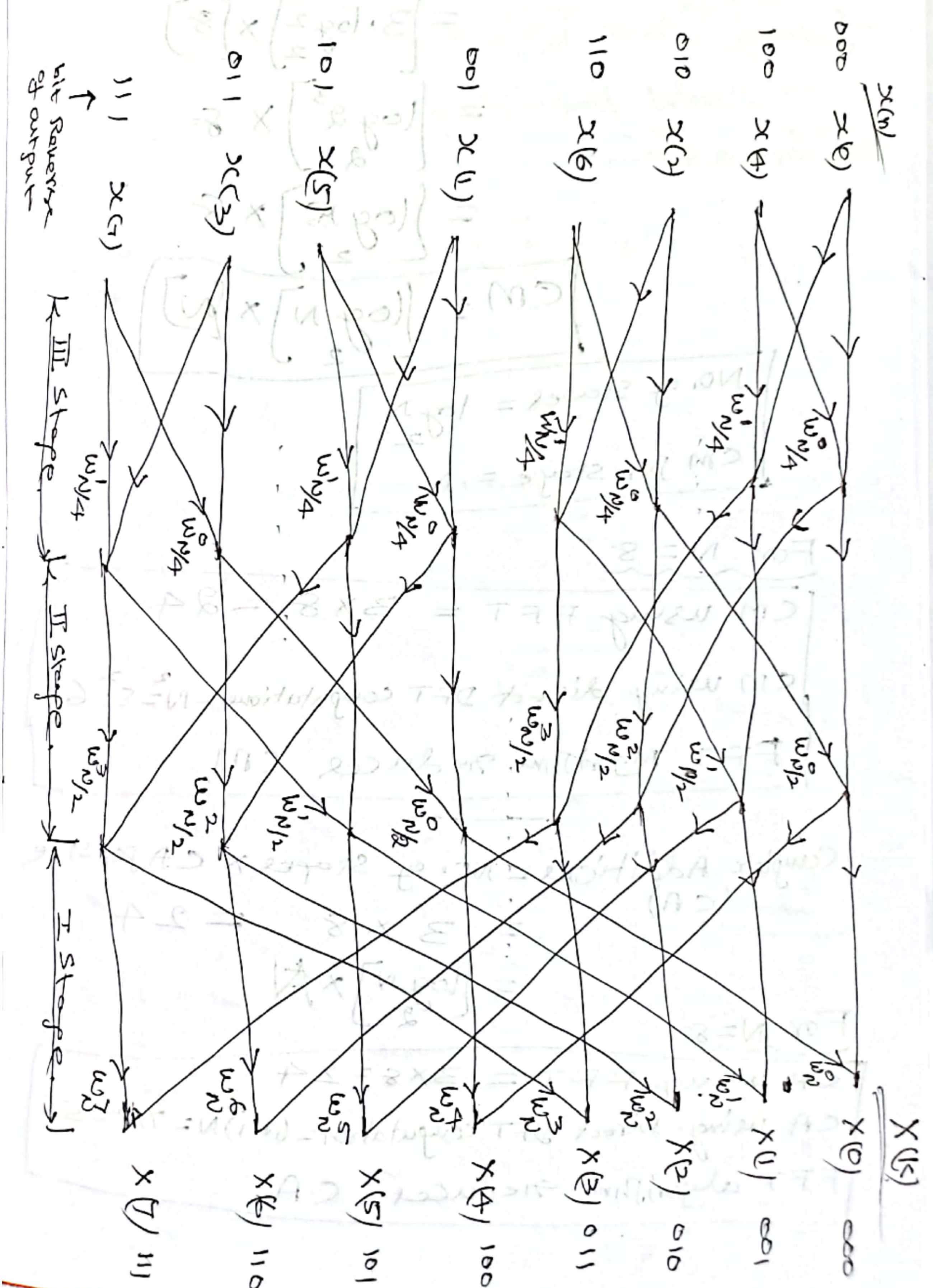


3rd Decomposition → III Stage

→ Decompose 2-pt. into 1 pt & 1 pt.



Radix 2, DIT-FFT Signal Flow graph



Complex multiplications (CM) = No. of stages × complex multiplications per stage

$$= 3 \times 8$$

$$= [3 \cdot \log_2 2] \times [8]$$

$$= [\log_2 2^3] \times 8$$

$$= [\log_2 8] \times 8$$

$$\boxed{CM = [\log_2 N] \times [N]}$$

No. of stages $= \log_2 N$
CM per stage $= N$

For $N = 8$

CM using FFT $= 3 \times 8 = 24$

CM using direct DFT computations $= N^2 = 8^2 = 64$

FFT Algorithm reduces CM

Complex Additions (CA) = No. of stages × CA per stage

$$= 3 \times 8 = 24$$

$$= [\log_2 N] \times N$$

For $N=8$

CA using FFT $= 3 \times 8 = 24$
CA using Direct DFT computation $= (N-1)N = 7 \times 8 = 56$
FFT algorithm reduces CA

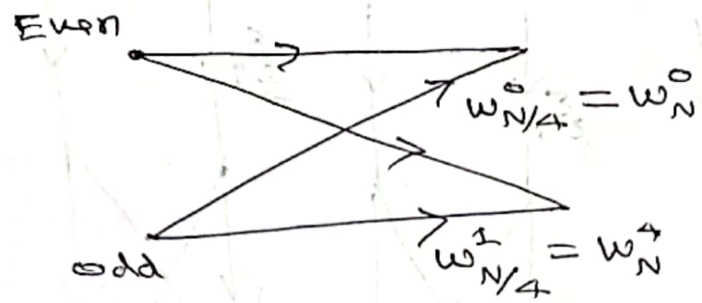
Butterfly structure:- It reduces further CA and CM in FFT algorithm.

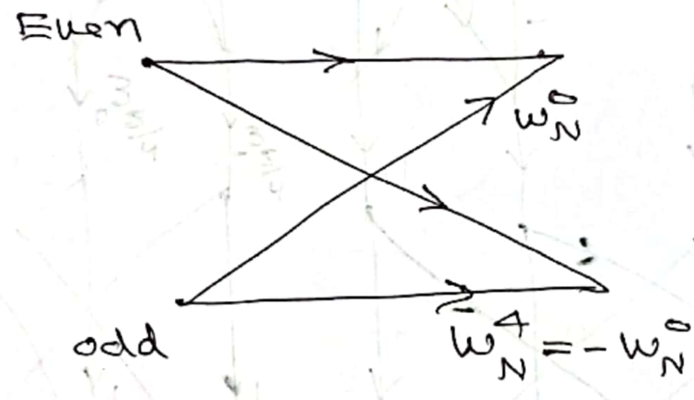
Symmetry property: $W_N^{k+\frac{N}{2}} = -W_N^k$

For $N=8$, $W_N^{k+\frac{8}{2}} = W_N^{k+4} = -W_N^k$.

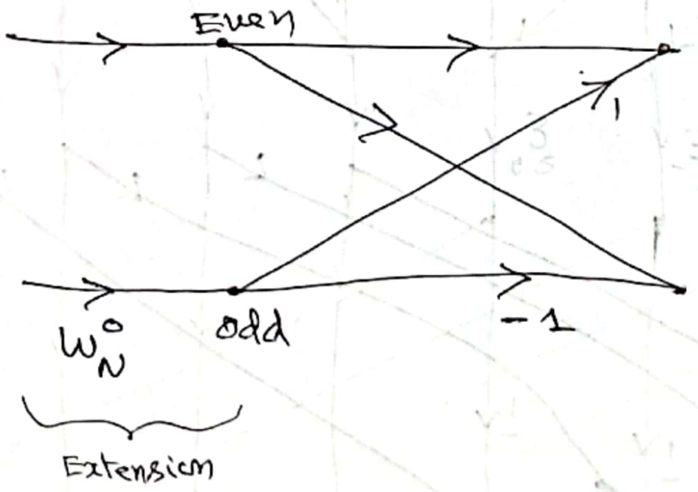
Consider one computational block

→ 2 multiplications





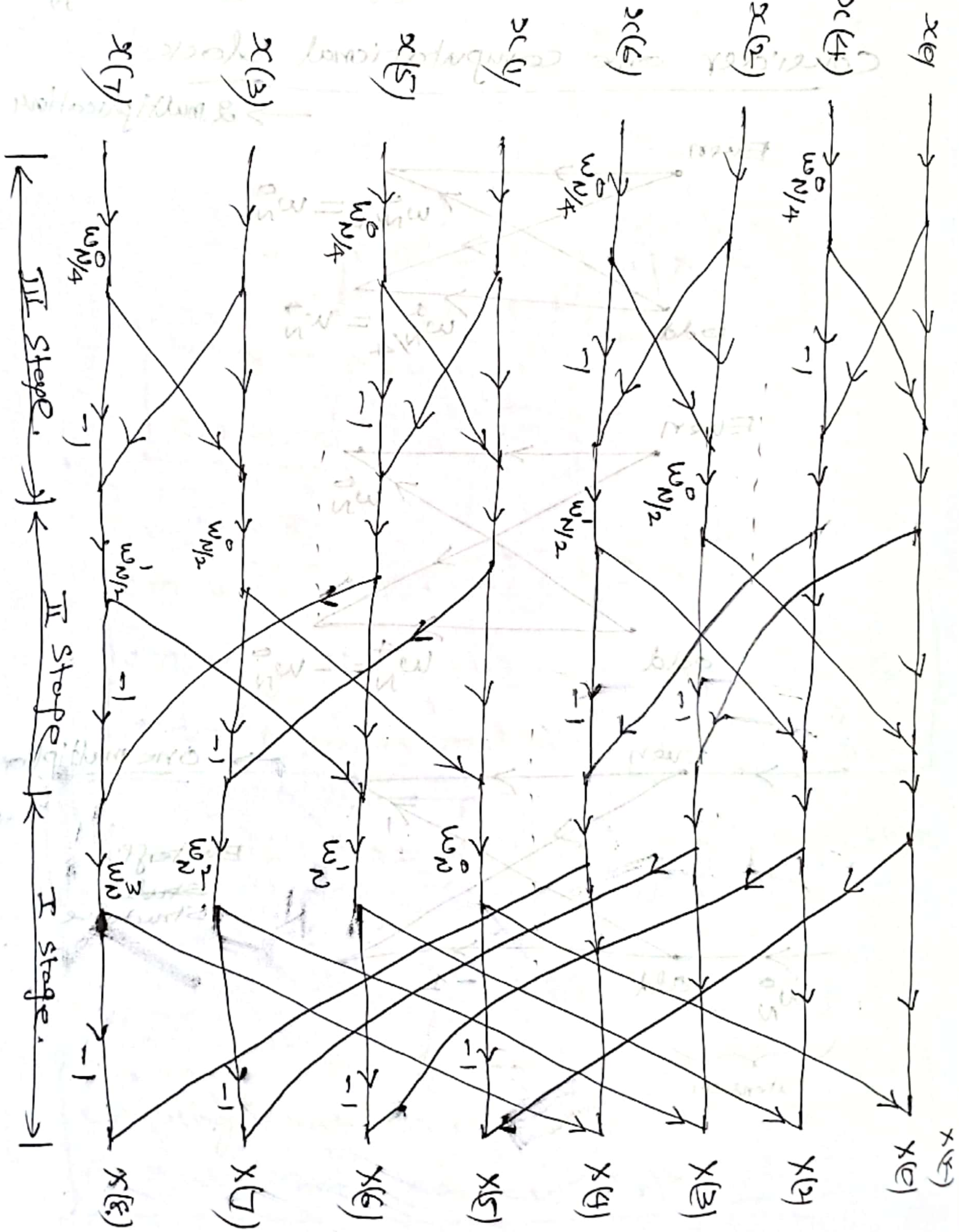
→ one multiplica



Butterfly ~~Stu~~ Structure

→ Radix-2, DIT-FFT using Butterfly Structure

→ Cooley-Tukey Algorithm



$CM = \text{No. of stages} \times CMPS$

$= 3 \times 4 = 12$

$= \left[\log_2 8\right] \times \left[\frac{8}{2}\right]$

$= \left[\log_2 N\right] \times \left[\frac{N}{2}\right]$

For $N=8$:

CM using FFT = 12

CM using Direct computation of DFT = 64

FFT algorithm reduces CM

$CA = \text{No. of stages} \times CAPS$

$= 3 \times 8 = 24$

$= \left[\log_2 8\right]\left[8\right]$

$= \left[\log_2 N\right]\left[N\right]$

For $N=8$

CA using FFT = 24

CA using Direct DFT computation = 56

∴ FFT algorithm reduces CA

In-place computation :-

once the output is computed, then the input is not required anymore, hence the output is stored in the same locations as the input. This leads to reduction in memory.

Features of DIT-FFT

1. It require $\lceil \log_2 N \rceil$ stages
2. It require $\left(\frac{N}{2}\right)$ CMPS
3. It require $CM = \lceil \log_2 N \rceil \lceil \frac{N}{2} \rceil$
4. In-place computation
5. Input is bit reverse of output.
6. In each stage, multiplication first, next addition.

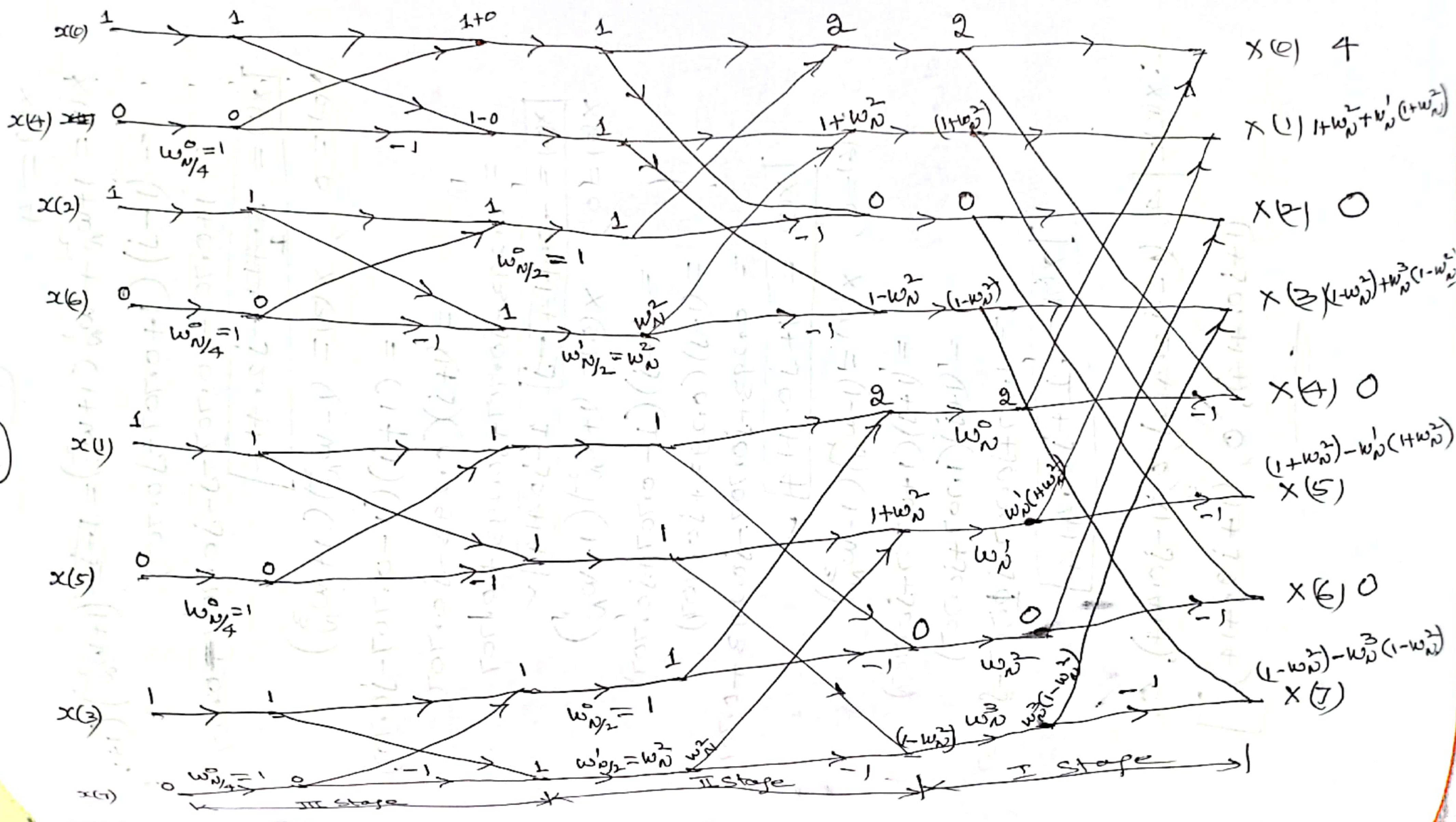
---

1. Find DFT of the sequence
$x(n) = [1, 1, 1, 1, 0, 0, 0, 0]$
using Radix 2-DIT-FFT. Show the intermidiate results on the signal flow graph.

= $W_{N/4}^{0} = W_N^{0} = 1$, $W_{N/2}^{0} = W_N^{0} = 1$

$W_{N/2}^{1} = W_N^{2} = -j$, $W_N^{1} = 0.707 - j0.707$

$W_N^{3} = -0.707 - j0.707$


x(0) 1
x(4) 0
x(2) 1
x(6) 0
x(1) 1
x(5) 0
x(3) 1
x(7) 0
$W^0_{N/4} = 1$
$W^0_{N/2} = 1$
$W^1_{N/2} = W^2_N$
W^2_N
1+0
1-0
$1+W^2_N$
$1-W^2_N$
$(1+W^2_N)$
$(1-W^2_N)$
W^0_N
W^1_N
W^2_N
W^3_N
$W^1_N(1+W^2_N)$
$W^3_N(1-W^2_N)$
-1
III Stage
II Stage
I Stage
X(0) 4
X(1) $1+W^2_N+W^1_N(1+W^2_N)$
X(2) 0
X(3) $(1-W^2_N)+W^3_N(1-W^2_N)$
X(4) 0
$(1+W^2_N)-W^1_N(1+W^2_N)$
X(5)
X(6) 0
$(1-W^2_N)-W^3_N(1-W^2_N)$
X(7)

$X(0) = 4$

$X(1) = 1 + W_N^2 + W_N^1(1 + W_N^2) = \cancel{1-j}\,(1+W_N^2)(1+W_N^1)$

$= (1-j)(1+0.707-j0.707)$

$= 1 + \cancel{0.707} - j0.707 - j - j0.707 \cancel{- 0.707}$

$\boxed{X(1) = \cancel{2.414}\ 1 - j2.414}$

$X(2) = 0$, $X(3) = (1 - W_N^2)(1 + W_N^3)$

$= (1+j)(1 - 0.707 - j0.707)$

$= (1+j)(0.293 - j0.707)$

$= 0.293 - j0.707 + j0.293 + 0.707$

$\boxed{X(3) = \cancel{-0.414 - j}\ 1 - j0.414}$ ✓

$X(4) = 0$, $X(5) = (1 + W_N^2)(1 - W_N^1)$

$= (1-j)(1 - 0.707 + j0.707)$

$= (1-j)(0.293 + j0.707)$

$= 0.293 + j0.707 - j0.293 + 0.707$

$\boxed{X(5) = 1 + j0.414}$ ✓

$X(6) = 0$, $X(7) = (1 - W_N^2)(1 - W_N^3)$

$= (1+j)(1 + 0.707 + j0.707)$

$= (1+j)(1.707 + j0.707)$

$= 1.707 + j0.707 + j1.707 - 0.707$

$\boxed{X(7) = 1 + j2.414}$

$\therefore X(k) = [4, (1 - j2.414), 0, (1 - j0.414), 0,$
$(1 + j0.414), 0, (1 + j2.414)]$

(2) compute DFT of the time domain sequence

$x(n) = \left[\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, -1, \frac{-1}{\sqrt{2}}, 0\right]$

using DIT-FFT. Provide the values at the intermediate stages

= Ans:

$X(K) = [0, (2.8 - j2.8), 0, 0, 0, 0, 0, (2.8 + j2.8)]$

(3) compute the 8-point DFT of the sequence $x(n) = [0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0]$ using Radix 2-DIT FFT

= Ans:-

$X(K) = [2, (0.5 - j1.2), 0, (0.5 - j0.2), 0, (0.5 + j0.2), 0, (0.5 + j1.2)]$

(4) Find the DFT of the sequence $x(n) = 2^n$ for $N = 8$, using DIT-FFT

$x(n) = [1, 2, 4, 8, 16, 32, 64, 128]$

= $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$

$\therefore X(K) = [225, (48.6 + j166.05), (-51 + j102), (-78.63 + j46.05), -85, (-78.63 - j46.05), (-51 - j102), (48.6 - j166.05)]$

(5) Find 8-point ~~DIT-FFT~~ DFT using DIT-FFT for $x(t) = \frac{(t+1)}{8}$ for $0 \le t \le \frac{8}{5}$ sec. Assume $x(t)$ is sampled at 5 samples / sec.

$= \quad x(t) = \left(\frac{t+1}{8}\right)$ for $0 \le t \le \frac{8}{5}$

$0 \le t \le 1.6$

$F_s = \cancel{8}\ 5$ Sample/sec

$F_s = 5\ Hz, \quad \therefore T_s = \frac{1}{F_s} = 0.2$ Sec.

Convert CT signal $x(t)$ into DT signal $x(n)$ using Sampling.

$$x(n) = x(t)\Big|_{t = n \cdot T_s} \qquad \cancel{n,0}\ \ n = 0 \text{ to } 7$$

$n=0, \quad x(0) = x(t)\Big|_{t=0} = \frac{t+1}{8} = \frac{1}{8} = \cancel{0.2}\ 0.125$

$n=1, \quad x(1) = x(t)\Big|_{t=1\times0.2} = \frac{0.2+1}{8} = 0.15$

$n=2, \quad x(2) = x(t)\Big|_{t=2\times0.2} = \frac{0.4+1}{8} = 0.175$

$n=3, \quad x(3) = x(t)\Big|_{t=3\times0.2} = \frac{0.6+1}{8} = 0.2$

$n=4, \quad x(4) = x(t)\Big|_{t=4\times0.2} = \frac{0.8+1}{8} = 0.225$

$n=5, \quad x(5) = x(t)\Big|_{t=5\times0.2} = \frac{1+1}{8} = 0.25$

$n=6, \quad x(6) = x(t)\Big|_{t=6\times0.2} = \frac{1.2+1}{8} = 0.275$

$n=7, \quad x(7) = x(t)\Big|_{t=7\times0.2} = \frac{1.4+1}{8} = 0.3$

$\therefore x(n) = [0.125, 0.15, 0.175, 0.2, 0.225,$

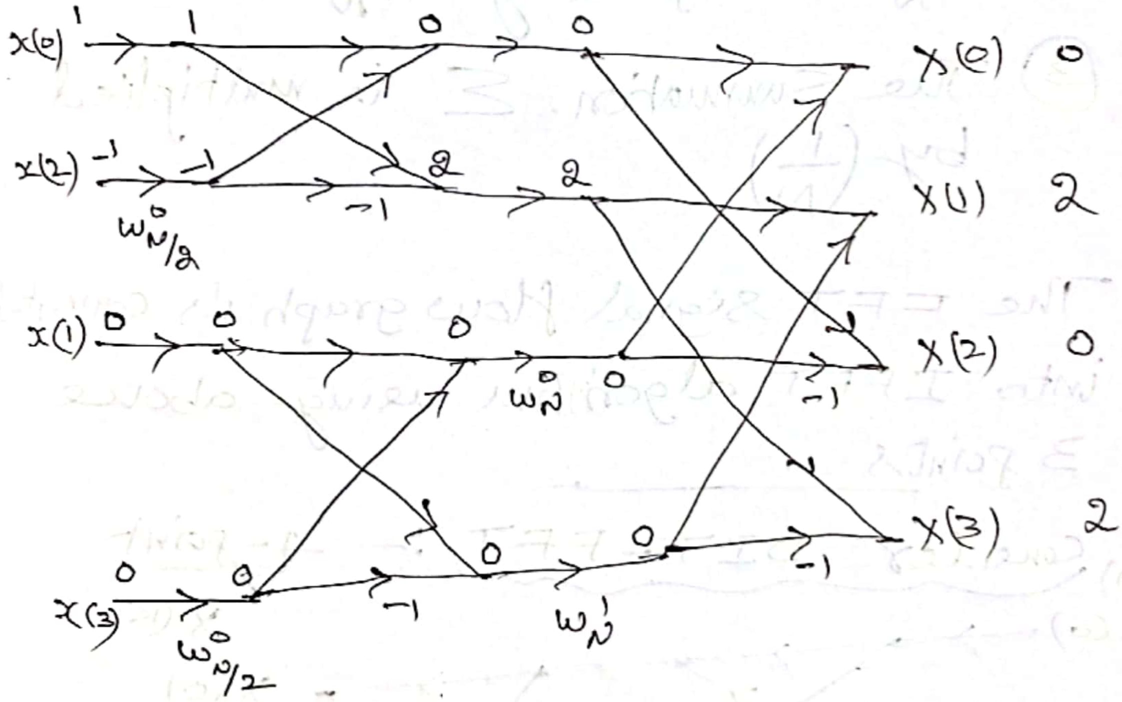
$0.25, 0.275, 0.3]$

Ans:- $X(K) = [1.7, (-0.1 + j0.24), (-0.1 + j0.1), (-0.1 + j0.041), (-0.1), (-0.1 - j0.041), (-0.1 - j0.1), (-0.1 - j0.24)]$

---

6) Find DFT of the sequence $x(n) = \cos\left(\frac{n\pi}{2}\right)$ for $N = 4$ using DIT-FFT algorithm

= $x(n) = [1, 0, -1, 0]$



$\therefore X(K) = [0, 2, 0, 2]$

Inverse Fast Fourier Transform (IFFT)

→ DIF-IFFT

Consider DFT: $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$ —— ①

IDFT: $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$ —— ②

Compare equations ① & ② to compute IFFT

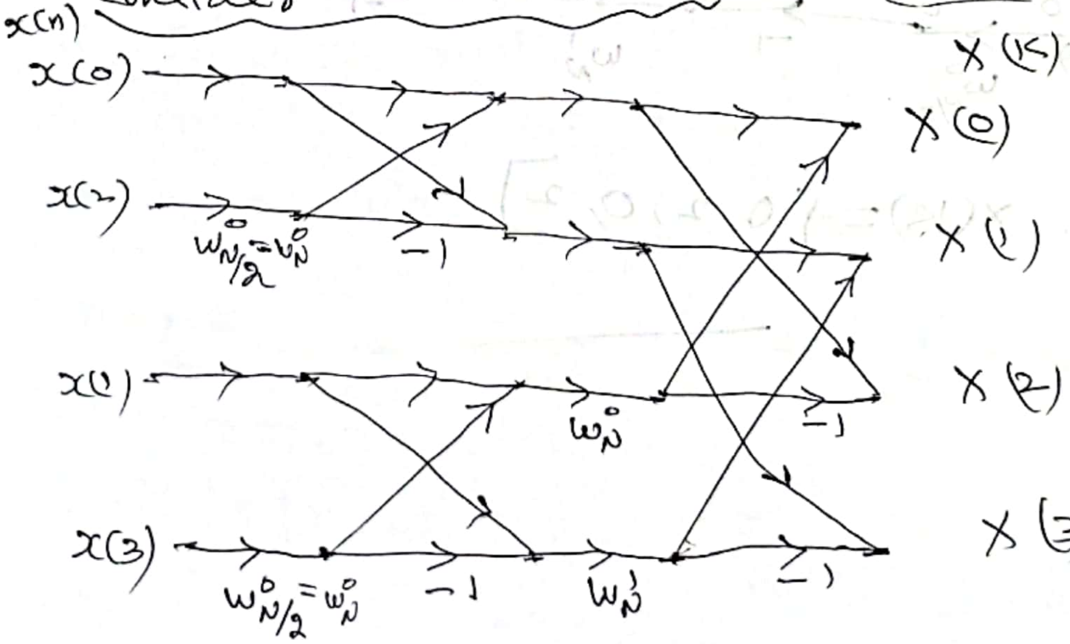
① $X(K)$ and $x(n)$ are interchanged

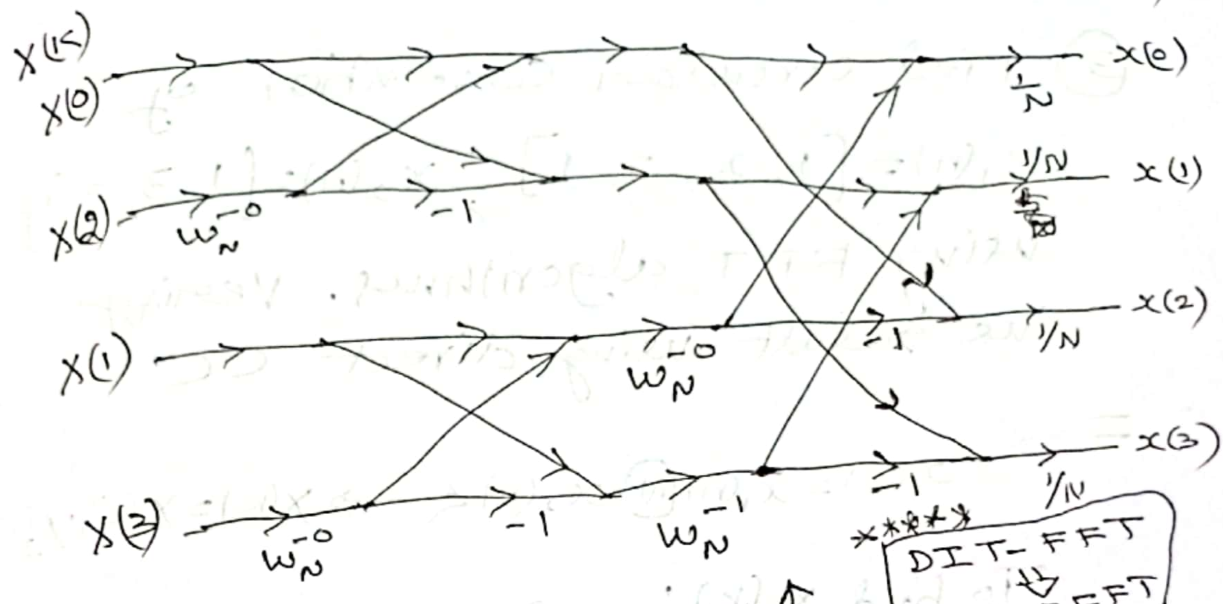
② W_N^K is replace by W_N^{-K}.

③ The summation Σ is multiplied by $\left(\frac{1}{N}\right)$

The FFT signal flow graph is converted into IFFT algorithm using above 3 points.

Consider DIF-FFT :- 4-point

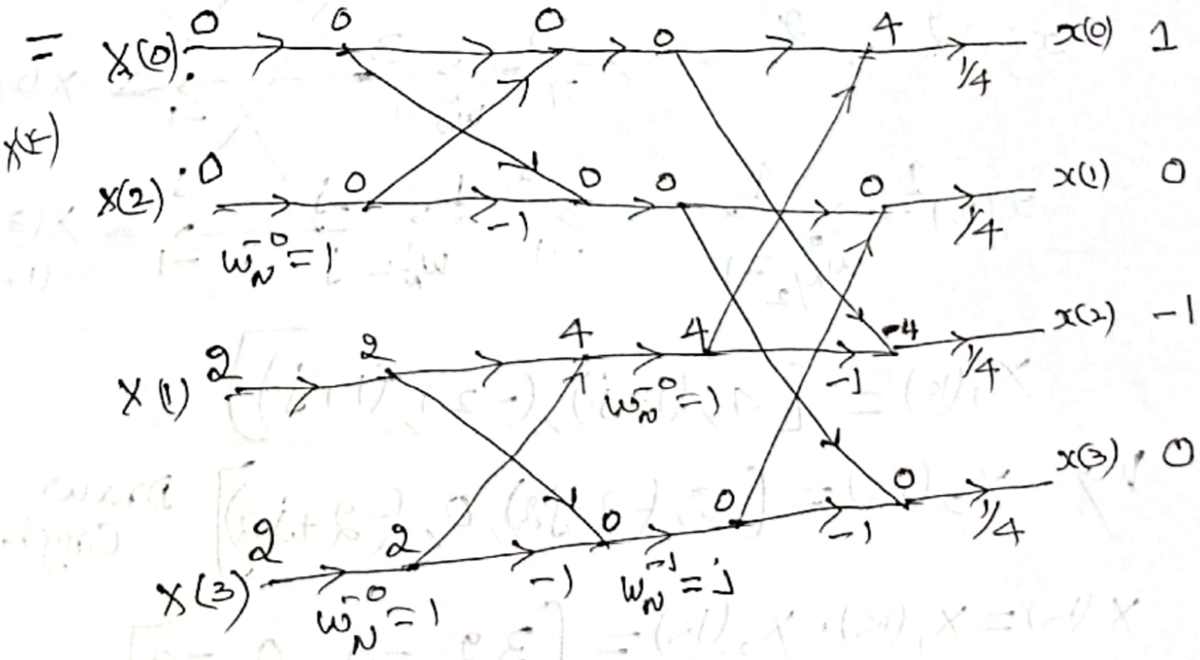




DIF-IFFT

① Find IDFT of $X(k) = [0, 2, 0, 2]$ using DIF-IFFT

=



$x(n) = [1, 0, -1, 0]$

② Find $x(n)$ for $X(k) = [1, -j2, -1, j2]$ using DIF-IFFT

=

3. Find circular convolution of

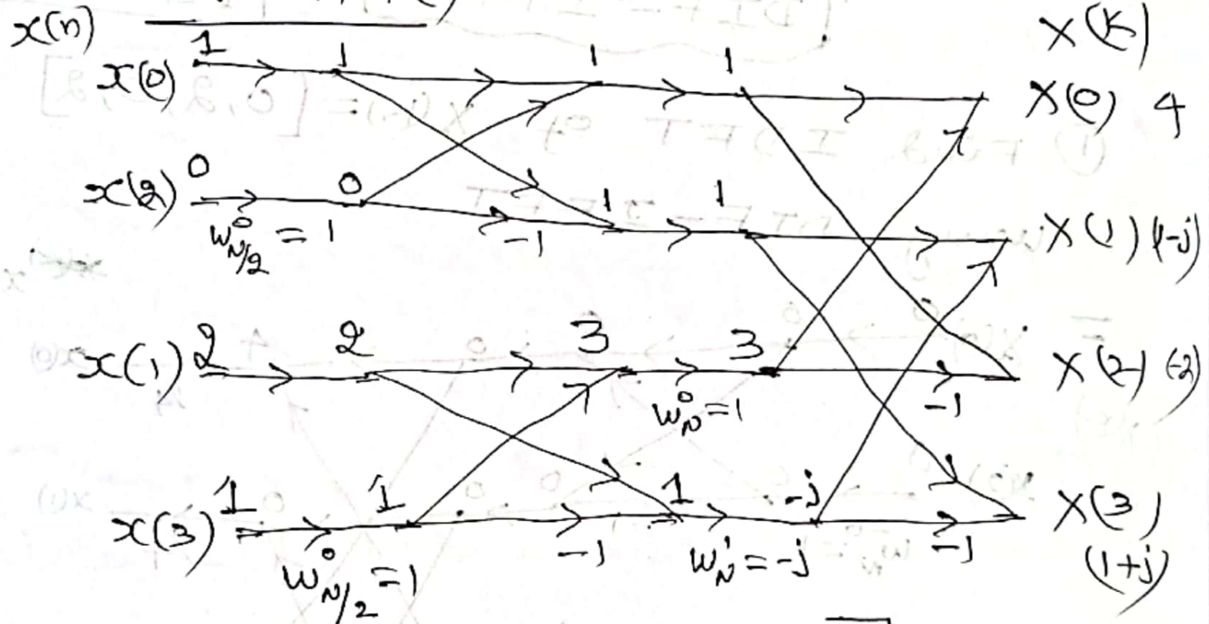
$x_1(n) = [1, 2, 0, 1]$, $x_2(n) = [1, 3, 3, 1]$

using FFT algorithms. Verify the result using direct CC

=

$$x(n) = x_1(n) \, (N) \, x_2(n) \longleftrightarrow X(k) = X_1(k) \cdot X_2(k)$$

To find $X_1(k)$:— DIT-FFT



$$X_1(k) = [4, (1-j), (-2), (1+j)]$$

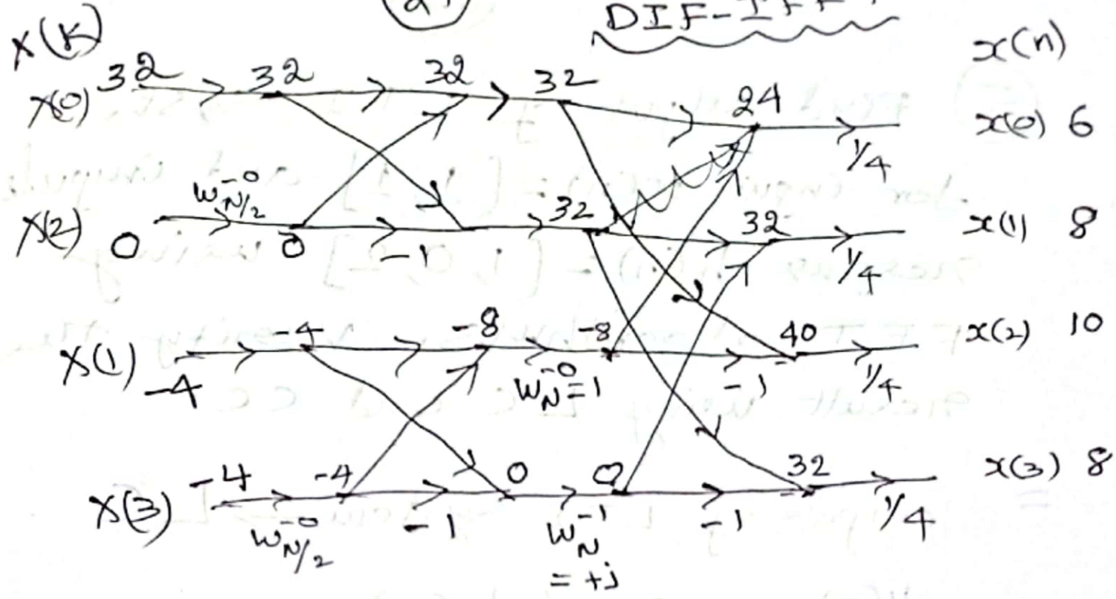
lly $X_2(k) = [8, (-2-j2), 0, (-2+j2)]$ Draw Graph.

$$X(k) = X_1(k) \cdot X_2(k) = [32, -4, 0, -4]$$

Take IDFT of $X(k)$ using IFFT.

(21)

DIF-IFFT



$x(n) = [6, 8, 10, 8]$

Verification :- cc

$$x(n) = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 3 & 1 & 1 & 3 \\ 3 & 3 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 10 \\ 8 \end{bmatrix}$$

(4) Determine IDFT of

$X(K) = [\underset{\uparrow}{0}, (2\sqrt{2}(1-j)), 0, 0, 0, 0, 0, 2\sqrt{2}(1+j)]$

using FFT algorithm

Ans: $x(n) = [\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0]$

5. Find output of LTI system for input $x(n) = [1, 1]$ and impulse response $h(n) = [1, 0, 2]$ using FFT algorithms. Verify the result using LC and CC

= output of LTI system → LC

$$y(n) = x(n) * h(n)$$

length → 4 → 2 → 3

To use FFT algorithms, LC must be converted into CC by padding zeros

$$y(n) = x(n) \, \textcircled{N} \, h(n) \longleftrightarrow Y(K) = X(K) \cdot H(K)$$

4   2 + 2 Zeros   3 + 1 Zero

Draw Signal Flow graphs

→ Find $X(K)$ using DIT-FFT algorithms

→ —//— $H(K)$ —//— —//—

→ Find $Y(K) = X(K) \cdot H(K)$

→ Find IFFT of $Y(K)$ using DIF-IFFT

Ans:- $y(n) = [1, 1, 2, 2]$

Verification:- Using Linear convolution (LC)

$x(n)$ \ $h(n)$	1	0	2
1	1	0	2
1	1	0	2

$y(n) = [1, 1, 2, 2]$

Using circular convolution (CC)

$$x(n) = [1, 1, 0, 0]$$
$$h(n) = [1, 0, 2, 0]$$

$$y(n) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$y(n) = [1, 1, 2, 2]$$

---

(6) Compute circular convolution using FFT and IFFT for the following sequences

$$x_1(n) = n \ \& \ x_2(n) = \cos\left(\frac{n\pi}{2}\right)$$

for $0 \leq n \leq 3$

=

$$x_1(n) = [0, 1, 2, 3]$$
$$x_2(n) = [1, 0, -1, 0]$$

→ $x(n) = x_1(n) \, (N) \, x_2(n) \longleftrightarrow X(k) = X_1(k) \cdot X_2(k)$

→ Find $X_1(k)$ using DIT-FFT

→ —"— $X_2(k)$

→ Find $X(k) = X_1(k) \cdot X_2(k)$

→ Find $x(n)$ using DIF-IFFT

Ans :— $x(n) = [-2, -2, 2, 2]$

Radix-2, Decimation in Frequency FFT (DIF-FFT)

Decompose Frequency component $X(K)$ into two parts even & odd

Consider N-pt. DFT $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$

Decompose $\sum_{n=0}^{N-1}$ into two equal parts

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{Kn} \quad \text{--- (1)}$$

Substitute $n = n + \frac{N}{2}$ in 2nd term only

Lower:- $n = \frac{N}{2}$ Upper:- $n = N-1$

$n + \frac{N}{2} = \frac{N}{2}$ $n + \frac{N}{2} = N-1$

$\boxed{n=0}$ $\boxed{n = \frac{N}{2}-1}$ — (2)

Substitute (2) in (1)

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{K\left(n+\frac{N}{2}\right)}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) \cdot W_N^{Kn} \cdot W_N^{K\frac{N}{2}}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{Kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{Kn} (-1)^K$$

$$\cancel{= \sum_{n=0}^{\frac{N}{2}-1} x(n) + x\left(n+\frac{N}{2}\right)}$$

$$\boxed{X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn}} \quad \text{--- (3)}$$

Decompose $X(k)$ into even $X(k)$ & odd $X(k)$

For Even $X(k)$:- substitute $k = 2k$ in eqn (3)

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (1)x\left(n+\frac{N}{2}\right)\right] W_N^{2kn}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] \cdot W_{N/2}^{kn}$$

$$\boxed{X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [g(n)] \cdot W_{N/2}^{kn}} \rightarrow N/2 \text{ point DFT Even} \quad \text{--- (4)}$$

Where $\boxed{g(n) = x(n) + x\left(n+\frac{N}{2}\right)}$ --- (5)

For odd $X(k)$:- Substitute $k = 2k+1$ in Equation (3)

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] \cdot W_N^{2kn} \cdot W_N^{1n}$$

$$\boxed{X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left\{\left[x(n) - x\left(n+\frac{N}{2}\right)\right] \cdot W_N^{n}\right\} \cdot W_{N/2}^{kn}}$$

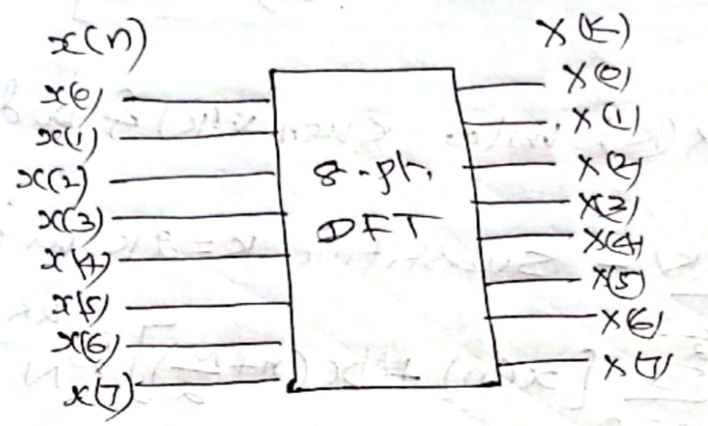
~~→ N/2 pt. DFT odd~~ ~~(6)~~

~~Where~~ $\cancel{h(n) = x(n) - x\left(n+\frac{N}{2}\right)}$ ~~(7)~~

$$\boxed{X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[h(n) \cdot W_N^{n}\right] W_{N/2}^{kn}} \rightarrow \frac{N}{2} \text{ pt. DFT odd} \quad \text{--- (6)}$$

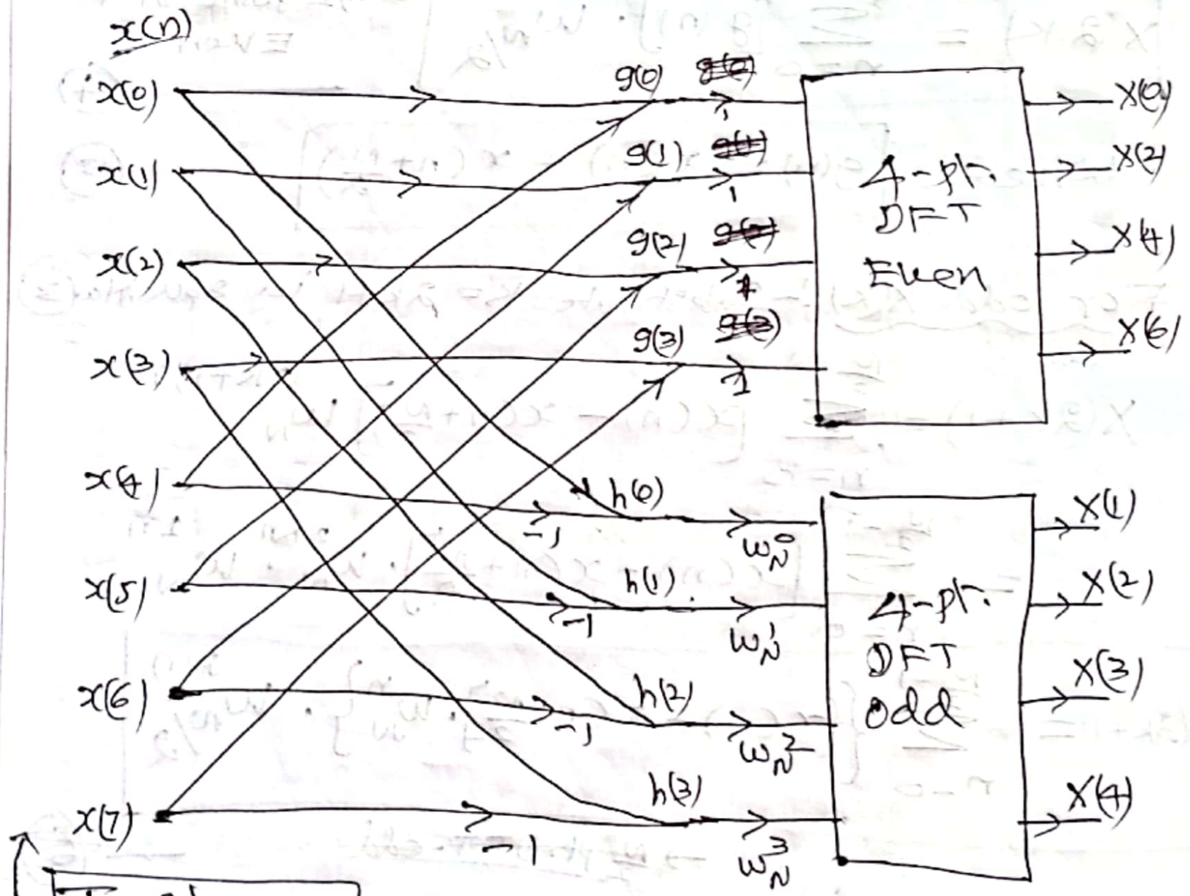
Where $\boxed{h(n) = x(n) - x\left(n+\frac{N}{2}\right)}$ --- (7)

Consider $N=8$, Representation of 8-pt. DFT



→ Devide 8-pt. DFT into two 4-pt. DFT's

→ Decompose X(K) into two → Even & odd
From Eqns. 4, 5, 6 & 7



I-stage

From Eqn 5 :- $g(n) = x(n) + x(n + \frac{N}{2})$ ← 8

$n=0$, $g(0) = x(0) + x(4)$

$n=1$, $g(1) = x(1) + x(5)$

$n=2$, $g(2) = x(2) + x(6)$

$n=3$, $g(3) = x(3) + x(7)$

From eqn (7), $h(n) = x(n) - x(n + \frac{N}{2})$ 8

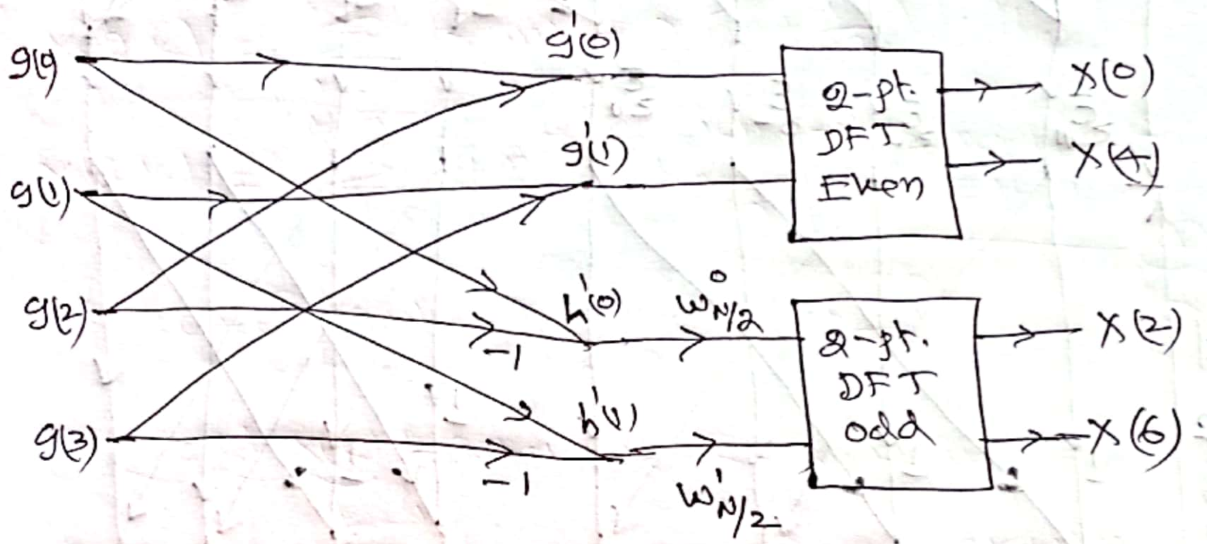
$n=0, \quad h(0) = x(0) - x(4)$

$n=1, \quad h(1) = x(1) - x(5)$

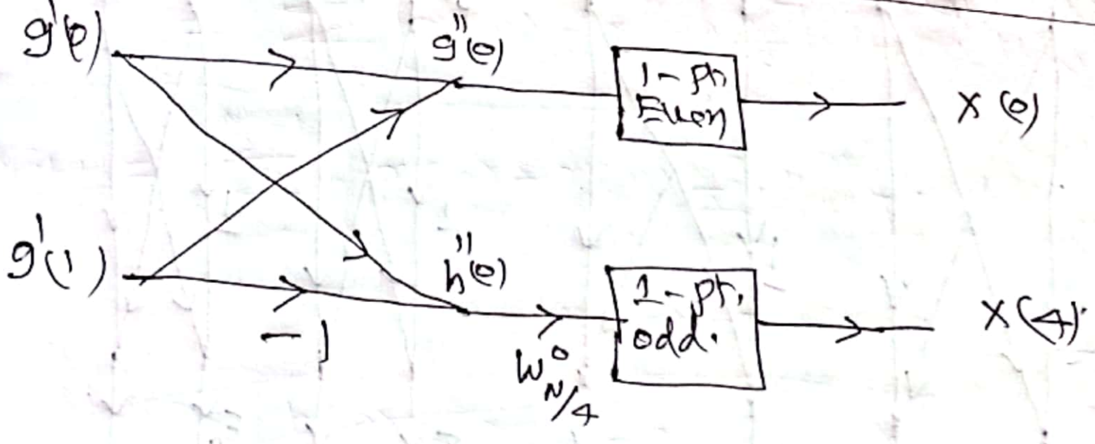
$n=2, \quad h(2) = x(2) - x(6)$

$n=3, \quad h(3) = x(3) - x(7)$

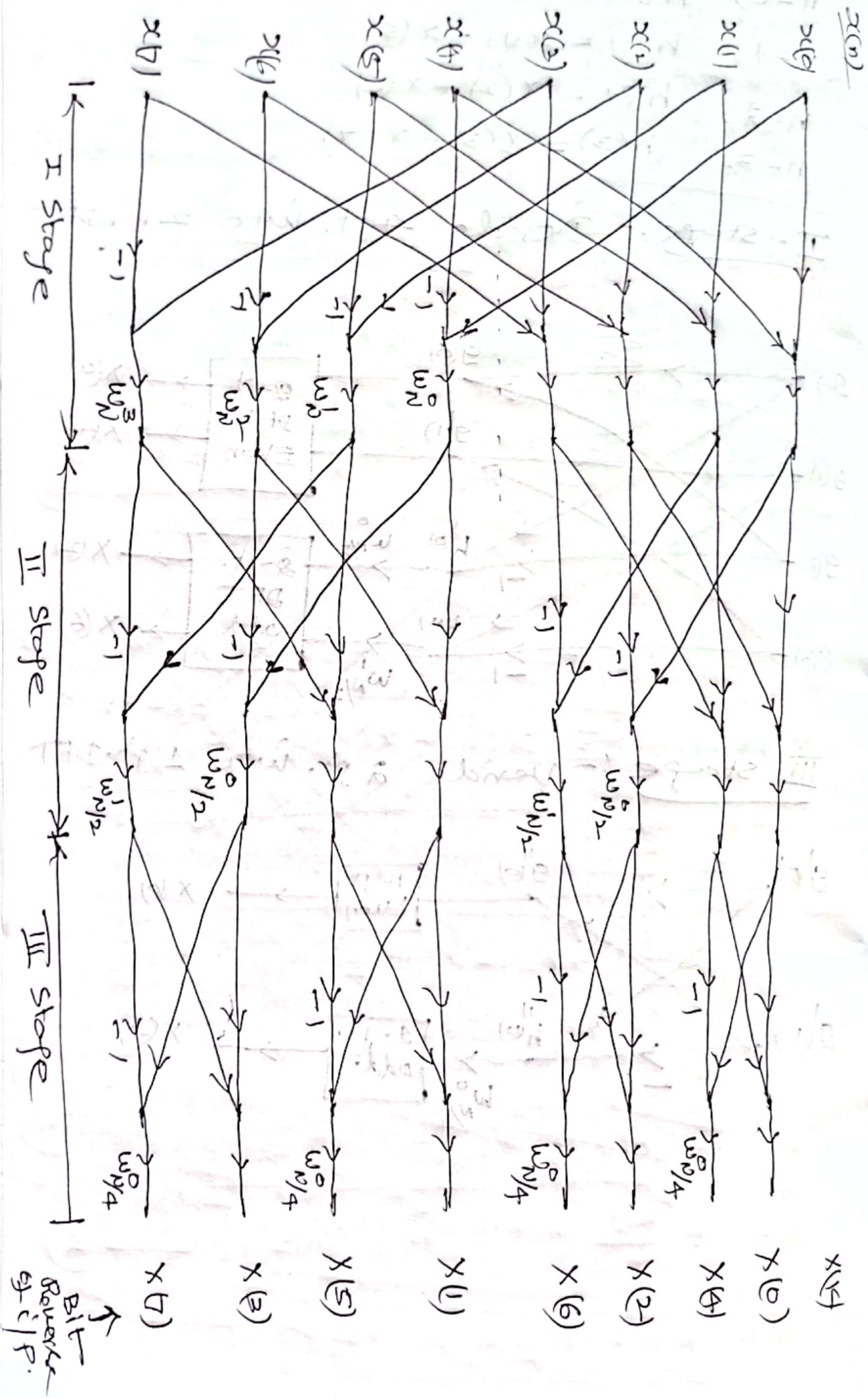
II - stage :- Devide 4 pt. into 2 pt. DFT



III stage :- Devide 2 pt. into 1 pt. DFT



Radix 2, DIF-FFT Algorithm



$CM = \text{No. of stages} \times CMPS$

$= 3 \times 4 = 12$

$= \left[\log_2 8\right]\left[\frac{8}{2}\right] = \left[\log_2 N\right]\left(\frac{N}{2}\right)$

For $N = 8$

CM using FFT = 12

CM using Direct DFT computation $= N^2 = 64$

→ FFT reduces CM

$CA = \text{No. of stages} \times CAPS$

$= 3 \times 8 = 24$

$= \left[\log_2 8\right]\left[8\right] = \left[\log_2 N\right]\left[N\right]$

For $N = 8$

CA using FFT = 24

CA using Direct computation of DFT $= (N-1)N$
$= 56$

→ FFT reduces CA

Features of DIF-FFT

1. It requires $\left[\log_2 N\right]$ stages
2. It require $\left(\frac{N}{2}\right)$ CMPS
3. It require $CM = \left[\log_2 N\right]\left(\frac{N}{2}\right)$
4. In-place computation
5. output is bit reverse of input
6. In each stage addition first, next multiplication.

① Develop the DIF-FFT algorithm for $N=8$, using the signal flow graph compute the 8-point DFT of the sequence $x(n) = \sin\left(\frac{n\pi}{2}\right)$

$\Rightarrow$ $x(n) = [0, 1, 0, -1, 0, 1, 0, -1]$

Ans: $X(K) = [0, 0, -j4, 0, 0, 0, j4, 0]$

② First five points of 8-point DFT of a real valued sequence is given by $X(k) = [0, (2+j2), (-j4), (2-j2), 0]$, Determine the remaining three points. Find the sequence $x(n)$ using DIF-FFT

$\Rightarrow$ Use symmetry property of DFT if the sequence is real

$$X(K) = X^*(N-K)$$ $* \to$ complex conjugate

$\therefore$ For $K=5$, $X(5) = X^*(8-5) = X^*(3)$
$= (2+j2)$

For $K=6$, $X(6) = X^*(8-6) = X^*(2) = j4$

For $K=7$, $X(7) = X^*(8-7) = X^*(1) = (2-j2)$

$\therefore$ remaining 3-points are

$[(2+j2), (j4), (2-j2)]$

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Full $X(K) = [0, (2+j2), (-j4), (2-j2), 0, (2+j2), (+j4), (2-j2)]$

Finding $x(n)$ using DIF-IFFT

write DIT-FFT $\Rightarrow$ DIF-IFFT by small modifications

Ans :- $x(n) = [1, 1, -1, -1, -1, 1, 1, -1]$

③ Find 4-point circular convolution of $x(n)$ & $h(n)$ using radix-2, DIF-FFT for $x(n) = [1, 1, 1, 1]$, $h(n) = [1, 0, 1, 0]$

= ~~To find~~ circular convolution

$y(n) = x(n) \; Ⓝ \; h(n) \xleftrightarrow{DFT} Y(K) = X(K) \cdot H(K)$

To find $X(K)$ using DIF-FFT

Draw 4-point DIF-FFT & compute

$X(K) = [4, 0, 0, 0]$

To find $H(K)$ using DIF-FFT

Draw 4-point DIF-FFT & compute

$H(K) = [2, 0, 2, 0]$

To find $Y(K)$

$Y(K) = X(K) \cdot H(K) = [8, 0, 0, 0]$

To find $y(n)$

Draw DIT-FFT algorithm and modify to draw DIF-IFFT

Ans: $y(n) = [2, 2, 2, 2]$

(4) (i) Find DFT of the sequence using DIF-FFT

$x_1(n) = [1, 2, -1, 2, 4, 2, -1, 2]$

(ii) If $x_2(n) = x_1(-n)$, then find $X_2(K)$

$\Rightarrow$ (i) ~~Use D~~ write DIF-FFT signal flow graph to find $X_1(K)$

Ans: $X_1(K) = [11, -3, 7, -3, -5, -3, 7, -3]$

(ii) To find $X_2(K)$:-

Using Time Reversal property

If $x_1(n) \xleftrightarrow{DFT} X_1(K)$

$$\boxed{x_2(n) = x_1(-n) \xleftrightarrow{DFT} X_1(-K) = X_2(K)}$$

$\Rightarrow X_2(K) = X_1(-K) = X_1(8-K)$

For $K=0$, $X_2(0) = X_1(8) = X_1(0) = 11$

$K=1$, $X_2(1) = X_1(7) = -3$

$K=2$, $X_2(2) = X_1(6) = 7$

$k=3, \quad X_2(3) = X_1(5) = -3$

$k=4, \quad X_2(4) = X_1(4) = -5$

$k=5, \quad X_2(5) = X_1(3) = -3$

$k=6, \quad X_2(6) = X_1(2) = 7$

$k=7, \quad X_2(7) = X_1(1) = -3$

~~$\therefore X_2(k) = [-3, 7, -3, -5, -3, 7, -3]$~~

$\therefore X_2(k) = [11, -3, 7, -3, -5, -3, 7, -3]$

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(5) Find $x(n)$ using DIF-IFFT for $X(K) = [0, 0, 4, 0, 0, 0, 0, 4]$.
Also find complex multiplications and additions in the signal flow graph.

$\Rightarrow$ Draw DIT-FFT algorithm and modify to convert it into DIF-IFFT to compute $x(n)$

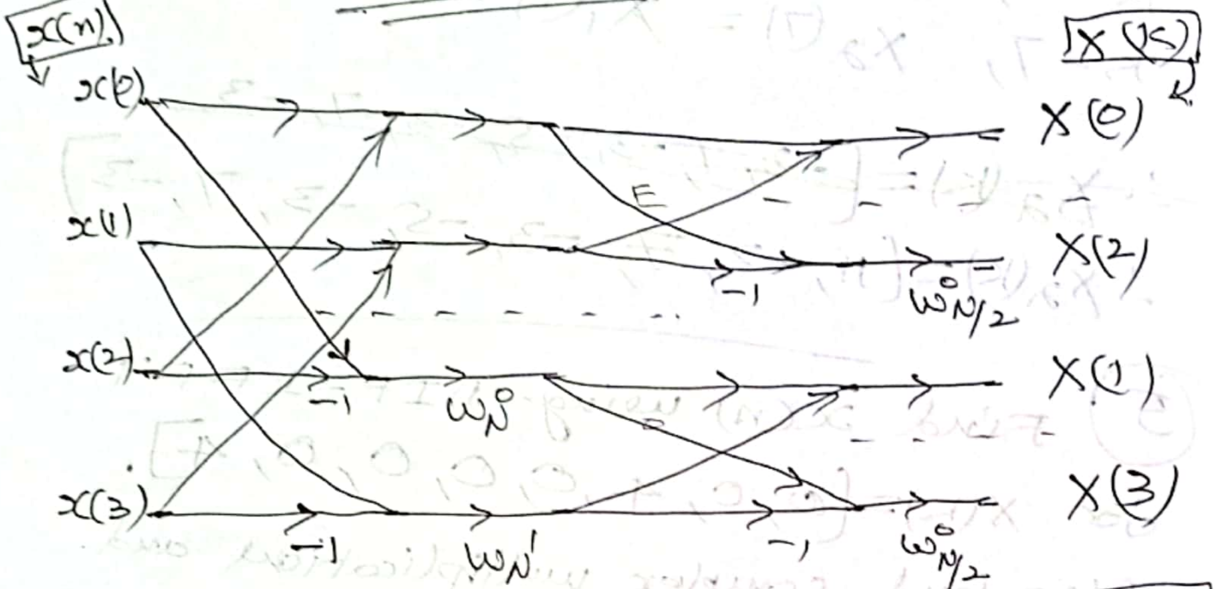
$x(n) = [8, (2.8 + j1.2), (-4 - j4), (-2.8 - j6.8), 0, (-2.8 + j6.8), (-4 + j4), (2.8 - j1.2)]$

Complex multiplications = 12
Complex additions = 24

Radix-2, DIT-IFFT Algorithm

For N = 4

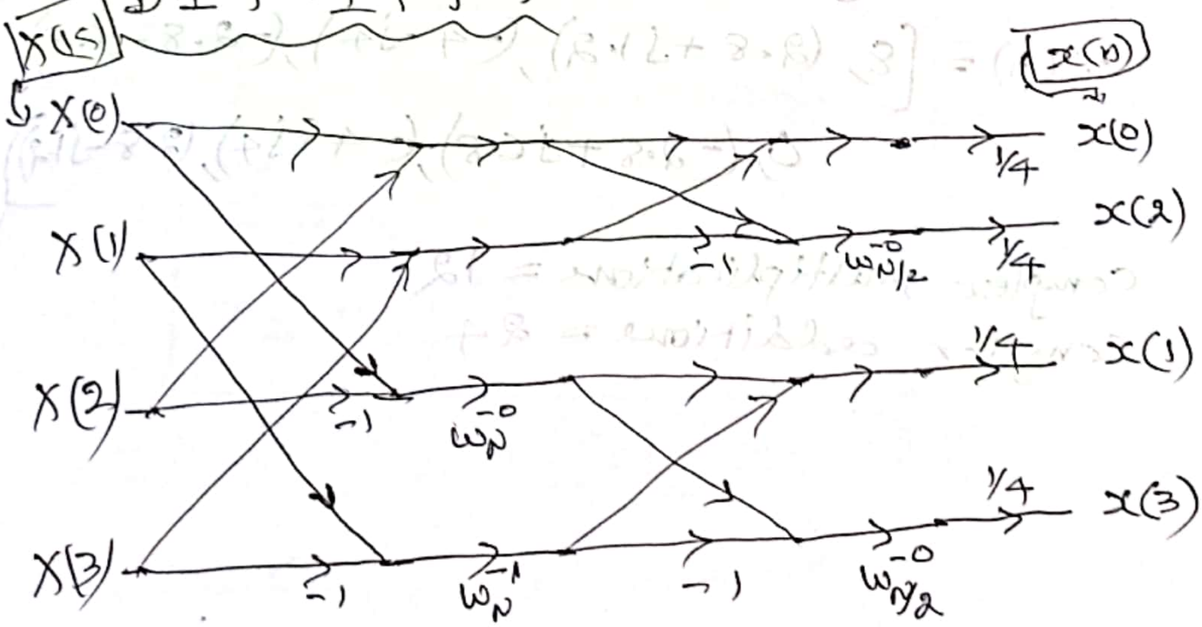
Write DIF-FFT signal flow graph.



Modify DIF-FFT To get $\boxed{\text{DIT-IFFT}}$

1. Interchang $x(n)$ & $X(k)$
2. Replace W_N^{kn} by W_N^{-kn}
3. Graph is multiplied by $\left(\frac{1}{N}\right)$

DIT-IFFT



(6) Find DFT $X(k)$ using DIF-FFT for $x(n) = n+1$ for $N=8$

=

(7) Find time sequence $x(n)$ using DIT-IFFT for

$X(k) = [\underset{\uparrow}{0}, [2\sqrt{2}(1-j)], 0, 0, 0, 0, 0, [2\sqrt{2}(1+j)]]$

=

(8) Use 8-point DIF Radix-2 FFT algorithm to find the DFT of the sequence $x(n) = [0.7, 1, 0.7, 0, -0.7, -1, -0.7, 0]$

=

(9) Find 8-point DFT ~~of the~~ using DIF-FFT signal graph. Given $x(n) = [1, -1, 1, -1, 1]$ for ~~log~~ length $N=5$.

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