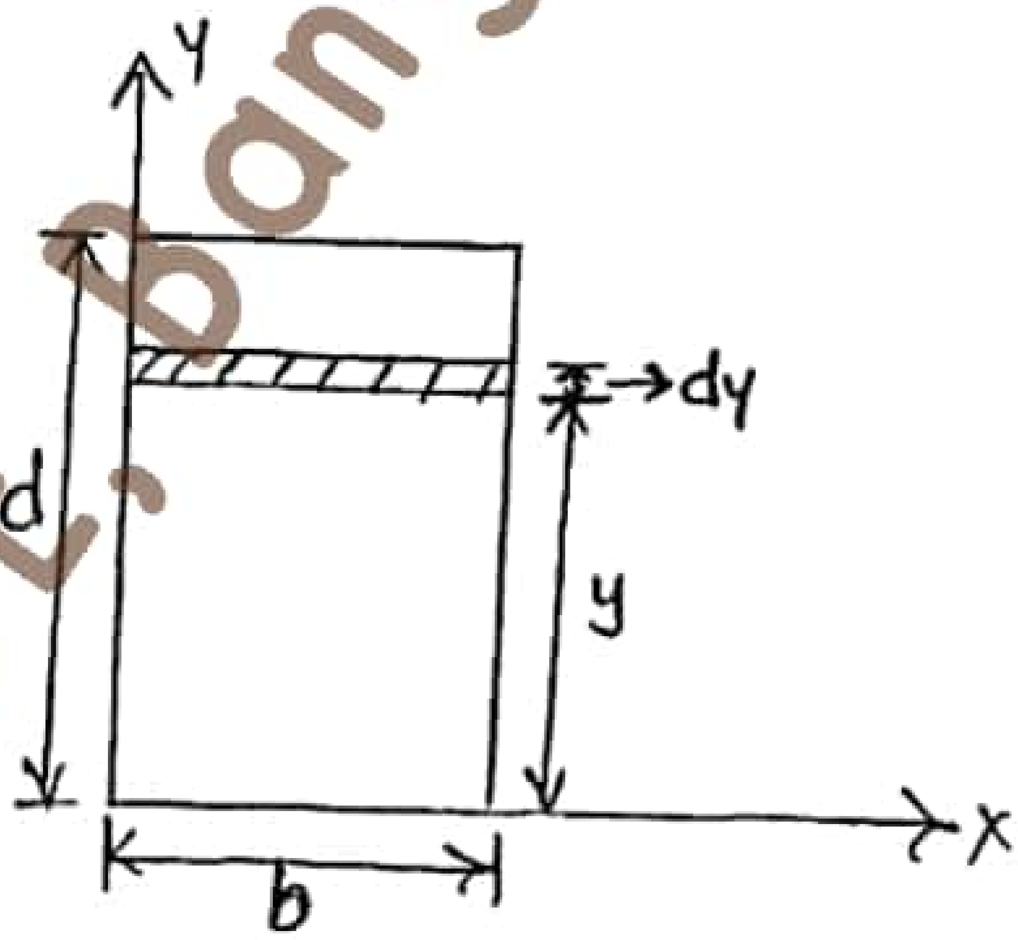
Centroid of Regular Geometrical Figures

1) Rectangle :-

Consider a rectangle of breadth 'b' and depth 'd' on x - y plane as shown in fig.

Let us consider an elemental area "da" of width "b" and depth "dy" at a distance 'y' from the x- axis.



Elemtal area $= da = b \times dy$

W.K.T $\quad \bar{y} = \dfrac{\int y \cdot da}{A}$

$$A\bar{y} = \int y \, da$$

$$= \int_0^d y \, b \, dy$$

$$= b \int_0^d y \, dy$$

$$= b \left[\frac{y^2}{2}\right]_0^d$$

Area of Rectangle

$A = b \times d$

$da = b \times dy$

$$A\bar{y} = \frac{bd^2}{2}$$

$$\cancel{b} \times \cancel{d} \; \bar{y} = \frac{\cancel{b} d^{\cancel{2}}}{2}$$

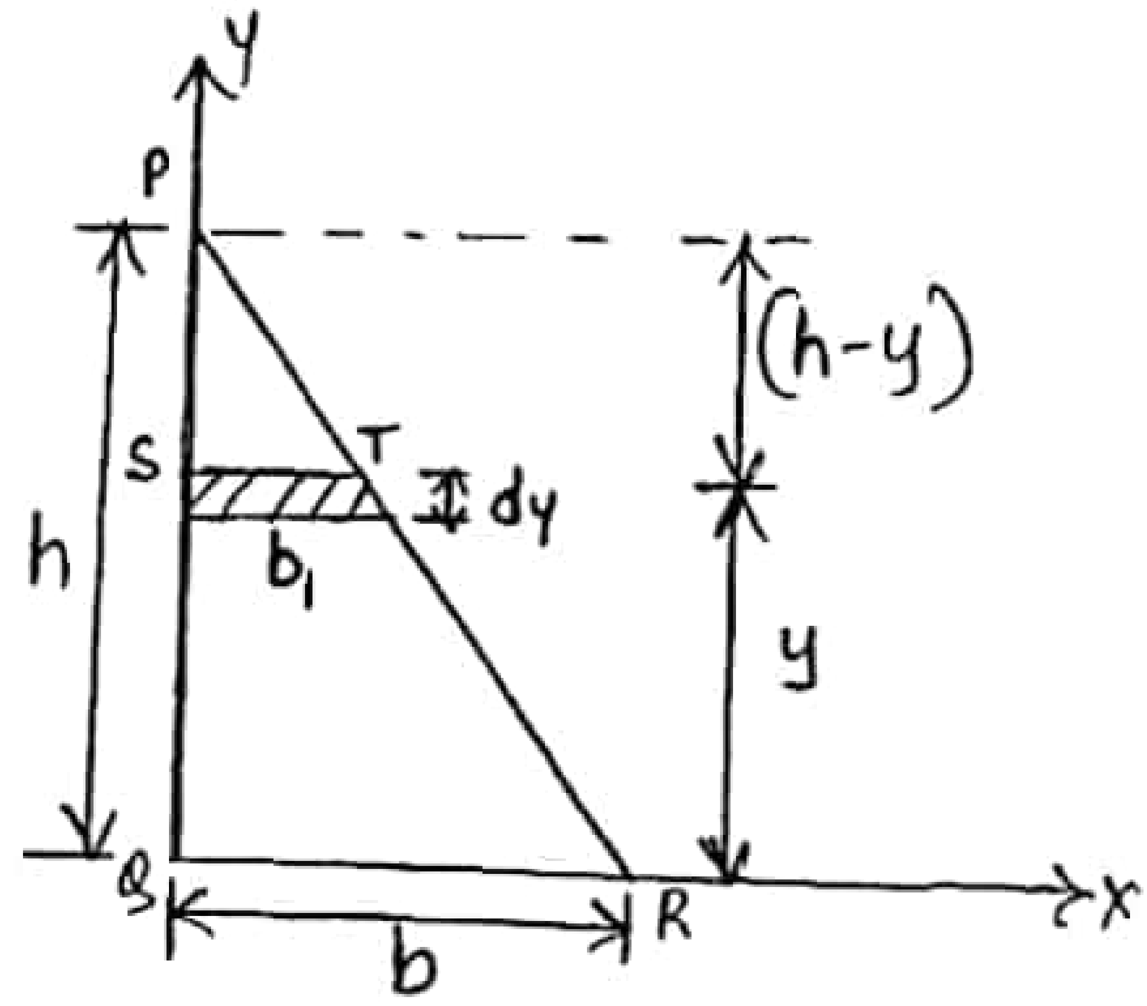
$$\therefore \boxed{\bar{y} = \frac{d}{2}} \qquad \text{III}^{ly} \qquad \boxed{\bar{x} = \frac{d}{2}}$$

2) Triangle

i) Right angle triangle

Consider a right angle Δ^{le} of base 'b' and height 'h' on xy plane as shown in fig.

Let us consider an elemental area of width b_1 and depth 'dy' at a distance y from the base (y- axis)



From similar Δ^{les} PQR & PST

$$\frac{b_1}{h-y} = \frac{b}{h}$$

$$\therefore \; b_1 = \frac{b(h-y)}{h}$$

Elemental area, $da = b_1 \times dy$ | Area of triangle, $A = \frac{1}{2} \times b \times h$

$$\therefore \; da = \frac{b(h-y)}{h}\,dy$$

W.K.T, $\bar{y} = \dfrac{\int y\,da}{A}$

$$A\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy$$

$$A\bar{y} = \frac{b}{h}\left[\int_0^h yh - \int_0^h y^2\right]dy$$

$$= \frac{b}{h}\left[\frac{hy^2}{2}\right]_0^h - \left[\frac{y^3}{3}\right]_0^h$$

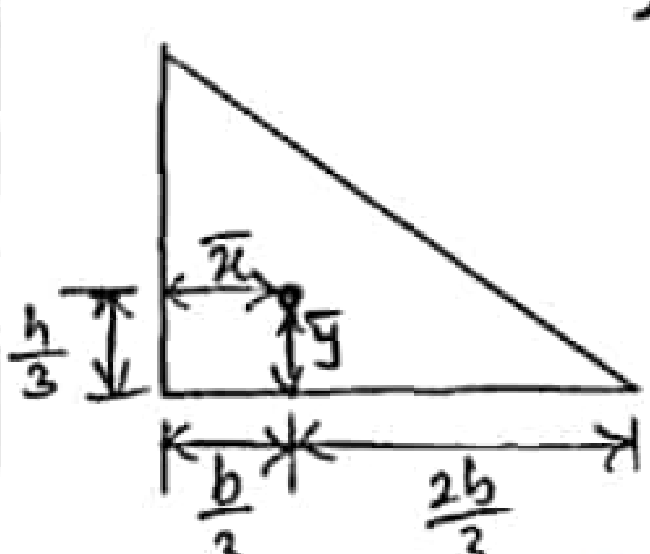
$$= \frac{b}{h}\left[\frac{h^3}{2} - \frac{h^3}{3}\right]$$

$$A\bar{y} = \frac{bh^{\cancel{3}}}{\cancel{h}6}$$

$$A\bar{y} = \frac{bh^2}{6}$$

| $A = \frac{1}{2} \times b \times h$

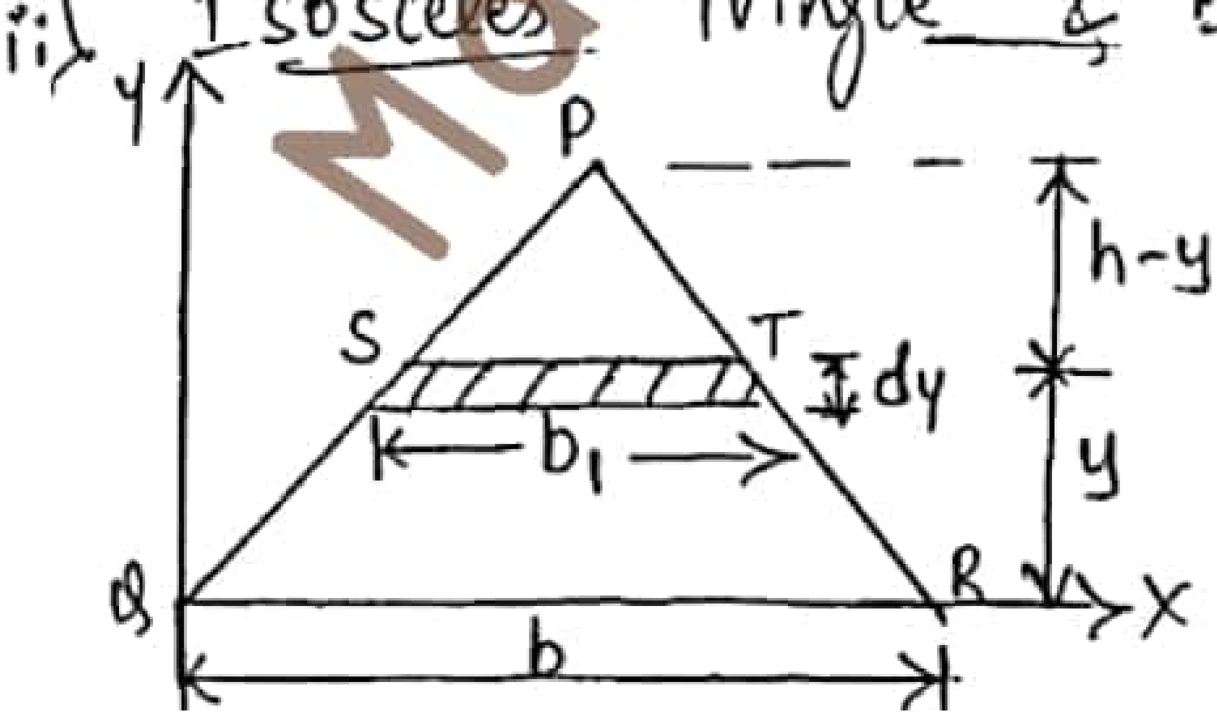
$$\frac{1}{2} \times \cancel{b} \times \cancel{h} \times \bar{y} = \frac{\cancel{b}\,h^{\cancel{2}}}{6}$$



$$\bar{y} = \frac{2h}{6} = \frac{h}{3}$$

$$\therefore \; \boxed{\bar{y} = \frac{h}{3}} \qquad \text{III}^{ly} \qquad \boxed{\bar{x} = \frac{b}{3}}$$

ii) Isosceles Tringle & Equilateral triangle

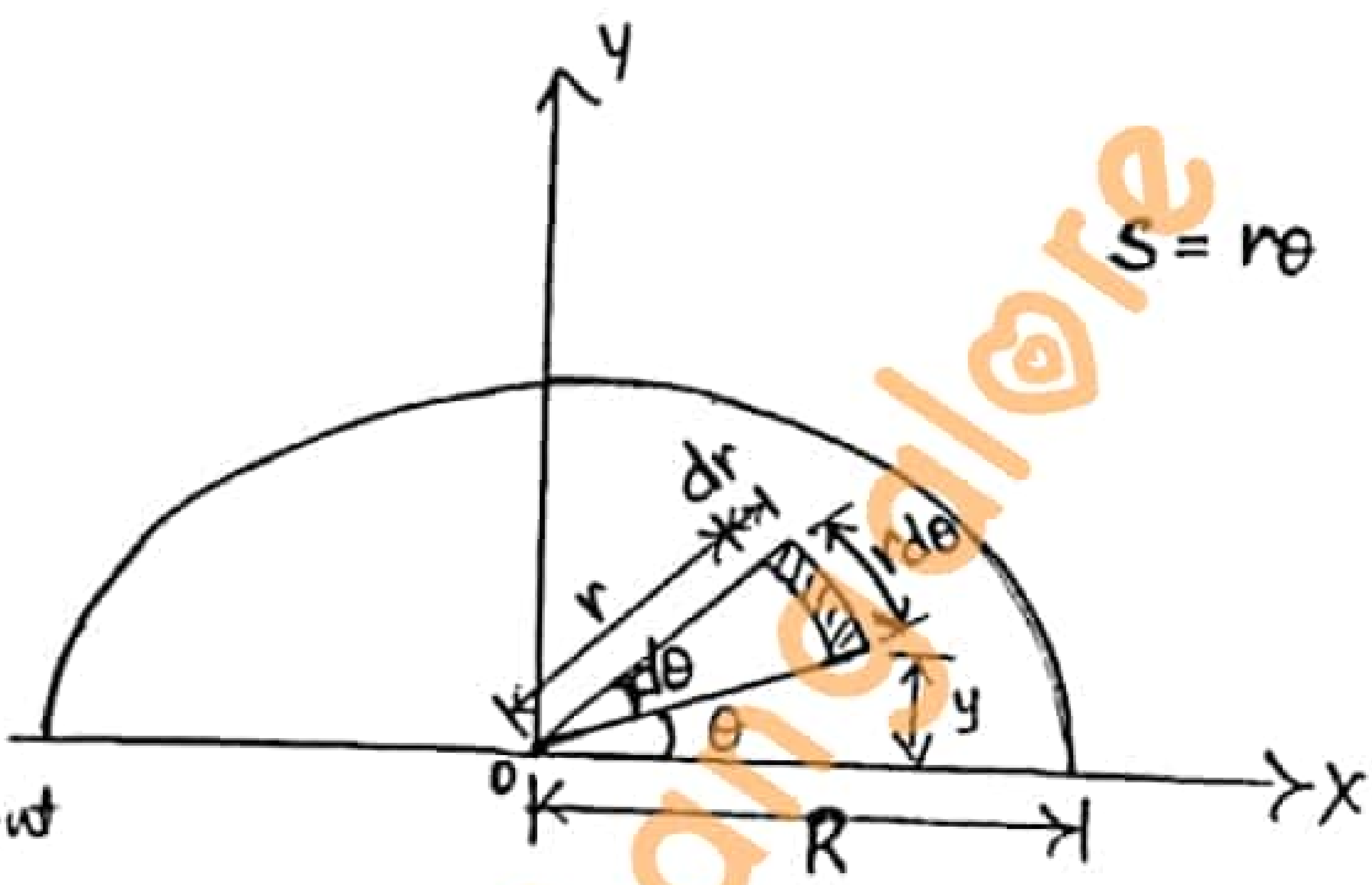


** Derivation part is same **

$$\therefore \; \boxed{\bar{y} = \frac{h}{3}} \quad \text{but} \quad \boxed{\bar{x} = \frac{b}{2}}$$

3) Semi circle

Consider a Semicircle of radius 'R' which is symmetrical about y-axis as shown in fig.



Let us consider an elemental area of sides $rd\theta$ & dr at a distance 'r' from O.

Semi circle is symmetrical about y-axis $\therefore$ $\boxed{\bar{x} = 0}$

$$\sin\theta = \frac{y}{r}$$

$$\boxed{y = r\sin\theta}$$

Elemental area = $da = rd\theta \times dr$ | Area of Semi circle, $A = \frac{\pi R^2}{2}$

W.K.T.,
$$\bar{y} = \frac{\int y\, da}{A}$$

$$A\bar{y} = \int_0^{\pi}\int_0^{R} y\, rd\theta\, dr$$

$$A\bar{y} = \int_0^{\pi}\int_0^{R} r\sin\theta\, rd\theta\, dr = \int_0^{\pi}\int_0^{R} r^2 \sin\theta\, d\theta\, dr$$

$$A\bar{y} = \int_0^{\pi} \sin\theta\, d\theta \int_0^{R} r^2 dr$$

$$= \int_0^{\pi} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^R = \int_0^{\pi} \left[\frac{R^3}{3}\right] \sin\theta\, d\theta$$

$$= \frac{R^3}{3}\int_0^{\pi} \sin\theta\, d\theta = \frac{R^3}{3}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{R^3}{3}\left[-\cos\pi + \cos 0\right]$$

$$= \frac{R^3}{3}\left[1+1\right] = \frac{2R^3}{3}$$

$$A = \frac{\pi R^2}{2}$$

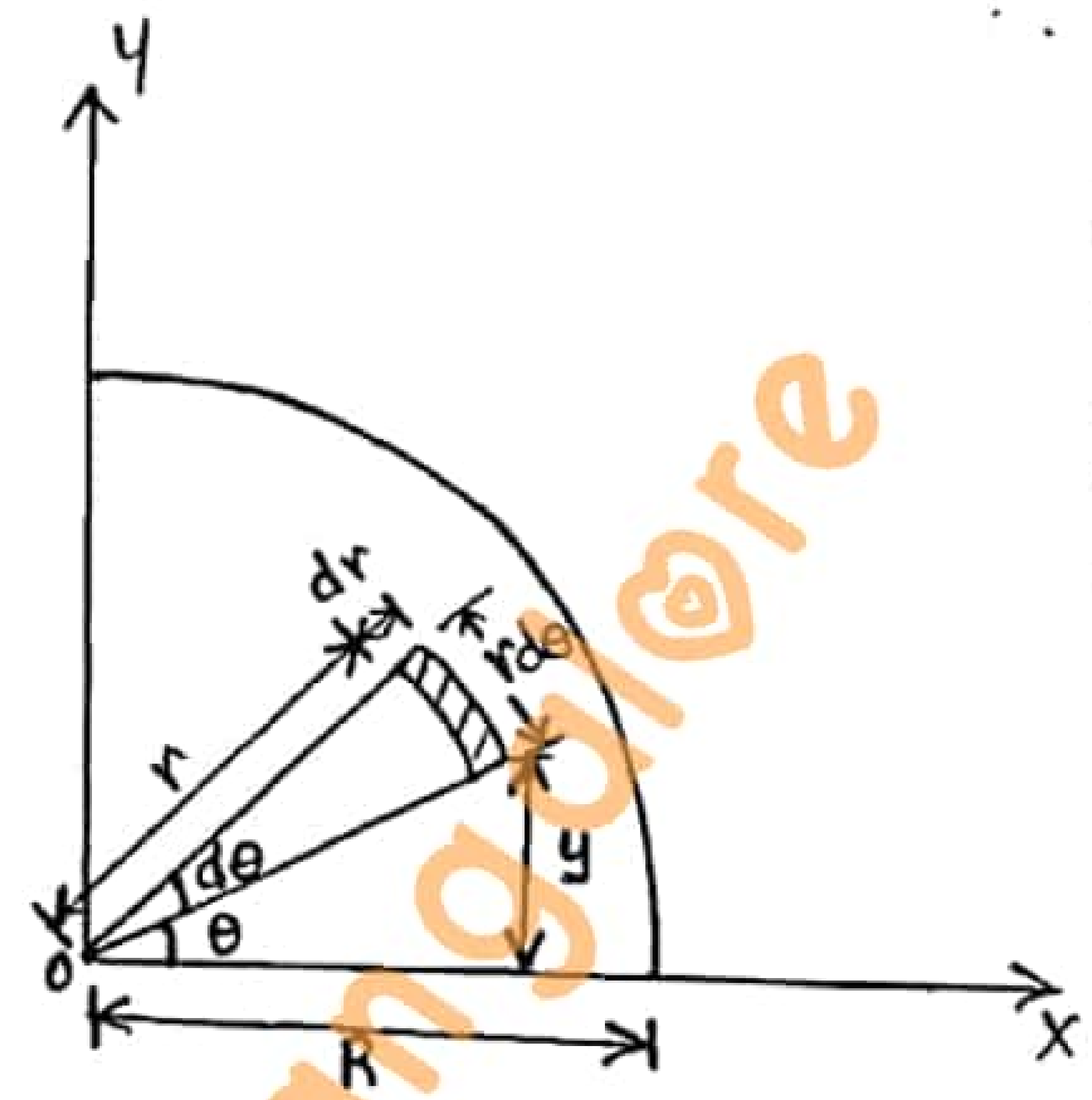
$$A\bar{y} = \frac{2R^3}{3}$$

$$\frac{\pi R^2}{2}\bar{y} = \frac{2R^3}{3}$$

$$\boxed{\therefore \bar{y} = \frac{4R}{3\pi}} \quad \& \quad \boxed{\bar{x} = 0}$$

4) <u>Quarter Circle</u>

Consider a quarter circle of a circle of radius 'R' (which is not symmetrical about any axis) on x-y plane as shown in fig.



Let us consider an elemental area of sides $rd\theta$ & dr at a distance of 'r' from o.

$$\sin\theta = \frac{y}{r}$$

$$\boxed{y = r\sin\theta}$$

Elemental area, $da = rd\theta \times dr$ | Area of quarter circle, $A = \frac{\pi R^2}{4}$

W.K.T,

$$\bar{y} = \frac{\int y\, da}{A}$$

$$A\bar{y} = \int_0^{\pi/2}\int_0^R r\sin\theta \times r\,d\theta \times dr$$

$$= \int_0^{\pi/2}\int_0^R r^2 \sin\theta\, d\theta\, dr$$

$$= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^R r^2 dr = \int_0^{\pi/2} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^R$$

$$= \int_0^{\pi/2} \sin\theta\, d\theta \left[\frac{R^3}{3}\right] = \frac{R^3}{3}\int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{R^3}{3}\left[\cos\theta\right]_0^{\pi/2}$$

$$= \frac{R^3}{3}\left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{R^3}{3}\left[0-(-1)\right]$$

$$A\bar{y} = \frac{R^3}{3}(1)$$

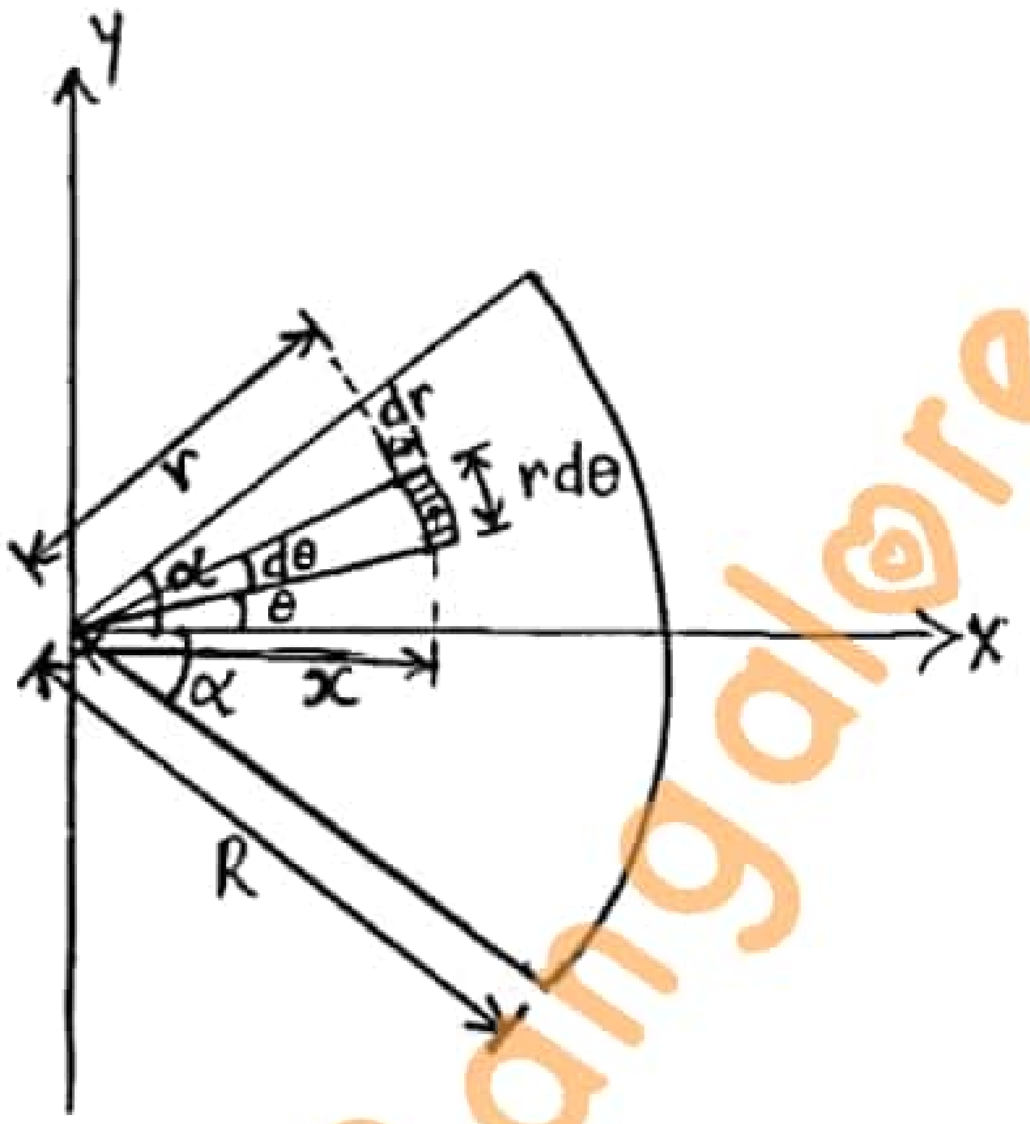
| $A = \frac{\pi R^2}{4}$

$$\frac{\pi \cancel{R^2}}{4} \times \bar{y} = \frac{R^{\cancel{3}}}{3}$$

$$\boxed{\bar{y} = \frac{4R}{3\pi}}$$ $\quad$ lll^{ly} $\quad$ $$\boxed{\bar{x} = \frac{4R}{3\pi}}$$

5) Sector :

Consider a sector of a circle symmetrical about x-axis, as shown in fig.



Let us consider an elemental area of sides $rd\theta$ & dr at a distance of r from O

$\cos\theta = \frac{x}{r}$ As fig is symm about x-axis

$\boxed{x = r\cos\theta}$ $\therefore \boxed{\bar{y} = 0}$

Elemental area $= \boxed{da = rd\theta \times dr}$

W.K.T, $\bar{x} = \frac{\int x\,da}{A}$

$$A\bar{x} = \int x\,da$$

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} (r\cos\theta)\,(rd\theta \times dr)$$

$$A\bar{x} = \int_{-\alpha}^{\alpha}\int_{0}^{R} (r^2 dr)\,(\cos\theta\, d\theta)$$

$$A\bar{x} = \left[\frac{r^3}{3}\right]_0^R \left[\sin\theta\right]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3} \times \left[\sin\alpha - (\sin -\alpha)\right]$$

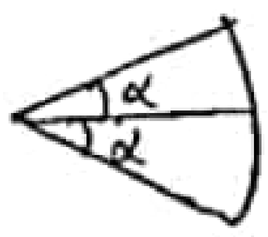
$$= \frac{R^3}{3}\left[\sin\alpha + \sin\alpha\right]$$

$$\boxed{A\bar{x} = \frac{R^3}{3}\, 2\sin\alpha} \longrightarrow (i)$$

Area of a sector of an included angle 'α' is

$$A = \frac{R^2\theta}{2}$$

θ = Total angle



$\therefore \boxed{\theta = 2\alpha}$

$$A = \frac{R^2\, 2\alpha}{2}$$

$$\therefore \boxed{A = R^2\alpha}$$

$$\therefore \quad R^2 \alpha \times \bar{x} = \frac{R^3}{3} 2 \sin \alpha$$

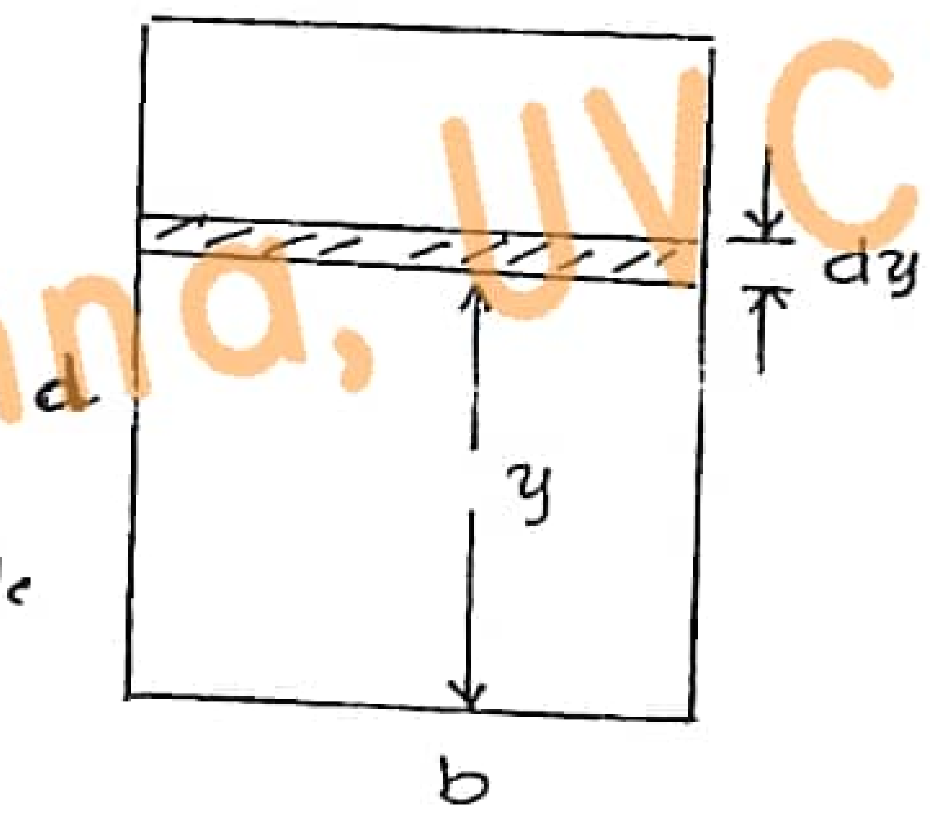
$$\therefore \quad \boxed{\bar{x} = \frac{2R \sin \alpha}{3\alpha}} \quad \& \quad \boxed{\bar{y} = 0}$$

Moment of Inertia of Various geometrical Sections

1. Moment of Inertia of a Rectangular section.

Consider a Rectangular lamina of base 'b' and of depth 'd' as shown in fig.

Consider a strip of thickness 'dy' placed parallel to the base at a distance 'y' from it.



Moment of inertia of the strip about the base of rectangle

$$= \text{area of strip} \times \text{distance}^2$$

$$= (b \cdot dy) \cdot y^2$$

$\therefore$ Moment of inertia of the whole rectangle about the base

$$I_{base} = \int_0^d (b \cdot dy) \cdot y^2 = b \int_0^d y^2 \cdot dy = b \left[\frac{y^3}{3} \right]_0^d$$

$$= \frac{b}{3} \left[d^3 - 0 \right] = \frac{bd^3}{3}$$

$$\boxed{I_{base} = \frac{bd^3}{3}}$$

Moment of inertia of the rectangular Section about its centroidal axis can be found using parallel axis theorem which says

$$I_{any\ axis} = I_G + Ah^2$$

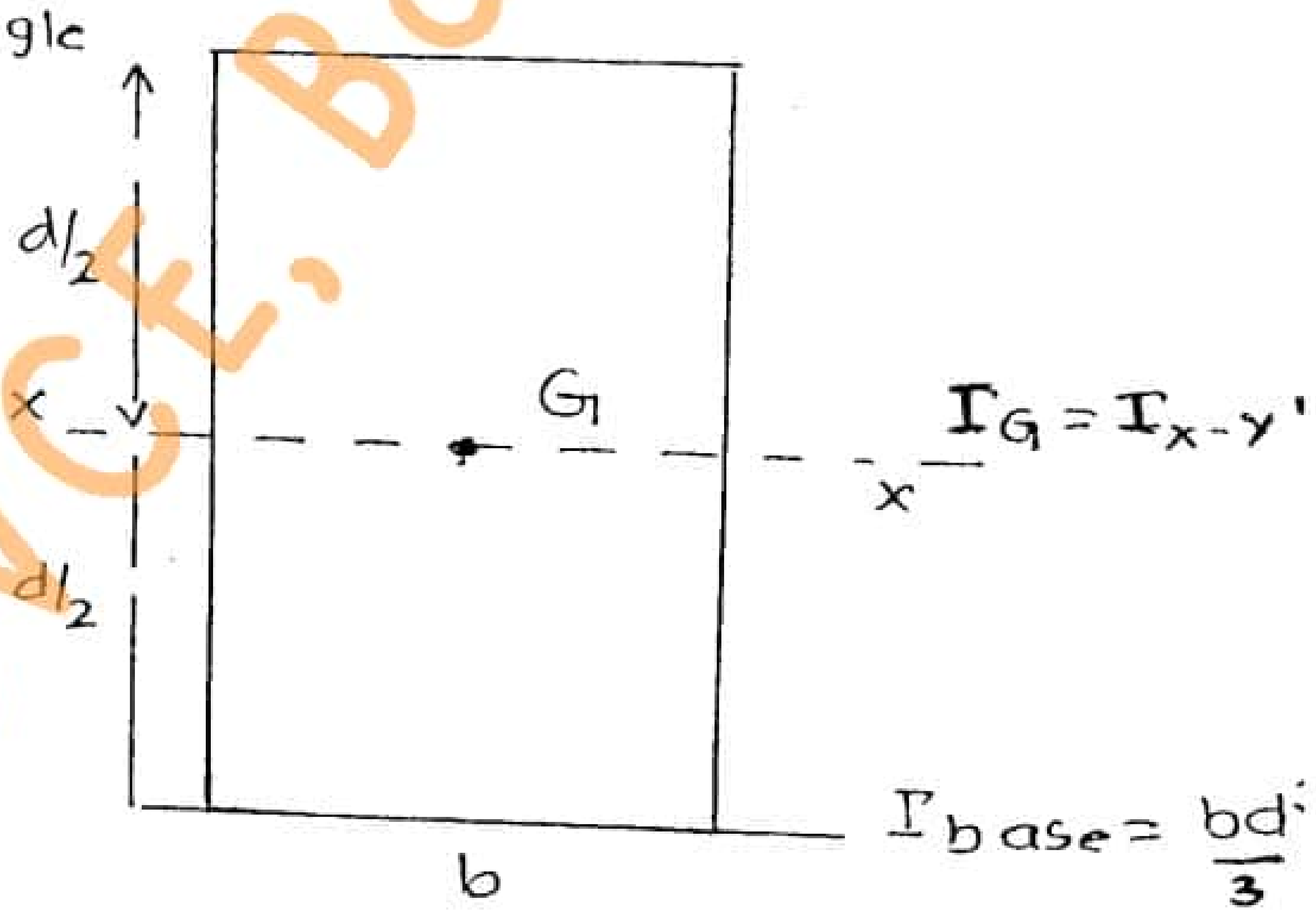
Where any axis = base of rectangle

I_G = I_{x-x} of the rectangle = ?

A = area of rectangle = bd

h = Distance b/n x-x axis and base of rectangle

$= d/2$



$$I_{base} = I_{x-x} + Ah^2$$

$$\frac{bd^3}{3} = I_{x-x} + bd\left(\frac{d}{2}\right)^2$$

$$= I_{x-x} + \frac{bd^3}{4}$$

$$I_{x-x} = \frac{bd^3}{3} - \frac{bd^3}{4}$$

$$= \frac{4bd^3 - 3bd^3}{12}$$

$$\boxed{I_{x-x} = \frac{bd^3}{12}}$$

|||ly,

$$\boxed{I_{y-y} = \frac{db^3}{12}}$$

Radius of Gyration (K)

It is defined as that distance from a given axis upto a point where the entire area is assumed to be concentrated

$$K = \sqrt{\frac{I}{A}}$$

Radius of gyration about x-x axis

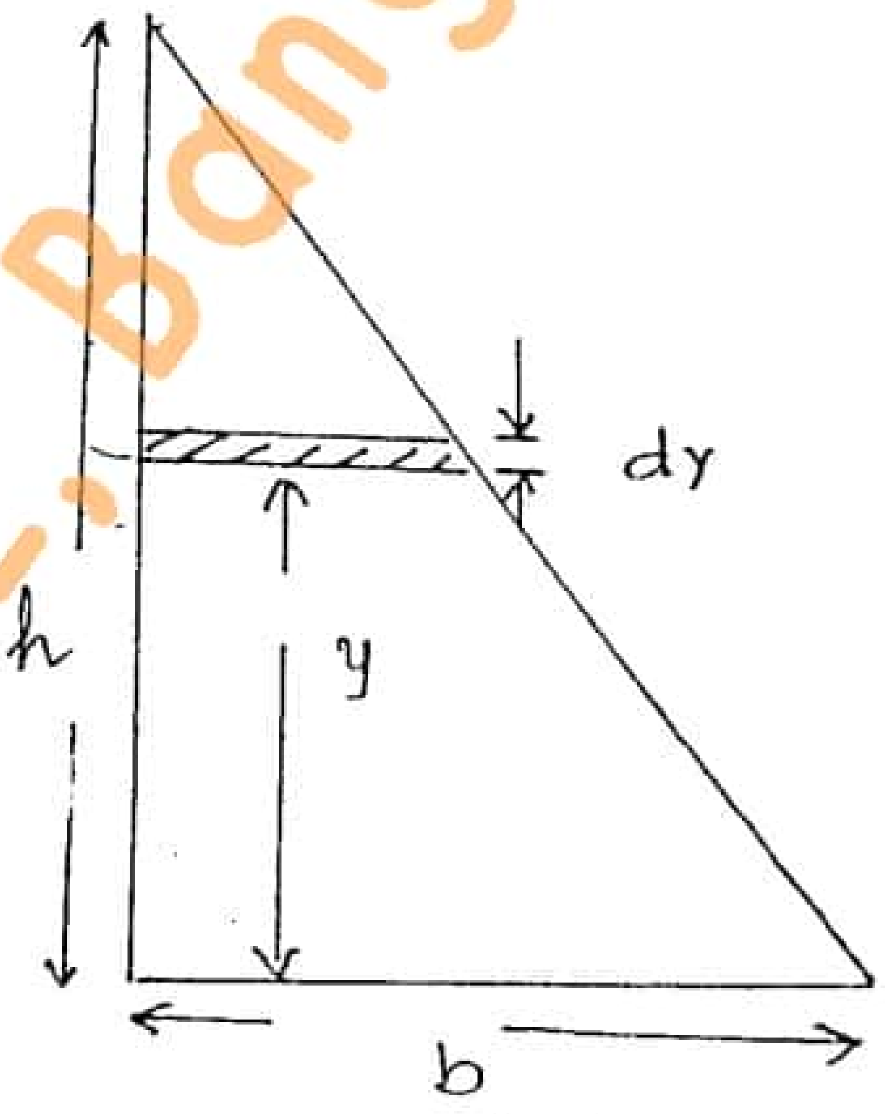
$$K_{x-y} = \sqrt{\frac{I_{x-x}}{A}}$$

Radius of gyration about y-y axis

$$K_{y-y} = \sqrt{\frac{I_{y-y}}{A}}$$

2. Moment of Inertia of a triangular lamina.

Consider a right angled triangle of base 'b' and of height 'h'. Let us consider a rectangular strip of thickness 'dy' placed parallel to the base at a distance 'y' from it, as shown in fig.



Width of the strip $= \frac{b}{h}(h-y)$ from similar property of triangles

Area of strip = width × thickness

$= \frac{b}{h}(h-y) \cdot dy$

∴ Moment of Inertia of the strip about the base of triangle

$= \text{Area of strip} \times \text{Distance}^2$

$= \frac{b}{h}(h-y) \cdot dy \cdot y^2$

∴ Moment of Inertia of the whole triangle about the base can be found by integrating the above eqn b/n the limits 0 to h.

$$I_{base} = \int_0^h \frac{b}{h}(h-y) \cdot dy \cdot y^2 = \frac{b}{h}\int_0^h (hy^2 - y^3)\, dy$$

$$= \frac{b}{h}\left[\frac{h \cdot y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$= \frac{b}{h}\left[\frac{h \cdot h^3}{3} - \frac{h^4}{4}\right]$$

$$= \frac{b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right] = \frac{b}{h}\left[\frac{4h^4 - 3h^4}{12}\right]$$

$$= \frac{b}{h} \times \frac{h^4}{12} = \frac{bh^3}{12}$$

∴ Moment of Inertia of the triangle about its base $= I_{base} = \frac{bh^3}{12}$

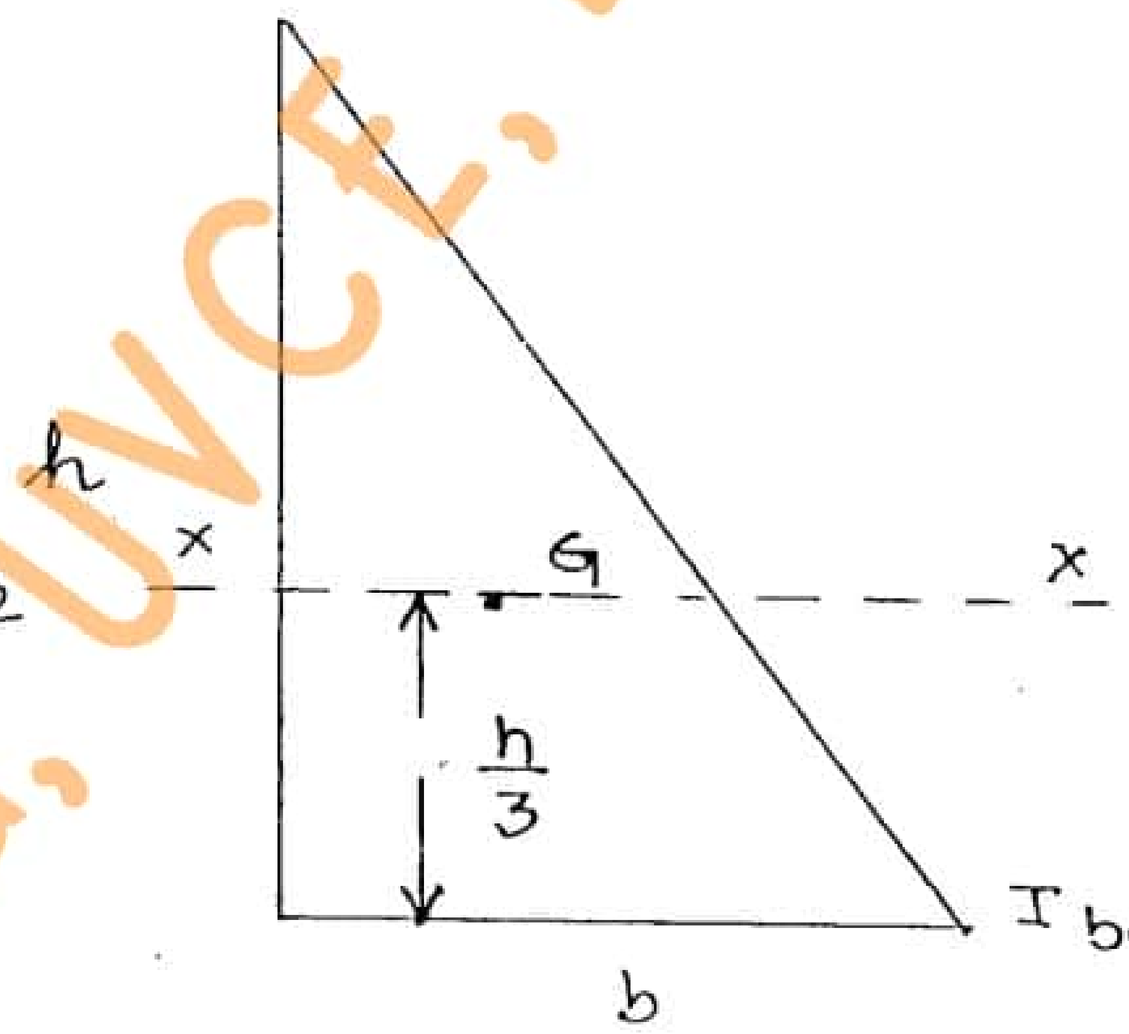
Moment of Inertia of a triangle about its centroidal axis can be found by making use of parallel axis theorem which says

$$I_{any\,axis} = I_G + Ah^2$$

where I_G = M.O.I. of a triangle about centroidal axis.

A = area of the triangle $= \frac{1}{2} \cdot b \cdot h$.

h = Distance b/n the centroidal axis and reference axis (ie base) $= \frac{h}{3}$



$I_{base} = \frac{bh^3}{12}$

$$\therefore I_{any\,axis} = I_G + Ah^2$$

$$I_{base} = I_{x-x} + Ah^2$$

$$\frac{bh^3}{12} = I_{x-x} + \frac{bh}{2}\left(\frac{h}{3}\right)^2$$

$$= I_{x-x} + \frac{bh^3}{18}$$

$$\therefore I_{x-x} = \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{3bh^3 - 2bh^3}{36}$$

$$\boxed{I_{x-x} = \frac{bh^3}{36}}$$

This is M.O.I. of the triangle about its centroidal axis.

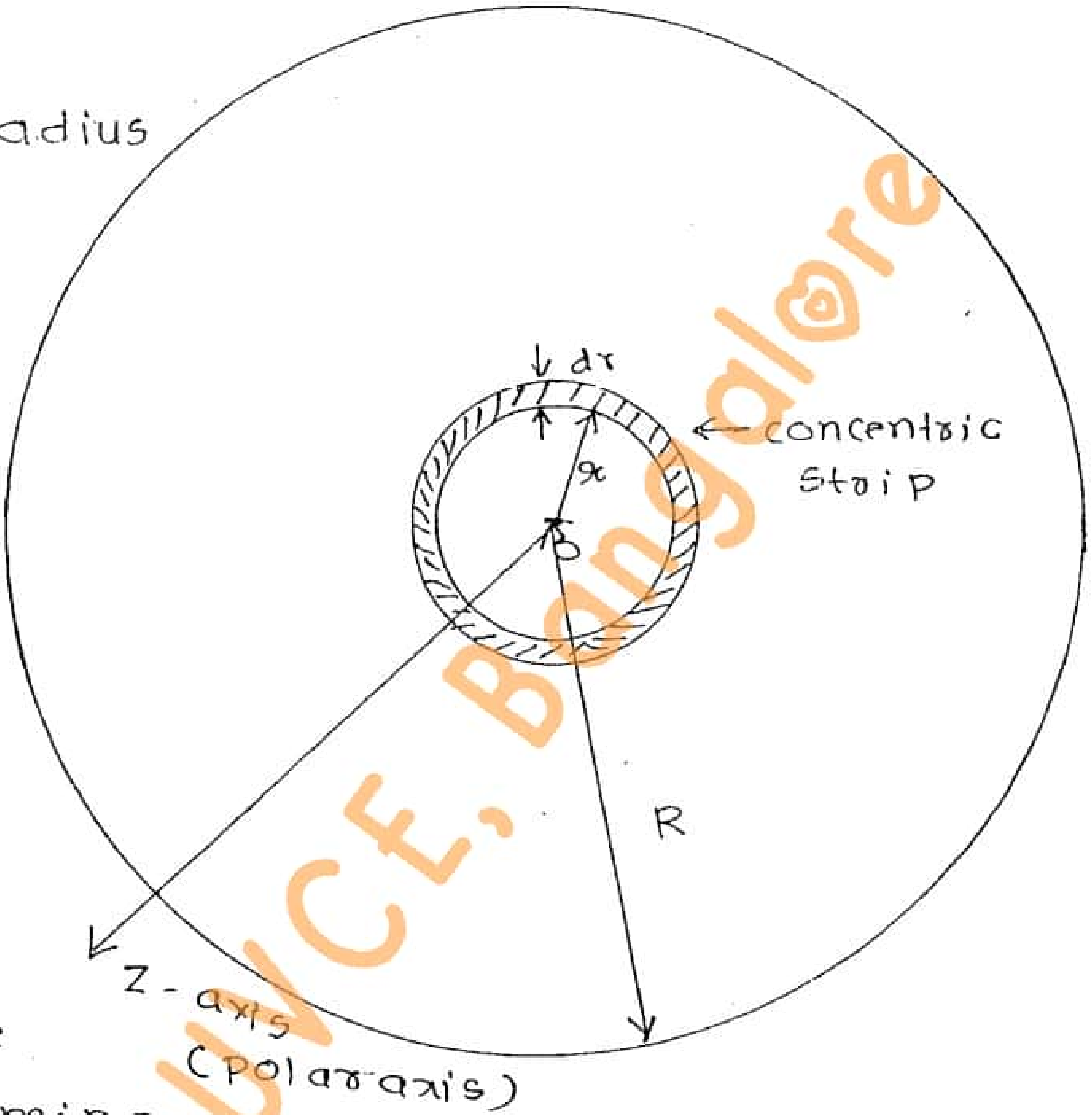
Note: 1. M.O.I of a triangle about its base $= \frac{bh^3}{12}$

2. M.O.I of a triangle about its centroidal axis $= \frac{bh^3}{36}$

3. Moment of Inertia of a circular sections:-

3.1. M.O.I of a circle

Consider a circular lamina of radius 'R'. Let us consider a circular concentric ring of thickness 'dr' at a distance 'r' from the centre of the circular lamina.



Consider Z axis (ie polar axis) through the centre of the circle (ie O) ⊥r to the plane of area of circular lamina.

Then area of the elemental ring = $2\pi r \cdot dr$

M.O.I of the ring about z-axis (ie polar axis) is

$$= \text{area of the ring} \times \text{Distance}^2$$

$$= 2\pi r \cdot dr \cdot r^2$$

$$= 2\pi r^3 dr$$

∴ M.O.I of the whole circular lamina about Z-axis can be found by integrating the above eqn b/n the limits 0 to R.

$$I_{\text{Polar axis}} = I_z = \int_0^R 2\pi r^3 dr = 2\pi \int_0^R r^3 dr$$

$$= 2\pi \left[\frac{r^4}{4} \right]_0^R$$

$$= \frac{2\pi}{4} \left[R^4 - 0 \right]$$

$$I_{\text{Polar axis}} = \frac{\pi R^4}{2} = I_{z\text{-axis}}$$

But from ⊥r axis theorem we know that

$$I_{\text{Polar axis}} = I_{z-z} = I_{x-x} + I_{y-y}$$

If a lamina is symmetrical about both axis and all 4 areas are equal, then $I_{x-x} = I_{y-y}$. Since circular section is symmetrical about both the axis and also all 4 areas are equal, then its I_{x-x} and I_{yy} values are equal

$$\therefore \quad I_p = I_z = I_{x-x} + I_{yy}$$

$$= 2I_{x-x} \quad \because I_{x-x} = I_{y-y}$$

$$\frac{\pi R^4}{2} = 2I_{x-x}$$

$$\therefore \quad \boxed{I_{x-x} = \frac{\pi R^4}{4} = I_{y-y}}$$

3.2 Moment of inertia of Semi-circular lamina:

M.O.I of a Semi-circular lamina can be found by making use of parallel axis theorem which says

$$I_{\text{any axis}} = I_G + Ah^2$$

where I_G = M.O.I of a semicircle about its centroidal axis (say x-x axis)

A = area of semicircle $= \frac{\pi R^2}{2}$

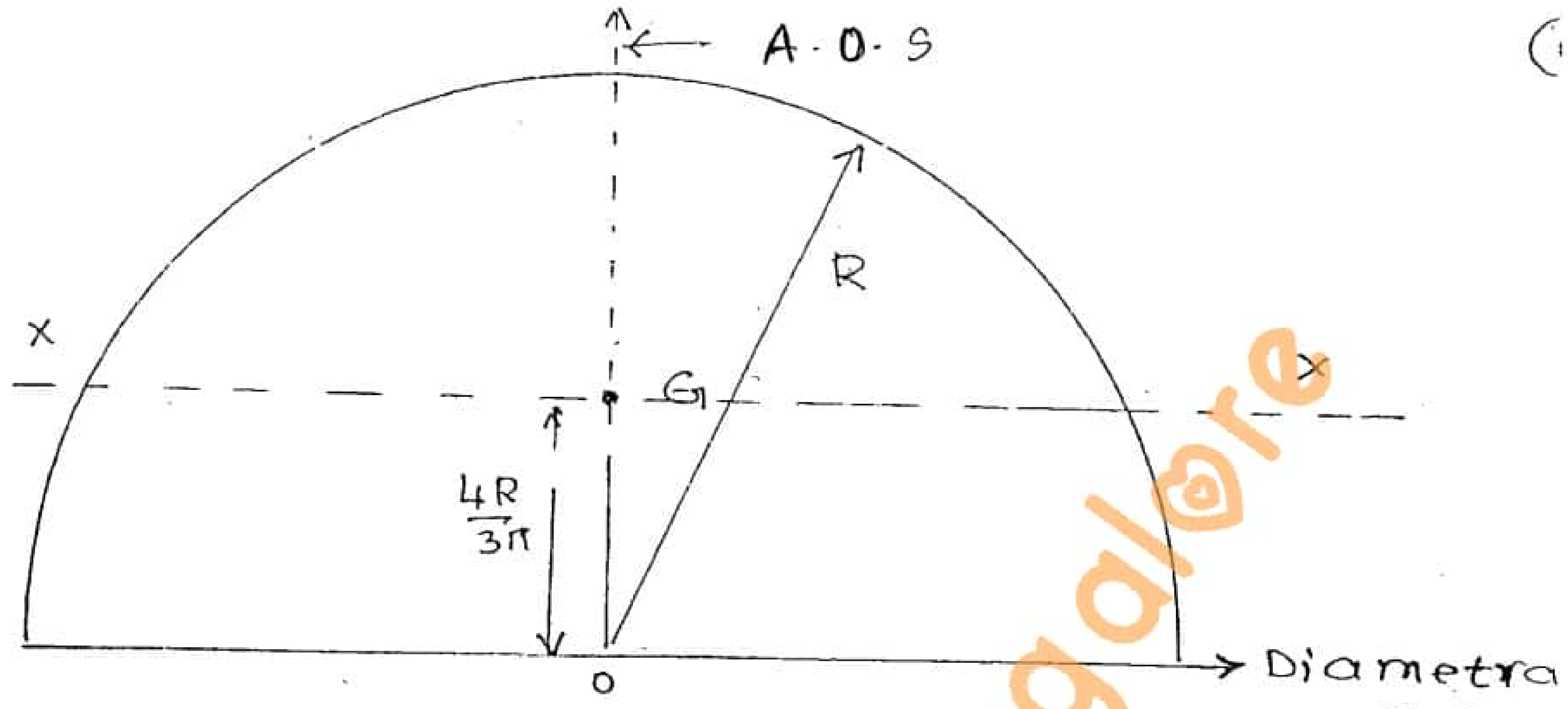
h = Distance b/n the centroidal axis (x-x) and reference axis (say base of semi-circle)

$$= \frac{4R}{3\pi}$$

M.O.I. of circle about its centroidal x-x axis

$$= \frac{\pi R^4}{4}$$

Then M.O.I of Semi-circle about the diametric axis (ie. base of semi-circle) $= \frac{\pi R^4}{8}$



Wkt, $I_{\text{any axis}} = I_G + Ah^2$

Let any axis be the diametral axis where the M.O.I of semicircle is $\frac{\pi R^4}{8}$

$$\therefore I_{\text{Diametral axis}} = I_{x-x} + Ah^2$$

$$\frac{\pi R^4}{8} = I_{x-x} + \frac{\pi R^2}{2}\left(\frac{4R}{3\pi}\right)^2$$

$$= I_{x-x} + \frac{16\pi R^4}{18\pi^2}$$

$$= I_{x-x} + \frac{16R^4}{18\pi}$$

$$\therefore I_{x-x} = \frac{\pi R^4}{8} - \frac{16R^4}{18\pi}$$

$$= R^4\left[\frac{\pi}{8} - \frac{16}{18\pi}\right]$$

$$= R^4(0.1097)$$

$$\boxed{I_{x-x} = 0.11 R^4} \qquad \because 0.1097 \simeq 0.11$$

$I_{y-y} = \frac{\pi R^4}{8}$ = M.O.I of semicircle about its centroidal y-y axis

NOTE:

The values of I_{x-x} and I_{y-y} for a semicircle are not constant. If the diametral axis of a semicircle is horizontal then $I_{x-x} = 0.11R^4$ and $I_{y-y} = \frac{\pi R^4}{8}$. If the diametral axis is vertical, then $I_{x-x} = \pi R^4/8$ and $I_{yy} = 0.11R^4$

3.3. M.O.I of a quarter of a circle :

Let us consider the quadrantic of radius 'R' as shown in fig. The M.O.I of a quadrant about its centroidal axis can be found by using parallel axis theorem

$$\therefore I_{any\,axis} = I_G + Ah^2$$

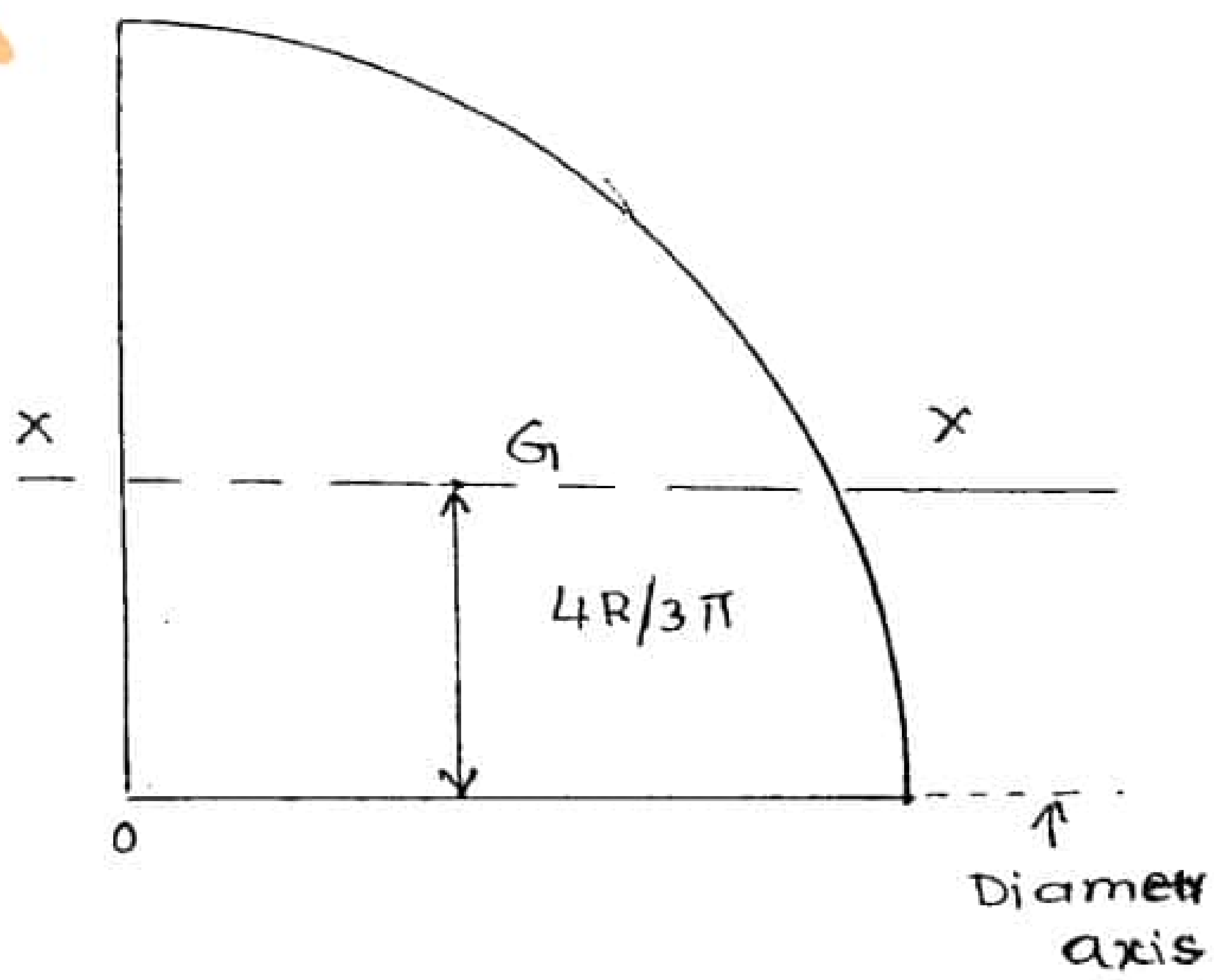
Where I_G = M.O.I of a quadrant about its centroidal axis (say x-x)

A = Area of quadrant

$= \frac{\pi R^2}{4}$

h = Distance b/n the centroidal axis (x-x) and the reference axis (say diametral axis)

$= \frac{4R}{3\pi}$



M.O.I of a circle about its centroidal x-x $= \frac{\pi R^2}{4}$

$\therefore$ Moment of Inertia of quadrant about the same axis (ie Diametral axis)

$$= \frac{\pi R^4}{16}$$

W.K.T.

$$I_{any\,axis} = I_G + Ah^2$$

$$I_{\substack{Diametral\\axis}} = I_{x-x} + Ah^2$$

$$\frac{\pi R^4}{16} = I_{x-x} + \frac{\pi R^2}{4}\left(\frac{4R}{3\pi}\right)^2$$

$$= I_{x-x} + \frac{16\pi R^4}{36\pi^2}$$

$$= I_{x-x} + \frac{16R^4}{36\pi}$$

$$\therefore I_{x-x} = \frac{\pi R^4}{16} - \frac{16R^4}{36\pi}$$

$$= R^4\left[\frac{\pi}{16} - \frac{16}{36\pi}\right]$$

$\therefore \quad I_{x-x} = R^4 (0.055)$

$$I_{x-x} = 0.055 R^4$$

The value of I_{y-y} in case of a quadrant is same as I_{x-x}

$\therefore \quad \boxed{I_{x-x} = I_{y-y} = 0.055 R^4}$

now