unit-4 Matrices

Rank of a matrix by elementary Transformations:

Elementary Transformations associated with a matrix:

The following are the elementary row Transformations of a matrix.

1. Interchange of any 2 rows.
2. Multiplication of any row by a non zero constant.
3. Addition to any row by a constant multiple of any other row.

Equivalent matrices:

Two matrices A and B of the same order are said to be equivalent if one matrix can be obtained from the other by a finite number of successive elementary row (column) transformations. It is denoted by ~~A~B~~ $A \sim B$.

Echelon form of a matrix:

A non-zero matrix A is said to be in row echelon form if the following conditions satisfies:

(i) All the zero rows are below non zero rows.

(ii) The first non zero entry in any non zero row is 1.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of a matrix:

The rank of a matrix A in its echelon form is equal to the number of non zero rows. It is denoted by $\rho(A)$.

Steps to be followed to solve the problems: Finding the rank of a matrix:

(i) In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry non zero, much preferably 1.

(ii) In case this entry is zero we can interchange with any suitable row to meet the requirement.

(iii) We then focus on the leading non zero entry (starting from the first row) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

(iv) Row echelon form will be achieved first and we can instantly write down the rank, which being the number of non-zero rows.

Examples:

Find the rank of the foll. matrices by elementary row transformations:

1. $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Ex: consider,

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to R_3 - R_1 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad R_3 \to R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having 2 non-zero rows.

Thus $\rho(A) = 2$

2. $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ (Ans: $\rho(A) = 2$)

3. $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ (Ans: $\rho(A) = 3$)

(Note:

1. Any $n \times n$ matrix A has rank $r < n$ iff $|A| = 0$ then A is singular.

Ex: 1, 2, 3 are singular matrices.)

Note:

1. The rank of every non-singular matrix of order 'n' is 'n'.
2. The rank of square matrix A of order 'n' can be less than 'n' iff A is singular matrix.
3. If there is a matrix A which has atleast one non-zero minor of order 'n' & there is no minor of A of order n+1, then the rank of A is 'n'.
4. The rank of every non-zero matrix is greater than or equal to one. The rank zero to every null matrix.

Defn:

A positive integer 'r' is said to be a rank of A if the foll. 2 condn are satisfied

(i) A has atleast one non-zero minor of order 'r'.

(ii) Every minor of A whose order is greater than 'r' is equal to zero.

Examples:

1. Find the rank of a matrix by using elementary transformations;

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 + C_1 \\ C_3 \to C_3 + 2C_1 \\ C_4 \to C_4 + 4C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\sim 1 \begin{bmatrix} 5 & 3 & 7 \\ 4 & 9 & 10 \\ 9 & 12 & 17 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$A \sim \begin{bmatrix} 5 & 3 & 7 \\ 9 & 12 & 17 \\ 9 & 12 & 17 \end{bmatrix}$$

$$A \sim \begin{vmatrix} 5 & 3 & 7 \\ 9 & 12 & 17 \\ 9 & 12 & 17 \end{vmatrix} = 0 \quad (\because \text{2 rows are equal } \det(A)=0)$$

Minor's of matrix A,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 4 & 9 \end{vmatrix} = 1(45-12) = 33 \neq 0$$

∴ The rank of matrix A is 3 i.e $\rho(A)=3$.

2. Find the value of 'a' in order that the matrix A is of rank 2, where

$$A = \begin{bmatrix} 6 & a & -1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

Sol^n: wkt, Any $n \times n$ matrix A has rank $r < n$ iff $|A| = 0$ then A is singular.

In this problem, it is given that the rank of matrix A is less than the order of the matrix which means the ~~matrix must be zero~~ matrix A is singular.

$\therefore |A| = 0$

$$\Rightarrow \begin{vmatrix} 6 & a & -1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 0$$

$\Rightarrow 6(6-4) - a(4-3) - 1(8-9) = 0$

$\Rightarrow 6(2) - a(1) - 1(-1) = 0$

$\Rightarrow 12 + 1 = a \Rightarrow \boxed{a = 13}$

3. Find the value of 'a' in order that the matrix A is of rank 3, where

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ a & 2 & 2 & 2 \\ 9 & 9 & a & 3 \end{bmatrix}$$

4. Find the inverse of the matrix A by using elementary transformations, where

(i) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol: The given matrix can be written as

$$A = IA$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \quad R_2 \to R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \quad \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + 2R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & +1 \\ 1 & 1 & +2 \\ 0 & 0 & 1 \end{bmatrix} A, \quad R_3 \to R_3 + 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A, \quad R_1 \to R_1 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A.$$

wkt, $AA^{-1} = I = A^{-1}A$ (orthogonal matrix).

Hence, $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

Normal form of a matrix:

A non zero matrix A can be reduced to one of the foll. 4 forms called the normal form of a matrix

(i) $[I_r]$ (ii) $[I_r \;\; 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

The nos 'r' is called rank of matrix A. i.e $\rho(A) = r$

Example:

1. Find the rank of the foll. matrices by reducing them to the normal form.

(i) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Apply: $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$ Apply $C_2 \to C_2 - C_1$
$C_3 \to C_3 + C_1$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$ $C_2 \to -\frac{1}{5} C_2$
$C_3 \to \frac{1}{6} C_3$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $C_3 \to C_3 - C_2$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which is in the normal form.

$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \rho(A) = 2.$

2. $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Examples:

1. Find the rank of the following matrices by reducing them to the normal form:

a) $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $R_2 \to 2R_1 + R_2$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_4$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$ $R_3 \to R_3 + 2R_2$, $R_4 \to R_4 - 3R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ $R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1 \\ C_4 \to C_4 - C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} C_3 \to C_3 - C_2 \\ C_4 \to C_4 - C_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to \frac{1}{2} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 3$$

b) $$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 \to R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} C_3 \to C_3 - C_1 \\ C_4 \to C_4 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} C_3 \to C_3 + 3C_2 \\ C_4 \to C_4 - C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

Linear equations:

Consider a set of 'm' linear equations in 'n' unknowns as follows,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where a_{ij}'s and b_i's are constants.

If $b_1, b_2, \ldots b_m$ are all zero, Then the system is called to be homogeneous. otherwise non-homogeneous.

The above system of equations can be written in the matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e $AX = B$

If $x_1 = x_2 = \cdots = x_n = 0$ is obviously a sol^n of the homogeneous system of eqns and is called Trivial solution.

If atleast one $x_i (i = 1, 2, \ldots, n)$ is not equal to zero then it is called a non-trivial solution.

The concept of the rank of a matrix helps us to conclude

(i) whether the system is consistent or not

(ii) whether the system possess unique sol^n or many solutions.

Condition for consistency & types of solⁿ:

Consider a system of 'm' equations in 'n' unknowns represented in the matrix form $AX = B$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called the coefficient matrix.

The matrix augmented to A with the extra column consisting of the elements of B is called the augmented matrix. It is denoted by $[A : B]$

i.e
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix eqn $AX = B$ is consistent if $\rho[A] = \rho[A : B]$.

Suppose, $\rho[A] = \rho[A:B] = r$, Then There is 2 types of solⁿ

(i) unique solⁿ:
$$\rho[A] = \rho[A:B] = r = n, \quad n \rightarrow \text{nos of unknowns}$$

(ii) Infinite solⁿ:
$$\rho[A] = \rho[A:B] = r < n, \text{ where } (n-r)$$
unknowns can take arbitrary values.

Note:

If $\rho[A] \neq \rho[A:B] \Rightarrow$ the system is inconsistent (doesnot posses a solⁿ).

Examples:

Test for consistency and solve:

1. $x+y+z=6$

$x-y+2z=5$

$3x+y+z=8$

Sol^n: ~~$[A:B]$~~ $[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$

consider,

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$\therefore \rho[A] = 3$ and $\rho[A:B] = 3$ i.e $r=3$

Also, nos of unknowns $n=3$

Since $\rho[A] = 3 = \rho[A:B]$ and $r = 3 = n$

$\therefore$ The given system of eq^{n}s is consistent and has unique sol^n.

Now $[A:B]$ can be written as linear eq^{n}s,

i.e $\quad x+y+z=6$

$\quad -2y+z=-1$

$\quad -3z=-9 \Rightarrow z=3$

$\therefore -2y+3=-1 \Rightarrow y=2$

$\therefore x+2+3=6 \Rightarrow x=1$

Thus, $x=1, y=2, z=3$ is the unique sol^n.

2. $\begin{cases} x+2y+3z=14 \\ 4x+5y+7z=35 \\ 3x+3y+4z=21 \end{cases}$ (Refer DSC for Problems)

3. Investigate the values of λ and μ
⇒ The system of eqns

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu, \text{ may have}$$

a) unique solⁿ
b) Infinite solⁿ
c) No solⁿ

Solⁿ: consider, the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 1 & 2 & 3 & : 10 \\ 1 & 2 & \lambda & : \mu \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & : 4 \\ 0 & 1 & \lambda-1 & : \mu-6 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & : 4 \\ 0 & 0 & \lambda-3 & : \mu-10 \end{bmatrix}$$

a) unique solⁿ:
we must have $\rho[A] = \rho[A:B] = 3$
$\Rightarrow \lambda - 3 \neq 0$
$\Rightarrow \lambda \neq 3$ irrespective of value of μ,
$\therefore \rho[A:B]$ is also 3.
Hence, the system will have unique solⁿ if $\lambda \neq 3$.

b) Infinite solⁿ:

we have $n=3$ and we need $\rho[A]=\rho[A:B]$

$\rho[A]=\rho[A:B]=r<3$

$\therefore$ we must have $r=2$

$\therefore \rho[A]=\rho[A:B]=2$ only when last row is completely zero. Hence $(\lambda-3)=0$ and $\mu-10=0$.

Hence, the system will have infinite solⁿ, if $\lambda=3$ and $\mu=10$.

c) No solution:

we must have $\rho[A] \neq \rho[A:B]$

By case ⓐ $\rho[A]=3$ if $\lambda \neq 3$

$\rho[A]=2$ if $\lambda=3$

If we impose $(\mu-10) \neq 0$ then

$\rho[A:B]=3$

Hence, the system has no solution if $\lambda=3$ and $\mu \neq 10$.

4. Find the non-trivial solution of the system

$x+3y-2z=0, \quad 2x-y+4z=0, \quad x-11y+14z=0$

5. Reduce the foll. matrix to its normal form & hence find its rank.

$$A=\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$$

6. Solve completely the system of equations.

$$2x+2y-2z=1$$
$$4x+4y-z=2$$
$$6x+6y-2z=3$$

1. $\left.\begin{aligned} x+3y-2z&=0 \\ 2x-y+4z&=0 \\ x-11y+14z&=0 \end{aligned}\right\} \Rightarrow AX=B,\quad B=0$

Sol^n: $A=\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ $\begin{aligned} R_2 &\to R_2-2R_1 \\ R_3 &\to R_3-R_1 \end{aligned}$

$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & -16 \end{bmatrix}$ $R_3 \to R_3-2R_2$

$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A)=2 < n$ (no of unknown)

$\therefore$ Given eqn is consistent & have non-Trivial sol^n.

$\Rightarrow x+3y-2z=0$

$\quad -7y+8z=0$

Let $z=k$, $y=+\frac{8k}{7}$, $x=+2k-3\cdot\frac{8k}{7}=\frac{14k-24k}{7}$

$x=-\frac{10k}{7}$

$\therefore k=0,1,2\ldots$

$\therefore$ Infinite sol^n.

2. $2w+3x-y-z=0$

$4w-6x-2y+2z=0$

$-6w+12x+3y-4z=0$

Sol^n: $A=\begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix}$ $\begin{aligned} R_2 &\to R_2-2R_1 \\ R_3 &\to R_3+3R_1 \end{aligned}$

$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 21 & 0 & -7 \end{bmatrix}$ $R_3 \to 4R_3+7R_2$

$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -12 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A)=2<4$

Eqns is consistent have infinite sol^n.

$2w+3x-y-z=0$

$-12x+4z=0$

~~Let z=k, x=k1, y=k2~~

Let $x=k_1$, $y=k_2$

$z=3k_1$, $w=\frac{-1}{2}(3k_1-k_2-3k_1)$

$w=\frac{k_2}{2}$

3. Test for consistency & solve

$2x - 3y + 7z = 5$

$3x + y - 3z = 13$

$2x + 19y - 47z = 32$

Solⁿ: $\begin{pmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 32 \end{pmatrix} \Rightarrow AX = B$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right] \quad \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & 81 & -27 & 11 \\ 0 & 22 & -54 & 27 \end{array}\right] \quad R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 0 & 5 \end{array}\right] \Rightarrow \begin{array}{l} \rho(A:B) = 3 \\ \rho(A) = 2 \end{array}$$

$\therefore \rho(A) < \rho(A:B) \therefore \rho(A) \neq \rho(A:B)$

Eqn is inconsistent and have no solution.

4. $5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

Solⁿ: $$[A:B] = \left[\begin{array}{ccc:c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to 5R_2 - 3R_1 \\ R_3 \to 5R_3 - 7R_1 \end{array}$$

$15 - 15 = 0$

$130 - 9 = 121$

$10 - 21 = -11$

$45 - 12 = 33$

$35 - 35$

$10 - 21$

$50 - 49$

$25 - 28$

$$\sim \left[\begin{array}{ccc:c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right] \quad R_3 \to 11R_3 + R_2$$

$$\sim \left[\begin{array}{ccc:c} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} \rho(A:B) = 2 \\ \rho(A) = 2 \end{array}$$

$\therefore \rho(A:B) = 2 = \rho(A) \longrightarrow$ Consistent

$r = 2 < 3 = n \longrightarrow$ infinite solⁿ.

Now $5x + 3y + 7z = 4$

$121y - 11z = 33 \Rightarrow 11y - z = 3$

let $z = k$, $y = \frac{3+k}{11}$, $x =$

5. $3x+3y+2z=1$

$x+2y=4$

$10y+3z=-2$

$2x-3y-z=5$

Sol^n: $[A:B]=\begin{bmatrix}3 & 3 & 2 & : & 1\\ 1 & 2 & 0 & : & 4\\ 0 & 10 & 3 & : & -2\\ 2 & -3 & -1 & : & 5\end{bmatrix}$ $R_2 \leftrightarrow R_1$

$\sim \begin{bmatrix}1 & 2 & 0 & : & 4\\ 3 & 3 & 2 & : & 1\\ 0 & 10 & 3 & : & -2\\ 2 & -3 & -1 & : & 5\end{bmatrix}$ $R_2 \to R_2 - 3R_1$, $R_4 \to R_4 - 2R_1$

$3-3=0$
$3-6=-3$
$2-0=2$
$1-12=-11$

$\sim \begin{bmatrix}1 & 2 & 0 & : & 4\\ 0 & -3 & 2 & : & -11\\ 0 & 10 & 3 & : & -2\\ 0 & -7 & -1 & : & -3\end{bmatrix}$ $R_3 \to 3R_3 + 10R_2$, $R_4 \to 3R_4 - 7R_2$

$2-2=0$
$-3-4=-7$
$-1-0=-1$
$5-8=-3$

$\sim \begin{bmatrix}1 & 2 & 0 & : & 4\\ 0 & -3 & 2 & : & -11\\ 0 & 0 & 29 & : & -116\\ 0 & 0 & -17 & : & 68\end{bmatrix}$ $R_4 \to 29R_4 - 17R_3$

$30-30=0$
$9+20=29$
$-6-110=-116$

$-21+21=0$
$-3-14=-17$
$-9+77=68$

$\sim \begin{bmatrix}1 & 2 & 0 & : & 4\\ 0 & -3 & 2 & : & -11\\ 0 & 0 & 29 & : & -116\\ 0 & 0 & 0 & : & 0\end{bmatrix} \Rightarrow \rho(A:B)=3$

$\rho(A)=3$

$r=3=n$ $\therefore$ ~~have infinitely many sol^n.~~

have unique sol^n

$\therefore$ $x+2y=4$

$-3y+2z=-11$

$29z=-116 \Rightarrow z=-4$

$-3y+2(-4)=-11$

$-3y=-11+8$

$y=\frac{-3}{-3}=1 \Rightarrow y=1$

$x+2(1)=4$

$x=2$

6. Find for what values of λ and μ, the equations $x+y+z=6$; $x+2y+3z=10$, $x+2y+\lambda z=\mu$.

Have (i) no sol^n

(ii) unique sol^n

(iii) Infinite nos of sol^n.

Solⁿ: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$

$\lambda-1-2$
$\mu-6-4$

(i) No solⁿ: $\rho(A:B) \neq \rho(A)$
$\therefore \rho(A) = 2$ i.e when $\lambda - 3 = 0 \Rightarrow \lambda = 3$ & $\mu \neq 10$

(ii) unique solⁿ: $\rho(A:B) = \rho(A) \Rightarrow r = n$
$r = 3 = n \implies \lambda \neq 3, \mu \neq 10$
$x + y + z = 6$
$y + 2z = 4$
$(\lambda-3)z = \mu - 10$

(iii) Infinite solⁿ $\lambda = 3, \mu = 10$

7. For what values of λ, the eqn
$x+y+z=1$; $x+2y+4z=\lambda$; $x+4y+10z=\lambda^2$
have a solⁿ & solve it.

Solⁿ: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \lambda \\ 1 & 4 & 10 & : & \lambda^2 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 3 & 9 & : & \lambda^2-1 \end{bmatrix}$ $R_3 \to R_3 - 3R_2$

$3-3$
$9-9$
$\lambda^2-1-3(\lambda-1)$
$= \lambda^2-1-3\lambda+3$
$\lambda^2-3\lambda+2$
$(\lambda-1)(\lambda-2)$

$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda-1 \\ 0 & 0 & 0 & : & (\lambda-1)(\lambda-2) \end{bmatrix}$ ~~$\rho(A) = 2$~~ ~~$\rho(A:B) \neq$~~

(i) $(\lambda-1)(\lambda-2) = 0$ $n = 3$
$\rho(A) = 2 = \rho(A:B) \implies r = 2 < 3 = n$
$\therefore$ Infinitely many solⁿ
$x + y + z = 1$
$y + 3z = \lambda - 1$

a) suppose $\lambda = 1$ and $z = k$
$y = -3k$
$x = 1 - k + 3k = 1 + 2k$

(or) b) $z = k$
$y = (\lambda-1) - 3k$
$x = 1 - k - ((\lambda-1) - 3k)$
$=$

ASSIGNMENT

8. Investigate the values of λ & μ so that eqn: $2x+3y+5z=9$; $7x+3y-2z=8$;
$2x+3y+\lambda z=\mu$
(i) no soln (ii) unique (iii) infinite soln.

9. Solve completely the system of eqn

a) $2x+2y-2z=1$
$4x+4y-z=2$
$6x+6y-2z=3$

b) $x+2y+3z=14$
$4x+5y+7z=35$
$3x+3y+4z=21$

10. Reduce the foll. into Normal form & find its Rank.

a) $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & -6 & -2 & 2 \\ -6 & 12 & 3 & -4 \end{bmatrix}$

Eigenvalues and eigenvectors:

In a square matrix A, if There exist a scalar λ (real or complex) and a non-zero column matrix X such That $AX = \lambda X$, Then λ is called an eigenvalue of A and X is called an eigenvector of A corresponding to an eigenvalue.

Note:

a) Each eigenvalue have eigenvector.

b) If eigenvalues of The system are unique whereas eigenvector need not be.

Solving eigenvalues and eigenvectors:

consider,

$$A\vec{X} = \lambda\vec{X}$$

$$A\vec{X} - \lambda\vec{X} = 0$$

$$A\vec{X} - \lambda I\vec{X} = 0$$

$$(A - \lambda I)\vec{X} = 0, \quad \vec{X} \neq 0$$

Suppose $A - \lambda I$ is invertible Then we can multiply $(A-\lambda I)^{-1}$ on both sides

$\Rightarrow \vec{X} = 0$ which is contradiction.

$\therefore (A - \lambda I)^{-1}$ cannot be invertible.

Thus by inverse of matrices, we have

$|A - \lambda I| = 0$ which forms characteristic polynomial and eqn formed is characteristic eqn and sol^n To This eqn will be eigenvalues.

Examples:

I. Find the eigenvalues & eigenvectors for the following:

1. $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Solⁿ: Consider the characteristic equation

$|A - \lambda I| = 0 \longrightarrow (1)$

$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 - 4 = 0$

$1 + \lambda^2 - 2\lambda - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0 \longrightarrow (2)$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3, -1$

Eqn (2) is the characteristic eqn

$\therefore$ The eigenvalues from (2) is

$\lambda = 3, -1$

To find eigenvectors: consider $(A - \lambda I)$.

put $\lambda = 3$ in $(A - \lambda I)\vec{X} = 0$,

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore -2x_1 + x_2 = 0$

$x_2 = 2x_1$

let $x_1 = k_1$

$\therefore x_2 = 2k_1$

$\therefore$ Eigenvector for $\lambda = 3$ is $k_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Put $\lambda = -1$ in $(A - \lambda I)\vec{X} = 0$, consider $A - \lambda I$

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore 2x_1 + x_2 = 0$

$x_2 = -2x_1$

let $x_1 = k_2$

$\therefore x_2 = -2k_2$

$\therefore$ The eigenvector for $\lambda = -1$ is

$$\begin{bmatrix} k_2 \\ -2k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

2. $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

Solⁿ: The characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 2 & 3 & 4-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-2-\lambda)(4-\lambda) = 0$

$\therefore \lambda = 1, -2, 4$ are the eigenvalues of the obtained characteristic equation.

To find eigenvectors:

Put $\lambda = 1$ in $(A - \lambda I)\vec{X} = 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 0 \\ 2 & 3 & 3 \end{bmatrix}$$

This gives the following linear equations

$$3x_1 \mp 3x_2 = 0$$

$$2x_1 + 3x_2 + 3x_3 = 0$$

let $x_2 = k_1$ $\therefore$ $3x_1 = -3x_2$

$$3x_1 = -3k_1$$

$$x_1 = -k_1$$

$$\therefore 3x_3 = -2x_1 - 3x_2$$

$$3x_3 = +2k_1 \mp 3k_1$$

$$3x_3 = -k_1$$

$$x_3 = \frac{-1}{3}k_1$$

$\therefore$ The eigenvector for $\lambda = 1$ is

$$\vec{X} = \begin{bmatrix} -k_1 \\ k_1 \\ -\frac{1}{3}k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ -1/3 \end{bmatrix}$$

Put $\lambda = -2$ in $(A - \lambda I)\vec{X} = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 3 & 6 \end{bmatrix}$$

This provides

$3x_1 = 0 \Rightarrow x_1 = 0$

$2x_1 + 3x_2 + 6x_3 = 0$

Let $x_2 = k_1$

$\therefore 6x_3 = -3k_1$

$x_3 = -\frac{1}{2}k_1$

$\therefore$ The eigenvector for $\lambda = -2$ is

$$\vec{X} = \begin{bmatrix} 0 \\ k_1 \\ -\frac{1}{2}k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$$

Put $\lambda = 4$ in $(A - \lambda I)\vec{X} = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 3 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

This yields,

$-3x_1 = 0 \Rightarrow x_1 = 0$

$3x_1 - 6x_2 = 0 \Rightarrow x_2 = 0$

$2x_1 + 3x_2 = 0$ and let $x_3 = k_1$

$\therefore$ The eigenvector for $\lambda = 4$ is

$$\vec{X} = \begin{bmatrix} 0 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

when $k_1 = 0, 1, 2, \ldots$

3. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solⁿ: The characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2(2-\lambda-1) + 1(2-3+\lambda) = 0$$

$$(2-\lambda)[6-3\lambda-2\lambda+\lambda^2-2] - 2(-\lambda+1) + (\lambda-1) = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\therefore$ The eigenvalues are $\lambda = 1, 1; 5$

To find eigenvectors:

put $\lambda = 5$ in $(A - \lambda I)\vec{X} = 0$

consider $A - \lambda I$

$$\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

By cross product ~~matrix~~ minor matrix method, (Cross multiplication)

$$\frac{x_1}{\begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}} = k_1$$

$$\frac{x_1}{6-2} = \frac{-x_2}{-3-1} = \frac{x_3}{2+2} = k_1$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k_1$$

$\therefore$ The eigenvector for $\lambda = 5$ is

$$\vec{X} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Put $\lambda = 1$ in $(A - \lambda I)\vec{X} = 0$

Consider $A - \lambda I$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives

$$x_1 + 2x_2 + x_3 = 0$$

Let $x_2 = 0$ and $x_3 = k_1$
$\Rightarrow x_1 = -k_1$ $\Big\}$ $\vec{X} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Let $x_2 = k_1$, $x_3 = 0$
$\Rightarrow x_1 = -2k_1$ $\Big\}$ $\vec{X} = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ The eigenvectors for $\lambda = 1$ are

$$\vec{X} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{X} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Assignment:

Find the eigenvalues & eigenvectors

a) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$

d) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ f) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Diagonalization of matrix:

A square matrix A is ~~called~~ said to be diagonalizable if it is similar to a diagonal matrix

i.e

if $\exists$ an invertible matrix P and a diagonal matrix D $\ni$ $P^{-1}AP = D$

(or) $A = PDP^{-1}$ where P and D are not unique.

Examples:

I. Check whether the following matrices are diagonalizable.

1. $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

Soln: Let $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$

The corresponding characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = 0$$

The eigenvalues are $\lambda = 1, 6$

To find eigenvectors:

put $\lambda = 1$ in $(A - \lambda I)\vec{X} = 0$

$$\vec{X} = K_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Put $\lambda = 6$ in $(A - \lambda I)\vec{X} = 0$

$$\vec{X} = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ The invertible matrix P is given by

$$P = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \qquad \left(\because P^{-1} = \frac{\text{adj } P}{|P|}\right)$$

Now,

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 8-6 & 4+2 \\ 6-9 & 3+3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -3 & 6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 30 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = D$$

$\therefore$ The matrix A is diagonalizable.

2. $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

Solⁿ: The characteristic equation is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

The eigenvalues are $\lambda = 1, -2, 3$

The eigenvectors are

for $\lambda = 1$, $\vec{X} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

for $\lambda = -2$, $\vec{X} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = 3$, $\vec{X} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

The invertible matrix P is

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 4 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 6 \\ -3 & 0 & -3 \end{bmatrix}$$

Now,

$$P^{-1}AP = -\frac{1}{6}\begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 6 \\ -3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 4 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

$\therefore$ A is diagonalizable.

Assignment:

I. Diagonalize and check for diagonizability for the following:

a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Quadratic forms:

A homogeneous polynomial of the second degree in any number of variables is called as Quadratic form.

Ex: a) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

b) $2x^2 + 10xy + 2y^2$

c) $4x^2 + 2xy - 3y^2$

In general: $ax^2 + bxy + cy^2$

Canonical form:

It is the simplest form obtained by a orthogonal transformation on diagonal matrix

i.e $Q = y^T D y$

Ex:

1) QF : $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$
   CF : $8y_1^2 + 2y_2^2 + 2y_3^2$

2) QF : $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$
   CF : $2y_1^2 + 3y_2^2 + 6y_3^2$

Reduction of a quadratic form to Canonical form by orthogonal transformation:

*) Convert the quadratic form into matrix form.

*) Find eigenvalues and eigenvectors

*) Check pairwise orthogonality of eigenvectors

*) Find D matrix

*) Compute $Q = y^T D y$

I. Convert the foll. into Matrix form

Formula:

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2}\text{coeff of } x_1x_2 & \frac{1}{2}\text{coeff of } x_1x_3 \\ \frac{1}{2}\text{coeff of } x_2x_1 & \text{coeff of } x_2^2 & \frac{1}{2}\text{coeff of } x_2x_3 \\ \frac{1}{2}\text{coeff of } x_3x_1 & \frac{1}{2}\text{coeff of } x_3x_2 & \text{coeff of } x_3^2 \end{bmatrix}$$

1. $2x^2 + 10xy + 2y^2$

Soln: Matrix form:

$$A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$$

2. $4x^2 + 2xy - 3y^2$

Soln: Matrix form:

$$A = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

3. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

Soln: Matrix form:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

4. $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

Soln: Matrix form:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

II. Reduction to Canonical form by orthogonal Transformation.

1. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

Solⁿ: The matrix form is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

The eigen values are $\lambda = 8, 2, 2$

The eigenvectors are:

for $\lambda = 8 \Rightarrow X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

for $\lambda = 2 \Rightarrow X_2 = k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and

$$X_3 = k\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

To check orthogonality b/w pair of eigenvectors

$$X_1^T X_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 - 1 + 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0 + 1 - 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2 - 1 - 1 = 0$$

∴ Eigenvectors are orthogonal to eachother

The invertible matrix P is

$$P = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Normalized matrix N is

$$N = \begin{bmatrix} \frac{2}{\sqrt{2^2+(-1)^2+1^2}} & \frac{0}{\sqrt{0^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+(-1)^2}} \\ \frac{-1}{\sqrt{2^2+(-1)^2+1^2}} & \frac{1}{\sqrt{0^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+(-1)^2}} \\ \frac{1}{\sqrt{2^2+(-1)^2+1^2}} & \frac{1}{\sqrt{0^2+1^2+1^2}} & \frac{-1}{\sqrt{1^2+1^2+(-1)^2}} \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Taking Transpose of N,

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Now,

$$N^T A N =$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{12}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} & 0 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \\ \frac{-4}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} & 0 + \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{3}{\sqrt{6}} & 0 - \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} & \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{6} + \frac{8}{6} + \frac{8}{6} & \frac{-2}{\sqrt{12}} + \frac{2}{\sqrt{12}} & \frac{4}{\sqrt{18}} - \frac{2}{\sqrt{18}} - \frac{2}{\sqrt{18}} \\ \frac{-8}{\sqrt{12}} + \frac{8}{\sqrt{12}} & \frac{2}{2} + \frac{2}{2} & \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} \\ \frac{16}{\sqrt{18}} - \frac{8}{\sqrt{18}} - \frac{8}{\sqrt{18}} & \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} & \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

Canonical form:

$$y^T D y$$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 8y_1 & 2y_2 & 2y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= 8y_1^2 + 2y_2^2 + 2y_3^2$ which is the required canonical form.

Q. 1. $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

Soln: MP:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(1-\lambda) - 1\right] + 1\left(-1(1-\lambda) - 0\right) + 0 = 0$$

$$(1-\lambda)\left[2 - 2\lambda - \lambda + \lambda^2 - 1\right] + 1(-1+\lambda) = 0$$

$$(1-\lambda)\left[\lambda^2 - 3\lambda + 1\right] - 1(1-\lambda) = 0$$

$$(1-\lambda)\left[\lambda^2 - 3\lambda + \cancel{1} - \cancel{1}\right] = 0$$

$$(1-\lambda)\left[\lambda^2-3\lambda\right]=0$$

$$(1-\lambda)\lambda(\lambda-3)=0$$

$$\lambda=0,1,3 \longrightarrow \text{eigenvalues.}$$

Eigenvectors:

$\lambda=0$, $(A-\lambda I)x=0$

$$\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x_1}{2-1}=\frac{-x_2}{-1-0}=\frac{x_3}{-1-0}$$

$$\frac{x_1}{1}=\frac{-x_2}{-1}=\frac{x_3}{-1}$$

$$X_1=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=k\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$\lambda=1$,

$$\begin{bmatrix}0 & -1 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x_1}{0-1}=\frac{-x_2}{0-0}=\frac{x_3}{-1-0}$$

$$\frac{x_1}{-1}=-\frac{x_2}{0}=\frac{x_3}{-1}$$

$$X_2=k\begin{bmatrix}-1\\0\\-1\end{bmatrix}$$

$\lambda=3$

$$\begin{bmatrix}-2 & -1 & 0\\ -1 & -1 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x_1}{2-1}=\frac{-x_2}{2-0}=\frac{x_3}{-1-0}$$

$$\frac{x_1}{1}=\frac{-x_2}{2}=\frac{x_3}{-1}$$

$$X_3=k\begin{bmatrix}1\\-2\\-1\end{bmatrix}$$

Verification

$$X_1^T X_2=\begin{bmatrix}1 & 1 & -1\end{bmatrix}\begin{bmatrix}-1\\0\\-1\end{bmatrix}=-1+1=0$$

$$X_2^T X_3=\begin{bmatrix}-1 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\-2\\-1\end{bmatrix}=-1+1=0$$

$$X_3^T X_1=\begin{bmatrix}1 & -2 & -1\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}=1-2+1=0$$

$\therefore$ Orthogonal vectors.

Modal matrix:

$$M=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & -2\\ -1 & -1 & -1\end{bmatrix}$$

Normalized modal matrix N:

$$N=\begin{bmatrix}\frac{1}{\sqrt3} & \frac{-1}{\sqrt2} & \frac{1}{\sqrt6}\\ \frac{1}{\sqrt3} & 0 & \frac{-2}{\sqrt6}\\ \frac{-1}{\sqrt3} & \frac{-1}{\sqrt2} & \frac{-1}{\sqrt6}\end{bmatrix},\quad N^T=\begin{bmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3}\\ \frac{-1}{\sqrt2} & 0 & \frac{-1}{\sqrt2}\\ \frac{1}{\sqrt6} & \frac{-2}{\sqrt6} & \frac{-1}{\sqrt6}\end{bmatrix}$$

Finding Diagonal matrix,

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \\ \frac{-1}{\sqrt2} & 0 & \frac{-1}{\sqrt2} \\ \frac{-1}{\sqrt6} & \frac{-2}{\sqrt6} & \frac{-1}{\sqrt6} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt3} & \frac{-1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & 0 & \frac{-2}{\sqrt6} \\ \frac{-1}{\sqrt3} & \frac{-1}{\sqrt2} & \frac{-1}{\sqrt6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \\ \frac{-1}{\sqrt2} & 0 & \frac{-1}{\sqrt2} \\ \frac{-1}{\sqrt6} & \frac{-2}{\sqrt6} & \frac{-1}{\sqrt6} \end{bmatrix} \begin{bmatrix} \cancel{\frac{1}{\sqrt3}} - \cancel{\frac{1}{\sqrt3}} & \frac{-1}{\sqrt2} & \frac{1}{\sqrt6} + \frac{2}{\sqrt6} \\ \frac{-1}{\sqrt3} + \frac{2}{\sqrt3} - \frac{1}{\sqrt3} & \frac{1}{\sqrt2} - \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} - \frac{4}{\sqrt6} - \frac{1}{\sqrt6} \\ \cancel{\frac{1}{\sqrt3}} - \cancel{\frac{1}{\sqrt3}} & -\frac{1}{\sqrt2} & -\frac{2}{\sqrt6} - \frac{1}{\sqrt6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \\ \frac{-1}{\sqrt2} & 0 & \frac{-1}{\sqrt2} \\ \frac{1}{\sqrt6} & \frac{-2}{\sqrt6} & \frac{-1}{\sqrt6} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt2} & \frac{3}{\sqrt6} \\ 0 & 0 & \frac{-6}{\sqrt6} \\ 0 & \frac{-1}{\sqrt2} & \frac{-3}{\sqrt6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\cancel{\frac{1}{\sqrt6}} + \cancel{\frac{1}{\sqrt6}} & \frac{3}{\sqrt{18}} - \frac{6}{\sqrt{18}} + \frac{3}{\sqrt{18}} \\ 0 & \frac{1}{\sqrt4} + \frac{1}{\sqrt4} & -\cancel{\frac{3}{\sqrt{12}}} + \cancel{\frac{3}{\sqrt{12}}} \\ 0 & \cancel{\frac{-1}{\sqrt{12}}} + \cancel{\frac{1}{\sqrt{12}}} & \frac{3}{\sqrt{36}} + \frac{12}{\sqrt{36}} + \frac{3}{\sqrt{36}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\cancel{2}}{\cancel{2}} & 0 \\ 0 & 0 & \frac{\cancel{18}\,3}{\cancel{6}\,1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

Reducing to C.F: $Y^T D Y$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_2^2 + 3y_3^2 \rightarrow CF$$

3. $2x^2 + 10xy + 2y^2$

Sol^n:

$$A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 25 = 0$$

$$4 + \lambda^2 - 4\lambda - 25 = 0$$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$\lambda^2 + 3\lambda - 7\lambda - 21 = 0$$

$$\lambda(\lambda+3) - 7(\lambda+3) = 0$$

$\lambda = -3, 7 \longrightarrow$ Eigenvalues

$\lambda = -3$, $(A - \lambda I)x = 0$

$$\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{5} = \frac{-x_2}{5} \text{ or } \frac{x_1}{1} = \frac{x_2}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 7$,

$$\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{+5} = \frac{-x_2}{-5} \text{ or } \frac{x_1}{1} = \frac{x_2}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

NMM:

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

DM:

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{2-5}{\sqrt{2}} & \frac{2+5}{\sqrt{2}} \\ \frac{5-2}{\sqrt{2}} & \frac{5+2}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-3}{\sqrt{2}} & \frac{7}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{7}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3-3}{(\sqrt{2})^2} & \frac{7-7}{(\sqrt{2})^2} \\ \frac{-3+3}{(\sqrt{2})^2} & \frac{7+7}{(\sqrt{2})^2} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix} = D$$

CF: $y^T D y$

$$= -3y_1^2 + 7y_2^2$$

Required CF.

Assignment:

1. Reduce into Canonical form:

a) $3x_1^2+5x_2^2+3x_3^2-2x_1x_2+2x_1x_3-2x_2x_3$

b) $2x_1^2+3x_2^2+5x_3^2+6x_1x_2+4x_1x_3+8x_2x_3$

c) $-16x^2+2xy-2y^2$

d) $3x_1^2-4x_1x_2+7x_2^2$