Unit - 2: Calculus - II

→ Taylor's and Maclaurin's expansion for single and two variable functions and applications.

→ Indeterminate forms: evaluation of limits by L'Hospital's rule.

→ Maxima & minima for single and two variable functions

→ Curve Tracing: General rules to trace Cartesian, polar and parametric curves.

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Taylor's Theorem:

Statement:

Let $f(x)$ be a function defined on $[a, b] \ni$

(i) $f^{(n-1)}(x)$ is continuous on $[a, b]$

(ii) $f^{(n-1)}(x)$ is differentiable on (a, b)

then $\exists\ c \in (a, b) \ni$

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f'''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(c)$$

Note:

1. If $(a, b) = (0, x)$ in Taylor's Theorem, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(c)$$

This is known as Maclaurin's expansion.

2. Denoting $R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$, where R_n is known as remainder after n Terms. As $n \to \infty$, the nos of Terms increase indefinitely and we have an infinite series expansion of $f(x)$ in powers of $(x-a)$ given by

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

This is called Taylor's series expansion of $f(x)$ about the point 'a'.

In particular $a = 0$, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

This is called Maclaurin's series expansion of $f(x)$.

3. for our convenience we can ~~rewrite~~ rewrite,

$$y(x) = y(a) + (x-a)\,y_1(a) + \frac{(x-a)^2}{2!}\,y_2(a) + \cdots$$

and

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \cdots$$

Examples:

1. Obtain Taylor's expansion of $\log_e x$ about $x=1$ upto the term containing fourth degree and hence obtain $\log_e(1.1)$

Soln: We have Taylor's expansion about $x=a$ given by

$$y(x) = y(a) + (x-a)\,y_1(a) + \frac{(x-a)^2}{2!}\,y_2(a) + \frac{(x-a)^3}{3!}\,y_3(a) + \frac{(x-a)^4}{4!}\,y_4(a) + \cdots \quad \to (1)$$

Given that, $y(x) = \log_e x$ and $a=1$

(1) becomes.

$$\log_e x = y(1) + (x-1)\,y_1(1) + \frac{(x-1)^2}{2!}\,y_2(1) + \frac{(x-1)^3}{3!}\,y_3(1) + \frac{(x-1)^4}{4!}\,y_4(1) + \cdots \quad \to (2)$$

Consider,

$$y(x) = \log_e(x) \qquad \therefore y(1) = \log_e 1 = 0$$

$$y_1(x) = \frac{1}{x} \qquad \therefore y_1(1) = \frac{1}{1} = 1$$

$$y_2(x) = -\frac{1}{x^2} \qquad \therefore y_2(1) = -1$$

$$y_3(x) = \frac{2}{x^3} \qquad \therefore y_3(1) = 2$$

$$y_4(x) = -\frac{2.3}{x^4} \qquad \therefore y_4(1) = -6$$

$$\vdots$$

Substituting in ② we obtain,

$$\log_e x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{\cancel{6}\,3}(\cancel{2}^{\,1})$$

$$+ \frac{(x-1)^4}{\cancel{24}\,4}(-\cancel{6}^{\,1}) + \cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Put $x = 1.1$ to obtain $\log(1.1)$

$$\log_e(1.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4} + \cdots$$

$$\log_e(1.1) = 0.0953$$

2. Expand $\tan^{-1} x$ in powers of $(x-1)$ upto the term containing fourth degree

Sol^n: We have

Taylor's expansion in powers of $(x-1)$,

$$y(x) = y(1) + (x-1)\, y_1(1) + \frac{(x-1)^2}{2!} y_2(1) + \frac{(x-1)^3}{3!} y_3(1) + \frac{(x-1)^4}{4!} y_4(1) + \cdots \longrightarrow (1)$$

consider,

$y(x) = \tan^{-1} x \qquad \therefore y(1) = \tan^{-1} 1 = \pi/4$

$y_1(x) = \dfrac{1}{1+x^2} \qquad \therefore y_1(1) = \dfrac{1}{2}$

$\Rightarrow (1+x^2)\, y_1 = 1$ (Diff)

$(1+x^2)\, y_2 + y_1 (2x) = 0 \qquad \therefore y_2(1) = -1/2$

again on differentiation

$(1+x^2)\, y_3 + y_2 (2x) + 2(x y_2 + y_1) = 0$

$\therefore y_3(1) = 1/2$

$|||^{ly}$ differentiating again

$(1+x^2)\, y_4 + y_3 (2x) + 2(x y_3 + y_2)$

$+ 2(x y_3 + y_2 + y_2) = 0$

$\therefore y_4(1) = 0$

$\vdots$

Substituting in (1) we obtain

$$\tan^{-1} x = \frac{\pi}{4} + \frac{1}{2}\left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \cdots \right)$$

which is the required series / expansion.

3. Expand $e^{\sin x}$ using Maclaurin's theorem upto the terms containing x^4.

Solⁿ: We have

Maclaurin's expansion

$$y(x) = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \cdots \longrightarrow (1)$$

Consider,

$$y(x) = e^{\sin x}, \quad \therefore y(0) = e^0 = 1$$

$$y_1(x) = \underbrace{e^{\sin x}}_{y} \cdot \cos x \quad \therefore y_1(0) = e^0 \cdot 1 = 1$$

$$y_2(x) = y(-\sin x) + \cos x \cdot y_1$$

$$\therefore y_2(0) = 1 \cdot (0) + 1 \cdot 1 = 1$$

$$y_3(x) = -(y \cos x + \sin x \cdot y_1) + (\cos x\, y_2 + y_1(-\sin x))$$

$$\therefore y_3(0) = 0$$

$$y_4(x) = -2[y_1 \cos x + \sin x\, y_2] - [y(-\sin x) + \cos x \cdot y_1] + y_2(-\sin x) + \cos x \cdot y_3$$

$$y_4(0) = -3$$

Substituting in (1), we obtain

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

4. Expand e^x about $x=0$, upto x^3.

Soln: We have,

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \cdots$$

Consider,

$y(x) = e^x \quad \therefore y(0) = 1$

$y_1(x) = e^x \quad \therefore y_1(0) = 1$

$\vdots$

$$\therefore y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Assignment:

1. Expand the following upto terms containing x^4:

a) $\sin x$ about $x=1$

b) $\cos x$ about $x=0$

c) $\log(\sec x)$ about $x=0$

d) $\tan x$ about $x=\frac{\pi}{4}$

e) $\log(1+x)$ about $x=0$

f) a^x about $x=0$

g) $e^{a\sin^{-1}x}$ about $x=0$

h) $\frac{x}{e^{x-1}}$ about $x=0$

i) $e^{\cos x}$ about $x=0$

Taylor's series for functions of two variables:

Let $f(x, y)$ be a function of two variables x, y. We can expand $f(x+h, y+k)$ in a series of ascending powers of h and k. The Taylor's series is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots \longrightarrow (1)$$

Eqn (1) is the Taylor's series expansion of $f(x, y)$ about the point (a, b).

Examples:

1. Expand $e^x \cos y$ about $(0, \frac{\pi}{2})$ upto the third term using Taylor series.

Sol^n: $f(x,y) = e^x \cos y$ $\qquad \therefore f(0, \frac{\pi}{2}) = e^0 \cdot \cos\frac{\pi}{2} = 0$

$\frac{\partial f}{\partial x} = f_x(x,y) = \cos y \cdot e^x$ $\qquad \therefore f_x(0, \frac{\pi}{2}) = 0$

$\frac{\partial f}{\partial y} = f_y(x,y) = e^x(-\sin y)$ $\qquad \therefore f_y(0, \frac{\pi}{2}) = -1$

$\frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = e^x \cos y$ $\qquad \therefore f_{xx}(0, \frac{\pi}{2}) = 0$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}(x, y) = -e^x \sin y \quad \therefore f_{xy}(0, \frac{\pi}{2}) = -1$

$\frac{\partial^2 f}{\partial y^2} = f_{yy}(x, y) = -e^x \cos y \quad \therefore f_{yy}(0, \frac{\pi}{2}) = 0$

$\frac{\partial^3 f}{\partial x^3} = f_{xxx}(x, y) = e^x \cos y \quad \therefore f_{xxx}(0, \frac{\pi}{2}) = 0$

$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}(x, y) = -e^x \sin y \quad \therefore f_{xxy}(0, \frac{\pi}{2}) = -1$

$\frac{\partial^3 f}{\partial x \partial y^2} = f_{xyy}(x, y) = -e^x \cos y \quad \therefore f_{xyy}(0, \frac{\pi}{2}) = 0$

$\frac{\partial^3 f}{\partial y^3} = f_{yyy}(x, y) = e^x \sin y \quad \therefore f_{yyy}(0, \frac{\pi}{2}) = 1$

We have

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$
$$+ \cdots \longrightarrow (1)$$

Here $a = 0$, $b = \frac{\pi}{2}$ and substituting f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{xxx}, f_{xyy}, f_{xxy} and f_{yy} we obtain,

$\therefore$ (1) can be written as

$$f(x,y) = 0 + \frac{1}{1!}\left[x.0 + \left(y-\frac{\pi}{2}\right)(-1)\right]$$
$$+\frac{1}{2!}\left[x^2.0 + 2(x)\left(y-\frac{\pi}{2}\right)(-1) + \left(y-\frac{\pi}{2}\right)^2(0)\right]$$
$$+\frac{1}{3!}\left[x^3.0 + 3x^2\left(y-\frac{\pi}{2}\right)(-1) + 3x\left(y-\frac{\pi}{2}\right)^2(0)\right.$$
$$\left.+\left(y-\frac{\pi}{2}\right)^3(1)\right] + \cdots$$
$$\therefore e^x \cos y = -y + \frac{\pi}{2} + \frac{1}{2}\left(-2xy + 2x\frac{\pi}{2}\right)$$
$$+\frac{1}{6}\left(-3x^2y + 3\frac{\pi}{2}x^2 + \left(y-\frac{\pi}{2}\right)^3\right) + \cdots$$
at $\left(0, \frac{\pi}{2}\right)$

2. Expand $\sin xy$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$ upto second degree terms.

Sol^n: $f(x,y) = \sin xy$ $\quad \therefore f\left(1, \frac{\pi}{2}\right) = 1$

$f_x(x,y) = y \cos xy$ $\quad \therefore f_x\left(1, \frac{\pi}{2}\right) = 0$

$f_y(x,y) = x \cos xy$ $\quad \therefore f_y\left(1, \frac{\pi}{2}\right) = 0$

$f_{xx}(x,y) = -y^2 \sin(xy)$ $\quad \therefore f_{xx}\left(1, \frac{\pi}{2}\right) = -\frac{\pi^2}{4}$

$f_{xy}(x,y) = -xy \sin xy + \cos xy$ $\quad \therefore f_{xy}\left(1, \frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f_{yy}(x,y) = -x^2 \sin xy$ $\quad \therefore f_{yy}\left(1, \frac{\pi}{2}\right) = -1$

We have,

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)\right.$$
$$\left. + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Put $a=1$, $b=\frac{\pi}{2}$ and all the partial derivatives we obtain,

$$f(x,y) = 1 + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) - \pi(x-1)\left(y-\frac{\pi}{2}\right)\right.$$
$$\left. + \left(y-\frac{\pi}{2}\right)^2\right] + \cdots$$

Assignment:

1. Expand

a) $e^x \cos y$ about $(0,0)$ upto third degree.

b) $e^x \sin y$ about $(1, \frac{\pi}{2})$ upto third degree.

c) e^{xy} about $(1,1)$ upto second degree.

d) $e^x \log(1+y)$ about $(0,0)$ upto second degree.

Indeterminate forms:

If an expression $F(x)$ at $x=a$ assumes forms like $0/0$, ∞/∞, $0\times\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ which donot represent any value are called indeterminate forms.

L' Hospital's rule (Theorem)

Statement:

If $f(x)$ and $g(x)$ are two functions ∋

(i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ i.e.

$f(a) = 0 = g(a)$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$,

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Note:

If $f'(a) = 0$ and $g'(a) = 0$ then we have

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(a)}{g''(x)}$ and so on.

Type-1: The rule can be applied directly in the case of forms $0/0$ and ∞/∞. In the cases of $\infty - \infty$ and $\infty \times 0$, we have to employ simple methods (taking LCM, using trignometric expressions, etc) to simplify the given expression in bringing it to the form $0/0$ or ∞/∞ so that the L' Hospital's rule can be employed.

Examples:

I. Evaluate the following limits:

1. $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

Solⁿ: let

$$k = \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L' Hospitals rule,

$$k = \lim_{x\to 0} \frac{xe^x + e^x - 1/(1+x)}{2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{0+1+1+1}{2} = \frac{3}{2}$$

2. $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$

Solⁿ: let

$$k = \lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)} \quad \left(\frac{0}{0}\right)$$

Applying L' Hospital's rule

$$= \lim_{x\to 0} \frac{+\sin x}{x\frac{1}{1+x} + \log(1+x)\cdot 1} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{x\left(\frac{-1}{(1+x)^2}\right) + \frac{1}{1+x} + \frac{1}{1+x}}$$

$$= \frac{1}{0 + \frac{1}{1} + \frac{1}{1}}$$

$$k = \frac{1}{2}$$

3. $\lim_{x\to 0} \frac{a^x - b^x}{x}$

Solⁿ: $k = \lim_{x\to 0} \frac{a^x - b^x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

Apply L' Hospitals rule

$$k = \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{1}$$

$$= \log a - \log b$$

$$k = \log\left(\frac{a}{b}\right)$$

4. $\lim_{x\to 0} \log_{\tan bx} \tan ax$

Solⁿ: $k = \lim_{x\to 0} \log_{\tan bx} \tan ax$

we have

$$\log_b a = \frac{\log a}{\log b}$$

$$\therefore k = \lim_{x\to 0} \frac{\log(\tan ax)}{\log(\tan bx)} \quad \left(\frac{-\infty}{-\infty} \text{ form}\right)$$

Applying L' Hospitals rule,

$$k = \lim_{x\to 0} \frac{\frac{1}{\tan ax} \cdot \sec^2 ax \cdot a}{\frac{1}{\tan bx} \sec^2 bx \cdot b}$$

$$= \lim_{x\to 0} \frac{a \sec^2 ax \cdot \tan bx}{b \sec^2 bx \tan ax}$$

$$= \lim_{x\to 0} \frac{a}{b} \cdot \frac{\cos^2 bx}{\cos^2 ax} \cdot \frac{\sin bx}{\cos bx} \cdot \frac{\cos ax}{\sin ax} \times \frac{2}{2}$$

$$= \frac{a}{b} \lim_{x\to 0} \frac{\sin 2bx}{\sin 2ax} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$k = \frac{a}{b} \lim_{x\to 0} \left(\frac{\cos 2bx \cdot 2b}{\cos 2ax \cdot 2a}\right) = \frac{\cancel{a}}{\cancel{b}} \cdot \frac{\cancel{2}\cancel{b}}{\cancel{2}\cancel{a}} = 1$$

5. $k = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$ $\left(\frac{-\infty}{\infty}\right)$

(Ans : $k = 0$)

6. $k = \lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$ ($\infty - \infty$ form)

We have to simplify,

$$k = \lim_{x \to 1} \left[\frac{x \log x - (x-1)}{(x-1)\log x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

Apply L' Hospital's rule,

$$k = \lim_{x \to 1} \left[\frac{x \cdot \frac{1}{x} + \log x - 1}{(x-1)\frac{1}{x} + \log x \cdot 1}\right]$$

$$= \lim_{x \to 1} \left[\frac{\log x \cdot x}{(x-1) + x\log x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \left[\frac{\log x + x \cdot \frac{1}{x}}{1 + x\frac{1}{x} + \log x}\right]$$

$$= \lim_{x \to 1} \left[\frac{\log x + 1}{2 + \log x}\right]$$

$$k = \frac{1}{2}$$

7. $k = \lim_{x \to 1} (1-x^2) \tan\left(\frac{\pi x}{2}\right)$ $(0 \times \infty)$

$$k = \lim_{x \to 1} \frac{(1-x^2)}{\cot \frac{\pi x}{2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{(-2x)}{-\operatorname{cosec}^2 \frac{\pi x}{2} \cdot \left(\frac{\pi}{2}\right)} = \frac{-2}{-\frac{\pi}{2}}$$

$$k = \frac{4}{\pi}$$

Assignment:

Evaluate the foll. limits:

1. $\lim_{x \to 0} \tan x . \log x$

2. $\lim_{x \to 0} \left[\frac{1}{x} - \frac{\log(1+x)}{x^2} \right]$

3. $\lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right) \right]$

4. $\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right)$

5. $\lim_{x \to \frac{\pi}{2}} (2x \tan x - \pi \sec x)$

Type - 2: The given expression or its simplified form will be in the $\left(\frac{0}{0}\right)$ form when $x = 0$ or as $x \to 0$ but will involve terms of the form $x^2 \sin x$, $x \sin^3 x$, $x \tan^2 x$, etc. In the event of applying the rule, the differentiation becomes tedious. We can modify such terms so as to involve $\left(\frac{\sin x}{x}\right)^k$ or $\left(\frac{\tan x}{x}\right)^k$ or $\left(\frac{x}{\sin x}\right)^k$ or $\left(\frac{x}{\tan x}\right)^k$ which can be separated out from the given expression. These terms become 1 as $x \to 0$ with the result we will be left with a simple expression (product gets eliminated) in the 0/0 form

Examples:

Evaluate the following limits:

1. $K = \lim_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$ $\left(\dfrac{0}{0}\right)$ form

Solⁿ: $K = \lim_{x\to 0} \dfrac{\tan x - x}{x^2 \cdot \dfrac{\tan x}{x} \cdot x}$

$$= \lim_{x\to 0} \frac{\tan x - x}{x^3} \; \overset{1}{\cancel{\lim_{x\to 0}\left(\frac{x}{\tan x}\right)}}$$

$$K = \lim_{x\to 0} \frac{\tan x - x}{x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

Applying L' Hospital's rule,

$$K = \lim_{x\to 0} \frac{(\sec^2 x - 1)}{3x^2}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan^2 x}{x^2}$$

$$= \frac{1}{3} \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2$$

$$K = \frac{1}{3}$$

2. Let $K = \lim_{x\to 0} \left[\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right]$ $(\infty - \infty)$ form

Solⁿ: $K = \lim_{x\to 0} \left[\dfrac{\sin^2 x - x^2}{x^2 \sin^2 x}\right]$ $\left(\dfrac{0}{0}\right)$ form

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \cdot \dfrac{\sin^2 x}{x^2} \cdot x^2}$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \; \overset{1}{\cancel{\lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right)}}$$

$$K = \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \cdot 1$$

Applying L'Hospital's rule;

$$k = \lim_{x\to 0} \frac{2\sin x \cdot \cos x - 2x}{4x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sin 2x - 2x}{4x^3} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos 2x \cdot 2 - 2x}{4 \cdot 3x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{2}(\cos 2x - 1)}{\cancel{4}_2 \cdot 3x^2}$$

$$= \lim_{x\to 0} \frac{-\cancel{2}\sin^2 x}{\cancel{2} \cdot 3x^2} = \frac{-1}{3} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2$$

$$k = \frac{-1}{3}$$

Assignment:

Evaluate the following:

1. $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$

2. $\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

Type – 3: 0^0, ∞^0, 1^∞

The ~~for~~ funⁿ will be of the form $[f(x)]^{g(x)}$ and we have to find the limit as $x \to a$.

let $k = \lim_{x \to a} [f(x)]^{g(x)}$

Taking logarithms on bothsides we have,

$$\log_e k = \lim_{x \to a} g(x) \log(f(x))$$

We can evaluate the limit on RHS as already discussed and let us suppose that the limit is equal to l.

i.e. $\log_e k = l$

$k = e^l$ which is the required limit.

Remark:

One of the common question is that why 1^∞ is indeterminate?

let $k = \lim_{x \to a} (f(x))^{g(x)}$ (1^∞ form)

$\log k = \lim_{x \to a} g(x) \log(f(x))$ ($\infty \times 0$) form

which is ~~indeed~~ indeterminate.

On the otherhand, c^∞ with $c \neq 1$

$\log k = \infty \times \log c = \infty$.

Examples:

Evaluate the following limits:

1. $k = \lim_{x \to 0} (\cos x)^{1/x^2}$ (1^∞ form)

Sol^n: Take log on b.s

$$\log k = \lim_{x \to 0} \log((\cos x)^{1/x^2})$$

$$\log_e k = \lim_{x\to 0} \frac{\log(\cos x)}{x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

Apply L'Hospital's rule,

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x}$$

$$\log_e k = -\frac{1}{2}\lim_{x\to 0} \frac{\tan x}{x} = -\frac{1}{2}$$

$$k = e^{-1/2} = 1/\sqrt{e}$$

2. $$k = \lim_{x\to a} \left[2-(x/a)\right]^{\tan\left(\frac{\pi x}{2a}\right)} \qquad (1^\infty \text{ form})$$

Solⁿ: $$\log k = \lim_{x\to a} \log\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$\log k = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\log\left(2-\frac{x}{a}\right) \qquad (\infty\times 0 \text{ form})$$

$$\log k = \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

Apply L'Hospital's rule,

$$\log k = \lim_{x\to a} \frac{\frac{1}{2-\frac{x}{a}}\cdot\left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}$$

$$= \lim_{x\to a} \frac{\frac{a}{2a-x}\left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}$$

$$= \frac{2}{\pi}\lim_{x\to a} \frac{-1\cdot a}{-(2a-x)\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}$$

$$\log k = \frac{2}{\pi}\,\frac{a}{a\operatorname{cosec}^2\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

$$k = e^{2/\pi}$$

3. $k = \lim_{x\to 1} (1-x^2)^{1/\log(1-x)}$ $\quad (0^0 \text{ form})$

Sol^n: $\log k = \lim_{x\to 1} \frac{1}{\log(1-x)} \log(1-x^2)$

$$\log k = \lim_{x\to 1} \frac{\log(1-x^2)}{\log(1-x)} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Apply L'Hospitals rule,

$$\log k = \lim_{x\to 1} \frac{\frac{1}{1-x^2}\cdot(-2x)}{\frac{1}{1-x}\,(1)}$$

$$= \lim_{x\to 1} \frac{-2x(1-x)}{(1-x^2)(1)}$$

$$\log k = \lim_{x\to 1} \frac{-2x}{1+x} = 1$$

$$k = e^1 = e$$

4. $k = \lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1}(x)\right)^{1/x}$ $\quad (0^0 \text{ form})$

Sol^n: $k = \lim_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1}x\right)^{1/x}$

$$= \lim_{x\to\infty} (\cot^{-1}x)^{1/x} \qquad \because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

$$k = \lim_{x\to\infty} \left(\tan^{-1}\left(\frac{1}{x}\right)\right)^{1/x}$$

Put $y = \frac{1}{x}$

As $x\to\infty$, $y\to 0$

$$\therefore k = \lim_{y\to 0} (\tan^{-1}y)^y \quad (0^0 \text{ form})$$

$$\log k = \lim_{y\to 0} \cancel{\log}\, y \log(\tan^{-1}y) \quad (0 \times -\infty)$$

$$= \lim_{y\to 0} \frac{\log(\tan^{-1}y)}{1/y} \quad \left(\frac{-\infty}{\infty}\right)$$

Apply L'Hospitals rule,

$$\log_e k = \lim_{y \to 0} \frac{\frac{1}{\tan^{-1} y} \cdot \frac{1}{1+y^2}}{-\frac{1}{y^2}}$$

$$= \lim_{y \to 0} \frac{-y^2}{\tan^{-1} y\,(1+y^2)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{y \to 0} \frac{-2y}{\tan^{-1} y\,(2y) + (1+y^2) \cdot \frac{1}{1+y^2}}$$

$$\log_e k = \lim_{y \to 0} \frac{-2y}{2y \cdot \tan^{-1} y + 1} = 0$$

$$k = e^0 = 1$$

5. $K = \lim_{x \to 0} (\cot x)^{\tan x}$ $\quad (\infty^0 \text{ form})$

Sol^n: $\log k = \lim_{x \to 0} \tan x \log(\cot x)$ $\quad (0 \times \infty \text{ form})$

$$\log k = \lim_{x \to 0} \frac{\log(\cot x)}{\cot x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

Apply L'Hospitals rule,

$$\log k = \lim_{x \to 0} \frac{\frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x)}{-\operatorname{cosec}^2 x}$$

$$\log k = \lim_{x \to 0} \tan x = 0$$

$$k = e^0 = 1$$

6. $K = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x}$ $\quad (\infty^0 \text{ form})$

Ans: $K = 1$

7. $K = \lim_{x \to 0} x^{\sin x}$ $\quad (0^0 \text{ form})$

Ans: $K = 1$

Assignment:

I. Evaluate the foll. limits:

a) $\lim_{x\to 1} x^{1/(1-x)}$

b) $\lim_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x}$

c) $\lim_{x\to \infty} \left(1+\frac{a}{x}\right)^{x}$

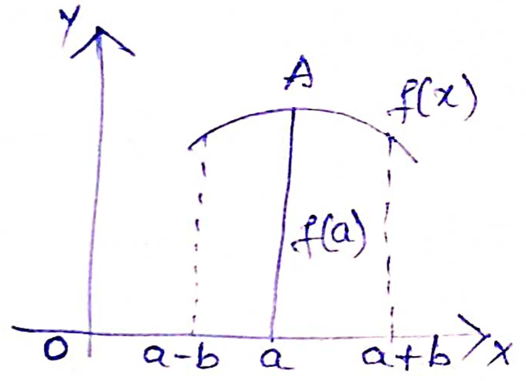
d) $\lim_{x\to \infty} \left(\frac{ax+1}{ax-1}\right)^{x}$

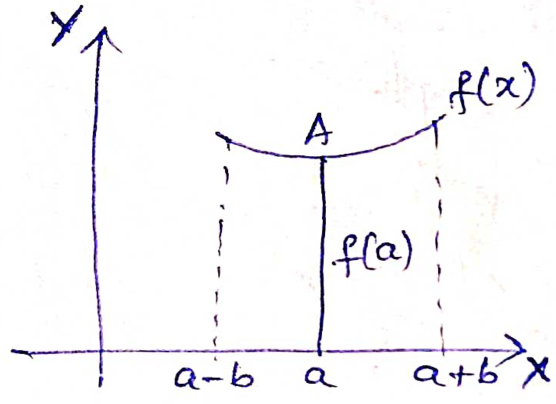
e) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

f) $\lim_{x\to 0} x^{\sin x}$

Maxima and Minima for one variables:

Def^n: A function $f(x)$ is said to have a maximum or a local maximum at the point $x=a$ where $a-b<a<a+b$, if $f(a) \geq f(a\pm b)$ for all sufficiently small positive b.





Def^n: A function $f(x)$ is said to have a minimum / local minimum at the point $x=a$ if $f(a) \leq f(a\pm b)$ where $a-b<a<a+b$ for all sufficiently small positive b.

Method of finding max^m and min^m values of a fun^n:

(i) Find $f'(x)$ and equate it to 0.

(ii) Solve $f'(x) = 0$.

(iii) Find $f''(x)$, and evaluate $f''(a)$

Then if

$f''(a) < 0$ Then $x = a$ is a point of local max^m.

$f''(a) > 0$ then $x = a$ is a point of local min^m

$f''(a) = 0$ then the sign of $f'(x)$ on left of 'a' and on the right of 'a' to arrive at result.

Examples:

1. Find The Max^m and min^m values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval $(0, 2)$.

Sol^n: $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$f'(x) = 12x^3 - 6x^2 - 12x + 6 = 0$

$12x^3 - x^2 - 2x + 6 = 0$

roots are $x = 1, -1, \frac{1}{2}$

Since $f(x)$ is defined b/w $(0, 2)$

the roots are $\{1, \frac{1}{2}\}$.

$f''(x) = 36x^2 - 12x - 12 \longrightarrow$ ⓐ

at $x = 1$,

$f''(1) = 36 - 12 - 12 = 36 - 24 > 0$

$\therefore x = 1$ is minimum point.

$f''\left(\frac{1}{2}\right) = \cancel{36}^{9} \cdot \frac{1}{\cancel{4}} - \cancel{12}^{6} \cdot \frac{1}{\cancel{2}} - 12 = 9 - 6 - 12 < 0$

$\therefore x = \frac{1}{2}$ is maximum point.

$f(1) = 3 - 2 - 6 + 6 + 1 = 2 \rightarrow$ min^M value

$f\left(\frac{1}{2}\right) = 3 \cdot \frac{1}{16} - 2\frac{1}{8} - 6\frac{1}{4} + 6\frac{1}{2} + 1$

$= 2.75 \rightarrow$ max^M value.

2. Find the max^M & min^M value of the fun^n $f(x) = \sin x\,(1 + \cos x)$ in $(0, \pi)$

Sol^n: Let $f(x) = \sin(1 + \cos x)$

$f'(x) = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$

$= 2\cos^2 x + \cos x - 1 = 0$

$\cos x = \dfrac{-1 \pm \sqrt{1 + 8}}{4}$

$\cos x = -1, \frac{1}{2}$ $\quad \therefore x = \pi, \frac{\pi}{3}$

We have

$f'(x) = \cos x\,(1 + \cos x) - \sin^2 x$

$f''(x) = \cos x\,(-\sin x) + (1 + \cos x)(-\sin x)$

$- 2\sin x \cdot \cos x$

Now,

- $f''(\pi) = 0 \quad \because \sin n\pi = 0, n \in N$

$$f''\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \left(1+\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) - 2\,\frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$= -\frac{\sqrt{3}}{4} - \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{2} < 0$$

$\therefore x = \frac{\pi}{3}$ is max^m point

$x = \pi$ is min^m point

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4} \longrightarrow max^m \text{ value}$$

$f(\pi) = 0 \longrightarrow$ doesn't exist.

Assignment:

1. Find the minima for
$f(x) = \cos 4x \,;\; 0 < x < \frac{\pi}{2}$

2. Find the max^m value for
$f(x) = \sin x + \cos x, \quad 0 < x < \frac{\pi}{2}$

Working procedure for finding extreme values of $f(x,y)$:

1. find the stationary points (x,y) ∋ $f_x = 0$. & $f_y = 0$

2. find the 2^{nd} order partial derivatives
$A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$
also compute the corresponding value of $AC - B^2$

3. (i) A stationary point (x_0, y_0) is a max^m pt if $AC - B^2 > 0$ and $A < 0$ then $f(x_0, y_0)$ is a max^m value.

(ii) A stationary point (x_1, y_1) is a min^m pt if $AC - B^2 > 0$ and $A > 0$ then $f(x_1, y_1)$ is a min^m value.

Note: If $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$ then the point (x,y) is called 'saddle point'.

Example:

1. Find the Extreme values of the fun^n:
$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Sol^n: Given $f(x,y) = x^3 + y^3 - 3x - 12y + 20$
$f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$.

To find stationary points; $f_x = 0$. & $f_y = 0$

$0 = 3x^2 - 3$ and $3y^2 - 12 = 0$

$x^2 = 1$ $\qquad$ $y^2 = 4$

$x = \pm 1$ $\qquad$ $y = \pm 2$

∴ The stationary points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$

let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$

	(1, 2)	(1, −2)	(−1, 2)	(−1, −2)
$A = 6x$	6	6	−6	−6
$B = 0$	0	0	0	0
$C = 6y$	12	−12	12	−12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
conclusion	min^m pt	Saddle pt	Saddle pt	Max^M. pt

$\therefore$ Max^m value of $f(x, y)$ is

$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$

and

Min^m value of $f(x, y)$ is

$f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

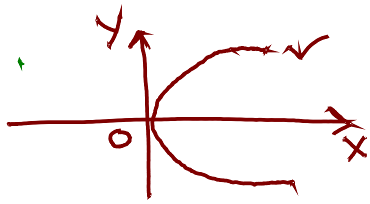
Thus max^m value is 38 and

min^m value is 2

Method of Tracing of curve in Cartesian form:-

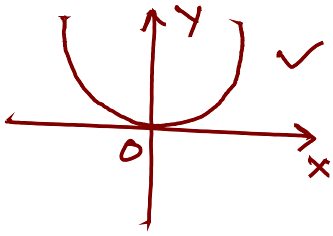
① Symmetry:-

(i) If the given equation of the curve contains even powers to 'y' only (or) 'y' is replaced by '-y' then the curve is symmetrical about x-axis.



Ex:- $y^2 = 4ax$ ✓

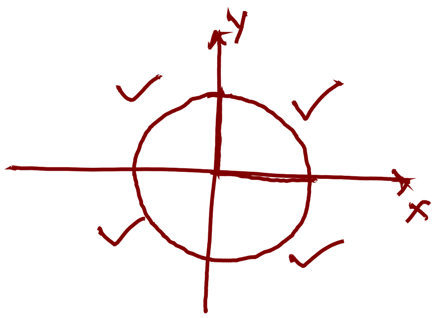
(ii) If the given equation of the curve contains even powers to 'x' only then the curve is symmetrical about y-axis.



Ex:- $x^2 = 4ay$

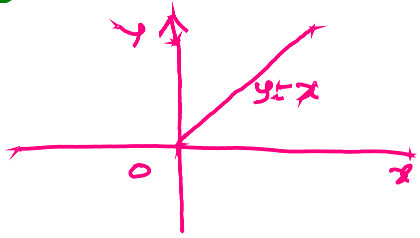
(iii) If the given equation of the curve contains even powers to both x, y then the curve is symmetrical about both the axes.
In this case, there are 4 equal portions of the curve exists one in each quadrant.

Ex:- $x^2 + y^2 = a^2$



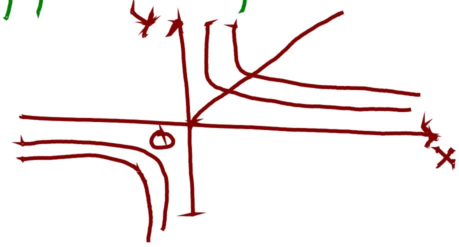
(iv) If the equation of the curve remains unchanged when x and y interchanged. Then the curve is symmetrical about the line $y=x$.

Ex :- $x^3+y^3=3axy$.



(v) If the eq of the curve remains unchanged when 'x' is changed to '$-x$' and 'y' is changed to '$-y$'. Then the curve is symmetrical about the origin. i.e., symmetrical in opposite quadrants.

Ex :- $xy=c$



② origin :- The given equation of the curve is satisfied by substituting $x=0, y=0$ Then the curve passes through origin.

Ex :- $y^2=4ax$ (passes through origin)

$y^2=4ax+c$ (not)

③ Intersection with Coordinate axes :-

(i) To get the points of intersection with y-axis, put $x=0$ in the given equation

(ii) To get the points of intersection with x-axis, put $y=0$ in the given equation.

Ex :- $y^2(a-x) = x^2(a+x)$

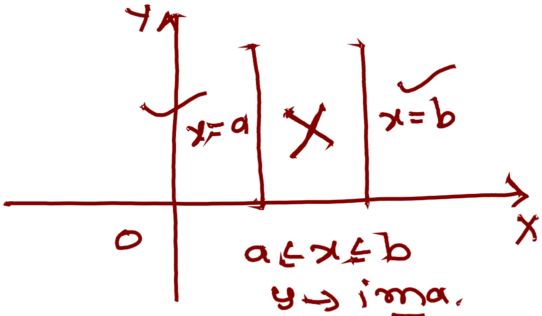
put $x=0 \Rightarrow y^2 a = 0 \Rightarrow y=0$

i.e., no intersection with y-axis

put $y=0 \Rightarrow x^2(a+x)=0 \Rightarrow x=0,-a$

i.e., The curve intersects the x-axis at $(-a,0)$.

(4) Region :- If possible, write the given equation of the curve in the form $y=f(x)$. Give the values to 'x' to make 'y' imaginary. Let 'y' be imaginary for the values of x between $x=a$ and $x=b$, no part of the curve exists between $x=a$ and $x=b$.



(5) Tangents :-

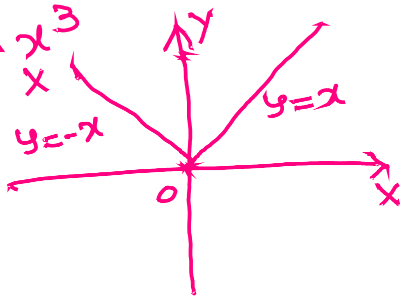
(i) If the equation of the curve passes through origin, the tangents at origin are given by equating the lowest degree terms occurring in the equation to zero.

Ex:- $y^2 a - \cancel{xy^2} = x^2 a + \cancel{x^3}$

$\Rightarrow \quad y^2 a - x^2 a = 0$

$\Rightarrow \quad y^2 = x^2$

$\Rightarrow \quad y = \pm x$.



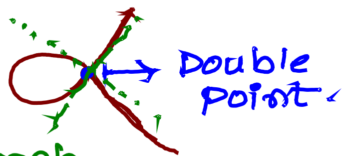
(ii) If the equation of the curve doesn't pass through origin, the tangents at any points are given by findind $\frac{dy}{dx}$ at that point and this indicates the direction of the tangent at that point.

NOTE :- If $\frac{dy}{dx} = 0$ ie, $\tan\psi = 0$ Then the tangent is parallel to x-axis.

•• If $\frac{dy}{dx} = \infty$, $\tan\psi = \infty$, $\psi = 90^\circ$ Then the tangent is parallel to y-axis.

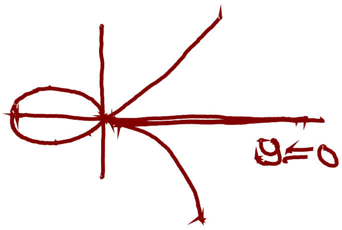
Double point :-

A point lying on two branches of the curve is called a Double point. At that point, the curve has two tangents one for each branch.



If the tangents are real and destinct, Then the double point is called NODE

If the tangents are real and coincide Then the double point is called CUSP.



⑥ Extension to Infinity :-

Give the values to x and find the value ~~of~~ 'x' where 'y' is infinity and vice verse.

⑦ Asymptotes :-

An Asymptote is a straight line which touches the curve at infinity.

Parallel asymptotes :-

(i) The asymptotes parallel to x-axis are obtained by equating the coefficents of highest powers of 'x' in the equation to zero. If the coefficents of highest power of 'x' is constant then there are no asymptotes parallel to x-axis.

(ii) The asymptotes parallel to y-axis are obtained by equating the coefficents of highest powers of 'y' to zero. If the coeff of highest power of 'y' is a constant then there are no asymptotes parallel to y-axis.

Oblique Asymptotes :- An asymptote which is neither parallel to x-axes nor to y-axes.

Let $y = mx + c$ be an asymptote. Substitute $y = mx + c$ in the given eq and equate the coeff of highest powers of 'x' to '0' and solve for 'm' and 'c'.

① Trace the curve $y^2(2a-x)=x^3$, $a>0$.

Sol :- Given $y^2(2a-x)=x^3$ —— ①

Symmetry :- The given equation of the curve contains even powers to y only so the curve is symmetrical about x-axis.

Origin :- The given equation of the curve is satisfied by substituting $x=0$, $y=0$.

$\therefore$ It passes through origin.

Intersection with the Coordinate axes :-

put $x=0$ in ①, $y^2(2a)=0 \Rightarrow y^2=0 \Rightarrow y=0$

$\therefore$ The eq ① doesn't intersects y-axis.

put $y=0$ in ①, $x^3=0 \Rightarrow x=0$

$\therefore$ The eq ① doesn't intersects x-axis

$\therefore$ The eq ① intersects only at origin.

Region :- Given eq can be written as

$$y^2=\frac{x^3}{2a-x} \Rightarrow y=\pm\sqrt{\frac{x^3}{2a-x}}$$

$\checkmark x=0, a, 2a, \ldots$

$x=-a, -2a, \ldots$

when $x>2a$, y is imaginary $\left(\frac{x^3}{2a-x}=-ve\right)$

when $x<0$, y is imaginary

$\therefore$ No part of the curve exists beyond the lines $x=2a$ and $x=0$. i.e., The curve exists only b/w the lines $x=0$ and $x=2a$.

Tangents :- The tangents at origin are obtained by equating the lowest degree terms to zero.

$$\therefore \quad y^2(2a)=0$$

$$\underbrace{y^2 2a}-xy^2=x^3$$

$$\Rightarrow \quad y^2=0$$

$$y^2(2a-x)=x^3$$

$$\Rightarrow \quad y=0,0$$

i.e., x-axis is tangent at origin.

The given equation has two tangents at the origin which are real and coinciding.

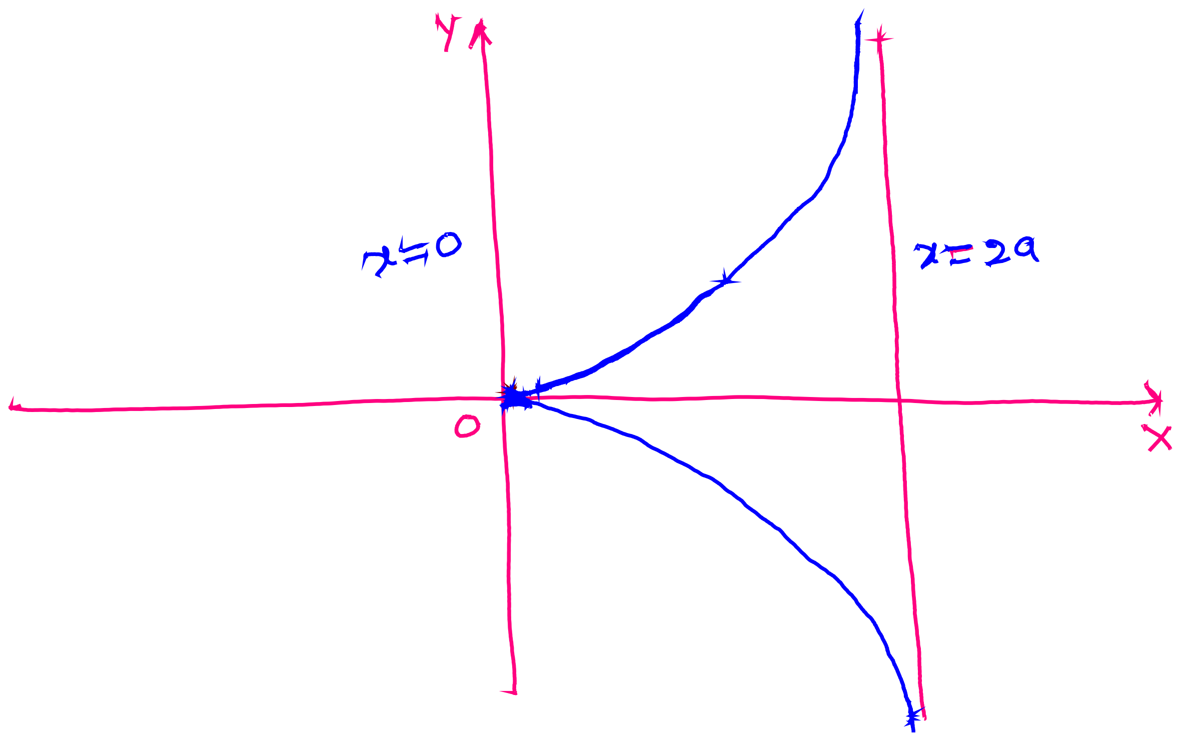
$\therefore$ The origin is CUSP.

Asymptotes :-

There are no asymptotes parallel to x-axis. since the coeff of highest power of x is constant

The asymptotes parallel to y-axis are obtained by equating the coeff of highest power of y to zero. $\therefore \; 2a-x=0 \Rightarrow x=2a$.

The shape of the curve :-



② Trace the curve $y^2(a-x) = x^2(a+x)$, $a>0$.

Given $y^2(a-x) = x^2(a+x)$ —①

Symmetry:– The eq ① is symmetrical about x-axes since even powers to 'y'.

Origin:– It passes through origin.

Intersection with coordinate axes:–

put $x=0$ in ① $\Rightarrow y^2 a = 0 \Rightarrow y=0$

∴ The eq ① doesn't intersects y-axis.

put $y=0$ in ① $\Rightarrow x^2(a+x)=0$

$\Rightarrow x=0$ or $x=-a$

∴ The curve intersects the x-axis at $(-a,0)$.

Region:– Given curve can be written as

$$y^2 = \frac{x^2(a+x)}{a-x} \Rightarrow y = \pm\sqrt{\frac{x^2(a+x)}{(a-x)}}.$$

when $x>a$, y is imaginary.

when $x<-a$, y is imaginary.

∴ The curve exists only between the lines $x=a$ and $x=-a$.

Tangents:–

(i) The tangents at origin are obtained by equating the lowest degree terms to zero

$$ay^2 - xy^2 = x^2 a + x^3$$

$$\Rightarrow ay^2 - x^2 a = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$$

∴ Origin is NODE.

(ii) Diff ① w.r.t x, we get $(y^2a - xy^2 = ax^2 + x^3)$

$$2y\frac{dy}{dx}a - x\,2y\frac{dy}{dx} - y^2 = 2ax + 3x^2$$

$$\Rightarrow \frac{dy}{dx}(2ay - 2xy) = 2ax + 3x^2 + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ax + 3x^2 + y^2}{2ay - 2xy}$$

At $(-a, 0)$, $\frac{dy}{dx} = \infty$

$\therefore$ The tangent is parallel to y-axis at $(-a, 0)$.

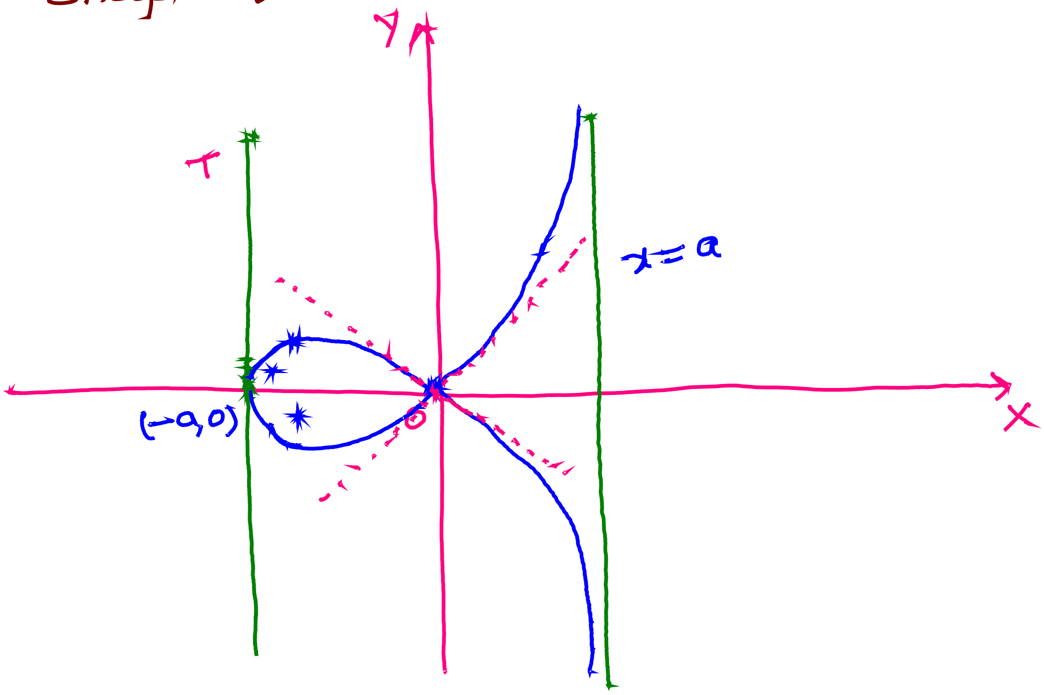
Asymptote :-

There are no asymptotes parallel to x-axis.

The asymptotes parallel to y-axis are obtained by equating the coeff of highest powers of y to zero. i.e., $a - x = 0$

$\Rightarrow x = a$.

The shape of the curve is



③ Trace the curve $x^{2/3}+y^{2/3}=a^{2/3}$, $a>0$

(ASTROID)

Given $x^{2/3}+y^{2/3}=a^{2/3}$ ——①

Symmetry:-

1. The eq ① remains same when 'x' & 'y' are interchanged. It is symmetrical about the line $y=x$.

2. The eq ① remains same when $x=-x$ and $y=-y$. It is symmetrical about the origin.

Origin:- It doesn't pass through origin.

Intersection with coordinate axes:-

put $y=0$ in ①, $x^{2/3}=a^{2/3} \Rightarrow x^2=a^2$
$\Rightarrow x=\pm a$

∴ The given curve intersects x-axis at $(a,0)$ and $(-a,0)$.

put $x=0$ in ①, $y^{2/3}=a^{2/3} \Rightarrow y^2=a^2 \Rightarrow y=\pm a$

∴ The given curve intersects y-axis at $(0,a)$ and $(0,-a)$.

Region:- eq ① can be written as

$$y^{2/3}=a^{2/3}-x^{2/3} \Rightarrow y=\left(a^{2/3}-x^{2/3}\right)^{3/2}$$

when $x>a$ and $x<-a$, y is imaginary.

∴ The curve lies b/w the lines $x=a$ and $x=-a$

Also $x = (a^{2/3} - y^{2/3})^{3/2}$ [from ①]

when $y > a$ and $y < -a$, x is imaginary

$\therefore$ The curve lies b/w $y = a$ and $y = -a$

Tangents :- Diff ① w.r.t x, we get

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}}$$

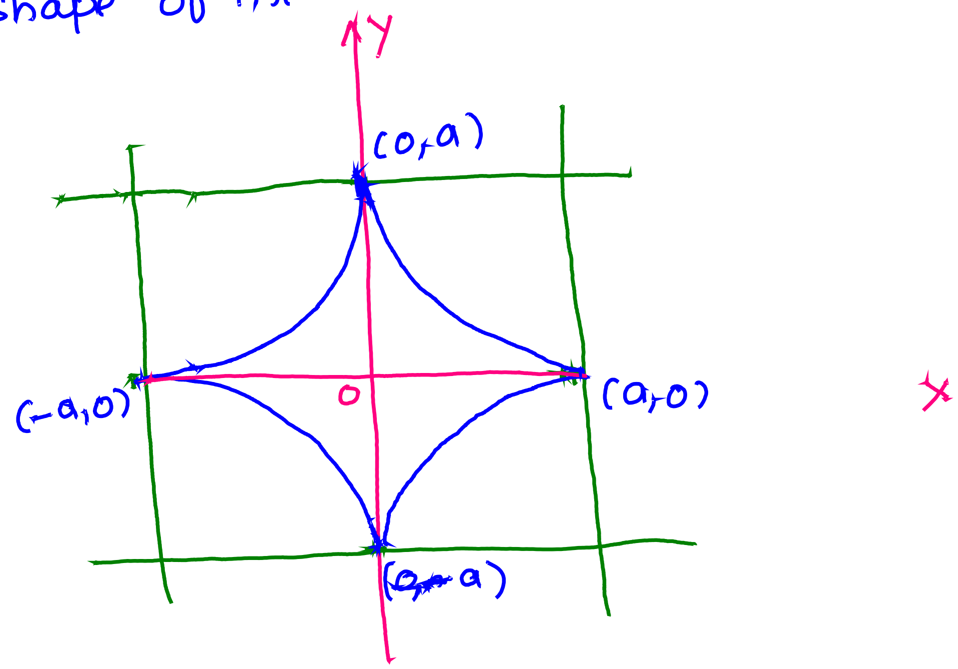
At $(\pm a, 0)$, $\frac{dy}{dx} = 0$

$\therefore$ Tangents to the curves at $(\pm a, 0)$ are parallel to x-axis.

At $(0, \pm a)$, $\frac{dy}{dx} = \infty$

$\therefore$ Tangents to the curve at $(0, \pm a)$ are parallel to Y-axis.

The shape of the curve is



④ Trace the curve $x^3+y^3=3axy$, $a>0$.

Given $x^3+y^3=3axy$ —①

Symmetry :- The given eq ① remains the sames when x and y interchanged.
∴ It is symmetrical about the line $y=x$.

Origin :- The curve passes through the origin.

Intersection with coordinate axes :-

put $y=0 \Rightarrow x=0$.
∴ The given curve doesn't intersects x-axis.

put $x=0 \Rightarrow y=0$
∴ The given curve doesn't intersects y-axis

put $y=x$ in ① we get $2x^3=3ax^2$
$\Rightarrow x=\frac{3}{2}a$

put $x=y$ in ①, we get $2y^3=3ay^2$
$\Rightarrow y=\frac{3}{2}a$

∴ The given eq of curve intersects the line $y=x$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

Tangents :-

(i) The tangents at origin are obtained by equating the lowest degree terms to zero. ∴ $3axy=0$
$\Rightarrow xy=0 \Rightarrow x=0, y=0$.

∴ origin is a NODE.

(ii) Diff ① w.r.t x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \frac{dy}{dx}(\cancel{3}y^2 - \cancel{3}ax) = \cancel{3}ay - \cancel{3}x^2$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $\frac{dy}{dx} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$

$\therefore \tan\psi = -1$

$\Rightarrow \psi = 135^0$

ie., The tangent to the curve at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ makes an angle 135^0 with the initial line.

Asymptotes:- There are no parallel asymptotes.

Oblique asymptotes:-

Let $y = mx + c$ be an asymptote.

Substituting in ①, we get

$$x^3 + (mx + c)^3 = 3ax(mx + c)$$

$$x^3 + m^3x^3 + c^3 + 3m^2x^2c + 3mxc^2 - 3amx^2 - 3axc = 0$$

$$\Rightarrow (1 + m^3)x^3 + (3m^2c - 3am)x^2 + (3mc^2 - 3ac)x + c^3 = 0$$

Equating the coeff of x^3 to zero

i.e, $1+m^3=0 \Rightarrow \underbrace{(m+1)}(\underbrace{m^2-m+1})=0$

$\Rightarrow \boxed{m=-1}$

Equating the coeff of x^2 to zero

$\cancel{3}m^2c - \cancel{3}am = 0$

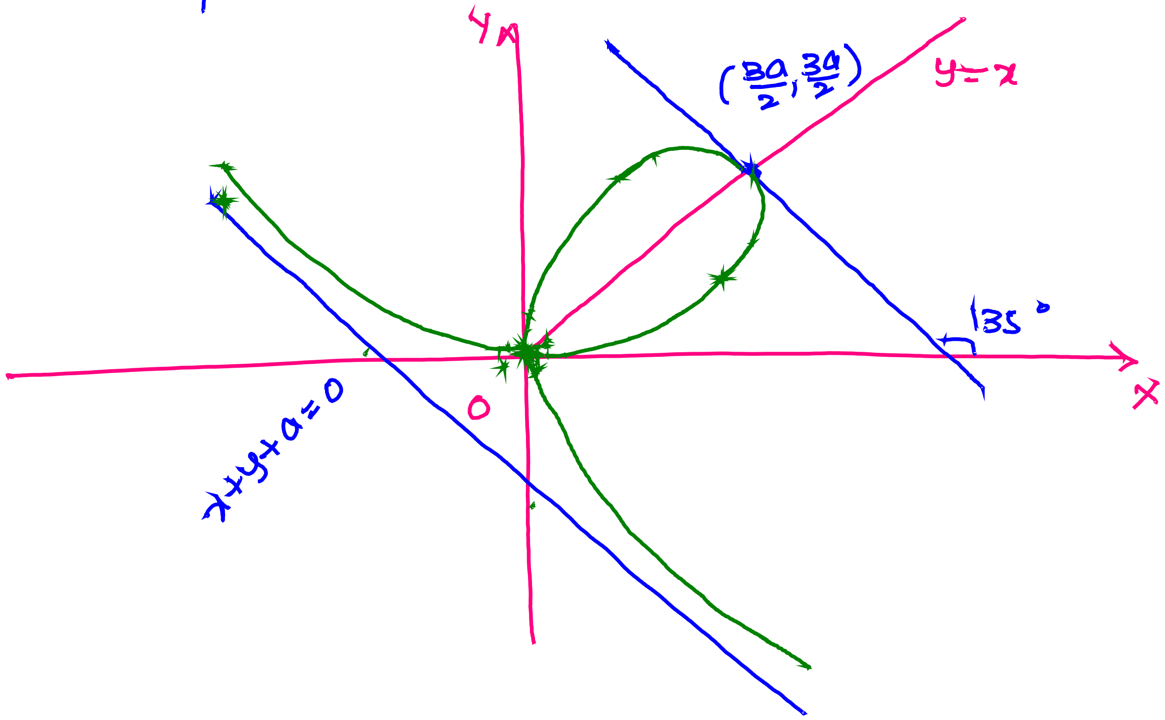
$\Rightarrow (-1)^2c - a(-1) = 0$

$\Rightarrow \boxed{c=-a}$

$\therefore$ $y=-x-a$ is an oblique asymptote

$\Rightarrow x+y+a=0$

The shape of the curve is



Method of Tracing of curve in polar form :-

Let $r = f(\theta)$ be an equation of curve in polar form.

Symmetry :-

(i) If the given equation of curve remains unchanged when 'θ' is changed to '$-\theta$' Then the curve is symmetrical about the initial line or $\theta = 0$ line or x-axis.

Ex :- $r = a(1 + \cos\theta)$

(ii) If the given equation of the curve remains same when 'r' is changed to '$-r$' (or) It contains even powers to 'r' Then the curve is symmetrical about the pole (or) origin.

Ex :- $r^2 = a^2 \cos 2\theta$; $r^2 = a^2 \sin 2\theta$

(iii) If the given equation of the curve remains same when 'r' is changed to '$-r$' and 'θ' is changed to '$-\theta$' (or) when 'θ' is changed $(\pi - \theta)$ Then the curve is symmetrical about $\theta = \pi/2$ line (or) y-axis.

Ex :- $r = a \sin 3\theta$

Pole :- Put $r=0$ in the given equation and find the value of 'θ'. If 'θ' is real, then the curve passes through the pole. This indicates the tangent to the curve at the pole.

Ex :- $r = a(1+\cos\theta),\ a>0$

put $r=0 \Rightarrow a(1+\cos\theta)=0$

$\Rightarrow 1+\cos\theta=0$

$\Rightarrow \cos\theta=-1 \Rightarrow \theta=\pi$

$\therefore$ $r=a(1+\cos\theta)$ passes through pole.

$\theta=\pi$ is a tangent to the curve at pole.

Region :- Find out the largest value of 'r' numerically. This indicates the curve lies within the radius 'r'.

Ex :- $r = a(1+\cos\theta),\ a>0.$

when $\theta=0 \Rightarrow \cos\theta=1 \therefore r=2a.$

ie., the curve lies within the radius $r=2a$.

Discussion for 'r' and 'θ' :-

Give certain values to 'θ' and find the corresponding values of 'r'. By using these points we can trace the curve.

Tangents :- The tangents at any points are given by finding $\tan\phi = r\frac{d\theta}{dr}$.
It will indicate the direction of the tangent at that point.

① Trace the curve $r = a(1+\cos\theta), a>0$.

"CARDIOID"

Given curve is $r = a(1+\cos\theta)$ —①

Symmetry :- The given eq of the curve ① remains same when 'θ' is replaced by '$-\theta$'.
∴ Eq of curve ① is symmetrical about the initial line or $\theta=0$ line.

Pole :- put $r=0$ in ①, we get

$$a(1+\cos\theta)=0$$
$$\Rightarrow 1+\cos\theta=0$$
$$\Rightarrow \cos\theta=-1 \Rightarrow \boxed{\theta=\pi}$$

∴ The given equation passes through pole.
$\theta=\pi$ is the tangent at pole.

Region :- when $\theta=0 \Rightarrow \cos\theta=1$
∴ $r=2a$ is the largest value.
∴ The entire curve lies within the radius $r=2a$.

Discussion for r and θ :-

θ	0	π/3	π/2	π
r	2a	3a/2	a	0

Tangents :- Diff ① w.r.t θ, we get

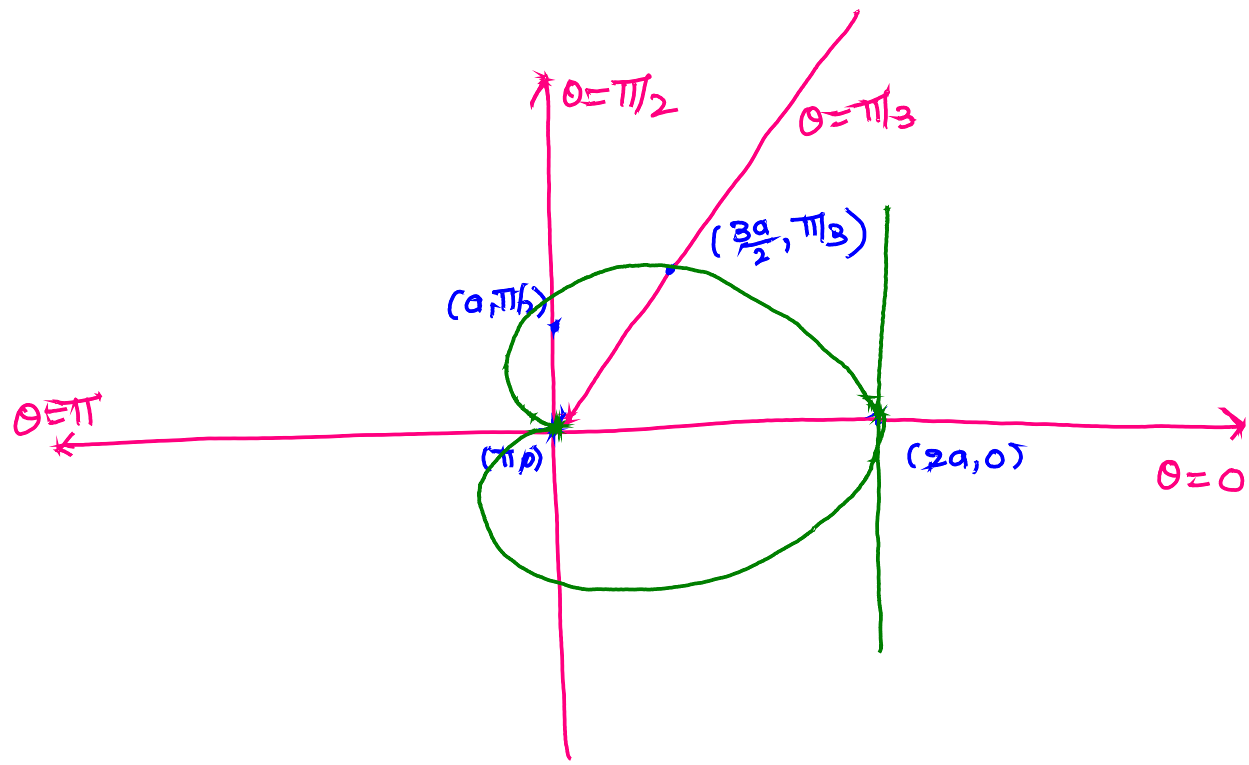
$$\frac{dr}{d\theta} = a(-\sin\theta). \Rightarrow \frac{d\theta}{dr} = \frac{-1}{a\sin\theta}.$$

Now $r\frac{d\theta}{dr} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta}$

when $\theta = 0 \Rightarrow \tan\phi = \infty \Rightarrow \phi = \pi/2$

The tangent at $\theta = 0$ is ⊥ to the initial line.

The shape of the curve is



② Trace the curve $r = a(1-\cos\theta)$.

Symmetry :- Initial line

Pole :- put $r=0 \Rightarrow a(1-\cos\theta)=0$
$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

$\therefore$ It passes through pole and $\theta=0$ is tangent at pole.

Region :- when $\theta=\pi$, $r=2a$.

$\therefore$ Entire curve lies within the radius $r=2a$.

Discussion for 'r' and 'θ' :-

θ	0	$\pi/3$	$\pi/2$	π
r	0	$a/2$	a	$2a$

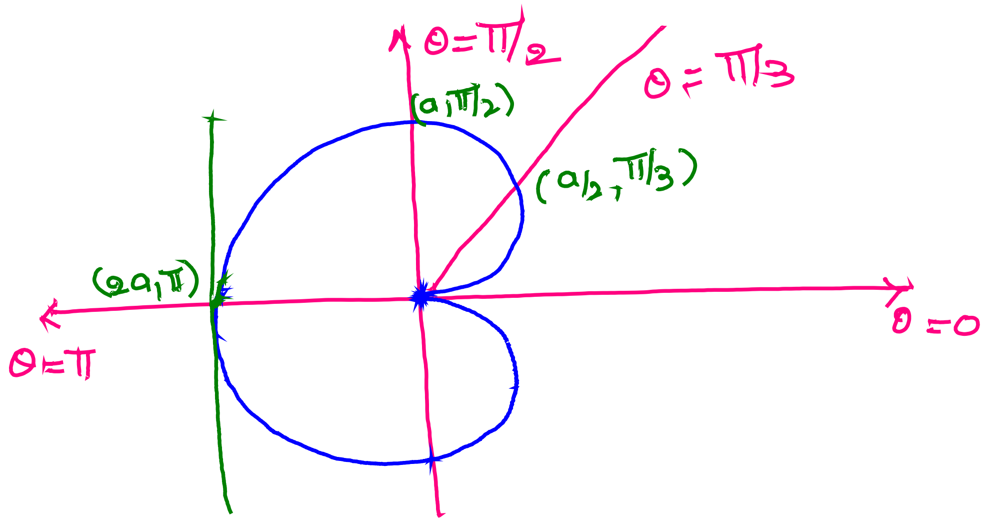
Tangents :- Diff $r = a(1-\cos\theta)$ w.r.t 'θ'

$$\frac{dr}{d\theta} = a\sin\theta \Rightarrow r\frac{d\theta}{dr} = \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$$

$$\Rightarrow \tan\phi = \frac{1-\cos\theta}{\sin\theta}$$

when $\theta=\pi$, $\tan\phi = \infty \Rightarrow \phi = \pi/2$.

$\therefore$ Tangent is $\perp$r to the initial line.

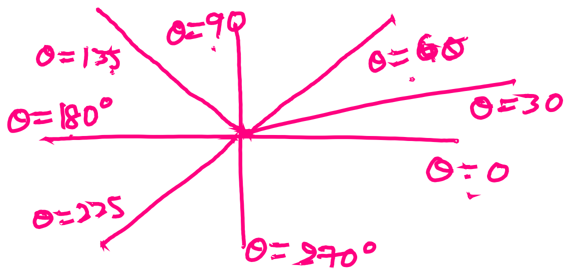


NOTE :- To trace the curves of the form $r = a\sin n\theta$ or $r = a\cos n\theta$ where 'n' is +ve integer.

Divide each quadrant into 'n' equal parts and give these value for 'θ' to find 'r'.

Ex :- $r = a\sin 3\theta$

Here $n = 3$



NOTE :-

(i) If 'n' is odd then we get 'n' loops

(ii) If 'n' is even then we get '2n' loops.

③ Trace the curve $r = a\sin 3\theta$.

Given $r = a\sin 3\theta$ —— ①

Symmetry :- The eq ① remains the same when 'r' is changed to $-r$ and 'θ' is changed to $-\theta$. ∴ It is symmetrical about $\theta = \pi/2$ line.

Pole :- put $r=0$ in ①, we get $a\sin 3\theta = 0$

$$\Rightarrow \sin 3\theta = 0$$

$$\Rightarrow 3\theta = 0 \Rightarrow \theta = 0$$

∴ eq ① passes through the pole and $\theta = 0$ is the tangent at the pole.

Region :- when $\sin 3\theta = 1$, $r = a$

$\therefore$ The entire curve lies within the radius 'a'.

Discussion for 'r' and 'θ' :-

Since $n = 3$, divide each quadrant into 3 equal parts.

θ	0°	30°	60°	90°	120°	150°	180°	210°
r	0	a	0	$-a$	0	a	0	$-a$

when $\theta = 210^\circ$, $r = -a$ ie., the point $(-a, 210^\circ)$ is same as $(a, 30^\circ)$.

$\therefore$ By giving the values of θ after 210°, the values of 'r' repeats.

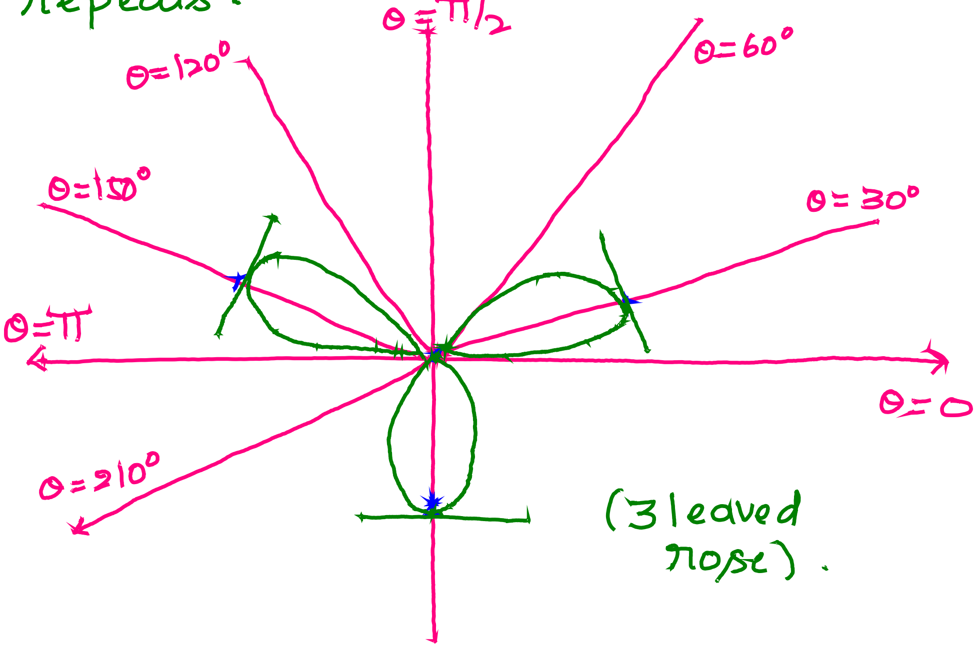
Tangents :-

Diff ① w.r.t 'θ'

$$\frac{dr}{d\theta} = 3a\cos 3\theta$$

$$\frac{r\,d\theta}{dr} = \frac{a\sin 3\theta}{3a\cos 3\theta}$$

$$\tan\phi = \frac{1}{3}\tan 3\theta$$



(3 leaved rose).

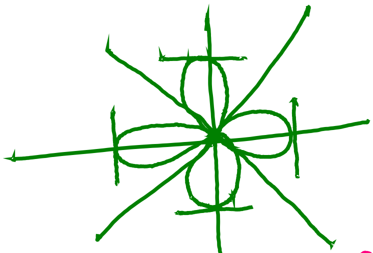
when $\theta = 30^\circ, 90^\circ, 150^\circ, 210^\circ$, $\tan\phi = \infty \Rightarrow \phi = \pi/2$.

$\therefore$ The tangent is ⊥r at the points $(a, 30)$, $(-a, 90)$, $(a, 150^\circ)$, $(a, 210)$.

Since $n = 3$ ie., odd, the curve has 3 loops.

④ Trace the curve $r = a\cos 2\theta$.

(left to students)



Method of tracing of curve in parametric form:-

The equation of the curve $x=\phi(t), y=\psi(t)$ is called the equation in parametric form.

Symmetry :-

(i) If $x=\phi(t)$ is an even function of 't' and $y=\psi(t)$ is an odd function of 't' then the curve is symmetrical about x-axis.

(ii) If $x=\phi(t)$ is an odd function of 't' and $y=\psi(t)$ is an even function of 't' then the curve is symmetrical about y-axis.

(iii) If both x, y are odd functions then the curve is symmetrical in opposite quadrants.

(iv) If both x, y are even functions then no reference drawn about symmetry.

Origin :- put $x=0$ and find 't'. For the same 't', if $y=0$ then the curve passes through pole. (origin).

Limits :- If possible, find the least and the greatest values of x, y. These values indicate in which region the curve lies (or) does not lie.

Discussion for t, x, y :- Give certain values for t and find $x; y$.

Tangents :- Find $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

If $\frac{dy}{dx} = 0$ at any point then the tangent is parallel to x-axis.

If $\frac{dy}{dx} = \infty$ at any point then the tangent is perpendicular to x-axis.

1. Trace the curve $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$.

Given $x = a(\theta - \sin\theta)$ —— (1)
$y = a(1 - \cos\theta)$ —— (2)

Symmetry :- Since 'y' is an even function of 'θ' the curve is symmetrical about y-axis.

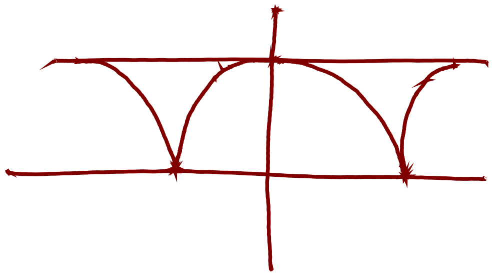
Origin :- when $\theta = 0$, both x, y are '0'
$\therefore$ The curve passes through origin.

Limits :- Since $-1 \le \cos\theta \le 1$, the greatest value of y is $2a$ when $\theta = \pi$, The least value of y is '0'.
$\therefore$ the curve entirely lies b/w the lines $y = 0$ and $y = 2a$.

Discussion for θ, x, y :-

$x = a(\theta + \sin\theta)$ — ①

$y = a(1 + \cos\theta)$ — ②



θ	0	π	2π
x	0	πa	$2\pi a$
y	0	$2a$	0

Tangents :-

Diff ① & ② w.r.t θ, we get $\frac{dx}{d\theta} = a(1-\cos\theta)$

$$\frac{dy}{d\theta} = a\sin\theta.$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)} = \frac{\cancel{2}\sin\theta/2\,\cos\theta/2}{\cancel{2}\sin^2\theta/2}$

$$\frac{dy}{dx} = \cot(\theta/2)$$

when $\theta = \pi$, $\frac{dy}{dx} = 0$

$\therefore$ At $(\pi a, 2a)$, the tangent is parallel to x-axis.

when $\theta = 2\pi$, $\frac{dy}{dx} = \infty$

$\therefore$ At ⊛ $(2\pi a, 0)$, the tangent is perpendicular to x-axis.

The shape of the curve is