Sequences and Series

Sequence: A seqn is a funⁿ whose domain is the set N of all natural nos and the range may be any set.

Real Sequence: A real seqn is a funⁿ whose domain is the set N of all natural nos & the range is a subset of R of all real nos i.e. $f: N \to \mathbb{R}$ is a real seqn. (or) $a: N \to R$ (or) $x: N \to R$

In general we denote a seqn by $a_1, a_2, a_3, \ldots$ and the n^{th} term of the seqn is denoted by a_n. $a_1, a_2, a_3, \ldots$ are called elements / Terms of the seqn & the seqn itself is denoted by $\{a_n\}$ (or) $\langle a_n \rangle$.

Ex: If $a_n = \frac{1}{x^n}$ then $\{a_n\} = \left\{\frac{1}{x^n}\right\}$

$$= \left\{\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \ldots\right\}$$

Constant Sequence: A seqn $\{a_n\}$ defined by $a_n = K \ \forall n \in N$ is called constant seqn.

Examples for sequence:

(i) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

(ii) $\left\{\frac{1}{2^{n-1}}\right\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$

(iii) $\left\{\frac{1}{n} - \frac{1}{n^2}\right\} = \left\{1-1, \frac{1}{2} - \frac{1}{4}, \frac{1}{3} - \frac{1}{9}, \ldots\right\}$

$$= \left\{0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \ldots\right\}$$

Bounded Sequences:

a) **Bounded below sequence:**

A seqn $\{a_n\}$ is said to be bounded below if $\exists$ a real nos 'm' $\ni$ $a_n \geq m \ \forall n \in N$

i.e if range of the seqn is bounded below.

b) Bounded above seqn:

A seqn $\{a_n\}$ is said to be bounded above if $\exists$ a real no 'M' $\ni a_n \leq M \; \forall n \in N$

i.e if range of the seqn is bounded above.

Here m & M are lower bound & upper bound of the seqn respl:

c) Bounded seqn:

A seqn $\{a_n\}$ is said to be bounded if it is bounded above & bounded below

i.e a seqn $\{a_n\}$ is bounded if $\exists$ 2 real nos 'm', & 'M' $\ni m \leq a_n \leq M \; \forall n \in N$

i.e if the range of seqn is bounded.

Note:

① If the seqn is not bounded then it is called as unbounded seqn.

② Every constant seqn is a bounded seqn

Example:

(i) $\{a_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{\infty}\}$

$m = 0, \quad M = 1$

Since all the elements of seqn are ≤ 1 and $>$ zero i.e $0 < a_n \leq 1, \forall n \in N$

$\therefore \{a_n\}$ is bounded seqn.

(ii) $\{a_n\} = \{(-1)^n \frac{1}{n}\} \longrightarrow$ bounded

$m = -1, \; M = 1/2 \quad (\because \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots 0\}$

(iii) $\{a_n\} = \{n\} = \{1, 2, 3, \dots \infty\}$

$\{a_n\}$ is bounded below i.e $m = 1$

but it is unbounded above

$\therefore \{a_n\}$ is unbounded.

Def^n:

① A real nos 'M' is said to be the least upper bound (or Supremum) of a seqn $\{a_n\}$ if

(i) M is upper bound of $\{a_n\}$ i.e $a_n \leq M$ $\forall n \in N$

(ii) Any nos less than M cannot be an upper bound of seqn $\{a_n\}$ i.e given an $\varepsilon > 0$ $\exists$ atleast one term $a_k \ni a_k > M - \varepsilon$.

② A real nos 'm' is said to be the greatest lower bound (or infimum) of a seqn $\{a_n\}$ if

(i) 'm' is lower bound of $\{a_n\}$ i.e $a_n \geq m$ $\forall n \in N$

(ii) Any nos greater than m cannot be an lower bound of seqn $\{a_n\}$ i.e given an $\varepsilon > 0$ $\exists$ atleast one term $a_k \ni a_k < m + \varepsilon$.

Ex: ① $\left\{1 + \frac{1}{n}\right\}$

$$= \left\{1+1, 1+\frac{1}{2}, 1+\frac{1}{3}, \dots 1+\frac{1}{\infty}\right\}$$

$$= \left\{2, \frac{3}{2}, \frac{4}{3}, \dots, 1\right\}$$

$\therefore$ Supremum $= 2$ & Infimum $= 1$

Limit of a Sequence:

A seqn $\{a_n\}$ is said to tend to a limit l as $n \to \infty$ if given $\varepsilon > 0$ however small $\exists$ a +ve integer 'm' depending upon $\varepsilon \ni |a_n - l| < \varepsilon$ $\forall n \geq m$.

i.e $\lim\limits_{n \to \infty} a_n = l$.

a) Convergent Seqn:

A seqn a_n is said to be convergent if seqn tends to a unique finite quantity.
i.e if for any $\varepsilon > 0$ $\exists$ a +ve integer 'm' depending on ε $\ni$ $|a_n - l| < \varepsilon$ $\forall$ $n \geq m$

i.e $\lim_{n\to\infty} a_n = l$.

b) Divergent Seqn:

A seqn a_n is said to be ~~dif~~ divergent if it diverges to $+\infty$ (or) $-\infty$.

i.e $\lim_{n\to\infty} a_n = +\infty$ (or) $\lim_{n\to\infty} a_n = -\infty$.

c) Oscillatory Seqn:

A seqn a_n is said to be oscillatory if it is neither convergent nor diverges to $+\infty$ (or) $-\infty$

Oscillatory seqn are 2 types:

(i) A bounded seqn which doesnot converges is said to oscilate finitely.

Ex: $\{a_n\} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$

(ii) A unbounded seqn which doesnot ~~diverge~~ diverge converge is said to oscilate infinitely.

Ex: $\{a_n\} = \{(-1)^n \cdot n\} = \{-1, 2, -3, 4, \ldots\}$

d) Null Seqn:

A seqn a_n is said to be a null seqn if it converges to zero.

i.e $\lim_{n\to\infty} a_n = 0$

Important points:

① Every convergent seqn has a unique limit. i.e a seqn cannot converge to more than one limit.

② Every convergent seqn is bounded.

③ If $\lim_{n\to\infty} a_n = l$ then $\lim_{n\to\infty} |a_n| = |l|$

④ In a convergent seqn if a finite nos of terms are removed the convergence of the seqn is not altered.

⑤ If $\lim_{n\to\infty} a_n = a$ & $a_n \geq 0 \ \forall n$ then $a \geq 0$.

Algebra of Sequences:

If $\lim_{n\to\infty} a_n = a$ & $\lim_{n\to\infty} b_n = b$ then

(i) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(ii) $\lim_{n\to\infty} (a_n - b_n) = a - b$

(iii) $\lim_{n\to\infty} (a_n . b_n) = ab$

(iv) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$, provided $b \neq 0$

Test the convergence of the foll seqn:

1. $a_n = \frac{2n^2+n+5}{n+3} \sin\frac{\pi}{n}$

Soln: let $a_n = \frac{2n^2+n+5}{n+3} . \sin\frac{\pi}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left\{ \frac{2n^2+n+5}{n+3} . \sin\frac{\pi}{n} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{\cancel{n^2}\left(2+\frac{1}{n}+\frac{5}{n^2}\right)}{\cancel{n}(1+3/n)} . \frac{\sin\frac{\pi}{n}}{\pi/n} . \frac{\pi}{\cancel{n}} \right\}$$

$$= \frac{(2+0+0)}{1+0} \cdot 1 \cdot \pi$$

$\lim_{n\to\infty} a_n = 2\pi \quad \therefore \{a_n\}$ converges.

2. $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$

Sol^n: let $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}$ (G.P formula)

$$a_n = \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$$

$$= \frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}} = \frac{3}{2}\left(1-\frac{1}{3^n}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^n}\right) = \frac{3}{2}(1-0)$$

$$= \frac{3}{2}$$

$\therefore \{a_n\}$ converges.

3. $n^2 + (-1)^n n$ (Ans: ∞, diverges)

4. $(1+\cos n\pi)$ (Ans: 0 & 2, oscillates $\therefore$ ~~not~~ donot converge)

5. $\frac{(-1)^{n-1}}{n}$ (Ans: 0, converges)

6. $\frac{(3n+1)(n+2)}{n(n-1)}$ (Ans: 3, converges)

7. $\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$ (Ans: $\frac{1}{3}$, converges)
(Hint: $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$)

8. $\sqrt{n}\left(\sqrt{n+4}-\sqrt{n}\right)$ (Ans: 2, converges)
(Hint: Rationalize)

9. $\frac{1^2+2^2+3^2+\cdots+n^2}{1^3+2^3+3^3+\cdots+n^3}$ (Ans: 0, converges
Hint: $\sum n^3 = \frac{n^2(n+1)^2}{4}$)

10. $\dfrac{2n^2 + 5\sin \pi/n}{n^2}$ (Ans : 2, converges)

11. $\dfrac{7+3.5^n}{6+4.5^n}$ (Ans : $\frac{3}{4}$, converges)

Monotonic Sequences:

a) A seqn a_n is said to be monotonically increasing if $a_{n+1} \geq a_n$, $\forall n \in N$

i.e $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq a_{n+1} \dots$

b) A seqn a_n is said to be monotonically decreasing if $a_{n+1} \leq a_n$, $\forall n \in N$

i.e $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq a_{n+1} \dots$

c) A seqn a_n is said to be monotonic, if it is either monotonically increasing (or) monotonically decreasing.

d) A seqn a_n is said to be strictly monotonically ↑ing if $a_{n+1} > a_n$ $\forall n \in N$.

e) A seqn a_n is said to be strictly monotonically ↓ing if $a_{n+1} < a_n$ $\forall n \in N$

f) A seqn a_n is said to be strictly monotonic if it is either strictly monotonically ↑ing (or) strictly monotonically ↓ing.

Properties of monotonic seqn:

1. A monotonic increasing seqn bounded above is convergent.

2. A monotonic decreasing seqn bounded below is convergent.

3. A monotonic increasing seqn which is not bounded above diverges to $+\infty$.

4. A monotonic decreasing seqn which is not bounded below diverges to $-\infty$.

5. Every monotonic seqn either converges (or) diverges.

Note:

(i) $\lim_{n\to\infty} n^{1/n} = 1$

(ii) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

1. P.T the $\{a_n\}$ defined by $a_1 = \sqrt{7}$ and $a_{n+1} = \sqrt{7+a_n}$ converges to the positive root of the equation $x^2 - x - 7 = 0$

Sol^n: Given $a_1 = \sqrt{7}$, $a_{n+1} = \sqrt{7+a_n}$

$\therefore \quad a_2 = \sqrt{7+a_1} = \sqrt{7+\sqrt{7}} > \sqrt{7}$

$\Rightarrow a_2 > a_1$

Suppose $a_n > a_{n-1}$

$7 + a_n > 7 + a_{n-1}$

$\Rightarrow \sqrt{7+a_n} > \sqrt{7+a_{n-1}}$

$\Rightarrow a_{n+1} > a_n$

$\therefore$ By induction,

$a_{n+1} > a_n \ \forall n$

$\Rightarrow \{a_n\}$ is monotonically $\uparrow$ing $\longrightarrow$ (1)

Now, $a_1 = \sqrt{7} < 7$.

Suppose $a_n < 7$

$\Rightarrow 7 + a_n < 7 + 7$

$\Rightarrow \sqrt{7+a_n} < \sqrt{14} < \sqrt{49}$

$\Rightarrow a_{n+1} < 7$

$\therefore$ By induction $a_n < 7 \ \forall n$

$\Rightarrow \{a_n\}$ is bounded above $\longrightarrow$ ②

From ① & ② $\{a_n\}$ is convergent.

let $\lim_{n\to\infty} a_n = l$

Now $a_{n+1} = \sqrt{7+a_n}$

$\Rightarrow a_{n+1}^2 = 7 + a_n$

$\Rightarrow \lim_{n\to\infty} (a_{n+1})^2 = \lim_{n\to\infty} (7+a_n)$

$\Rightarrow l^2 = 7 + l$

$\Rightarrow l^2 - l - 7 = 0$

$\Rightarrow l = \dfrac{1 \pm \sqrt{29}}{2}$

But $\dfrac{1-\sqrt{29}}{2} < 0$ where $a_n > 0 \ \forall n$

$\therefore l \neq \dfrac{1-\sqrt{29}}{2}$

Hence $\{a_n\}$ converges to $\dfrac{1+\sqrt{29}}{2}$ which is the positive root of the eqn

$x^2 - x - 7 = 0$

2. S.T the $\{a_n\}$ where $a_1 = 1$ and $a_n = \sqrt{2+a_{n-1}}$ $\forall n \geq 2$ is convergent & converges to 2.

3. S.T the $\{a_n\}$ defined by $a_1 = \sqrt{2}$ and ~~$a_{n+1} = \sqrt{2+a_n}$~~ $a_{n+1} = \sqrt{2a_n}$ converges to 2

Discuss the convergence of the foll. seqn whose n^{th} terms are given.

1. $\dfrac{\log(n+1)-\log n}{\sin 1/n}$

Solⁿ: $a_n = \dfrac{\log(n+1)-\log n}{\sin 1/n}$

$$= \frac{\log\left(\frac{n+1}{n}\right)}{\sin 1/n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\log\left(\frac{n+1}{n}\right)}{\frac{\sin 1/n}{1/n} \times \frac{1}{n}}$$

$$= \lim_{n\to\infty} \left\{ \frac{n \cdot \log\left(\frac{n+1}{n}\right)}{1 \cdot \cancel{\frac{\sin 1/n}{1/n}}} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{\log\left(\frac{n+1}{n}\right)^n}{1 \cdot \cancel{\frac{\sin 1/n}{1/n}}} \right\} = \log e$$

$\lim_{n\to\infty} a_n = 1 \quad \therefore$ convergent

2. $a_n = \left(1+\frac{1}{n}\right)^{\frac{n^2}{n+1}}$

$$= \left[\left(1+\frac{1}{n}\right)^n\right]^{n/n+1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{1+1/n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{\frac{1}{1+1/n}}$$

$= e \quad \therefore \{a_n\}$ is convergent.

3. $n[\log(n+1)-\log n]$ (Ans : 1)

4. $\left(\frac{n+1}{n-1}\right)^n$ (Ans : e^2)

5. $\left(\frac{n+1}{n}\right)^n (n+1)$ (Ans : ∞)

6. $\left(\frac{n+1}{n}\right)^{\frac{3n^2}{n+1}}$ (Ans : e^3)

1. Find the limit of the seqn
$0.4, 0.44, 0.444, \ldots$

Soln: let $a_1 = 0.4 = \frac{4}{10}$, $a_2 = 0.44 = \frac{44}{100}$,

$$a_3 = 0.444 = \frac{444}{1000}, \ldots$$

$$\therefore \{a_n\} = \left\{\frac{4}{10}, \frac{44}{100}, \frac{444}{1000}, \ldots\right\}$$

$$= 4\left\{\frac{1}{10}, \frac{11}{100}, \frac{111}{1000}, \ldots\right\}$$

$$= \frac{4}{9}\left\{\frac{9}{10}, \frac{99}{100}, \frac{999}{1000}, \ldots\right\}$$

$$= \frac{4}{9}\left\{\frac{10-1}{10}, \frac{100-1}{100}, \frac{1000-1}{1000}, \ldots\right\}$$

$$= \frac{4}{9}\left\{1-\frac{1}{10}, 1-\frac{1}{10^2}, 1-\frac{1}{10^3}, \ldots\right\}$$

$$\{a_n\} = \frac{4}{9}\left\{1-\frac{1}{10^n}\right\}$$

$$a_n = \frac{4}{9}\left\{1-\frac{1}{10^n}\right\}$$

$$\lim_{n\to\infty} a_n = \frac{4}{9}\lim_{n\to\infty}\left\{1-\frac{1}{10^n}\right\} = \frac{4}{9}$$

2. Find the limit of the seqn

(i) $0.5, 0.55, 0.555, \ldots$ (Ans: $\frac{5}{9}$)

(ii) $0.7, 0.77, 0.777, \ldots$ (Ans: $\frac{7}{9}$)

3. S.T the seq a_n defined by $a_1 = \sqrt{8}$, $a_{n+1} = \sqrt{8 + a_n}$ converges to the positive root of the equation $x^2 - x - 8 = 0$.

Series of Real numbers:

Defn:

If $\{a_n\}$ is a sequence of real numbers, then the sum of the terms of the seqn. which are infinite in number i.e., $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called an Infinite series. It is denoted by $\sum_{n=1}^{\infty} a_n$ (or) $\sum a_n$ and $a_1, a_2, a_3, \ldots, a_n \ldots$ are called the first, second, third, ... n^{th} Term of the series respl. If $\forall\, a_n > 0$ $\forall n$ then the series of positive terms.

Partial Sums:

If $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, $\ldots$,

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ are called partial Sums.

The seqn S_n is called the "Seqn of partial Sums". $\therefore$ To every infinite series $\sum a_n$, there corresponds a $\{S_n\}$ of its partial sums.

Defn:

a) Let $\sum a_n$ be a series and $\{S_n\}$ be the sequence of partial sums, Then the series $\sum a_n$ is convergent if the $\{S_n\}$ of its partial sums converges.
Thus $\sum a_n$ is convergent if $\lim_{n\to\infty} S_n = l =$ finite and unique.

b) The series $\sum a_n$ is divergent if the seqn $\{S_n\}$ is divergent.
Thus $\sum a_n$ is divergent if $\lim_{n\to\infty} S_n = +\infty$ (or) $-\infty$.

c) (i) The series $\sum a_n$ oscillates finitely if the seqn $\{S_n\}$ oscillates finitely.
Thus $\sum a_n$ oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges. if $\lim_{n\to\infty} S_n$ is finite but not unique.

(ii) The series $\sum a_n$ oscillates infinitely if the seqn $\{S_n\}$ of its partial sums oscillates infinitely. Thus $\sum a_n$ oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges. if $\lim_{n\to\infty} S_n$ oscillates between $+\infty$ and $-\infty$.

Examples:

1. Discuss the convergence of the series.

(i) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} + \dots$ to ∞

Soln: let $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Now, $a_1 = \frac{1}{1} - \frac{1}{2}$, $a_2 = \frac{1}{2} - \frac{1}{3}$, $\dots$, $a_n = \frac{1}{n} - \frac{1}{n+1}$

$\therefore S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$S_n = 1 - \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

$\therefore \{S_n\}$ converges to 1

$\Rightarrow \sum a_n$ converges to 1

(ii) $1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots$ to $+\infty$

sol^n: consider,

$S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum n^2$

$S_n = \frac{n(n+1)(2n+1)}{6}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6}\right) = +\infty$

$\therefore \{S_n\}$ diverges to $+\infty$

$\Rightarrow \sum a_n$ is divergent.

(iii) $\sum (-1)^n = -1 + 1 - 1 + 1 - \cdots$ Oscillates finitely

(iv) $\sum (-1)^n \cdot n = -1 + 2 - 3 + 4 - \cdots$ oscillates infinitely.

Important points:

a) A series of +ve Terms either converges or diverges to ∞

b) If a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (But not converge)

c) $\lim_{n\to\infty} a_n \neq 0$ $\Rightarrow \sum a_n$ is not convergent.

Ex: $\sum a_n = \sum \frac{1}{\sqrt{n}}$

Now $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

where $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$

$\therefore S_n > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$

i.e $S_n > \sqrt{n}$ $\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$ diverges.

Tests for convergence of Series:
There are 5 Types of Tests for convergence of the given series:

1. Comparison Tests.
2. P-Series/Harmonic series.
3. D'Alembert's Ratio Test.
4. Raabe's Test.
5. Cauchy's root Test.

1. Comparison Tests: There are 5 forms of:

Form: 1 – let $\sum a_n$ and $\sum b_n$ be two series of positive terms such that
(i) $\sum b_n$ is convergent and
(ii) $a_n \le k b_n \; \forall n$ except perhaps for the finite no of terms in the beginning where $k > 0$. Then $\sum a_n$ is also convergent.

Form: 2 – let $\sum a_n$ & $\sum b_n$ be 2 series of +ve terms. ∋
(i) $\sum b_n$ is divergent and.
(ii) $a_n \ge k b_n \; \forall n$ except perhaps for the finite no of terms in the beginning where $k > 0$. Then $\sum a_n$ is also divergent.

Form : 3 (limit form) – let $\sum a_n$ & $\sum b_n$ be 2 series of +ve terms and $\lim_{n\to\infty} \frac{a_n}{b_n}$ be a finite non-zero quantity. Then $\sum a_n$ & $\sum b_n$ both converge or diverge together.

Form : 4 – let $\sum a_n$ and $\sum b_n$ be 2 series of +ve terms. $\ni$

(i) $\sum b_n$ is convergent and

(ii) $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all values of n except perhaps for finite nos of terms in the beginning. Then $\sum a_n$ is also convergent.

Form : 5 – let $\sum a_n$ and $\sum b_n$ be 2 series of +ve terms $\ni$

(i) $\sum b_n$ is divergent and

(ii) $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$ $\forall n$ except perhaps for finite nos of terms in the beginning. Then $\sum a_n$ is also divergent.

2. P – Series (or) Harmonic series:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ is called the p-series.

Theorem: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

(i) convergent if $p > 1$

(ii) divergent if $p \leq 1$

Examples:

1. Test the convergence of the series

$$\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \cdots \infty$$

a) Sol^n: Consider the n^{th} term of the given series

$$a_n = \frac{n}{(2n-1)(2n)}$$

first factors $1, 3, 5 \ldots$
$\therefore AP = a + (n-1)d = 1 + (n-1)2$
$= 2n-1$

Second factor $2, 4, 6$
$\therefore AP = 2 + (n-1)2 = 2n$

$$\therefore \sum a_n = \sum \frac{n}{(2n-1)(2n)}$$

Now form the series,

$$\sum b_n = \sum \frac{n}{n^2} = \sum \frac{1}{n}$$

consider,

$$\frac{a_n}{b_n} = \frac{n}{(2n-1)(2n)} \times \frac{n}{1} = \frac{n^2}{n^2\left(2-\frac{1}{n}\right)(2)}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2\left(2-\frac{1}{n}\right)} = \frac{1}{4} \neq 0$$

Hence by comparison Test, both converges or diverges.

Now, $\sum b_n = \sum \frac{1}{n}$ is divergent using ~~pseries~~ P-series ($\because p=1$)

$\therefore \sum a_n$ is also divergent.

2 Test the convergence of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$$

Sol^n: The n^r of each term is $1, 3, 5 \ldots$

$\therefore AP = 1 + (n-1)2 = (2n-1)$

The D^r of first factor is $1, 2, 3 \ldots$

$AP = 1 + (n-1)1 = n$

II factor is $2, 3, 4 \ldots$

$AP = (2 + (n-1)1) = n+1$

III factor is $3, 4, 5 \ldots$

$AP = (3 + (n-1)1) = n+2$

Now, n^{th} term of series,

$$a_n = \frac{2n-1}{n(n+1)(n+2)}$$

$\therefore \sum a_n = \sum \frac{2n-1}{n(n+1)(n+2)}$

form the series b_n,

$\sum b_n = \sum \frac{n}{n^3} = \sum \frac{1}{n^2}$

Now,

$$\frac{a_n}{b_n} = \frac{2n-1}{n(n+1)(n+2)} \times \frac{n^2}{1} = \frac{n^3(2-1/n)}{n^3(1+1/n)(1+2/n)}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \neq 0$$

By using p-series, we have

$\sum b_n = \frac{1}{n^2}$ where $p > 1$

$\therefore \sum b_n$ is convergent.

$\Rightarrow \sum a_n$ is also convergent.

3. Test the convergence of the series

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots + \sqrt{\frac{n}{2n+2}} + \cdots$$

Solⁿ: $a_n = \sqrt{\frac{n}{2n+2}}$ $\therefore \sum a_n = \sum \sqrt{\frac{n}{2n+2}}$

Here in this problem, the degrees of n^r & d^r are same

$\therefore$ we find $\lim_{n\to\infty} a_n$

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n}{2n+2}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\sqrt{2+\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{2}} \neq 0$$

Since $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ is not convergent

$\therefore \sum a_n$ is divergent.

4. Discuss the convergence of the series

a) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$

Solⁿ: $a_n = \frac{n+1}{n^p} = \frac{n(1+\frac{1}{n})}{n^p} = \frac{(1+\frac{1}{n})}{n^{p-1}}$

Taking the form for b_n,

$b_n = \frac{1}{n^{p-1}}$

Now,

b $\frac{a_n}{b_n} = \left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$

c $\therefore \sum a_n$ and $\sum b_n$ behave alike

Now, $\sum b_n = \sum \frac{1}{n^{p-1}}$ converges if $p-1>1$ i.e $p>2$

diverges if $p-1 \leq 1$ i.e $p \leq 2$

$\Rightarrow \sum a_n$ converges if $p>2$ and

diverges if $p \leq 2$

5. Test the convergence of the series.

$\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots$

Solⁿ: The given series is $\sum_{2}^{\infty} \frac{1}{\log n}$

Since $\log n < n \quad \forall n > 1$

$\Rightarrow \frac{1}{\log n} > \frac{1}{n} \quad \forall n \geq 2$

Since each term in a series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is greater than the corresponding term of the series $\sum_{n=2}^{\infty} \frac{1}{n}$ which is divergent

$\therefore$ The given series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is divergent.

6. Test the convergence of the series:

(i) $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \cdots$ (Ans: divergent)

(ii) $\sum \frac{1}{n^p (n+1)^p}$ (Ans: convergent $p > \frac{1}{2}$, divergent $p \le \frac{1}{2}$)

(iii) $\sum \sqrt{n^4+1} - n^2$ (Ans: convergent)

(iv) $\sum \frac{1}{n^{1+1/n}}$ (Ans: divergent)

(v) $\frac{1 \cdot 2}{3 \cdot 4 \cdot 5} + \frac{2 \cdot 3}{4 \cdot 5 \cdot 6} + \frac{3 \cdot 4}{5 \cdot 6 \cdot 7} + \cdots$ (Ans: divergent)

(vi) $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$ (Ans: convergent)

(vii) $\frac{1}{1^p} + \frac{1}{3^p} + \frac{1}{5^p} + \frac{1}{7^p} + \cdots$ (Ans: convergent $p > 1$, divergent $p \le 1$)

3. D'Alembert's Ratio Test (or) Ratio test:

Let $\sum a_n$ be a series of positive terms $\ni \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l$. Then the series

(i) converges if $l < 1$
(ii) diverges if $l > 1$
(iii) the test fails if $l = 1$

Examples:

1. Discuss the convergence of the series
$$\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots + \frac{(n+1)!}{3^n} + \cdots$$

Sol^n: Let $a_n = \frac{(n+1)!}{3^n}$, $a_{n+1} = \frac{(n+2)!}{3^{n+1}}$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+2)!}{3^{n+1}} \cdot \frac{3^n}{(n+1)!} = \frac{(n+2)\cancel{(n+1)!}}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{3^n}}{\cancel{(n+1)!}}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{3}$$

$$\therefore \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+2}{3} = \infty > 1$$

$\therefore$ The given series is divergent.

2. Examine the convergence of the series

a) $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\cdots$

Solⁿ: let $a_n=\frac{1.2.3.4\cdots n}{3.5.7.9\cdots(2n+1)}$,

$$a_{n+1}=\frac{1.2.3\cdots n(n+1)}{3.5.7.9\cdots(2n+1)(2n+3)}$$

b) $\therefore \frac{a_{n+1}}{a_n}=\frac{n+1}{2n+3}=\frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(2+\frac{3}{n}\right)}$

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(2+\frac{3}{n}\right)}=\frac{1}{2}<1$

c) $\therefore$ Series is convergent.

3. Discuss the convergence of the series: $\sum\sqrt{\frac{n+1}{n^3+1}}\cdot x^n$

Solⁿ: let $a_n=\sqrt{\frac{n+1}{n^3+1}}\cdot x^n$, $a_{n+1}=\sqrt{\frac{n+2}{(n+1)^3+1}}\cdot x^{n+1}$

$\therefore \frac{\cancel{a_{n+1}}}{\cancel{a_n}}=\cancel{\sqrt{\frac{n+1}{n^3+1}}}\,\cancel{x^n}\cdot\cancel{\sqrt{\frac{(n+1)^3+1}{n}}}$

$\therefore \frac{a_{n+1}}{a_n}=\sqrt{\frac{n+2}{(n+1)^3+1}}\cdot\cancel{x^n}\cdot x\cdot\sqrt{\frac{n^3+1}{n+1}}\cdot\frac{1}{\cancel{x^n}}$

$$=\sqrt{\frac{\cancel{n^4}\left(1+2/n\right)\left(1+1/n^3\right)}{\cancel{n^4}\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{(n+1)^3}\right]\left(1+\frac{1}{n}\right)}}\cdot x$$

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1\cdot x$

$\therefore$ By ratio test,

if $x<1$ the given series is convergent

if $x>1$ the given series is divergent

if $x=1$ the test fails.

Now, when $x=1$,

$$a_n=\sqrt{\frac{n+1}{n^3+1}}$$

$$a_n = \sqrt{\frac{n(1+1/n)}{n^{3}(1+1/n^3)}} = \frac{1}{n}\sqrt{\frac{(1+1/n)}{(1+1/n^3)}}$$

Taking $b_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{\frac{1+1/n}{(1+1/n^3)}} = 1 \neq 0$$

$\therefore \sum b_n = \sum \frac{1}{n}$ is divergent using p-series $\because p = 1$

$\therefore$ The given series is also divergent.

Thus The series is convergent if $x < 1$,
divergent if $x \geq 1$

4. Examine the convergence of the series

$$\frac{1}{1.2.3} + \frac{x}{4.5.6} + \frac{x^2}{7.8.9} + \cdots$$

solⁿ: let $a_n = \dfrac{x^{n-1}}{(3n-2)(3n-1)3n}$,

I factors: $1, 4, 7, \ldots$
A.P $= 1 + (n-1)3$
$= 3n-2$

$2, 5, 6 \ldots$
AP $= 2 + (n-1)3$
$= 3n-1$

$3, 6, 9, \ldots$
AP $= 3 + (n-1)3$
$= 3n$

$$a_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{(3n-2)(3n-1)\,3n \cdot x}{(3n+1)(3n+2)(3n+3)}$$

$$= \frac{\cancel{n^3}(3-\frac{2}{n})(3-\frac{1}{n}) \cdot 3 \cdot x}{\cancel{n^3}(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{3.\cancel{3.3}}{3.\cancel{3.3}} \cdot x = x$$

By ratio Test,

The given series is convergent if $x < 1$,
divergent if $x > 1$ and
test fails if $x = 1$

Now, when $x = 1$,

$$a_n = \frac{1}{(3n-2)(3n-1)3n}$$ and Take $b_n = \frac{1}{n^3}$

a) $\therefore \frac{a_n}{b_n} = \frac{1}{(3n-2)(3n-1)3n} \times \frac{n^3}{1}$

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3.3.3} = \frac{1}{27} \neq 0$

By $\sum b_n = \sum \frac{1}{n^2}$ is convergent by p-series ($\because p>1$)

$\Rightarrow \sum a_n$ is also convergent.

$\therefore$ The given series is convergent if $x \le 1$, and divergent if $x > 1$.

5. Discuss the convergence of the foll. series,

(i) $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots$ (Ans: convergent)

(ii) $\frac{1}{2} + \frac{2!}{2^3} + \frac{3!}{2^5} + \frac{4!}{2^7} + \cdots$ (Ans: divergent)

(iii) $\sum \frac{n!}{n^n}$ (Ans: convergent)

(iv) $\sum \frac{5^n}{n^2+7}$ (Ans: Divergent)

(v) $\sum \frac{x^n}{n}$ $(x>0)$ (Ans: convergent if $x<1$, divergent if $x \ge 1$)

(vi) $1 + 2x + 3x^2 + 4x^3 + \cdots$ (Ans: ")

(vii) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots$ (Ans: convergent if $x \le 1$, divergent if $x > 1$

(viii) $\frac{x}{1.3} + \frac{x^2}{3.5} + \frac{x^3}{5.7} + \cdots$ (Ans: "

4. Raabe's Test

Let $\sum a_n$ be a series of +ve terms $\ni$ $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$, then

(i) $\sum a_n$ is convergent if $l > 1$

(ii) $\sum a_n$ is divergent if $l < 1$

Note:
(i) Raabe's Test is inconclusive when $l = 1$
(ii) Raabe's Test is stronger than ratio test & may succeed where ratio-test fails.

Examples:

1. Discuss the convergence of the series

(i) $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \cdots$ (Ans: $2 > 1$ $\therefore$ converges)

Sol^n: let $a_n = \frac{2n+3}{(2n-1)(2n)(2n+1)}$, $a_{n+1} = \frac{2n+5}{(2n+1)(2n+2)(2n+3)}$

Now,

$$\frac{a_{n+1}}{a_n} = \frac{(2n+5)}{(\cancel{2n+1})(2n+2)(2n+3)} \times \frac{(2n-1)(2n)(\cancel{2n+1})}{(2n+3)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left\{ \frac{n^3\left(2+\frac{5}{n}\right)\left(2-\frac{1}{n}\right)2}{n^3\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)\left(2+\frac{3}{n}\right)} \right\}$$

$$= \frac{2.\cancel{2}.\cancel{2}}{\cancel{2}.\cancel{2}.2} = 1$$

$\therefore$ Ratio Test fails

consider,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{(2n+2)(2n+3)^2}{(2n+5)(2n-1)2n} - 1\right)$$

$$= n\left[\frac{(2n+2)(2n+3)^2 - (2n+5)(2n-1)2n}{(2n+5)(2n-1)2n}\right]$$

$$= \frac{n\left[n^3\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right) - n^3.2\left(2+\frac{5}{n}\right)\left(2-\frac{1}{n}\right)\right]}{n^3\left(2+\frac{5}{n}\right)\left(2-\frac{1}{n}\right).2}$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \infty > 1$$

$\therefore$ By Raabe's Test the given series is convergent.

(ii) $\sum \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n} \cdot x^n$

Sol^n: let $a_n = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n} \cdot x^n$,

$$a_{n+1} = \frac{1.3.5 \cdots (2n-1)(2n+1)}{2.4.6 \cdots 2n.(2n+2)} x^{n+1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)}{(2n+2)} \cdot x = \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)} \cdot x = x$$

By ratio Test, the series is convergent if $x<1$ and divergent if $x>1$. The test fails when $x=1$.

when $x=1$, $a_n = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n}$

Applying Raabe's Test,

Now, $n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{2n+2}{2n+1} - 1\right)$

$$= n\left(\frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}\right)$$

$$= n\left(\frac{1}{2n+1}\right) = \frac{1}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1$$

$\therefore$ By Raabe's Test the given series is divergent.

Hence the series is convergent if $x<1$, and divergent if $x \geq 1$.

(iii) $\sum \frac{1.5.9 \cdots (4n-3)}{3.7.11 \cdots (4n-1)}$ (Ans: Divergent)

(iv) $\sum \frac{(n!)^2}{(2n)!} x^n \quad (x>0)$ (Ans: converges if $x<4$ diverges if $x \geq 4$)

(v) $\frac{3}{7} + \frac{3.6}{7.10} + \frac{3.6.9}{7.10.13} + \cdots$ (Ans: convergent).

5. Cauchy's Root Test:

If $\sum a_n$ is a +ve Term series $\ni \lim_{n\to\infty} a_n^{1/n} = l$,
Then the series
(i) converges if $l < 1$
(ii) diverges if $l > 1$
(iii) Test fails if $l = 1$

Examples:

1. Test the convergence of the series $\sum\left(\frac{n}{n+1}\right)^{n^2}$

Sol^n: let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

Now,
$$a_n^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^n$$
$$= \left(\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root Test, The given series is convergent.

2. Test the convergence of the series
$$\sum\left(\frac{nx}{n+1}\right)^n$$

Sol^n: let $a_n = \left(\frac{nx}{n+1}\right)^n$

$$\therefore a_n^{1/n} = \frac{nx}{n+1}$$
$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)} \cdot x$$
$$= x$$

By Cauchy's root test, series converges if $x<1$ diverges if $x>1$ and fails if $x=1$