I Semester

unit - I -

Calculus - I:

→ n^{th} derivative of std functions

→ Leibnitz's theorem and problems

→ Polar curves → angle b/w radius vector and Tangent

→ angle b/w two curves

→ Pedal equation

→ Curvature and radius of curvature

- → Cartesian
- → polar forms

→ Center & circle of curvature

- → evolutes
- → Singular points
- → asymptotes
- → involutes.

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unit - I } calculus
unit - II

unit - III — Sequence and Series

unit - IV — Matrices

unit - V — Multivariable calculus (Differentiation)

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unit - I — Successive differentiation

Successive differentiation:

It is a process of differentiating a given function successively n times and the results of such differentiation are called Successive derivatives.

If $y = f(x)$ (differentiable function)
we know that

$\frac{dy}{dx} = f'(x)$ is called as derivative of y wrt x. Symbolically, we write

$$y' = \frac{dy}{dx} = Dy = f'(x) \quad \text{where } D = \frac{d}{dx}$$
$$= y_1$$

The derivative of first derivative is called as the second derivative of y wrt x

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y = f''(x)$$
$$= y_2$$

where $D^2 = \frac{d^2}{dx^2}$

Thus in general the derivative of $(n-1)^{th}$ derivative of y wrt x is called as the n^{th} derivative of y wrt x
i.e

$$y^{(n)} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right) = D\left(D^{n-1}y\right) = \left(f^{(n-1)}(x)\right)'$$
$$= y_n$$

(Or)

$$y_n = \frac{d^ny}{dx^n} = D^ny = f^{(n)}(x)$$

where $D^n = \frac{d^n}{dx^n}$ represents the n^{th} derivative of y wrt x. and $y_1, y_2, \ldots$ are respectively called as the derivatives of first order, second order ...

Now $y = f(x)$ is equivalent to $y_0 = f^{(0)}(x)$ and known as derivative of order zero

If an expression represents the n^{th} derivative of a funⁿ obviously it must give 1^{st}, 2^{nd}, ... n^{th} derivatives. Question is, can we find n^{th} derivative of any funⁿ? ans is no. The ans actually is yes if ~~only~~ we are able to ~~notice~~ notice a sequential change from 1 derivative to other.

Ex. ① we have 1, 4, 9, 16, ... we try to figure the n^{th} term like suppose if we can rewrite it as

$$1^2, 2^2, 3^2, 4^2, \ldots = n^2$$

∴ The n^{th} term of seqⁿ is n^2

② The number 1, 2, 6, 24, 120, ...

$$\Rightarrow 1!, 2!, 3!, 4!, 5!, \ldots = n!$$

Therefore, we should have a seqⁿ or create a seqⁿ from 1 derivative to the succeeding derivative for deriving the n^{th} derivative of a given funⁿ which definite provide 1^{st}, 2^{nd}, 3^{rd}, ... derivative of the given funⁿ when $n = 1, 2, 3, \ldots$

n^{th} derivatives of some standard functions:

1. $y = x^n$

Solⁿ
$$y = x^n$$
$$y_1 = n x^{n-1}$$
$$y_2 = n(n-1) x^{n-2}$$
$$y_3 = n(n-1)(n-2) x^{n-3}$$
$$y_n = n(n-1)(n-2) \ldots x^{n-n}$$

$$y_n = n(n-1)(n-2) \ldots 2 \cdot 1$$
$$y_n = n!$$

2. $y = e^{ax}$

Solⁿ: $y = e^{ax}$

$$y_1 = a e^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

$$\vdots$$

$$y_n = a^n e^{ax}$$

3. $y = \frac{1}{ax+b}$

Solⁿ: $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$$y_1 = \frac{-1}{(ax+b)^2} \cdot a$$

$$y_2 = \frac{-1(-2)}{(ax+b)^3} \cdot a^2$$

$$y_3 = \frac{(-1)(-2)(-3)}{(ax+b)^4} a^3$$

$$\vdots$$

$$y_n = \frac{(-1)(-2)(-3)\cdots(-n)\ a^n}{(ax+b)^{n+1}}$$

$$y_n = \frac{(-1)^n\ 1.2.3\cdots n\ a^n}{(ax+b)^{n+1}}$$

$$y_n = \frac{(-1)^n\ n!\ a^n}{(ax+b)^{n+1}}$$

4. $y = \log(ax+b)$ (for students)

Solⁿ: $y_n = \frac{a^n (-1)^{n-1} (n-1)!}{(ax+b)^n}$

5. $y = \sin(ax+b)$

Sol^n $y = \sin(ax+b)$ $\left(\because \cos\theta = \sin\left(\frac{\pi}{2}+\theta\right)\right)$

$y_1 = \cos(ax+b) \cdot a$

$= a \cdot \sin\left(ax+b+\frac{\pi}{2}\right)$

$y_2 = a \cdot \cos\left(ax+b+\frac{\pi}{2}\right) \cdot a$

$= a^2 \sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$

$= a^2 \sin\left(ax+b+2\frac{\pi}{2}\right)$

$y_3 = a^2 \cdot \cos\left(ax+b+2\frac{\pi}{2}\right) \cdot a$

$= a^3 \sin\left(ax+b+3\frac{\pi}{2}\right)$

$\vdots$

$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

6. $y = \cos(ax+b)$

Sol^n: $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

7. $y = e^{ax}\sin(bx+c)$

Sol^n: $y = e^{ax}\sin(bx+c)$

$y_1 = e^{ax}\cos(bx+c) \cdot b + \sin(bx+c)\, e^{ax} \cdot a$

$y_1 = e^{ax}\left[b\cos(bx+c) + a\sin(bx+c)\right]$

Put $a = r\cos\theta$ and $b = r\sin\theta$

Then $r^2 = a^2+b^2$ and $\tan\theta = b/a$

$\Rightarrow r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

$\therefore \; y_1 = e^{ax}[r\cos\theta \sin(bx+c) + r\sin\theta\cos(bx+c)]$

$y_1 = e^{ax} \cdot r \sin(bx+c+\theta)$

$(\because \sin A \cos B + \cos A \sin B = \sin(A+B))$

$y_2 = r[e^{ax}\cos(bx+c+\theta).b + \sin(bx+c+\theta)ae^{ax}]$

$= re^{ax}[r\sin\theta\cos(bx+c+\theta) + r\cos\theta\sin(bx+c+\theta)]$

$y_2 = r^2 e^{ax} \sin(bx+c+2\theta)$

$y_3 = r^3 e^{ax} \sin(bx+c+3\theta)$

$\vdots$

$y_n = r^n e^{ax} \sin(bx+c+n\theta)$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

$y_n = (a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n\tan^{-1}(b/a))$

8. $y = e^{ax}\cos(bx+c)$

Solⁿ: $y_n = (a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n\tan^{-1}(b/a))$

(Hint: $\cos A\cos B - \sin A \sin B = \cos(A+B)$)

9. $y = a^{mx} \Rightarrow y_n = (m\log a)^n a^{mx}$

n^{th} derivatives of the functions:

1. $D^n(e^{ax}) = a^n e^{ax}$
2. $D^n(a^{mx}) = (m\log a)^n a^{mx}$
3. $D^n(ax+b)^m = n!\,a^n$ if $m=n$

4. $D^n(ax+b)^m = m(m-1)(m-2)\cdots(m-(n-1))\,a^n \cdot \cancel{a^{m-n}}$
$(ax+b)^{m-n}$ if $m>n$

5. $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

6. $D^n(\log(ax+b)) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

7. $D^n(\sin(ax+b)) = a^n \sin\left(\frac{n\pi}{2}+ax+b\right)$

8. $D^n(\cos(ax+b)) = a^n \cos\left(\frac{n\pi}{2}+ax+b\right)$

9. $D^n(e^{ax}\sin(bx+c)) = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\sin\left(n\tan^{-1}(b/a)+bx+c\right)$

10. $D^n(e^{ax}\cos(bx+c)) = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\cos\left(n\tan^{-1}\left(\frac{b}{a}\right)+bx+c\right)$

Examples:

1. $y = e^{3x} \Rightarrow y_n = 3^n e^{3x}$
2. $y = a^{3x} \Rightarrow y_n = (3\log a)^n a^{3x}$
3. $y = 3^{5x} \Rightarrow y_n = (5\log 3)^n 3^{5x}$
4. $y = \frac{1}{3x+2} \Rightarrow y_n = \frac{(-1)^n n!\cdot 3^n}{(3x+2)^{n+1}}$

5. $y = \log(2x+5)$

$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!\,2^n}{(2x+5)^n}$

6. $y = \frac{1}{(x+1)^2} \Rightarrow y = (x+1)^{-2}$

we have

$$D^n (ax+b)^m = m(m-1)(m-2)\cdots(m-(n-1))\,a^n (ax+b)^{m-n}$$

$\therefore m = -2,\ a = 1,\ b = 1$

$$y_n = -2(-3)(-4)\cdots(-2-(n-1))\,(x+1)^{-2-n} \cdot 1^n$$

$$= (-1)^n\, 2\cdot 3\cdot 4 \cdots (n+1)\,(x+1)^{-(n+2)}$$

$$y_n = \frac{(-1)^n (n+1)!}{(x+1)^{n+2}}$$

Assignment

7. $y = \cos(4x+3)$

$$y_n = 4^n \cos\left(\frac{n\pi}{2} + 4x + 3\right)$$

Assign

8. $y = \sin 6x$

$$y_n = 6^n \sin\left(\frac{n\pi}{2} + 6x\right)$$

Assign

9. $y = e^{2x} \cos 3x$

$$y_n = (\sqrt{13})^n e^{2x} \cos\left(n \tan^{-1}\left(\frac{3}{2}\right) + 3x\right)$$

Assign

10. $y = e^x \sin x$

$$y_n = (\sqrt{2})^n e^x \sin\left(\frac{n\pi}{4} + x\right)$$

11. $y = \sin^3 x$ $\quad \sin 3x = 3\sin x - 4\sin^3 x$

Solⁿ: $y = \frac{1}{4}(3\sin x - \sin 3x)$

$$y_n = \frac{1}{4}\left(3D^n(\sin x) - D^n(\sin 3x)\right)$$

$$= \frac{1}{4}\left[3\sin\left(x + \frac{n\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right]$$

12. $y = \cos^3 x$ $\quad \cos 3x = 4\cos^3 x - 3\cos x$

Solⁿ: $y_n = \frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right]$

13. $y = \cos^4 x$

Solⁿ: $y = (\cos^2 x)^2$

$y = \left(\frac{1+\cos 2x}{2}\right)^2$

$y = \frac{1}{4}(1+\cos 2x)^2$

$y = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$y = \frac{1}{4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)$

$y = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cdot\cos 4x$

$y_n = D^n\left[\frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{4} + \frac{1}{8}\right]$

$y_n = \frac{1}{2}2^n\cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n\cos\left(4x + \frac{n\pi}{2}\right) + 0$

$y_n = \frac{1}{2}2^n\cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n\cos\left(4x + \frac{n\pi}{2}\right)$

14. $y = \sin^4 x$

Solⁿ: $y_n = -\frac{1}{2}2^n\cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}4^n\cos\left(4x + \frac{n\pi}{2}\right)$

Assignment:

1. $y = \cos^2 x \sin^3 x$
2. $y = \sin^2 x \cos^3 x$
3. $y = \cos x \cos 2x \cos 3x$
4. $y = \sin x \sin 2x \sin 3x$
5. $y = \sinh 2x \cdot \cos 3x$
6. $y = \sinh 2x \cdot \sin 4x$
7. $y = e^{3x}\sin^2 x$

1. $y = \frac{x+3}{(x-1)(x+2)}$

Solⁿ: $\frac{x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$x+3 = A(x+2) + B(x-1)$

Put $x=1$ | Put $x=-2$
$4 = A(3)$ | $\cancel{4}\,1 = B(-3)$
$A = \frac{4}{3}$ | $B = -\frac{1}{3}$

$\therefore\ y = \frac{x+3}{(x-1)(x+2)} = \frac{4/3}{x-1} + \frac{\frac{-1}{3}}{x+2}$

$y_n = \frac{4}{3}\frac{(-1)^n n!\, 1^n}{(x-1)^{n+1}} - \frac{1}{3}\frac{(-1)^n n!\, 1^n}{(x+2)^{n+1}}$

$\left(\because\ y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}\right)$

2. $y = \frac{x^2}{(x-1)^2(x-2)}$

Solⁿ: $\frac{x^2}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$A = -3,\quad B = -1,\quad C = 4$

$y = -\frac{3}{x-1} + \frac{(-1)}{(x-1)^2} + \frac{4}{x-2}$

$y_n = -3\frac{(-1)^n n!\, 1^n}{(x-1)^{n+1}} - \frac{(-1)^n (n+1)!\, 1^n}{(x-1)^{n+2}}$
$+ 4\frac{(-1)^n n!\, 1^n}{(x-2)^{n+1}}$

Assignment:

1. $y = \frac{x+3}{(x+1)(x+2)}$

2. $y = \frac{x^2}{(x+1)(x-2)(x-3)}$

2. $y = \frac{x}{(x+4)(x-2)}$

4. $y = \frac{x+2}{(x+1)(3x+2)}$

Leibnitz Theorem

Statement:

If U and V are two functions of x and their n^{th} derivatives are exists then

$$D^n(UV) = (UV)_n = UV_n + {}^nC_1 U_1 V_{n-1} + {}^nC_2 U_2 V_{n-2} + \dots + {}^nC_{n-1} U_{n-1} V_1 + U_n V$$

where ${}^nC_n = {}^nC_0 = 1$ $\qquad \therefore {}^nC_r = \frac{n!}{(n-r)!\, r!}$

$${}^nC_1 = {}^nC_{n-1} = n$$

$${}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!}$$

$${}^nC_3 = {}^nC_{n-3} = \frac{n(n-1)(n-2)}{3!}$$

$${}^nC_4 = {}^nC_{n-4} = \frac{n(n-1)(n-2)(n-3)}{4!}$$

Find n^{th} derivative by using Leibnitz Theorem:

1. $y = x \sin 3x$

Soln: We have

$$D^n(UV) = (UV)_n = y_n$$

$$= UV_n + {}^nC_1 U_1 V_{n-1} + {}^nC_2 U_2 V_{n-2} + \dots$$

$U = x \qquad V = \sin 3x$

$U_1 = 1 \qquad V_n = 3^n \sin\left(3x + \frac{n\pi}{2}\right)$

$U_2 = 0 \qquad V_{n-1} = 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right)$

$$\therefore y_n = x \cdot 3^n \sin\left(3x + \frac{n\pi}{2}\right) + n\, 3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right)$$

Ans

2. $y = e^{3x}(2x+3)^3$

Sol^n: $y = e^{3x}(2x+3)^3$

$U = (2x+3)^3$ $\qquad$ $V = e^{3x}$

$U_1 = 3(2x+3)^2 \cdot 2 = 6(2x+3)^2$ $\qquad$ $V_n = 3^n e^{3x}$

$U_2 = 3 \cdot 2(2x+3) \cdot 2 = 24(2x+3)$ $\qquad$ $V_{n-1} = 3^{n-1} e^{3x}$

$U_3 = 3 \cdot 2 \cdot (2) \cdot 2 = 48$ $\qquad$ $V_{n-2} = 3^{n-2} e^{3x}$

$U_4 = 0$ $\qquad$ $V_{n-3} = 3^{n-3} e^{3x}$

$V_{n-4} = 3^{n-4} e^{3x}$

$$y_n = (2x+3)^3\, 3^n e^{3x} + n\, 6(2x+3)^2\, 3^{n-1} e^{3x} + \frac{n(n-1)}{2!} 24(2x+3)\, 3^{n-2} e^{3x} + \frac{n(n-1)(n-2)}{3!} 48\, 3^{n-3} e^{3x}$$

3. $x^2 e^x = y$

Sol^n: $y_n = x^2 e^x + 2n x e^x + n(n-1)e^x$.

4. $y = x \cos x$

Sol^n: $y_n = \cos\left(x + \frac{n\pi}{2}\right) x + n \cos\left(x + \frac{(n-1)\pi}{2}\right)$

5. $y = x^2 \log x$

Sol^n: $$y_n = \frac{x^2 (-1)^{n-1} (n-1)!\, 1^n}{x^n} + n\left(\frac{(-1)^{n-2}(n-2)!\, 1^{n-1}}{x^{n-1}}\right) 2x + \frac{n(n-1)}{2!}\left(\frac{(-1)^{n-3}(n-3)!\, 1^{n-2}}{x^{n-2}}\right) 2$$

1. If $y=\tan^{-1}x$, P.T

$(1+x^2)y_{n+2}+2(n+1)x\,y_{n+1}+n(n+1)y_n=0$

Sol^n: $y=\tan^{-1}x$

$y_1=\dfrac{1}{1+x^2}$

$(1+x^2)y_1=1$

Differentiating again w.r.t x we have

$(1+x^2)y_2+2x\,y_1=0$

Differentiating this n times we have

$D^n[(1+x^2)y_2]+2D^n[x\,y_1]=0 \rightarrow (1)$

Applying Leibnitz Theorem, i.e

$UV_n+{}^nC_1U_1V_{n-1}+{}^nC_2U_2V_{n-2}+\cdots$

$U=(1+x^2)$	$V=y_2$
$U_1=2x$	$V_n=y_{n+2}$
$U_2=2$	$V_{n-1}=y_{n+1}$
$U_3=0$	$V_{n-2}=y_n$

Eqn (1) becomes

$(1+x^2)y_{n+2}+{}^nC_1\,2x\,y_{n+1}+{}^nC_2\,2\,y_n+0$

$+2[x\,y_{n+1}+{}^nC_1\cdot1\cdot y_n+0]=0$

$U=x$	$V=y_1$
$U_1=1$	$V_n=y_{n+1}$
$U_2=0$	$V_{n-1}=y_n$

$(1+x^2)y_{n+2}+n\,2x\,y_{n+1}+\dfrac{n(n-1)}{\cancel{2}}\cancel{2}\,y_n$

$+2x\,y_{n+1}+2n\,y_n=0$

$(1+x^2)y_{n+2}+2x(n+1)y_{n+1}+(n^2-n+2n)y_n=0$

$(1+x^2)y_{n+2}+2(n+1)x\,y_{n+1}+n(n+1)y_n=0$

2. If $\sin^{-1}x = y$, P.T

$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$

Sol^n: We have,

$$y = \sin^{-1}x$$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = 1$$

Differentiating wrt x we get

$$\sqrt{1-x^2}\,y_2 + y_1 \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = 0$$

$$\sqrt{1-x^2}\,y_2 - \frac{x y_1}{\sqrt{1-x^2}} = 0$$

$$(1-x^2)\,y_2 - x y_1 = 0$$

Diff n times by applying leibnitz theorem,

$$D^n[(1-x^2)y_2] - D^n[x \cdot y_1] = 0$$

$$(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n + 0$$

$$-\left[x y_{n+1} + {}^nC_1 y_n \cdot 1\right] = 0$$

$$(1-x^2)y_{n+2} - n\,2x\,y_{n+1} - \cancel{2}\frac{n(n-1)}{\cancel{2}}y_n$$

$$-(x y_{n+1} + n y_n) = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$$

$U = 1-x^2 \quad V = y_2$

$U_1 = -2x \quad V_n = y_{n+2}$

$U_2 = -2 \quad V_{n-1} = y_{n+1}$

$U_3 = 0 \quad V_{n-2} = y_n$

$U = x \quad V = y_1$

$U_1 = 1 \quad V_n = y_{n+1}$

$U_2 = 0 \quad V_{n-1} = y_n$

3. If $y = \cos^{-1} x$ then P.T

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

4. If $y = (\sin^{-1} x)^2$, P.T

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

Solⁿ: $y = (\sin^{-1} x)^2$

$$y_1 = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}}$$

$\sqrt{1-x^2}\, y_1 = 2 \sin^{-1} x$ sq on b.s

$$(1-x^2) y_1^2 = 4 (\sin^{-1} x)^2$$

$$(1-x^2) y_1^2 = 4y$$

diff w.r.t 'x',

$$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) = 4y_1 \quad \div 2y_1$$

$$(1-x^2) y_2 - x y_1 = 2$$

diff n times and apply Leibnitz's theorem

$$(1-x^2) y_{n+2} + {}^nC_1 (-2x) y_{n+1} + {}^nC_2 (-2) y_n$$

$$-(x y_{n+1} + {}^nC_1 \cdot 1 \cdot y_n) = 0$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

5. If $y = (\cos^{-1}x)^2$, Then P.T

$$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0$$

$\cancel{6. \text{ If } y = (\sin^{-1}x)^2 \text{ Then P.T.}}$

$$\cancel{(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0}$$

6. If $y = \log(x + \sqrt{1+x^2})$, Then P.T

$$(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + n^2 y_n = 0$$

sol^n: $\quad y = \log(x + \sqrt{1+x^2})$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{\cancel{2}\sqrt{1+x^2}}\,\cancel{2}x\right)$$

$$y_1 = \frac{1}{\cancel{x+\sqrt{1+x^2}}}\left(\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\right)$$

$$y_1 = \frac{1}{\sqrt{1+x^2}}$$

$$\sqrt{1+x^2}\; y_1 = 1$$

diff we get

$$\sqrt{1+x^2}\; y_2 + y_1 \frac{1}{\cancel{2}}(1+x^2)^{1/2-1}(+\cancel{2}x) = 0$$

$$\sqrt{1+x^2}\; y_2 + \frac{y_1}{\cancel{2}}\, \frac{x}{\sqrt{1+x^2}} = 0$$

further conti...

7. If $y = e^{m\cos^{-1}x}$ P.T (5)

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Soln: $y = e^{m\cos^{-1}x}$

$y_1 = e^{m\cos^{-1}x} \cdot \frac{-m}{\sqrt{1-x^2}} = \frac{-my}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = -my$

diff again we get

$\sqrt{1-x^2}\, y_2 + y_1 \frac{1}{2\sqrt{1-x^2}}(-2x) = -my_1$

$(1-x^2)y_2 - xy_1 = -my_1\sqrt{1-x^2}$

$(1-x^2)y_2 - xy_1 = -m(-my)$

$(1-x^2)y_2 - xy_1 = m^2 y$

$(1-x^2)y_2 - xy_1 - m^2 y = 0$

$D^n[(1-x^2)y_2] - D^n[xy_1] - m^2 D^n[y] = 0$

Applying L.T,

$(1-x^2)y_{n+2} + nC_1(-2x)y_{n+1} + nC_2(-2)y_n + 0$

$-(xy_{n+1} + nC_1 \cdot 1 \cdot y_n + 0) - m^2 y_n = 0$

Simplification.

Q. If $y = a\cos(\log x) + b\sin(\log x)$, P.T

$$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1) y_n = 0$$

Solⁿ: $y = a\cos(\log x) + b\sin(\log x)$

differentiate w.r.t x

$$y_1 = a.(-\sin(\log x)).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$$

$$x y_1 = -a\sin(\log x) + b\cos(\log x)$$

diff again

$$x y_2 + y_1 = -a\cos(\log x)\frac{1}{x} + b(-\sin(\log x))\frac{1}{x}$$

$$x^2 y_2 + x y_1 = -(a\cos(\log x) + b\sin(\log x))$$

$$x^2 y_2 + x y_1 = -y$$

$$x^2 y_2 + x y_1 + y = 0$$

$$D^n(x^2 y_2) + D^n(x y_1) + D^n y = 0$$

$$x^2 y_{n+2} + {}^nC_1 (2x) y_{n+1} + {}^nC_2\, 2\, y_n + 0 +$$

$$x y_{n+1} + {}^nC_1 . 1 . y_n + y_n = 0$$

$$x^2 y_{n+2} + 2n x\, y_{n+1} + \frac{n(n-1)}{\cancel{2}} . \cancel{2} y_n + x\, y_{n+1} + n y_n + y_n = 0$$

$$x^2 y_{n+2} + (2n+1) x\, y_{n+1} + (n^2 \cancel{-n} \cancel{+n} + 1) y_n$$

$$x^2 y_{n+2} + (2n+1) x\, y_{n+1} + (n^2+1) y_n = 0$$

$$x^2 y_{n+1} + (2n+1) x\, y_{n+1} + (n^2+1) y_n$$

7. If $\cos^{-1}(y/b) = \log(x/n)^p$, P.T

$$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+p^2)\, y_n = 0$$

Soln: Given $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^p$

$$\cos^{-1}\left(\frac{y}{b}\right) = p\log\left(\frac{x}{n}\right)$$

$$\frac{y}{b} = \cos\left(p\left(\log\left(\frac{x}{n}\right)\right)\right)$$

$$\frac{y}{b} = \cos\left(p\left(\log x \cancel{+} - \log n\right)\right)$$

$$y = b\cos\left(p\left(\log x \cancel{+} - \log n\right)\right)$$

Diff w.r.t x we get

$$y_1 = -b\sin\left(p\left(\log x \cancel{+} - \log n\right)\right) p \cdot \frac{1}{x}$$

$$x y_1 = -bp\sin\left(p\left(\log x - \log n\right)\right)$$

Diff w.r.t x again

$$x y_2 + y_1 = -bp\cos\left(p\left(\log x - \log n\right)\right) \cdot p\frac{1}{x}$$

$$x^2 y_2 + x y_1 = -bp^2\cos\left(p\left(\log x - \log n\right)\right)$$

$$x^2 y_2 + x y_1 = -p^2 y$$

$$x^2 y_2 + x y_1 + p^2 y = 0 \qquad \text{continue...}$$

10. If $x = \sin t$ and $y = \cos pt$, P.T

$$(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - (n^2 - p^2)\, y_n = 0$$

Solⁿ: $x = \sin t$ $\qquad y = \cos pt$

$t = \sin^{-1} t$ $\quad \therefore y = \cos p \sin^{-1} x$

Now; $y = \cos(p \sin^{-1} x)$

$$y_1 = -\sin(p \sin^{-1} x) \cdot p \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = -p \sin(p \sin^{-1} x)$$

diff

$$\sqrt{1-x^2}\, y_2 + y_1 \frac{1}{2\sqrt{1-x^2}} (-2x) = -p\cos(p\sin^{-1}x) \cdot p \cdot \frac{1}{\sqrt{1-x^2}}$$

$$(1-x^2)\, y_2 - x y_1 = -p^2 \cos(p \sin^{-1} x)$$

$$(1-x^2)\, y_2 - x y_1 = -p^2 y$$

$$(1-x^2)\, y_2 - x y_1 + p^2 y = 0 \quad \text{conti - - -}$$

Assignment:

1. If $y^{1/m} + y^{-1/m} = 2x$ Then P.T
$$(x^2-1)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

2. If $y = (x + \sqrt{1+x^2})^m$ then P.T
$$(1+x^2)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

~~Polar curves and angle between the po~~ 6

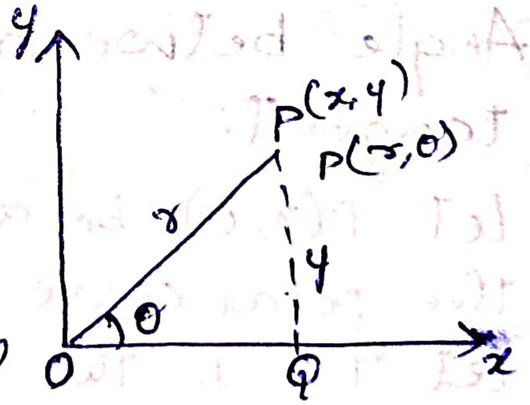
Polar curves

Polar curves: measuring a point from reference point and angle from the reference line.

Defⁿ:

- Initial reference point O in the plane is called as the pole
- A line OL drawn through O is called the initial line. If P is any given point in the plane, joining O and P there forms an angle at O.
- The length of OP denoted by r is called the radius vector of point P and the angle LOP denoted by θ measured in the anticlockwise direction is called the vectorial angle.
- The pair r and θ represented by $P=(r,\theta)$ or $P(r,\theta)$ are called as the Polar coordinates of the point P.

We will now build a relation b/w Cartesian and polar coordinates.



Let (x, y) and (r,θ) respl. represent the cartesian and polar coordinates of any point P in the plane where the origin O is taken as the pole and the x-axis is taken as the initial line.

From the figure we have $OQ = x$, $PQ = y$. Also, from right angled triangle OQP we have

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \quad \therefore x = r\cos\theta \longrightarrow (1)$$

$$\sin\theta = \frac{QP}{OP} = \frac{y}{r} \quad \therefore y = r\sin\theta \longrightarrow (2)$$

squaring and adding (1) and (2) we get

$$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$$

$$\therefore r = \sqrt{x^2+y^2} \longrightarrow (3)$$

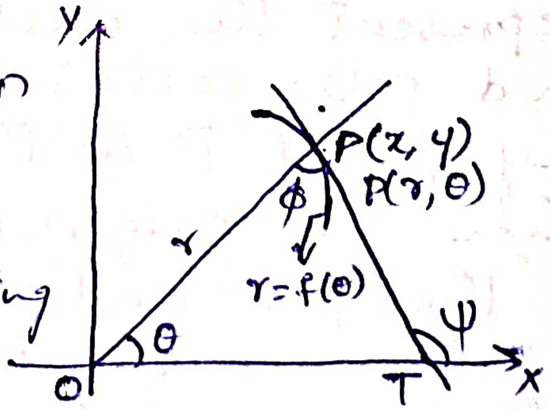
dividing (2) by (1), we get

$$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = y/x \Rightarrow \tan\theta = \frac{y}{x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow (4)$$

The relation (1) and (2) determine the cartesian coordinates in terms of polar cooordinates where as relations (3) and (4) determine the polar coordinates in terms of cartesian coordinates.

Angle between radius vector and tangent:



Let $P(r,\theta)$ be a point on the polar curve $r=f(\theta)$. Let 'PT' be the tangent to the curve at P meeting the initial line OX at T. Let $P\hat{T}X = \psi$. let the angle b/w the radius vector OP and the tangent PT be denoted by ϕ i.e

$\underline{\angle OPT} = \phi$.

If (x, y) are the cartesian coordinates of the point P, we have

$x = r\cos\theta$ and $y = r\sin\theta$ and

$\frac{dy}{dx}$ = slope of the Tangent PT

$= \tan\psi \quad (\because \psi = \theta + \phi$
Ext. angle = sum of int. opp. $\angle$s)

$$\frac{dy}{dx} = \tan(\theta + \phi)$$

$$\frac{dy}{dx} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi} \longrightarrow (1)$$

$$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta \frac{dr}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta \frac{dr}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

divide both n^r & d^r on RHS by $\frac{dr}{d\theta}\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{\cancel{\frac{dr}{d\theta}}\sin\theta}{\cancel{\frac{dr}{d\theta}}\cos\theta} + \frac{r\cos\theta}{\frac{dr}{d\theta}\cancel{\cos\theta}}}{\frac{\cancel{\frac{dr}{d\theta}}\cancel{\cos\theta}}{\cancel{\frac{dr}{d\theta}}\cancel{\cos\theta}} - \frac{r\sin\theta}{\frac{dr}{d\theta}\cos\theta}}$$

$$\frac{dy}{dx} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta} \longrightarrow (2)$$

Comparing (1) and (2), we get

$\tan\phi = r\frac{d\theta}{dr}$ → This is the formula for $\angle$ b/w radius vector OP & Tangent PT.

Note: 1) $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

Examples:

1. Find the angle b/w the radius vector and Tangent for the curve $r = a(1+\cos\theta)$. Also, find the slope of the curve at $\theta = \frac{\pi}{6}$.

Sol^n: $r = a(1+\cos\theta)$ d w.r.t θ

$$\frac{dr}{d\theta} = -a\sin\theta$$

The angle b/w the radius vector and tangent is given by

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= \cancel{a}(1+\cos\theta)\frac{1}{-\cancel{a}\sin\theta}$$

$$= \frac{1+\cos\theta}{-\sin\theta}$$

$$= \frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{-\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}$$

$1+\cos\theta = 2\cos^2\theta/2$

$\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$

$$\tan\phi = -\cot\theta/2$$

$$\tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\boxed{\phi = \frac{\pi}{2}+\theta/2}$$

At $\theta = \frac{\pi}{6}$, $\phi = \frac{\pi}{2}+\frac{\theta}{2} = \frac{\pi}{2}+\frac{\pi}{6\cdot 2} = \frac{\pi}{2}+\frac{\pi}{12}$

$$\phi = \frac{7\pi}{12}$$

$\psi = \theta+\phi = \frac{\pi}{6}+\frac{7\pi}{12} \Rightarrow \psi = \frac{3\pi}{4}$

$\therefore$ slope $= \tan\psi$ $= \tan 135°$ $= -1$

2. Find the angle b/w radius vector (7)
and Tangent for the curve $r = a(1-\cos\theta)$
and also find slope of the curve at
$\theta = \frac{\pi}{6}$.

Solⁿ: $\boxed{\phi = \theta/2}$ $\boxed{\text{Slope of the Tangent} = 1}$

3. Find the $\angle$ b/w radius vector and
tangent for the curve $r \cos^2 \theta/2 = a$
at $\theta = \frac{2\pi}{3}$

Solⁿ: $r \cos^2 \theta/2 = a$ d wrt θ

$$r \cdot \cancel{2}\cos\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\right)\cdot\frac{1}{\cancel{2}} + \cos^2\frac{\theta}{2}\frac{dr}{d\theta} = 0$$

$$-r \sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}\frac{dr}{d\theta} = 0$$

$$\cos^{\cancel{2}}\frac{\theta}{2}\frac{dr}{d\theta} = r\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}$$

$$\frac{dr}{d\theta} = \frac{r\sin\theta/2}{\cos\theta/2} = r\tan\theta/2$$

$$\therefore \frac{dr}{d\theta} = r\tan\frac{\theta}{2}$$

We have

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\tan\phi = \cancel{r}\cdot\frac{1}{\cancel{r}\tan\theta/2}$$

$$\tan\phi = \cot\theta/2$$

$$\tan\phi = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

At $\theta = \frac{2\pi}{3}$, $\phi = \frac{\pi}{2} - \frac{\cancel{2}\pi}{3\cdot\cancel{2}} = \frac{3\pi - 2\pi}{6}$

$$\phi = \frac{\pi}{6}$$

$\psi = \theta + \phi$, $\psi = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore$ Slope $= \tan\psi = \tan\frac{5\pi}{6} = -1/\sqrt{3}$

4. $\frac{2a}{r} = 1 - \cos\theta$ at $\theta = \frac{2\pi}{3}$

Sol^n: $\phi = -\theta/2$

$\psi = \frac{\pi}{3}$

slope of the Tangent $= \sqrt{3}$

5. $r = a(1+\sin\theta)$ at $\theta = \pi/2$

Sol^n: $\phi = \pi/2$ or $\frac{\pi}{4} + \frac{\theta}{2}$

$\psi = \pi$

slope $= 0$

6. Find the angle b/w radius vector and the Tangent

a) $r^2 \cos 2\theta = a^2$

Sol^n: $r^2 \cos 2\theta = a^2$

$\phi = \frac{\pi}{2} - 2\theta$

b) $r^m = a^m (\cos m\theta + \sin m\theta)$

Sol^n: d. w r T θ

$\cancel{m} r^{m-1} \frac{dr}{d\theta} = a^m (-\sin m\theta \cdot \cancel{m} + \cos m\theta \cdot \cancel{m})$

$\frac{dr}{d\theta} = \frac{a^m}{r^{m-1}} (\cos m\theta - \sin m\theta)$

$\therefore \tan\phi = r\frac{d\theta}{dr} = r \frac{r^{m-1}}{a^m(\cos m\theta - \sin m\theta)}$

$= \frac{r^m}{a^m(\cos m\theta - \sin m\theta)}$

$= \frac{\cancel{a^m}(\cos m\theta + \sin m\theta)}{\cancel{a^m}(\cos m\theta - \sin m\theta)}$

$= \frac{\cancel{\cos m\theta}(1 + \tan m\theta)}{\cancel{\cos m\theta}(1 - \tan m\theta)}$

$= \frac{\tan \pi/4 + \tan m\theta}{1 - \tan\frac{\pi}{4} \tan m\theta} = \tan\left(\frac{\pi}{4} + m\theta\right)$

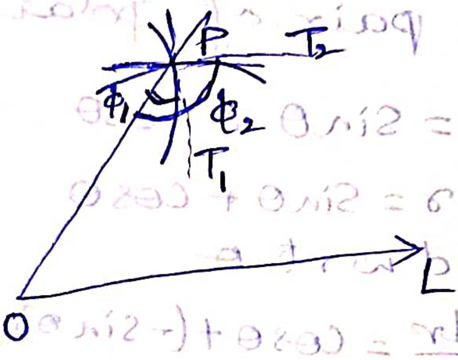
$\boxed{\text{Ans: } \phi = \frac{\pi}{4} + m\theta}$

Angle of intersection of two polar curves:

We know that the angle of intersection of any two curves is equal to the angle between the tangents drawn at the point of intersection of the two curves.

Let $r = f_1(\theta)$ and $r = f_2(\theta)$ be two curves intersecting at the point P.

Let PT_1 and PT_2 be the tangents drawn to the curves at the point P.



From the figure that ϕ_1 is the angle b/w radius vector OP and the tangent PT_1 and ϕ_2 is the angle b/w radius vector OP and the tangent PT_2.

Thus the angle b/w the two tangents is equal to $\phi_2 - \phi_1$.

$\therefore$ The acute angle of the intersection of the curves is equal to $|\phi_2 - \phi_1|$.

If $|\phi_2 - \phi_1| = \frac{\pi}{2}$ then the two curves intersect orthogonally.

Note: 1. Suppose $\phi_2 - \phi_1 = \frac{\pi}{2}$

$$\phi_2 = \frac{\pi}{2} + \phi_1$$

$$\tan\phi_2 = \tan\left(\frac{\pi}{2} + \phi_1\right) = -\cot\phi_1 = \frac{-1}{\tan\phi_1}$$

$$\tan\phi_1 \cdot \tan\phi_2 = -1$$

$\Rightarrow$ alternative cond^n for orthogonality of 2 polar curves.

2. If we are not able to obtain ϕ_1 and ϕ_2 explicitly then we have to write

$$|\tan(\phi_1 - \phi_2)| = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

~~than~~.

Examples:

Find the angle b/w two polar curves (or) find the angle of intersection of the pair of polar curves

1. $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

Sol^n: $r = \sin\theta + \cos\theta$

d w r t θ

$$\frac{dr}{d\theta} = \cos\theta + (-\sin\theta)$$

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\tan\phi_1 = r\frac{d\theta}{dr}$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$= \frac{\cos\theta(1 + \tan\theta)}{\cos\theta(1 - \tan\theta)}$$

$$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4} \cdot \tan\theta}$$

$$\tan\phi_1 = \tan\left(\frac{\pi}{4} + \theta\right)$$

$$\phi_1 = \frac{\pi}{4} + \theta$$

$r = 2\sin\theta$

d w r t θ

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$\tan\phi_2 = r\frac{d\theta}{dr}$$

$$= \frac{\cancel{2}\sin\theta}{\cancel{2}\cos\theta}$$

$$\tan\phi_2 = \tan\theta$$

$$\phi_2 = \theta$$

The angle b/w 2 polar curves $= |\phi_1 - \phi_2|$

$$= \left|\frac{\pi}{4} + \theta - \theta\right| = \frac{\pi}{4}$$

2. $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

Sol^n: $r = a(1-\cos\theta)$

d w r t θ

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\tan\phi_1 = r.\frac{d\theta}{dr}$$

$$= \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$$

$$= \frac{2\sin^2\theta/2}{\cancel{2}\sin\theta/2 . \cos\theta/2}$$

$$\tan\phi_1 = \tan\theta/2$$

$$\phi_1 = \theta/2$$

$r = 2a\cos\theta$

d w r t θ

$$\frac{dr}{d\theta} = -2a\sin\theta$$

$$\tan\phi_2 = r\frac{d\theta}{dr}$$

$$= \frac{\cancel{2a}\cos\theta}{-\cancel{2a}\sin\theta}$$

$$= -\cot\theta$$

$$\tan\phi_2 = \tan\left(\frac{\pi}{2}+\theta\right)$$

$$\phi_2 = \frac{\pi}{2}+\theta$$

The angle b/w 2 polar curves $= |\phi_1 - \phi_2|$

$$= \left|\frac{\theta}{2} - \frac{\pi}{2} - \theta\right| = \left|-\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\right| = \frac{\pi}{2}+\frac{\theta}{2} \rightarrow (1)$$

To find θ:

$r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$$\Rightarrow \cancel{a}(1-\cos\theta) = 2\cancel{a}\cos\theta$$

$$1 = 3\cos\theta$$

$$\cos\theta = \frac{1}{3}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\therefore |\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)$$

3. $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$

Sol^n: $\phi_1 = \frac{\pi}{2}+\theta$ $\qquad$ $\phi_2 = \frac{\pi}{2}+\theta/2$

$$|\phi_1 - \phi_2| = \theta/2 = \pi/6$$

4. $r = a(1-\sin\theta)$ and $r = a(1+\sin\theta)$

Sol^n: $r = a(1-\sin\theta)$ $\qquad$ $r = a(1+\sin\theta)$

$\frac{dr}{d\theta} = -a\cos\theta$ $\qquad$ $\frac{dr}{d\theta} = a\cos\theta$

$\tan\phi_1 = r\frac{d\theta}{dr}$ $\qquad$ $\tan\phi_2 = r\frac{d\theta}{dr}$

$= \frac{\cancel{a}(1-\sin\theta)}{-\cancel{a}\cos\theta}$ $\qquad$ $= \frac{\cancel{a}(1+\sin\theta)}{\cancel{a}\cos\theta}$

$\tan\phi_1 = \frac{-(1-\sin\theta)}{\cos\theta}$ $\qquad$ $\tan\phi_2 = \frac{1+\sin\theta}{\cos\theta}$

we have,

$$|\tan(\phi_1-\phi_2)| = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1 . \tan\phi_2}$$

$$= \frac{\frac{-1+\sin\theta}{\cos\theta} - \frac{(1+\sin\theta)}{\cos\theta}}{1+\frac{(-1+\sin\theta)}{\cos\theta}.\frac{(1+\sin\theta)}{\cos\theta}}$$

$$= \frac{\frac{-1+\cancel{\sin\theta}-1-\cancel{\sin\theta}}{\cos\theta}}{1+\frac{\sin^2\theta-1}{\cos^2\theta}}$$

$$= \frac{\frac{-2}{\cos\theta}}{\frac{\cos^2\theta+\sin^2\theta-1}{\cos^2\theta}} = \frac{-2\cos\theta}{1-1}$$

$|\tan(\phi_1-\phi_2)| = \infty$

$\phi_1-\phi_2 = \tan^{-1}\infty$

$$\boxed{\phi_1-\phi_2 = \frac{\pi}{2}}$$

5. $r = a\log\theta$ and $r = \dfrac{a}{\log\theta}$

Sol^n:

$r = a\log\theta$

$\dfrac{dr}{d\theta} = \dfrac{a}{\theta}$

$\tan\phi_1 = r\dfrac{d\theta}{dr}$

$= \dfrac{\cancel{a}\log\theta}{\dfrac{\cancel{a}}{\theta}}$

$\tan\phi_1 = \theta\log\theta$

$r = \dfrac{a}{\log\theta}$

$\dfrac{dr}{d\theta} = a(-1)(\log\theta)^{-2}\cdot\dfrac{1}{\theta}$

$\dfrac{dr}{d\theta} = \dfrac{-a}{\theta(\log\theta)^2}$

$\tan\phi_2 = r\dfrac{d\theta}{dr}$

$= \dfrac{\cancel{a}}{\log\theta} \times \dfrac{\theta(\log\theta)^{\cancel{2}}}{-\cancel{a}}$

$\tan\phi = -\theta\log\theta$

$\cancel{\tan\phi_1 - \tan\theta}$

$|\tan(\phi_1 - \phi_2)| = \left|\dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\cdot\tan\phi_2}\right|$

$= \left|\dfrac{\theta\log\theta + \theta\log\theta}{1 + (-\theta^2(\log\theta)^2)}\right|$

$|\tan(\phi_1 - \phi_2)| = \left|\dfrac{2\theta\log\theta}{1 - \theta^2(\log\theta)^2}\right| \longrightarrow (1)$

To find θ

$r = a\log\theta$ and $r = \dfrac{a}{\log\theta}$

$a\log\theta = \dfrac{a}{\log\theta} \Rightarrow \log^2\theta = 1$

$\Rightarrow \log\theta = 1$

$\theta = e^1$

$(1) \Rightarrow |\tan(\phi_1 - \phi_2)| = \left|\dfrac{2e\log e}{1 - e^2\log^2 e}\right|$

$|\tan(\phi_1 - \phi_2)| = \dfrac{2e}{1-e^2} \Rightarrow |\phi_1 - \phi_2| = \tan^{-1}\left(\dfrac{2e}{1-e^2}\right)$

$= 2\tan^{-1}e$

6. $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Sol^n: $\phi_1 = -2\theta$, $\phi_2 = 2\theta$

$$|\phi_1 - \phi_2| = 4\theta = \frac{\pi}{3}$$

Show that the following pairs of curves intersect eachother orthogonally

1. $r = a(1+\cos\theta)$ $\qquad$ $r = b(1-\cos\theta)$

Sol^n: $\frac{dr}{d\theta} = a(-\sin\theta)$ $\qquad$ $\frac{dr}{d\theta} = b\sin\theta$

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta} \qquad \tan\phi_2 = r\frac{d\theta}{dr} = \frac{\cancel{b}(1-\cos\theta)}{\cancel{b}\sin\theta}$$

$$\tan\phi_1 = \frac{(1+\cos\theta)}{-\sin\theta} \qquad \tan\phi_2 = \frac{(1-\cos\theta)}{\sin\theta}$$

Now,

$$\tan\phi_1 \cdot \tan\phi_2 = -\frac{(1+\cos\theta)}{\sin\theta}\frac{(1-\cos\theta)}{\sin\theta}$$

$$= -\frac{(1-\cos^2\theta)}{\sin^2\theta} = \frac{-\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}$$

$$= -1$$

Hence the curves intersect orthogonally.

2. $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$

3. $r^n = a^n(\cos n\theta)$ & $r^n = b^n \sin n\theta$

4. $r^2 \sin 2\theta = a^2$ & $r^2 \cos 2\theta = b^2$

5. $r = \frac{a}{1+\cos\theta}$ & $r = \frac{b}{1-\cos\theta}$

Assignment:

9

1. Find the angle of intersection of pair of curves:

a) $r^n = a^n \sec(n\theta + \alpha)$ & $r^n = b^n \sec(n\theta + \beta)$

b) $r = a(1+\cos\theta)$ & $r^2 = a^2 \cos 2\theta$

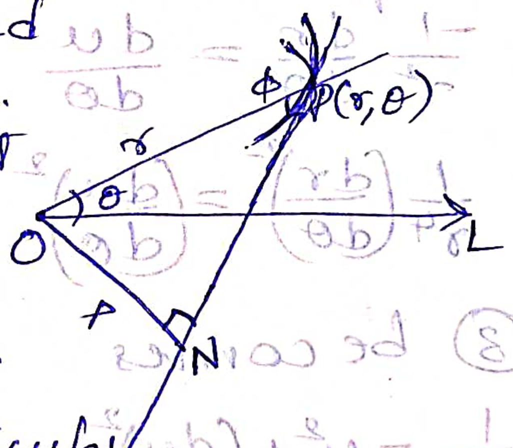
c) $r = a\theta/(1+\theta)$ & $r = a/(1+\theta^2)$

d) $r = a\theta$ & $r = a/\theta$

e) $r = 4\sec^2(\theta/2)$ & $r = 9\operatorname{cosec}^2\left(\frac{\theta}{2}\right)$

and Also check whether they are orthogonal to each other.

Length of the perpendicular from the pole to the Tangent:



Let O be the pole and OL be the initial line. Let $P(r,\theta)$ be any point on the curve and hence we have $OP = r$ and $\angle LOP = \theta$.

Draw $ON = p$ perpendicular from the pole on to the Tangent at P and let ϕ be the angle made by the radius vector with the Tangent.

From ~~the figure~~, the right angle ΔONP

$$\sin\phi = \frac{ON}{OP} = \frac{p}{r} \Rightarrow p = r\sin\phi$$

This expression is the basic expression for the length of the perpendicular p. Now we express p in terms of θ. We have,

$$\underbrace{p = r\sin\phi}_{(1)} \quad \text{and} \quad \underbrace{\cot\phi = \frac{1}{r}\frac{dr}{d\theta}}_{(2)}$$

Squaring ① ~~we~~ and taking reciprocal we get,

$$\frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{1}{\sin^2\phi}$$

$$= \frac{1}{r^2} \operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right) \quad (\text{using } ②)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \longrightarrow ③$$

Further, let $\frac{1}{r} = u$

diff w.r.t θ, we get

$$\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$\therefore$ ③ becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \longrightarrow ④$$

Eqn ③ & ④ are standard forms of p in terms of θ.

Pedal equation of a Polar curve:

The equation of the given curve $r = f(\theta)$ expressed in terms of p and r is called as pedal equation or p–r equation of the curve $r = f(\theta)$

Working rule: To find $p-r$ eqn:

Let $r = f(\theta)$ be the eqn of the curve

1. Find ϕ in terms of θ by using $\tan\phi = r\frac{d\theta}{dr}$
2. Eliminate θ from the eqn $r = f(\theta)$ with some simplification using $p = r\sin\phi$. Thus we get required pedal equation.

Note:

The pedal eqn of the curve $r = f(\theta)$ can be obtained easily by eliminating θ from $r = f(\theta)$ and

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Examples:

1. Find the pedal equation for the following curves:

1. $r^n = a^n \cos n\theta$

Soln: d w.r.t θ

$$n r^{n-1}\frac{dr}{d\theta} = -a^n \sin n\theta \cdot n$$

$$\frac{dr}{d\theta} = \frac{-a^n \sin n\theta \cdot r}{r^n}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$r\frac{dr}{d\theta} = -\frac{1}{\tan n\theta} = -\cot n\theta$$

$$r\frac{dr}{d\theta} = \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\tan\phi = \tan\left(\frac{\pi}{2} + n\theta\right)$$

$$\therefore \phi = \frac{\pi}{2} + n\theta \longrightarrow (1)$$

Consider, $p = r\sin\phi$

$$p = r\sin\left(\frac{\pi}{2} + n\theta\right)$$

$$p = r\cos n\theta \longrightarrow (2)$$

We have,

$$r^n = a^n \cos n\theta \Rightarrow \cos n\theta = \frac{r^n}{a^n} \longrightarrow (3)$$

Substituting (3) in (2),

$$p = r \cdot \frac{r^n}{a^n}$$

$$pa^n = r^{n+1}$$

$\therefore r^{n+1} = pa^n$ is the required pedal eqn.

2. $\frac{2a}{r} = 1 - \cos\theta$

sol^n: $\phi = -\frac{\theta}{2} \longrightarrow (1)$

using $p = r\sin\phi$

$$= r\sin\left(-\frac{\theta}{2}\right)$$

$$p = -r\sin\frac{\theta}{2} \longrightarrow (2)$$

We have

$$\frac{2a}{r} = 1 - \cos\theta$$

$$\frac{\cancel{2}a}{r} = \cancel{2}\sin^2\theta/2 \longrightarrow (3)$$

using (2) in (3)

$$\frac{a}{r} = \left(-\frac{p}{r}\right)^2$$

$$\frac{a}{r} = \frac{p^2}{r^2} \Rightarrow p^2 = ar$$

$\therefore p^2 = ar$ is the required p-r eqn.

3. $r = a(1 - \cos\theta)$

sol^n: $\phi = \theta/2$ and $r^3 = 2ap^2 \longrightarrow$ p-r eqn

4. $r = a\theta$ (10)

Sol^n: $\frac{dr}{d\theta} = a$

Now using $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}a^2$$

$$\frac{1}{p^2} = \frac{r^4 + r^2a^2}{r^6} = \frac{r^2+a^2}{r^4}$$

$$p^2 = \frac{r^4}{r^2+a^2} \Rightarrow p = \sqrt{\frac{r^4}{r^2+a^2}}$$

$$p = \frac{r^2}{\sqrt{r^2+a^2}}$$

5. $r = a + b\cos\theta \Rightarrow \cos\theta = \frac{r-a}{b}$

Sol^n: $\frac{dr}{d\theta} = -b\sin\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}b^2\sin^2\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\cancel{b^2}\cdot\frac{(r-a)^2}{\cancel{b^2}}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{r^2+a^2-2ar}{r^4}$$

$$= \frac{r^4 + r^2(r^2+a^2-2ar)}{r^6}$$

$$\frac{1}{p^2} = \frac{\cancel{r^2}(r^2+r^2+a^2-2ar)}{r^{\cancel{6}\,4}} = \frac{2r^2+a^2-2ar}{r^4}$$

$$p = \frac{r^2}{\sqrt{2r^2+a^2-2ar}}$$ ✗

$\boxed{\text{Ans: } p = \frac{r^2}{\sqrt{b^2-a^2-2ar}}}$

correct ans

6. $r^2 \cos 2\theta = a^2$

Soln: d w r t θ,

$$r^2(-\sin 2\theta)\,2 + \cos 2\theta\, 2r \frac{dr}{d\theta} = 0$$

$$\cos 2\theta \frac{dr}{d\theta} = r \sin 2\theta$$

$$\frac{dr}{d\theta} = \frac{r^3}{a^2} \sin 2\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{r^3}{a^2}\sin 2\theta\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{r^6}{a^4} \sin^2 2\theta \quad = (1-\cos^2 2\theta)$$

$$= \frac{1}{r^2} + \frac{r^2}{a^4}\left(1 - \frac{a^4}{r^4}\right)$$

$$\frac{1}{p^2} = \frac{r^2}{a^4}$$

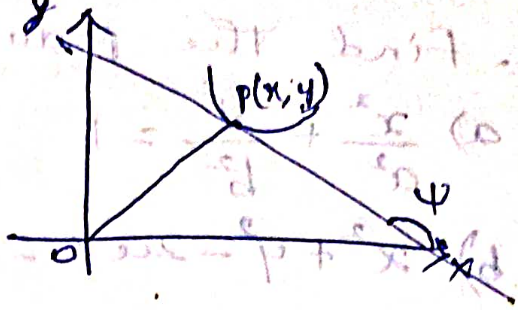
$$p = \frac{a^2}{r}$$

Assignment:

1. $\frac{2a}{r} = (1+\cos\theta)$
2. $r^m \cos m\theta = a^m$
3. $r = a \sin 3\theta$
4. $r^2 = a^2 \sin 2\theta + b^2 \cos 2\theta$
5. $r^n \sec n\theta = a^n$

Pedal eqn of the curve in Cartesian coordinate (11)

Let the eqn of the curve be $y = f(x) \longrightarrow (1)$



Eqn of the tangent at any point $p(x, y)$ is

$(Y - y) = f'(x)(X - x)$

$Y - y = f'(x) \cdot X - x f'(x)$

$(X f'(x) - Y) + (y - x f'(x)) = 0$

If p be the length of the $\perp^r$ from $(0, 0)$ to this tangent we have,

$$p = \frac{y - x f'(x)}{\sqrt{1 + (f'(x))^2}}$$

$$p = \frac{y - x f'(x)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \longrightarrow (2)$$

$\therefore aX + bY + C = 0$

$p = \frac{C}{\sqrt{a^2 + b^2}}$

$p = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$

$p = \frac{C}{\sqrt{a^2 + b^2}}$ (origin (0, 0))

Also, $r^2 = x^2 + y^2 \longrightarrow (3)$

Eliminating x and y from (1), (2) and (3) we obtain the required pedal eqn of the curve (1)

Examples:

1. Find the pedal eqn of the parabola

$y^2 = 4a(x + a)$

Soln: Given $y^2 = 4a(x + a) \longrightarrow (1)$

diff wrt 'x', we get

$\cancel{2}y \frac{dy}{dx} = \cancel{4}^{2}a$

$\frac{dy}{dx} = \frac{2a}{y}$

Eqn of the Tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y - y = \frac{2a}{y}(X - x)$$

$$yY - y^2 = 2aX - 2ax$$

$$2aX - yY + (y^2 - 2ax) = 0 \longrightarrow (2)$$

The length p of the $\perp$ from the origin to the Tangent (2) is given by

$$p = \frac{y^2 - 2ax}{\sqrt{(2a)^2 + (-y)^2}} \qquad (\because y^2 = 4a(x+a))$$

$$= \frac{4a(x+a) - 2ax}{\sqrt{4a^2 + y^2}}$$

$$= \frac{4ax + 4a^2 - 2ax}{\sqrt{4a^2 + 4ax + 4a^2}}$$

$$= \sqrt{\cancel{4a^2 + 2ax}}$$

$$= \frac{4a^2 + 2ax}{\sqrt{8a^2 + 4ax}}$$

$$p = \frac{\cancel{2}(2a^2 + ax)}{\cancel{2}\sqrt{2a^2 + ax}} \quad \text{sq on b.s}$$

$$p^2 = \frac{(2a^2 + ax)^{\cancel{2}}}{\cancel{2a^2 + ax}}$$

$$p = 2a^2 + ax = a(2a + x) \longrightarrow (3)$$

We have $r^2 = x^2 + y^2$

$$r^2 = x^2 + 4a(x + a)$$

$$r^2 = x^2 + 4ax + 4a^2 = (2a + x)^2$$

$$r = x + 2a \longrightarrow (4)$$

using (4) in (3),

$$p = a(\cancel{2a} + r - \cancel{2a}) \Rightarrow p = ar \longrightarrow (5)$$

required pedal eqn.

2. Find the pedal eqn of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Solⁿ: Given $x^{2/3} + y^{2/3} = a^{2/3} \longrightarrow (1)$

diff w.r.t 'x',

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

$$\frac{2}{3} y^{-1/3} \frac{dy}{dx} = -\frac{2}{3} x^{-1/3}$$

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

Eqn of the tangent to the curve at (x, y) is $(Y - y) = \frac{dy}{dx}(X - x)$

$$(Y - y) = -\frac{y^{1/3}}{x^{1/3}}(X - x)$$

$$\frac{Y}{y^{1/3}} - \frac{y}{y^{1/3}} = \frac{-X}{x^{1/3}} + \frac{x}{x^{1/3}}$$

$$\frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = x^{2/3} + y^{2/3} = a^{2/3}$$

$$\frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} - a^{2/3} = 0$$

The length of $\perp (0, 0)$ on this tangent is

$$p = \frac{a^{2/3}}{\sqrt{\left(\frac{1}{x^{1/3}}\right)^2 + \left(\frac{1}{y^{1/3}}\right)^2}} = \frac{a^{2/3}}{\sqrt{\frac{y^{2/3} + x^{2/3}}{x^{2/3} \cdot y^{2/3}}}} = \frac{a^{2/3}}{\sqrt{\frac{a^{2/3}}{x^{2/3} \cdot y^{2/3}}}} = a^{1/3} x^{1/3} y^{1/3}$$

$$p^2 = a^{2/3} x^{2/3} y^{2/3} \longrightarrow (2)$$

we have $r^2 = x^2 + y^2$

$$= (x^{\frac{2}{3}})^3 + (y^{\frac{2}{3}})^3$$

$$= (x^{\frac{2}{3}} + y^{\frac{2}{3}})^3 - 3x^{2/3} y^{2/3} (x^{2/3} + y^{2/3})$$

$$r^2 = (a^{\frac{2}{3}})^3 - 3 \underbrace{x^{2/3} y^{2/3} a^{2/3}}_{\text{from (2)}}$$

from (2)

$$r^2 = a^2 - 3p^2 \longrightarrow (3)$$

required pedal eqn.

Assignment

1. Find the pedal eqn for the foll:

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

b) $x^2+y^2-2ax=0$

c) $x=a\cos^3\theta$ and $y=a\sin^3\theta$

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Formulae connected with derivative of arc length:

1. Cartesian curve: $y=f(x)$

(i) $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (ii) $\sin\psi=\frac{dy}{ds}$

(iii) $\cos\psi=\frac{dx}{ds}$ (iv) $\tan\psi=\frac{dy}{dx}$

where ψ being the $\angle$ made by the tangent $P(x,y)$ on the curve with X-axis.

2. Parametric curve: $x=x(t)$, $y=y(t)$

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

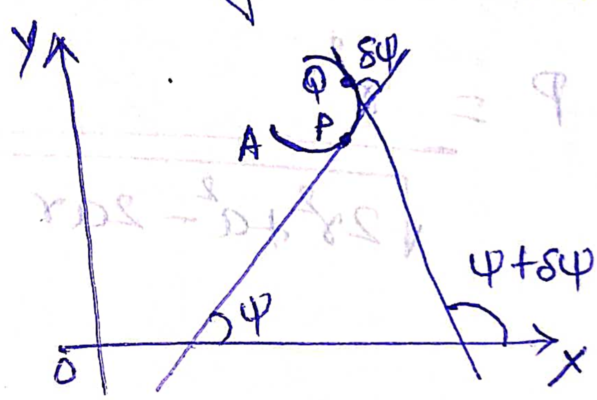
3. Polar curve: $r=f(\theta)$

(i) $\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

(ii) $\sin\phi=r\frac{d\theta}{ds}$ (iii) $\cos\phi=\frac{dr}{ds}$

Curvature and Radius of Curvature:

Consider a curve in XY plane and let A be a fixed point on it. Let P and Q be 2 points on the curve



Such that

$\widehat{AP} = s$ and $\widehat{AQ} = s + \delta s$ $\therefore$ $\widehat{PQ} = \delta s$

Let ψ and $\psi + \delta\psi$ resp/: be the angles made by the tangents at P and Q with x-axis. The angle $\delta\psi$ b/w the tangents is called the bending of the curve which depends on δs. $\delta\psi/\delta s$ is called as the mean curvature of the arc PQ. Also, the amount of bending of the curve at P is called as the curvature of the curve at P and is defined mathematically as

$$\lim_{\substack{\delta s \to 0 \\ (Q \to P)}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$ be denoted by K

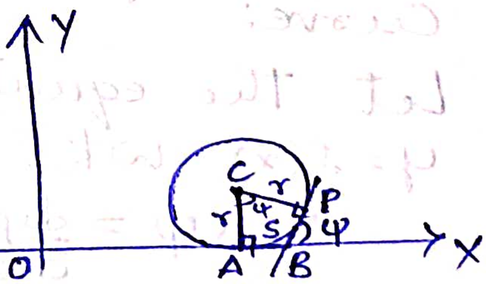
i.e Curvature $= K = \dfrac{d\psi}{ds}$.

if $K \neq 0$ then the reciprocal of the curvature is called as the radius of curvature and is denoted by ρ.

i.e Radius of Curvature $= \rho = \dfrac{1}{K} = \dfrac{ds}{d\psi}$

* The curvature of a circle is a constant!

Consider a circle of radius r having centre at the point C. Let A be a fixed point on the circle and P(x, y) be any point on the circle such that $\widehat{AP} = s$. Let ψ be the angle made by the tangent at P with the x-axis at the point B (interior angle being $\pi - \psi$). Clearly, CA = CP = r = radius.



We have from quadrilateral CABP,

$\therefore A\hat{C}P = \psi$

$$\hat{C} + \hat{A} + \hat{B} + \hat{P} = 2\pi$$

$$\hat{C} + \frac{\pi}{2} + \pi - \psi + \frac{\pi}{2} = 2\pi$$

Note :-

1. The curvature of a curve at a point is zero if and only if the curve doesnot bend at that point. A straight line doesnot bend at any point on it $\therefore$ the curvature of a st. line is zero at every point on it. A circle bends uniformly at every point on it $\therefore$ the curvature of circle is constant and $\neq 0$.

2. K and ρ is +ve or −ve according as $\frac{d\psi}{ds}$ is +ve or −ve.

Radius of curvature for different forms of curves:

(i) Radius of curvature for Cartesian curve:

Let the equation of the curve be $y = f(x)$ WKT

$$\tan\psi = \frac{dy}{dx}$$

diff w.r.t s, we have

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds}$$

$$(1+\tan^2\psi)\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds} \qquad \left(\because \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right.$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{1}{\rho} = \frac{d^2y}{dx^2}\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\frac{\rho}{1+\left(\frac{dy}{dx}\right)^2} = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}\left(1+\left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

where $\frac{dy}{dx} = y_1$

$\frac{d^2y}{dx^2} = y_2$

$$\boxed{\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}}$$

Note: The radius of curvature is independent of the choice of coordinate axes. Thus interchanging x and y gives us

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}.$$

This formula is particularly useful when tangent is $\perp^r$ to x-axis i.e $\frac{dy}{dx} = \infty$ or $\frac{dx}{dy} = 0$

1. Find the radius of curvature at any point on the curve

a) $x^{2/3}+y^{2/3}=a^{2/3}$

Sol^n: $x^{2/3}+y^{2/3}=a^{2/3}$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$y^{-1/3}\frac{dy}{dx}=-x^{-1/3}$$

$$\frac{dy}{dx}=\frac{-x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

diff again,

$$\cancel{\frac{d^2y}{dx^2}=-\left[\frac{y^{-1/3}\left(-\frac{1}{3}\right)x^{-2/3}-x^{-1/3}\frac{1}{3}y^{-2/3}\frac{dy}{dx}}{y^{-2/3}}\right]}$$

$$\frac{d^2y}{dx^2}=-\left[\frac{x^{1/3}\frac{1}{3}y^{-2/3}\frac{dy}{dx}-y^{1/3}\frac{1}{3}x^{-2/3}}{x^{2/3}}\right]$$

$$=-\frac{1}{3}\left[\frac{\cancel{x^{1/3}}\left(-\frac{y^{1/3}}{\cancel{x^{1/3}}}\right)y^{-2/3}-y^{1/3}\cdot x^{-2/3}}{x^{2/3}}\right]$$

$$=\frac{1}{3}\left[\frac{y^{-1/3}+\frac{y^{1/3}}{x^{2/3}}}{x^{2/3}}\right]=\frac{1}{3}\left[\frac{\frac{1}{y^{1/3}}+\frac{y^{1/3}}{x^{2/3}}}{x^{2/3}}\right]$$

$$=\frac{1}{3}\left[\frac{\frac{x^{2/3}+y^{2/3}}{x^{2/3}\cdot y^{1/3}}}{x^{2/3}}\right]=\frac{1}{3}\left[\frac{a^{2/3}}{x^{4/3}\cdot y^{1/3}}\right]$$

$$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{y^{2/3}}{x^{2/3}}\right]^{3/2}}{\frac{1}{3}\frac{a^{2/3}}{x^{4/3}y^{1/3}}}$$

$$= \frac{x^{2/3}+y^{2/3}}{x^{2/3}} \times 3\frac{x^{4/3}y^{1/3}}{a^{2/3}}$$

$$\rho = 3(axy)^{1/3}$$

b) $y = a\cosh\left(\frac{x}{a}\right)$

Sol^n: $y = a\cosh\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \cancel{a}\sinh\left(\frac{x}{a}\right)\cdot\frac{1}{\cancel{a}}$$

$$y_1 = \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$$

$$y_2 = \frac{d^2y}{dx^2} = \cosh\left(\frac{x}{a}\right)\cdot\frac{1}{a}$$

$$\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2} = 1$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left(1+\sinh^2\left(\frac{x}{a}\right)\right)^{3/2}}{\frac{1}{a}\cosh\left(\frac{x}{a}\right)} = \frac{a\left(\cosh^2\left(\frac{x}{2}\right)\right)^{3/2}}{\cosh\left(\frac{x}{a}\right)} = a\cosh^2\left(\frac{x}{a}\right)$$

$$\rho = a\left(\frac{y}{a}\right)^2$$

$$\rho = \frac{y^2}{a}$$

2. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$

Sol^n: $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$y_1 = \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{\sqrt{a}}{x\sqrt{x}}\right)$$

at $\left(\frac{a}{4}, \frac{a}{4}\right)$,

$$y_1 = -\frac{\sqrt{a/4}}{\sqrt{a/4}} = -1$$

$$y_2 = \frac{1}{2}\left[\frac{\sqrt{a}}{\frac{a}{4}\sqrt{\frac{a}{4}}}\right] = \frac{4}{a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{\frac{4}{a}} = \frac{2^{3/2} \cdot a}{2^2} = \frac{a}{\sqrt{2}}$$

Assignment:

1. Find the radius of curvature for the foll. curves

a) $xy = c^2$

b) $x^3 + y^3 = 3axy$ at $\left(\frac{3}{2}a, \frac{3}{2}a\right)$

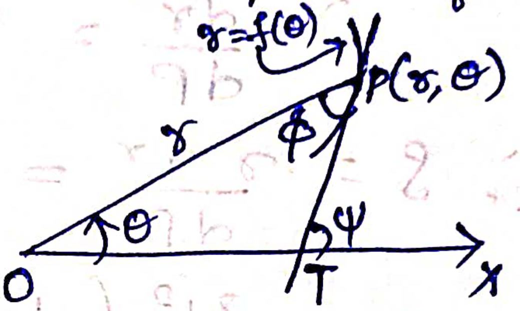
c) $9x^2 + 4y^2 = 36x$ at $(2, 3)$

d) $x^2 y = a(x^2 + y^2)$ at $(-2a, 2a)$

(Hint: $\frac{dy}{dx} = \infty \therefore \frac{dx}{dy} = 0 = y_1$)

(iv) Radius of curvature in polar form:

Suppose the equation to a curve is given in the polar form $r = f(\theta)$.



Let $P(r, \theta)$ be any point on the curve. PT be the tangent to the curve at P inclined at an angle ψ with initial line OX. ϕ be the angle b/w radius vector OP and the tangent PT.

WKT, $\psi = \theta + \phi$

diff wrt to s, we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \longrightarrow (1)$$

Also, we have

$$\tan\phi = r\frac{d\theta}{dr}$$

$$= r\Big/\left(\frac{dr}{d\theta}\right)$$

$$\tan\phi = \frac{r}{r_1} \quad \text{(2)} \quad \left(\text{where } r_1 = \frac{dr}{d\theta}\right)$$

diff (2) wrt 'θ', we get

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r \cdot r_2}{r_1^2} \quad \left(\text{where } r_2 = \frac{d^2r}{d\theta^2}\right)$$

$$\frac{d\phi}{d\theta} = \left(\frac{r_1^2 - r \cdot r_2}{r_1^2}\right)\frac{1}{\sec^2\phi}$$

$$\frac{d\phi}{d\theta} = \left(\frac{r_1^2 - r \cdot r_2}{r_1^2}\right)\frac{1}{1+\tan^2\phi}$$

$$= \frac{r_1^2 - r \cdot r_2}{r_1^2}\frac{1}{1+\frac{r^2}{r_1^2}}$$

$$= \frac{(r_1^2 - r \cdot r_2)}{\cancel{r_1^2}}\frac{\cancel{r_1^2}}{r_1^2 + r^2}$$

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - r \cdot r_2}{r_1^2 + r^2}$$

WKT,

$$\frac{d\phi}{d\theta} = \frac{d\phi}{ds}\cdot\frac{ds}{d\theta} = \frac{r_1^2 - r r_2}{r^2 + r_1^2}$$

$$\frac{d\phi}{ds} = \left(\frac{r_1^2 - r r_2}{r^2 + r_1^2}\right)\frac{d\theta}{ds} \longrightarrow (3)$$

Put eqn (3) in (1) we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \left(\frac{r_1^2 - r r_2}{r^2 + r_1^2}\right)\frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left[1 + \frac{r_1^2 - r r_2}{r^2 + r_1^2}\right]$$

$$= \frac{d\theta}{ds}\left[\frac{r^2 + r_1^2 + r_1^2 - r r_2}{r^2 + r_1^2}\right]$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds}\left[\frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2}\right] \longrightarrow (4)$$

we have,

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

(4) becomes,

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}\left[\frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2}\right]$$

$$\frac{d\psi}{ds} = \frac{1}{\sqrt{r^2+r_1^2}}\left[\frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}\right]$$

$$\frac{d\psi}{ds} = \frac{r^2+2r_1^2-rr_2}{(r^2+r_1^2)^{3/2}} \qquad \left(\because \frac{ds}{d\psi} = \rho\right)$$

$$\frac{1}{\rho} = \frac{r^2+2r_1^2-rr_2}{(r^2+r_1^2)^{3/2}}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \longrightarrow (5)$$

Eqn (5) is the radius of curvature in Polar form.

Examples:

1. Find the radius of curvature of the Cardioid $r = a(1+\cos\theta)$

Sol^n: $r = a(1+\cos\theta)$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$r_1 = -a\sin\theta$$

$$\frac{d^2r}{d\theta^2} = a(-\cos\theta)$$

$$r_2 = -a\cos\theta$$

The radius of curvature in polar form is given by

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2} \longrightarrow (1)$$

consider,

$$r^2 + r_1^2 = a^2(1+\cos\theta)^2 + (-a\sin\theta)^2$$
$$= a^2\left[1 + \cos^2\theta + 2\cos\theta + \sin^2\theta\right]$$
$$= a^2\left[1 + 1 + 2\cos\theta\right]$$
$$= a^2\left[2 + 2\cos\theta\right]$$
$$= 2a^2(1+\cos\theta)$$
$$= 2a^2 \cdot 2\cos^2\theta/2$$
$$r^2 + r_1^2 = 4a^2\cos^2\theta/2$$
$$(r^2 + r_1^2)^{3/2} = (2^2 a^2 \cos^2\theta/2)^{3/2}$$
$$= 2^3 a^3 \cos^3\theta/2$$

consider,

$$r^2 + 2r_1^2 - r r_2 = a^2(1+\cos\theta)^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)$$
$$= a^2\left[1 + \cos^2\theta + 2\cos\theta\right] + 2a^2\sin^2\theta + a^2(\cos\theta + \cos^2\theta)$$
$$= a^2(1 + 2 + 3\cos\theta) = a^2(3 + 3\cos\theta)$$

$= 3a^2(1+\cos\theta)$

$= 3a^2 \cdot 2\cos^2\theta/2$

$= 6a^2\cos^2\theta/2$

Now, (1) becomes

$$\rho = \frac{2^{\cancel{3}2} a^{\cancel{3}} \cos^{\cancel{3}}\theta/2}{\cancel{2}\cdot 3\cdot \cancel{a^2}\cancel{\cos^2\theta/2}}$$

$$\rho = \frac{4}{3}a\cos\theta/2$$

2. $r = a\theta$

Ans: $\dfrac{a(\theta^2+1)^{3/2}}{\theta^2+2}$

3. $r = a(1-\cos\theta)$

Ans: $\frac{4}{3}a\sin\theta/2$

4. $r^2\cos 2\theta = a^2$

Soln: $r^2\cos 2\theta = a^2$

$r^2(-\sin 2\theta)2 + \cos 2\theta \cdot 2rr_1 = 0$

$\cos 2\theta \cdot \cancel{2r}\, r_1 = \cancel{2}r^{\cancel{2}}\sin 2\theta$

$r_1 = r\dfrac{\sin 2\theta}{\cos 2\theta} = r\tan 2\theta$

$$r_1 = r\tan 2\theta$$

$$r_2 = r\sec^2 2\theta \cdot 2 + \tan 2\theta \cdot r_1$$

$$= 2r\sec^2 2\theta + \tan 2\theta \cdot r\tan 2\theta$$

$$r_2 = 2r\sec^2 2\theta + r\tan^2 2\theta$$

Consider,

$$r_1^2 + r^2 = r^2 + r^2\tan^2 2\theta$$

$$= r^2(1+\tan^2 2\theta)$$

$$= r^2\sec^2 2\theta$$

$$(r_1^2 + r^2)^{3/2} = r^3\sec^3 2\theta$$

Consider,

$$r^2 + 2r_1^2 - r r_2 = r^2 + 2r^2\tan^2 2\theta \cancel{- r \cdot r^3\sec^3 2\theta}$$

$$\cancel{= r^2 + r}$$

$$- r(2r\sec^2 2\theta + r\tan^2 2\theta)$$

$$= r^2 + 2r^2\tan^2 2\theta$$

$$- 2r^2\sec^2 2\theta - r^2\tan^2 2\theta$$

$$\cancel{= r^2 + 2r^2 - r^2\tan^2 2\theta}$$

$$= r^2 + r^2\tan^2 2\theta - 2r^2\sec^2 2\theta$$

$$= r^2[1 + \tan^2 2\theta - 2\sec^2 2\theta]$$

$$= r^2[\sec^2 2\theta - 2\sec^2 2\theta]$$

$$= r^2(-\sec^2 2\theta)$$

$$= -r^2\sec^2 2\theta$$

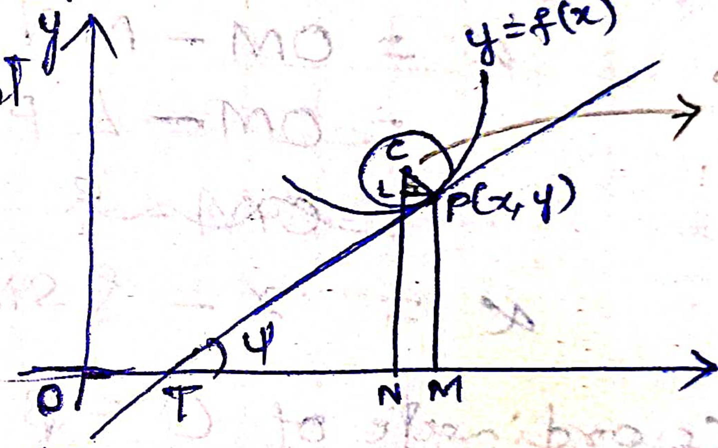
$$\rho = \frac{r^3 \sec^3 2\theta}{r^2 \sec^2 2\theta}$$

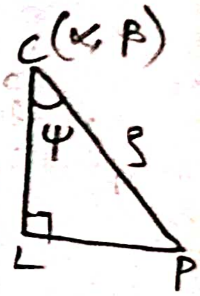
$$\rho = r \sec 2\theta = r \frac{1}{\cos 2\theta} = r \cdot \frac{r^2}{a^2} = \frac{r^3}{a^2}$$

Assignment:

1. Find The Radius of curvature
   a) $r^m = a^m \cos m\theta$
   b) $r(1+\cos\theta) = a$

Centre of curvature:

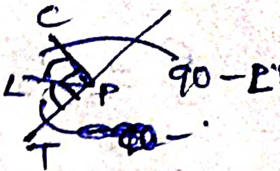




Let $P(x,y)$ be any point on a smooth curve $y=f(x)$ and ρ be the radius of curvature at P. Draw the normal to the curve at P and cut off $PC = \rho$ along the normal. The point C is called the Centre of curvature of the curve at P. The circle drawn with C as centre and ρ as the radius is called the circle of the curvature at P. ~~If Q is neigh~~ Thus, the centre of curvature may be taken as the point of intersection of normals.

Coordinates of the centre of curvature in Cartesian form:

From the fig,



$\hat{C} + \hat{P} = 90$ i.e $P\hat{C}L + C\hat{P}L = 90$

$\Rightarrow P\hat{C}L = 90 - C\hat{P}L$

$= 90 - (90 - L\hat{P}T)$

$P\hat{C}L = L\hat{P}T = PTX = \psi$

$\therefore P\hat{C}L = \psi$

x coordinate of $C = \alpha = ON$

$\alpha = OM - MN$ $\quad (\because OM = x$

$= OM - LP$ $\quad$ LP is perpendicular)

~~$= OM - \rho$~~

$\alpha = x - \rho \sin\psi \rightarrow (1)$

y coordinate of $C = \beta = CN$

$\beta = CN = CL + LN$

$\beta = \rho\cos\psi + y \rightarrow (2)$ $\quad (\because LN = PM = y)$

ΔCPL, $\hat{L} = 90°$

$\sin\psi = \frac{LP}{CP}$

$\rho \sin\psi = LP$

$\cos\psi = \frac{CL}{CP}$

$CL = \rho\cos\psi$

We have,

$\tan\psi = \frac{dy}{dx}$

$\therefore \cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}}$

$$\cos\psi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{\sqrt{1+y_1^2}} \longrightarrow (3)$$

$$\sin\psi = \frac{\sin\psi}{\cos\psi}\cdot\cos\psi$$

$$= \tan\psi\cdot\cos\psi$$

$$= \frac{dy}{dx}\cdot\frac{1}{\sqrt{1+y_1^2}}$$

$$\sin\psi = \frac{y_1}{\sqrt{1+y_1^2}} \longrightarrow (4)$$

Now,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \longrightarrow (5)$$

eqn (1) & (2) becomes,

$$\alpha = x - \frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{y_1}{(1+y_1^2)^{1/2}} \qquad (\because \alpha = x - \rho\sin\psi)$$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \longrightarrow (6)$$

$$\beta = y + \frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{1}{(1+y_1^2)^{1/2}} \qquad (\because \beta = \rho\cos\psi + y)$$

$$\beta = y + \frac{(1+y_1^2)}{y_2} \longrightarrow (7)$$

Thus, the coordinates of the centre of curvature are given by

$$C(\alpha, \beta) = C\left(x - \frac{y_1(1+y_1^2)}{y_2},\ y + \frac{(1+y_1^2)}{y_2}\right)$$

The eqn of circle of curvature is given by

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

Centre of curvature in parametric form:

The parametric form is given by $x = f(t)$ and $y = g(t)$. The coordinates of centre of curvature can be obtained ~~by~~ as follows,

$$\dot{x} = f'(t), \quad \dot{y} = g'(t),$$

$$\ddot{x} = f''(t), \quad \ddot{y} = g''(t),$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \quad \text{and} \quad \rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\sin\varphi = \frac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}$$

$$\sin\psi = \frac{\dot{y}/\dot{x}}{\sqrt{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2}}$$

$$= \frac{\dot{y}/\dot{x}}{\sqrt{\frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2}}} = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$\cos\psi = \frac{1}{\sqrt{1+\tan^2\psi}}$$

$$= \frac{1}{\sqrt{1+\frac{\dot{y}^2}{\dot{x}^2}}} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$\therefore \alpha = x - \rho\sin\psi$$

$$= x - \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}\left(\frac{\dot{y}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)$$

$$\alpha = x - \frac{\dot{y}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\beta = y + \rho\cos\psi$$

$$= y + \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}\left(\frac{\dot{x}}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right)$$

$$\beta = y + \frac{\dot{x}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$\therefore C(\alpha, \beta) = C\left(x - \frac{\dot{y}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},\ y + \frac{\dot{x}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\right)$$

~~Evolutes:~~
~~Examples:~~
~~1. Find the coordinates of centre of curvature~~

Evolutes:

As a point P moves along a given curve C_1, the centre of curvature corresponding to P describes another curve C_2. The curve C_2 is known as the evolute of the given curve C_1 and C_1 is known as Involute of C_2. Thus, the locus of the centre of the curvature of a curve is called its evolute and the curve itself is called the involute of its evolute.

Example:

1. Find the coordinates of centre of curvature at the point (x, y) on the parabola $y^2 = 4ax$. Hence obtain the evolute of the parabola.

Solⁿ: consider,

$$y^2 = 4ax$$
$$y = 2\sqrt{a}\sqrt{x}$$
$$y_1 = \cancel{2}\sqrt{a}\cdot\frac{1}{\cancel{2}\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$
$$y_2 = \sqrt{a}\left(-\frac{1}{2}x^{-3/2}\right)$$

coordinate of the centre of curvature are

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$
$$= x - \frac{\frac{\sqrt{\cancel{a}}}{\sqrt{x}}\left(1+\left(\frac{\sqrt{a}}{\sqrt{x}}\right)^2\right)}{-\frac{\sqrt{\cancel{a}}}{2}x^{-3/2}}$$
$$= x + 2\frac{x^{3/2}}{x^{1/2}}\left(\frac{x+a}{x}\right)$$
$$= x + 2x\left(\frac{x+a}{x}\right) = x + 2x + 2a$$
$$\alpha = 3x + 2a \longrightarrow (1)$$
$$\beta = y + \frac{(1+y_1^2)}{y_2}$$
$$= y + \frac{(1+a/x)}{-\frac{\sqrt{a}}{2}x^{-3/2}} = y + \frac{\left(\frac{x+a}{x}\right)}{-\frac{\sqrt{a}}{2}\cdot\frac{1}{x^{3/2}}}$$

$$\beta = y - \frac{2x^{1/2}}{\sqrt{a}}(x+a) \qquad (\because y = 2\sqrt{a}\sqrt{x})$$

$$= 2\sqrt{a}\sqrt{x} - \frac{2\sqrt{x}}{\sqrt{a}}(x+a)$$

$$= 2\sqrt{x}\left(\frac{\cancel{a} - x - \cancel{a}}{\sqrt{a}}\right) = \frac{-2\sqrt{x}.x}{\sqrt{a}}$$

$$\beta = -2a^{-1/2}x^{+3/2} \longrightarrow (2)$$

$$\therefore C(\alpha, \beta) = C\left(3x + 2a, -2a^{-1/2}x^{3/2}\right)$$

from α and β,

$$\alpha = 3x + 2a, \qquad \beta = -2a^{-1/2}x^{3/2}$$

$$3x = \alpha - 2a \qquad x^{3/2} = \frac{-\beta}{2a^{-1/2}}$$

$$x = \frac{\alpha - 2a}{3} \qquad x^{3/2} = -\frac{\beta}{2}a^{1/2} \longrightarrow (4)$$

$$x^{3/2} = \frac{(\alpha - 2a)^{3/2}}{3^{3/2}} \longrightarrow (3)$$

Equating (3) & (4), we have

$$\frac{(\alpha - 2a)^{3/2}}{3^{3/2}} = -\frac{\beta}{2}a^{1/2} \qquad \text{sqn on b.s}$$

$$\frac{(\alpha - 2a)^3}{3^3} = \frac{\beta^2}{4}a$$

$4(\alpha-2a)^3 = 27\beta^2 a$

~~$4(\alpha^3$.~~

$27a\beta^2 = 4(\alpha-2a)^3$

change α and β by x and y resp/;

$27ay^2 = 4(x-2a)^3 \longrightarrow (5)$

(5) is the eqn of evolute for the parabola.

2. Find the coordinates of centre of curvature at the point (x, y) for $y = 2at$ and $x = at^2$

Sol^n: Given $x = at^2$ and $y = 2at$

$\dot{x} = 2at$ $\qquad \dot{y} = 2a$

$\ddot{x} = 2a$ $\qquad \ddot{y} = 0$

The coordinates of centre of curvature is given by

$$C(\alpha, \beta) = C\left(x - \frac{\dot{y}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},\ y + \frac{\dot{x}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\right)$$

consider,

$$\alpha = x - \frac{\dot{y}(\dot{x}^2+\dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \longrightarrow (1)$$

$$\alpha = x - \frac{2a\left((2at)^2 + (2a)^2\right)}{2at(0) - 2a(2a)}$$

$$= x - \frac{2a\left(4a^2t^2 + 4a^2\right)}{-4a^2}$$

$$= x - 2a \cdot \frac{\cancel{4a^2}(t^2+1)}{-\cancel{4a^2}}$$

$$= x + 2at^2 + 2a \qquad (\because x = at^2)$$

$$= at^2 + 2at^2 + 2a$$

$$\alpha = 3at^2 + 2a \longrightarrow (1)$$

$$\beta = y + \frac{\dot{x}(\dot{x}^2 + \dot{y}^2)}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= y + \frac{2at\left((2at)^2 + (2a)^2\right)}{2at(0) - 2a \cdot 2a}$$

$$= y + \frac{2at \cdot \cancel{4a^2}(t^2+1)}{-\cancel{4a^2}}$$

$$= y - 2at^3 - 2at \qquad (\because y = 2at)$$

$$= \cancel{2at} - 2at^3 - \cancel{2at}$$

$$\beta = -2at^3$$

$$C(\alpha, \beta) = (3at^2 + 2a, \; -2at^3)$$

$\alpha = 3at^2 + 2a.$

$3at^2 = \alpha - 2a$

$t^2 = \frac{\alpha - 2a}{3a}$

cubing on b.s

$t^6 = \frac{(\alpha-2a)^3}{(3a)^3} \longrightarrow (3)$

$\beta = -2at^3$

$t^3 = -\frac{\beta}{2a}$

Squring on b.s

$t^6 = \frac{+\beta^2}{(2a)^2} \longrightarrow (4)$

equating (3) and (4), we have

$$\frac{(\alpha-2a)^3}{27a^3} = \frac{\beta^2}{4a^2}$$

$$4(\alpha-2a)^3 = \beta^2 27a$$

$$27a\beta^2 = (4(\alpha-2a)^3)$$

Replace α by x & β by y we get

$$27ay^2 = 4(x-2a)^3$$

3. Find the centre of curvature for $y = \log\sec x$ at $\left(\frac{\pi}{3}, \log 2\right)$

Sol^n: $y = \log \sec x$

$y_1 = \frac{1}{\sec x} \cdot \sec x \cdot \tan x \Rightarrow y_1 = \tan x$

$y_2 = \sec^2 x$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\alpha = x - \tan x$$

at $\left(\frac{\pi}{3}, \log 2\right)$,

$$\alpha = \frac{\pi}{3} - \tan\frac{\pi}{3}$$

$$\alpha = \frac{\pi}{3} - \sqrt{3}$$

$$\beta = y + \frac{(1+y_1^2)}{y_2}$$

$$\beta = y + 1$$

at $\left(\frac{\pi}{3}, \log 2\right)$

$$\beta = \log 2 + 1$$

$$\therefore C(\alpha, \beta) = C\left(\frac{\pi}{3} - \sqrt{3}, \log 2 + 1\right)$$

4. Find the circle of curvature for $xy = a^2$ at (a, a).

Sol^n: $xy = a^2$

$$x \cdot \frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} = -y$$

$$y_1 = \frac{-y}{x}$$

$$y_2 = -\left(\frac{x y_1 - y(1)}{x^2}\right)$$

$$y_2 = \frac{y - x y_1}{x^2}$$

$$y_2 = \frac{y - x\cdot\left(\frac{-y}{x}\right)}{x^2} = \frac{2y}{x^2}$$

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\left(-\frac{\cancel{y}}{\cancel{x}}\right)\left(1+\frac{y^2}{x^2}\right)}{\frac{2\cancel{y}}{\cancel{x^2}}}$$

$$\alpha = x + \frac{(x^2+y^2)x}{2x^2} \quad \text{at } (a,a)$$

$$\alpha = a + \frac{\cancel{(a^2+a^2)} a}{\cancel{2a^2}} = 2a$$

$$\beta = y + \frac{(1+y_1^2)}{y_2}$$

$$\beta = 2a$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \sqrt{2}a$$

Now, The eqn of circle at (α, β) is

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

$$(x-2a)^2 + (y-2a)^2 = (\sqrt{2}a)^2$$

$x^2+y^2-4ax-4ay+6a^2=0$ is the circle of curvature.

Assignment:

1. Find the coordinates of centre of curvature and evolute for the foll:

a) $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

c) $x = a\cos\theta$, $y = b\sin\theta$

d) $x^2 = 4ay$

Singular points:

A point $P(x, y)$ on the curve $f(x, y) = 0$ is said to be singular point if both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at this point.

At a singular point $P(x, y)$ of a curve

(i) The Tangent To The curve at P contains (22)
no point in the neighbourhood of P
of The curve other than P. In this
case P is called an isolated point
or conjugate point.

(ii) The Tangent to the curve at P crosses
the curve at P in This case Point is the
P of ~~inflection~~ inflection

(iii) The curve crosses itself at P then
P is called as multiple point of the
curve.

Double point:
A point through which there pass 2
branches of curve is called a double
point. A curve has 2 Tangents at
a double point one for each branch.
~~Thus, There are 3 different types of
singular points of a curve, namely~~
~~→ Isolated / conjugate point~~
~~→ point of inflection~~
~~→ Multiple point.~~

Classification of double points:
There are 3 types depending on the
nature of the Tangents:

1. When 2 Tangents at the double points are real & distinct then the point is said to be 'Node'.

2. When 2 tangents at the double points are real & coincident then the point is said to be 'cusp'.

3. If there are no other real points of the curve in a nbd of this point and the tangents at a double point are imaginary then the point is called isolated point / Conjugate point.

Nature of double point for the curve $f(x,y)=0$ can be obtained by finding ~~deel~~ determinant.

i.e determinant (Δ) is given by

$$\cancel{\Delta = b^2 - \Delta}$$

$$\Delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial^2 f}{\partial x^2}\right)$$

$$\left[\text{(or) } \Delta = b^2 - 4ac\right]$$

(i) $\Delta > 0$ then point is Node

(ii) $\Delta = 0$ then " " Cusp

(iii) $\Delta < 0$ " " " isolated

Examples:

1. Find the singular point on the curve
$$x^3+x^2+y^2-x-4y+3=0$$

Solⁿ: The given curve is
$$f=x^3+x^2+y^2-x-4y+3=0$$
$$\frac{\partial f}{\partial x}=3x^2+2x-1$$
$$\frac{\partial f}{\partial y}=2y-4$$

The singular points are obtained by solving $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$\therefore\ 3x^2+2x-1=0$ $\qquad$ $2y-4=0$

$x=\frac{1}{3},\ x=-1$ $\qquad$ $y=2$

$\therefore$ The singular points are $\left(\frac{1}{3},2\right)$ & $(-1,2)$

2. Determine the position and nature of the double point of the curve
$$x^3+2x^2+2xy-y^2+5x-2y=0$$

Solⁿ: The given curve is
$$f=x^3+2x^2+2xy-y^2+5x-2y=0$$
$$\frac{\partial f}{\partial x}=3x^2+4x+2y+5$$
$$\frac{\partial f}{\partial y}=2x-2y-2$$

The double points are obtained by solving the equations $\frac{\partial f}{\partial x}=0$ & $\frac{\partial f}{\partial y}=0$

$$2x-\cancel{2y}-2=0$$
$$3x^2+4x+\cancel{2y}+5=0$$
$$\overline{3x^2+6x+3=0} \div 3$$
$$x^2+2x+1=0$$
$$(x+1)^2=0$$
$$\boxed{x=-1,-1}$$

$$2x-2y-2=0$$
$$2(-1)-2y-2=0$$
$$-2y-2=2$$
$$-2y=4$$
$$\boxed{y=-2}$$
$$\boxed{y=-2}$$

$\therefore$ The possible double points are $(-1,-2)$ $(-1,-2)$

To find the nature of double point:

$$\frac{\partial^2 f}{\partial x^2}=6x+4 \quad \text{at } (-1,-2)$$
$$=6(-1)+4$$
$$\frac{\partial^2 f}{\partial x^2}=-2$$

$$\frac{\partial^2 f}{\partial y^2}=-2 \quad \text{at } (-1,-2)$$
$$\frac{\partial^2 f}{\partial y^2}=-2$$
$$\frac{\partial^2 f}{\partial x\partial y}=2$$

$$\Delta=\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2-\left(\frac{\partial^2 f}{\partial x^2}\right)^2\left(\frac{\partial^2 f}{\partial y^2}\right)^2$$
$$=4-4$$
$$\Delta=0$$

$\therefore$ The double point $(-1,-2)$ is a cusp.

Assignment:

1. Determine the position and nature of double points of the curves
   a) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$
   b) $(2y + x + 1)^2 - 4(1 - x)^5 = 0$

2. S.T the curve $y^2 = (x-a)^2(x-b)$ has at $x = a$, an isolated point, if $a < b$, a node, ~~and~~ if $a > b$ and if $a = b$ a cusp.

Asymptotes:

A straight line is said to be an asymptote of an infinite branch of a curve if ~~as~~ the point P recedes to infinity along the branch, the $\perp^r$ distance of P from the st. line ~~tet~~ tends to zero.

Examples:

1. Find the asymptotes $\parallel^{le}$ to the coordinate axes

$$x^2y^2 - a^2x^2 = a^2y^2$$

Sol^n:
$$x^2y^2 - a^2x^2 = a^2y^2$$
$$x^2(y^2 - a^2) = a^2y^2$$

Co-efficient of highest power of x is (y^2-a^2)

Equating this to zero

$$y^2-a^2=0$$
$$y^2=a^2$$
$$y=\pm a$$

are the asymptotes $\parallel$ to x-axis.

Rewrite the eqn as

$$x^2y^2-a^2y^2=a^2x^2$$
$$y^2(x^2-a^2)=a^2x^2$$

Equating to zero the coefficient of highest power of y.

$$x^2-a^2=0$$
$$\Rightarrow x^2=a^2$$
$$\Rightarrow x=\pm a$$

are the asymptotes $\parallel$ to y-axis.

Assignment:

1. Find the asymptotes $\parallel$ to coordinate axes

a) $y^3-x^2y=x^2+1$

b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$