Unit-5 Multivariable Calculus (Differentiation)

Partial differentiation:

It is done on a funⁿ with more than 1 variable i.e on funⁿ of the form $z = f(x, y)$.

The Partial derivatives of 1st order are denoted by $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ or f_x, f_y

The 2nd order partial derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Note:
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Examples:

1. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for the foll:

① $z = x^2 + y^2 - 3xy$

$$\frac{\partial z}{\partial x} = 2x - 3y, \quad \frac{\partial z}{\partial y} = 2y - 3x$$

② $z = \frac{x}{y} + \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{y} - \frac{y}{x^2}, \quad \frac{\partial z}{\partial y} = \frac{-x}{y^2} + \frac{1}{x}$$

③ $z = x^y + y^x$ $\quad \left(\because \frac{d}{dx}(a^x) = a^x \log a\right)$

$$\frac{\partial z}{\partial x} = y x^{y-1} + y^x \log y, \quad \frac{\partial z}{\partial y} = x^y \log x + x y^{x-1}$$

④ $z = \tan^{-1}\left(\frac{y}{x}\right)$

⑤ $z = e^{x/y}$

⑥ $z = x \cdot \cos^{-1}\left(\frac{x}{y}\right)$

⑦ $u = e^{ax} \sin by$

2. Verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ for the above ①–⑥ problems.

If $u = x^2y + y^2z + z^2x$ S.T. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$

Homogeneous fun^n:

A fun^n $z = f(x,y)$ is said to be homogeneous fun^n of ~~order~~ degree 'n' if it can be expressed in the form of $z = x^n f\left(\frac{y}{x}\right)$.

Ex:

1) $f(x,y) = \frac{x^4 - y^4}{x - y}$

$$= \frac{x^{\cancel{4}\,3}\left(1 - \frac{y^4}{x^4}\right)}{\cancel{x}\left(1 - \frac{y}{x}\right)} = \frac{x^3\left(1 - \left(\frac{y}{x}\right)^4\right)}{\left(1 - \frac{y}{x}\right)}$$

$f(x,y) = x^3 f\left(\frac{y}{x}\right)$

$\therefore f(x,y)$ is a homo. fun^n of ~~order~~ degree 3.

② $f(x,y) = \frac{x^3 + y^3}{x - y}$

$$= \frac{x^{\cancel{3}\,2}\left(1 + \left(\frac{y}{x}\right)^3\right)}{\cancel{x}\left(1 - \frac{y}{x}\right)} = \frac{x^2\left(1 + \left(\frac{y}{x}\right)^3\right)}{\left(1 - \frac{y}{x}\right)}$$

$f(x,y) = x^2 f\left(\frac{y}{x}\right) \longrightarrow$ fun^n of ~~oder~~ degree 2.

Euler's theorem for homogeneous fun^n:

statement:

If $u = f(x,y)$ is a homogeneous fun^n of degree 'n' then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Extension of Euler's Theorem: If $u = f(x,y)$ is H.F of deg. 'n' then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

1. Verify Euler's Theorem for $u = x^4 \log\left(\frac{y}{x}\right)$

Given $u = x^4 \log\left(\frac{y}{x}\right) \longrightarrow ① \quad = x^4 f\left(\frac{y}{x}\right)$

$\therefore u$ is a homo. fun^n of order 4.

we have to prove,

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 4u \longrightarrow ②$$

Diff eqn ① partially w.r.t 'x'

$$\frac{\partial u}{\partial x} = x^{\cancel{4}2} \cdot \frac{1}{\frac{y}{x}} \cancel{y}\left(\frac{-1}{x^2}\right) + \log\left(\frac{y}{x}\right) \cdot 4x^3$$

$$\frac{\partial u}{\partial x} = -x^3 + 4x^3 \log\left(\frac{y}{x}\right) \longrightarrow ③$$

Diff eqn ① partially w.r.t 'y'

$$\frac{\partial u}{\partial y} = x^4 \frac{1}{\frac{y}{\cancel{x}}} \cdot \frac{1}{\cancel{x}}(1)$$

$$\frac{\partial u}{\partial y} = \frac{x^4}{y} \longrightarrow ④$$

Substitute ③ and ④ in ② we get

$$x\left(-x^3 + 4x^3 \log\left(\frac{y}{x}\right)\right) + \cancel{y} \cdot \frac{x^4}{\cancel{y}} = 4u$$

$$-\cancel{x^4} + 4x^4 \log\left(\frac{y}{x}\right) + \cancel{x^4} = 4u$$

$$\Rightarrow \quad 4x^4 \log\left(\frac{y}{x}\right) = 4u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 4x^4 \log\left(\frac{y}{x}\right) = 4u.$$

2. Verify E.T for $u = \frac{x^2+y^2}{x-y}$ (Ans)

Given $u = \dfrac{x^2+y^2}{x-y} \longrightarrow ①$

$$u = \frac{x^{\cancel{2}}\left(1 + \left(\frac{y}{x}\right)^2\right)}{\cancel{x}\left(1 - \frac{y}{x}\right)}$$

$$u = \frac{x\left(1 + \frac{y}{x}\right)}{\left(1 - \frac{y}{x}\right)} = x \cdot f\left(\frac{y}{x}\right)$$

$\therefore u$ is a homo. funⁿ of order 1

$\therefore$ we have to prove,

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1u. \longrightarrow (2)$$

Diff (1) partially wrt 'x',

$$\frac{\partial u}{\partial x} = \frac{(x-y)(2x) - (x^2+y^2)(1)}{(x-y)^2} = \frac{2x^2 - 2xy - x^2 - y^2}{(x-y)^2} = \frac{x^2-y^2-2xy}{(x-y)^2} \to (3)$$

Diff (1) partially wrt 'y',

$$\frac{\partial u}{\partial y} = \frac{(x-y)(+2y) - (x^2+y^2)(-1)}{(x-y)^2} = \frac{2xy - 2y^2 + x^2 + y^2}{(x-y)^2}$$

$$= \frac{x^2 - y^2 + 2xy}{(x-y)^2} \longrightarrow (4)$$

Substitute (3) & (4) in (2) we get

$$x\left(\frac{x^2-y^2-2xy}{(x-y)^2}\right) + y\left(\frac{x^2-y^2+2xy}{(x-y)^2}\right) = 1u$$

$$\frac{x^3 - xy^2 - 2x^2y + x^2y - y^3 + 2xy^2}{(x-y)^2} = 1u$$

$$\frac{x^3 - y^3 + xy^2 - x^2y}{(x-y)^2} = 1u$$

$$\frac{x^3 + xy^2 - y^3 - x^2y}{(x-y)^2} = 1u$$

$$\frac{x(x^2+y^2) - y(y^2+x^2)}{(x-y)^2} = 1u$$

$$\frac{(x^2+y^2)\cancel{(x-y)}}{(x-y)^{\cancel{2}}} = 1u$$

$$\frac{x^2+y^2}{x-y} = 1u.$$

Hence verified.

3. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ S.T $x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u.$

Solⁿ: Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}$$

let $z = \tan u$

$$= \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = \frac{x^2\left(1+\frac{y^3}{x^3}\right)}{\left(1-\frac{y}{x}\right)}$$

$$z = x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ Z is homo. funⁿ of order 2.

$\therefore$ According to Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$x.\sec^2 u.\frac{\partial u}{\partial x} + y.\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2.\frac{\sin u}{\cancel{\cos u}}.\cos^{\cancel{2}} u$$

$$= 2\sin u.\cos u$$

$$\therefore x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

4. If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ S.T $x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u.$

5. If $u = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$ S.T $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0$

6. Verify E.T for The funⁿ

$$z = x^3 - 2x^2y + 3xy^2 + y^3$$

7. If $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

8. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$

Total differential and total derivative:

① let $z = f(x, y)$ then total differential of z denoted by dz defined as

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

let $u = f(x, y, z)$ then total differential

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

② let $z = f(x, y)$ and $x = f_1(t)$ & $y = f_2(t)$ then total derivative of z denoted by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

let $u = f(x, y, z)$ then Total derivative of u with $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

③ let $z = f(x, y)$ and $x = f_1(u, v)$ & $y = f_2(u, v)$ Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Examples:

1. Find the total differential of the foll fun^n:

1. $z = \log(x^2 + y^2)$

Sol^n: The total differential is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{1}{x^2+y^2} \cdot 2x \cdot dx + \frac{1}{x^2+y^2} \, 2y \, dy$$

$$dz = \frac{2}{x^2+y^2}(x \cdot dx + y \cdot dy)$$

2. $z = x^3 y + 2xy^3$

Sol^n: Total differential is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (3x^2 y + 2y^3)\, dx + (x^3 + 2x \cdot 3y^2)\, dy$$

$$dz = (3x^2 y + 2y^3)\, dx + (x^3 + 6xy^2)\, dy$$

3. $f(x, y, z) = \log(x + y + z)$

Sol^n: $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \frac{1}{x+y+z}(dx + dy + dz)$$

Find $\frac{df}{dt}$ by using partial differentiation for the foll fun^n:

(i) $f(x, y, z) = \log(x^2 + y^2 + z^2)$ where $x = e^t$, $y = \sin t$, $z = \cos t$ at $t = 0$.

(ii) $f(x, y) = x^2 + 3xy + y^2$ where $x = 2at$, $y = at^2$ (Ans: $8a^2 t + 18a^2 t^2 + 4a^2 t^3$)

Sol^n

(i) Given $f(x, y, z) = \log(x^2 + y^2 + z^2)$, $z = \cos t$ and $x = e^t$, $y = \sin t$,

Total derivative,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{1}{x^2+y^2+z^2} \cdot 2x \cdot e^t + \frac{1}{x^2+y^2+z^2} \cdot 2y \cdot \cos t + \frac{1}{x^2+y^2+z^2} \cdot 2z(-\sin t)$$

$$= \frac{2x \cdot e^t}{x^2+y^2+z^2} + \frac{2y \cos t}{x^2+y^2+z^2} - \frac{2z \sin t}{x^2+y^2+z^2}$$

$$= \frac{1}{e^{2t} + \sin^2 t + \cos^2 t}\left(2e^t \cdot e^t + \cancel{2\sin t \cdot \cos t} - \cancel{2\cos t \cdot \sin t}\right)$$

$$= \frac{2e^{2t}}{e^{2t} + 1}$$

at $t = 0$, $= \frac{2e^0}{e^0 + 1} = \frac{2}{1+1} = \frac{2}{2} = 1.$

1. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, S.T
$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$$

Sol: Given $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

put $X = 2x-3y$, $Y = 3y-4z$, $Z = 4z-2x$

$\therefore u = f(X, Y, Z)$

~~Total derivatives is are~~

Partial derivatives are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z} \longrightarrow (3)$$

Eqn (1), (2), (3) will be

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}(2) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}(-3) + \frac{\partial u}{\partial Y}(3) + \frac{\partial u}{\partial Z}(0)$$

$$\frac{\partial u}{\partial Z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-4) + \frac{\partial u}{\partial Z}(4)$$

consider, LHS

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z}$$

$$= 6\left(2\cdot\frac{\partial u}{\partial X} - 2\frac{\partial u}{\partial Z}\right) + 4\left(-3\frac{\partial u}{\partial X} + 3\frac{\partial u}{\partial Y}\right) + 3\left(-4\frac{\partial u}{\partial Y} + 4\frac{\partial u}{\partial Z}\right)$$

$$= \cancel{12\frac{\partial u}{\partial X}} - \cancel{12\frac{\partial u}{\partial Z}} - \cancel{12\frac{\partial u}{\partial X}} + \cancel{12\frac{\partial u}{\partial Y}} - \cancel{12\frac{\partial u}{\partial Y}} + \cancel{12\frac{\partial u}{\partial Z}}$$

$= 0 = RHS.$

Hence proved.

2. If $u = f(x-y, y-z, z-x)$, S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

3. If $u = f(x, z, y/z)$, S.T $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0$

Jacobians : is a functional determinant.

Def^n:
let u & v are fun's of 2 independent variables x & y with continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

Then the 2^{nd} order determinant

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called as Jacobian of u & v w.r.t x & y

It is denoted by $J\frac{(u, v)}{(x, y)}$ or $\frac{\partial(u, v)}{\partial(x, y)}$

i.e $J\frac{(u, v)}{(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$

Examples:

1. If $u = 3x + 5y$, $v = 4x - 3y$ S.T $\frac{\partial(u, v)}{\partial(x, y)} = -29$

Sol^n: consider,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix} = -9 - 20$$

$$\frac{\partial(u, v)}{\partial(x, y)} = -29.$$

2. If $u = 2x - 3y$, $v = 5x + 4y$, S.T $\frac{\partial(u, v)}{\partial(x, y)} = 23$

3. If $x = r\cos\theta$, $y = r\sin\theta$, S.T $\frac{\partial(x, y)}{\partial(r, \theta)} = r$.

4. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^3\sin 2\theta$.

chain rule of Jacobian

composite funⁿ: If u, v are funⁿ of x and y and x, y are funⁿ of r & θ. Then u, v are connected to r, θ and u, v are called composite funⁿ of r, θ.

Acc. to chain rule of Jacobian,
if J_1 is a Jacobian of funⁿ of u & v w.r.t x & y and J_2 is a Jacobian of funⁿ of x & y w.r.t u, v Then $J_1 \cdot J_2 = 1$

Example:
suppose $x = r\cos\theta \longrightarrow ①$
$y = r\sin\theta \longrightarrow ②$

~~adding~~ squaring & adding eqn ① & ②,
$x^2+y^2 = r^2 \longrightarrow ③$

divide ② by ①
$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow ④$$

Now, acc to chain rule of Jacobian,
if $J_1(r,\theta)$ w.r.t x, y and $J_2(x,y)$ w.r.t r, θ then $J_1 \times J_2 = 1$

i.e $\frac{\partial(r,\theta)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 1$

Now,
$$J_1 = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}\right]\frac{1}{r} = \frac{1}{r} = J_1$$

(from ③ & ④)
$r^2 = x^2+y^2$
$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$
$\frac{\partial r}{\partial y} = \frac{y}{r}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot y \cdot \frac{-1}{x^2}$
$= \frac{x^2}{x^2+y^2} \times \frac{y}{x^2}$
$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$,

$\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$

$$J_2 = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

(from ① & ②)

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$J_2 = r$$

Now,

$$J_1 \times J_2 = \frac{1}{r} \times r = 1$$

$$J_1 \times J_2 = 1$$

Hence proved.

1. If $x = uv$, $y = \frac{u}{v}$ Then P.T $JJ' = 1$

sol^n: let $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

Given

$x = uv \longrightarrow ①$

$y = \frac{u}{v} \longrightarrow ②$

multiply ① and ②,

$xy = u \cdot v \cdot \frac{u}{v}$

$xy = u^2 \Rightarrow u = \sqrt{xy} \longrightarrow ③$

divide ① and ②

$\frac{x}{y} = \frac{uv}{\frac{u}{v}}$

$\frac{x}{y} = v^2 \Rightarrow v = \sqrt{\frac{x}{y}} \longrightarrow ④$

consider,

$$J = J_1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

(from ③ & ④)

$$= \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} \\ \frac{1}{2}\frac{1}{\sqrt{xy}} & \frac{-1}{2y}\sqrt{\frac{x}{y}} \end{vmatrix}$$

$$J_1 = -\frac{1}{4y} - \frac{1}{4y} = \frac{-2}{4y} = \frac{-1}{2y}$$

$$J' = J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad (\text{from } ① \text{ \& } ②)$$

$$= \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$J_2 = -\frac{u}{v} - \frac{u}{v} = \frac{-2u}{v}$$

Taking product of J_1 and J_2

$$\Rightarrow J_1 \times J_2 = \frac{-1}{\cancel{2}y} \times \frac{-\cancel{2}u}{v} \quad (\because y = \frac{u}{v})$$

$$= \frac{\cancel{v}}{u} \times \frac{\cancel{u}}{\cancel{v}}$$

$$J_1 \times J_2 = 1$$

$$\therefore JJ' = 1$$

Hence proved.

2. If $x = u(1-v)$, $y = uv$ P.T $JJ' = 1$

3. If $u = x+y$, $v = x-y$ P.T $JJ' = 1$

4. If $x = e^u \cos v$, $y = e^u \sin v$ P.T $JJ' = 1$

5. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, S.T

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$$

6. If $x = r\sin\theta.\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

S.T $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$.

PARTIAL DIFFERENTIATION

Functions of two or more variables

If $z = f(x, y)$ is called a function of two variables, x and y are Independent variables and z is dependent variable

Partial derivatives

Let $z = f(x,y)$ be a function of two variables x and y.

The first order partial derivative of z wrt x, denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or z_x or f_x is defined as

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

From the above definition, we understand that $\frac{\partial z}{\partial x}$ is the ordinary derivative of z w.r.t. x, treating y as constant.

The first order partial derivative of z w.r.t y, denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or z_y or f_y is defined as

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

From the above definition, we understand that $\frac{\partial z}{\partial y}$ is the ordinary derivative of z w.r.t y, treating x as constant

The partial derivatives $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or z_{xx} or f_{xx}

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } z_{yy} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } z_{yx} \text{ or } f_{yx}$$

and $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or z_{xy} or f_{xy}

are known as second order partial derivatives.

In all ordinary cases, It can be verified that (2)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

The third and higher order partial derivative of $f(x,y)$ are defined in an analogous way. Also, the second and higher order partial derivatives of more than two independent variables are defined similarly.

A note on rules of partial differentiation:

All the rules of differentiation applicable to functions of a single independent variable are applicable for partial differentiation also; the only difference is that while differentiating partially w.r.t one independent variable all other independent variables are treated as constants.

(1) If $u = x^y$, then show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Soln we have $u = x^y$, Taking log on b.s we get

$$\Rightarrow \log u = \log x^y$$

$$\Rightarrow \log u = y \log x \rightarrow (*)$$

Differentiating (*) partially w.r.t x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{y}{x}$$

$$\therefore \frac{\partial u}{\partial x} = u\frac{y}{x} = x^y \frac{y}{x} = y x^{y-1}$$

$\frac{d}{dx}(a^x) = a^x \log a$

Differentiating partially w.r.t y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = y.\, x^{y-1}\log x + x^{y-1}.1$$

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1}[1 + y\log x].$$

$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = x^{y-1}[y\log x + 1] \rightarrow (1)$$

Differentiating partially w.r.t y, we get

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$\frac{1}{u}\frac{\partial u}{\partial y} = \log x$

$\frac{\partial u}{\partial y} = u \log x = x^y \log x$

differentiating partially wrt x, we get

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = y\, x^{y-1} \log x + x^y \frac{1}{x}$

$\frac{\partial^2 u}{\partial x \partial y} = y\, x^{y-1} \log x + x^{y-1}$

$u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = x^{y-1}\left[y \log x + 1\right] \rightarrow ②$

From ① & ② $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \underline{u_{xy} = u_{yx}}$

<u>or</u>

$u = x^y$ $\left[\frac{d}{dx} a^x = a^x \log a\right]$

$\therefore \frac{\partial u}{\partial x} = y\, x^{y-1}$ ... Treating y as a constant $\left[\frac{d}{dx} x^n = n \cdot x^{n-1}\right]$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1} \cdot 1 + y\, x^{y-1} \log x$

$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1}\left[1 + y \log x\right] \rightarrow ①$

$\frac{\partial u}{\partial y} = x^y \log x$ Treating x as a constant.

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = y \cdot x^{y-1} \log x + x^y \frac{1}{x}$

$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1}\left[1 + y \log x\right] \rightarrow ②$

From ① & ②,

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

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2. If $U = e^x(x\cos y - y\sin y)$, then show that (4)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Soln

$$\therefore \quad U = e^x(x\cos y - y\sin y) \rightarrow (1)$$

differentiate (1) partially w.r.t x, we get

$$\frac{\partial U}{\partial x} = e^x[\cos y - 0] = e^x\cos y + (x\cos y - y\sin y)e^x$$
$$+ (x\cos y - y\sin y)e^x$$

$$\frac{\partial U}{\partial x} = e^x[\cos y + x\cos y - y\sin y]$$

differentiate again partially w.r.t x, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} = e^x[\cos y] + (\cos y + x\cos y - y\sin y)e^x$$
$$= e^x\cos y + e^x\cos y + x\cos y\, e^x - y\sin y\, e^x$$

$$\frac{\partial^2 U}{\partial x^2} = e^x[\cos y + \cos y + x\cos y - y\sin y]$$
$$\frac{\partial^2 U}{\partial x^2} = e^x[2\cos y + x\cos y - y\sin y] \rightarrow (2)$$

differentiate (1) partially w.r.t y, we get

$$\frac{\partial U}{\partial y} = e^x(x(-\sin y) - (y\cos y + \sin y)) + 0$$
$$\frac{\partial U}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

differentiate again partially w.r.t y, we get

$$\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) = e^x(-x\cos y - [y(-\sin y) + \cos y(1)] - \cos y]$$
$$= e^x(-x\cos y + y\sin y - \cos y - \cos y)$$

$$\frac{\partial^2 U}{\partial y^2} = e^x(-2\cos y - x\cos y + y\sin y) \rightarrow (3)$$

Adding (2) & (3), we get

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = e^x[2\cos y + x\cos y - y\sin y] +$$
$$e^x[-2\cos y - x\cos y + y\sin y]$$
$$= e^x(\cancel{2\cos y} + \cancel{x\cos y} - \cancel{y\sin y} - \cancel{2\cos y} - \cancel{x\cos y} + \cancel{y\sin y})$$
$$= 0$$

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(3) If. $u = e^x(x\cos y - y\sin y)$; show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. (5)

Soln
$$\frac{\partial u}{\partial x} = e^x(\cos y) + (x\cos y - y\sin y)e^x \to (1)$$
$$\frac{\partial u}{\partial y} = e^x\left(-x\sin y - (y\cos y + \sin y)\right] + 0$$
$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) \to (2)$$

differentiate (1) partially wrt y; we get
$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -e^x\sin y + e^x\left(-x\sin y - (y\cos y + \sin y \cdot 1)\right) + 0$$
$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = -e^x\sin y - e^x x\sin y - e^x y\cos y - e^x\sin y$$
$$u_{yx} = \frac{\partial^2 u}{\partial y \partial x} = -e^x(2\sin y + x\sin y + y\cos y) \to (3)$$

differentiate (2) partially wrt x, we get
$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = e^x(-\sin y) + (-x\sin y - y\cos y - \sin y)e^x$$
$$= -e^x\sin y - x\sin y e^x - y\cos y e^x - \sin y e^x$$
$$\frac{\partial^2 u}{\partial x \partial y} = -e^x(2\sin y + x\sin y + y\cos y) \to (4)$$

From (3) & (4), we have
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

(4) If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then show that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Soln
$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \to (1)$$

differentiate (1) partially wrt x, we get
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y} + \frac{1}{1+(y/x)^2}\, y\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y} - \frac{1}{1+\frac{y^2}{x^2}}\, \frac{y}{x^2}$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{\cancel{y}}} \cdot \frac{1}{\cancel{y}} - \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \frac{y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \rightarrow ②$$

differentiate partially ① w.r.t y, we get

$$\frac{\partial u}{\partial y} = \sqrt{\frac{1}{1-\left(\frac{x}{y}\right)^2}}\; x\cdot\left(-\frac{1}{y^2}\right) + \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}$$

$$= -\sqrt{\frac{1}{\frac{y^2-x^2}{y^2}}}\;\frac{x}{y^2} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}$$

$$= -\frac{1}{\frac{\sqrt{y^2-x^2}}{\cancel{y}}}\cdot\frac{x}{y^{\cancel{2}}} + \frac{1}{\frac{x^2+y^2}{x^{\cancel{2}}}}\cdot\frac{1}{\cancel{x}}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \rightarrow ③$$

adding ② & ③, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

⑤ If $u = x^2y + y^2z + z^2x$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x+y+z)^2$$

Soln $\quad u = x^2y + y^2z + z^2x$

$$\frac{\partial u}{\partial x} = 2xy + z^2, \quad \frac{\partial u}{\partial y} = x^2 + 2yz, \quad \frac{\partial u}{\partial z} = y^2 + 2zx$$

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$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = 2xy+z^2+x^2+2yz+y^2+2zx$$
$$= (x+y+z)^2$$

6) If $z=f(y/x)$, then show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$

soln
$$z=f(y/x)$$
$$\frac{\partial z}{\partial x} = f'(y/x)\cdot y\left(-\frac{1}{x^2}\right)$$
$$x\frac{\partial z}{\partial x} = -\frac{y}{x}f'(y/x) \rightarrow ①$$
$$\frac{\partial z}{\partial y} = f'(y/x)\cdot\frac{1}{x}$$
$$y\frac{\partial z}{\partial y} = f'(y/x)\frac{y}{x} \rightarrow ②$$
From ① & ②,
$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \cancel{-\frac{y}{x}f'(y/x)}+\cancel{\frac{y}{x}f'(y/x)}=0$$

7) If $u=f(x+ay)+g(x-ay)$, then show that
$$\frac{\partial^2 u}{\partial y^2}=a^2\frac{\partial^2 u}{\partial x^2}.$$

soln
$$u=f(x+ay)+g(x-ay)$$
$$u_x=\frac{\partial u}{\partial x} = f'(x+ay).1+g'(x-ay).1$$
$$u_{xx}=\frac{\partial^2 u}{\partial x^2} = f''(x+ay).1+g''(x-ay).1 \rightarrow ①$$
$$\frac{\partial u}{\partial y}=u_y= f'(x+ay).a+g'(x-ay)(-a)$$
$$\frac{\partial^2 u}{\partial y^2}=u_{yy} = f''(x+ay).a^2+g''(x-ay)a^2 \rightarrow ②$$
From ① & ②, we have
$$\frac{\partial^2 u}{\partial y^2} = [f''(x+ay)+g''(x-ay)]a^2$$
$$= a^2\frac{\partial^2 u}{\partial x^2}.$$
$$\therefore \frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad //$$

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Homogeneous function

A function $u = f(x, y)$ is said to be a homogeneous function of degree n if it can be expressed in the form $x^n g(y/x)$ or $y^n g(x/y)$, g being any arbitrary function.

Or

if it can be expressed in the form $x^n \phi(y/x)$ or $y^n \phi(x/y)$, ϕ being any arbitrary function.

ex :- ① $u = x^2 + y^2 + 3xy$

$$x^2\left(1 + (y/x)^2 + 3(y/x)\right) = x^2 \phi(y/x) \Rightarrow \text{Homogeneous function of degree } 2$$

② $u = \dfrac{x^4 + y^4}{x+y} \Rightarrow u = \dfrac{x^{\cancel{4}\,3}\left(1 + (y/x)^4\right)}{\cancel{x}\left(1 + y/x\right)} = x^3 \phi(y/x)$

$\Rightarrow$ u is homogeneous function of degree 3

③ $u = \dfrac{x^2 + y^2}{\sqrt{x+y}} \Rightarrow \dfrac{x^2\left(1 + (y/x)^2\right)}{x^{1/2}\sqrt{1 + y/x}} = x^{2 - 1/2}\left[\dfrac{\left(1 + (y/x)^2\right)}{\sqrt{1 + y/x}}\right]$

$$= x^{3/2}\,\frac{\left(1 + (y/x)^2\right)}{\sqrt{1 + y/x}}$$

$$= x^{3/2}\phi(y/x)$$

$\Rightarrow$ u is a Homogeneous function of degree 3/2.

④ $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x+y}\right)$ Here u is not a homogeneous function

$\therefore z = \sin u = \dfrac{x^2 + y^2}{x+y} = \dfrac{x^{\cancel{2}}\left(1 + (y/x)^2\right)}{\cancel{x}\left(1 + y/x\right)} = x\phi(y/x)$

z is a homogeneous function of degree 1

⑤ $u = \log\left(\dfrac{x^3 + y^3}{x+y}\right)$ Here u is not a homogeneous function

$\therefore z = e^u = \dfrac{x^3 + y^3}{x+y} = \dfrac{x^{\cancel{3}}\left(1 + (y/x)^3\right)}{\cancel{x}\left(1 + y/x\right)} = x^2\phi(y/x)$

z is a homogeneous function of degree 2.

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Euler's theorem

Statement : If $u = f(x, y)$ is a homogeneous function of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Verify Euler's theorem for the following functions.

1. $u = x^3 + y^3 + 3x^2y + 2y^2x$

soln $u = x^3 + y^3 + 3x^2y + 2y^2x$

$$u = x^3\left[1 + (y/x)^3 + 3(y/x) + 2(y/x)^2\right] = x^3\phi(y/x)$$

$\Rightarrow$ u is homogeneous of degree 3 $\therefore n = 3$

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy + 2y^2$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3x^2 + 4yx$$

We have Euler's theorem : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

$$L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(3x^2 + 6xy + 2y^2) + y(3y^2 + 3x^2 + 4yx)$$

$$= 3x^3 + 6x^2y + 2y^2x + 3y^3 + 3x^2y + 4y^2x$$

$$= 3x^3 + 9x^2y + 6y^2x + 3y^3$$

$$= 3(x^3 + 3x^2y + 2y^2x + y^3)$$

$$= 3(x^3 + y^3 + 3x^2y + 2y^2x)$$

$$= 3u = R.HS = nu = 3u \quad \text{since } n = 3$$

Hence Euler's theorem is verified.

2. $z = x^3 - 2x^2y + 3xy^2 + y^3$.

soln $z = x^3 - 2x^2y + 3xy^2 + y^3$

$$z = x^3\left[1 - 2(y/x) + 3(y/x)^2 + (y/x)^3\right] = x^3\phi(y/x)$$

$\Rightarrow$ z is homogeneous of degree 3 $\therefore n = 3$

$$\frac{\partial z}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$\frac{\partial z}{\partial y} = -2x^2 + 6xy + 3y^2$$

We have Euler's theorem: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ (10)

$$LHS = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x(3x^2 - 4xy + 3y^2) + y(-2x^2 + 6xy + 3y^2)$$

$$= 3x^3 - 4x^2y + 3xy^2 - 2yx^2 + 6xy^2 + 3y^3$$

$$= 3x^3 + 3y^3 - 6x^2y + 9xy^2$$

$$= 3(x^3 + y^3 - 2x^2y + 3xy^2)$$

$$= 3z \quad \text{since } n = 3$$

$$= RHS$$

3. $u = y^n \log(x/y)$

This is a homogeneous function of degree n by the definition

$$\therefore \frac{\partial u}{\partial x} = y^n \cdot \frac{1}{(x/y)} \cdot \frac{1}{y} = \frac{y^n}{x} \quad -(1)$$

$$\frac{\partial u}{\partial y} = y^n \cdot \frac{1}{(\cancel{x}/y)}\left(-\frac{\cancel{x}}{y^2}\right) + ny^{n-1}\log(x/y)$$

i.e., $\frac{\partial u}{\partial y} = -y^{n-1} + ny^{n-1}\log(x/y) \rightarrow (2)$

From (1) and (2), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{x}\frac{y^n}{\cancel{x}} + y(-y^{n-1} + ny^{n-1}\log(x/y))$$

$$= \cancel{y^n} - \cancel{y^n} + ny^n\log(x/y)$$

$$= ny^n\log(x/y) = nu$$

4. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Soln $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ by data

$$\sin u = \frac{x^2+y^2}{x+y} = \frac{x^{\cancel{2}}(1 + (y/x)^2)}{\cancel{x}(1 + y/x)} = x\left\{\frac{1 + (y/x)^2}{1 + y/x}\right\}$$

i.e., $\sin u = x\,\phi(y/x)$

$\Rightarrow$ $\sin u$ is homogeneous function of degree 1.

Apply Euler's theorem for the function $\sin u$ we have,

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$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = n\cdot\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = n\sin u \div \cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\sin u}{\cos u} = n\tan u \; ; \; n=1$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

5. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Soln $\quad u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ by data.

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} = \frac{x^{\cancel{3}2}\left(1+(y/x)^3\right)}{\cancel{x}\left(1-y/x\right)} = x^2\,\phi(y/x)$$

$\Rightarrow$ $\tan u$ is homogeneous function of degree 2.

Applying Euler's theorem for the function $\tan u$ we have,

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = n\cdot\tan u; \; n=2$$

i.e., $$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u} = 2\frac{\frac{\sin u}{\cos u}}{1/\cos^2 u}$$

$$= 2\frac{\sin u}{\cancel{\cos u}} \times \cos^{\cancel{2}} u = 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

6. If $u = \cot^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin 2u$$

Soln $\quad u = \cot^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ by data

$$\Rightarrow \quad \cot u = \frac{x^3+y^3}{x+y} = \frac{x^{\cancel{3}2}(1+(y/x)^3)}{x(1+y/x)} = x^2\phi(y/x) \quad (12)$$

$\Rightarrow$ $\cot u$ is homogeneous function of degree 2.

Applying Euler's theorem for the function $\cot u$ we have

$$x\frac{\partial}{\partial x}\cot u + y\frac{\partial}{\partial y}\cot u = n.\cot u \quad n=2$$

$$-x\,\mathrm{cosec}^2 u\frac{\partial u}{\partial x} - \mathrm{cosec}^2 u\, y\frac{\partial u}{\partial y} = 2\cot u \div -\mathrm{cosec}^2 u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\cot u}{-\mathrm{cosec}^2 u} = \frac{2\frac{\cos u}{\sin \cancel{u}}}{-\frac{1}{\sin^{\cancel{2}} u}}$$

$$= \frac{2\cos u}{-1/\sin u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2\cos u \times \sin u = -\sin 2u$$

7. If $u = \sin^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$

Soln $\quad u = \sin^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ by data

$$\Rightarrow \quad \sin u = \frac{x^3+y^3}{x+y} = \frac{x^{\cancel{3}2}(1+(y/x)^3)}{\cancel{x}(1+y/x)} = x^2\phi(y/x)$$

$\Rightarrow$ $\sin u$ is homogeneous function of degree 2

Apply Euler's theorem for the function $\sin u$ we have,

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = n\cdot\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = n\sin u \div \cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\tan u \quad ; \quad n=2$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u.$$

(8) If $U = \log\left(\frac{x^3+y^3}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

Soln We cannot put the given U in the form $x^n\phi(y/x)$

$$\therefore \quad e^U = \frac{x^3+y^3}{x+y} = \frac{x^3(1+(y/x)^3)}{x(1+y/x)} = x^2\left\{\frac{1+(y/x)^3}{1+y/x}\right\}$$

i.e. $e^U = x^2\phi(y/x)$

$\Rightarrow$ e^U is homogeneous function of degree 2

$\therefore$ $n = 2$

Now applying Euler's theorem for the homogeneous function e^U we have

$$x\frac{\partial}{\partial x}(e^U) + y\frac{\partial}{\partial y}(e^U) = ne^U$$

i.e., $x e^U\frac{\partial u}{\partial x} + y e^U\frac{\partial u}{\partial y} = 2e^U \div e^U$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$$

(9) If $U = \log_e\left(\frac{x^4+y^4}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

Soln. We cannot put the given U in the form $x^n\phi(y/x)$

$$\therefore \quad e^U = \frac{x^4+y^4}{x+y} = \frac{x^{\cancel{4}3}(1+y^4/x^4)}{\cancel{x}(1+y/x)} = x^3\left(\frac{1+(y/x)^4}{1+y/x}\right)$$

i.e., $e^U = x^3 g(y/x) \Rightarrow e^U$ is homogeneous of degree 3 $\therefore$ $n=3$

Now applying Euler's theorem for the homogeneous function e^U we have

$$x\frac{\partial}{\partial x}(e^U) + y\frac{\partial}{\partial y}e^U = ne^U$$

i.e., $xe^U\frac{\partial u}{\partial x} + ye^U\frac{\partial u}{\partial y} = 3e^U \div e^U$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

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TOTAL DIFFERENTIATION AND TOTAL DERIVATIVE

If $U = f(x,y)$ then the total differential or the exact differential of U is defined as

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy. \quad \text{—(1)}$$

Further, if $U = f(x,y)$ where $x(t)$, $y = y(t)$ i.e., x and y are themselves functions of an independent variable t, then total derivative of U is given by

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} \quad \text{— (2)}$$

Similarly, the total differential of a function $U = f(x,y,z)$ is defined by

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \text{—(3)}$$

Further, if $u = f(x,y,z)$ and if $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of U is given by

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}. \quad \text{—(4)}$$

Find the total differential of the following functions.

1. $f = x\cos y - y\cos x$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= \frac{\partial}{\partial x}(x\cos y - y\cos x)dx + \frac{\partial}{\partial y}(x\cos y - y\cos x)dy$$

$$df = (\cos y + y\sin x)dx + (-x\sin y - \cos x)dy$$

2. $f(x,y,z) = e^{xyz}$

soln $f(x,y,z) = e^{xyz}$

$$\frac{\partial f}{\partial x} = e^{xyz}(yz), \quad \frac{\partial f}{\partial y} = e^{xyz}.(xz),$$

$$\frac{\partial f}{\partial z} = e^{xyz}(xy).$$

$\therefore$ Total differential of $f(x,y,z)$ is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
$$= e^{xyz}(yz)\,dx + e^{xyz}(xz)dy + e^{xyz}(xy)\,dz$$
$$df = e^{xyz}\,(yz\,dx + xz\,dy + xy\,dz)$$

③ Find $\frac{dz}{dt}$ if

(i) $z = xy^2 + x^2y$, where $x = at^2$, $y = 2at$

(ii) $z = \tan^{-1}(y/x)$, where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$.

Soln :(i) consider $z = xy^2 + x^2y$

$$\frac{\partial z}{\partial x} = y^2 + 2xy \;\&\; \frac{\partial z}{\partial y} = 2xy + x^2$$

Since $x = at^2$ & $y = 2at$, we have

$$\frac{dx}{dt} = 2at \;,\; \frac{dy}{dt} = 2a$$

Hence
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2xy + y^2)\,2at + (2xy + x^2)\,2a$$
$$= (y^2 + 2xy)\,y + (2xy + x^2)2a$$
$$= y^3 + 2xy^2 + 2xy \times 2a + 2ax^2$$
$$\frac{dz}{dt} = y^3 + 2xy^2 + 4axy + 2ax^2$$

To get $\left(\frac{dz}{dt}\right)$ explicitly in terms of t,

we substitute $x = at^2$ & $y = 2at$, ∴ we get

$$\frac{dz}{dt} = (2at)^3 + 2\,(at^2)\,(2at)^2 + 4a \times 2at \times at^2 + 2a(at^2)^2$$
$$= 8a^3t^3 + 2at^2 \times 4a^2t^2 + 8a^3t^3 + 2a\,(a^2t^4)$$
$$= 16a^3t^3 + 8a^3t^4 + 2a^3a^4$$
$$= 16a^3t^3 + 10a^3t^4$$
$$= 2a^3t^3\,(8 + 5t) \text{ or } 2a^3\,(8t^3 + 5t^4)$$

(ii) $z = \tan^{-1}(y/x)$, where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

Soln:- $\frac{\partial z}{\partial x} = \frac{-1}{1+(y/x)^2} \cdot \frac{y}{x^2} = \frac{-y}{\frac{x^2+y^2}{\cancel{x^2}}} \frac{1}{\cancel{x^2}} = \frac{-y}{x^2+y^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}$.

Since $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$, we have

$\frac{dx}{dt} = e^t + e^{-t} = y$, $\frac{dy}{dt} = e^t - e^{-t} = x$.

Hence $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= \frac{-y}{x^2+y^2} y + \frac{x}{x^2+y^2} x$

$= \frac{x^2-y^2}{x^2+y^2}$.

Substituting $x = e^t - e^{-t}$ & $y = e^t + e^{-t}$, we get

$\frac{dz}{dt} = \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$

$= \frac{e^{2t} + e^{-2t} - 2e^t e^{-t} - (e^{2t} + e^{-2t} + 2e^t e^{-t})}{e^{2t} + e^{-2t} - \cancel{2e^t e^{-t}} + (e^t)^2 + (e^{-t})^2 + \cancel{2e^t e^{-t}}}$

$= \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} - 2e^t e^{-t} - \cancel{e^{2t}} - \cancel{e^{-2t}} - 2e^t e^{-t}}{2e^{2t} + 2e^{-2t}}$

$= \frac{-\cancel{4}\, 2}{\cancel{2}(e^{2t} + e^{-2t})}$

$\therefore \frac{dz}{dt} = \frac{-2}{e^{2t} + e^{-2t}}$ //

4. $e^x [x\sin y + y\cos y]$

Soln:- Let $z = f(x,y) = e^x[x\sin y + y\cos y]$ then

$$\frac{\partial z}{\partial x} = e^x[\sin y] + (x\sin y + y\cos y)e^x$$

$$\frac{\partial z}{\partial x} = e^x((1+x)\sin y + y\cos y)$$

and $$\frac{\partial z}{\partial y} = e^x(x\cos y + y(-\sin y) + \cos y)$$

$$\frac{\partial z}{\partial y} = e^x((1+x)\cos y - y\sin y)$$

Hence $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

i.e., $$dz = e^x[(1+x)\sin y + y\cos y]dx + e^x[(1+x)\cos y - y\sin y]dy$$

5. If $Z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = 2t$, $y = 1-t^2$, show that $\frac{dz}{dt} = \frac{2}{1+t^2}$.

Soln:- $z = \tan^{-1}(x/y)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+x^2/y^2}\cdot\frac{1}{y} = \frac{1}{\frac{y^2+x^2}{\cancel{y^2}}}\frac{1}{\cancel{y}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\,x\left(-\frac{1}{y^2}\right) = \frac{-x}{\frac{y^2+x^2}{\cancel{y^2}}}\cdot\cancel{y^2} = \frac{-x}{x^2+y^2}$$

Since $x = 2t$, $y = 1-t^2$, we have

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = -2t$$

Hence $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

$$= \frac{y}{x^2+y^2}2 - \frac{x}{x^2+y^2}(-2t)$$

$$\frac{dz}{dt} = \frac{2y}{x^2+y^2} + \frac{2xt}{x^2+y^2} = \frac{2y+x^2}{x^2+y^2}$$

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$$\frac{dz}{dt} = \frac{2(1-t^2)+(2t)^2}{(2t)^2+(1-t^2)^2}$$

$$= \frac{2-2t^2+4t^2}{4t^2+1-2t^2+t^4}$$

$$\frac{dz}{dt} = \frac{2+2t^2}{t^4+t^2+1} = \frac{2(\cancel{1+t^2})}{(t^2+1)^{\cancel{2}}} = \frac{2}{1+t^2}$$

DIFFERENTIATION OF IMPLICIT FUNCTIONS

Let $U = f(x,y)$ and let y be a function of x and also $f(x,y)=c$, c being a constant.

i.e., $U = f(x,y)$ where $y = y(x)$. Hence by the rule of the total derivative,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

i.e., $$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Since $U = f(x,y) = c$ we have $\frac{du}{dx} = 0$ and the above equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\partial u/\partial x \Big/ \frac{\partial u}{\partial y} = \frac{u_x}{u_y}$$

Thus we can say that if $U = f(x,y) = c$ then

$$\boxed{\frac{dy}{dx} = -\frac{u_x}{u_y} \quad \text{or} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Remark: It is a known fact that a function $f(x,y)=c$ is called an implicit function and we are conversant in finding the derivative of an implicit function. Here we have a formula for $\frac{dy}{dx}$ in terms of partial derivatives and the same can be successively applied for higher order derivatives.

① If $x^y + y^x =$ constant or $x^y + y^x = 2$, find $\frac{dy}{dx}$.

Soln:- Let $u = f(x,y) = x^y + y^x = 2$

$$\therefore \quad u_x = \frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} = y x^{y-1} + y^x \log y$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = x^y \log x + x y^{x-1}$$

we have $\quad \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{[y x^{y-1} + y^x \log y]}{[x^y \log x + x y^{x-1}]}$ //

② If $x^2 + xy + y^2 = 1$, find $\frac{d}{dx}(x^2 y)$

Soln $\quad \frac{d}{dx}(x^2 y) = x^2 \frac{dy}{dx} + y(2x) \Rightarrow 2xy + x^2 \frac{dy}{dx}$ —①

$$f(x,y) = x^2 + xy + y^2 - 1 = 0$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x+y}{2y+x} \rightarrow ②$$

From ① & ②

$$\frac{d}{dx}(x^2 y) = 2xy + x^2 \left[-\frac{2x+y}{2y+x}\right]$$

$$= 2xy - \frac{x^2(2x+y)}{2y+x} \quad //$$

③ If $u = x \log(xy)$ & $x^3 + y^3 - 3axy = 1$ find $\frac{du}{dx}$

Soln $\quad u = x \log(xy)$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

$$= \frac{\cancel{x}}{x\cancel{y}}\cancel{y} + \log(xy) + \frac{x}{\cancel{x}y}\cancel{x}\frac{dy}{dx} = 1 + \frac{x}{y}\frac{dy}{dx} + \log(xy).$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = -\left(\frac{x^2 - ay}{y^2 - ax}\right)$$

$$\frac{du}{dx} = 1 + \frac{x}{y} - \left(\frac{x^2 - ay}{y^2 - ax}\right) + \log(xy)$$

$$= 1 + \log(xy) + x/y \left[\frac{ay - x^2}{-ax + y^2}\right]$$

COMPOSITE FUNCTIONS

If $u = f(x,y)$ where $x = x(r,s)$ and $y = y(r,s)$ then u is a composite function of two independent variables r, s. Therefore in principle we should we able to differentiate u w.r.t r and also w.r.t s partially. Thus we have the following chain rules for the two partial derivatives. It is convenient to write the rule having the data analysed in the following format.

$$u \to (x,y) \to (r,s) \Rightarrow u \to (r,s) \begin{cases} \partial u/\partial r \\ \partial u/\partial s \end{cases}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \; ; \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \to (5)$$

NOTE: 1) The rules (2) & (5) can be established from the basic limit form definition of a partial derivative.

(2) The rules (2) & (5) can be extended to functions involving more than two independent variables.

(3) The rules (2) & (3) can be successively applied for getting higher order derivatives of the given function.

(4) The symbol $\to$ is used only to indicate the composition of the variables so that the associated rule can be written conveniently.

① If $z = f(x,y)$, where $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Soln: $z \to (x,y) \to (u,v) \Rightarrow z \to (u,v)$

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \; ;$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

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i.e., $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y} (-e^{-u}) \rightarrow (1)$ (21)

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \rightarrow (2)$$

Consider LHS $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$

$$= \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} - \left[-e^{-v}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} e^{v} \right]$$

$$= \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v}$$

$$= \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{v})$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

LHS = RHS //

(2) If $z = f(x,y)$, where $x = r\cos\theta$, $y = r\sin\theta$,
show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

Soln $z \rightarrow (x,y) \rightarrow (r,\theta) \Rightarrow z \rightarrow (r,\theta)$

$\therefore \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$; $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

i.e $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \cos\theta + \frac{\partial z}{\partial y} \sin\theta \rightarrow (1)$

and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot (-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$

$$\frac{\partial z}{\partial \theta} = r\left[-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta \right]$$

or $\frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta \rightarrow (2)$

squaring and adding (1), (2) and collecting suitable terms we have,

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(22)

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta + \cancel{2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta} - \cancel{2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta}$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\sin^2\theta+\cos^2\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad [\because \cos^2\theta+\sin^2\theta=1]$$

R.H.S = L.HS

(3) If $z=f(x,y)$ where $x=e^u\cos v$, $y=e^u\sin v$

Show that $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$.

Soln $\quad z \to (x,y) \to (u,v) \Rightarrow z \to (u,v)$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

i.e., $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u\cos v + \frac{\partial z}{\partial y}\cdot e^u\sin v$

and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^u(-\sin v) + \frac{\partial z}{\partial y}\cdot e^u\cos v$

consider LHS

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^u\cos v\left(\frac{\partial z}{\partial x}\,{-\sin v}\,e^u + \frac{\partial z}{\partial y}e^u\cos v\right) + e^u\sin v\left(\frac{\partial z}{\partial x}\cdot e^u\cos v + \frac{\partial z}{\partial y}\cdot e^u\sin v\right)$$

$$= \cancel{-(e^u)^2\cos v\sin v\frac{\partial z}{\partial x}} + (e^u)^2\cos^2 v\frac{\partial z}{\partial y} + \cancel{(e^u)^2\sin v\cos v\frac{\partial z}{\partial x}} + (e^u)^2\sin^2 v\frac{\partial z}{\partial y}$$

$$= e^{2u}(\cos^2 v+\sin^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y} \quad [\underbrace{\cos^2 v+\sin^2 v}_{=1}]$$

LHS = RHS.

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④ If $z=f(x,y)$ where $x=u^2-v^2$ & $y=2uv$ (22)

Show that $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2(u^2+v^2)\frac{\partial z}{\partial x}$

soln : $z \to (x,y) \to (u,v) \Rightarrow z \to (u,v)$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$; $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

i.e., $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot 2u + \frac{\partial z}{\partial y}\cdot 2v$; $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot(-2v) + \frac{\partial z}{\partial y}\, 2u$

LHS

$$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\left[\frac{\partial z}{\partial x}2u + \frac{\partial z}{\partial y}2v\right] - v\left[\frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}\cdot 2u\right]$$

$$= 2u^2\frac{\partial z}{\partial x} + 2uv\cancel{\frac{\partial z}{\partial y}} + 2v^2\frac{\partial z}{\partial x} - 2uv\cancel{\frac{\partial z}{\partial y}}$$

$$= 2(u^2+v^2)\frac{\partial z}{\partial x} = \text{RHS}$$

⑤ If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

→ Here we need to convert the given function u in to a composite function.

Let $u = f(p,q,r)$ where $p=\frac{x}{y}$, $r=\frac{z}{x}$, $q=\frac{y}{z}$

i.e., $\{u \to (p,q,r) \to (x,y,z)\} \Rightarrow u \to (x,y,z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{1}{y} + \frac{\partial u}{\partial r}\cdot z\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial q}\cdot 0$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{x}{y} + \frac{\partial u}{\partial r}\left(-\frac{z}{x}\right) \to (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{-x}{y^2} + \frac{\partial u}{\partial q}\cdot\frac{1}{z} + \frac{\partial u}{\partial r}\cdot 0$$

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$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial p} + \frac{y}{z}\frac{\partial u}{\partial q} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}\cdot\frac{-y}{z^2} + \frac{\partial u}{\partial r}\cdot\frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}\frac{z}{x} \rightarrow (3)$$

Adding (1), (2) & (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}\frac{\partial u}{\partial p}} + \cancel{\left(-\frac{z}{x}\right)\frac{\partial u}{\partial r}} \cancel{-\frac{x}{y}\frac{\partial u}{\partial p}}$$
$$\cancel{+\frac{y}{z}\frac{\partial u}{\partial q}} \cancel{-\frac{y}{z}\frac{\partial u}{\partial q}} \cancel{+\frac{z}{x}\frac{\partial u}{\partial r}}$$
$$= 0$$

(6) If $u = f(x-y,\ y-z,\ z-x)$, Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Soln Let $u = f(p, q, r)$ where $p = x-y$, $q = y-z$, $r = z-x$

i.e., $u \rightarrow (p, q, r) \rightarrow (x, y, z) \Rightarrow u \rightarrow (x, y, z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot 1 + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot(-1) + \frac{\partial u}{\partial q}\cdot 1 + \frac{\partial u}{\partial r}\cdot 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}\cdot(-1) + \frac{\partial u}{\partial r}\cdot 1$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \rightarrow (3)$$

Adding (1), (2) and (3) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial q}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

(4) If $u = f(2x-3y, 3y-4z, 4z-2x)$, show that

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$$

Soln Let $u = f(p, q, r)$ where $p = 2x-3y$, $q = 3y-4z$

$r = 4z-2x$

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot 2 + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot(-2)$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}$$

$$6\frac{\partial u}{\partial x} = 12\frac{\partial u}{\partial p} - 12\frac{\partial u}{\partial r} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\cdot(-3) + \frac{\partial u}{\partial q}\cdot 3 + \frac{\partial u}{\partial r}\cdot 0$$

$$\frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial p} + 3\frac{\partial u}{\partial q} \Rightarrow 4\frac{\partial u}{\partial y} = -12\frac{\partial u}{\partial p} + 12\frac{\partial u}{\partial q} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}\cdot(-4) + \frac{\partial u}{\partial r}\cdot 4$$

$$\frac{\partial u}{\partial z} = -4\frac{\partial u}{\partial q} + 4\frac{\partial u}{\partial r} \Rightarrow 3\frac{\partial u}{\partial z} = -12\frac{\partial u}{\partial q} + 12\frac{\partial u}{\partial r} \rightarrow (3)$$

Adding (1), (2) and (3) we get

$$6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = \cancel{12\frac{\partial u}{\partial p}} - \cancel{12\frac{\partial u}{\partial r}} - \cancel{12\frac{\partial u}{\partial p}} + \cancel{12\frac{\partial u}{\partial q}}$$

$$- \cancel{12\frac{\partial u}{\partial q}} + \cancel{12\frac{\partial u}{\partial r}} = 0$$

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(8) If $u = f\left(\frac{x}{z}, z, y/z\right)$, then show that (86)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = t\frac{\partial u}{\partial t}$$

Soln:- Let $u = f\left(\frac{x}{z}, z, \frac{y}{z}\right)$ where $s = \frac{x}{z}$, $t = z$, $r = y/z$

$$u \to (s,t,r) \to (x,y,z) \Rightarrow u \to (x,y,z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{1}{z} + \frac{\partial u}{\partial t}\cdot 0 + \frac{\partial u}{\partial r}\cdot 0$$

$$x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{x}{z} \to (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\cdot 0 + \frac{\partial u}{\partial r}\cdot\frac{1}{z}$$

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial r} \to (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{-x}{z^2} + \frac{\partial u}{\partial t}\cdot 1 + \frac{\partial u}{\partial r}\cdot\left(-\frac{1}{z^2}\right)y$$

$$z\frac{\partial u}{\partial z} = -\frac{x}{z}\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}z + \left(-\frac{y}{z}\frac{\partial u}{\partial r}\right) \to (3)$$

Adding (1), (2) & (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial s}\cdot\frac{x}{z}} + \cancel{\frac{y}{z}\frac{\partial u}{\partial r}} - \cancel{\frac{x}{z}\frac{\partial u}{\partial s}} + \frac{\partial u}{\partial t}z - \cancel{\frac{y}{z}\frac{\partial u}{\partial r}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = z\frac{\partial u}{\partial t} = t\frac{\partial u}{\partial t}$$

(9) If $u = f(x, z, y/z)$ P.T $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z}$ (27)

$= x\frac{\partial u}{\partial s} - z\frac{\partial u}{\partial t}$.

Soln Let $u = f(x, z, y/z) = f(s, t, r)$ where $s = x$, $t = z$

$r = y/z$. $u \to (x, y, z) \leftarrow^{(x,y,z)} \leftarrow (s, t, r) \leftarrow u$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot 1 + \frac{\partial u}{\partial t}\cdot 0 + \frac{\partial u}{\partial r}\cdot 0$$

$$x\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial s} \to (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\cdot 0 + \frac{\partial u}{\partial r}\cdot\frac{1}{z}$$

$$y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial r} \to (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\cdot 1 + \frac{\partial u}{\partial r}\cdot(-1/z^2)y$$

$$z\frac{\partial u}{\partial z} = z\frac{\partial u}{\partial t} + \left(-\frac{y}{z}\right)\frac{\partial u}{\partial r} \to (3)$$

$\therefore$ consider LHS

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = x\frac{\partial u}{\partial s} - \cancel{\frac{y}{z}\frac{\partial u}{\partial r}} - z\frac{\partial u}{\partial t} + \cancel{\frac{y}{z}\frac{\partial u}{\partial r}}$$

$$= x\frac{\partial u}{\partial s} - z\frac{\partial u}{\partial t}.$$

Assignment

(10) If $z = f(x, y)$ where $x = u^2 + v^2$, $y = 2uv$ show that $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2(u+v)(u-v)\frac{\partial z}{\partial x}$

or $u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = 2(u^2 - v^2)\frac{\partial z}{\partial x}$

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JACOBIANS

Let u and v be functions of two independent variables x & y. The Jacobian (J) of u and v w.r.t x and y is symbolically represented and defined as follows.

$$J\left(\frac{u,v}{x,y}\right) \text{ or } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly if u, v, w are functions of three independent variables x, y, z then

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

NOTE : If u and v are functions of x and y and if $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

① If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$

soln $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta\, r\cos\theta - (-r\sin\theta)\sin\theta$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r$$

$J = r$

② If $x = uv$ $y = \frac{u}{v}$ find $\frac{\partial(x,y)}{\partial(u,v)}$

soln $x = uv$ $y = \frac{u}{v}$

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$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \cancel{v}\left(-\frac{u}{v^{\cancel{2}}}\right) - \frac{u}{v}$$

$$= -\frac{2u}{v} \quad //$$

(3) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Soln By data $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} \end{array}$$

$$= \left(-\frac{yz}{x^2}\right)\left(-\frac{zx}{y^2} \times -\frac{xy}{z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{x\cancel{y}}{z^{\cancel{2}}} \cdot \frac{\cancel{z}}{\cancel{y}} - \frac{x}{\cancel{y}} \times \frac{\cancel{y}}{z}\right)$$
$$+ \frac{y}{x}\left(\frac{\cancel{z}}{y} \times \frac{x}{\cancel{z}} + \frac{\cancel{z}x}{y^{\cancel{2}}} \times \frac{\cancel{y}}{\cancel{z}}\right)$$

$$= -\frac{yz}{x^2}\left\{\frac{x^2}{\cancel{yz}} - \frac{x^2}{\cancel{yz}}\right\} - \frac{z}{x}\left\{-\frac{x}{z} - \frac{x}{z}\right\}$$
$$+ \frac{y}{x}\left\{\frac{x}{y} + \frac{x}{y}\right\}$$

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$$= -\frac{yz}{x^2}(0) - \frac{\cancel{z}}{\cancel{x}}\left(-\frac{2x}{\cancel{z}}\right) + \frac{\cancel{y}}{\cancel{x}} \times \frac{2x}{\cancel{y}}$$

$$= 2+2 = 4$$

Thus $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

④ If $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$, evaluate the jacobian of x,y,z w.r.t ρ,ϕ,z.

or

find $\partial(x,y,z)/\partial(\rho,\phi,z)$

soln $x=\rho\cos\phi$, $y=\rho\sin\phi$, $z=z$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{array}$$

$$= +\cos\phi\,(\rho\cos\phi) - (-\rho\sin\phi)(\sin\phi)$$

$$= \rho\cos^2\phi + \rho\sin^2\phi = \underline{\underline{\rho}}$$

⑤ If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$

show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Soln

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \end{array}$$

$$= \sin\theta\cos\phi\,(0-(-r\sin\theta)\,r\sin\theta\cos\phi)$$
$$-\; r\cos\theta\cos\phi\,(0 - r\sin\theta\cos\theta\cos\phi)$$
$$-r\sin\theta\sin\phi\,(-r\sin\theta\sin\theta\sin\phi - r\cos\theta\sin\phi\cos\theta)$$

$$= r^2\sin\theta\cos^2\phi\sin^2\theta + r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\sin^2\phi\cos^2\theta$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\sin^2\phi\cos^2\theta$$

$$= r^2\sin\theta\,(\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \sin^2\phi\cos^2\theta)$$

$$= r^2\sin\theta\,(\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta(\sin^2\phi + \cos^2\phi))$$

$$= r^2\sin\theta\,(\sin^2\theta\,(1) + \cos^2\theta(1))$$

$$= r^2\sin\theta\quad(1)$$

$$= r^2\sin\theta$$

Thus $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

(6) If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$, show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Soln $\quad u = \dfrac{x}{y-z}, \quad v = \dfrac{y}{z-x}, \quad w = \dfrac{z}{x-y}$

$$\frac{\partial u}{\partial x} = \frac{1}{y-z} \qquad \frac{\partial v}{\partial x} = \frac{-y}{(z-x)^2} \qquad \frac{\partial w}{\partial x} = \frac{z}{(x-y)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{x}{(y-z)^2}. \qquad \frac{\partial v}{\partial y} = \frac{1}{z-x} \qquad \frac{\partial w}{\partial y} = \frac{-z}{(x-y)^2}.$$

$$\frac{\partial u}{\partial z} = \frac{-x}{(y-z)^2} \qquad \frac{\partial v}{\partial z} = \frac{y}{(z-x)^2} \qquad \frac{\partial w}{\partial z} = \frac{1}{x-y}.$$

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We have
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{y-z} & \frac{x}{(y-z)^2} & \frac{-x}{(y-z)^2} \\ \frac{-y}{(z-x)^2} & \frac{1}{z-x} & \frac{y}{(z-x)^2} \\ \frac{z}{(x-y)^2} & \frac{-z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

(It is very difficult to expand directly. We shall use properties of determinants)

Taking $\frac{1}{(y-z)^2}$, $\frac{1}{(z-x)^2}$, $\frac{1}{(x-y)^2}$ as common factors respectively from first, second & third rows (aiming to avoid denominators) we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{(y-z)^2}\,\frac{1}{(z-x)^2}\,\frac{1}{(x-y)^2} \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} y-z & x & -x \\ -y & z-x & y \\ z & -z & x-y \end{vmatrix} \end{array}$$

$$= \frac{1}{(y-z)^2}\,\frac{1}{(z-x)^2}\,\frac{1}{(x-y)^2}\Big[(y-z)\left[(z-x)(x-y)+zy\right] - x(-y(x-y)-zy) + -x(yz-z(z-x)\Big]$$

$$= \frac{1}{(y-z)^2}\,\frac{1}{(z-x)^2}\,\frac{1}{(x-y)^2}\Big[(y-z)\left[zx-zy-x^2+xy+zy\right] - x(-yx+y^2-zy)+(-x)(yz-z^2+zx)$$

$$= \frac{1}{(y-z)^2}\,\frac{1}{(z-x)^2}\,\frac{1}{(x-y)^2}\Big[yzx - yx^2 + xy^2 - z^2x + zx^2 - xyz + x^2y - xy^2 + xyz - xyz + xz^2 - zx^2\Big]$$

$$= \underline{\underline{0}}$$

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(7) If $U = 2xy$, $V = x^2 - y^2$, where $x = r\cos\theta, y = r\sin\theta$. (22)

Evaluate $\frac{\partial(U,V)}{\partial(r,\theta)}$

Soln We have a composite function

$(U,V) \to (x,y) \to (r,\theta) \Rightarrow (U,V) \to (r,\theta)$

By the composite function property of Jacobians we have,

$$\frac{\partial(U,V)}{\partial(r,\theta)} = \frac{\partial(U,V)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} \to (1)$$

$U = 2xy \quad \therefore \quad \frac{\partial U}{\partial x} = 2y \qquad \frac{\partial U}{\partial y} = 2x$

$V = x^2 - y^2 \quad \therefore \quad \frac{\partial V}{\partial x} = 2x \qquad \frac{\partial V}{\partial y} = -2y$

$$\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2 = -4(x^2+y^2) \to (2)$$

Also $x = r\cos\theta \quad \therefore \quad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$y = r\sin\theta \quad \therefore \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(1) = r \to (3)$$

Using the results (2) & (3) in (1) we get,

$$\frac{\partial(U,V)}{\partial(r,\theta)} = -4(x^2+y^2) \cdot r = -4(r^2\cos^2\theta + r^2\sin^2\theta)$$

Thus $\frac{\partial(U,V)}{\partial(r,\theta)} = -4r^3$

Assignment

(8) If $U = x^2 - y^2$, $V = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$ then determine the Jacobian $\frac{\partial(U,V)}{\partial(r,\theta)}$. Ans:- $4r^3$

NOTE: COMPOSITE FUNCTIONS PROPERTY OF JACOBIANS

If u and v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$$

9) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln We have

$$u = \frac{x+y}{1-xy}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(1-xy)1 - (x+y)(-y)}{(1-xy)^2} = \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)\cdot 1 - (x+y)(-x)}{(1-xy)^2} = \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$v = \tan^{-1}x + \tan^{-1}y$$

$$\therefore \frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

Now
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2}\left[\frac{\cancel{1+y^2}}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{\cancel{1+x^2}}\right]$$

$$= \frac{1}{(1-xy)^2}(0) = 0$$

Thus $\frac{\partial(u,v)}{\partial(x,y)} = 0$

(10) If $U = x+3y^2-z^3$, $V = 4x^2yz$, $\omega = 2z^2 - xy$, evaluate that $\dfrac{\partial(U,V,\omega)}{\partial(x,y,z)}$ at $(1,-1,0)$.

or

If $U = x+3y^2-z^3$, $V = 4x^2yz$, $\omega = 2z^2 - xy$ find $\dfrac{\partial(U,V,\omega)}{\partial(x,y,z)}$ at $(1,-1,0)$.

Soln $U = x+3y^2-z^3$, $V = 4x^2yz$, $\omega = 2z^2-xy$

$$\frac{\partial(U,V,\omega)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix}$$

$$= \begin{array}{c} \quad + \qquad - \qquad + \\ \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix} \end{array}$$

$$= 1(4x^2z \times 4z - (-4x^2y \times x))$$
$$- 6y(8xyz \times 4z + y \times 4x^2y)$$
$$-3z^2(-8x^2yz + 4x^2yz)$$
$$= 1(16x^2\cancel{z^2} + 4x^3y) - 6y(32\cancel{y}xz^2 + 4x^2y^2)$$
$$-3z^2(-\cancel{4x^2}yz)$$

At $(1,-1,0)$

$$\frac{\partial(U,V,\omega)}{\partial(x,y,z)} = 4(+1)^3(-1) - 6(-1)(4\,1^2(-1)^2)$$
$$= 4(-1) + 6\times 4 = -4+24 = 20$$

or

$$\begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

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It will be easier if the elements of the determinant are evaluated at $(1,-1,0)$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ at } (1,-1,0) = \begin{vmatrix} \overset{+}{1} & \overset{-}{-6} & \overset{+}{0} \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(0-4)+6(0+4) = -4+24 = \underline{\underline{20}}$$

(11) If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$

Soln $x = u(1-v)$; $y = uv$

$$\frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = -u \qquad \frac{\partial y}{\partial v} = u$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u - (-uv) = (1-v)u + uv$$

$$= u - \cancel{uv} + \cancel{uv} = u$$

$$\therefore \boxed{J = u}$$

Next we shall express u and v in terms of x and y.

By data $x = u - uv$ and $y = uv$

Hence $u = x+y$; $v = \frac{y}{x+y}$ $\therefore \frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 1$

$$\frac{\partial v}{\partial x} = \frac{(x+y)\cdot 0 - y(1)}{(x+y)^2} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)\cdot 1 - y(1)}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$J' = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^{\cancel{2}}}$$

$$J' = \frac{1}{x+y} = \frac{1}{u}$$

Thus $J' = \frac{1}{u}$ Hence $J \cdot J' = u \cdot \frac{1}{u} = 1$

Thus $J J' = 1$

(12) If $x = uv$, $y = \frac{u}{v}$, then prove that $JJ' = 1$

Soln $x = uv$, $y = \frac{u}{v}$

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$J = -\frac{u\cancel{v}}{v^{\cancel{2}}} - \frac{u}{v} = -\frac{2u}{v}$$

Next we shall express u and v in terms of x and y

By data $x = uv$ and $y = \frac{u}{v}$

Hence $u^2 = xy$ $\quad$ $v^2 = x/y$

or $\therefore u = \sqrt{xy}$ $\quad$ or $v = \sqrt{x/y}$

$u^2 = xy$ $\quad$ $v^2 = x/y$

$$2u \frac{\partial u}{\partial x} = y \qquad 2v \frac{\partial v}{\partial x} = \frac{1}{y}$$

$$\frac{\partial u}{\partial x} = \frac{y}{2u} \qquad \frac{\partial v}{\partial x} = \frac{1}{2vy}$$

$$\frac{\partial u}{\partial x} = \frac{y}{2\sqrt{xy}} \qquad \frac{\partial v}{\partial x} = \frac{1}{2\frac{\sqrt{x}}{\cancel{\sqrt{y}}}\cancel{y}\sqrt{y}}$$

$$\frac{\partial u}{\partial x} = \frac{y}{2\sqrt{xy}} \qquad \frac{\partial v}{\partial x} = \frac{1}{2\sqrt{xy}}.$$

$x = uv$ $\quad$ $y = \frac{u}{v}$

$u = x/v$

$u = x/u/y$

$u = \frac{x \times y}{u}$

$u^2 = xy$

$y = \frac{u}{v}$

$(v)^2 = \left(\frac{u}{y}\right)^2 \Rightarrow v^2 = \frac{x\cancel{y}}{y^{\cancel{2}}}$

$v^2 = x/y$

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$$2u\frac{\partial u}{\partial y} = x \qquad 2v\frac{\partial v}{\partial y} = -x/y^2$$

$$\frac{\partial u}{\partial y} = \frac{x}{2u} = \frac{x}{2\sqrt{xy}} \qquad \frac{\partial v}{\partial y} = \frac{-x}{y\sqrt{y}\,\cancel{y^2}} \times 2 \times \sqrt{x}\cancel{\sqrt{y}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{2\sqrt{xy}} \qquad \frac{\partial v}{\partial y} = -\frac{x}{y2\sqrt{xy}}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2\sqrt{xy}} & \frac{-x}{2y\sqrt{xy}} \end{vmatrix}$$

$$= \frac{-xy}{4(\sqrt{xy})^2 y} - \frac{x}{4xy}$$

$$= -\frac{\cancel{x}y}{4\cancel{x}y^2} - \frac{\cancel{x}}{4\cancel{x}y} = \frac{-1}{4y} - \frac{1}{4y}$$

$$= -\frac{\cancel{2}}{\cancel{4}_2 y} = \frac{-1}{2y} = \frac{-1}{2 \times \frac{u}{v}} = \frac{-v}{2u}$$

Thus $J' = \frac{-v}{2u}$ Hence $J.J' = -\frac{2u}{v} \times \frac{-v}{2u} = 1$

Thus $JJ' = 1$

(19) If $u = x+y$, $v = x-y$, prove that $JJ' = 1$

Soln $u = x+y \qquad v = x-y$

$$\frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 1, \ \frac{\partial v}{\partial x} = 1, \ \frac{\partial v}{\partial y} = -1$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$J = -1-1 = -2$$

we shall now express x and y in terms of u, v.

We have $x+y=u$ and $x-y=v$

$x+y=u \rightarrow ①$ $x-y=v \rightarrow ②$

adding ① & ②, we get

$2x = u+v$

$x = \frac{u+v}{2}$

Subtracting ① & ②, we get

$\cancel{x}+y-\cancel{x}+y = u-v$

$2y = u-v$

$y = \frac{u-v}{2}$

$\therefore \quad x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}$

$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \frac{\partial y}{\partial v} = \frac{-1}{2}$

$$\therefore J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$J' = -\frac{1}{4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

Hence $J \cdot J' = -\cancel{2} \times -\frac{1}{\cancel{2}} = 1$

(14) If $x = e^u \cos v$, $y = e^u \sin v$, prove that $JJ' = 1$

soln $x = e^u \cos v$; $y = e^u \sin v$

$\frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$

$\frac{\partial x}{\partial v} = -e^u \sin v \qquad \frac{\partial y}{\partial v} = e^u \cos v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix}$$

$$J = e^{2u}\cos^2 v + e^{2u}\sin^2 v = e^{2u} \cdot 1 = e^{2u}$$

Again consider $x = e^{u}\cos v$, $y = e^{u}\sin v$

$\therefore x^2+y^2 = e^{2u}$ or $2u = \log(x^2+y^2)$

$\therefore u = \frac{1}{2}\log(x^2+y^2)$

Also $\frac{y}{x} = \frac{\cancel{e^{u}}\sin v}{\cancel{e^{u}}\cos v} = \tan v$

$\Rightarrow v = \tan^{-1}(y/x)$

Consider $u = \frac{1}{2}\log(x^2+y^2)$

$\therefore \frac{\partial u}{\partial x} = \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+y^2} = \frac{x}{x^2+y^2}$; $\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}y}{x^2+y^2} = \frac{y}{x^2+y^2}$

Also $v = \tan^{-1} y/x$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2/x^2}\cdot\frac{1}{x} = \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}}\frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}.$$

Now $J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$J' = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{2}}} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

Hence $J.J' = \cancel{e^{2u}}\cdot\frac{1}{\cancel{e^{2u}}} = 1$

Thus $\underline{\underline{JJ' = 1}}$

(15) $u = x(1-y)$ & $v = xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$ & $\frac{\partial(x,y)}{\partial(u,v)}$ & verify $JJ' = 1$

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soln $U = x(1-y) \quad V = xy$

$$\frac{\partial U}{\partial x} = 1-y, \ \frac{\partial U}{\partial y} = -x, \quad \frac{\partial V}{\partial x} = y, \ \frac{\partial V}{\partial y} = x$$

$$J = \frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$$

$$= (1-y)x + xy = x - \cancel{yx} + \cancel{xy} = x$$

$$J = x.$$

$$J' = \frac{\partial(x,y)}{\partial(U,V)}$$

$U = x(1-y)$ & $V = xy$

$U+V = x(1-y) + xy$

$U+V = x - \cancel{xy} + \cancel{xy}$

$\therefore \ U+V = x \qquad y = \frac{V}{x} = \frac{V}{U+V}$

$$\therefore \ x = U+V \qquad y = \frac{V}{U+V}$$

$$\frac{\partial x}{\partial U} = 1, \ \frac{\partial x}{\partial V} = 1, \quad \frac{\partial y}{\partial U} = \frac{(U+V)\cdot 0 - V(1)}{(U+V)^2} = \frac{-V}{(U+V)^2}$$

$$\frac{\partial y}{\partial V} = \frac{(U+V)\cdot 1 - V\cdot 1}{(U+V)^2} = \frac{U + \cancel{V} - \cancel{V}}{(U+V)^2} = \frac{U}{(U+V)^2}$$

$$J' = \frac{\partial(x,y)}{\partial(U,V)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-V}{(U+V)^2} & \frac{U}{(U+V)^2} \end{vmatrix}$$

$$J' = \frac{U}{(U+V)^2} + \frac{V}{(U+V)^2} = \frac{\cancel{U+V}}{(U+V)^{\cancel{2}}} = \frac{1}{x}$$

Hence $J \cdot J' = \cancel{x} \cdot \frac{1}{\cancel{x}} = 1$

(16) If $U = \frac{xy}{z}$, $V = \frac{yz}{x}$, $w = \frac{zx}{y}$ verify that $JJ' = 1$

soln $U = \frac{xy}{z}$, $V = \frac{yz}{x}$, $w = \frac{zx}{y}$

$$\therefore \quad J = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \partial V/\partial y & \partial V/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = 4 \quad \text{(see in (8))}$$

consider $V = \frac{yz}{x}$, $U = \frac{xy}{z}$, $\omega = \frac{zx}{y}$

$U = \frac{xy}{z}$, $V = \frac{yz}{x}$, $\omega = \frac{zx}{y}$.

$\therefore$ $U = x.\frac{x}{\omega}$, $V = \frac{\cancel{z} y y}{U}$, $\omega = z^2/V$

$\Rightarrow$ $U\omega = x^2$, $VU = y^2$, $\omega V = z^2$

$\therefore x = \sqrt{U\omega}$ $y = \sqrt{VU}$, $z = \sqrt{\omega V}$

$$J' = \frac{\partial(x,y,z)}{\partial(U,V,\omega)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial U} & \frac{\partial z}{\partial V} & \frac{\partial z}{\partial \omega} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\omega}{2\sqrt{U\omega}} & 0 & \frac{U}{2\sqrt{U\omega}} \\ \frac{V}{2\sqrt{VU}} & \frac{U}{2\sqrt{VU}} & 0 \\ 0 & \frac{\omega}{2\sqrt{V\omega}} & \frac{V}{2\sqrt{V\omega}} \end{vmatrix}$$

$$= \frac{1}{2\sqrt{UW} \times 2\sqrt{VU} \times 2\sqrt{VW}} \begin{vmatrix} \omega & 0 & U \\ V & U & 0 \\ 0 & \omega & V \end{vmatrix}$$

$$= \frac{1}{8\sqrt{U}.\sqrt{\omega}.\sqrt{V}.\sqrt{U}.\sqrt{V}.\sqrt{\omega}} \left(\omega(UV) + U(V\omega)\right)$$

$$= \frac{1}{4\cancel{8}U\cancel{V}\omega} \cancel{2}U\cancel{V}\omega = \frac{1}{4}$$

$\therefore J.J' = 1$

Assignment

(17) If $U = \frac{yz}{x}$, $V = \frac{zx}{y}$, $\omega = \frac{xy}{z}$, verify that $JJ' = 1$

(1) If $z = e^{ax+by} f(ax-by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Soln

$$z = e^{ax+by} f(ax-by) \rightarrow (1)$$

differentiate (1) partially wrt x, we get

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by) \cdot a + f(ax-by) e^{ax+by} \cdot a$$

$$= a\left[e^{ax+by} f'(ax-by) + e^{ax+by} f(ax-by)\right] \rightarrow (2)$$

Differentiate (1) partially wrt y, we get

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} \cdot b \rightarrow (3)$$

Now $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$ by using (2) & (3) becomes

$$= ba\left[e^{ax+by} f'(ax-by) + e^{ax+by} f(ax-by)\right]$$
$$+ a\left[-e^{ax+by} f'(ax-by) b + b f(ax-by) e^{ax+by}\right]$$

$$= \cancel{ab e^{ax+by} f'(ax-by)} + ba\, e^{ax+by} f(ax-by)$$
$$- \cancel{ab e^{ax+by} f'(ax-by)} + ab\, e^{ax+by} f(ax-by)$$

$$= 2ab\, e^{ax+by} f(ax-by)$$

$$\therefore b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz \quad (\text{using } (1))$$

(2) If $u = e^{xyz}$, Show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2) e^{xyz}$

Soln

given $u = e^{xyz}$

Take log on both sides

$$\log u = \log e^{xyz} = xyz \log e = xyz \quad | \because \log e = 1$$

$$\log u = xyz$$

Diff wrt x, partially

$$\frac{1}{u}\frac{\partial u}{\partial x} = yz \Rightarrow \frac{\partial u}{\partial x} = uyz = e^{xyz} yz$$

Diff wrt y, partially

$$\frac{\partial^2 u}{\partial y \partial x} = e^{xyz} \cdot z + yz\, e^{xyz} xz$$

$$\frac{\partial^2 u}{\partial y \partial x} = e^{xyz}\left[z + xyz^2\right]$$

Diff wrt z, partially

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = e^{xyz}\left[1 + 2xyz\right] + (z + xyz^2) e^{xyz} \cdot xy$$
$$= e^{xyz}\left[1 + 2xyz + xyz + x^2y^2z^2\right]$$

$$\therefore \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[1+3xyz+x^2y^2z^2\right]$$

(3) If $u=\log r$, where $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, S.T

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}$$

Soln Given $r^2=(x-a)^2+(y-b)^2+(z-c)^2 \rightarrow (1)$

Differentiating partially wrt 'x' we get

$$\Rightarrow 2r\frac{dr}{dx}=2(x-a) \quad \text{or} \quad \frac{dr}{\partial x}=\frac{x-a}{r} \rightarrow (2)$$

Similarly, $\frac{\partial r}{\partial y}=\frac{y-b}{r}$ and $\frac{\partial r}{\partial z}=\frac{z-c}{r}$

Now $u=\log r$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{1}{r}\frac{\partial r}{\partial x}=\frac{1}{r}\left(\frac{x-a}{r}\right), \text{ from (1)}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{x-a}{r^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right)=\frac{r^2(1)-(x-a)2r\left(\frac{\partial r}{\partial x}\right)}{r^4}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=\frac{r^2-2(x-a)^2}{r^4}, \text{ from (2)}$$

Similarly, $\frac{\partial^2 u}{\partial y^2}=\frac{r^2-2(y-b)^2}{r^4}$, $\frac{\partial^2 u}{\partial z^2}=\frac{r^2-2(z-c)^2}{r^4}$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3r^2-2\left[(x-a)^2+(y-b)^2+(z-c)^2\right]}{r^4}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3r^2-2r^2}{r^4} \text{ (from (1))}=\frac{r^2}{r^4}=\frac{1}{r^2}$$

Homogeneous function

1. verify the Euler's theorem for the function

$$u=\frac{x^2+y^2}{x-y}$$

Soln Given

$$u=\frac{x^2+y^2}{x-y}=\frac{x^2\left(1+(y/x)^2\right)}{x(1-y/x)}=x\phi(y/x)$$

$\therefore$ u is a homogeneous function of degree 1 $(n=1)$

Diff u wrt x

$$\frac{\partial u}{\partial x}=\frac{(x-y)(2x)-(x^2+y^2)(1)}{(x-y)^2}=\frac{2x^2-2xy-x^2-y^2}{(x-y)^2}$$

$$\frac{\partial u}{\partial x}=\frac{x^2-2xy-y^2}{}$$

$$x\frac{\partial u}{\partial x} = \frac{x^3 - xy^2 - 2x^2y}{(x-y)^2} \rightarrow ①$$

Diff u w.r.t y

$$\frac{\partial u}{\partial y} = \frac{(x-y)2y - (x^2+y^2)(-1)}{(x-y)^2} = \frac{2xy - 2y^2 + x^2 + y^2}{(x-y)^2} = \frac{x^2 - y^2 + 2xy}{(x-y)^2}$$

$$y\frac{\partial u}{\partial y} = \frac{x^2y - y^3 + 2xy^2}{(x-y)^2} \rightarrow ②$$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

LHS

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3 - xy^2 - 2x^2y}{(x-y)^2} + \frac{x^2y - y^3 + 2xy^2}{(x-y)^2}$$

$$= \frac{x^3 - \cancel{xy^2} - 2x^2y + x^2y - y^3 + 2xy^2}{(x-y)^2}$$

$$= \frac{x^3 + xy^2 - x^2y - y^3}{(x-y)^3} = \frac{x(x^2+y^2) - y(x^2+y^2)}{(x-y)^3}$$

$$= \frac{(x^2+y^2)\cancel{(x-y)}}{(x-y)^{\cancel{3}\,2}} = \frac{x^2+y^2}{(x-y)^2} = u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Hence, the Euler's theorem is verified

② Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ where $\log u = \frac{x^3+y^3}{3x+4y}$.

Soln Let $z = \log u = \frac{x^3+y^3}{3x+4y} = \frac{x^3(1+(y/x)^3)}{x(3+4y/x)} = x^2\left(\frac{1+(y/x)^3}{3+4y/x}\right)$

$\therefore$ z is a homogeneous function of degree 2 in x & y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \rightarrow ①$$

But $\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x}$ & $\frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$

Hence (i) becomes

$$x\cdot\frac{1}{u}\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\frac{\partial u}{\partial y} = 2\log u \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$$

③ If $u = \sin^{-1}\left(\frac{x+2y+3z}{x^8+y^8+z^8}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln:- Here u is not a homogeneous function. We therefore, write

$$w = \sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = \frac{x(1+2y/x+3z/x)}{x^8(1+(y/x)^8+(z/x)^8)}$$

$$= x^{-7}\left(\frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}\right)$$

Thus w is a homogeneous function of degree -7 in x,y,z. Hence by Euler's theorem

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = (-7)w \quad \ldots (1)$$

But $\frac{\partial w}{\partial x} = \cos u\frac{\partial u}{\partial x}$, $\frac{\partial w}{\partial y} = \cos u\frac{\partial u}{\partial y}$, $\frac{\partial w}{\partial z} = \cos u\frac{\partial u}{\partial z}$

$\therefore$ (1) becomes $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

$\boxed{\text{or}}$

$$\sin u = \frac{x+2y+3z}{x^8+y^8+z^8} = \frac{x(1+2y/x+3z/x)}{x^8(1+(y/x)^8+(z/x)^8)}$$

$$= x^{-7}\left(\frac{1+2(y/x)+3(z/x)}{1+(y/x)^8+(z/x)^8}\right)$$

$\Rightarrow$ $\sin u$ is homogeneous function of degree -7.

Apply Euler's theorem for the function $\sin u$ we have

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u + z\frac{\partial}{\partial z}\sin u = n\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y(+\cos u)\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u$$

*. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Soln Let $v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$ and $w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$ $\ldots$ (i)

$$u = u(v,w)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right)$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad \ldots \text{(ii)} \quad [\text{using (i)}]$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0)$$ (using (i)) or $y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}$ ...(iii)

Similarly $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right)$ [using (i)]

(or) $z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w}$

Adding (ii), (iii) and (iv), we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

* If $x = r\cos\theta$; $y = r\sin\theta$, $z = z$ find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Soln

$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial x}{\partial z} = 0$

$y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$, $\frac{\partial y}{\partial z} = 0$

$z = z \Rightarrow \frac{\partial z}{\partial r} = 0$, $\frac{\partial z}{\partial \theta} = 0$, $\frac{\partial z}{\partial z} = 1$

Now
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

* If $x+y = u$ and $y = uv$ prove that $J\cdot J' = 1$

Soln

$u = x+y$ and $y = uv$

$\Rightarrow$ $x + uv = u \Rightarrow x = u(1-v)$ and $y = uv$

$\Rightarrow$ $\frac{\partial x}{\partial u} = 1-v \Rightarrow \frac{\partial y}{\partial u} = v$ $\quad \frac{\partial x}{\partial v} = -u$ $\quad \frac{\partial y}{\partial v} = u$.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v)+uv$$
$= u = x+y$

$J' = 1/u$

$u = x+y$

$v = y/x+y$

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial x} = -y/(x+y)^2$

$\partial v/\partial y = x/(x+y)^2$

$$J' = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$J' = 1/u$ [see page No 36, 37]

$$\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} = u\cdot\frac{1}{u} = 1$$

$J\cdot J' = 1$ Hence proved.

* If $u = x + y^2/x$ and $v = y^2/x$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln $u = x + y^2/x \Rightarrow \frac{\partial u}{\partial x} = 1 - y^2/x^2, \frac{\partial u}{\partial y} = 2y/x$

$v = y^2/x \Rightarrow \frac{\partial v}{\partial x} = -y^2/x^2, \frac{\partial v}{\partial y} = 2y/x$

Now $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-y^2/x^2 & 2y/x \\ -y^2/x^2 & 2y/x \end{vmatrix}$

$= (1 - y^2/x^2)(2y/x) + 2y^3/x^3 = \frac{2y}{x} - \frac{2y^3}{x^3} + \frac{2y^3}{x^3} = \frac{2y}{x}$.

* If $u = \sin(x^2+y^2)$, find $\frac{du}{dx}$, where $a^2x^2 + b^2y^2 = c^2$.

soln Given $u = \sin(x^2+y^2)$

WKT $\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$= \cos(x^2+y^2) \cdot 2x + \cos(x^2+y^2) \cdot 2y\frac{dy}{dx}$

$\frac{du}{dx} = 2\left(x + y\frac{dy}{dx}\right)\cos(x^2+y^2) \rightarrow ①$

But $a^2x^2 + b^2y^2 = c^2$

Diff wrt x, we get

$2a^2x + 2yb^2\frac{dy}{dx} = 0 \Rightarrow \cancel{2}b^2y\frac{dy}{dx} = -\cancel{2}a^2x$

$\Rightarrow \frac{dy}{dx} = \frac{-a^2x}{b^2y}$

$\therefore ① \Rightarrow \frac{du}{dx} = 2\left[x + \cancel{y}\left(\frac{-a^2x}{b^2\cancel{y}}\right)\right]\cos(x^2+y^2)$

$= 2\left[x - \frac{a^2}{b^2}x\right]\cos(x^2+y^2)$

$\frac{dy}{dx} = 2x\left[1 - \frac{a^2}{b^2}\right]\cos(x^2+y^2)$.

* Assignment

If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)}$ & $J' = \frac{\partial(r,\theta,z)}{\partial(x,y,z)}$. Also verify $J \cdot J' = 1$.

* If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

soln We have $\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2}$, $\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$, $\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$

$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$, $\frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2}$, $\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$

$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$, $\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$, $\frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$

$$\therefore \frac{\partial(y_1 y_2 y_3)}{\partial(x_1, x_2 x_3)} = \begin{vmatrix} \partial y_1/\partial x_1 & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_2 x_1 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_1 \end{vmatrix} = \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4$$

Assignment

* Find the constants a and b such that the surfaces $ax^2 + y^2 = bz$ and $3x^2 - 2y^2 - 3z^2 + 8 = 0$ are orthogonal at the point $(-1, 2, 1)$.

Assignment

* Check whether the functions $u = e^x + \log y + xyz$, $v = \log x + e^y + xyz$, $w = xyz$ are functionally dependent.

Gradient, Divergence and curl.

Scalar and vector point functions.

* If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a scalar point function and we say that a scalar field ϕ has been defined in R.

Examples: 1. $\phi = x^2 + y^2 + z^2$ 2. $\phi = xy^2 z^3$

* If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called a vector point function and we say that a vector field $\vec{A}$ has been defined in R.

Ex:- 1. $\vec{A} = x^2 i + y^2 j + z^2 k$ 2. $\vec{A} = xyz i + yz j + zk$

* Operators:- ① The vector differential operator ∇ (read as "Nabla" or "Del") is defined by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \sum \frac{\partial}{\partial x} i$$

② The Laplacian operator ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \partial^2/\partial y^2 + \partial^2/\partial z^2 = \sum \partial^2/\partial x^2.$$

Note :- ① Laplacian: If $\phi(x,y,z)$ is a continuously differentiable scalar function and $\vec{A}(x,y,z)$ is a continuously differentiable vector function, we can define the laplacian for ϕ as well as for $\vec{A}$ as follows,

Laplacian of $\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$.

Laplacian of $\vec{A} = \nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$.

② If ϕ is a scalar function, the equation $\nabla^2\phi = 0$ is called a Laplace's equation and a function which satisfies Laplace's equation is called a harmonic function.

Also $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called a laplace's equation in two dimensions.

* Gradient of a scalar field :-

If $\phi(x,y,z)$ is a continuously differentiable scalar function then the gradient of ϕ (grad ϕ) is defined to be $\nabla\phi$. i.e., grad $\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$.

Obviously $\nabla\phi$ is a vector quantity

* Divergence of a vector field :-

If $\vec{A}(x,y,z)$ is a continuously differentiable vector function then divergence of $\vec{A}$ (div $\vec{A}$) is defined to be $\nabla\cdot\vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$, where a_1, a_2, a_3 are all functions of x,y,z then we have

div $\vec{A} = \nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot(a_1 i + a_2 j + a_3 k)$

i.e., div $\vec{A} = \nabla\cdot\vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$

Clearly div $\vec{A}$ is a scalar quantity

Note:- ① Divergence of a constant vector is 0

② $\nabla\cdot\vec{A} \neq \vec{A}\cdot\nabla$

③ If divergence of $\vec{A}$ is 0, then $\vec{A}$ is said to be solenoidal. i.e div. $\vec{A} = 0 \Rightarrow \vec{A}$ is solenoidal.

* Curl of a vector field

If $\vec{A}(x,y,z)$ is a continuously differentiable vector function then curl of $\vec{A}$ (curl $\vec{A}$) is defined to be $\nabla\times\vec{A}$.

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

Clearly, curl $\vec{A}$ is a vector quantity.

* Note: ① If $\vec{A}$ is a constant function, then curl $\vec{A} = 0$

② If curl $\vec{A} = 0$ (or) $\nabla \times \vec{A} = 0$, then $\vec{A}$ is irrotational

Vector identities

1. Show that (i) curl (grad ϕ) = 0 i.e, $\nabla \times (\nabla \phi) = 0$ ii) div (curl $\vec{A}$) = 0

(i) proof :- Let ϕ be a scalar point function of x, y, z.

$$\text{grad}\,\phi = \nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\text{curl}\,(\text{grad}\,\phi) = \nabla \times (\nabla \phi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) - j\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + k\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right)$$

$$\nabla \times (\nabla \phi) = 0$$

Thus curl (grad ϕ) = 0, for any scalar function ϕ

ii) div (curl $\vec{A}$) = 0 i.e., $\nabla \cdot (\nabla \times \vec{A}) = 0$.

Proof: Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z.

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\nabla \times \vec{A} = i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial}{\partial x} a_3 - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial}{\partial x} a_2 - \frac{\partial a_1}{\partial y}\right)$$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{A}) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[i\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - j\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + k\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 a_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 a_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 a_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 a_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 a_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 a_1}{\partial z \partial y}} = 0$$

$\Rightarrow \nabla\cdot(\nabla\times\vec{A})=0$

i.e, $\text{div}\cdot(\text{curl}\,\vec{A})=0$.

2. prove that $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

soln or $\text{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot(\text{curl}\,\vec{A}) - \vec{A}(\text{curl}\,\vec{B})$.

Let $\vec{A} = A_1 i + A_2 j + A_3 k$ and $\vec{B} = B_1 i + B_2 j + B_3 k$

$$\nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\vec{B}\cdot(\nabla\times\vec{A}) = B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

|||ly

$$\vec{A}\cdot(\nabla\times\vec{B}) = A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

consider

$$\vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}(\nabla\times\vec{B}) = \Sigma\left[-B_2\frac{\partial A_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} + A_2\frac{\partial B_3}{\partial x} - A_3\frac{\partial B_2}{\partial x}\right]$$

$$= \Sigma\left[B_3\frac{\partial A_2}{\partial x} + A_2\frac{\partial B_3}{\partial x} - \left(A_3\frac{\partial B_2}{\partial x} + B_2\frac{\partial A_3}{\partial x}\right)\right]$$

$$= \Sigma\left[\frac{\partial}{\partial x}(A_2B_3) - \frac{\partial}{\partial x}(A_3B_2)\right]$$

$$= \Sigma\frac{\partial}{\partial x}[A_2B_3 - A_3B_2] = \nabla\cdot\Sigma\frac{\partial}{\partial x}(A_2B_3 - A_3B_2)\,i$$

$$= \nabla\cdot(\vec{A}\times\vec{B})$$

$$\Rightarrow \nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}(\nabla\times\vec{B}).$$

Hence proved.

Problems

① If $\vec{F} = 3x\hat{i} + 5y\hat{j} + 6z\hat{k}$, find the curl $\vec{F}$.

soln given $\vec{F} = 3x\hat{i} + 5y\hat{j} + 6z\hat{k}$

$F_1 = 3x \quad F_2 = 5y \quad F_3 = 6z$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x & 5y & 6z \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(6z) - \frac{\partial}{\partial z}(5y)\right) - \hat{j}\left(\frac{\partial}{\partial x}6z - \frac{\partial}{\partial z}3x\right) + \hat{k}\left(\frac{\partial}{\partial x}5y - \frac{\partial}{\partial y}3x\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$\Rightarrow \text{curl}\,\vec{F} = \nabla\times\vec{F} = 0$

Since curl $\vec{F}$ or $\nabla\times\vec{F} = 0$, then $\vec{F}$ is irrotational.

② Find the divergence and curl of the vector

$\vec{F} = (xyz + y^2z)\hat{i} + (3x^2 + y^2z)\hat{j} + (xz^2 - y^2z)\hat{k}$.

Soln given $\vec{F} = (xyz + y^2z)\hat{i} + (3x^2+y^2z)\hat{j} + (xz^2 - y^2z)\hat{k}$.

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[(xyz+y^2z)\hat{i} + (3x^2+y^2z)\hat{j} + (xz^2-y^2z)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xyz+y^2z) + \frac{\partial}{\partial y}(3x^2+y^2z) + \frac{\partial}{\partial z}(xz^2-y^2z)$$

$$\nabla \cdot \vec{F} = yz + 2yz + 2xz - y^2$$

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz+y^2z & 3x^2+y^2z & xz^2-y^2z \end{vmatrix}$$

$$\therefore \hat{i}\left(\frac{\partial}{\partial y}(xz^2-y^2z) - \frac{\partial}{\partial z}(3x^2+y^2z)\right) - \hat{j}\left[\frac{\partial}{\partial x}(xz^2-y^2z) - \frac{\partial}{\partial z}(xyz+y^2z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x^2+y^2z) - \frac{\partial}{\partial y}(xyz+y^2z)\right]$$

Assignment $\nabla \times \vec{F} = (-2yz - y^2)\hat{i} - (z^2 - xy - y^2)\hat{j} + (6x - xz + 2yz)\hat{k}$.

(3) Find the divergence and curl of the vector

soln same as 4 $\quad \vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$.

(4) Find $\operatorname{div}\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \operatorname{grad}(x^3+y^3+z^3-3xy)$

or $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$.

Soln Let $\phi = x^3+y^3+z^3-3xyz$

$$\vec{F} = \operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \vec{F} = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

Now $\quad \operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

$$\text{i.e.,} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left((3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k\right)$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

Thus $\operatorname{div}\vec{F} = 6x+6y+6z = 6(x+y+z)$

Also $\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & 3z^2-3xy \end{vmatrix}$

$$= i\left(\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right\} - j\left(\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right\} + k\left\{\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right\}$$

$= i\{-3x-(-3x)\} - j\{-3y-(-3y)\} + k\{-3z-(-3z)\}$

i.e., $\text{curl}\,\vec{F} = 0$ Thus $\text{div}\,\vec{F} = 6(x+y+z)$; $\text{curl}\,\vec{F} = \vec{0}$

(5) If $\vec{F} = \text{grad}\,(x^3y + y^3z + z^3x - x^2y^2z^2)$ find the $\text{div}\,\vec{F}$ & $\text{curl}\,\vec{F}$ at $[1, 2, 3]$.

Soln given $\vec{F} = \text{grad}\,(x^3y + y^3z + z^3x - x^2y^2z^2)$

$$= \nabla\cdot(x^3y + y^3z + z^3x - x^2y^2z^2)$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^3y + y^3z + z^3x - x^2y^2z^2)$$

$$\vec{F} = (3x^2y + z^3 - 2xy^2z^2)\,i + (x^3 + 3y^2z - 2x^2yz^2)\,\hat{j} + (y^3 + 3z^2x - 2x^2y^2z)\,\hat{k}$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(3x^2y + z^3 - 2xy^2z^2) + \frac{\partial}{\partial y}(x^3 + 3y^2z - 2x^2yz^2) + \frac{\partial}{\partial z}(y^3 + 3z^2x - 2x^2y^2z)$$

$$\text{div}\,\vec{F} = 6yx - 2y^2z^2 + 6yz - 2x^2z^2 + 6zx - 2x^2y^2$$

$$\text{div}\,\vec{F}\Big|_{(1,2,3)} = 12 - 72 + 36 - 18 + 18 - 8 = -32$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ 3x^2y + z^3 - 2xy^2z^2 & x^3 + 3y^2z - 2x^2yz^2 & y^3 + 3z^2x - 2x^2y^2z \end{vmatrix}$$

$$= i\,(3y^2 - 4x^2yz - (3y^2 - 4x^2yz)] - \hat{j}\,(3z^2 - 4xy^2z - (3z^2 - 4xy^2z)] + \hat{k}\,[(3x^2 - 4xyz^2) - (3x^2 - 4xyz^2)]$$

$$\text{curl}\,\vec{F} = 0$$

(6) Find the constant a such that $\vec{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$ is solenoidal.

Soln Given $\vec{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$

Since $\vec{F}$ is solenoidal $\Rightarrow$ $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = 0$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x} y(ax^2+z) + \frac{\partial}{\partial y} x(y^2-z^2) + \frac{\partial}{\partial z} 2xy(z-xy) = 0$$

$$\Rightarrow 2ayx + 2yx + 2xy = 0$$

$$\Rightarrow 2ayx + 4xy = 0$$

$$\Rightarrow \cancel{2}a\cancel{y}\cancel{x} = -\overset{2}{\cancel{4}}\cancel{x}\cancel{y}$$

$$\boxed{a = -2}$$

Assignment Solns

Check whether the functions $u = e^x + \log y + xyz$, $v = \log x + e^y + xyz$, $w = xyz$ are functionally dependent

Soln

Consider $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} e^x + yz & \frac{1}{y} + xz & xy \\ \frac{1}{x} + yz & e^y + xz & xy \\ yz & zx & xy \end{vmatrix} = xy \begin{vmatrix} e^x + yz & \frac{1}{y} + xz & 1 \\ \frac{1}{x} + yz & e^y + xz & 1 \\ yz & zx & 1 \end{vmatrix}$$

$$= xy \begin{vmatrix} e^x & 1/y & 1 \\ 1/x & e^y & 1 \\ 0 & 0 & 1 \end{vmatrix} + xy \begin{vmatrix} yz & xz & 0 \\ yz & xz & 0 \\ yz & xz & 0 \end{vmatrix}$$

$$= xy \left[e^{x+y} - \frac{1}{xy} \right] + 0 = xy e^{x+y} - 1 \neq 0$$

Thus u, v, w are not functionally dependent.

Note:- Let u, v & w be the functions of three independent variables x, y & z. If the functions u, v & w are functionally dependent, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Find the constants a & b such that $\frac{\partial(\ldots)}{\partial(x,y,z)}$ ...