Linear differential Equations

A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

Thus, an ordinary linear differential equation of order n is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X \dots (1)$$

where the coefficients $P_1, P_2, \dots, P_n$, and X are function of x in general.

If $X=0$, then the equation (1) is called a homogeneous equation; otherwise, it is a non-homogeneous equation.

Linear differential equations with constant co-efficients

An ordinary linear differential equation of order n with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x) \dots (1)$$

where $a_1, a_2, \dots a_n$ are constants and $f(x)$ is a known function of the independent variable x.

Homogeneous equations of second and higher order with constant co-efficients

We first consider linear homogeneous equations of second order with constant co-efficients. These equations are of the form

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = 0 \quad -- (1)$$

where P and Q are constants.

We often write y' or Dy for $\frac{dy}{dx}$ and y'' or $D^2 y$ for $\frac{d^2 y}{dx^2}$. Thus, equation (1) is rewritten in the following two alternative forms:

$$y'' + Py' + Qy = 0 \quad -(2)$$

$$(D^2 + PD + Q)y = 0 \quad -(3).$$

Equation (3) is written in the compact form:

$$F(D)y = 0 \quad \rightarrow (4)$$

where $F(D) = D^2 + PD + Q \rightarrow (5)$

$$= \frac{d^2}{dx^2} + P\frac{d}{dx} + Q \quad - (6)$$

is a second order linear differential operator.

The solution of (2) consists of two parts

Part 1: Solution obtained by solving $(D^2+PD+Q)y=0$ which we call it as Complementary function (C.F).

Part 2: Solution obtained by solving $(D^2+PD+Q)y = X$ where X is a function of x or constant & this is called particular Integral (P.I)

Thus the complete solution is

y = Complementary function + particular Integral

$y = C.F + P.I.$

Methods of finding Complementary Function (C.F)

Step 1: Find Auxiliary Equation (A.E) $f(m)=0$ by writing $D=m$ in $f(D)$ of equation (2).

Step 2: Find the roots of the A.E i.e. values of m. Let the roots are $m_1, m_2, \ldots, m_n$.

Step 3: required C.F is obtained as per the roots stated below

Roots of A.E	Complementary function
(1) All roots $m_1, m_2, \ldots, m_n$ are real and distinct	$c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}$

(2) If $m_1 = m_2 = m_3 = \cdots = m_n = m$ be the roots of the AE associated with an n^{th} order equation then $y = (c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1})e^{mx}$ is the general solution. (Real and repeated)

③ If $m_1 = m_2 = m_3 = m$ and the remaining roots are real and distinct then the general solution is of the form $y = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$

④ If the roots of A.E. are complex i.e., $\alpha \pm i\beta$ then C.F is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

⑤ If the root is purely imaginary i.e., $\pm i\beta$ $(\alpha = 0)$ then $y = c_1 \cos\beta x + c_2 \sin\beta x$.

⑥ If the complex root $(\alpha \pm i\beta)$ is repeated n times then the general solution is given by

$$y = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2 + \dots + c_{n-1} x^{n-1})\cos\beta x + (c_1' + c_2' x + c_3' x^2 + \dots + c_{n-1}' x^{n-1})\sin\beta x\right].$$

① Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$

Soln The given equation can be written as

$(D^2 + D - 2)y = 0$ where $D = \dfrac{d}{dx}$.

∴ Auxiliary equation is $m^2 + m - 2 = 0$ i.e., $(m-1)(m+2) = 0$ $\Rightarrow m = 1, -2$

∴ The roots of this equation are $m_1 = 1$ & $m_2 = -2$, which are real and distinct.

Thus the general solution of this equation is

$y = c_1 e^{x} + c_2 e^{-2x}$.

② Solve $(D^2 + 10D + 25)y = 0$

Soln A.E is $m^2 + 10m + 25 = 0$

$m^2 + 10m + 25 = 0$

$m^2 + 5m + 5m + 25 = 0$

$m(m+5) + 5(m+5) = 0$

$(m+5)^2 = 0$

$m = -5, -5$

or $m = \dfrac{-10 \pm \sqrt{\cancel{100 - 4\times 25}}}{2} = \dfrac{-\cancel{10}^{5}}{\cancel{2}}$

$m = -5, -5$

$(m+5)^2 = 0$

Since the roots are real & repeated, the general solution of the given equation is given by

$y = (c_1 + c_2 x)e^{-5x}$.

③ Solve $y'' - 6y' + 13y = 0$

Soln The given equation can be written as

$(D^2 - 6D + 13)y = 0$ where $D = d/dx$

A.E is $m^2 - 6m + 13 = 0$ i.e $m = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4\times1\times13}}{2\times1}$

$\boxed{ax^2+bx+c=0 \quad x = -b \pm \sqrt{b^2-4ac}/2a}$

$m = \dfrac{6 \pm \sqrt{36 - 52}}{2} = 6 \pm \sqrt{-16}/2 = 6 \pm i4/2$

$m = 3 \pm i2$

Here the roots are complex i.e., $m_1 = 3+2i$ & $m_2 = 3-2i$

$\therefore$ The general solution is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

④ Solve $(D^2+16)y=0$

Soln A.E is $m^2+16=0$

$m^2 = -16$

$m = \pm\sqrt{-16} = \pm 4i$

The general solution is

$$y = e^{0x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$y = c_1 \cos 4x + c_2 \sin 4x.$$

⑤ Solve $(D^3-2D^2+4D-8)y=0$

Soln. A.E is $m^3-2m^2+4m-8=0$ this cannot be factorized. We shall use synthetic division method by first finding a root by inspection

$m=2$ is a root (by inspection)

2	1	-2	4	-8
	0	2	0	8
	1	0	4	$\boxed{0}$

i.e $m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m=\pm\sqrt{-4}$.

$m = \pm 2i$

The roots are $2, \pm 2i$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + (c_2 \cos 2x + c_3 \sin 2x)$$

⑥ Solve $(D^4+8D^2+16)y=0$

Soln A.E is $m^4+8m^2+16=0$

$(m^2+4)^2=0$

The roots are $m = \pm 2i, \pm 2i$

$\therefore$ The general soln of the given equation is

$$y = (c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x.$$

⑦ $(D^4+2D^3+3D^2+2D+1)y=0$

Soln AE is $m^4+2m^3+3m^2+2m+1=0$

$\Rightarrow (m^4+2m^3+m^2)+(2m^2+2m+1)=0$

or $(m^2+m)^2+2(m^2+m)+1=0$

or $(m^2+m+1)^2=0$

$\Rightarrow \quad m = \frac{-1 \pm \sqrt{-3}}{2}, \frac{-1 \pm \sqrt{-3}}{2}$ i.e $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Hence the general solution of the equation is

$$y = e^{-1/2 x}\left[(c_1+c_2x)\cos\left(\frac{\sqrt{3}}{2}x\right) + (c_3+c_4x)\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

(8) Solve $(D^3 - D^2 + 4D - 4)y = 0$

Soln A.E is $m^3 - m^2 + 4m - 4 = 0$

$m = 1$ is a root (by inspection)

$$\begin{array}{c|cccc} 1 & 1 & -1 & 4 & -4 \\ & 0 & 1 & 0 & 4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

i.e., $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm\sqrt{-4} = \pm 2i$

The roots are $1, +2i, -2i$

$\therefore$ The general solution is

$$y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x.$$

(9) Solve $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

Soln A.E is $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

$m = 1$ is a root (by inspection)

$$\begin{array}{c|ccccc} 1 & 1 & -2 & 5 & -8 & 4 \\ & 0 & 1 & -1 & 4 & -4 \\ \hline & 1 & -1 & 4 & -4 & \boxed{0} \end{array}$$

i.e, $m^3 - m^2 + 4m - 4 = 0$

$m = 1$ is a root (by inspection)

$$\begin{array}{c|cccc} 1 & 1 & -1 & 4 & -4 \\ & 0 & 1 & 0 & 4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

$m^2 + 4 = 0$

$m^2 = -4 \Rightarrow m = \pm\sqrt{-4} = \pm 2i$

The roots are $1, 1, \pm 2i$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x.$$

(10) Solve $(D^4 + 5D^3 + 6D^2 - 4D - 8)y = 0$

Soln A.E is $m^4 + 5m^3 + 6m^2 - 4m - 8 = 0$

By inspection $m = 1$ is a root. Using synthetic division $m = 1$

$$\begin{array}{c|ccccc} 1 & 1 & 5 & 6 & -4 & -8 \\ & 0 & 1 & 6 & 12 & 8 \\ \hline -2 & 1 & 6 & 12 & 8 & \boxed{0} \\ & 0 & -2 & -8 & -8 & \\ \hline -2 & 1 & 4 & 4 & \boxed{0} & \\ & 0 & -2 & -4 & & \\ \hline & 1 & 2 & \boxed{0} & & \end{array}$$

$\therefore m^3 + 6m^2 + 12m + 8 = 0$

$m = -2$

$\therefore m^2 + 4m + 4 = 0$

$\therefore m + 2 = 0$

$\boxed{m = -2}$

Thus the roots of the A.E are $1, -2, -2, -2$

Hence the general solution is

$$y=c_1e^{x}+(c_2+c_3x+c_4x^2)e^{-2x}$$

Assignment

1. Solve : $(D^3+3D+2)y=0$ Ans $y=(c_1+c_2x)e^{x}+c_3e^{-2x}$
2. Solve the equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$

   Ans $y=e^{3x}(c_1\cos 2x+c_2\sin 2x)$

Non-homogeneous equations of second-order with constant coefficients

Working Rule.

1. Given the equation $\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=f(x)\ldots(1)$ find the complimentary function of homogeneous part of the differential equation (i.e. by taking $f(x)=0$)
2. Find a function $v(x)$ that satisfies the equation (1). This is a particular integral (PI)
3. write down the general solution (or complete solution) of Equation (1) as $y=C.F+P.I$

Now we proceed to consider a general method of constructing a particular integral. Inverse operator and particular integral.

A particular integral of the equation $\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=f(x)\ldots(1)$ is a function of $v(x)$ that satisfies the equation.

Thus, if we rewrite equation (1) as $(D^2+a_1D+a_2)y=f(x)$ i.e. $F(D)y=f(x)$ then a PI $v(x)$ is a function such that $[F(D)]v(x)=f(x)\ldots(2)$ i.e., $v(x)=\frac{1}{F(D)}f(x)\ldots(3)$ where $\frac{1}{F(D)}$ is the inverse of the linear differential operator (FD).

This inverse operator is defined through the condition that the relations (2) and (3) are equivalent to one another. It can be proved that $\frac{1}{F(D)}$ is also a linear operator in the sense that $\frac{1}{F(D)}(f(x_1)+f(x_2))=\frac{1}{F(D)}f(x_1)+\frac{1}{F(D)}f(x_2)$

Specific forms of particular integrals.

Type I: When $f(x) = e^{ax}$

$$\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}$$ where $F(a) \neq 0$ (D being replaced by a)

$$\frac{1}{F(D)} e^{ax} = \frac{x e^{ax}}{F'(a)}$$ ($F(a) = 0$, $F'(a) \neq 0$)

$$\frac{1}{F(D)} e^{ax} = \frac{x^2 e^{ax}}{F''(a)}$$ ($F'(a) = 0$, $F''(a) \neq 0$) and so on.

① $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 5e^{2x}$$

Soln: the given equation can be written as $(D^2 - 6D + 9)y = 5e^{2x}$

AE of the given equation is $m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$(m-3)^2 = 0$

$\therefore$ the roots are $m = 3, 3$

$\therefore$ CF : $y_c = (c_1 + c_2 x)e^{3x}$

now $$P.I = y_p = \frac{1}{D^2 - 6D + 9} \cdot 5e^{2x} = \frac{5}{2^2 - 6\times 2 + 9} e^{2x} = 5e^{2x}$$

Hence the complete solution of the given equation is

$$y = y_c + y_p = (c_1 + c_2 x)e^{3x} + 5e^{2x}$$

② Solve $y'' + 4y' + 3y = 2e^{-3x}$

Soln: The given equation can be written as

$$(D^2 + 4D + 3)y = 2e^{-3x}$$

AE of the given equation is $m^2 + 4m + 3 = 0$

$m^2 + 3m + m + 3 = 0$

$m(m+3) + 1(m+3) = 0$

$(m+3)(m+1) = 0$

$\therefore$ The roots are $m = -3, -1$

Therefore $C.F = c_1 e^{-3x} + c_2 e^{-x}$

$$P.I = \frac{1}{D^2 + 4D + 3} 2e^{-3x}$$ Replace D by -3

$$= \frac{1}{(-3)^2 + 4\times -3 + 3} 2e^{-3x} = \frac{2}{9 - 12 + 3} e^{-3x}$$ (Denominator $= 0$)

[Here $F(a) = 0$. Incase of failure $\frac{1}{F(D)} e^{ax} = x\frac{e^{ax}}{F'(a)}$ $\boxed{F'(a) \neq 0}$

$$= \frac{2x e^{-3x}}{2D + 4} = \frac{2x e^{-3x}}{2\times -3 + 4} = \frac{\cancel{2} x e^{-3x}}{-\cancel{2}} = -x e^{-3x}$$

$P.I = -xe^{-3x}$

$\therefore$ complete solution $y = C_1e^{-3x} + C_2e^{-x} - xe^{-3x}$

(3) Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x + e^{3x}$

soln A.E of the given equation is $m^2 - 6m + 9 = 0$

$\Rightarrow m^2 - 3m - 3m + 9 = 0 \Rightarrow m(m-3) - 3(m-3) = 0 \Rightarrow m = 3, 3$

Thus $C.F = (C_1 + C_2x)e^{3x}$

$$P.I = \frac{1}{D^2 - 6D + 9}(e^x + e^{3x})$$

$$= \frac{1}{D^2-6D+9}e^x + \frac{1}{D^2-6D+9}e^{3x}$$

$$= \frac{1}{1-6+9}e^x + \frac{1}{3^2-6\times3+9}e^{3x} = \frac{1}{4}e^x + \frac{1}{0}e^{3x} \quad (D^r = 0 \text{ for } D = 3)$$

$$= \frac{1}{4}e^x + \frac{xe^{3x}}{2D-6} = \frac{1}{4}e^x + \frac{x^2e^{3x}}{2} \quad \left[2^{nd} \text{ term } \frac{1}{F(D)}e^{ax} = \frac{x^2e^{ax}}{F''(a)}\right]$$

$\therefore$ complete solution

$$y = (C_1 + C_2x)e^{3x} + \frac{1}{4}e^x + \frac{x^2}{2}e^{3x}.$$

(4) Solve $(D^3 + D^2 + D + 1)y = e^{3x+4}$

soln A.E of the given equation is $m^3 + m^2 + m + 1 = 0$

i.e $m^2(m+1) + 1(m+1) = 0 \Rightarrow (m^2+1)(m+1) = 0$

$\Rightarrow m = \pm i, -1$

$CF = C_1\cos x + C_2\sin x + C_3e^{-x}$

$$P.I = \frac{1}{D^3 + D^2 + D + 1}e^{3x+4} = \frac{1}{27+9+3+1}e^{3x+4} = \frac{1}{40}e^{3x+4}.$$

$\therefore$ General solution $y = C.F + P.I = C_1\cos x + C_2\sin x + C_3e^{-x} + \frac{1}{40}e^{3x+4}$

(5) solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = (e^x + 1)^2$

Soln The AE of the given d.e is $m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0$

$\Rightarrow m = -2, -3 \quad \therefore CF = C_1e^{-2x} + C_2e^{-3x}$

$$P.I = \frac{1}{D^2+5D+6}(e^x+1)^2 = \frac{1}{D^2+5D+6}(e^{2x} + 2e^x + 1)$$

$$= \frac{1}{D^2+5D+6}e^{2x} + \frac{1}{D^2+5D+6}2e^x + \frac{1}{D^2+5D+6}e^{0x}$$

$$= \frac{1}{4+10+6}e^{2x} + \frac{2}{1+5+6}e^x + \frac{1}{6}. = \frac{1}{20}e^{2x} + \frac{1}{6}e^x + \frac{1}{6}$$

$\therefore y = CF + PI = C_1e^{-2x} + C_2e^{-3x} + 1/20\, e^{2x} + 1/6\, e^x + 1/6$ //

(6) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -3\sinh x$

Soln The givend. Equation is $(D^2+4D+5)y = -3\sinh x$

AE of the d.E is $m^2+4m+5=0$

$$m = \frac{-4 \pm \sqrt{16-4\times1\times5}}{2} = \frac{-4\pm\sqrt{-4}}{2}$$

$$m = -2 \pm i$$

$$C.F = e^{-2x}(c_1\cos x + c_2\sin 2x)$$

$$PI = \frac{1}{D^2+4D+5} - 3\sinh x = \frac{-3}{D^2+4D+5}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= -\frac{3}{2}\left[\frac{e^x}{D^2+4D+5} - \frac{e^{-x}}{D^2+4D+5}\right]$$

$$= -\frac{3}{2}\left[\frac{e^x}{1+4+5} - \frac{e^{-x}}{1-4+5}\right] = -\frac{3}{2}\left[\frac{e^x}{10} - \frac{e^x}{2}\right]$$

$$= \frac{3}{20}(5e^{-x} - e^x)$$

Therefore the complete solution is given by

$$y = e^{-2x}(c_1\cos x + c_2\sin 2x) + \frac{3}{20}(5e^{-x} - e^x)$$

(7) Solve : $(D^2-4)y = 3\cosh^2 x$

Soln Here, the A.E is $m^2-4=0 \Rightarrow m = \pm 2$

So that $CF = c_1e^{2x} + c_2e^{-2x}$

Also $f(x) = 3\cosh^2 x = 3\left\{\frac{e^x+e^{-x}}{2}\right\}^2 = \frac{3}{4}(e^{2x}+e^{-2x}+2)$

So that

$$P.I = \frac{1}{F(D)} f(x) = \frac{1}{D^2-4}\frac{3}{4}(e^{2x}+e^{-2x}+2)$$

$$= \frac{3}{4}\left\{\underbrace{\frac{1}{D^2-4}}_{F(D)}e^{2x} + \underbrace{\frac{1}{D^2-4}}_{F(D)}e^{-2x} + \underbrace{\frac{1}{D^2-4}}_{F(D)}2e^{0x}\right\}$$

$$= \frac{3}{4}\left\{\frac{x}{F'(2)}e^{2x} + \frac{1}{F'(-2)}xe^{-2x} + \frac{2}{F(0)}e^{0x}\right\}$$

$$= \frac{3}{4}\left\{\frac{xe^{2x}}{2D} + \frac{xe^{-2x}}{2D} + \frac{2}{-4}\right\}$$

$$= \frac{3}{4}\left\{\frac{xe^{2x}}{4} - \frac{xe^{-2x}}{4} - \frac{1}{2}\right\}$$

$$= \frac{3}{16}(xe^{2x} - xe^{-2x} - 2)$$

Therefore, the general solution of the given Equation is

$$y = C.F + P.I$$

$$= c_1e^{2x} + c_2e^{-2x} + \frac{3}{16}(xe^{2x} - xe^{-2x} - 2)$$

(8) Solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$.

Soln Here the A.E is $F(m) = m^2 - 4 = 0$, whose roots are $m_1 = 2$ & $m_2 = -2$. Therefore,

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

Next, we note that

$$f(x) = \cosh(2x-1) + 3^x$$
$$= \frac{1}{2}\{e^{(2x-1)} + e^{-(2x-1)}\} + e^{x\log 3}$$
$$= \frac{1}{2}e^{2x-1} + \frac{1}{2}e^{-(2x-1)} + e^{x\log 3}$$

Therefore,

$$P.I = \frac{1}{2}\frac{1}{D^2-4}e^{2x-1} + \frac{1}{2}\frac{1}{D^2-4}e^{-(2x-1)} + \frac{1}{D^2-4}e^{x\log 3}$$
$$= \frac{1}{2}\frac{xe^{2x-1}}{2D_2} + \frac{1}{2}\frac{xe^{-(2x-1)}}{2D_{-2}} + \frac{1}{(\log 3)^2-4}3^x$$
$$P.I = \frac{xe^{2x-1}}{8} - \frac{1}{8}xe^{-(2x-1)} + \frac{3^x}{(\log 3)^2-4}$$

$\therefore$ Complete solution is

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{8}\left(e^{2x-1} - e^{-(2x-1)}\right) + \frac{3^x}{(\log 3)^2-4}$$
$$= c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

Assignment

1. Solve $(D^2-6D+9)y = 6e^{3x} + 7e^{-2x} - \log 2$.

Ans $y = (c_1+c_2x)e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$

2. $y'' + 2y' + y = \cosh\left(\frac{x}{2}\right)$

Ans $y = (c_1+c_2x)e^{-x} + \frac{2}{9}e^{x/2} + 2e^{-x/2}$.

Type 2: When $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{F(D^2)}\sin(ax+b) = \frac{1}{F(-a^2)}\sin(ax+b), \text{ provided } F(-a^2) \neq 0$$

If $F(-a^2) = 0$, $\frac{1}{F(D^2)}\sin(ax+b) = x \cdot \frac{1}{F'(-a^2)}\sin(ax+b)$, provided $F'(-a^2) \neq 0$.

If $F'(-a^2) = 0$, $\frac{1}{F(D^2)}\sin(ax+b) = x^2 \cdot \frac{1}{F''(-a^2)}\sin(ax+b)$, provided $F''(-a^2) \neq 0$ and so on.

Similarly, $\frac{1}{F(D^2)}\cos(ax+b) = \frac{1}{F(-a^2)}\cos(ax+b)$, provided $F(-a^2) \neq 0$

If $F(-a^2)=0$, $\frac{1}{F(D^2)}\cos(ax+b)= x\cdot\frac{1}{F'(-a^2)}\cos(ax+b)$, provided $F'(-a^2)\neq 0$

If $F'(-a^2)=0$, $\frac{1}{F(D^2)}\cos(ax+b)= x^2\frac{1}{F''(-a^2)}\cos(ax+b)$, provided $F''(-a^2)\neq 0$ and so on.

solve:

① $(D^2-8D+9)y = 8\sin 5x$

Soln A.E is $m^2-8m+9=0$

$$m= \frac{-(-8)\pm\sqrt{(-8)^2-4\times 9}}{2} = \frac{8\pm\sqrt{64-36}}{2} = \frac{8\pm\sqrt{28}}{2}$$

$$= \frac{8\pm\sqrt{7\times 4}}{2} = \frac{8\pm 2\sqrt{7}}{2} = 4\pm\sqrt{7}$$

whose roots are $m_1=4+\sqrt{7}$ and $m_2=4-\sqrt{7}$.

Therefore,

$$C.F = c_1 e^{(4+\sqrt{7})x} + c_2 e^{(4-\sqrt{7})x}$$

$$P.I = \frac{1}{F(D)}f(x) = \frac{1}{D^2-8D+9}8\sin 5x \qquad \text{Replace } D^2 \text{ by } (-5^2)$$

$$= \frac{8\sin 5x}{-5^2-8D+9} = \frac{8\sin 5x}{-25-8D+9} = \frac{8\sin 5x}{-8D-16} = \frac{8\sin 5x}{-8(D+2)}$$

$$= -\frac{1}{D+2}\sin 5x = -\frac{1}{D+2}\frac{D-2}{D-2}\sin 5x = -\frac{(D-2)}{D^2-4}\sin 5x$$

$$= \frac{-(D-2)}{-5^2-4}\sin 5x = \frac{(D-2)\sin 5x}{29} = \frac{1}{29}[D\sin 5x - 2\sin 5x]$$

$$= \frac{1}{29}[5\cos 5x - 2\sin 5x]$$

Therefore, the general solution of the given equation is $y = C.F + P.I$

$$= c_1 e^{(4+\sqrt{7})x} + c_2 e^{(4-\sqrt{7})x} + \frac{1}{29}[5\cos 5x - 2\sin 5x]$$

② Solve $y''+3y'+2y = 4\cos^2 x$

Soln Here, the A.E is

$$m^2+3m+2=0$$
$$m^2+m+2m+2=0$$
$$m(m+1)+2(m+1)=0$$
$$\Rightarrow (m+2)(m+1)=0 \Rightarrow m=-1,-2$$

Therefore, $C.F = c_1 e^{-x} + c_2 e^{-2x}$

WKT $4\cos^2 x = 2\cdot 2\cos^2 x = 2(1+\cos 2x)$

$$P.I = \frac{2}{D^2+3D+2}(1+\cos 2x)$$

$$= \frac{2}{D^2+3D+2}e^{0.x} + \frac{2}{D^2+3D+2}\cos 2x$$

$$= \frac{2}{2} + \frac{2}{-2^2+3D+2}\cos 2x = 1+\frac{2}{3D-2}\cos 2x$$

$$= 1+\frac{2}{3D-2}\times\frac{(3D+2)}{(3D+2)}\cos 2x = 1+\frac{2(3D+2)\cos 2x}{(3D)^2-2^2 = 9\times -2^2-4} = 1+\frac{6D(\cos 2x)+4\cos 2x}{-36-4}$$

$= 1 + \frac{6x - 2\sin 2x + 4\cos 2x}{-40} = 1 + \frac{\cancel{2}(-6\sin 2x + 2\cos 2x)}{-\cancel{40}\,20}$

$= 1 - \frac{1}{20}(-6\sin 2x + 2\cos 2x) = 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$

Therefore, the general solution of the given equation is $y = C.F + P.I = c_1 e^{-x} + c_2 e^{-2x} + 1$
$\frac{1}{10}(3\sin 2x - \cos 2x)$.

(3) Solve $y'' + 4y' - 12y = e^{2x} - 3\sin 2x$

Soln Here the A.E is $(D^2 + 4D - 12)y = e^{2x} - 3\sin 2x$

$m^2 + 4m - 12 = 0 \Rightarrow m^2 - 2m + 6m - 12 = 0$

$\Rightarrow m(m-2) + 6(m-2) = 0 \Rightarrow (m-2)(m+6) = 0$

$m = -6, 2$

$C.f = c_1 e^{-6x} + c_2 e^{2x}$

Next.

$P.I = \frac{1}{D^2 + 4D - 12}(e^{2x} - 3\sin 2x)$

$= \frac{1}{D^2 + 4D - 12} e^{2x} - \frac{3}{D^2 + 4D - 12}\sin 2x$

$= \frac{1}{4 + 8 - 12 = 0.} e^{2x} - \frac{3}{-2^2 + 4D - 12}\sin 2x$

$= \frac{1}{2D + 4} x e^{2x} - \frac{3}{4D - 16}\sin 2x$

$= \frac{x}{8} e^{2x} - \frac{3}{4(D-4)} \times \frac{D+4}{D+4}\sin 2x$

$= \frac{xe^{2x}}{8} - \frac{3}{4}\frac{(D+4)\sin 2x}{D^2 - 16} = \frac{xe^{2x}}{8} - \frac{3}{4}\frac{(D+4)\sin 2x}{-2^2 - 16}$

$= \frac{xe^{2x}}{8} + \frac{3}{80}[D\sin 2x + 4\sin 2x]$

$= \frac{xe^{2x}}{8} + \frac{3}{80}[2\cos 2x + 4\sin 2x]$

$P.I = \frac{xe^{2x}}{8} + \frac{3}{\cancel{80}\,40} \times \cancel{2}[\cos 2x + 2\sin 2x]$

Therefore, the general solution is

$y = C.F + P.I$

$= c_1 e^{-6x} + c_2 e^{2x} + \frac{1}{8} x e^{2x} + \frac{3}{40}(\cos 2x + 2\sin 2x]$

(4) Solve : $y'' + 2y' + y = e^{2x} - \cos^2 x$

Soln:- Here, the A.E is $m^2+2m+1 = (m+1)^2=0$ which yields $C.F = (C_1+C_2x)e^{-x}$

Next, $P.I = \frac{1}{D^2+2D+1} e^{2x} - \cos^2 x = \frac{1}{D^2+2D+1} e^{2x} - \frac{1}{D^2+2D+1}\cos^2 x$

$$P.I = \frac{1\, e^{2x}}{4+8+1} - \frac{1}{D^2+2D+1}\frac{(1+\cos 2x)}{2}$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2}\frac{1}{D^2+2D+1}e^{0x} - \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{2}\frac{1}{-2^2+2D+1}\cos 2x$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{2}\frac{1}{2D-3}\times\frac{2D+3}{2D+3}\cos 2x$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{2}\frac{(2D+3)}{4D^2-9}\cos 2x = \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{2}\frac{(2D+3)\cos 2x}{-4\times 2^2-9}$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{2}\frac{2D\cos 2x + 3\cos 2x}{-25}$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} + \frac{1}{2\times 25}(-2\times 2\sin 2x + 3\cos 2x)$$

$$= \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{50}(4\sin 2x - 3\cos 2x)$$

Hence the General solution of the given equation is

$$y = (C_1+C_2x)e^{-x} + \frac{1}{13}e^{2x} - \frac{1}{2} - \frac{1}{50}(4\sin 2x - 3\cos 2x)$$

(5) Solve $(D^2+a^2)y = \cos ax$

Soln Here the A.E is $m^2+a^2=0 \Rightarrow m^2=-a^2 \Rightarrow m=\pm ia$

Therefore $C.F = C_1\cos ax + C_2\sin ax$

Also, $P.I = \frac{1}{D^2+a^2}\cos ax = \frac{1}{-a^2+a^2=0}\cos ax = \frac{x}{2D}\cos ax$

$\boxed{\frac{1}{D}\cos x = \int \cos ax}$

$$= \frac{x}{2}\frac{1}{D}\cos ax = \frac{x}{2}\frac{\sin ax}{a}$$

$$y = C.F + P.I = C_1\cos ax + C_2\sin ax + \frac{x}{2a}\sin ax.$$

(6) Solve $(D^2+a^2)y = \sin ax$

Soln $C.F = C_1\cos ax + C_2\sin ax$

$$P.I = {}^1/_{D^2+a^2}\sin ax = \frac{1}{-a^2+a^2=0}\sin ax = \frac{x}{2D}\sin ax$$

$$= \frac{x}{2}\frac{1}{D}\sin ax = -\frac{x}{2}\frac{\cos ax}{a}$$

$$y = C.F + P.I = C_1\cos ax + C_2\sin ax - x/_{2a}\cos ax.$$

(7) Solve : $y'' + y = \sin x \sin 2x$

Soln Here the AE is $(m^2+1) = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$C.F = C_1 \cos x + C_2 \sin x$

Also $f(x) = \sin x \sin 2x = -\frac{1}{2}[\cos 3x - \cos(-x)]$

$\sin A . \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$= \frac{1}{2}[\cos x - \cos 3x]$. Therefore,

$$P.I = \frac{1}{D^2+1}\frac{1}{2}[\cos x - \cos 3x] = \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1^2+1}\cos x - \frac{1}{-3^2+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D}\cos x - \frac{1}{-8}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{1}{8}\cos 3x\right] = \frac{1}{16}\left[4x\sin x + \cos 3x\right]$$

Therefore, the general solution of the given equation is $y = C.F + P.I$.

$$= C_1\cos x + C_2 \sin x + \frac{1}{16}(4x\sin x + \cos 3x).$$

(8) Solve $(D^2+4)y = e^x + \sin 2x$

Soln Here the A.E is $m^2+4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$

$C.F = C_1 \cos 2x + C_2 \sin 2x$

$$P.I = \frac{1}{D^2+4}(e^x + \sin 2x) = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{5}e^x + \frac{1}{-2^2+4=0}\sin 2x = \frac{1}{5}e^x + \frac{x}{2D}\sin 2x$$

$$= \frac{1}{5}e^x + \frac{x}{2}\frac{1}{D}\sin 2x = \frac{1}{5}e^x + \frac{x}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$P.I = \frac{1}{5}e^x - \frac{x\cos 2x}{4}$$

Therefore, the general solution of the given equation is $y = C.F + P.I$

$$= C_1\cos 2x + C_2 \sin 2x + \frac{1}{5}e^x - \frac{x\cos 2x}{4}.$$

(9) Solve $(D^2-4)y = e^x + \sin 2x$

Soln Here the A.E is $m^2-4=0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$.

$C.F = C_1 e^{2x} + C_2 e^{-2x}$

$$P.I = \frac{1}{D^2-4}(e^x + \sin 2x) = \frac{1}{D^2-4}e^x + \frac{1}{D^2-4}\sin 2x$$

$= \frac{1}{1-4}e^x + \frac{1}{-2^2-4}\sin 2x = \frac{1}{-3}e^x - \frac{1}{8}\sin 2x$

Therefore, the general solution of the given equation is $y = C.F + P.I = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x - \frac{1}{8}\sin 2x$.

Assignment.

① Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos 3x$ Ans:- $y = (c_1 + c_2x)e^x - \frac{1}{50}(3\sin 3x + 4\cos 3x)$

② Solve $y'' + 9y = \cos 2x \cos x$

Ans $y = c_1\cos 3x + c_2\sin 3x + \frac{x}{12}\sin 3x + \frac{\cos x}{16}$

Type 3: particular integral when $f(x)$ is a polynomial in x i.e., $f(x) = P_m(x)$.

Suppose $f(x) = P_m(x)$, where m is a polynomial of degree m. In this case, P.I. is given by

$$\psi(x) = \frac{1}{F(D)}\{P_m(x)\} \rightarrow ①$$

To evaluate the right-hand side, we write $\frac{1}{F(D)}$ as $[F(D)]^{-1}$ and expand it is ascending powers of D by using binomial expansion to get an expression of the form

$$\frac{1}{F(D)} = \{a_0 + a_1D + a_2D^2 + \dots\} \rightarrow ②$$

Then ① becomes

$$\psi(x) = \{a_0 + a_1D + a_2D^2 + \dots\}\{P_m(x)\} \dots ③$$

which can be simplified by using the fact that $D = \frac{d}{dx}$.

Binomial Expansions.

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

① Solve $(D^2 + D - 2)y = 1 + x - x^2$

Soln Here, the A.E is $m^2 + m - 2 = (m+2)(m-1) = 0$

Therefore $C.F = c_1e^{-2x} + c_2e^x \rightarrow ①$

Next $P.I = \frac{1}{F(D)}\{1 + x - x^2\} - ②$

We note that $\frac{1}{F(D)} = \frac{1}{(D+2)(D-1)} = \frac{1}{3}\left\{\frac{1}{D-1} - \frac{1}{D+2}\right\}$ → By partial fraction.

$$= -\tfrac{1}{3}\left\{\frac{1}{1-D} + \frac{1}{2(1+D/2)}\right\}$$

$\left[\text{since } \frac{1}{D-1} = \frac{-1}{1-D} \text{ \& } \frac{-1}{D+2} = -\frac{1}{2+D} = -\frac{1}{2(1+D/2)}\right]$

$$= -\tfrac{1}{3}\left\{(1-D)^{-1} + \tfrac{1}{2}(1+D/2)^{-1}\right\}$$

$$= -\frac{1}{3}\left\{(1+D+D^2+D^3+\cdots) + \frac{1}{2}\left(1-D/2+\frac{D^2}{4}-\frac{D^3}{8}+\cdots\right)\right\}$$

Using this in (2), we get

$$P.I = \frac{-1}{3}\left\{(1+D+D^2+D^3+\cdots)+\frac{1}{2}\left(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}+\cdots\right)\right\}(1+x-x^2)$$

$$= \frac{-1}{3}\left\{(1+D+D^2+D^3+\cdots)(1+x-x^2)+\frac{1}{2}\left(1-D/2+\frac{D^2}{4}-\frac{D^3}{8}+\cdots\right)(1+x-x^2)\right\}$$

$$= \frac{-1}{3}\left\{(1+x-x^2)+(1-2x)-2+0+\frac{1}{2}\left((1+x-x^2)-\frac{1}{2}(1-2x)+\frac{1}{4}(-2)+0\right)\right\}$$

$$= \frac{-1}{3}\left\{\cancel{1}+x-x^2+\cancel{1}-2x-\cancel{2}+\frac{1}{2}+\frac{x}{2}-\frac{x^2}{2}-\cancel{\frac{1}{4}}+\frac{x}{2}-\cancel{\frac{1}{4}}\right\}$$

$$= -\frac{1}{\cancel{3}}\left\{-\frac{\cancel{3}x^2}{2}\right\} = \frac{x^2}{2}.$$ Therefore, the general solution of the given equation is

$$y = C.F + P.I = c_1e^{-2x} + c_2e^{x} + \frac{x^2}{2}.$$

(2.) $y'' + y' = x^2+2x+4$

soln A.E $m^2+m=0 \Rightarrow m(m+1)=0 \Rightarrow m=0$ or $m=-1$

$$C.F = c_1e^{0x}+c_2e^{-x} = c_1+c_2e^{-x}$$

$$P.I = \frac{1}{F(D)}(x^2+2x+4) = \frac{1}{D(D+1)}(x^2+2x+4)$$

$\left[\frac{1}{D(D+1)} = \frac{1}{D}-\frac{1}{D+1} \text{ by partial fraction}\right]$

$$= \left(\frac{1}{D}-\frac{1}{D+1}\right)(x^2+2x+4) = \frac{1}{D}(x^2+2x+4)-(1+D)^{-1}(x^2+2x+4)$$

$$= \int(x^2+2x+4)\,dx - (1-D+D^2-D^3+\cdots)(x^2+2x+4)$$

$$= \frac{x^3}{3}+x^2+4x-\left\{(x^2+2x+4)-(2x+2)+2\right\}$$

$$= \frac{x^3}{3}+\cancel{x^2}+4x-\cancel{x^2}-\cancel{2x}-4+\cancel{2x}+\cancel{2}-\cancel{2}$$

$$P.I = \frac{x^3}{3}+4x-4$$

Hence the general solution is

$$y = C.F + PI = c_1+c_2e^{-x}+\frac{x^3}{3}+4x-4$$

(3) Solve $y''-6y'+25y = e^{2x}+\sin x + x$

Soln A.E $m^2-6m+25=0$ $\quad m=\frac{-(-6)\pm\sqrt{(-6)^2-4\times25}}{2\times1}$

$m=3\pm4i$ $\quad =\frac{6\pm\sqrt{36-100}}{2}=\frac{6\pm\sqrt{-64}}{2}$

$C.F=e^{3x}(c_1\cos4x+c_2\sin4x)\rightarrow ①$ $\quad =\frac{6\pm\sqrt{-8^2}}{2}=\frac{6\pm8i}{2}$

$P.I=\frac{1}{F(D)}(e^{2x}+\sin x+x)$ $\quad =3\pm4i$

$=\frac{1}{F(D)}e^{2x}+\frac{1}{F(D)}\sin x+\frac{1}{F(D)}x$

We note that

$\frac{1}{F(D)}e^{2x}=\frac{1}{D^2-6D+25}e^{2x}=\frac{1}{4-12+25}e^{2x}=\frac{1}{17}e^{2x}$

$\frac{1}{F(D)}\sin x=\frac{1}{D^2-6D+25}\sin x=\frac{1}{-1^2-6D+25}\sin x=\frac{1}{24-6D}\sin x$

$=\frac{-1}{6(D-4)}\sin x=\frac{-1}{6(D-4)}\frac{(D+4)}{(D+4)}\sin x=\frac{-(D+4)}{6(D^2-16)}\sin x$

$=\frac{-(D+4)\sin x}{6(-1^2-16)}=\frac{-1}{6\times-17}\, D\sin x+4\sin x$

$=\frac{1}{102}(\cos x+4\sin x)$

$\frac{1}{F(D)}x=\frac{1}{D^2-6D+25}x=\frac{1}{25\left[1+\left(\frac{D^2}{25}-\frac{6D}{25}\right)\right]}x$

$=\frac{1}{25}\left[1+\left(\frac{D^2}{25}-\frac{6D}{25}\right)\right]^{-1}x$

$=\frac{1}{25}\left[1-\left(\frac{D^2}{25}-\frac{6D}{25}\right)+\left(\frac{D^2}{25}-\frac{6D}{25}\right)^2\cdots\right]x$

$=\frac{1}{25}\left[x+0+\frac{6}{25}\right)=\frac{1}{25}\left(x+\frac{6}{25}\right)$

$P.I=\frac{1}{17}e^{2x}+\frac{1}{102}(\cos x+4\sin x)+\frac{1}{25}\left(x+\frac{6}{25}\right)$

$y=C.F+P.I$

$=e^{3x}(c_1\cos4x+c_2\sin4x)+\frac{1}{17}e^{2x}+\frac{1}{102}(\cos x+4\sin x)$
$+\frac{1}{25}\left(x+\frac{6}{25}\right)$

(4) Solve $y''-4y=x^2$

Soln A.E is $m^2-4=0\Rightarrow m^2=4\Rightarrow m=\pm\sqrt{4}=\pm2$

C.F $c_1e^{2x}+c_2e^{-2x}$

$P.I=\frac{1}{F(D)}x^2=\frac{1}{D^2-4}x^2=\frac{1}{-4(1-D^2/4)}x^2$

$$= -\frac{1}{4}\left(1-D^2/4\right)^{-1} x^2$$

$$= -\frac{1}{4}\left(1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\cdots\right) x^2$$

$$P.I = -\frac{1}{4}\left(x^2+\frac{\cancel{2}}{\cancel{4}2}\right) = -\frac{1}{4}x^2-\frac{1}{8} = -\frac{1}{8}\left(2x^2+1\right)$$

$$\therefore \quad y = C.F + P.I = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}(2x^2+1)$$

(5) Solve $y'' + 3y' + 2y = 1 + 3x + x^2$

or

Solve $(D^2+3D+2)y = 1+3x+x^2$

Soln Here, the A.E is $m^2+3m+2 = (m+1)(m+2)=0$ whose roots are $-1, -2$, therefore,

$$C.F = c_1 e^{-x} + c_2 e^{-2x}.$$

Next, $P.I = \frac{1}{F(D)}(1+3x+x^2) = \frac{1}{D^2+3D+2}(1+3x+x^2).$

We note that

$$\frac{1}{D^2+3D+2} = \frac{1}{2}\left[\frac{1}{1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)}\right] = \frac{1}{2}\left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\left(\frac{3D}{2}+\frac{D^2}{2}\right)^2-\cdots\right]$$

$$= \frac{1}{2}\left[1-\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}+\frac{3D^3}{2}+\frac{D^4}{4}+\cdots\right]$$

$$= \frac{1}{2}\left[1-\frac{3D}{2}+\frac{7D^2}{4}+\cdots\right]$$

Therefore,

$$P.I = \frac{1}{2}\left(1-\frac{3D}{2}+\frac{7D^2}{4}+\cdots\right)(1+3x+x^2)$$

$$= \frac{1}{2}\left\{(1+3x+x^2)-\frac{3}{2}(3+2x)+\frac{7}{4}(2)\right\}$$

$$= \frac{1}{2}\left\{1+\cancel{3}x+x^2-\frac{9}{2}-\frac{\cancel{6}3}{\cancel{2}}x+\frac{7}{2}\right\}$$

$$= \frac{1}{2}x^2.$$

Hence, the general solution of the given equation is

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}x^2.$$

(6) Solve : $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Here the A.E is $(m-2)^2=0$, so that

$$C.F = (C_1+C_2x)e^{2x} \quad \ldots(1)$$

Next,

$$P.I = \frac{1}{F(D)}\{8(e^{2x}+\sin 2x+x^2)\}$$

$$= 8\left\{\frac{1}{F(D)}e^{2x}+\frac{1}{F(D)}\sin 2x+\frac{1}{F(D)}x^2\right\} \quad (2)$$

We have

$$\frac{1}{F(D)}e^{2x} = \frac{1}{(D-2)^2}e^{2x} = \frac{1}{(2-2)^2=0}e^{2x} = \frac{x}{2(D-2)}e^{2x}$$

$$= \frac{x}{2(2-2)=0}e^{2x} = \frac{x^2}{2}e^{2x} \rightarrow (3)$$

$$\frac{1}{F(D)}\sin 2x = \frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-2^2-4D+4}\sin 2x$$

$$= -\frac{1}{4}\frac{1}{D}(\sin 2x) = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{\cos 2x}{8} \rightarrow (4)$$

$$\frac{1}{F(D)}x^2 = \frac{1}{(D-2)^2}x^2 = \frac{1}{\ldots} = \frac{1}{4}(1-D/2)^{-2}x^2$$

$$= \frac{1}{4}\left\{1+\cancel{2}\left(\frac{D}{\cancel{2}}\right)+3\left(\frac{D}{2}\right)^2+4\left(\frac{D}{2}\right)^3+\ldots\right\}x^2$$

$$= \frac{1}{4}\left\{x^2+2x+\frac{3}{\cancel{4}2}\times\cancel{2}+0\right\} = \frac{1}{4}\left\{x^2+2x+\frac{3}{2}\right\}$$

$$= \frac{1}{8}(2x^2+4x+3) \ldots (5)$$

using (3) to (5) in (2), we get

$$P.I = 4x^2e^{2x}+\cos 2x+2x^2+4x+3 \rightarrow (6)$$

Therefore, the general solution of the given equation is $y = C.F + P.I$

$$= (C_1+C_2x)e^{2x}+4x^2e^{2x}+\cos 2x+2x^2+4x+3.$$

Assignment

1. Solve $(2D^2+2D+3)y = x^2+2x-1$

Ans $y = e^{-(x/2)}\left(C_1\cos\frac{\sqrt{5}}{2}x+C_2\sin\frac{\sqrt{5}}{2}x\right)+\frac{1}{3}\left(x^2+\frac{2}{3}x-\frac{25}{9}\right)$

2. Solve : $y''+5y'+6y = x^2+e^{-2x}$

Ans $y = C_1e^{-3x}+C_2e^{-2x}+\frac{1}{6}\left(x^2-\frac{5}{3}x+\frac{19}{18}\right)+xe^{-2x}$

Type 4 : particular integral when $f(x)=e^{ax}V$ where V is a function of x then

$$\frac{e^{ax}V}{F(D)} = e^{ax}\frac{1}{F(D+a)}V.$$

① Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^{x}\cos x$

Soln Here, the A.E is $m^2-2m+4=0$,

$$m = \frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times4}}{2} = \frac{2\pm\sqrt{4-16}}{2}$$

$$m = \frac{2\pm\sqrt{-12}}{2} = \frac{2\pm\sqrt{-(4\times3)}}{2} = \frac{2\pm2\sqrt{-3}}{2}$$

$$= 1\pm i\sqrt{3}$$

Therefore, $C.F = e^{x}(C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x)$

Next,

$$P.I = \frac{1}{F(D)}(e^{x}\cos x) = e^{x}\left\{\frac{1}{F(D+1)}\cos x\right\}$$

$$= e^{x}\frac{1}{(D+1)^2-2(D+1)+4}\cos x = e^{x}\frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-2+4}\cos x$$

$$= e^{x}\frac{1}{D^2+3}\cos x = \frac{e^{x}}{-1^2+3}\cos x = \frac{1}{2}e^{x}\cos x.$$

Therefore, the general solution of the given equation is

$$y = C.F + PI$$

$$= e^{x}\{c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + \tfrac{1}{2}\cos x\}$$

② Solve $(D^2-2D+5)y = e^{2x}\sin x$

Soln Here, the A.E is $m^2-2m+5=0$,

$$m = \frac{-(-2)\pm\sqrt{4-4\times5}}{2} = \frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{-16}}{2} = 1\pm i2$$

Therefore, $C.F = e^{x}(C_1\cos2x + C_2\sin2x)$

Next,

$$P.I = \frac{1}{F(D)}e^{2x}\sin x = e^{2x}\frac{1}{F(D+2)}\sin x$$

$$= e^{2x}\frac{1}{(D+2)^2-2(D+2)+5}\sin x = e^{2x}\frac{1}{D^2+2D+5}\sin x$$

$$= e^{2x}\frac{1}{D^2+4D+\cancel{4}-2D-\cancel{4}+5}$$

$$= e^{2x}\frac{1}{-1^2+2D+5}\sin x = e^{2x}\frac{1}{2D+4}\sin x = \frac{e^{2x}}{2}\frac{1}{D+2}\sin x$$

$$= \frac{e^{2x}}{2}\frac{1}{D+2}\times\frac{D-2}{D-2}\sin x = \frac{e^{2x}}{2}\frac{(D-2)\sin x}{D^2-2^2}$$

$$= \frac{e^{2x}}{2} \frac{(D-2)\sin x}{-1^2-2^2} = -\frac{1}{10} e^{2x}(\cos x - 2\sin x)$$

(3) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x + x$.

Soln Here, the AE is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$.

Therefore, $C.F. = (C + C_2 x)e^x$

Next, $P.I = \frac{1}{F(D) = D^2-2D+1}(xe^x + x) = e^x \frac{1}{F(D+1)} x + \frac{1}{F(D)} x$.

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x + \frac{1}{D^2 - 2D + 1} x$$

$$= e^x \frac{1}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - \cancel{2} + \cancel{1}} x + [1 + (D^2 - 2D)]^{-1} x$$

$$= e^x \frac{1}{D^2} x + [1 - (D^2 - 2D) + \ldots] x$$

$$= e^x \frac{x^3}{6} + (x - 0 + 2) = \frac{1}{6} x^3 e^x + (x+2)$$

or
$\frac{1}{(D-1)^2} x$
$= (D-1)^{-2} x$
$= (1 + 2D + 3D^2 + \ldots) x$
$= x + 2$.

(4) Solve: $\frac{d^2y}{dx^2} + 4y = 2e^x \sin^2 x$.

Soln Here, $F(D) = D^2 + 4$, which leads to

$C.F = C_1 \cos 2x + C_2 \sin 2x$.

Also, $P.I = \frac{1}{F(D)}(2e^x \sin^2 x) = \frac{1}{D^2+4} \cancel{2} e^x \frac{(1-\cos 2x)}{\cancel{2}}$

$$= \frac{1}{D^2+4} e^x - \frac{1}{D^2+4} e^x \cos 2x = \frac{1}{5} e^x - e^x \frac{1}{(D+1)^2+4} \cos 2x$$

$$= \frac{1}{5} e^x - e^x \frac{1}{D^2+2D+1+4} \cos 2x = \frac{1}{5} e^x - e^x \frac{1}{-2^2+2D+5} \cos 2x$$

$$= \frac{1}{5} e^x - e^x \frac{1}{2D+1} \cos 2x = \frac{1}{5} e^x - e^x \frac{(2D-1)}{(2D+1)(2D-1)} \cos 2x$$

$$= \frac{1}{5} e^x - e^x \frac{(2D\cos 2x - \cos 2x)}{4D^2 - 1}$$

$$= \frac{1}{5} e^x - e^x \frac{2D\cos 2x - \cos 2x}{4[-(2)^2] - 1} = \frac{1}{5} e^x - e^x \frac{(-4\sin 2x - \cos 2x)}{-17}$$

$$= \frac{1}{5} e^x + \frac{1}{17} e^x [-4\sin 2x - \cos 2x]$$

Therefore, the general solution of the given equation is $y = C.F + P.I$

$$= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5} e^x - \frac{1}{17} e^x (4\sin 2x + \cos 2x)$$

(5) $y''(x) - 2y'(x) + 5y = e^x \sin x$.

Soln The given equation can be written as

$$(D^2-2D+5)y = e^x \sin x$$

A.E is $m^2-2m+5=0$

whose roots are $1\pm 2i$. Therefore,

$$C.F = e^x(c_1\cos 2x + c_2\sin 2x)$$

Next,

$$P.I = \frac{1}{F(D)} e^x \sin x = e^x \frac{1}{F(D+1)} \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+5} \sin x$$

$$= e^x \frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-2+5} \sin x = e^x \frac{1}{D^2+4} \sin x$$

$$= e^x \frac{1}{-1^2+4} \sin x = \frac{e^x}{3} \sin x$$

Therefore, the general solution of the given equation is $y = C.F + P.I$

$$= e^x(c_1\cos 2x + c_2\sin 2x) + \frac{e^x}{3}\sin x$$

$$= e^x\left(c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\sin x\right)$$

Assignment

① Solve : $(D^2-5D+6)y = e^{2x}x^3$

Ans : $c_1e^{3x} + c_2e^{2x} - e^{2x}\left[\frac{x^4}{4} + x^3 + 3x^2 + 6x + 6\right]$

Types : To Evaluate $\frac{1}{F(D)}(xV)$ where V is a function of x then $\frac{1}{F(D)}xV = x\frac{1}{F(D)}V + \left[\frac{d}{dD}\frac{1}{F(D)}\right]V$

$$= x\frac{1}{F(D)}V - \frac{F'(D)}{(F(D))^2}V$$

① Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

→ Here $F(D) = D^2+4$, which leads to

A.E is $m^2+4=0$

$m^2=-4$

$m=\pm\sqrt{-4}$

$m=\pm 2i$

$$C.F = c_1\cos 2x + c_2\sin 2x$$

Now $P.I = \frac{1}{D^2+4} x\sin x$.

$$= \frac{x}{D^2+4}\sin x + \left(\frac{d}{dD}\frac{1}{D^2+4}\right)\sin x$$

$$= x\frac{1}{-1^2+4}\sin x + \frac{(D^2+4)\cdot 0 - 2D}{(D^2+4)^2}\sin x$$

$$= \frac{x}{3}\sin x - \frac{2D\sin x}{(-1^2+4)^2} = \frac{x}{3}\sin x - \frac{2\cos x}{9}$$

$$y = C.F + P.I = c_1\cos 2x + c_2\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$$

② solve: $y'' - 2y' + y = xe^x\sin x$

Soln:- The given equation can be written as

$$(D^2 - 2D + 1)y = xe^x\sin x$$

A.E is $m^2 - 2m + 1 = 0$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$\therefore \quad C.F = (c_1 + c_2x)e^x$$

$$P.I = \frac{1}{(D-1)^2}xe^x\sin x = e^x\frac{1}{(D+\cancel{1}-\cancel{1})^2}x\sin x$$

$$= e^x\frac{1}{D^2}x\sin x = e^x\frac{1}{D}\left[\frac{1}{D}x\sin x\right]$$

$$= e^x\frac{1}{D}\left[x\int\sin x\,dx - \int\left(1\int\sin x\,dx\right)dx\right]$$

$$= e^x\frac{1}{D}\left[x(-\cos x) + \sin x\right]$$

$$= e^x\left[x\int -\cos x\,dx - \int\left(1\int -\cos x\,dx\right)dx + \int\sin x\,dx\right]$$

$$= e^x\left[-x\sin x - 2\cos x\right] = -e^x\left[x\sin x + 2\cos x\right]$$

$$\therefore \quad y = C.F + P.I = (c_1 + c_2x)e^x - e^x\left[x\sin x + 2\cos x\right]$$

or

$$P.I = \frac{1}{(D-1)^2}xe^x\sin x = e^x\frac{1}{D^2}x\sin x$$

$$= e^x\left[x - \frac{2D}{D^2}\right]\frac{\sin x}{D^2} = e^x\left[x - \frac{2}{D}\right]\frac{\sin x}{-1}$$

$$= e^x\left[x\sin x + \frac{2}{D}\sin x\right] \qquad \left[\because \frac{1}{D}\sin x = -\cos x\right]$$

$$= e^x\left[-x\sin x + 2(-\cos x)\right]$$

$$\therefore \quad P.I = -e^x(x\sin x + 2\cos x)$$

we have

$$\frac{1}{F(D)}xV = x\frac{1}{F(D)}V + \left[\frac{d}{dD}\frac{1}{F(D)}\right]V$$

$$= x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V$$

$$= \left[x - \frac{F'(D)}{F(D)}\right]\frac{V}{F(D)}$$

Second order ordinary linear differential equations with variable coefficients

The general form of the linear equation of second order with variable coefficients is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q & R are functions of x.

Working Rule for finding complete solution when an Integral of C.F is known.

(a) Write the given equation in the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

(i.e., making the coefficients of $\frac{d^2y}{dx^2}$ as unity)

(b) If an integral u of C.F is given, assume $y = uv$ to be complete solution of the given equation.

(c) Put $y = uv$ in the given equation, then the equation reduces to the form

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}.$$

(d) Put $\frac{dv}{dx} = z$ and solve the linear differential equation obtained (if $(R=0)$, the equation reduces to the "variable separable" equation & hence solve.)

(e) In the obtained solution put $\frac{dv}{dx} = z$ and integrate w.r.t x, then we get v. Thus $y = uv$ is the required complete solution of (1).

NOTE: If an integral of C.F is not given, one can obtain it in certain cases by using the following table.

① If $1+P+Q=0$, then an integral of C.F is e^{x}.

② If $1-P+Q=0$, then an integral of C.F is e^{-x}.

③ If $a^2+aP+Q=0$, then an integral of C.F is e^{ax}.

④ If $P+Qx=0$, then an integral of C.F is x.

5. If $2+2px+qx^2=0$, then an integral of C.F is x^2.

Problems:

1. Solve $x^2y''+xy'-9y=0$ given that $y=x^3$ is a part of the complimentary function.

Soln: Dividing the equation throughout by x^2

$$y''+\frac{1}{x}y'-\frac{9}{x^2}y=0\ldots (1) \quad \text{Here } P=\frac{1}{x},\ Q=\frac{-9}{x^2},\ R=0$$

Given $u(x)=x^3$. We seek the solution of (1) in the form $y=uv$, where v satisfies the equation

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}.$$

i.e., $$\frac{d^2v}{dx^2}+\left(\frac{1}{x}+\frac{2}{x^3}3x^2\right)\frac{dv}{dx}=0$$

$$\Rightarrow \frac{d^2v}{dx^2}+\frac{7}{x}\frac{dv}{dx}=0 \rightarrow (2)$$

Put $\frac{dv}{dx}=z \Rightarrow \frac{d^2v}{dx^2}=\frac{dz}{dx}$

$\therefore (2) \Rightarrow \frac{dz}{dx}+\frac{7}{x}z=0 \rightarrow (3)$

$$I.F=e^{\int 7/x\,dx}=e^{7\log x}=e^{\log x^7}=x^7$$

Hence the solution of eq (3) is given by

$$z.(I.F)=\int Q.(I.F)\,dx+c_1$$

$$z\,x^7=\int 0.x^7\,dx+c_1$$

$z=c_1x^{-7}$ but $z=\frac{dv}{dx}$ $\therefore \frac{dv}{dx}=c_1x^{-7}$.

on integration

$v=\frac{-c_1}{6x^6}+c_2$ $\therefore$ Solution of the given equation is $y=uv$

i.e., $y=x^3\left(\frac{-c_1}{6x^6}+c_2\right)$ i.e., $y=\frac{-c_1x^3}{6x^{6}}+c_2x^3$

or $\boxed{y=Ax^{-3}+Bx^3}$ $\left(A=\frac{-c_1}{6},\ B=c_2\right)$

2. Solve $x^2\frac{d^2y}{dx^2}-2x(x+1)\frac{dy}{dx}+2(x+1)y=x^3$ given that x is a part of complimentary function.

Soln: The given equation is not in standard form, dividing throughout by x^2 we get the

Standard form $\frac{d^2y}{dx^2} - \frac{2(x+1)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x \rightarrow (1)$

Here $P = -\frac{2(x+1)}{x}$, $Q = \frac{2(x+1)}{x^2}$, $R = x$

Let $y = uv = xv$ be the solution of (1), then

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

i.e., $\frac{d^2v}{dx^2} + \left(\frac{-2(x+1)}{x} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x}{x}$ i.e., $\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \rightarrow (2)$

put $\frac{dv}{dx} = z$ then (2) becomes $\frac{dz}{dx} - 2z = 1 \rightarrow (3)$ which is a linear equation $\therefore$ I.F $= e^{-\int 2\,dx} = e^{-2x}$

$\therefore$ Solution of (3) is $z e^{-2x} = \int 1 \cdot e^{-2x}\,dx + c_1$

i.e., $z e^{-2x} = -\frac{e^{-2x}}{2} + c_1$ or $z = -\frac{1}{2} + c_1 e^{2x}$

But $z = \frac{dv}{dx}$ $\therefore$ $\frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x}$ on integration

$$v = -\frac{1}{2}x + c_1\frac{e^{2x}}{2} + c_2$$

$\therefore$ Solution of (1) is $y = uv$, i.e., $y = x\left[-\frac{1}{2}x + c_1\frac{e^{2x}}{2} + c_2\right]$

(3) Solve $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$

Soln:- Dividing the equation throughout by $(x+2)$, we get

$$y'' - \left(\frac{2x+5}{x+2}\right)y' + \frac{2}{x+2}y = \left(\frac{x+1}{x+2}\right)e^x$$

Here $P = -\left(\frac{2x+5}{x+2}\right)$

$Q = 2/x+2$

$R = x+1/x+2$

By inspection we check $2^2 + 2P + Q = 4 + 2\left(-\frac{2x+5}{x+2}\right) + \frac{2}{x+2}$

$$= \frac{4(x+2) - 2(2x+5) + 2}{x+2} = \frac{4x+8-4x-10+2}{x+2} = 0$$

$\therefore$ e^{2x} is a part of C.F (Notice if $a^2 + aP + Q = 0$ then e^{ax} is a part of C.F)

Let $y = uv = e^{2x}v$ be the complete solution of (1)

Then (1) reduces to $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

i.e., $\frac{d^2v}{dx^2} + \left(-\frac{2x+5}{x+2} + \frac{2}{\cancel{e^{2x}}}\cdot 2\cancel{e^{2x}}\right)\frac{dv}{dx} = \left(\frac{x+1}{x+2}\right)\frac{e^x}{e^{2x}}$ or

$$\frac{d^2v}{dx^2} + \left(\frac{-2x-5+4x+8}{x+2}\right)\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x} \rightarrow (2)$$

Put $\frac{dv}{dx} = z$, then (2) becomes $\frac{dz}{dx} + \frac{2x+3}{x+2} z = \frac{x+1}{x+2} e^{-x}$ ---(3)

$I.F = e^{\int \frac{2x+3}{x+2} dx} = e^{\int (2 - 1/x+2) dx}$
$= e^{2x - \log(x+2)}$
$= e^{2x} \cdot e^{\log \frac{1}{x+2}}$
$= \frac{e^{2x}}{x+2}$

$\therefore$ Solution of (3) is

$$z \cdot \frac{e^{2x}}{x+2} = \int \frac{x+1}{x+2} e^{-x} \frac{e^{2x}}{x+2} dx + c_1$$

$2x+3 | x+2$
divisor
$2 \to$ quotient
$x+2 \overline{) 2x+3}$
$\underline{2x+4}$
$-1 \to R$
$2 - 1/x+2$
$q + r/d$

i.e., $$z \cdot \frac{e^{2x}}{x+2} = \int e^{x} \frac{x+1}{(x+2)^2} dx + c_1 = \int e^{x} \frac{(x+2)-1}{(x+2)^2} dx + c_1$$

$$= \int e^{x} \left[\frac{1}{x+2} - \frac{1}{(x+2)^2} \right] dx + c_1$$

$$z \frac{e^{2x}}{x+2} = e^{x} \frac{1}{x+2} + c_1 \quad \left(\because \int e^{x} [f(x) + f'(x)] dx = e^{x} f(x) \right]$$

$$\therefore z = e^{-x} \frac{x+2}{x+2} + c_1 (x+2) e^{-2x}$$

$z = e^{-x} + c_1 (x+2) e^{-2x}$ but $z = \frac{dv}{dx}$

$\frac{dv}{dx} = e^{-x} + c_1 (x+2) e^{-2x}$ on integration

$$v = -e^{-x} + c_1 \left[-(x+2) \frac{e^{-2x}}{2} - (1) \frac{e^{-2x}}{4} \right] + c_2$$

$$= -e^{-x} - c_1 \frac{e^{-2x}}{4} [2x+4+1] + c_2 = -e^{-x} - \frac{c_1}{4} e^{-2x} (2x+5) + c_2$$

$\therefore$ complete solution of (1) is $y = uv$

i.e., $$y = e^{2x} \left[-e^{-x} - \frac{c_1}{4} e^{-2x} (2x+5) + c_2 \right]$$

$$y = c_2 e^{2x} - e^{x} - \frac{c_1}{4} (2x+5)$$

(4) Solve $\frac{d^2y}{dx^2} - (\cot x) \frac{dy}{dx} - (1 - \cot x) y = e^{x} \sin x$

soln: The given d.e is in standard form.

Here $P = -\cot x$, $Q = -(1 - \cot x)$, $R = e^{x} \sin x$

By inspection $1 + P + Q = 1 - \cot x - 1 + \cot x = 0$

therefore e^{x} is apart of C.F.

Let $y = uv = e^{x} v$ be the complete solution of the given d.e.

Then it reduces to $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u} \frac{du}{dx} \right) \frac{dv}{dx} = \frac{R}{u}$

i.e., $\frac{d^2v}{dx^2} + \left(-\cot x + \frac{2}{e^x} e^x\right)\frac{dv}{dx} = \frac{e^x \sin x}{e^x}$

i.e., $\frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$ Put $\frac{dv}{dx} = z$

Then the above equation becomes

$$\frac{dz}{dx} + (2-\cot x)z = \sin x$$

$$I.F = e^{\int(2-\cot x)dx} = e^{2x - \log\sin x} = e^{2x} \cdot e^{\log(\sin x)^{-1}}$$

$$= e^{2x} e^{\log\left(\frac{1}{\sin x}\right)} = \frac{e^{2x}}{\sin x}$$

$\therefore$ Solution $z \cdot \frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\, dx + c_1$

$z \cdot \frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$ or $z = \frac{\sin x}{2} + c_1 \sin x\, e^{-2x}$

But $z = \frac{dv}{dx}$ $\therefore$ $\frac{dv}{dx} = \frac{\sin x}{2} + c_1 \sin x\, e^{-2x}$ on integration

$$v = \frac{-1}{2}\cos x + c_1 \int e^{-2x} \sin x\, dx + c_2$$

$\therefore v = \frac{-1}{2}\cos x + \frac{c_1}{5} e^{-2x}(-2\sin x - \cos x) + c_2$. $\left[\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

Thus the complete soln of the given equation is

$$y = uv = e^x\left[-\frac{1}{2}\cos x + \frac{c_1}{5} e^{-2x}(-2\sin x - \cos x) + c_2\right]$$

i.e., $y = -\frac{1}{2}e^x \cos x - c_1 \frac{e^{-x}}{5}(2\sin x + \cos x) + c_2 e^x$.

Assignment

① Solve $x^2y'' + xy' - y = 2x^2$ $(x>0)$ given that $\frac{1}{x}$ is a part of its complementary function.

Ans $y = 2/3\, x^2 + c_1 x/2 + c_2/x$.

Method of Variation of parameters.

Working Rule.

① Bring the given differential equation to the standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

② Find the C.F of the given equation & let it be $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x)$

(3) Assume the complete solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ in the form $y=V_1(x)\phi_1(x)+V_2(x)\phi_2(x)$

(4) Write down the equation

$$V_1'(x)\phi_1(x)+V_2'(x)\phi_2(x)=0 \text{ \& }$$
$$V_1'(x)\phi_1'(x)+V_2'(x)\phi_2'(x)=R$$ and solve them for $V_1'(x)$ & $V_2'(x)$.

(5) Integrate $V_1'(x)$ and $V_2'(x)$, obtained in step 4, to get $V_1(x)$ and $V_2(x)$. put this solution in

$$y=V_1(x)\phi_1(x)+V_2(x)\phi_2(x)$$

(1) By the method of variation of parameters,

Solve : $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$

Soln. Here the A.E is $m^2-1=0$, whose roots are ± 1.

Therefore, the CF is $\phi(x)=c_1e^x+c_2e^{-x}$ — (1)

We seek the general solution in the form

$$y=V_1(x)\phi_1(x)+V_2(x)\phi_2(x)$$
i.e., $y=V_1(x)e^x+V_2(x)e^{-x}$ → (2)

write down the equation in the form:

$$V_1'(x)e^x+V_2'(x)e^{-x}=0 \rightarrow (3)$$
$$V_1'(x)e^x-V_2'(x)e^{-x}=\frac{2}{1+e^x} \rightarrow (4)$$

Adding these equations, we get

$$\cancel{2}V_1'(x)e^x=\cancel{2}/1+e^x$$
$$V_1'(x)=\frac{e^{-x}}{1+e^x}=\frac{1}{e^x(1+e^x)} \rightarrow (5)$$

Then equation (3) gives.

$$\frac{1}{\cancel{e^x}(1+e^x)}\cancel{e^x}+V_2'(x)e^{-x}=0$$
$$V_2'(x)=-\frac{1}{e^{-x}(1+e^x)}=\frac{-e^x}{1+e^x} \rightarrow (6)$$

Now, (5) yields

$$V_1(x)=\int\frac{dx}{e^x(1+e^x)}+C_1$$

put $e^x=z$

$\therefore e^x\,dx=dz$

$dx=dz/z$.

$$V_1(x) = \int \frac{dz/z}{z(1+z)} = \int \frac{1}{z^2(1+z)}\, dz + C_1$$

now Consider

$$\frac{1}{z^2(1+z)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{1+z} \quad x'y \quad z^2(z+1)$$

$$1 = Az(1+z) + B(1+z) + Cz^2$$

$$1 = Az(1+z) + B(1+z) + Cz^2$$

put $z=0$ put $z=-1$

$B=1$ $\boxed{C=1}$

$$1 = Az + Az^2 + B + Bz + Cz^2$$

$$A + C = 0$$

$$\boxed{A=-1}$$

$$\therefore V_1(x) = \int \left(\frac{-1}{z} + \frac{1}{z^2} + \frac{1}{1+z}\right) dz + C_1$$

$$= -\log z + \left(-\frac{1}{z}\right) + \log(1+z) + C_1$$

$$= -\tfrac{1}{z} + \log\left(\frac{1+z}{z}\right) + C_1$$

$$= -e^{-x} + \log(1+e^{-x}) + C_1$$

again $\int V_2'(x) = -\int \frac{e^x}{1+e^x}\, dx$

$$V_2(x) = -\log(1+e^x) + C_2$$

Substituting for $V_1(x)$ & $V_2(x)$ in (2), we get

$$y = \left[-e^{-x} + \log(1+e^{-x}) + C_1\right]e^x + \left[-\log(1+e^x) + C_2\right]e^{-x}$$

$$= C_1 e^x + C_2 e^{-x} - 1 + e^x \log(1+e^{-x}) - e^{-x}\log(1+e^x).$$

(2) By the method of variation of parameters, solve: $y'' + a^2 y = \tan ax$

Soln For the given equation, the C.F is

$(D^2+a^2)y = 0$

A.E $m^2 + a^2 = 0 \Rightarrow m^2 = -a^2 \Rightarrow m = \pm ia$

$$\phi(x) = c_1 \cos ax + c_2 \sin ax \rightarrow (1)$$

Let the solution be

$$y = V_1(x)\phi_1(x) + V_2(x)\phi_2(x)$$

$$= V_1(x)\cos ax + V_2(x)\sin ax \rightarrow (2)$$

write down the equation in the form

$$V_1'(x)\cos ax + V_2'(x)\sin ax = 0 \rightarrow (3)$$

$$-V_1'(x)\, a\sin ax + V_2'(x)(a\cos ax) = \tan ax$$

$$-V_1'(x)\sin ax + V_2'(x)\cos ax = \tfrac{1}{a}\tan ax \rightarrow (4)$$

Multiplying ③ by $\cos ax$ & ④ by $\sin ax$ and taking the difference, we get

$$v_1'(x)\cos^2 ax + \cancel{v_2'(x)}\sin ax\cos ax = 0$$

$$\underset{+}{-}v_1'(x)\sin^2 ax \underset{(-)}{+} \cancel{v_2'(x)}\sin ax\cos ax = \underset{-}{1/a}\tan ax\sin ax$$

$$v_1'(x)[\cos^2 ax+\sin^2 ax] = -1/a\,\sin ax\tan ax$$

$$v_1'(x) = -\frac{1}{a}\sin ax\tan ax \rightarrow ⑤$$

using ⑤ in ③, we get

$$-\frac{1}{a}\sin ax\tan ax\cos ax + v_2'(x)\sin ax = 0$$

$$-\frac{1}{a}\sin ax\frac{\sin ax}{\cancel{\cos ax}}\cancel{\cos ax} + v_2'(x)\sin ax = 0$$

$$v_2'(x)\cancel{\sin ax} = \frac{1}{a}\sin^{\cancel{2}} ax$$

$$v_2'(x) = \frac{1}{a}\sin ax \rightarrow ⑥$$

Integrating ⑤ and ⑥, we obtain

$$\int v_1'(x) = -\frac{1}{a}\int \sin ax\tan ax\,dx + c_1$$

$$v_1(x) = -\frac{1}{a}\int\frac{\sin^2 ax}{\cos ax}dx + c_1$$

$$= -\frac{1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx + c_1$$

$$= -\frac{1}{a}\int(\sec ax-\cos ax)dx + c_1$$

$$= \frac{1}{a}\int(\cos ax-\sec ax)dx + c_1$$

$$= \frac{1}{a}\left[\frac{1}{a}\sin ax - \frac{1}{a}\log|\sec ax+\tan ax|\right] + c_1$$

$$v_1(x) = \frac{1}{a^2}\left[\sin ax - \log|\sec ax+\tan ax|\right] + c_1$$

$$v_2(x) = -\frac{1}{a^2}\cos ax + c_2$$

Putting these in to ②, we get

$$y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\cancel{\cos ax\sin ax}$$
$$-\frac{1}{a^2}\cos(ax)\log|\sec ax+\tan ax|$$
$$-\frac{1}{a^2}\cancel{\cos ax\sin ax}$$

$$y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos(ax)\log|\sec ax+\tan ax|$$

This is the general solution of the given equation.

(3) Using the method of variation of parameter, Solve: $\frac{d^2y}{dx^2}+y = \sec x \tan x$.

Soln For the given differential equation, the C.F is

$$\phi(x) = c_1 \cos x + c_2 \sin x ,$$

We seek the general solution in the form

$$y = v_1(x)\cos x + v_2(x)\sin x \rightarrow (1)$$

write down the equation in the form

$$v_1'(x)\cos x + v_2'(x)\sin x = 0 \rightarrow (2)$$

$$-v_1'(x)\sin x + v_2'(x)\cos x = \sec x \tan x \rightarrow (3)$$

Solving these equations, we get

$$v_2'(x) = \tan x , \quad v_1'(x) = -\tan^2 x = 1-\sec^2 x$$

$$v_1'(x)\cos x \sin x + v_2'(x)\sin^2 x = 0$$
$$-v_1'(x)\sin x \cos x + v_2'(x)\cos^2 x = \sec x \tan x \cos x$$

$$v_2'(x) = \sec x \tan x \cos x = \frac{1}{\cos x}\tan x \cos x$$
$$\boxed{v_2'(x) = \tan x}$$

$$v_1'(x)\cos x + \tan x \sin x = 0$$
$$v_1'(x) = -\tan x \sin x / \cos x$$
$$v_1'(x) = -\tan^2 x$$

Integrating these, we get

$$v_1(x) = x - \tan x + c_1$$
$$v_2(x) = \log(\sec x) + c_2$$

Putting these in to (1), we get

$$y = c_1\cos x + c_2 \sin x + x\cos x - \sin x + (\sin x)\log \sec x .$$

(4) By the method of variation of parameters, Solve: $y'' - 2y' + y = e^x \log x$

Soln Here, the A.E is $m^2-2m+1=0$, whose roots are 1, 1. Therefore, C.F is

$$\phi(x) = (c_1 + c_2 x) e^x \rightarrow (1)$$

We seek the general solution in the form

$$y = v_1(x)e^x + v_2(x)\, x e^x \rightarrow (2)$$

write down the equation in the form

$$v_1'(x)e^x + v_2'(x)\, xe^x = 0$$
$$v_1'(x)e^x + v_2'(x)(1+x)e^x = e^x \log x .$$

or

$$v_1'(x) + v_2'(x)x = 0 \rightarrow (3)$$
$$v_1'(x) + v_2'(x)(1+x) = \log x \rightarrow (4)$$

Solving these equations, we get

$$v_2'(x) = \log x , \quad v_1'(x) = -x\log x$$

Therefore, $v_1(x) = \int -x \log x \, dx + c_1$

$$= -\left[\log x \int x\,dx - \int\left(\frac{1}{x}\int x\,dx\right)dx\right] + c_1$$

$$V_1(x) = -\frac{x^2}{2}\log x + \int \frac{1}{x}\frac{x^2}{2}dx + c_1$$

$$V_1(x) = -\frac{x^2}{2}\log x + \frac{1}{4}x^2 + c_1$$

$$V_2(x) = \int \log x + c_1 = \log x\, x - x + c_2 = x(\log x - 1) + c_2$$

putting these in to ②, we get

$$y = e^x\left[-\frac{1}{2}x^2\log x + \frac{1}{4}x^2 + c_1\right] + xe^x\left[x(\log x - 1) + c_2\right]$$

$$y = e^x\left[c_1 + c_2 x - \frac{3x^2}{4} + \frac{1}{2}x^2\log x\right].$$

This is the general solution of the given differential equation.

5. By the method of variation of parameters, solve $y'' - 2y' + 2y = e^x \tan x$

Soln Here, the A.E is $m^2 - 2m + 2 = 0$, whose roots are $1 \pm i$. Therefore, the C.F is

$$\phi(x) = e^x(c_1\cos x + c_2\sin x) \quad \text{—①}$$

We seek the general solution in the form

$$y = V_1(x)e^x\cos x + V_2(x)e^x\sin x \quad \text{—②}$$

write down the equation in the form

$$V_1'(x)e^x\cos x + V_2'(x)e^x\sin x = 0$$

$$V_1'(x)e^x(\cos x - \sin x) + V_2'(x)e^x(\sin x + \cos x) = e^x\tan x$$

or

$$V_1'(x)\cos x + V_2'(x)\sin x = 0 \rightarrow ③$$

$$V_1'(x)(\cos x - \sin x) + V_2'(x)(\sin x + \cos x) = \tan x \rightarrow ④$$

Using ③, equation ④ becomes

$$-V_1'(x)\sin x + V_2'(x)\cos x = \tan x \rightarrow ⑤$$

Solving equations ③ & ⑤, we get

$$V_1'(x) = -\sin x\tan x = -\frac{\sin^2 x}{\cos x}$$

& $V_2'(x) = \sin x$

These yield

$$V_1(x) = \sin x - \log|\sec x + \tan x| + c_1$$

$$V_2(x) = \cos x + c_2$$

Substituting these in to ②, we get

$$y = e^x\{c_1\cos x + c_2\sin x - (\cos x)\log|\sec x + \tan x|\}$$

this is the general solution of the given equation.

Assignment

① By the method of variation

$$V_1'(x)\cancel{\cos x}\sin x + V_2'(x)\sin^2 x = 0$$

$$-V_1'(x)\sin x\cos x + V_2'(x)\cos^2 x = \frac{\tan x}{\cos x}$$

$$V_2'(x) = \tan x\cos x$$

$$V_2'(x) = \sin x$$

$$V_1'(x)\cos x + \sin^2 x = 0$$

$$V_1'(x) = -\frac{\sin^2 x}{\cos x} = -\sin x\tan x$$

of parameters, solve: $y'' + a^2 y = \sec ax$ (34)

Ans $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} (\cos ax) \log(\cos ax) + \frac{1}{a} x \sin ax$

(2) By the method of variation of parameters, solve:

$$y'' + 2y' + y = e^{-x} \log x$$

Ans
$$y = e^{-x} \left[c_1 + c_2 x - \frac{3}{4} x^2 + \frac{1}{2} x^2 \log x \right]$$

Method of undetermined coefficients

We consider a method of determining a P.I. of the equation $L(D)y = X$, which avoids the use of the inverse difference operator $1/L(D)$. This method, known as the method of undetermined coefficients, is suitable only when the equation is with constant coefficients & X is in some particular forms. The working rule for the method for different cases is as follows:

Working Rule.

Case 1. Suppose $X = p_n(x)$, where $p_n(x)$ is a polynomial of degree n in x. In this case, we assume a P.I. $\varphi(x)$ in the form

$\varphi(x) = K_1 x^n + K_2 x^{n-1} + \dots + K_n x + K_{n+1}$ Here $K_1, K_2, \dots K_n, K_{n+1}$ are constants to be determined by using the fact that $L\{\varphi(x)\} = X$.

Case 2: Suppose $X = e^{ax} p_n(x)$, where 'a' is a known constant & $p_n(x)$ is as in Case 1. In this case, we assume $\varphi(x)$ in the form

$$\varphi(x) = e^{ax} \left(K_1 x^n + K_2 x^{n-1} + \dots + K_n x + K_{n+1} \right) \rightarrow (2)$$

Here $K_1, K_2, \dots K_{n+1}$ are constants to be determined by using the fact that $L\{\varphi(x)\} = X$.

Case 3: Suppose $X = e^{ax} p_n(x) \{ \alpha \cos bx + \beta \sin bx \}$, where a, b, α, β are known constants and $p_n(x)$ is as in case 1. In this case, we assume $\varphi(x)$ in the form

$$\varphi(x) = e^{ax} \left[(\cos bx) \{ K_1 x^n + K_2 x^{n-1} + \dots + K_n x + K_{n+1} \} + (\sin bx) \{ K_1' x^n + K_2' x^{n-1} + \dots + K_n' x + K_{n+1}' \} \right]$$

where $K_1, K_2 \dots K_n, K_{n+1}, K_1', K_2', \dots K_n', K_{n+1}'$ are $\rightarrow$ (3)

constants to be determined by using the fact that $L\{\psi(x)\} = x$.

(1) By the method of undetermined coefficients, solve the equation

$$y'' - 2y' + 5y = 25x^2 + 12$$

soln For the given equation, the A.E is

$L(m) = m^2 - 2m + 5 = 0$ whose roots are $1 \pm 2i$.

Therefore, the C.F. is

$$\phi(x) = e^x (c_1 \cos 2x + c_2 \sin 2x) \rightarrow (1)$$

In view of the form of the right-hand side of the given equation, we take a P.I in the form $\psi(x) = k_1 x^2 + k_2 x + k_3 \rightarrow (2)$ [case 1]

This gives $\psi'(x) = 2k_1 x + k_2$, $\psi''(x) = 2k_1$ so that

$$\begin{aligned} L\{\psi(x)\} &= \psi''(x) - 2\psi'(x) + 5\psi(x) \\ &= 2k_1 - 2(2k_1 x + k_2) + 5(k_1 x^2 + k_2 x + k_3) \\ &= 5k_1 x^2 + (5k_2 - 4k_1)x + (2k_1 - 2k_2 + 5k_3) \end{aligned}$$

Therefore, $\psi(x)$ satisfies the given equation, i.e., $L\{\psi(x)\} = 25x^2 + 12$, if

$$5k_1 x^2 + (5k_2 - 4k_1)x + (2k_1 - 2k_2 + 5k_3) = 25x^2 + 12$$

Equating the corresponding coefficients we get

$5k_1 = 25$, $5k_2 - 4k_1 = 0$, $2k_1 - 2k_2 + 5k_3 = 12$.

These give $k_1 = 5$, $k_2 = 4$, $k_3 = 2$. For these values of k_1, k_2, k_3, (2) becomes

$$\psi(x) = 5x^2 + 4x + 2 \rightarrow (3)$$

Therefore, the general solution of the given equation is

$$\begin{aligned} y &= C.F + P.I = \phi(x) + \psi(x) \\ &= e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2 \end{aligned}$$

(2) By the method of undetermined coefficients, solve the equation: $y'' - 2y' - 3y = e^{2x}$.

soln For the given equation, the AE is

$L(m) \equiv m^2 - 2m - 3 = 0$, whose roots are $3 \,\&\, -1$.

Therefore, the C.F is

$$\phi(x) = c_1 e^{3x} + c_2 e^{-x} \rightarrow (1)$$

In view of the form of the right-hand side of the given equation, we take a p.I in the form

$$\psi(x) = Ke^{2x} \rightarrow (2)$$

[see case 2. Here X is of the form $e^{ax} p_n(x)$ where $a=2$ & $p_n(x)=1$.]

Then $\psi'(x) = 2Ke^{2x}$, $\psi''(x) = 4Ke^{2x}$ so that

$$L\{\psi(x)\} = \psi''(x) - 2\psi'(x) - 3\psi(x)$$

$$= \cancel{4Ke^{2x}} - \cancel{4Ke^{2x}} - 3Ke^{2x} = -3Ke^{2x}$$

Accordingly, $\psi(x)$ satisfies the given equation, i.e, $L\{\psi(x)\} = e^{2x}$, if $-3Ke^{2x} = e^{2x}$, or $K = -1/3$. For this K equn (2) becomes $\psi(x) = -1/3\, e^{2x}$. Therefore, the general solution of the given equation is

$$y = C.F + P.I = \phi(x) + \psi(x) = c_1e^{3x} + c_2e^{-x} - 1/3\, e^{2x}.$$

3. By the method of undetermined coefficients, solve the equation

$$y'' - y' - 2y = 10\sin x$$

Soln For the given equation, the A.E is $L(m) = m^2 - m - 2 = 0$, whose roots are -1 & 2. Therefore, the C.F is $\phi(x) = c_1e^{-x} + c_2e^{2x} \rightarrow (1)$

In view of the form of the RHS of the given equation, we take a PI in the form

$$\psi(x) = K_1\cos x + K_2\sin x \rightarrow (2)$$

[case 3. Here $a=0$, $P_n(x)=1$, $\alpha=0$, $\beta=10$, $b=1$]

Then $\psi'(x) = -K_1\sin x + K_2\cos x$ &

$\psi''(x) = -K_1\cos x - K_2\sin x$ so that

$$L\{\psi(x)\} = \psi''(x) - \psi'(x) - 2\psi(x)$$

$$= -(3K_1 + K_2)\cos x - (3K_2 - K_1)\sin x$$

Therefore, $\psi(x)$ satisfies the given equation i.e., $L\{\psi(x)\} = 10\sin x$, if

$$-(3K_1 + K_2)\cos x - (3K_2 - K_1)\sin x = 10\sin x$$

This yields $(3K_1 + K_2) = 0$ & $3K_2 - K_1 = -10$, so that $K_1 = 1$ & $K_2 = -3$. For these values of K_1 & K_2, (2) becomes

$$\psi(x) = \cos x - 3\sin x \rightarrow (3)$$

Hence, the general solution of the given equation

is $y = C.F + P.I = \phi(x) + \psi(x)$
$= c_1 e^{-x} + c_2 e^{2x} + \cos x - 3\sin x$.

(4) By using the method of undetermined coefficients solve the following differential equations

i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$ ii) $\frac{d^2y}{dx^2} + 4y = x^2 + e^{-x}$

iii) $y'' + y' - 2y = x + \sin x$

(i) soln For the given equation, the A.E is $m^2 - 2m = 0$ whose roots are $m_1 = 0$ & $m_2 = 2$. Therefore the C.F is $\phi(x) = c_1 + c_2 e^{2x} \rightarrow (1)$

we seek a P.I in the form

$$\psi(x) = e^x (K_1 \sin x + K_2 \cos x) \rightarrow (2)$$

This gives

$$\psi'(x) = e^x (K_1 \cos x - K_2 \sin x) + (K_1 \sin x + K_2 \cos x)e^x$$

$$\psi'(x) = e^x [(K_1 - K_2)\sin x + (K_1 + K_2)\cos x]$$

$$\psi''(x) = e^x [(K_1 - K_2)\cos x - (K_1 + K_2)\sin x] + e^x [(K_1 - K_2)\sin x + (K_1 + K_2)\cos x]$$

$$\psi''(x) = e^x [2K_1 \cos x - 2K_2 \sin x]$$ so that

$$\psi''(x) - 2\psi'(x) = -2e^x \{K_1 \sin x + K_2 \cos x\}$$

Therefore, $\psi(x)$ satisfies the given d.e if $-2K_1 = 1$, $K_2 = 0$. using these in (2), we get

$$\psi(x) = -\frac{1}{2} e^x \sin x$$

$$\therefore y = \phi(x) + \psi(x) = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x.$$

ii) soln For the given equation, the C.F is

$$\phi(x) = c_1 \cos 2x + c_2 \sin 2x \rightarrow (1)$$

We seek a P.I in the form

$$\psi(x) = (K_1 x^2 + K_2 x + K_3) + K_4 e^{-x} \rightarrow (2)$$

This gives

$$\psi'(x) = 2K_1 x + K_2 - K_4 e^{-x},$$

$$\psi''(x) = 2K_1 + K_4 e^{-x},$$

so that

$\psi''(x) + 4\psi(x) = 4k_1x^2 + 4k_2x + (2k_1+4k_3) + 5k_4e^{-x}$

Therefore, $\psi(x)$ satisfies the given differential equation if $4k_1=1$, $4k_2=0$, $2k_1+4k_3=0$, $5k_4=1$; that is, $k_1=1/4$, $k_2=0$, $k_3=-1/8$, $k_4=1/5$. For these values, (2) becomes

$$\psi(x) = \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{5}e^{-x}$$

$$\therefore y = \phi(x) + \psi(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x^2 - \frac{1}{8} + \frac{1}{5}e^{-x}.$$

iii) Soln Here, the A.E is $m^2+m-2=0 \Rightarrow (m+2)(m-1)=0$ which yields the C.F as

$$\phi(x) = c_1e^{-2x} + c_2e^{x} \rightarrow (1)$$

We seek a P.I in the form

$$\psi(x) = (k_1x+k_2) + (k_3\sin x + k_4\cos x) \rightarrow (2)$$

This gives $\psi'(x) = k_1 + k_3\cos x - k_4\sin x$

$$\psi''(x) = -k_3\sin x - k_4\cos x.$$

Accordingly,

$$\psi''(x) + \psi'(x) - 2\psi(x)$$
$$= -2k_1x + (k_1-2k_2) - (3k_3+k_4)\sin x + (k_3-3k_4)\cos x$$

Hence, $\psi(x)$ satisfies the given differential equation if $-2k_1=1$, $k_1-2k_2=0$, $-(3k_3+k_4)=1$, $k_3-3k_4=0$; that is, $k_1=-1/2$, $k_2=-1/4$, $k_3=-3/10$, $k_4=-1/10$.

Using these values, (2) becomes

$$\psi(x) = -\frac{1}{2}x - \frac{1}{4} - \frac{3}{10}\sin x - \frac{1}{10}\cos x.$$

$$\therefore y = \phi(x) + \psi(x)$$
$$= c_1e^{-2x} + c_2e^{x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x).$$

Assignment

1. Solve the following differential Equations by the method of undetermined coefficients:

(1) $y'' + y' - 2y = x + \cos 2x$

Ans:- $c_1e^{(1+\sqrt{17})x/2} + c_2e^{(1-\sqrt{17})x/2} - \frac{1}{4}x + \frac{1}{16} - \frac{2}{17}\cos 2x - \frac{1}{34}\sin 2x$

(2) $(D^2-4D+3)y = 20\cos x$

Ans: $c_1e^{x} + c_2e^{3x} + 2\cos x - 4\sin x$.