Double Integrals

① Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dy\,dx.$

Soln $\int_0^5 \int_0^{x^2} (x^3+xy^2)\,dy\,dx$

$$\int_0^5 \left[x^3y + \frac{xy^3}{3}\right]_0^{x^2} dx = \int_0^5 \left(x^5+\frac{x^7}{3}\right)dx = \frac{x^6}{6}+\frac{x^8}{24}\Big|_0^5$$

$$= \frac{5^6}{6}+\frac{5^8}{24} = 5^6\left(\frac{1}{6}+\frac{25}{24}\right) = \frac{15625}{6}+\frac{390625}{24}$$

$$= 2604.16+16276.04 = 18880.2$$

Triple Integrals

① Evaluate $I = \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$

Soln
$$= \int_{-1}^1 \left\{\int_0^z \left[\int_{x-z}^{x+z} (x+y+z)\,dy\right]dx\right\}dz$$

$$= \int_{-1}^1 \left[\int_0^z \left[xy+\frac{y^2}{2}+zy\right]_{y=x-z}^{y=x+z} dx\right]dz$$

$$= \int_{-1}^1 \left\{\int_0^z x(x+z)+\frac{(x+z)^2}{2}+z(x+z) - \left[x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right]dx\right\}dz$$

$$= \int_{-1}^1 \left\{\int_0^z (x+z)(x+z)+\frac{(x+z)^2}{2} - \left[(x+z)(x-z)+\frac{(x-z)^2}{2}\right]dx\right\}dz$$

$$= \int_{-1}^1 \left\{\int_0^z (x+z)\left(\cancel{x}+z-(x-z)\right)+\frac{1}{2}\left\{(x+z)^2-(x-z)^2\right\}dx\right\}dz$$

$$= \int_{-1}^1 \left\{\int_0^z (x+z)(2z)+\frac{1}{2}\left\{x^2+2xz+z^2-[x^2-2zx+z^2]\right\}dx\right\}dz$$

$$= \int_{-1}^1 \left\{\int_0^z 2xz+2z^2+\frac{1}{\cancel{2}}(\overset{2}{\cancel{4}}xz)\,dx\right\}dz = \int_{-1}^1 \left\{\int_0^z (4xz+2z^2)\,dx\right\}dz$$

$$= \int_{-1}^1 \left(\frac{\overset{2}{\cancel{4}}x^2z}{\cancel{2}}+2z^2x\right)_0^z dz = \int_{-1}^1 \left[2x^2z+2z^2x\right]_0^z dz = \int_{-1}^1 (2z^3+2z^3-0)\,dz$$

$$= \int_{-1}^1 4z^3\,dz = \frac{\cancel{4}z^4}{\cancel{4}}\Big|_{-1}^1 = 1-(-1)^4 = 0.$$

② Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Soln
$$= \int_{x=0}^a \int_{y=0}^x \int_{z=0}^{x+y} e^x.e^y.e^z\,dz\,dy\,dx$$

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$$= \int_{x=0}^{a} e^{x} \left\{ \int_{y=0}^{x} e^{y} \left(\int_{z=0}^{x+y} e^{z}\,dz \right) dy \right\} dx$$ ②

$$= \int_{x=0}^{a} e^{x} \left\{ \int_{y=0}^{x} e^{y} \left[e^{z} \right]_{0}^{x+y} dy \right\} dx$$

$$= \int_{x=0}^{a} e^{x} \left\{ \int_{y=0}^{x} e^{y} \left[e^{x+y} - e^{0} \right] dy \right\} dx$$

$$= \int_{x=0}^{a} e^{x} \left\{ \int_{y=0}^{x} \left(e^{x} e^{2y} - e^{y} \right) dy \right\} dx$$

$$= \int_{x=0}^{a} e^{x} \left\{ \left[\frac{e^{x} e^{2y}}{2} - e^{y} \right]_{0}^{x} \right\} dx$$

$$= \int_{x=0}^{a} e^{x} \left\{ \frac{e^{x} e^{2x}}{2} - e^{x} - \left(\frac{e^{x}}{2} - 1 \right) \right\} dx$$

$$= \int_{x=0}^{a} e^{x} \left\{ \frac{e^{3x}}{2} - e^{x} - \frac{e^{x}}{2} + 1 \right\} dx = \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right) dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^{x} \right) dx = \left. \frac{e^{4x}}{8} - \frac{3}{4} e^{2x} + e^{x} \right|_{0}^{a}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \left(\frac{1}{8} - \frac{3}{4} + 1 \right) = \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$

$$= \frac{1}{8} \left(e^{4a} - 6e^{2a} + 8e^{a} - 3 \right)$$

③ Evaluate $I = \int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$

Soln

$$I = \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{1-x^2}} y \left[\int_{0}^{\sqrt{1-x^2-y^2}} z\,dz \right] dy \right\} dx$$

$$= \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{1-x^2}} y \left[\frac{z^2}{2} \right]_{0}^{\sqrt{1-x^2-y^2}} dy \right\} dx$$

$$= \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{1-x^2}} y \left(\frac{\left(\sqrt{1-x^2-y^2}\right)^2}{2} \right) dy \right\} dx$$

$$= \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{1-x^2}} y \left(\frac{1-x^2-y^2}{2} \right) dy \right\} dx$$

$$= \int_{0}^{1} x \left\{ \int_{0}^{\sqrt{1-x^2}} \left(\frac{y - yx^2 - y^3}{2} \right) dy \right\} dx$$

$$= \frac{1}{2} \int_{0}^{1} x \left\{ (1-x^2) \frac{y^2}{2} - \frac{y^4}{4} \right\}_{0}^{\sqrt{1-x^2}} dx$$

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③

$$= \frac{1}{2}\int_0^1 x\left\{\frac{(1-x^2)(\sqrt{1-x^2})^2}{2} - \frac{(\sqrt{1-x^2})^4}{4}\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{(1-x^2)^2}{2} - \frac{1}{4}(1-x^2)^2\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{2(1-x^2)^2-(1-x^2)^2}{4}\right\}dx = \frac{1}{8}\int_0^1 x\,(1-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^1 x\,(1+x^4-2x^2)\,dx = \frac{1}{8}\int_0^1 (x+x^5-2x^3)\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}+\frac{x^6}{6}-\frac{\cancel{2}x^4}{\cancel{4}2}\right]_0^1 = \frac{1}{8}\left[\cancel{\frac{1}{2}}+\frac{1}{6}-\cancel{\frac{1}{2}}\right] = \frac{1}{48}.$$

④ Evaluate $\int_0^1\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$.

Soln

$$I = \int_0^1 x\left\{\int_0^{\sqrt{a^2-x^2}} y\left[\int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\right]dy\right\}dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{a^2-x^2}} y\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\right\}dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{a^2-x^2}} y\left[\frac{(\sqrt{a^2-x^2-y^2})^2}{2}\right]dy\right\}dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{a^2-x^2}} y\left(\frac{a^2-x^2-y^2}{2}\right)dy\right\}dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{a^2-x^2}}\left(\frac{ya^2-yx^2-y^3}{2}\right)dy\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{(a^2-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right\}_0^{\sqrt{a^2-x^2}}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{(a^2-x^2)(\sqrt{a^2-x^2})^2}{2} - \frac{(\sqrt{a^2-x^2})^4}{4}\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left\{\frac{(a^2-x^2)^2}{2}-\frac{1}{4}(a^2-x^2)^2\right\}dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{2(a^2-x^2)^2-(a^2-x^2)^2}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 x\,\frac{(a^2-x^2)^2}{4}\,dx = \frac{1}{8}\int_0^1 x\,(a^4-2a^2x^2+x^4)\,dx$$

$$= \frac{1}{8}\int_0^1 (xa^4-2a^2x^3+x^5)\,dx = \frac{1}{8}\int_0^1 (xa^4-2a^2x^3+x^5)\,dx$$

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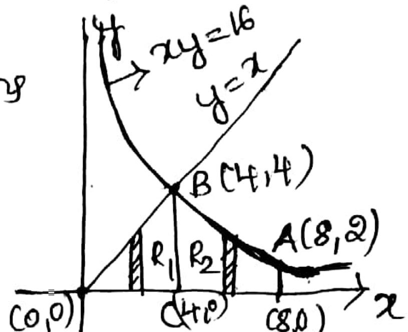
(4)

$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{\cancel{2}a^2x^4}{\cancel{4}2} + \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{a^4}{2} - \frac{a^2}{2} + \frac{1}{6}\right]$$

Evaluation of $\iint_R f(x,y)\,dx\,dy$ over the specific region R

(1) Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$



Soln By solving the given equations i.e., $x=y$ and $xy=16$

Now $x^2=16 \Rightarrow x=\pm\sqrt{16}=\pm 4$.

[In first quadrant] $x=4$ This gives $y=4$ & hence the two curves intersect at the points B(4,4).

The region of integration is as shown in figure

From Figure, we observe that R is made up of two parts R_1 & R_2. In R_1, x varies from 0 to 4, and for each x, y varies from 0 to x. In R_2, x varies from 4 to 8, and, for each x, y varies from 0 to $16/x$. Thus

$$I = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dx\,dy + \int_4^8\int_0^{16/x} x^2\,dx\,dy$$

$$= \int_0^4 x^2\,dx\,[y]_0^x + \int_4^8 x^2\,dx\,[y]_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 x^{\cancel{2}}\times\frac{16}{\cancel{x}}\,dx = \int_0^4 x^3\,dx + \int_4^8 16x\,dx$$

$$= \frac{x^4}{4}\Big|_0^4 + \frac{\cancel{16}8x^2}{\cancel{2}}\Big|_4^8 = \frac{4^{\cancel{4}3}}{\cancel{4}} + (8\times 8^2 - 8\times 4^2)$$

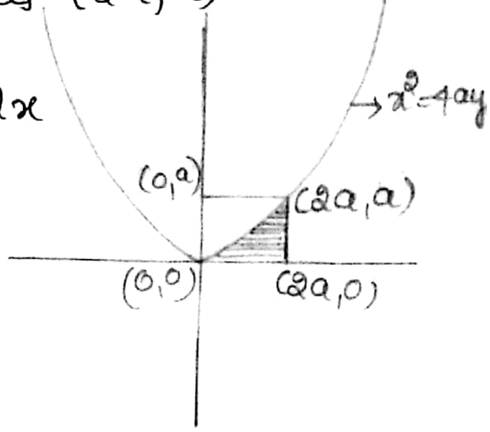
$$= 64 + 512 - 128 = 448.$$

(2) Evaluate $\iint_R xy\,dx\,dy$ where R is the domain bounded by x-axis Ordinate $x=2a$ and the curve $x^2=4ay$.

Soln $x^2 = 4ay$ is a parabola symmetrical about the y-axis. The point of intersection of this curve with $x = 2a$ is to be found.

Hence $(2a)^2 = 4ay$ or $4a^2 = 4ay$ $\therefore y = a$

The point of intersection is $(2a, a)$



$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_{y=0}^{x^2/4a} dx$$

$$= \frac{1}{2}\int_{x=0}^{2a} x\left[\left(\frac{x^2}{4a}\right)^2\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{2a} \frac{x^5}{16a^2}\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2}\,\frac{(2a)^6}{6} = \frac{\cancel{64}^{32}a^6}{\cancel{32}\times\cancel{6}_3\,a^2} = \frac{a^4}{3}$$

Thus $I = \frac{a^4}{3}$

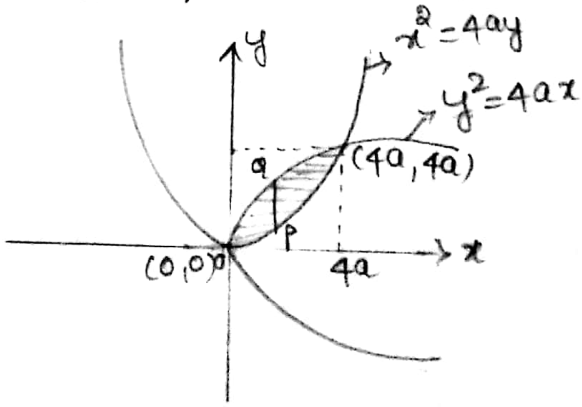
NOTE: Alternative form of $I = \int_{y=0}^{a}\int_{x=\sqrt{4ay}}^{2a} xy\,dx\,dy$; $I = \frac{a^4}{3}$

(3) Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Soln The given region R is as shown in Figure.

Solving the given equations, we find that the two parabolas intersect at the points $(0,0)$ and $(4a, 4a)$.

put $x = \frac{y^2}{4a}$ in $x^2 = 4ay$



$\Rightarrow x^2 = 4ay$

$\Rightarrow \left(\frac{y^2}{4a}\right)^2 = 4ay \Rightarrow \frac{y^4}{16a^2} = 4ay$

$\Rightarrow y^3 = 64a^3 \Rightarrow y = (64a^3)^{1/3}$

$= (4^3a^3)^{1/3} = 4a$

$\Rightarrow y = 4a$

$\therefore y^2 = 4ax \Rightarrow (4a)^2 = 4ax \Rightarrow x = 4a$

$\therefore$ The point of intersection is $(4a, 4a)$

Therefore, in the region bounded by these parabolas, x increases from 0 to $4a$, and, for each x, y

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increases from a point P on the parabola $x^2=4ay$ to a point Q on the parabola $y^2=4ax$. We find that, at P, $y=(x^2/4a)$ and, at Q, $y=\sqrt{4ax}$. Hence

$$\iint_R y\,dx\,dy = \int_{x=0}^{4a}\left\{\int_{y=x^2/4a}^{\sqrt{4ax}} y\,dy\right\}dx$$

$$= \int_{x=0}^{4a}\left[\frac{y^2}{2}\right]_{y=x^2/4a}^{\sqrt{4ax}} dx = \frac{1}{2}\int_{x=0}^{4a}\left[4ax-\left(\frac{x^2}{4a}\right)^2\right]dx$$

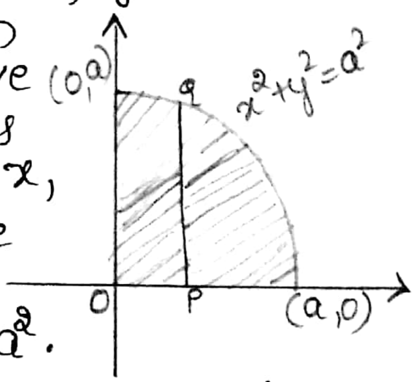
$$= \frac{1}{2}\int^{4a}\left(4ax-\frac{x^4}{16a^2}\right)dx = \frac{1}{2}\left[\frac{\cancel{4}^{2}ax^2}{\cancel{2}}-\frac{x^5}{16\times 5a^2}\right]_{x=0}^{4a}$$

$$= \frac{1}{2}\left[2a(4a)^2-\frac{(4a)^5}{16\times 5a^2}\right] = \frac{1}{2}\left[32a^3-\frac{\cancel{4}\times\cancel{4}\times 4\times 4\,a^{\cancel{5}3}}{\cancel{16}\times 5\,\cancel{a^2}}\right] = 16a^3-\frac{64a^3}{10}$$

$$= \frac{160a^3-64a^3}{10} = \frac{96a^3}{10} = \frac{48a^3}{5}.$$

(4) Evaluate $\iint_R xy\,dx\,dy$ where R is the quadrant of the circle $x^2+y^2=a^2$, where $x\geq 0$, $y\geq 0$.

Soln. Here, the region of integration is shown in Figure. We observe that, in this region, x increases (varies) from 0 to a, and, for each x, y varies from a point P on the x-axis to a point Q on the first Quadrant of the circle C: $x^2+y^2=a^2$.



We find that, at Q, $y=\sqrt{a^2-x^2}$. Thus, for each x, y increases from 0 to $\sqrt{a^2-x^2}$

Hence,

$$\iint_R xy\,dx\,dy = \int_{x=0}^{a}\left\{\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\right\}dx$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx = \int_{x=0}^{a} x\left[\frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{a}(a^2x-x^3)\,dx = \frac{1}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right] = \frac{a^4}{8}$$

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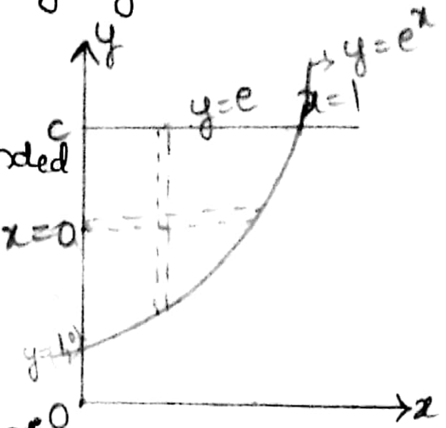
Change of order of Integration

(1) Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration.



Soln The region of integration is bounded by the lines $x=0$, $x=1$, $y=e$ and the curve $y=e^x$.

i.e., $x \to 0$ to 1, $y \to e^x$ to e

On changing the order of integration, the variable x varies from 0 to $\log y$, for each value of x the variable y varies from 1 to e

Thus $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y} = \int_1^e \int_0^{\log y} \frac{dx\,dy}{\log y}$

$= \int_1^e \frac{dy}{\log y}[x]_0^{\log y} = \int_1^e \frac{\cancel{\log y}}{\cancel{\log y}}\,dy$

$= y|_1^e = e-1.$

when $x=0$ & $x=1$
$\Rightarrow y=e^0=1$
$y=e^1=e$
$y=e^x$
$x=\log y$

(2) Assig Evaluate $I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ by changing the order of integration.

(3) Change the order of integration in the integral $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$ and hence evaluate it

Soln Here x varies from 0 to a, and, for each x, y varies from $y=x/a$ to $y=\sqrt{x/a}$. Hence the lower value of y lies on the curve $y=x/a$ (which is straight line) and the upper value of y lies on the curve $y^2=x/a$ (which is a parabola). The region R of integration is as shown in figure.

From figure, we observe that, in R, y varies from 0 to 1 and, for each y, x varies from ay^2 to ay.

Hence $I = \int_{x=0}^{a} \int_{y=x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy$

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$$I = \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_{ay^2}^{ay} dy$$

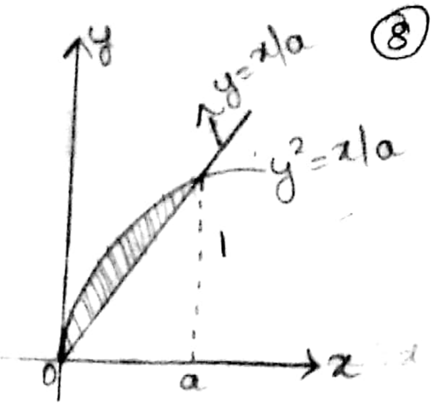
$$= \int_0^1 \left[\frac{(ay)^3}{3} + y^2(ay) - \left(\frac{(ay^2)^3}{3} + y^2(ay^2)\right)\right] dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \frac{a^3y^4}{12} + \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5}\Big|_0^1$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{7a^3 - 4a^3}{84} + \frac{5a-4a}{20}$$

$$= \frac{3a^3}{84} + \frac{a}{20} = \frac{a^3}{28} + \frac{a}{20}$$



$y = x/a \qquad y^2 = x/a$

$x = ay$

$\Rightarrow y^2 = \cancel{a}y/\cancel{a}$

$y^2 - y = 0$

$y(y-1) = 0$

$y=0 \quad y=1$

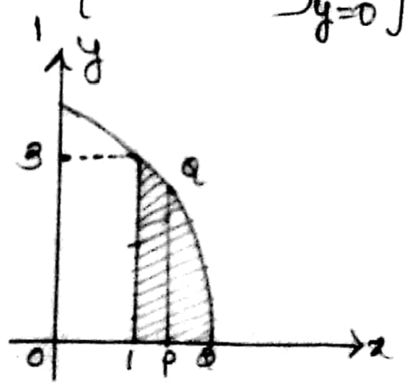
(4) Change the order of integration in the integral $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ and hence evaluate

Soln Here, y varies from 0 to 3, and, for each y, x varies from $x=1$ to $x=\sqrt{4-y}$, or $x^2 = 4-y$. thus, for each y, the lower value of x lies on the line $x=1$ and the upper value lies on the parabola $x^2 = 4-y$. The region R of integration is shown in figure. From figure, we note that, in R, x varies from 1 to 2, and, for each x, y varies from a point P on the x-axis to the point Q on the parabola $x^2 = 4-y$. thus, for each x, y varies from 0 to $4-x^2$.

Therefore,

$$\int_{y=0}^{3}\int_{x=1}^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_{x=1}^{2}\int_{y=0}^{4-x^2} (x+y)\,dy\,dx = \int_1^2 \left\{\left[xy + \frac{y^2}{2}\right]_{y=0}^{y=4-x^2}\right\} dx$$

$$= \int_1^2 \left\{x(4-x^2) + \frac{1}{2}(4-x^2)^2\right\} dx$$



$$= \int_1^2 \left\{4x - x^3 + \frac{1}{2}(16 - 8x^2 + x^4)\right\} dx$$

$$= \left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{x^4}{4} + \frac{1}{2}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\right]_1^2$$

$$= 2(2)^2 - \frac{2^4}{4} + \frac{1}{2}16(2) - \frac{8(2)^3}{2\times 3} + \frac{1}{2}\frac{2^5}{5} - \left(2 - \frac{1}{4} + \frac{1}{2}\left(16 - \frac{8}{3} + \frac{1}{5}\right)\right)$$

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$$= 8-4+16-\frac{32}{3}+\frac{16}{5}-\left(2-\frac{1}{4}+8-\frac{4}{3}+\frac{1}{10}\right) = \frac{20}{1}-\frac{32}{3}+\frac{16}{5}-\left(10-\frac{1}{4}-\frac{4}{3}+\frac{1}{10}\right)$$ 9

$$= \frac{300-160+48}{15} - \left(\frac{600-15-80+6}{60}\right) = \frac{188}{15} - \frac{511}{60} = \frac{752-511}{60} = \frac{241}{60}$$

(5) Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{xa}} x(x^2+y^2)\,dy\,dx$ by changing the order of integration.

soln given limits are

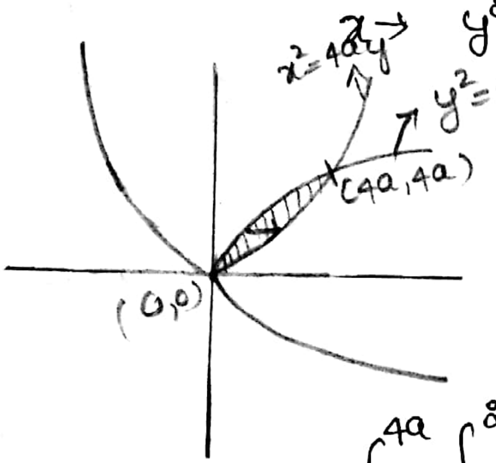
$x \to 0$ to $4a$

$y \to \frac{x^2}{4a}$ to $2\sqrt{xa}$

After changing the order of integration

$y \to 0$ to $4a$

$x \to y^2/4a$ to $2\sqrt{ay}$



$x = \frac{(4a)^2}{4a} = 4a$

$\boxed{x=4a}$

$x=0$ $\quad$ $x=4a$

$y = x^2/4a$ $\quad$ $y=2\sqrt{xa}$

$x^2 = 4ay$ $\quad$ $y^2 = 4ax$

$\left(\frac{y^2}{4a}\right)^2 = 4ay$ $\quad$ $x = y^2/4a$

$y^4 = 64a^3 y$

$y^3 = 64a^3$

$y = (64a^3)^{1/3} = 4a$

$\boxed{y=4a}$

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} x(x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{4a}\int_{y^2/4a}^{2\sqrt{ay}} (x^3+xy^2)\,dx\,dy = \int_{y=0}^{4a}\left[\frac{x^4}{4}+\frac{x^2}{2}y^2\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_{y=0}^{4a}\left[\frac{(2\sqrt{ay})^4}{4} + \frac{(2\sqrt{ay})^2}{2}y^2 - \left(\frac{1}{4}\left(\frac{y^2}{4a}\right)^4 + \frac{1}{2}y^2\left(\frac{y^2}{4a}\right)^2\right)\right]dy$$

$$= \int_{y=0}^{4a}\left\{\frac{2^{\cancel{4}\,2}(ay)^2}{\cancel{4}} + \frac{\cancel{4}ay^3}{\cancel{2}} - \left[\frac{1}{4}\frac{y^8}{(4a)^4} + \frac{1}{2}y^2\frac{y^4}{16a^2}\right]\right\}dy$$

$$= \int_{y=0}^{4a}\left(4a^2y^2 + 2ay^3 - \frac{1}{4^5a^4}y^8 - \frac{1}{32a^2}y^6\right)dy$$

$$= 4a^2\frac{y^3}{3} + \frac{2ay^4}{4} - \frac{1}{1024a^4}\frac{y^9}{9} - \frac{1}{32a^2}\frac{y^7}{7}\Bigg|_0^{4a}$$

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$$= \frac{4}{3}a^2(4a)^3 + \frac{2a}{4}(4a)^4 - \frac{1}{1024a^4}\frac{(4a)^9}{9} - \frac{1}{32a^2}\frac{(4a)^7}{7}$$

$$= \frac{256}{3}a^5 + 128a^5 - \frac{4^9a^5}{1024\times 9} - \frac{1\times 4^7 a^5}{32\times 7}$$

$$= \frac{256}{3}a^5 + 128a^5 - \frac{4^9 a^5}{4^5\times 9} - \frac{4^7 a^5}{4\times 4\times 2\times 7}$$

$$= \frac{256}{3}a^5 + 128a^5 - \frac{256a^5}{9} - \frac{512}{7}a^5$$

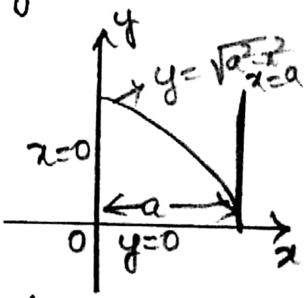
$$= \frac{5376a^5 + 8064a^5 - 1792a^5 - 4608a^5}{63}$$

$$= \frac{7040}{63}a^5$$

Change of Variables

(1) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$ by changing to polar co-ordinates.

Soln The region of integration is bounded by the curves $y=0$ (the x-axis) and $y=\sqrt{a^2-x^2}$ or $x^2+y^2=a^2$ & held between the vertical lines $x=0$ & $x=a$. The region is positive quadrant of the circle $x^2+y^2=a^2$.



To change in to polar coordinate.
put $x=r\cos\theta$ $y=r\sin\theta$ $dx\,dy = r\,dr\,d\theta$
$\Rightarrow x^2+y^2=r^2$. The limits are $r=0$ to $r=a$
& $\theta = 0$ to $\pi/2$

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

put $a^2-r^2=t$
$-2r\,dr = dt$
$r\,dr = -\frac{dt}{2}$
when $r=0$, $t=a^2$
$r=a$, $t=0$

$$= \frac{1}{-2}\int_{\theta=0}^{\pi/2}\int_{t=a^2}^{0}\sqrt{t}\,dt\,d\theta = \frac{-1}{2}\int_0^{\pi/2} \left.\frac{t^{3/2}}{3/2}\right|_{a^2}^{0} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} 0-(a^2)^{3/2}\, d\theta$$

$$= \frac{1}{3}a^3\,[\theta]_0^{\pi/2}$$

$$= \frac{1}{3}a^3\frac{\pi}{2}$$

$$= \frac{\pi a^3}{6}$$

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Area and Volume

(1) Show that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16a^2}{3}$

Soln The region between the given parabolas is depicted in Fig. In this region, x varies from 0 to $4a$ and, for each x, y varies from a point on the parabola $x^2=4ay$ to a point on the parabola $y^2=4ax$; that is, from $y=x^2/4a$ to $y=2\sqrt{ax}$. Hence the required area is

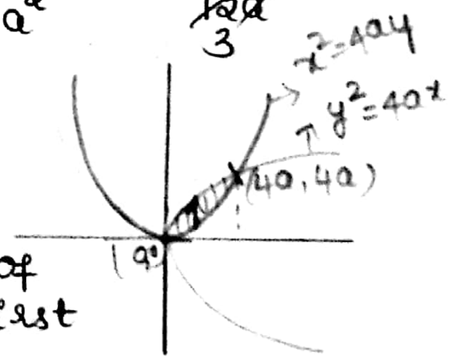
$$A=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx=\int_0^{4a}[y]_{x^2/4a}^{2\sqrt{ax}}dx=\int_0^{4a}\left(2\sqrt{ax}-\frac{x^2}{4a}\right)dx$$

$$=2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{4a}-\frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$=\frac{4}{3}\sqrt{a}\left[(4a)^{3/2}\right]-\frac{1}{12a}(4a)^3=\frac{4}{3}\sqrt{a}\left[(4)^{3/2}a^{3/2}\right]-\frac{1}{12a}\,4\times4\times4\,a^{3}$$

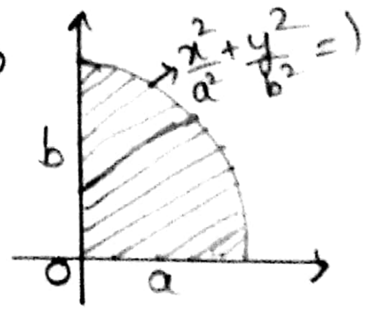
$$=\frac{4}{3}\sqrt{a}\times 8a^{3/2}-\frac{16}{3}a^2=\frac{32}{3}a^2-\frac{16}{3}a^2$$

$$A=\frac{16a^2}{3}$$



(2) Find the area bounded by the arc of the ellipse $x^2/a^2+y^2/b^2=1$ in the first quadrant.

Soln In the given region, x varies from 0 to a & , for each x, y varies from 0 to a point on the ellipse; i.e to the point for which $y=b(1-x^2/a^2)^{1/2}$.



Hence, the required area is

$$A=\int_{x=0}^{a}\int_{y=0}^{b(1-x^2/a^2)^{1/2}} dy\,dx=\int_{x=0}^{a} b\left(1-\frac{x^2}{a^2}\right)^{1/2}dx$$

$$=\frac{b}{a}\int_{x=0}^{a}(a^2-x^2)^{1/2}dx=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}x/a\right]_0^a$$

$$=\frac{b}{a}\cdot\frac{a^2}{2}\sin^{-1}1=\frac{b}{\cancel{a}}\,\frac{a^{\cancel{2}}}{2}\,\frac{\pi}{2}$$

$$=\frac{\pi}{4}ab.$$

$$\left[\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}(x/a)\right]$$

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NOTE: Area, volume

1. $\iint_R dx\,dy =$ Area of the region R in the cartesian form.

2. $\iiint_V dx\,dy\,dz =$ Volume of a solid

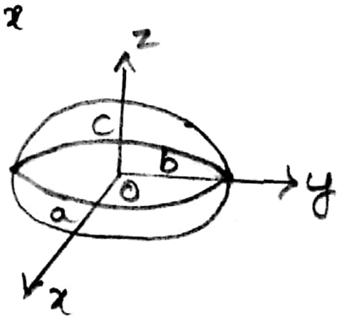
3. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Soln. Required volume is equal to 8 times the volume of that part of the ellipsoid which lies in the first octant. In this part z varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

For $z=0$, y varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

For $z=0$, $y=0$, x varies from 0 to a

$$\therefore V = 8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}}\int_{z=0}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx$$



$$= 8c\int_0^a\left\{\int_{y=0}^{b\sqrt{1-x^2/a^2}}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2} dy\right\}dx$$

$$= 8c\int_0^a\left\{\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{\left(\sqrt{1-x^2/a^2}\right)^2-(y/b)^2}\,dy\right\}dx \qquad \text{put } \sqrt{1-\frac{x^2}{a^2}}=K$$

$y/b = t$

$$= 8c\int_0^a\left\{\int_{t=0}^{K} b\sqrt{K^2-t^2}\,dt\right\}dx$$

$dy = b\,dt$
$y=0 \quad t=0$
$y=b\sqrt{1-x^2/a^2}=bK$
$\Downarrow\ t=K$

$$= 8bc\int_0^a\left[\left\{\frac{t\sqrt{K^2-t^2}}{2}+\frac{K^2}{2}\sin^{-1}\frac{t}{K}\right\}\right]_0^K dx$$

$$= 8bc\int_0^a\left(\frac{K^2}{2}\sin^{-1}1\right)dx = 8bc\int_0^a\frac{1}{2}\left(1-\frac{x^2}{a^2}\right)\pi/2\,dx$$

$$= 2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left(a-\frac{a^3}{3a^2}\right) = \frac{4\pi}{3}abc.$$

4. Find the volume of the parallelepiped (A, B, C) where $A=(1,2,3)$, $B=(3,0,6)$, $C=(7,1,9)$.

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LINE INTEGRAL

1. If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C F\,dR$ where c is the curve in the xy plane $y = 2x^2$ from (0,0) to (1,2)

Soln Since the particle moves in the xy-plane (z=0), we take $R = x\hat{i} + y\hat{j}$. Then $\int_C F\,dR$, where C is the parabola $y = 2x^2$

$$= \int_C (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \int_C (3xy\,dx - y^2\,dy) \quad (1)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, (1) becomes

$$= \int_{x=0}^{1} \left(3x \times 2x^2\,dx - (2x^2)^2\,d(2x^2)\right) = \int_0^1 6x^3\,dx - 4x^4 \times 4x\,dx$$

$$= \int_{x=0}^{1} (6x^3\,dx - 16x^5\,dx) = \frac{6x^4}{4} - \frac{16x^6}{6}\Big|_0^1 = \frac{\cancel{6}^{3}}{\cancel{4}_{2}} - \frac{\cancel{16}^{8}}{\cancel{6}_{3}} = \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9-16}{6} = -7/6.$$

2. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve $y = x^3$ from the point (1,1) to the point (2,8) if
$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$.

Soln $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$

$\vec{F}\cdot d\vec{r} = (5xy - 6x^2)\,dx + (2y - 4x)\,dy$

$y = x^3$; $dy = 3x^2\,dx$

$\vec{F}\cdot d\vec{r} = (5x\,x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2\,dx$

$$\int_C \vec{F}\,d\vec{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \frac{\cancel{6}x^6}{\cancel{6}} + \frac{\cancel{5}x^5}{\cancel{5}} - \frac{\cancel{12}^{3}x^4}{\cancel{4}} - \frac{\cancel{6}^{2}x^3}{\cancel{3}}\Big|_1^2 = x^6 + x^5 - 3x^4 - 2x^3\Big|_1^2$$

$$= 2^6 + 2^5 - 2^4\times 3 - 2\times 2^3 - (1+1-3-2)$$

$$= 64 + 32 - 48 - 16 - (-3)$$

$$= 35.$$

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(3) Find the total work done by a force $\vec{F} = 2xy\hat{i} - 4z\hat{j} + 5x\hat{k}$ along the curve $x = t^2$, $y = 2t+1$, $z = t^3$ from the point $t=1$ to $t=2$.

Soln Take $r = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = t^2\hat{i} + (2t+1)\hat{j} + t^3\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$$

$$\vec{F} = 2xy\hat{i} - 4z\hat{j} + 5x\hat{k}$$

$$= 2t^2(2t+1)\hat{i} - 4(t^3)\hat{j} + 5t^2\hat{k}$$

$$= (4t^3 + 2t^2)\hat{i} - 4t^3\hat{j} + 5t^2\hat{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = (4t^3+2t^2)2t - 4t^3\cdot 2 + 5t^2 \times 3t^2$$

$$= 8t^4 + 4t^3 - 8t^3 + 15t^4$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = 23t^4 - 4t^3$$

$$\int_C \vec{F}\, d\vec{r} = \int_C \left(\vec{F}\,\frac{d\vec{r}}{dt}\right) dt = \int_1^2 (23t^4 - 4t^3)\, dt$$

$$= \left.\frac{23t^5}{5} - \frac{4t^4}{4}\right|_1^2$$

$$= \frac{23}{5}(32) - 16 - \frac{23}{5} + 1 = 127.6$$

Green's theorem

Two continuous function $p(x,y)$ and $q(x,y)$ having continuous partial derivatives in the region R of the xy plane bounded by a simple closed curve C.

$$\int p\,dx + q\,dy = \iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\, dy$$

(1) By using Green's theorem, evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle in the xy plane bounded by the lines $y=0$, $x = \pi/2$ and $y = 2x/\pi$.

soln $p = y - \sin x$, $q = \cos x$

$\partial p/\partial y = 1$ $\quad \partial q/\partial x = -\sin x$

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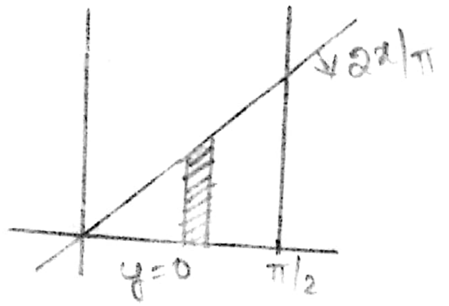
By Green's theorem,

$$\int P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int (y - \sin x)\,dx + \cos x\,dy = \iint_R (-\sin x - 1)\,dx\,dy$$



$$= -\iint_R (\sin x + 1)\,dx\,dy$$

y varies from 0 to $2x/\pi$

x varies from 0 to $\pi/2$

$$= -\int_0^{\pi/2}\int_0^{2x/\pi} (\sin x + 1)\,dy\,dx$$

$$= -\int_0^{\pi/2} (\sin x + 1)\left[y\right]_0^{2x/\pi} dx$$

$$= -\int_0^{\pi/2} (\sin x + 1)(2x/\pi)\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi/2} (2x\sin x + 2x)\,dx$$

$$= -\frac{1}{\pi}\left[2\{x(-\cos x) - 1\times -\sin x\} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_0^{\pi/2}$$

$$= -\frac{1}{\pi}\left[-2x\cos x + 2\sin x + x^2\right]_0^{\pi/2}$$

$$= -\frac{1}{\pi}\left[-\frac{\cancel{2}\pi}{\cancel{2}}\overset{0}{\cancel{\cos}}\pi/2 + 2\sin\pi/2 + \frac{\pi^2}{4} - (0)\right]$$

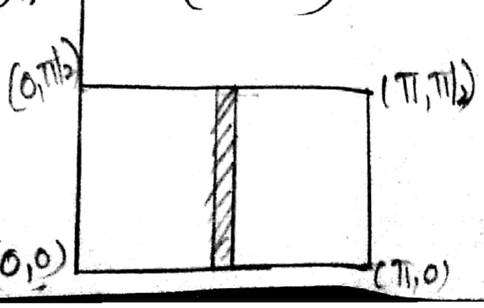
$$= -\frac{1}{\pi}\left[2 + \frac{\pi^2}{4}\right] = -\left[\frac{2}{\pi} + \frac{\pi}{4}\right]$$

$$= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

② Using Green's theorem evaluate $\int_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$ and $(0,\pi/2)$

Soln $P = e^{-x}\sin y \qquad Q = e^{-x}\cos y$

$\frac{\partial P}{\partial y} = e^{-x}\cos y \qquad \frac{\partial Q}{\partial x} = -e^{-x}\cos y$



y varies from 0 to $\pi/2$

x varies from 0 to π

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By Green's theorem,

$$\int_C p\,dx + q\,dy = \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$$

$$\int e^{-x}\sin x\,dx + e^{-x}\cos y\,dy = -2\iint e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\pi}\left[\int_0^{\pi/2} e^{-x}\cos y\,dy\right]dx$$

$$= -2\int_0^{\pi} e^{-x}\sin y\Big]_0^{\pi/2}dx$$

$$= -2\int_0^{\pi} e^{-x}\left[\sin\frac{\pi}{2} - \sin 0\right]dx$$

$$= -2\int_0^{\pi} e^{-x}\,dx = -2\frac{e^{-x}}{-1}\Big|_0^{\pi}$$

$$= 2\left[e^{-x}\right]_0^{\pi} = 2e^{-\pi} - 2$$

$$= 2\left[e^{-\pi} - 1\right]$$

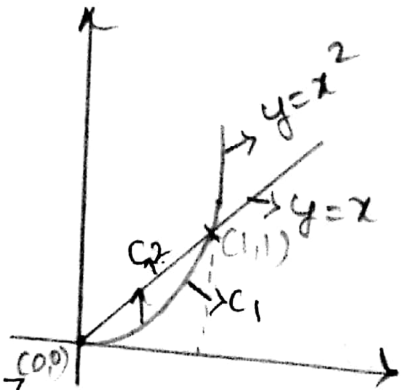
(3) Verify the Green's theorem for $\int_C [(xy+y^2)dx + x^2dy]$, where C is bounded by $y=x$ & $y=x^2$.

Soln By green's theorem

$P = xy+y^2$

$Q = x^2$

$$\text{RHS} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1\int_{x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_0^1 xy - \frac{\cancel{2}y^2}{\cancel{2}}\Big|_{x^2}^{x} dx = \int_0^1 \left[x^2 - x^2 - (x^3 - x^4)\right]dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \frac{x^5}{5} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{5} - \frac{1}{4}$$

$$= \frac{4-5}{20} = -\frac{1}{20} \rightarrow (1)$$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$y = 1$

$\Rightarrow$ $P = xy + y^2$, $Q = x^2$

$\frac{\partial P}{\partial y} = x + 2y$; $\partial Q/\partial x = 2x$.

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$$\int_C (xy+y^2)\,dx + x^2 dy = \int_{C_1} (xy+y^2)\,dx + x^2 dy + \int_{C_2} (xy+y^2)\,dx + x^2 dy$$

$$\int_{C_1} (xy+y^2)\,dx + x^2 dy = \int_0^1 \{x(x^2)+(x^2)^2\}\,dx + x^2 d(x^2) = \int_0^1 (3x^3+x^4)\,dx$$

$$= \frac{19}{20} \quad \text{⑤}$$

$$\int_{C_2} (xy+y^2)\,dx + x^2 dy = \int_1^0 [(x^2+x^2)\,dx + x^2 dx] = -\int_0^1 3x^2 dx = -1$$

$$\therefore \int_C (xy+y^2)\,dx + x^2 dy = \frac{19}{20} - 1 = \frac{-1}{20} \rightarrow (2)$$

From (1) & (2), we note that

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$ The Green's theorem is thus verified for the given integral.

<u>The Stokes theorem</u>

<u>Statement</u>: The line integral of the tangential component of a vector $\vec{F}$ taken around a simple curve C is equal to the surface integral of the normal component of the curl $\vec{F}$ taken over any surface S having C as its boundary

i.e. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$

(1) Evaluate by stoke's theorem $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, $z=3$

Soln $\oint_C \sin z\,dx - \cos x\,dy + \sin y\,dz = \oint_C (\sin z\,\hat{i} - \cos x\,\hat{j} + \sin y\,\hat{k})\cdot(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k})$

$= \oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \sin z\,\hat{i} - \cos x\,\hat{j} + \sin y\,\hat{k}$.

Therefore given integral is $\oint_C \vec{F}\cdot d\vec{r}$

By stokes theorem, we have $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$

Now $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}$

The boundary of the given rectangle is $0 \le x \le \pi$, $0 \le y \le 1$ lies on the plane $z=3$ which is parallel to xy-plane. Therefore the unit normal drawn to S is $\hat{k}$. i.e. $\hat{n} = \hat{k}$

$\therefore (\nabla\times\vec{F})\cdot\hat{n} = (\cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k})\cdot\hat{k} = \sin x$

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$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_S \sin x\,ds = \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy$$

$$= \int_0^1 -\cos x\Big|_0^{\pi} dy = -\int_0^1 (\cos\pi - \cos 0)\,dy$$

$$= 2\int_0^1 dy = 2\,y\Big|_0^1 = 2$$

Gauss Divergence theorem

Statement: Let S be the closed boundary surface of a region of volume V. Then for a continuously differentiable vector field $\vec{F}$ defined on V and on S

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$ where $\hat{n}$ is the outward drawn unit normal vector at any point of S.

① Using Guass divergence theorem, evaluate $\iint_S (x\hat{i}+y\hat{j}+z^2\hat{k})\cdot\hat{n}\,ds$ where S is the closed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1$.

Soln Given $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$ $\therefore$ $\operatorname{div}\vec{F} = 1+1+2z = 2(1+z)$

By divergence theorem, we have $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$

$$= \iiint_V 2(1+z)\,dV = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{1} 2(1+z)\,dz\,dy\,dx$$

$$= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[z+\frac{z^2}{2}\right]_0^1 dy\,dx = 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1+1/2)\,dy\,dx$$

$$= 3\int_{-1}^{1} y\Big]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 3\int_{-1}^{1}\left(\sqrt{1-x^2}+\sqrt{1-x^2}\right)dx$$

$$= 6\int_{-1}^{1}\sqrt{1-x^2}\,dx = 12\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 12\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1 = 12\left[\frac{1}{2}\cdot\sin^{-1}1\right]$$

$$= 6\cdot\frac{\pi}{2} = 3\pi$$

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