PARTIAL DIFFERENTIAL EQUATIONS

Definition:- An equation containing one or more partial derivative of an unknown function of two or more independent variables is known as a partial differential equation.

E.g. 1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

2) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$

3) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$

4) $\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \frac{\partial z}{\partial y} = 1$

NOTE:

* A differential equation is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

e.g. 1) $\frac{dy}{dx} = f(x)$, here "x" is an independent variable and "y" is a dependent variable.

2) $\frac{d^2y}{dx^2} = -3x + 4$

* Ordinary Differential equation is an equation that contains only one independent and one or more of its derivatives with respect to the variable.

eg. 1) $y', y'', \ldots, y^{n}, \ldots$ with respect to x

2) $y' = x^2 - 1$

Order of a partial differential Equation: ②

The Order of a pde is defined as Order of the highest partial derivative occuring in the pde

In above examples of pdes ① & ② are of the first Order. ④ is of the Second Order & ③ is of the third order.

Degree of a pde: The degree of a pde is the degree of the highest Order derivative which occurs in it after the equation has been rationalized i.e., made free from radicals and fractions so far as derivative are concerned.

In the above Examples of pdes ①, ② & 3 are of first degree, ④ is of the second degree while $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$ is of second degree.

Note: In this chapter, we consider the case of two independent variable we usually assume them to be x, y & assume z to be the dependent variable. We adopt the following notations throughout the study of pdes

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad \&$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$\parallel^{ly}$, we can write

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} \text{ \& so on}$$

Classification of First Order pde:

1. **Linear Equation**: A first order pde is said to be a linear equation if it is linear in p, q and z i.e., if it is of the form $P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$

eg. $yp - xq = xyz + x$ & $p+q = z+xy$.

2. **Semi-linear Equation**: A first order pde is said to be a semi-linear equation if it is linear in p & q and the coefficient of p and q the functions of x and y only i.e., if it is of the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

e.g., $e^x p - yxq = xz^2$ & $yp + xq = \frac{x^2z^2}{y^2}$.

3. **Quasi Linear Equation**: A first order pde is said to be a quasi-linear equation if it is linear in p & q, i.e., if it is of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z).$$

e.g., $(x^2+z^2)p - xyq = z^3x + y^2$ & $x^2zp + y^2zp = xy$

4. **Non-Linear Equation**: pdes of the form $f(x,y,z,p,q) = 0$ which do not come under the above three types are said to be non-linear Equations.

eg. $pq = z$, $p^2+q^2 = 1$ & $x^2p^2 + y^2q^2 = z^2$ are all non-linear pde.

Formation of pde :

By Elimination of Arbitrary constants:

(a) By Elimination of arbitrary constants

Consider an Equation $F(x,y,z,a,b)=0 \rightarrow (1)$ where a & b denote arbitrary constants. Let z be regarded as function of two independent variables x & y.

Differentiating (1) w.r.t x & y partially, we get

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \quad \& \quad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \rightarrow (2)$$

Eliminating two constants a & b from three equation of (1) & (2). We shall obtain an equation of the form $f(x,y,z,p,q)=0$ which is a pde of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables. The above procedure of elimination will give rise to pdes of higher order than the first.

1. Form the pde by eliminating arbitrary constants a and b from $z = ax+by+ab$

Soln: Given $z = ax+by+ab \rightarrow (1)$

Differentiate (1) partially wrt x & y, we obtain

$$\frac{\partial z}{\partial x} = a = p \rightarrow (2) \qquad \frac{\partial z}{\partial y} = b = q \rightarrow (3)$$

Substituting p & q for a & b in eq (1), we get the required pde as

$$z = px+qy+pq.$$

2. $z = ax+y$ where a is the only arbitrary constant and x, y are two independent variables.

Soln: Given $z = ax + y \rightarrow (1)$

Diff (1) partially w.r.t x, we get $\frac{\partial z}{\partial x} = a \rightarrow (2)$

Diff (1) partially w.r.t y, we get $\frac{\partial z}{\partial y} = 1 \rightarrow (3)$

Eliminating a between (1) & (2), we get

$$z = \frac{\partial z}{\partial x} x + y \Rightarrow z = px + y = xp + y \rightarrow (4)$$

Since (3) does not contain arbitrary constant
so (3) is also a pde under consideration.
Thus we get two pdes (3) & (4).

3. Eliminate a and b from $az + b = a^2 x + y$.

Soln: Given $az + b = a^2 x + y \rightarrow (1)$

Diff (1) partially w.r.t x & y, we get

$$a\frac{\partial z}{\partial x} = a^2 \; - (2) \qquad a\frac{\partial z}{\partial y} = 1 \rightarrow (3)$$

Eliminating a from (2) & (3), we have

$$a\frac{\partial z}{\partial x} \cdot a\frac{\partial z}{\partial y} = a^2 \cdot 1 \Rightarrow \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \text{ which}$$
$$\Rightarrow pq = 1$$

is the unique pde of order one

4. Eliminate a, b & c from $z = ax + by + cxy$.

Soln: Given $z = ax + by + cxy \; - (1)$

Diff (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a + cy \rightarrow (2) \qquad \frac{\partial z}{\partial y} = b + cx \rightarrow (3)$$

Diff (2) partially w.r.t x & (3) w.r.t y, we get

$$\frac{\partial^2 z}{\partial x^2} = 0 \rightarrow (4) \qquad \frac{\partial^2 z}{\partial y^2} = 0 \rightarrow (5)$$

Diff (2) w.r.t y, we get $\frac{\partial^2 z}{\partial x \partial y} = c \rightarrow (6)$

consider (2) & (3)

$$\Rightarrow x\left(\frac{\partial z}{\partial x}\right) = ax + cxy \quad \& \quad y\left(\frac{\partial z}{\partial y}\right) = by + cxy$$

$$\therefore x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = ax + cxy + by + cxy$$

using ① & ⑥, we get

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) \to ⑦$$

Here we get three pde given by ④, ⑤ & ⑦ which are all of order two.

(b) <u>By Elimination of arbitrary function ϕ</u>:

Let $\phi(u,v) = 0$, where u & v are functions of x, y and z $\to$ ①

We treat z as dependent variable and x & y as independent variables so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0 \quad \& \quad \frac{\partial x}{\partial y} = 0 \to ②$$

Differentiating ① partially wrt x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \cancel{\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x}} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) +$$

$$\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \cancel{\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x}} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\cdot\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) \to ③$$

Similarly differentiating ① partially wrt y we get

$$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \to ④$$

Eliminating ϕ with the help of (3) & (4) we get

$$\frac{\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}}$$

$$\Rightarrow \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

$$\Rightarrow \quad Pp + Qq = R \cdot \rightarrow (5)$$

where $P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$

$Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$

$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$

Thus we obtain a Linear p.d.e of first order and of first degree in p and q.

1. Form a pde by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$. What is the order of this pde?

Soln: Given $\phi(x+y+z, x^2+y^2-z^2) = 0 \rightarrow (1)$

Let $u = x+y+z$ & $v = x^2+y^2-z^2 \rightarrow (2)$

Then (1) becomes $\phi(u,v) = 0 \rightarrow (3)$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\cdot\frac{\partial v}{\partial z}\right) = 0 \rightarrow (4)$$

From (2), $\left.\begin{array}{l}\frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 1, \ \frac{\partial u}{\partial z} = 1, \ \frac{\partial v}{\partial x} = 2x, \\ \frac{\partial v}{\partial y} = 2y, \ \frac{\partial v}{\partial z} = -2z\end{array}\right\} \rightarrow (5)$

From (4) & (5),

$$\frac{\partial \phi}{\partial u}(1+p) + \frac{\partial \phi}{\partial v}\left(2x + p(-2z)\right) = 0$$

$$\Rightarrow \frac{\left(\frac{\partial\phi}{\partial u}\right)}{\left(\frac{\partial\phi}{\partial v}\right)} = \frac{-2(x-pz)}{(1+p)} \rightarrow ⑥$$

Again, differentiating ③ partially wrt y, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}(1+q)+\frac{\partial\phi}{\partial v}\left(2y+q(-2z)\right)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}(1+q)+2\left(\frac{\partial\phi}{\partial v}\right)(y-qz)=0 \quad \text{by ⑤}$$

$$\Rightarrow \frac{\left(\frac{\partial\phi}{\partial u}\right)}{\left(\frac{\partial\phi}{\partial v}\right)} = \frac{-2(y-qz)}{(1+q)} \rightarrow ⑦$$

Eliminating ϕ from ⑥ & ⑦, we obtain

$$\frac{(x-pz)}{(1+p)} = -\frac{(y-qz)}{(1+q)}$$

$$\Rightarrow (1+q)(x-pz) = (1+p)(y-qz)$$

$\Rightarrow (y+z)p-(x+z)q = x-y$ which is the desired pde of first order.

2. Form pde by eliminating the arbitrary function from $z=f(x+it)+g(x-it)$, where $i=\sqrt{-1}$.

Soln: Given $z=f(x+it)+g(x-it) \rightarrow ①$

Differentiating ① twice partially wrt x, we get

$$\frac{\partial z}{\partial x} = f'(x+it)+g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it)+g''(x-it) \rightarrow ②$$

III'y $\frac{\partial z}{\partial t} = i[f'(x+it) - g'(x-it)]$

$$\frac{\partial^2 z}{\partial t^2} = -(f''(x+it) + g''(x-it)) \rightarrow (3)$$

From (2)+(3), we get

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$. which is the required pde

Exercise

1. Derive a pde (by eliminating the constants) from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

2. Form the pde (by eliminating the arbitrary function) from
   a) $z = (x+y)\phi(x^2-y^2)$
   b) $z = f(x+at) + g(x-at)$
   c) $f(x^2+y^2, z-xy) = 0$

3. Find the DE of all planes which are at a constant distance "a" from the origin.

4. Form a PDE by eliminating a, b, c from the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

5. Form pde from $z = xf_1(x+t) + f_2(x+t)$

6. Form the pde (by eliminating the arbitrary function) from $F(xy+z^2, x+y+z) = 0$

7. Form pde from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

8. Form pde from the solutions
   i) $z = f(x) + e^y g(x)$ ii) $z = \frac{1}{x}[F(x-at) + F(x+at)]$

9. Find the DE of all spheres whose center lies on the z-axis

10. Find the DE of all spheres of radius 'a' units having their centers in the xy-plane

Solution of non homogeneous pde by direct integration

Defn If each term of the equation contains either the dependent variable or one of its derivatives it is said to be homogeneous otherwise, non-homogeneous.

eg. 1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is homogeneous

2) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u + xy$ is non-homogeneous.

1. Solve the equation $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Soln: The given equation may be rewritten as

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y \rightarrow (1)$$

Integrate (1) w.r.t x, we get

$$\frac{\partial z}{\partial y} = \frac{x^3}{3}y + f(y) \rightarrow (2)$$

where $f(y)$ is an arbitrary function.

Integrate (2) w.r.t y, we get

$$z = \frac{x^3}{3}\frac{y^2}{2} + \int f(y)\,dy + g(x)$$

$$z = \frac{x^3 y^2}{6} + F(y) + g(x) \rightarrow (3)$$

where $F(y) = \int f(y)\,dy$ & $g(x)$ is an arbitrary function of x. Expression (3) is a general solution of the given equation. This solution contains two arbitrary function $F(y)$ and $g(x)$.

(11)

* Since y is treated as a constant in pde wrt x, the constant of integration is taken as an arbitrary function of y while integrating a function of x & y wrt x. III'ly, while integrating a function of x & y wrt y the constant of integration is taken as a function of x.

2. Solve the equation $\frac{\partial^2 z}{\partial y^2} = \sin xy$.

Soln The given equation may be written as

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \sin xy \rightarrow (1)$$

Integrating (1) with respect to y, we get

$$\frac{\partial z}{\partial y} = \frac{-\cos xy}{x} + f(x) \rightarrow (2)$$

where $f(x)$ is an arbitrary function of x.

Integrate (2) wrt y, we get

$$z = -\frac{1}{x^2}\sin xy + yf(x) + g(x) \rightarrow (3)$$

where $g(x)$ is an arbitrary function of x

Expression (3) is a general solution of the given equation.

3. Solve the equation $xy\frac{\partial^2 z}{\partial x \partial y} - x\frac{\partial z}{\partial x} = y^2$.

Soln Dividing throughout by x, the equation may be rewritten as

$$y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial x} = \frac{y^2}{x}$$

$$\Rightarrow y\frac{\partial p}{\partial y} - p = \frac{y^2}{x}$$

$$\Rightarrow \frac{1}{y}\frac{\partial p}{\partial y} - \frac{1}{y^2}p = \frac{1}{x}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{p}{y}\right) = \frac{1}{x} \rightarrow (1)$$

Integrating (1) wrt y, we get

$$\frac{p}{y} = \frac{1}{x}\cdot y + f(x)$$

$\Rightarrow \frac{1}{y}\frac{\partial z}{\partial x} = \frac{y}{x} + f(x) \Rightarrow \frac{\partial z}{\partial x} = y^2/x + y f(x) \rightarrow (2)$

Integrate (2) w.r.t x, we get

$z = y^2 \log x + y\int f(x)\,dx + g(y)$

$z = y^2 \log x + y\,F(x) + g(y)$

is a general solution of the given equation

Here $F(x) = \int f(x)\,dx$ & $g(y)$ are arbitrary functions of x and y respectively.

4\. Solve the Equation $\frac{\partial^2 u}{\partial x \partial y} = 2x\left(\frac{\partial u}{\partial y} + 1\right)$

Soln :- put $\frac{\partial u}{\partial y} = v$ in the given Eqn, it becomes

$\frac{\partial v}{\partial x} - 2x\,v = 2x \rightarrow (1)$

If y is treated as a constant, this equation is a first-order ordinary linear DE in which v is the Dependent Variable & x is the independent variable.

eqn (1) is of the form
$\frac{dy}{dx} + Py = Q$ which is a linear diff equation
Leibnitz's solution of the equation is
$y(I.F) = \int Q(I.F)\,dx + C$
$I.F = e^{\int P\,dx}$

For this equation

$I.F = e^{\int(-2x)dx} = e^{-x^2}$

$\therefore$ A Solution of (1) is

$v.e^{-x^2} = \int (2x)(e^{-x^2})\,dx + f(y)$

where $f(y)$ is an arbitrary function of y.

But $v = \frac{\partial u}{\partial y}$

(2) becomes

$$\frac{\partial u}{\partial y} \cdot e^{-x^2} = \int (2x) \cdot (e^{-x^2})\, dx + f(y)$$

$$= 2\int x \cdot e^{-x^2} dx + f(y)$$

$$\boxed{\begin{aligned} &\int x \cdot e^{-x^2} dx \\ &= \int e^{-t} dt/2 \quad \text{put } x^2 = t,\ 2x\,dx = dt,\ x\,dx = \tfrac{dt}{2} \\ &= -e^{-t}/2 = -e^{-x^2}/2 \end{aligned}} = -\cancel{2}\frac{e^{-x^2}}{\cancel{2}} + f(y)$$

$$\frac{\partial u}{\partial y} \cdot e^{-x^2} = -e^{-x^2} + f(y)$$

$$\frac{\partial u}{\partial y} = -1 + e^{x^2} f(y) \rightarrow (3)$$

Integrate (3) w.r.t y, we get

$$u = -y + e^{x^2}\int f(y)\, dy + g(x) \rightarrow (4)$$

where $g(x)$ is an arbitrary function of x &

$F(y) = \int f(y)\, dy$.

Expression (4) is a general solution of the given Equation.

5. Solve : $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = -\sin y - x\cos y$.

<u>Soln</u> put $q = \frac{\partial z}{\partial y}$, the given Equation becomes

$$\frac{\partial q}{\partial y} - xq = -(\sin y + x\cos y) \rightarrow (1)$$

If x is treated as a constant, this equation is a first-order ordinary linear DE in which q is the dependent variable & y is the independent variable.

For this Equation $I.F = e^{\int(-x)\,dy}$

$$I.F = e^{-xy}$$

Eq ① is of the form
$$\frac{dx}{dy} + Px = Q$$
$$I.F = e^{\int P dy}$$
$$\therefore x(I.F) = \int Q(I.F)dy + C.$$
Here $x = v$.

$\therefore$ A solution of ① is
$$v e^{-xy} = \int -(\sin y + x\cos y) e^{-xy} dy + f(x)$$
$$= \int \left\{ \frac{d}{dy}\left(e^{-xy}\cos y\right)\right\} dy + f(x)$$

$$\frac{d}{dy}(e^{-xy}\cos y) = -e^{-xy}\sin y + \cos y\, e^{-xy}(-x)$$
$$= -e^{-xy}[\sin y + x\cos y]$$

$$v e^{-xy} = e^{-xy}\cos y + f(x)$$
$$\Rightarrow \frac{\partial z}{\partial y} = \cos y + e^{xy} f(x) \rightarrow ②$$

Integrate ② w.r.t y, we get
$$z = \sin y + \left(\frac{e^{xy}}{x}\right) f(x) + g(x)$$
This is a general Solution of the given Equation, this Solution contains two arbitrary function $f(x)$ & $g(x)$.

6. Solve the equation $\frac{\partial^2 z}{\partial x^2} = x+y$, given that $z = y^2$ when $x=0$ and $\frac{\partial z}{\partial x} = 0$ when $x=2$.

Soln: Given $\frac{\partial^2 z}{\partial x^2} = x+y \rightarrow ①$

Integrate ① w.r.t x, we get
$$\frac{\partial z}{\partial x} = \frac{x^2}{2} + yx + f(y) \rightarrow ②$$
where $f(y)$ is an arbitrary function

of y

Integrate ② w.r.t x, we get

$$z = \frac{x^3}{6} + \frac{yx^2}{2} + xf(y) + g(y) \rightarrow ③$$

where $g(y)$ is an arbitrary function of y.

Equation ③ is a general solution of the ①

Given that $\frac{\partial z}{\partial x} = 0$ when $x = 2$

using this condition in ②, we get

$$0 = \frac{4}{2} + 2y + f(y)$$

$$f(y) = -2(1+y) \rightarrow ④$$

using this in ③ we get

$$z = \frac{x^3}{6} + \frac{yx^2}{2} + x(-2(1+y)) + g(y)$$

$$z = \frac{x^3}{6} + \frac{x^2y}{2} - 2x(1+y) + g(y) \rightarrow ⑤$$

It is also given that $z = y^2$ when $x = 0$.

using this in ⑤, we get $g(y) = y^2$. putting this back in to ⑤, we get

$$z = \frac{x^3}{6} + \frac{x^2y}{2} - 2x(1+y) + y^2$$

is the required solution

7. Solve $\frac{\partial^2 u}{\partial x^2} = x + y$.

Soln: Integrating w.r.t x

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + yx + f(y)$$

again Integrating w.r.t x, we get

$$u = \frac{x^3}{6} + \frac{x^2y}{2} + xf(y) + h(y)$$ is the required solution

8. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Soln: Integrating wrt x

$$\int \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = \int \cos(2x+3y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

Again integrating wrt x

$$\frac{\partial z}{\partial y} = \frac{1}{2}\left[\frac{-\cos(2x+3y)}{2}\right] + x\cdot f(y) + h(y)$$

$$\int \cos(2x+3y) = \frac{\sin(2x+3y)}{2}$$

Integrating wrt y.

$$z = -\frac{1}{4}\frac{\sin(2x+3y)}{3} + xF(y) + H(y) + k(x)$$

required soln

$F(y) = \int f(y)\,dy$

$H(y) = \int h(y)\,dy$.

Exercise

1. Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$
2. Solve $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, when $x=0$, $z=e^y$ & $\dfrac{\partial z}{\partial x} = 1$
3. $\dfrac{\partial^2 z}{\partial x^2} = xy$. Subject to the condition $\dfrac{\partial z}{\partial x} = \log(1+y)$ when $x=1$ & $z=0$ when $x=0$
4. Solve $\dfrac{\partial u}{\partial x} = 6xy + z^3$
5. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$ given that $z=0$ when $x=0$ & $\dfrac{\partial z}{\partial x} = x$ when $y=1$
6. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ given that $\dfrac{\partial z}{\partial y} = -2\sin y$ when $x=0$ & $z=0$ when y is an odd multiple of $\pi/2$.
7. Solve the equation $\dfrac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 = \cos(2x-y)$

given that $z=0$ when $y=0$ & $\frac{\partial z}{\partial y}=0$ when $x=0$

8. Solve the equation $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cot x$ given that $u=0$ when $t=0$ & $\frac{\partial u}{\partial t}=0$ at $x=0$.

METHOD OF SEPARATION OF VARIABLES:

It involves a solution which breaks up into a product of functions each of which contains only one of the variables.

1. Solve by the method of separation of variables

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Soln: Suppose that the given Equation has Solution of the form $z(x,y) = X(x)\,Y(y) \to (1)$, where X is a function of x alone & Y that of y alone.

Substituting this value of z in the given equation, we have.

$$X''Y - 2X'Y + XY' = 0 \quad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy}$$

Separating the variables, we get

$$\frac{X''-2X'}{X} = \frac{-Y'}{Y} \to (2)$$

Since x and y are independent variables, therefore (2) can only be true if each side is Equal to the same constant, a (say).

$\therefore$ $\frac{X''-2X'}{X} = a$ i.e., $X''-2X'-aX=0 \to (3)$

and $\frac{-Y'}{Y} = a$ i.e, $Y'+aY=0 \to (4)$

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To solve the Ordinary linear Equation ③, the auxiliary Equation is

$$m^2-2m-a=0 \quad \text{where } m = 1\pm\sqrt{1+a}$$

$\therefore$ The solution of ③ is

$$X = c_1 e^{[1+\sqrt{1+a}]x} + c_2 e^{[1-\sqrt{1+a}]x}$$

and the solution of ④ is $Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (1), we get

$$z = \left\{c_1 e^{[1+\sqrt{1+a}]x} + c_2 e^{[1-\sqrt{1+a}]x}\right\}. c_3 e^{-ay}$$

ie., $z = \left\{ k_1 e^{[1+\sqrt{1+a}]x} + k_2 e^{[1-\sqrt{1+a}]x}\right\} e^{-ay}$

which is the required Complete Solution

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2. Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$.

Solution: Assume the solution $u(x,t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X'-X)T = 2XT'$$

or $$\frac{X'-X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$\therefore$ $X' - X - 2kX = 0$ or $\frac{X'}{X} = 1+2k \rightarrow ①$

and $\frac{T'}{T} = k \rightarrow ②$

Integrate Eq ① w.r.t x, we get

$$\log X = (1+2k)x + \log c.$$

$\Rightarrow$ $X = c$ ie $ce^{(1+2k)x}$.

From ②, $\log T = kt + \log c'$

$\Rightarrow$ $T = c'e^{kt}$.

or $X' - (2K+1)X = 0$

$$\boxed{\begin{array}{l} \text{A.E is } m-(2K+1)=0 \\ \therefore m = 2K+1 \\ X = c e^{(1+2K)x} \end{array}}$$

$\therefore T' = KT$

$T' - KT = 0$

A.E $m - K = 0$

$\boxed{m = K}$

$\boxed{T = c' e^{Kt}}$

Thus $u(x,t) = XT = cc' e^{(1+2K)x} \cdot e^{Kt} \rightarrow (3)$

Now. $6e^{-3x} = u(x,0) = cc' e^{(1+2K)x}$

$\therefore$ $cc' = 6$ and $1+2K) = -3$ or $K = -2$.

Substituting these values in (3), we get

$u(x,t) =$ $\therefore$ $u = 6e^{-3x} \cdot e^{-2t}$.

$\Rightarrow$ $u = 6e^{-(3x+2t)}$ which is the required solution.

A number of problems in engineering give rise to the following well-known partial differential Equations:

a) Heat (Diffusion) Equation : $\nabla^2 u = \left(\frac{1}{K}\right)\left(\frac{\partial u}{\partial t}\right)$

* 1-Dimensional : $\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{K}\right)\left(\frac{\partial u}{\partial t}\right)$

* 2-Dimensional ; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{K}\right)\left(\frac{\partial u}{\partial t}\right)$

* 3-Dimensional: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(\frac{1}{K}\right)\left(\frac{\partial u}{\partial t}\right)$

Note: 1) $c^2 = K$ in heat equation. (some times)

2) ∇^2 is called the Laplacian operator & is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. A function which satisfies Laplace Equation is called

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harmonic function.

b) Wave Equation: $\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

* One-dimensional: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2}\right)$

* 2-dimensional: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{1}{c^2}\right)\left(\frac{\partial^2 u}{\partial t^2}\right)$

* 3-dimensional: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(\frac{1}{c^2}\right)\left(\frac{\partial^2 u}{\partial t^2}\right)$

c) Laplace's (or harmonic) Equation: $\nabla^2 u = 0$

* 2-dimensional: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

* 3-dimensional: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

Solution of Heat Equation:

Assume that a solution of $\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2}\right)\left(\frac{\partial u}{\partial t}\right)$ is of the form $u(x,t) = X(x)\,T(t)$ where X is a function of x alone and T is a function of t only.

Substituting this in ①, we get

$$X''T = \frac{1}{c^2}T'X$$

$$\Rightarrow XT' = c^2 X''T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} \rightarrow ②$$

Clearly the left side of ② is a function of x only & the right side is a function of t alone. Since x and t are independent variables, ② can hold good if each side is equal to a constant K (say). Then ② leads to the odes

$$\frac{X''}{X} = K \Rightarrow X'' = KX \Rightarrow \frac{d^2X}{dx^2} - KX = 0 \rightarrow ③$$

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$\rightarrow \frac{dT}{dt} - kc^2 T = 0 \rightarrow (4)$ (58)

Solving (3) and (4), we get

a) when k is positive and $= p^2$, say

$X'' - p^2 X = 0$

A.E $m^2 - p^2 = 0$

$(m^2 - p^2) = 0 \Rightarrow m = \pm\sqrt{p^2}$, $m = \pm p$

$X = c_1 e^{px} + c_2 e^{-px}$

$T' - kc^2 T = 0$

$m - kc^2 = 0$

$m = kc^2$

$T = c_3 e^{kc^2 t}$, $T = c_3 e^{p^2 c^2 t}$

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2 p^2 t}$$

b) when k is negative & $= -p^2$, say

$$X = c_4 \cos px + c_5 \sin px, \quad T = c_6 e^{-c^2 p^2 t}$$

$X'' + p^2 X = 0$

A.E $m^2 + p^2 = 0$

$m^2 = -p^2$

$m = \pm\sqrt{-p^2} = \pm\sqrt{-1}\sqrt{p^2}$ $i = \sqrt{-1}$

$m = \pm ip$ if the roots are complex $X = e^{ax}(A\cos bx + B\sin bx)$

$X = e^{0x}(c_4 \cos px + c_5 \sin px)$

$T' + p^2 c^2 T = 0$

$m + p^2 c^2 = 0$

$m = -p^2 c^2$

$T = c_6 e^{-c^2 p^2 t}$

c) when k is zero. $X = c_7 x + c_8$, $T = c_9$

$X'' = 0$ if the roots are repeated

$m^2 = 0$ $X = (c_1 + c_2 x)e^{m_1 x}$

$m = 0, 0$

$X = (c_7 x + c_8) e^{0 \cdot x}$

$X = (c_7 x + c_8)$

$T' = 0$

$m = 0$

$T = c_9$

$\therefore$ The various possible solution of the heat equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t} \rightarrow (5)$$

$$u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t} \rightarrow (6)$$

$$u = (c_7 x + c_8) c_9 \rightarrow (7)$$

When we deal with heat transfer problems, it should be noted that temperature u must decrease with increase of time

i.e., $u(x,t) \to 0$, as $t \to \infty$.

So, such a condition is satisfied by the solution for heat transfer problems, and is:

$u = [c_1 \cos px + c_2 \sin px] c_3 e^{-c^2 p^2 t}$, $(k<0)$

General Solution of two-Dimensional Laplace Equation

Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \to (1)$

Suppose that (1) has solutions of the form $u(x,y) = X(x)\,Y(y) \to (2)$ where X and Y are respectively functions of x & y

From (2), $\frac{\partial^2 u}{\partial x^2} = X''Y$ & $\frac{\partial^2 u}{\partial y^2} = XY'' \to (3)$

Substitute (3) in (1) we get

$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} \to (4)$$

Since the LHS of (4) depends only on x and the RHS depends only on y. Both sides of (4) must be equal to some constant, say μ then leads to two ode $\frac{X''}{X} = \mu$ i.e. $X'' - \mu X = 0$ &

$-\frac{Y''}{Y} = \mu$ i.e. $Y'' + \mu Y = 0 \to (5)$ whose solutions depends only on the value of μ.

There are three cases arise & they are

Case (1): When $\mu = 0$, (5) reduces to $X'' = 0$ & $Y'' = 0$ on solving these, we get $X = A_1 x + B_1$ and

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$Y = C_1 y + D_1$

$\therefore$ soln of ① is $u(x,y) = (A_1 x + B_1)(C_1 y + D_1) \to 6$.

case 2 : When $\mu = \lambda^2$, i.e., positive. Here $\lambda \neq 0$ then ⑤ reduces to $X'' - \lambda^2 X = 0$ & $Y'' + \lambda^2 Y = 0$.

on solving these, $X = A_2 e^{\lambda x} + B_2 e^{-\lambda x}$ and

$$Y = C_2 \cos\lambda y + D_2 \sin\lambda y$$

$\therefore$ soln of ① is $u(x,y) = (A_2 e^{\lambda x} + B_2 e^{-\lambda x})(C_2 \cos\lambda y + D_2 \sin\lambda y) \to$ ⑦

case 3 : When $\mu = -\lambda^2$, i.e, negative. Here $\lambda \neq 0$. Then ⑤ reduces to $X'' + \lambda^2 X = 0$ & $Y'' - \lambda^2 Y = 0$

on solving these, $X = A_3 \cos\lambda x + B_3 \sin\lambda x$ &

$$Y = C_3 e^{\lambda y} + D_3 e^{-\lambda y}$$

$\therefore$ ① is $u(x,y) = (A_3 \cos\lambda x + B_3 \sin\lambda x)(C_3 e^{\lambda y} + D_3 e^{-\lambda y}) \to$ ⑧

out of the above mentioned three types of solution ⑤, ⑥ & ⑦, we must select an appropriate solution which suits the physical nature of the problem and given boundary condition

Note: Consider a DE of type $y'' + py' + qy = 0$, where p, q are some constant coefficients. For each of the equation we can write the so-called characteristic (auxiliary) equation as $K^2 + pK + q = 0$. The general solution of the homogeneous DE depends on the roots of the characteristic quadratic equation There are the following option:

* Discriminant of the characteristic quadratic equation $D>0$. Then the roots of the characteristic equations K_1 & K_2 are real and distinct. In this case the GS is given by the following function $y(x) = c_1 e^{K_1 x} + c_2 e^{K_2 x}$. where c_1 & c_2 are arbitrary real no's.

* Discriminant of the CQE $D=0$. Then the roots are real and equal. It is said in this case that there exists one repeated root K_0 of order 2. The GS of the DE has the form
$$y(x) = (c_1 x + c_2)\, e^{K_1 x}$$

* Discriminant of the CQE $D<0$, such an equation has complex roots $K_1 = \alpha + \beta i$, $K_2 = \alpha - \beta i$.
The general solution is written as
$$y(x) = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right].$$

[illegible]

A linear partial differential equation of the first order, commonly known as Lagrange's linear equation is of the form

$$Pp + Qq = R$$

where P, Q and R are functions of x, y, z. This equation is called a quasi linear equation. When P, Q and R are independent of z it is known as linear equation. Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u,v) = 0$ where u, v are some functions of x, y, z.

To solve the equation $Pp + Qq = R$

a) Form the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

b) Solve these simultaneous equation by giving $u = a$ and $v = b$ as its solutions.

c) Write the complete solution as $\phi(u,v) = 0$ or $u = f(v)$.

① Solve $y^2 z p + x^2 z q = y^2 x$.

Soln: Given equation can be written as

$y^2 z p + x^2 z q = y^2 x$

The subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\Rightarrow \frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

consider the first two fractions,

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$\Rightarrow x^2 dx = y^2 dy$$

On integration, we get $x^3 - y^3 = a \rightarrow ①$

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Again the first and third fractions give

$$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$$\Rightarrow x\,dx = z\,dz$$

On integration, $x^2 - z^2 = b \rightarrow (2)$

$\therefore$ The complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Or

$$\phi(x^3 - y^3, x^2 - z^2) = 0.$$

2. Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$.

Solution. Here the subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using multipliers x, y and z, we get each

$$\text{fraction} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lzy} + \cancel{lyz} - \cancel{mxz}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ which on integration gives $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$

$$\Rightarrow x^2 + y^2 + z^2 = a \rightarrow (1)$$

Again using multipliers l, m and n, we get

$$\text{each fraction} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$ which on integration gives $lx + my + nz = b \rightarrow (2)$

From (1) & (2), the required solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz).$$

$\ldots)p + 2xyq = 2xz$

Soln: The Subsidiary Equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

From the last two fractions, we have

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

which on integration gives

$$\log y = \log z + \log a$$

$$\log y - \log z = \log a$$

$$\log(y/z) = \log a.$$

$$\Rightarrow \quad y/z = a \rightarrow (1)$$

Using multipliers x, y & z, we have

$$\text{each fraction} = \frac{xdx+ydy+zdz}{x(x^2-y^2-z^2)+2xy^2+2xz^2}$$

$$= \frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2}.$$

$$= \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}.$$

$$= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}.$$

choose $\quad \dfrac{xdx+ydy+zdz}{\cancel{x}(x^2+y^2+z^2)} = \dfrac{dz}{2\cancel{x}z}$

$$\Rightarrow \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\Rightarrow \quad \log(x^2+y^2+z^2) = \log z + \log b$$

$$\Rightarrow \quad \log(x^2+y^2+z^2) - \log z = \log b$$

$$\Rightarrow \quad \log\left[\frac{x^2+y^2+z^2}{z}\right] = \log b.$$

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$$\Rightarrow \frac{x^2+y^2+z^2}{z} = b \rightarrow (2)$$

From (1) & (2), the required solution is

$$x^2+y^2+z^2 = zf(y/z)$$

4. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Soln: The Subsidiary Equation are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

using the multipliers $\frac{1}{x}, \frac{1}{y}$ & $\frac{1}{z}$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} \ldots}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}.$$

$$0 = \frac{1}{x}dx + \frac{dy}{y} + \frac{dz}{z}.$$

which on integration gives

$$\log x + \log y + \log z = \log a$$

or

$$xyz = a \quad \rightarrow (1)$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we ge[t]

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$\therefore \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$, which on integratio[n] gives

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \quad \rightarrow (2)$$

Hence from (1) & (2), the complete Solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$. (29)

Soln: Here the Subsidiary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \rightarrow ①$$

Each of these equations $=\frac{dx-dy}{x^2-y^2-(y-x)z}=\frac{dy-dz}{y^2-z^2-x(z-y)}$

$$\Rightarrow \frac{d(x-y)}{x^2-y^2-yz+xz}=\frac{d(y-z)}{y^2-z^2-xz+xy}$$

$$\Rightarrow \frac{d(x-y)}{x^2-y^2-xy+\underline{xy}-yz+\underline{xz}}=\frac{d(y-z)}{y^2-z^2-xz+\underline{zy}+\underline{zy}-zy}$$

$$\Rightarrow \frac{d(x-y)}{x(x+y+z)-y(y+x+z)}=\frac{d(y-z)}{y(y+x+z)-z(z+x+y)}$$

$$\Rightarrow \frac{d(x-y)}{(x-y)\cancel{(x+y+z)}}=\frac{d(y-z)}{\cancel{(x+y+z)}(y-z)}$$

$$\Rightarrow \frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$$

on Integration, we get

$$\log(x-y)=\log(y-z)+\log c.$$

or $\frac{x-y}{y-z}=c. \rightarrow ②$

using multipliers x, y, z we have each fraction $=\frac{xdx+ydy+zdz}{x^3-xyz+y^3-xyz+z^3-xyz}$

$a^3+b^3+c^3-3abc$
$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$$=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}.$$

$$=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} \rightarrow ③$$

$$\frac{dx+dy+dz}{(x^2+y^2+z^2-yz-\cancel{zx}-xy)}$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x+y+z} = dx+dy+dz$$

$$\Rightarrow \int (x\,dx+y\,dy+z\,dz) = \int (x+y+z)\,d(x+y+z)$$

$$\Rightarrow x^2+y^2+z^2 = (x+y+z)^2+2c'$$

or $xy+yz+zx+c'=0 \rightarrow (5)$

from (2) & (5), $\dfrac{x-y}{y-z} = f(xy+yz+zx)$.

Exercise

Solve the following Equation:

1) $xp+yq=3z$

2) $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$.

3) $p\cos(x+y)+q\sin(x+y)=z$

4) $(z-y)p+(x-z)q=y-x$.

5) $pyz+qzx=xy$

6) $(y+z)p-(z+x)q=x-y$.

* Write the given Equation in the form $f(x,y,z,p,q)=0$. This can be done by bringing all the terms of the given Equation to one side and denoting the entire expression by f'.

* Write down the charpit's auxiliary Equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

* Select two proper fractions so that the resulting integral may come out to be simplest relation involving at least one of p or q.

* Using the relation obtained in previous step, with the given Equation determine p & q. putting these values of p & q in $dz = pdx + qdy$ and on integration gives the complete Integral of the given Equation.

① Find the complete integral of

$$p(1+q^2) + (b-z)q = 0$$

Sol: The given Equation can be written as

$$p(1+q^2)+(b-z)q = 0 \rightarrow ①$$

$$f = p(1+q^2)+(b-z)q$$

$$\Rightarrow \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial p} = 1+q^2, \quad \frac{\partial f}{\partial q} = b-z+2pq$$

$\therefore$ The charpit's auxiliary Equation are

$$\frac{dp}{0+p(-q)} = \frac{dq}{0+q(-q)} = \frac{dz}{-p(1+q^2)-q(b-z+2pq)} = \frac{dx}{-(1+q^2)} = \frac{dy}{-(b-z+2pq)}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{-q^2} = \frac{dz}{-p-pq^2-qb+qz-2pq^2} = \frac{dx}{-(1+q^2)} = \frac{dy}{z-b-2pq}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{-q^2} = \frac{dz}{-3pq^2-p-qb+qz} = \frac{dx}{-(1+q^2)} = \frac{dy}{z-b-2pq}$$

consider the first two ratios

$$\frac{dp}{\cancel{+}p\cancel{q}} = \frac{dq}{\cancel{+}q^{\cancel{2}}}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

on integration, we get

$$\log p = \log q - \log c$$

$$\Rightarrow \log p + \log c = \log q$$

$$\Rightarrow pc = q \rightarrow ②$$

② in ① $\Rightarrow$ $p(1+p^2c^2) + (b-z)pc = 0$

$$\Rightarrow 1+p^2c^2 = -(b-z)c$$

$$\Rightarrow p^2c^2 = -(b-z)c-1$$

$$\Rightarrow p^2c^2 = (z-b)c-1$$

$$dz = \frac{\sqrt{c(z-b)-1}}{c}\,dx + \sqrt{c(z-b)-1}\,dy$$

$$dz = \frac{\sqrt{c(z-b)-1}}{c}\,dx + \sqrt{c(z-b)-1}\,dy$$

$$dz = \sqrt{c(z-b)-1}\left[\frac{dx}{c} + dy\right)$$

or $$\frac{c\,dz}{\sqrt{c(z-b)-1}} = dx + c\,dy$$

Integrating,

$2\sqrt{c(z-b)-1} = x + cy + a$ is the complete integral.

2. Find the complete Integral of $px + qy = pq$

Soln: The given equation can be written as

$$f = px + qy - pq = 0 \rightarrow (1)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = p, \ \frac{\partial f}{\partial y} = q, \ \frac{\partial f}{\partial z} = 0, \ \frac{\partial f}{\partial p} = x - q, \ \frac{\partial f}{\partial q} = y - p.$$

∴ The Charpit's auxiliary equations are

$$\frac{dp}{p + p\cdot 0} = \frac{dq}{q + q\cdot 0} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{q-x} = \frac{dy}{p-y}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px + pq - qy + pq} = \frac{dx}{q-x} = \frac{dy}{p-y}$$

$$\Rightarrow dp/p = \frac{dq}{q} = \frac{dz}{2pq - px - py} = \frac{dx}{q-x} = \frac{dy}{p-y}$$

Consider the first two ratios

$$\frac{dp}{p} = \frac{dq}{q}$$

on integration, $\log p = \log q + \log a$

$$\Rightarrow p = aq \rightarrow (2)$$

(2) in (1) $\Rightarrow$ $aqx + qy - aq^2 = 0$

$\Rightarrow$ $ax + y - aq = 0$

$\Rightarrow$ $aq = ax + y$

$\Rightarrow$ $p = ax + y$

$\Rightarrow$ $q = \dfrac{ax+y}{a}$

consider $dz = p\,dx + q\,dy$

$= (ax+y)dx + \left(\dfrac{ax+y}{a}\right)dy$

$\Rightarrow$ $a\,dz = (ax+y)(dy + a\,dx)$

on integration,

$az = \dfrac{(ax+y)^2}{2} + b_1$

$\int [f(x)]^n f'(x) = \dfrac{[f(x)]^{n+1}}{n+1}$

$\Rightarrow$ $2az = (ax+y)^2 + 2b_1$ is the complete Integral

$\Rightarrow$ $2az = (ax+y)^2 + b$

3. Find the complete integral of

$pxy + pq + qy = yz$ by charpit's method

Soln: The given Equation can be written as

$$f = pxy + pq + qy - yz = 0 \rightarrow (1)$$

$\Rightarrow$ $\dfrac{\partial f}{\partial x} = py$, $\dfrac{\partial f}{\partial y} = q + px - z$, $\dfrac{\partial f}{\partial z} = -y$, $\dfrac{\partial f}{\partial p} = xy + q$,

$\dfrac{\partial f}{\partial q} = p + y$

$\therefore$ The charpit's auxiliary Equations are

$$\frac{dp}{py - py0} = \frac{dq}{px + q - qy} = \frac{dz}{-p(xy+q) - q(p+y)} =$$

$(a+y)$ $\quad$ $(1+y)$

The first relation gives $dp = 0$

on integration, $p = a \rightarrow (2)$

Using in (1), $axy + aq + qy - yz = 0$

$\Rightarrow q(a+y) = yz - axy$

$\Rightarrow q = \dfrac{yz - axy}{a+y}$

$\therefore dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + \dfrac{(yz - axy)}{a+y}dy$

$\Rightarrow \dfrac{dz - a\,dx}{z - ax} = \dfrac{y}{a+y}dy \Rightarrow \dfrac{dz - a\,dx}{z - ax} = \left(1 - \dfrac{a}{a+y}\right)dy$

on integration,

$\Rightarrow \log(z - ax) = y - a\log(a+y) + b$ is the required complete integral

Exercise

1. Find the complete integral of $p^2+q^2 - 2px - 2qy + 2xy = 0$ by Charpits method
2. Find the complete integral of $p^2x + q^2y = z$ by Charpit's method.
3. $(p^2+q^2)y = qz$
4. $q = px + p^2$
5. $px + qy + pq = 0$

Exercise I[st] Assignment

1. Derive a pde (by eliminating the constants) from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow (1)$

Soln: diff (1) with respect to x & y partially

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \qquad 2\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{a^2} \qquad \frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$p = \frac{x}{a^2} \rightarrow (2) \qquad q = \frac{y}{b^2} \rightarrow (3)$$

multiply by x and y in (2) & (3) we get

$$px = \frac{x^2}{a^2} \rightarrow (4) \qquad qy = \frac{y^2}{b^2} \rightarrow (5)$$

adding (4) & (5) we get

$$xp + yq = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$\Rightarrow xp + yq = 2z$ is the required PDE

2. Form the pde (by eliminating the arbitrary function) from

a) $z = (x+y)\phi(x^2-y^2) \rightarrow (1)$

Soln:- Differentiating z partially with respect to x and y,

$$\frac{\partial z}{\partial x} = p = (x+y)\phi'(x^2-y^2).2x + \phi(x^2-y^2) \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = q = (x+y)\phi'(x^2-y^2)(-2y) + \phi(x^2-y^2) \rightarrow (3)$$

From (1), we have $\phi(x^2-y^2) = \frac{z}{x+y}$

Substitute in (2) and (3) we get

$$p = (x+y)\phi'(x^2-y^2).2x + \frac{z}{x+y}$$

$$\Rightarrow p - \frac{z}{x+y} = 2x(x+y)\phi'(x^2-y^2) \rightarrow (4)$$

$$q - \frac{z}{x+y} = -2y(x+y)\,\phi'(x^2-y^2) \rightarrow (5)$$

Divide (4) and (5) we get

$$\frac{p - \frac{z}{x+y}}{q - z/x+y} = \frac{\cancel{2}x\cancel{(x+y)}\cancel{\phi'(x^2-y^2)}}{-\cancel{2}y\cancel{(x+y)}\cancel{\phi'(x^2-y^2)}}$$

$$\frac{\frac{(x+y)p - z}{\cancel{x+y}}}{\frac{q(x+y)-z}{\cancel{x+y}}} = -\frac{x}{y}$$

$\Rightarrow y[(x+y)p - z] = -x[q(x+y) - z]$

$\Rightarrow pxy + y^2p - zy = -x^2q - qxy + zx$

$\Rightarrow p\cancel{xy} + \underline{y^2p} + \underline{x^2q} + qxy - zy - zx = 0$

$\Rightarrow x(py+qx) + y(py+qx) - z(x+y) = 0$

$\Rightarrow (x+y)(py+qx) - z(x+y) = 0$

$\Rightarrow py + qz = z$ is required soln

$\Rightarrow \left(\frac{\partial z}{\partial x}\right)y + \left(\frac{\partial z}{\partial y}\right)z = z$

b) $z = f(x+at) + g(x-at)$

Soln:- given $z = f(x+at) + g(x-at) \rightarrow (1)$

differentiating (1) twice partially wrt x & y we get

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at) \rightarrow (2)$$

$$\frac{\partial z}{\partial t} = a[f'(x+at) - g'(x-at)]$$

$$\frac{\partial^2 z}{\partial t^2} = a^2[f''(x+at) + g''(x-at)] \rightarrow (3)$$

Substituting (2) in (3) we get

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

c) $f(x^2+y^2, z-xy) = 0$

Soln:- Let $u = x^2+y^2$; $v = z - xy$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = 2y;$$

$$\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y = p - y;$$

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and $\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x = q - x$

Now $f(u,v) = 0 \rightarrow (1)$

Diff [1] partially wrt x,

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial f}{\partial u} \cdot 2x + \frac{\partial f}{\partial v}(p-y) = 0 \rightarrow (2)$$

Again Diff (1) partially wrt 'y', we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

$$\therefore \frac{\partial f}{\partial u} \cdot 2y + \frac{\partial f}{\partial v}(q-x) = 0 \rightarrow (3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from [2] and [3]

we get

$$\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix} = 0$$

$$\therefore x(q-x) - y(p-y) = 0$$

$$\Rightarrow xq - yp = x^2 - y^2$$

is the required partial differential Equation

Form the pde from $z = x f_1(x+t) + f_2(x+t)$

Sln:- Given $z = x f_1(x+t) + f_2(x+t) \rightarrow (1)$

differentiating (1) partially wrt x & t, we get

$$\frac{\partial z}{\partial x} = x f_1'(x+t) + f_1(x+t).1 + f_2'(x+t).1$$

$$\frac{\partial z}{\partial x} = x f_1'(x+t) + f_1(x+t) + f_2'(x+t) \rightarrow (2)$$

$$\frac{\partial z}{\partial t} = x f_1'(x+t) + f_2'(x+t) \rightarrow (3)$$

diff [2] partially w.r.t x and [3] w.r.t t, we get

$$\frac{\partial^2 z}{\partial x^2} = x f_1''(x+t) + f_1'(x+t) + f_1'(x+t) + f_2''(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = x f_1''(x+t) + 2 f_1'(x+t) + f_2''(x+t) \rightarrow (4)$$

$$\frac{\partial^2 z}{\partial t^2} = x f_1''(x+t) + f_2''(x+t) \rightarrow 5$$

differentiate 3 wrt x we get

$$\frac{\partial^2 z}{\partial x \partial t} = x f_1''(x+t) + f_1'(x+t) + f_2''(x+t) \rightarrow 6$$

4 + 5

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 2f_1'(x+t) + x f_1''(x+t) + f_2''(x+t) + x f_1''(x+t) + \cancel{f_1'(x+t)} + f_2''(x+t)$$

$$= 2\left[f_1'(x+t) + x f_1''(x+t) + f_2''(x+t)\right]$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 2\frac{\partial^2 z}{\partial x \partial t}}$$

Form the pde (by eliminating the arbitrary function) from

$F(xy+z^2, x+y+z) = 0$

Soln: Given $F(xy+z^2, x+y+z) = 0 \rightarrow 1$

Let $U = xy+z^2$ & $V = x+y+z \rightarrow 2$

Then 1 becomes $F(U,V) = 0 \rightarrow 3$

$$\frac{\partial F}{\partial U}\left(\frac{\partial U}{\partial x} + p\frac{\partial U}{\partial z}\right) + \frac{\partial F}{\partial V}\left(\frac{\partial V}{\partial x} + p\cdot\frac{\partial V}{\partial z}\right) = 0 \rightarrow 4$$

From 2, $\left.\begin{array}{l}\frac{\partial U}{\partial x} = y, \quad \frac{\partial U}{\partial y} = x, \quad \frac{\partial U}{\partial z} = 2z, \quad \frac{\partial V}{\partial x} = 1 \\ \frac{\partial V}{\partial y} = 1, \quad \frac{\partial V}{\partial z} = 1\end{array}\right\} \rightarrow 5$

From 4 & 5,

$$\frac{\partial F}{\partial U}\left(y + 2pz\right) + \frac{\partial F}{\partial V}\left(1+p\right) = 0$$

$$\Rightarrow \frac{\left(\frac{\partial F}{\partial U}\right)}{\left(\frac{\partial F}{\partial V}\right)} = -\frac{(1+p)}{(y+2pz)} \rightarrow 6$$

Again, differentiating 3 partially wrt y, we ge

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad (8)$$

$$\Rightarrow \frac{\partial F}{\partial u}\left(x+2qz\right)+\frac{\partial F}{\partial v}\left(1+q\right)=0 \text{ by } (5)$$

$$\Rightarrow \frac{\left(\frac{\partial F}{\partial u}\right)}{\left(\frac{\partial F}{\partial v}\right)}=\frac{-(1+q)}{(x+2qz)} \rightarrow (7)$$

Eliminating F from (6) & (7), we obtain

$$\frac{(1+p)}{(y+2pz)}=\left(\frac{1+q}{x+2qz}\right)$$

$$\Rightarrow (x+2qz)(1+p)=(1+q)(y+2pz)$$

$$\Rightarrow x+px+2qz+\cancel{2pqz}=y+2pz+qy+\cancel{2qpz}$$

$$\Rightarrow p(x-2z)+q(2z-y)=y-x.$$

$$\Rightarrow (x-2z)p+(2z-y)q=y-x.$$

form pde from the solutions

$z = f(x) + e^{y}.g(x) \to (1)$

differentiate (1) partially w.r.t y we get

$$\frac{\partial z}{\partial y} = e^{y}.g(x) \to (2)$$

differentiate (2) partially w.r.t y we get

$$\frac{\partial^2 z}{\partial y^2} = e^{y}.g(x) \to (3)$$

From (2) & (3) we have

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$$ This is the required PDE

ii) $z = \frac{1}{r}\left[F(r-at) + F(r+at)\right] \to (1)$

differentiate partially w.r.t 't' we get

$$\frac{\partial z}{\partial t} = \frac{1}{r}\left(F'(r-at)(-a) + F'(r+at).a\right]$$

$$\frac{\partial z}{\partial t} = \frac{a}{r}\left[F'(r+at) - F'(r-at)\right]$$

partially, differentiating again w.r.t 't' we get

$$\frac{\partial^2 z}{\partial t^2} = \frac{a}{r}\left[F''(r+at)a - F''(r-at)(-a)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{a^2}{r}\left[F''(r-at) + F''(r+at)\right] \to (2)$$

Differentiate [1] partially w.r.t 'r', we get

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r-at).1 + F'(r+at).1\right] + \left[F(r-at) + F(r+at)\right].\left(\frac{-1}{r^2}\right)$$

$$r^2\frac{\partial z}{\partial r} = -\left[F(r-at) + F(r+at)\right] + r\left[F'(r-at) + F'(r+at)\right] \to (3)$$

Differentiate [3] w.r.t 'r'

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial z}{\partial r}\right) = -\left[F'(r-at) + F'(r+at)\right] + r\left[F''(r-at) + F''(r+at)\right] + \left[F'(r-at) + F'(r+at)\right].$$

$\frac{x}{r})+c$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial z}{\partial r}\right) = r\left[F''(r-at)+F''(r+at)\right] \rightarrow [4]$$

From [2], $\left[F''(r-at)+F''(r+at)\right] = \frac{r}{a^2}\left(\frac{\partial^2 z}{\partial t^2}\right)$

Substituting in [4], we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial z}{\partial t r}\right) = r\cdot\frac{r}{a^2}\left(\frac{\partial^2 z}{\partial t^2}\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial z}{\partial r}\right) = \frac{r^2}{a^2}\left(\frac{\partial^2 z}{\partial t^2}\right)$$

is the required PDE.