Calculus of Variation

Let us consider a second order differential equation with constant co-efficient

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X.$$

where X is a function of 'x'

i.e., $aD^2y + bDy + cy = x$ where $\frac{dy}{dx} = Dy$

$$[aD^2 + bD + c]y = x$$

Then the Auxillary Eqn is given by

$am^2 + bm + c = 0$ Replace $D = m$

which has two roots, the roots may be real & different (or) real and repeated (Equal) (or) Imaginary.

Case(i): Suppose the roots are real and different.

Say m_1 & m_2

Complementary function (C.F) $= C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Case(ii): Suppose the roots are real and repeated (Equal)

say $m_1 = m_2 = m$

$$C.F = (C_1 + x C_2) e^{mx}$$

Case (iii): Suppose the roots are imaginary (or) Complex.

say $m = \alpha \pm i\beta$.

$C.F = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

The other part of a solution is obtained by considering the right hand side 'X' is called the particular integral and $y = C.F + P.I$ is the complete solution of the given equation.

Different cases to find the particular integral.

Case(i): when $X = e^{ax}$

$$P.I = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$$

i.e, Replace D by a

with $f(a) \neq 0$.

Case(ii): when $X = e^{ax}$ & $f(a) = 0$

$$P.I = \frac{1}{f(D)} e^{ax} = \frac{x\, e^{ax}}{f'(D)}$$ Provided $f'(a) \neq 0$

$$= \frac{x\, e^{ax}}{f'(a)}$$

Suppose $f'(a) = 0$

then. $$P.I = \frac{x^2}{f''(D)} e^{ax} = \frac{x^2}{f''(a)} e^{ax}$$ provided $f''(a) \neq 0$

Case(iii): when $X = \sin ax$ (or) $\cos ax$

$$P.I = \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$$

Replace $D^2 = -a^2$

Provided $f(-a^2) \neq 0$.

Ily

$$P.I = \frac{1}{f(D^2)} \cos ax = \frac{1}{f(-D^2)} \cos ax$$

Replace $D^2 = -a^2$

provided $f(-a^2) \neq 0$

Case (iv): when $X = \sin ax$ (or) $\cos ax$ & $f(-a^2) = 0$

$$P.I = \frac{1}{D^2+a^2} \sin ax$$

$$= \frac{-x}{2a} \cos ax$$

Ily

$$P.I = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

Note: Let us consider a second order differential equation with variable co-efficients.

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = x$$

put $x = e^t \Rightarrow t = \log x$. $\frac{d}{dt} = D$.

and $x^2 \frac{d^2y}{dx^2} \Rightarrow D(D-1)y$

& $x \frac{dy}{dx} = Dy$.

Calculus of Variations

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The problem of finding a function $y(x)$ for which the functional $I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$ is extremeum (maxim or minima). under the Condition $y(x_1) = y_1$ and $y(x_2) = y_2$ is called variational problem or fundamental problems of Calculus of variation

Variation of a function:

Consider a function $f(x, y, y')$, where 'x' is independent variable, $y = y(x)$ a dependent variable & $y' = y'(x) = \frac{dy}{dx}$

Suppose we give small increment to y and y' so that they become respectively, $y + h\alpha(x)$, $y' + h\alpha'(x)$.

where h = small parameter independent of x.

Now, By using Taylor's expansion for two variable fun.

we have

$$f(x, y + h\alpha(x), y' + h\alpha'(x)) = f(x, y, y') + \left(h\alpha \frac{\partial}{\partial y} + h\alpha' \frac{\partial}{\partial y'}\right) f + \frac{1}{2!}\left(h\alpha \frac{\partial}{\partial y} + h\alpha' \frac{\partial}{\partial y'}\right)^2 f + \cdots$$

Neglecting second and higher degree terms. we have

$$f(x, y + h\alpha(x), y' + h\alpha'(x)) - f(x, y, y') = h\alpha \frac{\partial f}{\partial y} + h\alpha' \frac{\partial f}{\partial y'}$$

$$\Rightarrow \boxed{\delta f = h\alpha \frac{\partial f}{\partial y} + h\alpha' \frac{\partial f}{\partial y'}} \longrightarrow (1)$$

Here δf is called the variation of f.

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ating

Applying $f = y$ in ①, we get

$$\delta y = h\alpha \frac{\partial y}{\partial y} + h\alpha' \frac{\partial y}{\partial y'} = h\alpha \cdot 1 + h\alpha' \cdot 0$$

$$\Rightarrow \boxed{\delta y = h\alpha} \longrightarrow ②$$

Again Applying $f = y'$ in ①, we get

$$\delta y' = h\alpha \frac{\partial y'}{\partial y} + h\alpha' \frac{\partial y'}{\partial y'} = h\alpha \cdot 0 + h\alpha' \cdot 1$$

$$\Rightarrow \boxed{\delta y' = h\alpha'} \longrightarrow ③$$

Using ② & ③ in ① we have

$$\boxed{\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y'}$$

geometrically $y(x)$ and $y(x) + h\alpha(x)$ represents two neighbouring curves. variation in f represents the change in f from curve to curve.

Note: $\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

Soln: let us $\delta\left(\frac{dy}{dx}\right) = \delta y' = h\alpha'$ (using ③)

$$= h \cdot \frac{d\alpha}{dx} = \frac{d}{dx}(h\alpha)$$

$$= \frac{d}{dx}(\delta y). \text{ (by using ②)}$$

Thus $\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

Functional

Let 'S' be a set of functions of a single variable defined over an interval (x_1, x_2). Then any function which assigns to each function in 'S' a unique real value is called a functional

i.e., functional is a mapping from functions to real numbers. $I(y) : S \longrightarrow R$

Now, consider a function of the form $f(x, y, y')$ where y' is the derivative of y w.r.t 'x'.

and $x \in (x_1, x_2)$.

The integral $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is a functional.

Ex: ① $\int_0^1 x + (y')^2\, dx$ ② $\int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$

Note:

① If $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ then $\delta \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_{x_1}^{x_2} \delta f(x, y, y')\, dx$

i.e., the variational of a functional associated with $f(x, y, y')$ is equal to the functional associated with variation of f.

Soln: Let $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is a functional

Since the value of I depends on y and y'.

we have $\delta I = \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial y'} \delta y'$

Since I is in the form of an integral $\frac{\partial I}{\partial y}$ and $\frac{\partial I}{\partial y'}$,

By using Leibnitz rule for differentiation under the integral sign. Hence we have

$$\delta I = \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial y} [f(x, y, y')] \, dx \right\} \delta y + \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial y'} [f(x, y, y')] \, dx \right\} \delta y'$$

i.e., $$\delta I = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \delta y \, dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \delta y' \, dx$$

$$\Rightarrow \delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right] dx = \int_{x_1}^{x_2} \delta f \, dx$$

$$\therefore \delta \int_{x_1}^{x_2} f(x, y, y') \, dx = \int_{x_1}^{x_2} \delta f(x, y, y') \, dx$$

2). δ and $\frac{d}{dx}$: δ and $\int$ are Commutative with each other.

Problems

1) If u & v are functions of x, y, y' prove the following

a) $\delta(cu) = c\,\delta u$, 'c' being a constant

b) $\delta(c) = 0$

c) $\delta(c_1 u \pm c_2 v) = c_1 \delta u \pm c_2 \delta v$

d) $\delta(uv) = u\,\delta v + v\,\delta u$

e) $\delta\left(\frac{u}{v}\right) = \frac{v\,\delta u - u\,\delta v}{v^2}, \quad (v \neq 0)$

Soln: Let $u(x, y, y')$ and $v(x, y, y')$ be any two funs.

and we have, for any function $f(x, y, y')$

$$\delta f = \frac{\partial f}{\partial y} \cdot \delta y + \frac{\partial f}{\partial y'} \cdot \delta y'$$

(a) $$\delta(cu) = \frac{\partial}{\partial y}(cu)\,\delta y + \frac{\partial}{\partial y'}(cu)\,\delta y'$$

$$= c\frac{\partial u}{\partial y}\,\delta y + c\frac{\partial u}{\partial y'}\,\delta y'$$

$$= c\left[\frac{\partial u}{\partial y}\,\delta y + \frac{\partial u}{\partial y'}\,\delta y'\right] = \underline{\underline{c \cdot \delta u}}$$

b) Taking $f(x, y, y') = c. \Rightarrow \frac{\partial f}{\partial y} = 0 = \frac{\partial f}{\partial y'}$

$\therefore$ $\underline{\underline{\delta c = 0}}$

c) $$\delta(c_1 u \pm c_2 v) = \frac{\partial}{\partial y}(c_1 u \pm c_2 v)\,\delta y + \frac{\partial}{\partial y'}(c_1 u \pm c_2 v)\,\delta y'$$

$$= \left(c_1\frac{\partial u}{\partial y} \pm c_2\frac{\partial v}{\partial y}\right)\delta y + \left(c_1\frac{\partial u}{\partial y'} \pm c_2\frac{\partial v}{\partial y'}\right)\delta y'$$

$$= \left(c_1 \frac{\partial u}{\partial y} \pm c_2 \frac{\partial v}{\partial y}\right) \delta y + \left(c_1 \frac{\partial u}{\partial y'} \pm c_2 \frac{\partial v}{\partial y'}\right) \delta y'$$

$$= c_1 \left(\frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial y'} \delta y'\right) \pm c_2 \left(\frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial y'} \delta y'\right)$$

$$= c_1 \delta u \pm c_2 \delta v$$

d) Let $f = uv$

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y'$$

$$= \left(u \frac{\partial v}{\partial y} + v \frac{\partial u}{\partial y}\right) \delta y + \left(u \frac{\partial v}{\partial y'} + v \frac{\partial u}{\partial y'}\right) \delta y'$$

$$= u \left(\frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial y'} \delta y'\right) + v \left(\frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial y'} \delta y'\right)$$

$$\delta f = u \, \delta v + v \, \delta u$$

Thus $\underline{\delta(uv) = u \delta v + v \delta u}$

e) Let $f = \frac{u}{v}$

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y'$$

$$\therefore \delta\left(\frac{u}{v}\right) = \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2} \cdot \delta y + \frac{v \frac{\partial u}{\partial y'} - u \frac{\partial v}{\partial y'}}{v^2} \delta y$$

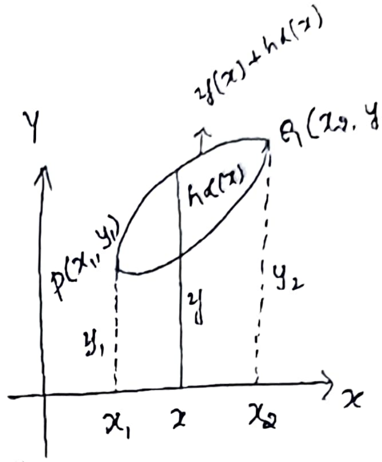
$$= \frac{1}{v^2} \left\{ v \left(\frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial y'} \delta y'\right) - u \left(\frac{\partial v}{\partial y} \cdot \delta y + \frac{\partial v}{\partial y'} \delta y'\right) \right\}$$

$$= \frac{1}{v^2} (v \, \delta u - u \, \delta v)$$

Thus $\boxed{\delta\left(\frac{u}{v}\right) = \frac{v \, \delta u - u \, \delta v}{v^2}}$

Euler's Equation.

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ to be an extremum is that $\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$



Proof:

Let I be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Also, let $y = y(x) + h\alpha(x) \longrightarrow (1)$

be the neighbouring curve joining these points so that we must have $\alpha(x_1) = 0$ at P and $\alpha(x_2) = 0$ at Q $\rightarrow (2)$

when $h = 0$ these two curves coincide thus making I is an ~~...~~ extremum.

i.e., $I = \int_{x_1}^{x_2} f(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x))\, dx$

is an extremum when $h = 0$.

This requires $\frac{dI}{dh} = 0$ when $h = 0$, treating I to be a function of h.

By using Leibnitz rule for differentiation under the integral sign we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h} f(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x))\, dx$$

(x, y, y') Applying chain rule for the partial derivative in R.H.S (11)

& we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h} \right] dx \longrightarrow (3)$$

But h is independent of x and hence $\frac{\partial x}{\partial h} = 0$

Let us consider eqn (1) and differentiate w.r.t 'x'

$$\therefore \; y' = y'(x) + h\,\alpha'(x) \longrightarrow (4)$$

Also, we have from (1), $\frac{\partial y}{\partial h} = \alpha(x)$ and from (4)

$$\frac{\partial y'}{\partial h} = \alpha'(x)$$

Using these results in (3) we have,

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x) \right] dx \longrightarrow (5)$$

Integrating the second term by parts

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \left[\frac{\partial f}{\partial y'}\alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \right\}$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \frac{\partial f}{\partial y'}\alpha(x_2) - \frac{\partial f}{\partial y'}\alpha(x_1) \right\} - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

~~But, from eqn (2)~~, using eqn (2) we have

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \alpha(x)\,dx$$

But we have already stated that $\frac{dI}{dh}$ must be zero when $h = 0$ for I to be an extremum, & $\alpha(x)$ is arbitrary

we have $\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$

1) Show that the Euler's Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ can be put in the following forms using

(a) $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0$

(b) $\frac{d}{dx}\left[f - y' \frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial x}$

further obtain the particular forms of Euler's Equation in the following cases.

(c) f is independent of x.

(d) f is independent of y.

(e) f is independent of x and y.

Soln: we have Euler's Equation. $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$

(a) It is evident that $\frac{\partial f}{\partial y'}$, is a function of x, y, y'

Since f is a function of x, y, y'

Let $z = \frac{\partial f}{\partial y'}$ so that $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{dz}{dx} \rightarrow ①$

Using the Expression for the total derivative we have

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} + \frac{\partial z}{\partial y'}\frac{dy'}{dx}$$

i.e., $$\frac{dz}{dx} = \frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial^2 f}{\partial y \partial y'} y' + \frac{\partial^2 f}{\partial y'^2} y'' \rightarrow ②$$

$\left(\because \text{By using } eq^n ① \text{ \& } \frac{dy}{dx} = y'\right)$

Con..

By using Eqn (1) we can write Euler's Equation in the form

$$\frac{\partial f}{\partial y} - \frac{dx}{dx} = 0$$

Thus $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0$ (∵ By using Eqn (2))

b) By using the Expression for the total derivative.

we have,

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \qquad \left(\because \frac{dy}{dx} = y'\right)$$

i.e., $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \rightarrow (3)$

Also $\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + y'' \frac{\partial f}{\partial y'} \rightarrow (4)$

Now, (3) - (4) will give

$$\frac{df}{dx} - \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \cancel{\frac{\partial f}{\partial y'} y''} - y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \cancel{y'' \frac{\partial f}{\partial y'}}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' - y' \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \frac{\partial f}{\partial x} + y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]$$

$$= \frac{\partial f}{\partial x} + y'(0)$$

∵ By Euler's Eqn

i.e., $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$= \frac{\partial f}{\partial x}$$

Thus $$\boxed{\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial x}}$$

ng these

$\frac{\partial f}{\partial y}$ -

(c) f is independent of x.

i.e., $\frac{\partial f}{\partial x} = 0$ and, the result (b) becomes

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

Thus $$\boxed{\left[f - y'\frac{\partial f}{\partial y'}\right] = k,}$$ where 'k' is a constant.

(d) given 'f' is independent of y

i.e., $\frac{\partial f}{\partial y} = 0$ and Hence Euler's equation in the form

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \boxed{\frac{\partial f}{\partial y'} = k.}$$

where 'k' is a constant.

(e) ~~Suppose~~ 'f'

given 'f' is independent of x and y.

i.e., $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

Differentiating these partially w.r.t y', we obtain

$$\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x}\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y}\right) = 0$$

(CP) $$\frac{\partial^2 f}{\partial x \partial y'} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial y'} = 0$$

(5)

sing these along with $\frac{\partial f}{\partial y}=0$ in the equation

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0 \quad \text{(By using (a))}$$

we obtain, $-\frac{\partial^2 f}{\partial y'^2} y'' = 0$

$\Rightarrow y'' = 0$ provided $\frac{\partial^2 f}{\partial y'^2} \neq 0$.

OR $\boxed{\frac{d^2 y}{dx^2} = 0}$

Procedure for Solving a Variational problem

Step 1: given $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$, Solving a variational problem is to find the function 'y' such that 'I' is Extremum. The Condition $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Step 2: We adopt, Euler's eqn for the function $f(x, y, y')$ which will result in an ordinary differential eqn in y.

Step 3: we solve for y using a suitable method which results in the required 'y' with ~~arbitrarily~~ arbitrary constants.

Problems

1) find the Extremal of the functional $\int_{x_1}^{x_2} (y' + x^2 y'^2)\, dx$ find the x_2

Sol^n: Let $f(x, y, y') = y' + x^2 y'^2$

Consider Euler's Equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \longrightarrow (1)$$

Hence $f = y' + x^2 y'^2$

(Differentiate partially with respect to y. i.e., treating x and y' are constants)

$$\Rightarrow \frac{\partial f}{\partial y} = 0$$

$$\& \quad \frac{\partial f}{\partial y'} = 1 + 2x^2 y'$$

Now Eq^n (1) becomes.

$$0 - \frac{d}{dx}(1 + 2x^2 y') = 0$$

$$\Rightarrow \frac{d}{dx}(1 + 2x^2 y') = 0$$

Integrating w.r.t 'x' we get

$$\Rightarrow \int \frac{d}{dx}(1 + 2x^2 y') = k_1 \Rightarrow 1 + 2x^2 y' = k_1$$

$$\Rightarrow 2x^2 y = k_1 - 1 \Rightarrow y' = \left(\frac{k_1 - 1}{2}\right)\frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{k_1 - 1}{2}\right)\frac{1}{x^2}$$

Again integrating w.r.t x

$$\Rightarrow \int dy = \left(\frac{k_1 - 1}{2}\right)\int \frac{1}{x^2}\, dx$$

where

$$\Rightarrow y = \left(\frac{k_1 - 1}{2}\right)\left(\frac{-1}{x}\right) + c_2 \Rightarrow \boxed{y = \frac{c_1}{x} + c_2}$$

'dx

find the function 'y' which makes the integral

$\int_{x_1}^{x_2} (1+xy'+xy'^2)\,dx$ is an Extremum

Soln: Let $f(x,y,y') = 1+xy'+xy'^2$

Euler's Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \longrightarrow (1)$

Hence $f = 1+xy'+xy'^2 \Rightarrow \frac{\partial f}{\partial y}=0$ & $\frac{\partial f}{\partial y'} = x+2xy'$

Now Eqn (1) becomes

$\Rightarrow 0 - \frac{d}{dx}(x+2xy')=0 \Rightarrow \frac{d}{dx}(x+2xy')=0$

Integrating w.r.t 'x' we get

$\int \frac{d}{dx}(x+2xy') = k_1$

$\Rightarrow x+2xy' = k_1 \Rightarrow y' = \frac{dy}{dx} = \frac{k_1-x}{2x}$

$\Rightarrow \frac{dy}{dx} = \frac{k_1}{2x} - \frac{1}{2}$

again integrating w.r.t 'x'

$\Rightarrow \int dy = \int\left(\frac{k_1}{2x}-\frac{1}{2}\right)dx \Rightarrow y = \frac{k_1}{2}\log x - \frac{1}{2}x + c_2$

$\Rightarrow y = c_1 \log x - \frac{1}{2}x + c_2$

3) find the Extremal of the functional $\int_{x_1}^{x_2} (y^2+y'^2+2ye^x)\,dx$

Soln: Given $f(x,y,y') = y^2+y'^2+2ye^x$

and Euler's Eqn $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \longrightarrow (1)$

Hence $f = y^2+y'^2+2ye^x \Rightarrow \frac{\partial f}{\partial y} = 2y+2e^x$ & $\frac{\partial f}{\partial y'} = 2y'$

then eqn ① becomes $(2y+2e^x) - \frac{d}{dx}(2y') = 0$

$\Rightarrow 2y + 2e^x - 2y'' = 0 \Rightarrow y'' - y = e^x$

$\Rightarrow (D^2-1)y = e^x$ where $D = \frac{d}{dx}$

By Auxillary eqn

$m^2 - 1 = 0$ ∵ Replace D by m

$\Rightarrow m = \pm 1$

Hence $C.F = y_c = C_1 e^x + C_2 e^{-x}$

(Complementory function)

$P.I = \frac{e^x}{D^2-1}$ Replace D by 1 $\Rightarrow P.I = \frac{e^x}{1-1}$

Dr=0

then $y_p = \frac{x e^x}{2D} = \frac{xe^x}{2}$ Again Replace D = 1

Thus $y = C_1 e^x + C_2 e^{-x} + \frac{x e^x}{2}$

②

4) find the Extremal of the functional $\int_1^2 (x^2 y'^2 + 2y^2 + 2xy)\,dx$

Until & unless specified at x=1 & x=2 Y becomes Zero.

i.e y(1)=0 & y(2)=0

Soln: given $f(x,y,y') = x^2y'^2 + 2y^2 + 2xy$

Euler's Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \longrightarrow ①$

Here $f = x^2y'^2 + 2y^2 + 2xy$

Also $\frac{\partial f}{\partial y} = 4y + 2x$ $\frac{\partial f}{\partial y'} = 2x^2y'$

Eqn ① becomes $4y + 2x - \frac{d}{dx}(2x^2y') = 0 \Rightarrow 4y + 2x - 2x^2y'' - 4xy' = 0$

$\Rightarrow \cancel{-4y+2x}$

$\Rightarrow -2x^2y'' - 4xy' + 4y = -2x$

$\Rightarrow x^2y'' + 2xy' - 2y = x \longrightarrow (2)$

By heart

Put $x = e^t$ $t = \log x$

Then $xy' = Dy$, $x^2y'' = D(D-1)y$. where $D = \frac{d}{dt}$

Now Eqn (2) becomes

$[D(D-1) + 2D - 2]y = e^t$

i.e., $[D^2 - D + 2D - 2]y = e^t$

By Auxillary Eqn $m^2 + m - 2 = 0$

$\Rightarrow (m-1)(m+2) = 0$

$\therefore m = 1, -2.$

Thus $y_c = C_1 e^t + C_2 e^{-2t}$ where $t = \log x$

$y_c = C_1 e^{\log x} + C_2 e^{\log x^{-2}} \Rightarrow \underline{y_c = C_1 x} + C_2 x^{-2}$

$y_p = \dfrac{e^t}{D^2 + D - 2}$ Replace $D = 1$ $\Rightarrow y_p = \dfrac{e^t}{0}$

But $y_p = P.I = \dfrac{te^t}{2D+1} = \dfrac{te^t}{2(1)+1} = \dfrac{te^t}{3} = \dfrac{x\log x}{3}$

Replace $D=1$

Thus $y = C.F + P.I = C_1 x + C_2 x^{-2} + \dfrac{x\log x}{3} \longrightarrow (3)$

But $y(1) = 0 \Rightarrow y = 0$ at $x = 1$

Eqn (3) becomes $0 = C_1 + C_2 + \dfrac{\log 1}{3} \Rightarrow C_1 = -C_2 \longrightarrow (4)$

& $y(2) = 0 \Rightarrow y = 0$ at $x = 2$

$$0 = 2c_1 + \frac{c_2}{4} + \frac{2\log 2}{3}$$

$$0 = 2c_1 - \frac{c_1}{4} + \frac{2\log 2}{3} \Rightarrow 0 = \frac{8c_1 - c_1}{4} + \frac{2\log 2}{3}$$

$$\Rightarrow \frac{-7c_1}{4} = \frac{2\log 2}{3} \Rightarrow c_1 = \frac{-8\log 2}{21} \rightarrow (5)$$

Substitute (4) & (5) in (3) we get

$$y = -\frac{8\log 2}{21}x + \frac{8\log 2}{21}x^{-2} + \frac{x\log x}{3}$$

find the curve on which the functional $\int_0^1 ((y')^2 + 12xy)\,dx$ with $y(0)=0$ and $y(1)=1$ can be determined.

Soln: Let $I = \int_0^1 ((y')^2 + 12xy)\,dx$

Let $f(x, y, y') = (y')^2 + 12xy$

Consider Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow (1)$

Here $\frac{\partial f}{\partial y} = 12x \qquad \frac{\partial f}{\partial y'} = 2y'$

Eqn (1) becomes $(12x) - \frac{d}{dx}(2y') = 0$

$\Rightarrow 12x - 2y'' = 0 \Rightarrow y'' = 6x$

i.e., $\frac{d^2y}{dx^2} = 6x$ integrating w.r.t 'x' we get

$\frac{dy}{dx} = 6\frac{x^2}{2} + C_1 \Rightarrow \frac{dy}{dx} = 3x^2 + C_1$

Again integrating w.r.t 'x' we get

$y = 3\frac{x^3}{3} + C_1x + C_2$

$\Rightarrow y = x^3 + C_1x + C_2 \rightarrow (2)$

Using the condition $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$

Eqn (2) becomes

$0 = 0 + 0 + C_2$

$\boxed{C_2 = 0}$

$1 = 1 + C_1 + C_2$

$\boxed{C_1 = 0}$

Thus eqn (2) becomes $y = x^3$ is the required curve

(7) Solve the variational problem $\delta \int_0^1 (x+y+y'^2)\,dx = 0$ under $y(0)=1$ and $y(1)=2$.

Soln: Let $f(x,y,y') = x+y+y'^2$

$\delta I = 0$ is Equivalent to the Euler's Equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow (1)$$

Here $f = x+y+y'^2$

$$\frac{\partial f}{\partial y} = 1 \quad \& \quad \frac{\partial f}{\partial y'} = 2y'$$

Eqn (1) becomes $1 - \frac{d}{dx}(2y') = 0 \Rightarrow 1 - 2y'' = 0$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}$$

Integrating w.r.t x we get $\frac{dy}{dx} = \frac{x}{2} + C_1$

and again integrating w.r.t x we get

$$y = \frac{x^2}{4} + C_1x + C_2 \rightarrow (2)$$

Using the Conditions, $y=1$ at $x=0$ and $y=2$ at $x=1$,

Eqn (2) becomes $1 = 0+0+C_2$ $\quad$ $2 = \frac{1}{4} + C_1 + 1$

$\boxed{C_2 = 1}$ $\quad$ $1 - \frac{1}{4} = C_1 \Rightarrow C_1 = \frac{3}{4}$

Thus eqn (2) becomes $y = \frac{x^2}{4} + \frac{3}{4}x + 1$

(or) $4y = x^2 + 3x + 4$

Solve the variational problem

$$\delta \int_0^{\pi/2} (y^2 - y'^2)\, dx = 0 ; \quad y(0) = 0 , \; y(\pi/2) = 2$$

Sol^n: Let $f(x, y, y') = y^2 - y'^2$

Consider the Euler's eq^n $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow ①$

Here $f = y^2 - y'^2$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial y'} = -2y'$$

Eq^n ① becomes $2y - \frac{d}{dx}(-2y') = 0$

$$\Rightarrow y + \frac{d}{dx}(y') = 0 \Rightarrow y + y'' = 0$$

$$\Rightarrow (D^2 + 1) y = 0$$

Auxillary eq^n $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y = c_1 \cos x + c_2 \sin x \rightarrow ②$$

Using the conditions $y = 0$ at $x = 0$ & $y = 2$ at $x = \pi/2$

Eq^n ② becomes

$0 = c_1 + 0$ & $2 = c_1(0) + c_2$

$\boxed{c_1 = 0}$ $\boxed{c_2 = 2}$

$$\underline{y = 2 \sin x}$$

** 9) Show that the functional $\int_{x_1}^{x_2} (y^2 + x^2 y')\, dx$ assumes extreme values on the straight line $y = x$

$\Rightarrow 1 = 4$

Sol^n: given $f = y^2 + x^2 y'$

$\rightarrow \quad \frac{\partial f}{\partial y} = 2y \qquad \frac{\partial f}{\partial y'} = x^2$

Consider Euler's Eq^n $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow (1)$

Eq^n(1) becomes $\quad 2y - \frac{d}{dx}(x^2) = 0$

$\Rightarrow 2y - 2x = 0$

$\Rightarrow 2y = 2x \Rightarrow \underline{y = x}$ is the St line

**

(II) find the Extremal functional $\int_0^1 (3x + \sqrt{y'})\,dx$, $y(1) = 5$, $y(2) = 7$

Sol^n: Note: when 'f' is independent of y

$\Rightarrow \frac{\partial f}{\partial y} = 0$

then the Euler's Eq^n becomes

$\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{Constant}$

Sol^n: given $f = 3x + (y')^2$

which is independent of $y \Rightarrow \frac{\partial f}{\partial y} = 0$

$\therefore \frac{\partial f}{\partial y'} = \text{Constant} \Rightarrow \frac{1}{2\sqrt{y'}} = c \Rightarrow 1 = 2c\sqrt{y'}$

Squaring on b.s

$\Rightarrow 1 = 4c^2 y' \Rightarrow y' = \frac{1}{4c^2}$

$\Rightarrow \frac{dy}{dx} = c_1 \Rightarrow \boxed{y = c_1 x + c_2} \rightarrow (1)$

given $y(1) = 5 \Rightarrow y = 5$ at $x = 1$

Eqn (1) becomes $5 = c_1 + c_2 \rightarrow (2)$

Also $y(2) = 7 \Rightarrow y = 7$ at $x = 2$

Eqn (1) becomes $7 = 2c_1 + c_2 \rightarrow (3)$

Solving (2) & (3)

$$\begin{aligned} c_1 + \cancel{c_2} &= 5 \\ 2c_1 + \cancel{c_2} &= 7 \\ \hline -c_1 &= -2 \Rightarrow \boxed{c_1 = 2} \end{aligned}$$

~~$c_1 = 5$~~ $c_1 + c_2 = 5$

$c_2 = 5 - c_1 = 5 - 2$

$\boxed{c_2 = 3}$

$\therefore \boxed{y = 2x + 3}$

(12) find the extremal functional $I = \int_1^2 \frac{\sqrt{1+y'^2}}{x} dx$, $y(1) = 0$, $y(2) = 1$.

(OR)

Show that the equation of the curve joining the points $(1,0)$ & $(2,1)$ for which $I = \int_1^2 \frac{\sqrt{1+y'^2}}{x} dx$ is an extremum is a circle

Soln:

Given $f = \dfrac{\sqrt{1+(y')^2}}{x}$

Euler's Equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0 \longrightarrow (1)$

Also $\dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{\partial y'} = \dfrac{1}{x}\dfrac{1}{2\sqrt{1+(y')^2}} \times 2y' = \dfrac{y'}{x\sqrt{1+(y')^2}}$

Now Eqn (1) becomes

$$0 - \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+(y')^2}}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+(y')^2}}\right) = 0$$

Integrating w.r.t 'x'

$$\Rightarrow \frac{y'}{x\sqrt{1+(y')^2}} = C_1 \Rightarrow y' = C_1\left(x\sqrt{1+(y')^2}\right)$$

Squaring on both side we get

$$\Rightarrow y'^2 = C_1^2 x^2 (1+(y')^2) \Rightarrow y'^2 - C_1^2 x^2 (y')^2 = C_1^2 x^2$$

$$\Rightarrow y'^2 = \frac{C_1^2 x^2}{1-C_1^2 x^2} \Rightarrow y' = \frac{\sqrt{C_1^2 x^2}}{\sqrt{1-C_1^2x^2}} = \frac{C_1 x}{\sqrt{1-C_1^2x^2}}$$

i.e., $\dfrac{dy}{dx} = \dfrac{C_1 x}{\sqrt{1-C_1^2x^2}}$

Again integrating w.r.t 'x'

$$\Rightarrow \int dy = \int \frac{C_1 x}{\sqrt{1-C_1^2x^2}}\,dx + C_2 \longrightarrow (2)$$

put $1 - C_1^2x^2 = z \Rightarrow dz = -2C_1^2 x\,dx$

$$\Rightarrow C_1 x\,dx = -\frac{dz}{2C_1}$$

Eqn (2) becomes $y = \displaystyle\int \frac{-dz}{2C_1\sqrt{z}} + C_2 \Rightarrow y = \frac{-1}{2C_1} 2\sqrt{z} + C_2$

$$y = \frac{-\sqrt{1-C_1^2x^2}}{C_1} + C_2 \longrightarrow (3)$$

Since the curve passes through (1,0) and (2,1) we have

i.e. $y=0$ at $x=1$

then Eqn (3) becomes

$$0 = \frac{-\sqrt{1-C_1^2}}{C_1} + C_2 \Rightarrow \frac{\sqrt{1-C_1^2}}{C_1} = C_2$$

Squaring on b.s

$$\Rightarrow \frac{1-C_1^2}{C_1^2} = C_2^2 \Rightarrow \frac{1}{C_1^2} - 1 = C_2^2 \rightarrow (4)$$

Also $y=1$ at $x=2$

then Eqn (4) becomes

$$1 = \frac{-\sqrt{1-C_1^2(4)}}{C_1} + C_2 \Rightarrow (1-C_2) = \frac{-\sqrt{1-C_1^2(4)}}{C_1}$$

Squaring on b.s

$$\Rightarrow (1-C_2)^2 = \frac{1-4C_1^2}{C_1^2} \Rightarrow (1-C_2)^2 = \frac{1}{C_1^2} - 4$$

$$\Rightarrow (1-C_2)^2 = C_2^2 + 1 - 4 \qquad \left(\because \text{ By using Eqn (4)} \quad \frac{1}{C_1^2} = C_2^2 + 1\right)$$

$$\Rightarrow (1-C_2)^2 = C_2^2 - 3$$

$$\Rightarrow 1 + \cancel{C_2^2} - 2C_2 = \cancel{C_2^2} - 3 \Rightarrow -2C_2 = -3-1$$

$$\Rightarrow -2C_2 = -4 \Rightarrow \boxed{C_2 = +2}$$

Hence, Eqn (4) becomes $\frac{1}{C_1^2} = C_2^2 + 1 \Rightarrow \frac{1}{C_1^2} = 4+1$

$$\Rightarrow \frac{1}{5} = C_1^2 \Rightarrow \boxed{C_1 = \frac{1}{\sqrt{5}}}$$

Substituting these values in ③ we get

$$y = \frac{-\sqrt{1-(1/5)x^2}}{1/\sqrt{5}} + 2 \Rightarrow y = \frac{\frac{-\sqrt{5-x^2}}{\sqrt{5}}}{1/\sqrt{5}} + 2$$

$$\Rightarrow y = -\sqrt{5-x^2} + 2 \Rightarrow (y-2) = -\sqrt{5-x^2}$$

Squaring on b.s

$$\Rightarrow (y-2)^2 = (5-x^2)$$

$$\Rightarrow x^2 + (y-2)^2 = 5$$

which is a circle with Centre (0, 2) & radius $\sqrt{}$

1) Show that the Extremal of $I = \int_{x_1}^{x_2} \sqrt{y(1+(y')^2)}\, dx$ is a Parabola

Soln: given $f = \sqrt{y(1+(y')^2)}$

which is independent of 'x' and Corresponding Euler's Eqn is $f - y'\frac{\partial f}{\partial y'} = A. \longrightarrow ①$

But $\frac{\partial f}{\partial y'} = \frac{1 \times y \cancel{2} y'}{\cancel{2}\sqrt{y(1+(y')^2)}} = \frac{yy'}{\sqrt{y(1+(y')^2)}}$

Eqn ① becomes

$$\sqrt{y(1+(y')^2)} - y'\frac{yy'}{\sqrt{y(1+(y')^2)}} = A$$

$$y(1+(y')^2) - y(y')^2 = A\sqrt{y(1+(y')^2)}$$

$$\Rightarrow y + y(y')^2 - y(y')^2 = A\sqrt{y(1+(y')^2)}$$

$$\Rightarrow y = A\sqrt{y(1+(y')^2)}$$

Squaring on b.s, we get

$$\Rightarrow y^2 = A^2(y(1+(y')^2)) \Rightarrow y = A^2(1+(y')^2)$$

$$\Rightarrow y - A^2 = A^2(y')^2 \Rightarrow (y')^2 = \frac{y-A^2}{A^2}$$

$$\Rightarrow y' = \frac{\sqrt{y-A^2}}{A} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{y-A^2}}{A}$$

Integrating on b.s.

$$\Rightarrow \int dy = \frac{1}{A}\int\sqrt{y-A^2}\,dx \quad \cancel{\Rightarrow \int\frac{dy}{\sqrt{y-A^2}} = \frac{1}{A}x + B}$$

$$\Rightarrow A\int\frac{dy}{\sqrt{y-A^2}} = \int dx$$

$$\Rightarrow A\cdot 2\sqrt{y-A^2} = x - B$$

Squaring on b.s

$$\Rightarrow A^2\cdot 4(y-A^2) = (x-B)^2$$

$$\Rightarrow (x-B)^2 = 4A^2(y-A^2)$$

$$\Rightarrow (x-B)^2 = 4a(y-A^2) \quad \text{where } A^2 = a$$

which represents a parabola

2) find the Equation of the curve joining the points $(0,1)$ and $(2,3)$ for which $I = \int_0^2 \frac{\sqrt{1+(y')^2}}{y}\,dx$ is a minimum

Soln: given $f = \frac{\sqrt{1+(y')^2}}{y} \longrightarrow ①$

which is independent of 'x', and the corresponding

Euler's eqn is $f - y'\frac{\partial f}{\partial y'} = A \longrightarrow ②$

By using eqn ① $\frac{\partial f}{\partial y'} = \frac{1}{y}\frac{1 \times 2y'}{2\sqrt{1+(y')^2}} = \frac{y'}{y\sqrt{1+(y')^2}}$

then eqn ② becomes

$$\frac{\sqrt{1+(y')^2}}{y} - \frac{(y')^2}{y\sqrt{1+(y')^2}} = A$$

$\times^{ly}$ $\sqrt{1+(y')^2}$ on both side

$$\frac{1+(y')^2}{y} - \frac{(y')^2}{y} = A\sqrt{1+(y')^2}$$

$\Rightarrow 1 + \cancel{(y')^2} - \cancel{(y')^2} = Ay\sqrt{1+(y')^2} \Rightarrow 1 = Ay\sqrt{1+(y')^2}$

Squaring on both side

$\Rightarrow 1 = A^2y^2(1+(y')^2)$

$\Rightarrow 1 = A^2y^2 + A^2y^2(y')^2 \Rightarrow (y')^2 = \frac{1-A^2y^2}{A^2y^2}$

$\Rightarrow y' = \frac{\sqrt{1-A^2y^2}}{Ay} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-A^2y^2}}{Ay}$

$$\Rightarrow \int \frac{Ay}{\sqrt{1-A^2y^2}}\,dy = \int dx$$

put $1-A^2y^2 = t \Rightarrow dt = -A^2 2y\,dy$

$$\Rightarrow Ay\,dy = \frac{-dt}{2A}$$

$$\int \frac{-dt}{2A\sqrt{t}} = \int dx \Rightarrow \frac{-1}{2A}\,2\sqrt{t} = x-B$$

$\left| \int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\right.$

$$\Rightarrow -\frac{\left(\sqrt{1-A^2y^2}\right)}{A} = (x-B)$$

$$\Rightarrow -\left(\sqrt{1-A^2y^2}\right) = A(x-B)$$

Squaring on b.s

$$\Rightarrow 1-A^2y^2 = A^2(x-B)^2 \Rightarrow \frac{1}{A^2}-y^2 = (x-B)^2$$

$$\boxed{(x-B)^2+y^2 = R^2} \rightarrow (3) \quad \text{where } R^2 = 1/A^2$$

which represents a circle with Centre $(B, 0)$ & radius R.

Since $y=1$ at $x=0$, then Eqn (3) becomes

$$(0-B)^2+1 = \frac{1}{A^2}$$

$$\Rightarrow B^2 = \frac{1}{A^2}-1 \rightarrow (4)$$

IIIly $y=3$ at $x=2$ in Eqn (3) becomes

$$(2-B)^2+9 = \frac{1}{A^2}$$

$$\Rightarrow (2-B)^2+9 = B^2+1$$

$\left(\text{By using Eqn (4) } \frac{1}{A^2} = B^2+1\right)$

$$\Rightarrow 4+\cancel{B^2}-4B+9 = \cancel{B^2}+1$$

$$\Rightarrow -4B+12 = 0$$

$$\Rightarrow -4B = -12$$

$$\boxed{B=3} \rightarrow (4)$$

Eqn ④ becomes

$$B^2 = \frac{1}{A^2} - 1 \Rightarrow \frac{1}{A^2} = 9 + 1$$

$$\Rightarrow \frac{1}{10} = A^2 \rightarrow ⑤$$

Substitute eqn ④ & ⑤ in ③ we get

$$(x-3)^2 + y^2 = 10$$

③ Find the Extremal of the functional $I = \int_0^{\pi/2} (y^2 - (y')^2 - 2y\sin x)\,dx$

$$I = \int_0^{\pi/2} \left(y^2 - (y')^2 - 2y\sin x\right) dx$$ under the end conditions

$$y(0) = y(\pi/2) = 0.$$

Ans: $\boxed{y = \frac{-x}{2}\cos x}$

Hints : $P.I = \frac{1}{D^2 + a^2}\sin ax = \frac{-x}{2a}\cos ax.$

Geodesics :

Geodesics on a Surface is a curve along which the distance between two points on the Surface is a minimum.

Problems :

1) Prove that the shortest distance between two points in a plane is along the straight line joining them (or) Prove that the geodesics on a plane are straight lines.

Proof: Let $y = y(x)$ be a curve joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in a plane.

We know that the arc length between P and Q is given by

$$S = \int_{x_1}^{x_2} \frac{ds}{dx}\,dx = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

i.e., $$S = I = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx.$$

Now Geodesic on a plane is a curve $y = f(x)$, for which 'S' is a minimum.

i.e., 'S' should satisfy Euler's formula

Let $f(x, y, y') = f = \sqrt{1+y'^2}$

Also, we have $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \longrightarrow (1)$

then $\frac{\partial f}{\partial y} = 0$ & $\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+y'^2}} \times 2y' = \frac{y'}{\sqrt{1+y'^2}}$

Now eqn ① becomes.

$$-\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \Rightarrow \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

Integrating on both sides we get.

$$\int \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \Rightarrow \frac{y'}{\sqrt{1+y'^2}} = A$$

$$\Rightarrow y' = A\sqrt{1+y'^2} \Rightarrow y'^2 = A^2(1+y'^2)$$

$$\Rightarrow y'^2 - A^2 y'^2 = A^2 \Rightarrow y'^2(1-A^2) = A^2$$

$$\Rightarrow y'^2 = \frac{A^2}{1-A^2} \Rightarrow y' = \frac{A}{\sqrt{1-A^2}}$$

Again integrating on b.s

$$\Rightarrow \int \frac{dy}{dx} = \int \frac{A}{\sqrt{1-A^2}}$$

$$\Rightarrow \int dy = \frac{A}{\sqrt{1-A^2}} \int dx \qquad \text{where } m = \frac{A}{\sqrt{1-A^2}}$$

$$\Rightarrow y = mx + c$$

Hence the geodesic on a plane is a Straight line

Find the geodesics on a surface given that the arc length on the surface is $S = \int_{x_1}^{x_2} \sqrt{x(1+(y')^2)}\, dx$.

Soln: given $f = \sqrt{x(1+(y')^2)}$

which is independent of y.

$\therefore$ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \xrightarrow{\text{reduces to}} ①$

Also $\frac{\partial f}{\partial y} = 0$ & $\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{x(1+(y')^2)}} \times 2y'x = \frac{xy'}{\sqrt{x(1+(y')^2)}}$

Now Eqn ① becomes

$$-\frac{d}{dx}\left[\frac{xy'}{\sqrt{x(1+(y')^2)}}\right] = 0 \Rightarrow \frac{d}{dx}\left[\frac{xy'}{\sqrt{x(1+(y')^2)}}\right] = 0$$

Integrating w.r.t x

$$\Rightarrow \int \frac{d}{dx}\left[\frac{xy'}{\sqrt{x(1+(y')^2)}}\right] = c \Rightarrow \frac{xy'}{\sqrt{x(1+(y')^2)}} = c$$

$$\Rightarrow \frac{\cancel{\sqrt{x}}\sqrt{x}\, y'}{\cancel{\sqrt{x}}\sqrt{1+(y')^2}} = c \Rightarrow \frac{\sqrt{x}\, y'}{\sqrt{1+(y')^2}} = c$$

Squaring on b. s.

$$\Rightarrow \frac{x(y')^2}{1+(y')^2} = c^2 \Rightarrow x(y')^2 = c^2(1+(y')^2)$$

$$\Rightarrow x(y')^2 - c^2(y')^2 = c^2$$

$$\Rightarrow (x-c^2)(y')^2 = c^2$$

$$\Rightarrow (y')^2 = \frac{c^2}{(x-c^2)} \Rightarrow y' = \frac{c}{\sqrt{x-c^2}}$$

i.e., $\frac{dy}{dx} = \frac{c}{\sqrt{x-c^2}}$

Again integrating on b.s

$$\Rightarrow \int dy = c\int \frac{dx}{\sqrt{x-c^2}} \Rightarrow y = c\,2\sqrt{x-c^2} + c_1$$

$$\Rightarrow (y-c_1) = 2c\sqrt{x-c^2}$$

Squaring on b.s we get

$$\Rightarrow (y-c_1)^2 = 4c^2(x-c^2)$$ is the required

geodesic, which is a parabola

3) Geodesic on a Sphere

(OR)

"Show that the geodesic on a sphere of radius 'a' are its great circles"

Proof: The element of arc length in Spherical Co-ordinate System (r, θ, ϕ) is given by.

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

Since $r = a \Rightarrow dr = 0$

$$\therefore (ds)^2 = a^2(d\theta)^2 + a^2\sin^2\theta\,(d\phi)^2$$

$$\Rightarrow (ds)^2 = a^2\left[(d\theta)^2 + \sin^2\theta\,(d\phi)^2\right]$$

$$(ds)^2 = a^2(d\theta)^2\left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]$$

$$\Rightarrow ds = a\sqrt{1+\sin^2\theta(\phi')^2}\,(d\theta)$$ where $\phi' = \frac{d\phi}{d\theta}$

$$S = a\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\,(\phi')^2}\,d\theta$$

Now 'S' is minimum for the above functional

Hence $f = \sqrt{1+\sin^2\theta\,(\phi')^2}$

i.e., $f = \sqrt{1+(y')^2\sin^2 x}$ where $x=\theta$ & $\phi = y$

which is independent of 'y' $\Rightarrow \frac{\partial f}{\partial y} = 0$,

Then the Euler's Eq^n becomes $\frac{\partial f}{\partial y'} = \text{constant.} \longrightarrow (1)$

Also $$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+(y')^2\sin^2 x}} \times 2y'\sin^2 x$$

Eq^n (1) becomes $$\frac{y'\sin^2 x}{\sqrt{1+(y')^2\sin^2 x}} = b$$

$$\Rightarrow y'\sin^2 x = b\sqrt{1+(y')^2\sin^2 x}$$

Squaring on both sides we get

$$(y')^2\sin^4 x = b^2\left[1+(y')^2\sin^2 x\right]$$

$$\Rightarrow (y')^2\left[\sin^4 x - b^2\sin^2 x\right] = b^2$$

$$\Rightarrow (y')^2 = \frac{b^2}{\sin^4 x - b^2\sin^2 x} \Rightarrow y' = \frac{b}{\sqrt{\sin^4 x - b^2\sin^2 x}}$$

$$\Rightarrow y' = \frac{b}{\sin^2 x\sqrt{1-b^2\operatorname{cosec}^2 x}} = \frac{b\operatorname{cosec}^2 x}{\sqrt{1-b^2\operatorname{cosec}^2 x}}$$

$$\Rightarrow y' = \frac{b\,\text{cosec}^2 x}{\sqrt{1-b^2-b^2\cot^2 x}} \Rightarrow y' = \frac{b\,\text{cosec}^2 x}{\sqrt{(\sqrt{1-b^2})^2-(b\cot x)^2}}$$

3 Now

Integrating w.r.t 'x' we get

$$y = \int \frac{b\,\text{cosec}^2 x\,dx}{\sqrt{(\sqrt{1-b^2})^2-(b\cot x)^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1}x + c}$$

put $b\cot x = t$

$-b\,\text{cosec}^2 x\,dx = dt$

i.e., $$y = \int \frac{-dt}{\sqrt{(\sqrt{1-b^2})^2 - t^2}} \Rightarrow y = \cos^{-1}\left(\frac{t}{\sqrt{1-b^2}}\right) + c$$

$$\Rightarrow (y-c) = \cos^{-1}\left(\frac{t}{\sqrt{1-b^2}}\right) \Rightarrow \cos(y-c) = \frac{t}{\sqrt{1-b^2}}$$

$$\Rightarrow \cos(y-c) = \frac{b\cot x}{\sqrt{1-b^2}}$$

$y = \phi$

$x = \theta$

where $k = \frac{b}{\sqrt{1-b^2}}$

$$\Rightarrow k\cot\theta = \cos(\phi - c)$$

$$\Rightarrow k\cot\theta = \cos\phi\cos c + \sin\phi\sin c$$

$$\Rightarrow \cot\theta = \cos\phi\frac{\cos c}{k} + \sin\phi\frac{\sin c}{k}.$$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} = A\cos\phi + B\sin\phi$$

$$\Rightarrow \cos\theta = A\sin\theta\cos\phi + B\sin\theta\sin\phi$$

$$\Rightarrow a\cos\theta = A(a\sin\theta\cos\phi) + B(a\sin\theta\sin\phi) \rightarrow (2)$$

In Spherical Co-ordinates we have

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Here $r = a$.

Now eqn ② becomes

$$z = Ax + By$$

which is a plane passing through the origin and the intersection of $z = Ax + By$ and the sphere $x^2+y^2+z^2=a^2$ is the great circle.

Thus the geodesics on a sphere are great circles.

3) find the geodesic on Right circular cone.

Hints : Consider the right circular cone with vertex at the origin and along z-axes. the semi vertical angle 'α' whose equation in Spherical Co-ordinates is $\theta = \alpha$.

The element of arc length ds in spherical Co-ordinate (r, θ, ϕ) is given by

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta (d\phi)^2$$

Since $\theta = \alpha \implies d\theta = 0$

$$\therefore S = \int_{\phi_1}^{\phi_2} \sqrt{(r')^2 + r^2 \sin^2\alpha}\; d\phi$$

Ans : $\frac{c}{y} = \sin\alpha \cos\left[(\sin\alpha)x + b\right]$

Q. Show that ~~Catem~~ Catenary is the curve which when rotated about a line generates a surface of minimum area.

Solution: Let the curve from a point A to B revolves around x - axis

Then the area of the surface is given by

$$S = \int_A^B 2\pi y\, ds = \int_A^B 2\pi y \frac{ds}{dx}\, dx$$

$$\Rightarrow S = \int_A^B 2\pi y \sqrt{1+(y')^2}\, dx$$

Since 2π is a constant, Here $f = y\sqrt{1+(y')^2}$

which is independent of x

$\therefore$ the Euler's eqn becomes

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

$$\Rightarrow y\sqrt{1+(y')^2} - y'\frac{y \cdot \cancel{2} y'}{\cancel{2}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{y(1+(y')^2) - y(y')^2}{\sqrt{1+(y')^2}} = c \Rightarrow y + \cancel{y(y')^2} - \cancel{y(y')^2} = c\sqrt{1+(y')^2}$$

$$\Rightarrow y = c\sqrt{1+(y')^2}$$

Squaring on b.s.

$$\Rightarrow y^2 = c^2(1+(y')^2) \Rightarrow y^2 - c^2 = c^2(y')^2$$

When

$$y' = \frac{\sqrt{y^2-c^2}}{c} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c} \Rightarrow \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

Integrating w.r.t x:

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\int dx \qquad \left(\text{where } c_1 = \frac{1}{c}\right)$$

$$\Rightarrow \cosh^{-1}\left(\frac{y}{c}\right) = c_1 x + b$$

$$\Rightarrow \frac{y}{c} = \cosh(c_1 x + b)$$

$$\Rightarrow \boxed{y = c\cosh(c_1 x + b)}$$

which is a Catenary

OR

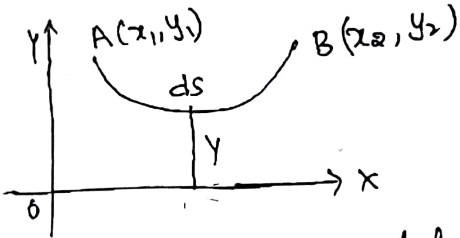
$$\cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + b$$

$$\frac{y}{c} = \cosh\left(\frac{x}{c}+b\right)$$

(or) $\boxed{y = c\cosh\left(\frac{x}{c}+b\right)}$

Hanging Cable problem:

1) If a cable hangs freely under gravity from the fixed points, then show that the shape of the curve is a catenary.



Soln:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two fixed points from which the cable is suspended. If ds is the arc element of the cable and ρ is the density, then the mass of the element $\rho\,ds$ and the potential energy is given by $\rho\,ds\,g\,y$ (along x-axis)

(g is acceleration due to gravity).

The total potential energy V is given by $V = \int_A^B \rho\,ds\,g\,y$

$$\Rightarrow V = \int_A^B \rho g\, y\, ds = \rho g \int_A^B y \frac{ds}{dx}\, dx$$

i.e., $V = \rho g \int_A^B y \cdot \sqrt{1+(y')^2}\, dx$

Since ρg is a constant, Here $f = y\sqrt{1+(y')^2}$

which is independent of 'x'.

then the Euler's Eqn becomes $f - y' \frac{\partial f}{\partial y'} = \text{constant}$.

which is independent of 'x'

$\therefore$ The Euler's Eqn becomes

$$f - y' \frac{\partial f}{\partial y'} = \text{Constant}$$

$$\Rightarrow y\left(\sqrt{1+(y')^2}\right) - y' \frac{y \cdot \cancel{2} y'}{\cancel{2}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{y(1+(y')^2) - y(y')^2}{\sqrt{1+(y')^2}} = c$$

$$\Rightarrow y = c\sqrt{1+(y')^2}$$

Squaring on b.s

$$\Rightarrow y^2 = c^2(1+(y')^2) \Rightarrow y^2 - c^2 = c^2(y')^2 \Rightarrow (y')^2 = \frac{y^2-c^2}{c^2}$$

$$\Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\Rightarrow \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\, dx$$

integrating w.r.t 'x'

$$\int \frac{dy}{\sqrt{y^2 - c^2}} = \frac{1}{c} \int dx$$

$$\Rightarrow \cosh^{-1}\left(\frac{y}{c}\right) = \frac{x}{c} + b$$

$$\Rightarrow \frac{y}{c} = \cosh\left(\frac{x}{c} + b\right)$$

$$\Rightarrow \boxed{y = c \cosh\left(\frac{x}{c} + b\right)}$$

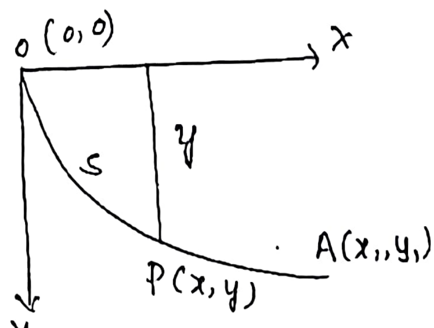
which is Catenary.

(Brachistochrone problem)

Curve of quickest descent :

To find the equation of the plane curve on which a particle in the absence of the friction, will slide from one point to the another point in shortest time under the action of gravity.
(The curve will be a cycloid)

Proof :



Let the particle start from the point 'O' (initially at rest) and slide along the curve OA.
Let $P(x, y)$ be the position of the particle at a time 't and the arc $OP = s$.

The particle is moving under gravity only.

Thus we have gain in kinetic energy = loss in potential energy at P

$\Rightarrow$ K.E at P - K.E at O = P.E at P

$$\Rightarrow \frac{1}{2}\cancel{m}\left(\frac{ds}{dt}\right)^2 - 0 = \cancel{m}gy \Rightarrow \left(\frac{ds}{dt}\right)^2 = 2gy$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{2gy}$$

Time taken by the particle to move from O to A

is $T = \int_0^T dt = \int_0^T \frac{dt}{ds}\, ds = \int_0^T \frac{1}{\sqrt{2gy}}\, ds$

$$\Rightarrow T = \int_0^{x_1} \frac{1}{\sqrt{2gy}} \frac{ds}{dx}\, dx$$

But $\frac{dt}{ds} = \frac{1}{\sqrt{2gy}}$

$$\Rightarrow T = \int_0^{x_1} \frac{1}{\sqrt{2gy}} \sqrt{1+(y')^2}\, dx = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\, dx$$

We need to find $y(x)$ such that T is minimum.

Here $f(x, y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = f$

which is independent of x

Hence we can take Euler's equation in the form

$$f - y'\frac{\partial f}{\partial y'} = C$$

in potentia

at P

$$\Rightarrow \frac{\sqrt{1+(y')^2}}{\sqrt{y}} - y' \frac{\cancel{2}y'}{\sqrt{y}\cdot\cancel{2}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{(1+\cancel{(y')^2}) - \cancel{(y')^2}}{\sqrt{y}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+(y')^2}} = c \Rightarrow 1 = c\sqrt{y}\sqrt{1+(y')^2}$$

Squaring on b.s

A

$$\Rightarrow 1 = c^2(y)(1+(y')^2)$$

$$\Rightarrow \frac{1}{c^2} = y + y(y')^2$$

$$\Rightarrow a = y + y(y')^2 \qquad \frac{1}{c^2} = a$$

$$\Rightarrow a - y = y(y')^2$$

$$\Rightarrow (y')^2 = \frac{a-y}{y} \Rightarrow y' = \frac{\sqrt{a-y}}{\sqrt{y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{a-y}}{\sqrt{y}}$$

$$\Rightarrow \frac{\sqrt{y}}{\sqrt{a-y}}\,dy = dx$$

Again integrating on b.s

$$\Rightarrow x = \int \frac{\sqrt{y}}{\sqrt{a-y}}\,dy$$

put $y = a\sin^2\theta \Rightarrow dy = 2a\sin\theta\cos\theta\,d\theta$

$$\therefore \quad x = \int \frac{\sqrt{a \sin^2\theta}}{\sqrt{a - a\sin^2\theta}} \; a\, 2 \sin\theta \cos\theta \, d\theta$$

$$x = \int \frac{\sqrt{a}\, 2a \cdot \sin\theta \sin\theta \cos\theta}{\sqrt{a(1-\sin^2\theta)}} \, d\theta$$

$1 - \sin^2\theta = \cos^2\theta$

$$x = 2a \int \frac{\cancel{\sqrt{a}} \cdot \sin^2\theta \cancel{\cos\theta}}{\cancel{\sqrt{a}} \cancel{\cos\theta}} \, d\theta$$

$1 - \cos 2\theta = \frac{\sin^2\theta}{2}$

$$x = 2a \int \sin^2\theta \, d\theta$$

$$x = 2a \int \frac{(1-\cos 2\theta)}{2} \, d\theta = a \int (1-\cos 2\theta) \, d\theta$$

$$x = a \left[\theta - \frac{1}{2} \sin 2\theta\right] + c$$

$$x = \frac{a}{2} \left[2\theta - \sin 2\theta\right] + c$$

when $\theta = 0$, $x = 0 \Rightarrow c = 0$

$$\therefore \quad x = \frac{a}{2}(2\theta - \sin 2\theta) \quad \& \quad y = a\sin^2\theta = \frac{a}{2}(1 - \cos 2\theta)$$

Let $\frac{a}{2} = b$ & $2\theta = \phi$

i.e., $x = b(\phi - \sin\phi)$ and $y = b(1 - \cos\phi)$

$\Rightarrow x = b(\phi - \sin\phi)$, $y = b(1 - \cos\phi)$

which represents a cycloid