Partial Differentiation

Denoted by $\frac{\partial f}{\partial x}$ function variable

$y = f(x, t)$

$y = x^2 + t^2$ ← Variable

variable

$$\frac{\partial y}{\partial x} = 2x \quad \Big| \quad \frac{\partial y}{\partial t} = 2t$$

Homogeneous functions:-

Any function $f(x, y)$ which can be expressed in the form $x^n \phi\left(y/x\right)$ is called homogeneous f^n of degree n

Euler's Theorem:-

If $u(x, y)$ is a homogeneous f^n of degree n in x & y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Total differentiation :-

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Differentiation of implicit f^n

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

where $\Rightarrow \dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$

Jacobian :-

If u & v are f^n of the two independant variables x & y then determinant
$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$
is called Jacobian denoted by $\dfrac{\partial(u,v)}{\partial(x,y)} = J\left(\dfrac{u,v}{x,y}\right)$

If u, v & w are f^n of the three indebendant variables x, y, z then determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$
is called Jacobian denoted by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left(\frac{u,v,w}{x,y,z}\right)$$

Properties of Jacobian:-

1. $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = JJ^{-1} = 1$$ u,v are functions of x,y

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = JJ^{-1} = 1$$ u,v are functions of x,y,z

2. If u & v are f^n of r & s where r & s are f^n of x & y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$ } Similar to chain rule

3. If f^n u, v, w of three independent variables x, y, z are not independent then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

i.e., if $J\left(\frac{u,v,w}{x,y,z}\right) = 0 \rightarrow$ Dependent

$J\left(\frac{u,v,w}{x,y,z}\right) \neq 0 \rightarrow$ Independent

Jacobian of implicit functions:-

If variables x, y, u, v are connected by implicit f^n

$f_1(x, y, u, v) = 0$ & $f_2(x, y, u, v) = 0$

then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^n \frac{\partial(f_1, f_2)/\partial(x,y)}{\partial(f_1, f_2)/\partial(u,v)}$$

If it is x, y connected with u, v then $n = 2$
& if x, y, z are connected with u, v, w then $n = 3$

Vector Diff^n :-

The ∇ operator

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

If f is a f^n there are 2 cases
i) f is scalar
ii) f is vector

If f is a Scalar:-

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

↓

Gradient of f
(grad f)

If f is a Vector

↳ $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$\nabla \cdot f$ — dot product

↓

Divergence of f
(div f)

$\nabla \times f$ — cross product

↓

Curl of f
(curl f)

Normal Vector & unit Normal Vector :-

$\vec{N} = \text{grad } f \rightarrow$ Normal vector

$\hat{N} = \dfrac{\text{grad } f}{|\text{grad } f|} = \dfrac{\vec{N}}{|\vec{N}|} \rightarrow$ Unit normal vector

Directional Derivative :-

Directional derivative in $\vec{A}$ direction $= (\text{grad } f) \cdot \hat{A}$

Unit vector of $\vec{A}$

Angle b/w two scalar f^n :- $\rightarrow$ at point (a, b, c)

$$\cos\theta = \frac{(\text{grad } f)_{(a,b,c)} \cdot (\text{grad } g)_{(a,b,c)}}{|\text{grad } f| \; |\text{grad } g|}$$

Substituting the points in both the gradients

Tangent Vector :-

Suppose $\vec{r} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

then tangent vector $\boxed{\vec{T} = \dfrac{d\vec{r}}{dt}}$