Rank of Matrix:-

The no. of non-zero rows in Echelon form

or

Order of largest minor

Finding Rank

Echelon form   Normal form

Echelon form:- Rank ρ = No. of non-zero rows

Given matrix is reduced to Echelon form by elemantary row transformations.

Example:-
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Echelon form}}$$

$\rho(A) = 3$

Normal form:- Given matrix is reduced to Echelon form by row transform^n. Later column transform^n are applied to reduce to one of 4 forms given below

$$I_r \,;\, \begin{bmatrix} I_r & 0 \end{bmatrix} ;\, \begin{bmatrix} I_r \\ 0 \end{bmatrix} ;\, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = r \leftarrow$ order of identity matrix

Solution of system of equations:-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{B}$$

$$AX = B$$

$B = 0$ → Homogeneous eq^n → Trivial sol^n

$B \neq 0$ → Non-homogeneous eq^n → Non-trivial sol^n

Test for consistancy:-

$A:B$ → Augmented Matrix

Rank of augmented matrix: $\rho(A:B)$; Rank of coefficient matrix: $\rho(A)$

$\rho(A:B) = \rho(A)$ → Consistent

$\rho(A:B) \neq \rho(A)$ → Not Consistent

Consistent → $\rho(A:B) = \rho(A) = r = n$ → Unique sol^n (n → no. of unknowns)

→ $\rho(A:B) = \rho(A) = r < n$ → ∞ sol^n

Characteristic Polynomial / Equation:-

$|A - \lambda I| \rightarrow$ Characteristic Polynomial

$|A - \lambda I| = 0 \rightarrow$ Characteristic Eq^n

Eigen Value & Eigen Vector:-

In a square matrix A, if $\exists$ a scalar λ (R or C) & a non-zero column matrix $X \ni AX = \lambda X$, then λ is called Eigen Value of A & X is called eigen vector corresponding to an eigen value

Short trick to find characteristic eq^n:-

For (2×2) matrix:-

Trace of A → sum of diagonal elements

$$\lambda^2 - Tr(A)\lambda + det(A) = 0$$

↳ determinant of A

For 3×3 matrix:-

$$\lambda^3 - Tr(A)\lambda^2 + \left(\text{sum of minors along diagonal}\right)\lambda - det(A) = 0$$

Cayley-Hamilton Theorem:-

A square matrix always satisfies its characteristic eq^n.

Diagonalization of matrix:-

A square matrix A is said to be diagonalizable if it is similar to a diagonal matrix

i.e $A = PDP^{-1} \Rightarrow D = P^{-1}AP$

$P \rightarrow$ Matrix where each row is each eigen vector

$D \rightarrow$ Matrix in which all elements are zero except the diagonal, which are the eigen values

Quadratic form to Matrix form:-

$$A = \begin{bmatrix} \text{Coeff of } x_1^2 & \frac{1}{2}\text{Coeff of } x_1x_2 & \frac{1}{2}\text{Coeff of } x_1x_3 \\ \frac{1}{2}\text{Coeff of } x_1x_2 & \text{Coeff of } x_2^2 & \frac{1}{2}\text{Coeff of } x_2x_3 \\ \frac{1}{2}\text{Coeff of } x_1x_3 & \frac{1}{2}\text{Coeff of } x_2x_3 & \text{Coeff. of } x_3^2 \end{bmatrix}$$

Canonical form:-

$y^T D y \longrightarrow$ Canonical form

where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$