Polarimeter Analysis

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1 Calibrating and Correcting the Retardance of a Waveplate

1.1 Importance of Waveplate Calibration

Our method involves rotating a quarter waveplate in front of a linear polarizer to extract the Stokes vectors of a beam of light. Since we are relatively insensitive to (polarization independant) losses, we can assume an ideal linear polarizer – that is we can assume that the transmission of the orthoganally polarized light is negligably small. However, we are likely not justified in assuming an ideal quarter waveplate. Supposing even that we have narrowband (or even monochormatic) light, a waveplate will typically have a wavelength-dependant phase delay. For a quarter waveplate, the relative delay between its fast and slow axes is $\delta = \frac{2n+1}{2}\pi$, where n is an integer. The wavelength dependance can be minimized for a so-called "zero-order" waveplate[1], i.e. on for which $n = 0$, is used. Even for zero-order waveplates however, there is still *some* wavelength dependance and so we can not assume that we have a real QWP for all angles.

A waveplate with fast axis horizontal will create a relative phase shift δ between the horizontal and vertical polarization components. If we are working with fully polarized states, the Jones formalism may be used and the corresponding Jones matrix is given by[2]

$$\begin{pmatrix} E_h \\ E_v \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} E_h \\ E_v \end{pmatrix} = \begin{pmatrix} E_h \\ e^{i\delta} E_v \end{pmatrix} \tag{1}$$

1.2 A Method for Measuring δ

A simple method to measure the phase delay δ is to rotate the waveplate in between two polarizers. The polarizers can either be crossed or parallel. Experimentally it may be easier to align cross polarizers since it is easier to measure a minimum than a maximum. Suppose that the first polarizer is aligned horizontally. Regardless of the input light, it will be completely polarized horizontally with some amplitude E_0 and intensity $I \propto |E_0|^2$. Since it is completely polarized, we may work in the simpler Jones formalism. The output field is then:

$$\vec{E}_{out} = P_h W_{\delta,\theta} \vec{E}_{in} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E_0 \\ 0 \end{pmatrix} = A\vec{E}_0 \tag{2}$$

. . . where $W_{\delta,\theta}$ is the matrix corresponding to a waveplate of phase shift δ rotated an angle θ with respect to the horizontal. The output intensity is then

$$I_\delta(\theta) = |A|^2 I_0 \tag{3}$$

With crossed polarizers, the intensity is similarly

$$I_{\delta,\perp}(\theta) = |B|^2 I_0 \tag{4}$$

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[1] Actually there are two classes of zero-order waveplates, "true" and "effective" zero order. For true zero order, the physical delay between a monochromatic beam two polarizations is $\lambda/4$. Since this is very hard to make, typically two orthogonally oriented waveplates with order m and $m+1$ are cemented together, advancing the beam by $(m+1)$ and delaying by m quarter wavelengths, which has the same effect at the precise wavelength.

[2] There is also a global phase shift applied to both beams $e^{i\phi}$ which is equivalent to simple propagation over a distance $L = 2\pi\phi\lambda$. This is not relevant here. Also note that some author prefer to share the phase equally between the two polarizations so the each component gets a phase factor $e^{\pm i\delta/2}$.

The parameters A and B can be obtained via rotating the waveplate in equation 1:

$$P_h W_{\delta,\theta} = R(\theta) P_h W_{\delta,0} R(-\theta)$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta e^{i\delta} & \cos\theta\sin\theta\left(1 - e^{i\delta}\right) \\ \cos\theta\sin\theta\left(1 - e^{i\delta}\right) & \sin^2\theta + \cos^2\theta e^{i\delta} \end{pmatrix} \tag{5}$$

Thus the intensity as a function of angle is:

$$I_\delta(\theta) = \left(\cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos\delta\right) I_0 \tag{6}$$

Note that if $\delta = \frac{2n+1}{2}\pi$ (with integer n), then $I_\delta(\theta) = I_0$ which is interesting. If $\delta = n\pi$ then $I_\delta(\theta) = \left(\cos^2\theta - \sin^2\theta\right)^2 = \cos^2 2\theta$ and we have perfect "visibility". Note as well that with the crossed polarizer result we have $|B|^2 = \sin 4\theta \sin\frac{\delta}{2}$

To extract the phase δ, we may align the polarizers in our setup and observe the resultant trace. An example of this is shown in figure 1. There will be maxima and minima which can be used to extract the phase: taking the derivative of equation 6 with respect to θ:

$$\frac{d}{d\theta} I_\delta(\theta) = \left[2\cos\delta\sin 2\theta\cos 2\theta - 4\cos\theta\sin\theta\left(\cos^2\theta - \sin\theta^2\right)\right] I_0$$
$$= \left[\sin 4\theta\left(\cos\delta - 1\right)\right] I_0 \tag{7}$$

Setting this to zero we have either that $\delta = 0$ (as a special case - recall δ is fixed) or that $\theta = \frac{n\pi}{4}$. For even n, the third term in equation 6 vanishes and either the sine or cosine gives 1 (the other is 0). This is the maximum at $I = I_0$. For odd n the result is a minimum. In this case $\sin\theta = \cos\theta = 1/\sqrt{2}$ and we have.

$$\frac{I_{min}}{I_{max}} \equiv \eta = \frac{1 + \cos\delta}{2} \tag{8}$$

We can thus find the phase delay of a waveplate as:

$$\boxed{\delta = \arccos\left(2\eta - 1\right)} \tag{9}$$

Figure 1: Using a red LED to calibrate our QWP. We find that for this wavelength ($\lambda \approx 630$ nm) we have a $\lambda/3.0$ plate! An overall phase of 0.69 rad was added to equation 6 to match the trigger origin of the motor.
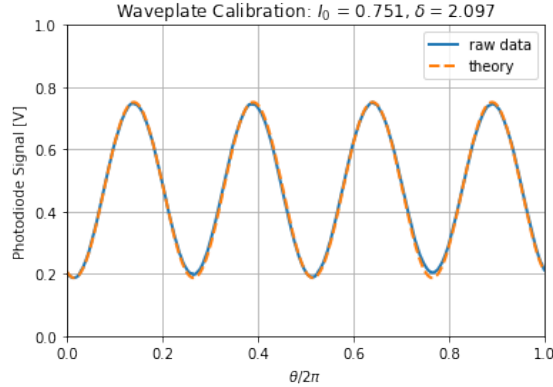
1.3 Correcting for Non-Ideal Waveplates

Given that we know the true phase shift of our waveplate, can we still extract the Stokes parameters as before? It turns out that we can . . .

2 Expected Intensity, Given Stokes Vector

2.1 Mueller Matrix of our system

Consider a monochromatic field with vacuum wavelength λ. A generic waveplate, with phase retardance $\delta/2$ and fast axis horizontal performs the transformation:

$$E_x \to e^{i\delta/2} E_x \tag{10a}$$

$$E_y \to e^{-i\delta/2} E_y \tag{10b}$$

Since $S_0 = |E_x|^2 + |E_y|^2$ and $S_1 = |E_x|^2 - |E_y|^2$, neither are affected by the waveplate as expected intuitively for a horizontal/vertical fast/slow axis. However, both S_2 and S_3 are affected. To see this, we can use Euler's identity:

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\begin{aligned}
S_2 &= E_x E_y^* + E_x^* E_y \\
&\to e^{i\delta} E_x E_y^* + e^{-i\delta} E_x^* E_y \\
&= \cos\delta \left(E_x E_y^* + E_x^* E_y \right) + i\sin\delta \left(E_x E_y^* - E_x^* E_y \right) \\
S_2 &\to \cos\delta S_2 + \sin\delta S_3
\end{aligned}
$$

Similarly, $S_3 \to \cos\delta S_3 - \sin\delta S_2$. Writing this as a matrix equation yeilds the Mueller matrix for a arbitrary waveplate:

$$W_{\delta,0^\circ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & \sin\delta \\ 0 & 0 & -\sin\delta & \cos\delta \end{pmatrix} \tag{11}$$

To find the matrix corresponding to a waveplate rotated an angle θ to the horizontal we apply the rotation matrix[3]:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

. . . so that

$$W_{\delta,\theta} = R(\theta) W_{\delta,0^\circ} R(-\theta) \tag{13}$$

The end result of this is then sent through a linear polarizer aligned horizontally, having matrix

$$P_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

We can just blast away at the whole expression now but many of the terms will be redundant for our purposes. Recall that all that we measure is the intesity, givent by S_0. Note that, writing the rotated waveplate as some abstract matrix, we have that the stokes vector after passage through the system is

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[3]Note that the 2θ comes about in taking the square to find the intensity.

$$\vec{S}' = \begin{pmatrix} I(\theta) \\ S_1'(\theta) \\ S_2'(\theta) \\ S_3'(\theta) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \tag{15}$$

From this, we can just take the first component of the output Stokes vector to find:

$$I(\theta) = \frac{W_{11} + W_{21}}{2} S_0 + \frac{W_{12} + W_{22}}{2} S_1 + \frac{W_{13} + W_{23}}{2} S_2 + \frac{W_{41} + W_{24}}{2} S_3 \tag{16}$$

so we just need to find the first two rows of the rotated waveplate. Actually it isn't too much more work doing the whole calculation but this gives the answer in terms of what we calculate. Grinding out equation 13, we get:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \cos\delta \sin^2 2\theta & \sin 2\theta \cos 2\theta (1 - \cos\delta) & -\sin 2\theta \sin\delta \\ 0 & \sin 2\theta \cos 2\theta (1 - \cos\delta) & \sin^2 2\theta + \cos\delta \cos^2 2\theta & \cos 2\theta \sin\delta \\ 0 & \sin\delta \sin 2\theta & -\sin\delta \cos 2\theta & \cos\delta \end{pmatrix} \tag{17}$$

Finally, plugging this back into equation 16, we have the desired result of the intensity as a function of Stokes parameters:

$$I(\theta) = \frac{1}{2} S_0 + \frac{\cos^2 2\theta + \cos\delta \sin^2 2\theta}{2} S_1 + \frac{\sin 2\theta \cos 2\theta (1 - \cos\delta)}{2} S_2 - \frac{\sin 2\theta \sin\delta}{2} S_3 \tag{18}$$

Finally, by employing trigonometric half angle formulae ($\sin 2x = 2 \sin x \cos x$, $2 \sin^2 x = 1 - \cos 2x$, and $2 \cos^2 x = 1 + \cos 2x$) we arrive at:

$$\boxed{I(\theta) = \frac{1}{2} \left[S_0 + \left(\frac{1 + \cos\delta}{2} \right) S_1 \right] - \left(\frac{\sin\delta}{2} S_3 \right) \sin 2\theta + \left(\frac{1 - \cos\delta}{4} S_1 \right) \cos 4\theta + \left(\frac{1 - \cos\delta}{4} S_2 \right) \sin 4\theta} \tag{19}$$

Note that in the limit of an ideal quarter waveplate ($\delta = \pi/2$) this becomes

$$I(\theta) = \frac{2S_0 + S_1}{4} - \frac{S_3}{2} \sin 2\theta + \frac{S_1}{4} \cos 4\theta + \frac{S_2}{4} S_2 \sin 4\theta \tag{20}$$

as is found in the literature.

3   Extracting the Stokes Vectors

The intensity through a rotating waveplate of retardance δ followed by a polarizer aligned horizontally is given by:

$$I(\theta) = \left[\frac{1 + \cos\delta}{2} \left(S_0 + \frac{1}{2} S_1 \right) \right] + \left[\frac{\sin\delta}{2} S_3 \right] \sin 2\theta + \left[\frac{1 - \cos\delta}{4} S_1 \right] \cos 4\theta + \left[\frac{1 - \cos\delta}{4} S_2 \right] \sin 4\theta$$
$$\equiv A + B \sin 2\theta + C \cos 4\theta + D \sin 4\theta \tag{21}$$

Using the orthogonality of trigonometric functions, we can write:

$$S_1 = \frac{4}{\pi(1 - \cos\delta)} \int_0^{2\pi} I(\theta) \cos 4\theta d\theta \tag{22a}$$

$$S_2 = \frac{4}{\pi(1 - \cos\delta)} \int_0^{2\pi} I(\theta) \sin 4\theta d\theta \tag{22b}$$

$$S_3 = \frac{2}{\pi \sin\delta} \int_0^{2\pi} I(\theta) \sin 2\theta d\theta \tag{22c}$$

$$S_0 = \frac{1}{\pi(1 + \cos\delta)} \int_0^{2\pi} I(\theta) d\theta - \frac{1}{2} S_1 \tag{22d}$$

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4 A Few Results from Fourier Theory

4.1 Orthogonality of Sinusoidal Functions

Let T be some interval and define $\omega_0 \equiv 2\pi/T$. Consider sines and cosines with frequency equal to some integer multiple of ω_0:

$$f_{\omega_0} \in \{\sin(n\omega_0 t), \cos(m\omega_0 t)\} \ldots n, m \in \mathbb{Z} \tag{23}$$

These functions are orthogonal when integrated over a period T, specifically:

$$\int_0^T \sin(n\omega_0 t) \sin(m\omega_0 t)\, dt = \frac{\pi}{\omega_0}\delta_{mn} \tag{24a}$$

$$\int_0^T \cos(n\omega_0 t) \cos(m\omega_0 t)\, dt = \frac{\pi}{\omega_0}\delta_{mn} \tag{24b}$$

$$\int_0^T \sin(n\omega_0 t) \cos(m\omega_0 t)\, dt = 0 \tag{24c}$$

To see this, it is helpful to use the identities:

$$\sin(n\omega_0 t) \sin(m\omega_0 t) = \frac{1}{2}\left[\cos((n-m)\omega_0 t) - \cos((n+m)\omega_0 t)\right] \tag{25a}$$

$$\cos(n\omega_0 t) \cos(m\omega_0 t) = \frac{1}{2}\left[\cos((n-m)\omega_0 t) + \cos((n+m)\omega_0 t)\right] \tag{25b}$$

$$\sin(n\omega_0 t) \cos(m\omega_0 t) = \frac{1}{2}\left[\sin((n+m)\omega_0 t) - \sin((n-m)\omega_0 t)\right] \tag{25c}$$

If $n = m$, the identities give $(1-\cos 2n\omega_0 t)/2 = \sin^2 n\omega_0 t$, $(1+\cos 2n\omega_0 t)/2 = \cos^2 n\omega_0 t$, and $(\sin 2n\omega_0 t)/2$ resectively. Regardless of whether or not $n = m$, the frequency arguments to the sine and cosine terms are still just integer multiples of ω_0, since $2n$, $n - m$, and $n + m$ are just some other integer. Thus every term in 25 can be written as $\sin(N\omega_0 t)$ or $\cos(N\omega_0 t)$. This means that the integral of these terms over a period vanishes:

$$\int_0^{T=2\pi/\omega_0} \sin(N\omega_0 t)dt = -\frac{1}{N\omega_0}\left[\cos(2N\pi) - \cos 0\right] = \frac{1-1}{N\omega_0} = 0$$

$$\int_0^{T=2\pi/\omega_0} \cos(N\omega_0 t)dt = \frac{1}{N\omega_0}\left[\sin(2N\pi) - \sin 0\right] = \frac{0-0}{N\omega_0} = 0$$

The only possible non-zero term is thus the constant term, occuring when $n = m$, for which:

$$\int_0^T \frac{1}{2}dt = \frac{T}{2} = \frac{\pi}{\omega_0} \tag{26}$$

Thus, almost all terms in equation 24a, written using eq. 25 are identically zero. The only exception is when $n = m$ in 24aa and 24ab, for which equation 26 holds. Written in terms of the dirac delta function then, this completes the proof.