AA/EE/ME 548: Linear Multivariable Control

Lecture 04

4/9/2025

Announcements/reminders

- Sign-up sheet for homework/worked example
  - Those who signed up for week 1&2, you have 1 week to submit your worked example
- Assignment deadlines should be posted on

Information on course project

- Understanding of varied background and experiences. Can be more research-y or pedagogical.
  - Directly related to on-going research
  - Motivated by a real-world system and apply techniques covered/mentioned in this course
  - Implement a technique described in a paper (and try adding a spin to it)
  - Compare and contrast different techniques and do a trade-off study
  - Okay to combine with other course projects*
- *Needs to connect with topics covered in the course
  - Anti-example: Train an end-to-end controller.
    - But incorporate control-theoretic concepts into the training process to improve controller performance & compare to vanilla E2E controller OK.
- Okay (and encouraged) to use any homework code to help your project.
  - Okay to use open-source code, as long as you properly acknowledge and reference it.
- How to read and write a paper: https://faculty.washington.edu/kymleung/assets/courses/aa598/2024/lecture02.pdf

This week

- ~~Intro to optimization~~
  - ~~https://uw-ctrl.github.io/lmc-book/lectures/optimization.html~~

- Control Barrier Functions and Control Lyapunov Functions (connection with Lyapunov Theory from 547)
  - https://uw-ctrl.github.io/lmc-book/lectures/control-certificates.html
  - Summarize Lyapunov theory with slides
  - CBF on board

Stability

Type	**Definition**	**Notes**
Lyapunov Stability	Trajectories stay close to equilibrium if started nearby	No guarantee of convergence
Asymptotic Stability	Trajectories stay close **and** converge to equilibrium	Includes Lyapunov stability
Exponential Stability	Trajectories converge to equilibrium **at an exponential rate**	Stronger than asymptotic stability
Local Stability	Stability properties hold **near** the equilibrium	Common in nonlinear systems
Global Stability	Stability properties hold **for all** initial conditions	Often harder to prove

Proving stability

- Linear systems: , λ_i eigenvalues of A, for all i
  - $\dot{x} = Ax$: $\mathrm{Re}(\lambda_i) < 0$
  - $x_{t+1} = Ax_t$: $|\lambda_i| < 1$

- What about nonlinear systems $\dot{x} = f(x)$?

Lyapunov function

Definition (Lyapunov function)

Given a general nonlinear dynamical system $\dot{x} = f(x)$, $x \in \mathcal{X} \subset \mathbb{R}^n$. Without loss of generality, let $x = 0$ be an equilibrium state. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a scalar function that is continuous, and has continuous first derivatives. Then the origin $x = 0$ is Lyapunov stable if:

- $V(0) = 0$
- $V(x) > 0 \, \forall x \in \mathcal{X} \setminus \{0\}$
- $\dot{V}(x) = \nabla V(x)^T f(x) \leq 0 \, \forall x \in \mathcal{X}$

- Can shift to a non-zero equilibrium point
- Intuitively, V is an "energy-like" function
- Intuitively, need the energy to be non-increasing along the trajectory

Lyapunov stability for linear systems

- For linear system, $\dot{x} = Ax$, a quadratic Lyapunov function is usually a good candidate
  - Spring force is a linear in displacement, energy is quadratic in velocity, potential energy is quadratic in displacement
  - $V(x) = x^T P x,\ P$ is positive definite

- $\dot{V}(x) = \dot{x}^T P x + x^T P\, \dot{x} \leq 0 \;\Rightarrow x^T (A^T P + PA) x \leq 0$

  $\Rightarrow A^T P + \; PA$ is PD

Lyapunov stability theory for linear continuous time invariant systems.

Theorem (Lyapunov stability theorem for linear systems)

For a linear system $\dot{x} = Ax$, the origin $x = 0$ is asymptotically stable if and only if for any symmetric positive definite matrix Q, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

What about discrete time linear systems?

- $x_{t+1} = Ax_t,$
  - $V(x) = x^T P x,\ P$ is positive definite

- $V(x_{t+1}) - V(x_t) \leq 0 \ldots$