Euler equations in 2D cylindrical coordinates

Jack Coughlin

January 23, 2024

The compressible Euler equations are

$$\partial_t(\rho) + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$
$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0, \tag{2}$$
$$\partial_t(E) + \nabla \cdot ((E+p)\boldsymbol{u}) = 0. \tag{3}$$

In cylindrical coordinates in the r, z plane, the del operators are

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}},$$
$$\nabla \cdot \boldsymbol{g} = \frac{1}{r}\frac{\partial (r g_r)}{\partial r} + \frac{\partial g_z}{\partial z}.$$

Expanding the Euler equations, then, we get the following:

$$\partial_t \rho + \frac{1}{r}\frac{\partial (r\rho u_r)}{\partial r} + \frac{\partial(\rho u_z)}{\partial z} = 0,$$
$$\partial_t(\rho \boldsymbol{u}) + \frac{1}{r}\frac{\partial(r\rho u_r \boldsymbol{u})}{\partial r} + \frac{\partial(\rho u_z \boldsymbol{u})}{\partial z} + \frac{\partial p}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{z}} = 0,$$
$$\partial_t E + \frac{1}{r}\frac{\partial(r(E+p)u_r)}{\partial r} + \frac{\partial((E+p)u_z)}{\partial z} = 0.$$

The issue with this set of equations is that the terms with a factor of $1/r$ cannot be put in conservation form. For example, if we imagine discretizing the continuity equation as written directly in a finite difference scheme, we would have

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\frac{1}{\color{red}{r}}\frac{(r\rho u_r)_{i+1/2} - (r\rho u_r)_{i-1/2}}{\Delta r}.$$

The terms at the cell faces $i \pm 1/2$ present no difficulty, but the question is at what point is the factor of $1/r$ colored in red supposed to be evaluated?

To deal with this problem, one approach is to expand the $\frac{\partial}{\partial r}$ terms, and move all non-derivative terms to the right-hand side:

$$\partial_t \rho + \frac{\partial(\rho u_r)}{\partial r} + \frac{\partial(\rho u_z)}{\partial z} = -\frac{\rho u_r}{r}, \tag{4}$$
$$\partial_t(\rho \boldsymbol{u}) + \frac{\partial(\rho u_r \boldsymbol{u})}{\partial r} + \frac{\partial(\rho u_z \boldsymbol{u})}{\partial z} + \frac{\partial p}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{z}} = -\frac{\rho u_r \boldsymbol{u}}{r}, \tag{5}$$
$$\partial_t E + \frac{\partial((E+p)u_r)}{\partial r} + \frac{\partial((E+p)u_z)}{u_z} = -\frac{(E+p)u_r}{r}. \tag{6}$$

Another approach is to multiply through by r:

$$\partial_t(r\rho) + \frac{\partial(r\rho u_r)}{\partial r} + \frac{\partial(r\rho u_z)}{\partial z} = 0, \tag{7}$$

$$\partial_t(r\rho\boldsymbol{u}) + \frac{\partial(r\rho u_r\boldsymbol{u})}{\partial r} + \frac{\partial(r\rho u_z\boldsymbol{u})}{\partial z} + r\frac{\partial p}{\partial r}\hat{\boldsymbol{r}} + \frac{\partial(rp)}{\partial z}\hat{\boldsymbol{z}} = 0, \tag{8}$$

$$\partial_t(rE) + \frac{\partial(r(E+p)u_r)}{\partial r} + \frac{\partial(r(E+p)u_z)}{\partial z} = 0. \tag{9}$$

We now observe that the momentum equation can be rewritten as

$$\partial_t(r\rho\boldsymbol{u}) + \frac{\partial(r(\rho u_r\boldsymbol{u} + p\hat{\boldsymbol{r}}))}{\partial r} + \frac{\partial(r(\rho u_z\boldsymbol{u} + p\hat{\boldsymbol{z}}))}{\partial z} = p\hat{\boldsymbol{r}}$$