Day 2, Session 2: Logs/Exponentiation

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Exponentiation

- A mathematical operation corresponding to repeated multiplication
- The second in the order of operations! (P**E**MDAS)
- Composed of two numbers: a base, b, and an exponent, n
- $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$

Positive vs negative exponents

- Exponents correspond to multiplication
- Positive exponent: multiplication, e.g. $2^2 = 2 \times 2$
- Negative exponent: multiplication of reciprocals, i.e. $2^{-2} = \frac{1}{2} \times \frac{1}{2}$

Properties of exponents

- For any base b and any n an integer:
  - $b^0 = 1$
  - $b^1 = b$
  - $b^{n+1} = b^n \times b$
- For $b \neq 0$ and any n an integer:
  - $b^n = b^{n+1}/b$
  - $b^{-n} = 1/b^n$

Exponent identities

- For all $b, c \neq 0$:
  - $b^{m+n} = b^m \times b^n$
  - $b^{m \times n} = (b^m)^n$
  - $(b \times c)^n = b^n \times c^n$

Example: integer exponent properties and identities

- Take $b = 2$
- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 2 \times 2$
- $2^3 = 2^2 \times 2 = 8$
- $(2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = 36$ (check: $2 \times 3 = 6, 6^2 = 36$)

Rational exponents (roots)

- nth root of b: the number x such that $x^n = b$
- Written as $b^{1/n}$ or $\sqrt[n]{b}$
- Some identities (for b positive):
  - $b = (b^n)^{1/n}$
  - $b^{m/n} = (b^m)^{1/n} = \sqrt[n]{b^m}$

Exponential function

- An important constant: e, approximately 2.718
- Useful as a base for powers
- Define $\exp(x) = e^x$
- Useful identity: $\exp(x + y) = e^{x+y} = \exp(x) \times \exp(y)$

Exercise: exponents and the exponential function

1. What is the result of x^2 multiplied by x^3?
2. $(x^{-2})^4 = ?$
3. $\exp(x - y) = ?$

Solutions: exponents and the exponential function

1. $x^2 \times x^3 = x^5$, since we add the exponents when we multiply
2. $(x^{-2})^4 = x^{-2\times 4} = x^{-8}$
3. $\exp(x - y) = e^{x-y} = e^x \times e^{-y} = e^x/e^y = \exp(x)/\exp(y)$

Logarithms

- Exponents correspond to multiplication
- Addition is easier than multiplication
- Logarithms (logs) transform multiplication into addition!

Logs: definition

- Defined as the inverse operation of exponentiation
- Takes a base b and a number x
- The log of x to base b is the number y such that $b^y = x$
- Written as $\log_b(x) = y$
- Natural log: $\log_e(x)$, commonly written $\log(x)$

Logs: definition

- Undefined for $x \leq 0$
- Log is an increasing function: as x increases, $\log_b(x)$ increases
- $\log_b(b) = 1$

Logs: identities

- Multiplication: $\log_b(xy) = \log_b(x) + \log_b(y)$
- Division: for $y \neq 0$, $\log_b(x/y) = \log_b(x) - \log_b(y)$
- Powers: $\log_b(x^p) = p \log_b(x)$
- Roots: for $p \neq 0$, $\log_b(x^{1/p}) = \log_b(x)/p$
- Inverse function: $\log_b(b^x) = x \log_b(b) = x$

Example: log identities

- Multiplication: $\log(2 \times 3) = \log(2) + \log(3) = \log(6)$
- Logs of numbers < 1 are negative:
  $\log(2/3) = \log(2) - \log(3) < 0$
- Power: $\log(x^2) = 2\log(x)$

Common bases, changing base

- The three most common bases: e, 10, and 2
- e — common in mathematics
- 10 — common for calculating numbers in the decimal system
- 2 — common in computer science
- Changing between bases: $\log_b(x) = \dfrac{\log_k(x)}{\log_k(b)}$

$\exp(\cdot)$ and $\log(\cdot)$

- Recall $\exp(x) = e^x$
- Natural log: $\log(x) = \log_e(x)$
- So $x = \log[\exp(x)]$! And $x = \exp[\log(x)]$!

Exercise: logarithms

1. $\log(xy) = ?$

2. $\log(x/y) = ?$

3. $\log[\exp(2x)] = ?$

4. $\exp[\log(x^2)] = ?$

Solutions: logarithms

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(x/y) = \log(x) - \log(y)$
3. $\log[\exp(2x)] = 2x$
4. $\exp[\log(x^2)] = \exp[2\log(x)] = \exp(2)\exp[\log(x)] =$
   $x\exp(2)$

Uses of logarithms in statistics

- Transformation of the data — look at a multiplicative relationship rather than an additive relationship
- Logistic regression, Poisson regression
- For more, see BIOST 512/513!