Error in drainage equation VIC?

The master version of VIC contains the following statement that is used to compute vertical drainage between two model layers:

```
Q12 = init_moist - pow(pow(init_moist - resid_moist, 1.0 - expt) -
                  Ksat / pow(max_moist - resid_moist, expt) * (1.0 - expt),
                  1.0  / (1.0 - expt)) - resid_moist;
```

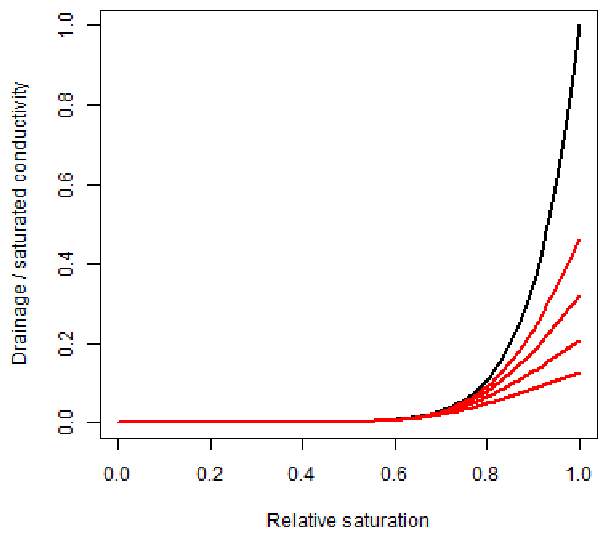
I translated this to the following equation:

$$Drainage = \theta - \theta_r - \left[(\theta - \theta_r)^{1-c} - \frac{K_s(1-c)}{(\theta_s - \theta_r)^c}\right]^{1/(1-c)} \quad (1)$$

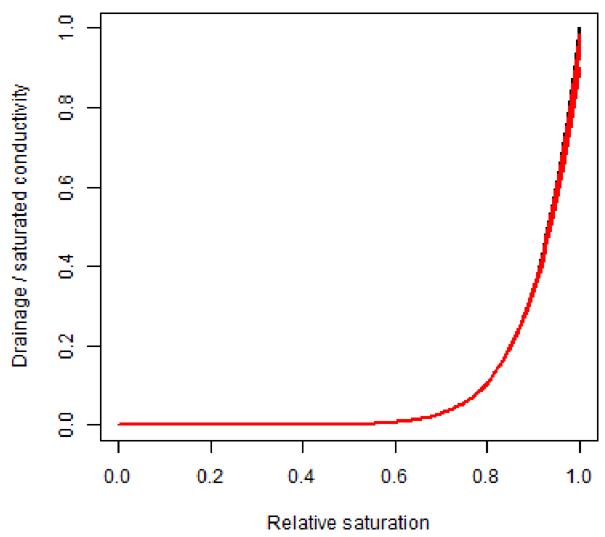
Where c in the equation equals expt in the computer code. I expected the equation of Brooks-Corey but my colleague Lisanne Nauta found the following explanation for the replacement on github (https://github.com/UW-Hydro/VIC/commit/59c9f5dbcf0835e6a3ee77eb1820f6349228ec27): "Modify layer drainage Q12 calculation - use exact integral (instead of numerical solution) in order to avoid unreasonably-strong soil moisture oscilation when the Brook & Corey curve is steep".

There are at least two reasons why I have doubts about the modified equation:

1) drainage does not become zero if water content equals residual water content, nor is drainage equal to K_s if water content equals saturated water content.
2) In the code units are in mm and mm per time step, i.e. θ, θ_r and θ_s are in mm (for the whole layer) and K_s and drainage are in mm per time step. The problem is that with these units Eq. 1 drainage is a function of layer depth since θ, θ_r and θ_s are proportional to layer depth. I demonstrate the effect of layer depth in the graph below, where I put Drainage / Ks on the vertical axis and relative saturation (the ratio of (θ - θ_r) and (θ_s - θ_r)) on the horizontal axis. The black line corresponds to the Brooks-Corey equation. The red lines show Eq.1 for four layer depths (4000, 2000, 1000 and 500 mm from top to bottom). I set fractional (of the total volume) θ_r = 0.1, θ_s = 0.45, c = 10, Ks = 400 mm/d and used a time step of 1 day.



I then produced another graph, only changing Ks from 400 to 4 mm/d. The new graph is shown below



Now all four red lines almost coincide with the Brooks-Corey line, suggesting that 1) Eq.1 is a good approximation of Brooks-Corey for small values of K_s (e.g. 4 mm/d) and 2) I made no error is producing the graphs. However, values for mean K_s in Table 3 of Cosby et al. (1984) vary between roughly 100 and 4000 mm/d, so 4 mm/d is far below this range.

I produced more graphs, which show that Equation 1 more or less coincides with Brooks-Corey, not only for small K_s but also for thick layers and small time steps. That makes me start wondering whether Eq. 1 is valid if θ is not the water content at the beginning but the water content at the end of the time step (implicit scheme?).

So, perhaps in the end Eq. 1 is correct but this is difficult to see. I that case, some openly accessible explanation including the assumptions made and the derivation of the equation (which I could not find) could help taking away the unpleasant feeling that something might be wrong with this for VIC crucial equation.

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December 22, 2021