ECE 650, Fall 2015
A Polynomial-Time Reduction from VERTEX-COVER to CNF-SAT

VERTEX-COVER is the following problem:

- Input: An undirected graph $G = (V, E)$, and an integer $k \in [0, |V|]$.
- Output: True, if G has a vertex cover of size k, false otherwise.

CNF-SAT is the following problem:

- Input: a propositional logic formula, F, in Conjunctive Normal Form (CNF).
  That is, $F = c_1 \wedge c_2 \wedge \ldots \wedge c_m$, for some positive integer m. Each such c_i is called a "clause". A clause $c_i = l_{i,1} \vee \ldots \vee l_{i,p}$, for some positive integer p. Each such $l_{i,j}$ is called a literal. A literal $l_{i,j}$ is either an atom, or the negation of an atom.
- Output: True, if F is satisfiable, false otherwise.

We present a polynomial-time reduction from VERTEX-COVER to CNF-SAT. A polynomial-time reduction is an algorithm that runs in time polynomial in its input. In our case, it takes as input G, k and produces a formula F with the property that G has a vertex cover of size k if and only if F is satisfiable.

The use of such a reduction is that given an instance of VERTEX-COVER that we want to solve, G, k, we can transform it to F, and provide F as input to a SAT-solver. The true/false answer from the SAT solver is the answer to the instance of VERTEX-COVER. Assuming the SAT solver works efficiently (for some characterization of "efficient"), we now have an efficient way of solving VERTEX-COVER.

The reduction

Given G, k where $G = (V, E)$, denote $|V| = n$. Assume that the vertices are named $1, \ldots, n$. Construct F as follows.

- Adopt $n \times k$ atoms, each denoted $x_{i,j}$ where $i \in [1, n]$ and $j \in [1, k]$. The mindset behind this is the following. We see a vertex cover of size k as a list of k vertices. The meaning of $x_{i,j}$ is that $x_{i,j}$ is true if and only if the vertex i is the jth vertex in the vertex cover.
- Adopt the following as the clauses.
  - $\forall i \in [1, k]$, a clause $x_{1,i} \vee x_{2,i} \vee \ldots \vee x_{n,i}$. Such a clause ensures that (at least) one of the n vertices is the ith vertex in the vertex cover.
  - $\forall m \in [1, n], \forall p, q \in [1, k]$ with $p < q$, a clause $\neg x_{m,p} \vee \neg x_{m,q}$. Such a clause ensures that the vertex m cannot be both the pth and qth vertex in the vertex cover, for $p \neq q$.
  - $\forall m \in [1, k], \forall p, q \in [1, n]$ with $p < q$, a clause $\neg x_{p,m} \vee \neg x_{q,m}$. Such a clause ensures that the mth vertex in the vertex cover cannot be mapped into by both p and q, for $p \neq q$.
  - For each edge <i, j> $\in E$, a clause $x_{i,1} \vee x_{i,2} \vee \ldots \vee x_{i,k} \vee x_{j,1} \vee x_{j,2} \vee \ldots \vee x_{j,k}$. Such a clause ensures that at least one of i and j is in the vertex cover.

The number of clauses that the reduction outputs is therefore: $k + n\binom{k}{2} + k\binom{n}{2} + |E|$.