ESS 411/511 Geophysical Continuum Mechanics Class #28

Highlights from Class #27 – Abigail Thienes
Today's highlights on Wednesday – Zoe Krauss

Today

- More on Moments
- Constitutive Relations

For Wednesday and Friday please read

- Ed's notes on Seismic Moment
- Ed's notes on constitutive relations
- Ed's notes on classical fluids
- Ed's notes on elastic waves

(on right sidebar at)
https://courses.washington.edu/ess511/NOTES/)

Problem Sets

Problem Set #5 has been graded
- Results are on Canvas
- I will return annotated paper late today

Problem Set #6
- Coming back soon …

For Problems Lab on Thursday
- Study Questions for take-at-home Final exam have been posted.

I plan to prepare some notes about the issues encountered on the Mid-term and on Problem Sets #4 and #5

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Warm-up (break-out rooms)

- What exactly are constitutive relations?
- Why are they useful?
- What are some examples?

Class-prep: Constitutive relations

Please read
- Raymond notes on stress and moments
- Ed's notes on constitutive relations

Both are on the class web site at

https://courses.washington.edu/ess511/NOTES/notes.html

Assignment

In general, a linear relation between two second-order tensors f_{ij} and g_{ij} requires a fourth-order tensor coefficient c_{ijkm} with 81 components.

$$f_{ij} = c_{ijkm}\, g_{km}$$

However, stress σ_{ij} and strain ε_{km} can be related with just 2 scalar coefficients λ and μ (the Lamé constants) in Hooke's Law for linear elasticity.

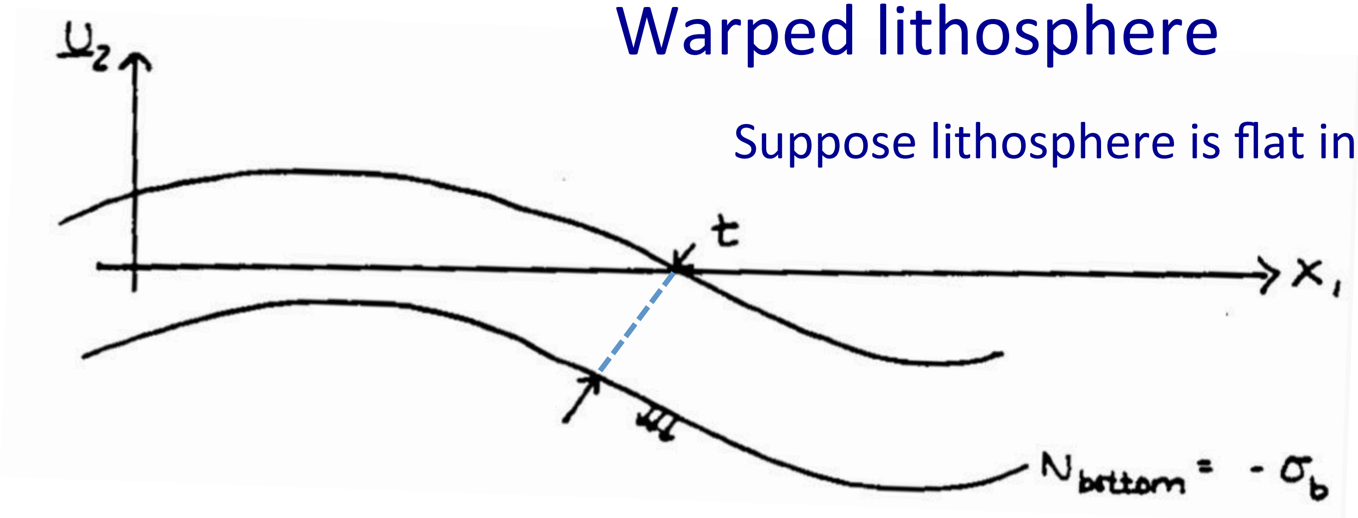
$$\sigma_{ij} = \lambda \delta_{ij}\, \varepsilon_{kk} + 2\mu\, \varepsilon_{ij}$$

In a paragraph, outline
- the properties of stress and strain tensors that allow much of this simplification,
- the conservation laws that are invoked,
- and any additional assumptions.
- What do the Lamé constants represent, and what are their units?

Moments of interest in the Earth

- Rock overhangs
- Snow cornices
- Length scale of support by bending lithosphere
- Earthquake moment magnitude

Warped lithosphere



Suppose lithosphere is flat in relaxed state

Now lithosphere is warped into a sinusoid

- e.g. by loading with a big ice sheet

$$u_2(x_1) = a \sin kx_1 \qquad \lambda = 2\pi/k$$

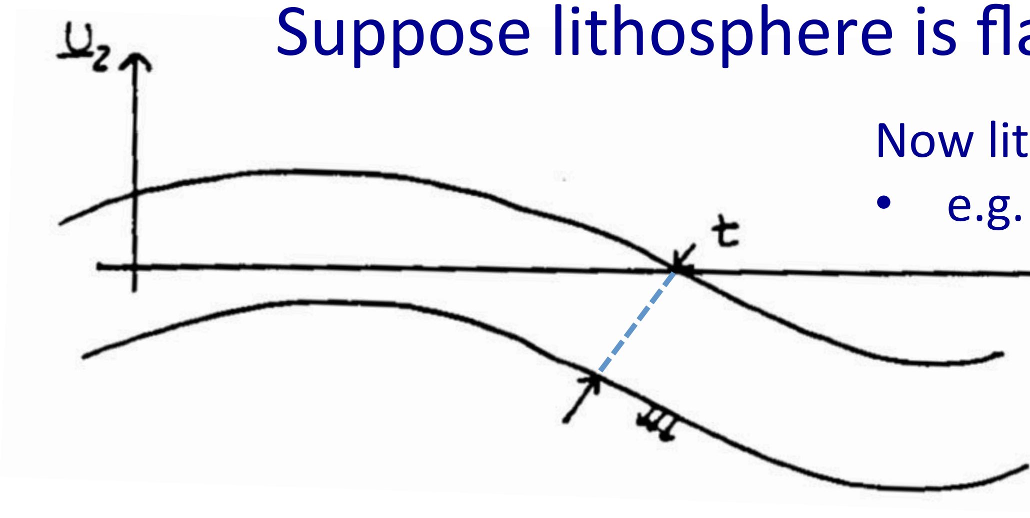
- Far from any edge, $P_1 = P_2 = 0$
- $N_{top} = 0$
- $N_{bot} = \sigma_b$

$$\frac{\partial^4 u_2}{\partial x_1^4} = k^4 a \sin kx_1 = \frac{1}{D} F_2(x_1)$$

σ_b results from 2 effects

- weight of the overlying lithosphere
- bending stresses in the lithosphere, which tries to relax
  - pull the lithosphere up and away from the mantle in the hollows,
  - push down on the mantle under the crests.

Suppose lithosphere is flat in relaxed state



Now lithosphere is warped into a sinusoid

- e.g. by loading with a big ice sheet

$$u_2(x_1) = a \sin kx_1 \qquad \lambda = 2\pi/k \qquad (1)$$

$N_{bottom} = -\sigma_b$

- Far from any edge, $P_1 = P_2 = 0$
- $N_{top} = 0$
- $N_{bot} = \sigma_b$

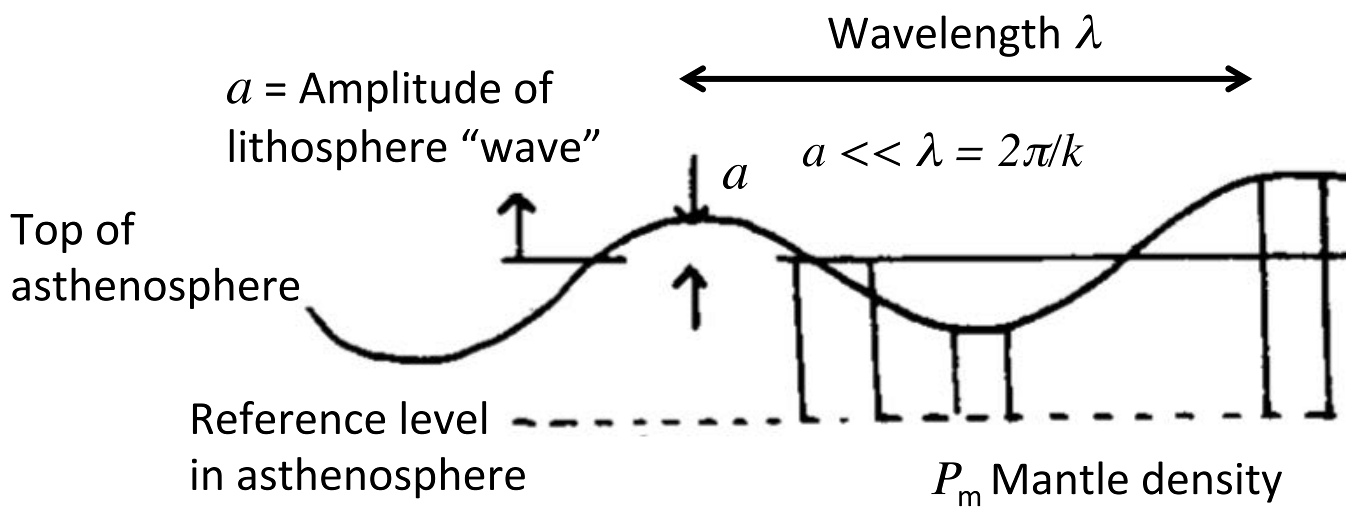
$\Rightarrow$ $F_2 = \sigma_b - \rho_c g t$ (2)

so $$\frac{\partial^4 u_2}{\partial x_1^4} = k^4 a \sin kx_1 = \frac{1}{D} F_2(x_1) \qquad (3)$$

σ_b varies with x_1 and results from 2 effects

- weight of the overlying lithosphere
- bending stresses in the lithosphere, which tries to relax and flatten
  - pull the lithosphere up from the mantle in the hollows,
  - push down on the mantle under the crests.

From (2) and (3) $$\sigma_b = \rho_c g t + D k^4 a \sin kx_1 \qquad (4)$$



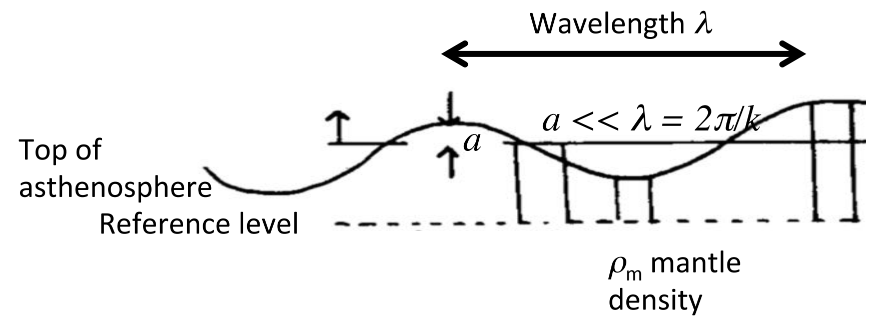
At the reference level, first account for
weight of the mantle "bumps": $\rho_m g\, a \sin(kx_1)$
weight of the lithosphere with thickness t: $\rho_c g\, t$

$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + \text{const}$$

Now account for the bending moments trying to flatten the lithophere
At the reference level, there are no horizontal stress gradients (why?)

$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + Dk^4 a \sin kx_1 + \text{const}$$

$$= (\rho_m g + Dk^4) a \sin kx_1 + \rho_c g t + \text{const}$$



The flexural rigidity of the lithosphere adds a restoring force additional to the force coming from the topography

$$\sigma = +\rho_m g a \sin kx_1 + \rho_c g t + Dk^4 a \sin kx_1 + \text{const}$$

$$= (\rho_m g + Dk^4) a \sin kx_1 + \rho_c g t + \text{const}$$

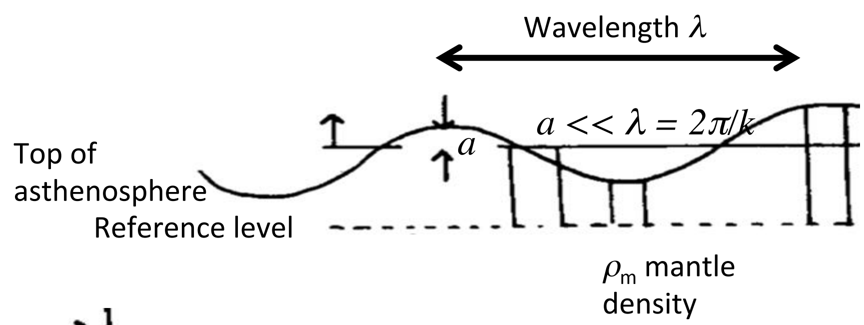
Bending stresses in the lithosphere become dominant $k^4 D > \rho_m g$ or $k > \left[\frac{\rho_m g}{D}\right]^{\frac{1}{4}}$

In terms of the wavelength λ, $\lambda = \frac{2\pi}{k} < 2\pi\left[\frac{D}{\rho_m g}\right]^{\frac{1}{4}}$

Short wavelengths can be supported, but long wavelength waves just sag into the mantle based on their weight

At the reference level in the asthenosphere, there are no horizontal stress gradients (why?)

Supportable wavelengths



Bending stresses in the lithosphere become dominant $k^4 D > \rho_m g$ or $k > \left[\frac{\rho_m g}{D}\right]^{\frac{1}{4}}$

Flexural rigidity $D = Mt^3/12$

Elastic modulus $M = E/(1 - \upsilon^2) \approx 10^{10}$ Pa

ρ_m g ~0.3× 10^5 Pa m-1, and with t = 100 km, $D = 10^{24}$ Pa m^3

So bending stresses become important for λ<500 km

At the reference level in the asthenosphere, there are no horizontal stress gradients (why?)

How to measure earthquake size?

(A) – by amount of energy released

e.g. Richter Magnitude $M = C_1 \log E + C_2$ E = energy released

But how to measure the energy?

Energy goes into

1) elastic waves, acoustic waves.
   – speed of rupture matters
2) production of fault gouge (crushing rock)
   – rock type matters.
3) frictional heating of fault zone
   – normal and shear stresses matter
   – coefficient of friction matters
4) potential energy (uplift & subsidence)

Only 1) is readily available to seismologists

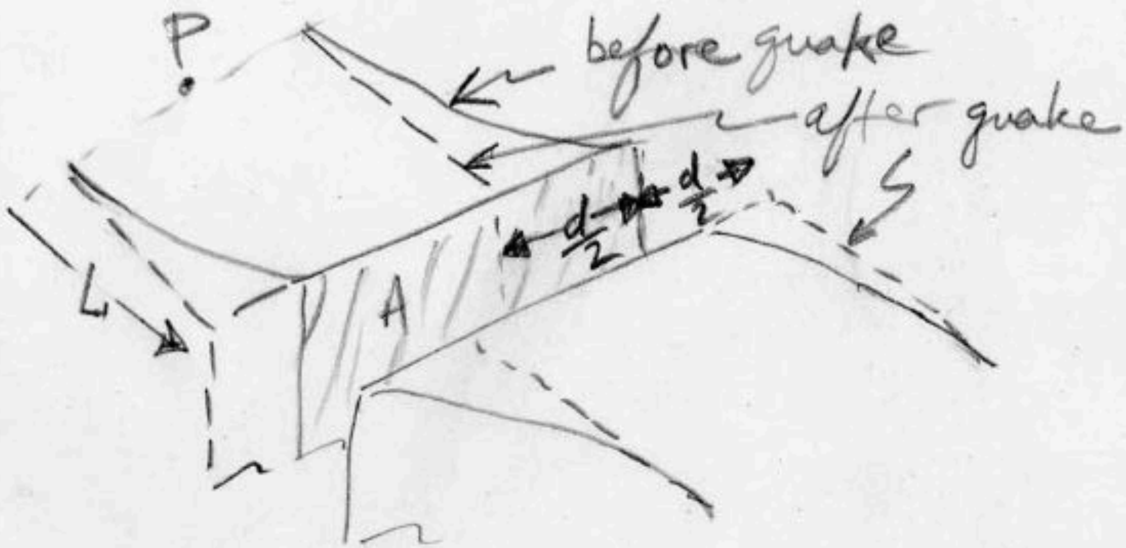
Maybe 4), after geodetic resurvey.

So getting a measure of total energy release can be difficult.

Estimate strain energy released?

Maybe instead of measuring the energy directly, we can estimate the energy release based on the strain energy, which we can estimate from a few parameters of the rock and the fault.

Prior to the quake, elastic energy was stored in the rock around the fault.



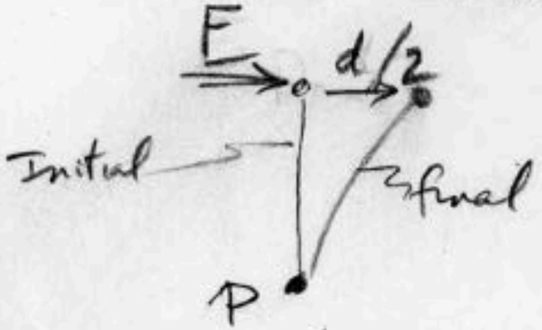
$\frac{d}{2}$ = slip during quake (one side of fault)

A = area of fault rupture.

L = distance away from fault where permanent offset is small.

Estimate strain energy in a spring?

How do we put energy into a "spring", e.g. a metal ruler?



Ruler is pinned at P.

Force $\underline{F}$ applied to end of ruler causes displacement $\underline{d}/2$

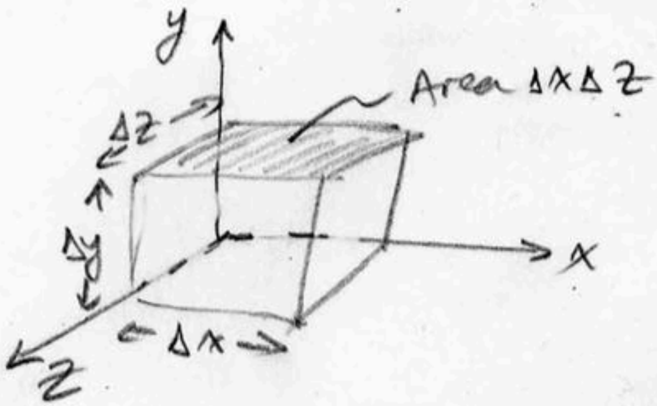
When force is released, spring returns to initial state.

Energy stored in bent spring is $E = \underline{F} \cdot \underline{d}/2$

(1 Spring represents one side of fault.)

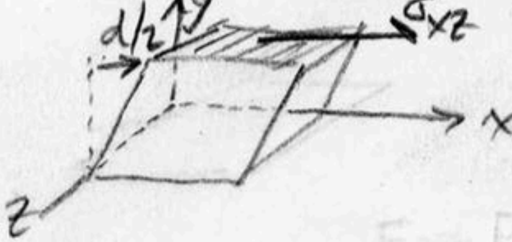
Energy released by a spring

Now lets use a block of a continuum as the spring



Apply a force to the surface with area $\Delta x \Delta z$

Elastic block deforms with displacement $\underline{d}/2$ (one side of a fault)



$\underline{E} = \underline{F} \cdot \underline{d}/2$

The force is $|\underline{F}| = \sigma_{xy}\, \Delta x \Delta z$ required to bring rock to failure limit.

The energy is $E = Fd = (\sigma_{xy}\, \Delta x \Delta z) d$

The shear strain is $\epsilon_{xy} \sim d/\Delta y$ inside the block.

Energy per unit volume

In general

$$\text{So } E = (\sigma_{xy}\,\Delta x\,\Delta z)(\epsilon_{xy}\,\Delta y)$$

$$= \sigma_{xy}\,\epsilon_{xy}(\Delta x\,\Delta y\,\Delta z) = \sigma_{xy}\,\epsilon_{xy}\,V$$

$\frac{\text{Energy}}{\text{Volume}}$ $\quad E/V = \sigma_{xy}\,\epsilon_{xy}$

Stored elastic strain energy in a volume is

$$E = \int_V \sigma_{xy}\,\epsilon_{xy}\,dV$$

Our fault has stored up elastic strain energy, and it releases it during a quake ($\sigma_{xy} \to 0$).

$$E = \int_V \sigma_{xy}\,\epsilon_{xy}\,dV$$

For elastic continuum

$\sigma_{xy} \sim \mu\,\epsilon_{xy}$

μ is an elastic constitutive parameter

$$= \int_V \mu\,\epsilon_{xy}^2\,dV$$

Simple, but not maybe simple enough …

For the fault: $\epsilon_{xy} \sim \frac{d}{L}$ strain

$dV \sim AL$ volume strained.

so $E = \int_V \mu\, \epsilon_{xy}^2\, dV \sim \mu \left(\frac{d}{L}\right)^2 AL$

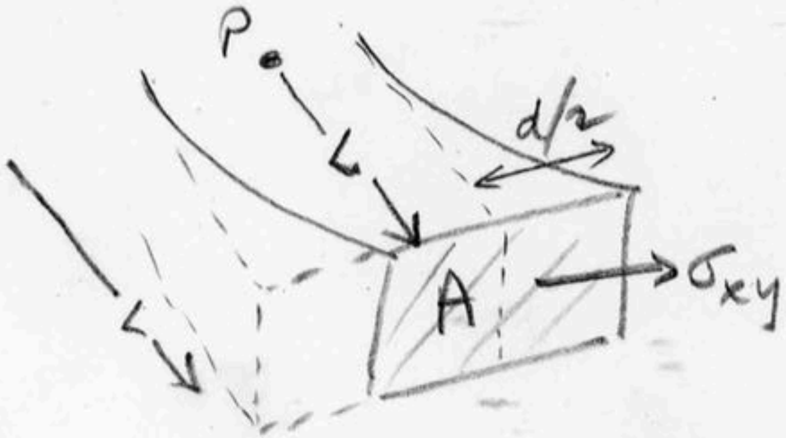
$= \mu A d \left(\frac{d}{L}\right)$

We can estimate μ (rock elasticity)
A (rupture area)
d (fault motion)

But estimating L is tricky.
So let's look for a better way to estimate earthquake size.

(B) Moment Magnitude

Shear traction σ_{xy} brought fault to point of failure.
(σ_{xy} corresponds to P_2 in Raymond notes)



L is moment arm about P.
Moment created by σ_{xy} (and released during quake) is

$$M_1 = FL = (\sigma_{xy} A) L$$

$\sigma_{xy} = \mu \epsilon_{xy}$ - constitutive relation of rock

$\epsilon_{xy} = \frac{d/2}{L}$ - strain

So, on one side of fault
Moment release is

$$M_1 = (\mu \epsilon_{xy}) A L = \frac{\mu d A L}{2L} = \frac{\mu d A}{2}$$

Both sides of the fault release moment

Moment release is

$$M = 2M_1 = 2\left(\mu \frac{Ad}{2}\right) = \mu A d.$$

$$M = \boxed{\text{elastic Modulus}} \times \boxed{\text{Fault area}} \times \boxed{\text{Slip}}$$

All are easily estimated.

$$\boxed{M = \mu A d}$$

We don't need to know where P is located.