ESS 411/511 Geophysical Continuum Mechanics Class #11

Highlights from Class #10 – Brandon Lomahoynaya
Today's highlights on Monday – Zoe Krauss

For Monday class
- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.8

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas
- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

I hope those among you who are US citizens are planning to vote.
There is information at uw.edu/studentlife/vote

ESS 411/511 Geophysical Continuum Mechanics

## Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

# ESS 511 Term Projects

For those of you taking this class as ESS 511, it is time for you to give a 60-second outline of your ideas so far about your term topic.

- Andrew
- Barrett
- Maleen
- Zoe
- Madeleine L

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**Warm-up (break-out rooms)**

Class-prep questions from Wednesday

## Choosing eigenvectors

Suppose you have found the 3 eigenvalues $\lambda_{(q)}$ for 3x3 tensor $t_{ij}$, and then the 3 corresponding eigenvectors $\pm n_j^{(q)}$. The sign ambiguity can arise because the tensor projects an eigenvector parallel to itself, but the projection vector $t_{ij}\, n_j$ can point in either direction, i.e. either parallel to $n_j$ or antiparallel.

So the transformation matrix is

$$\begin{bmatrix} \pm n_1^{(1)} & \pm n_2^{(1)} & \pm n_3^{(1)} \\ \pm n_1^{(2)} & \pm n_2^{(2)} & \pm n_3^{(2)} \\ \pm n_1^{(3)} & \pm n_2^{(3)} & \pm n_3^{(3)} \end{bmatrix}$$

where each row is one of the eigenvectors.

- What criterion would you use to decide which sign combination of signs to use on each eigenvector?
- Suppose in a different problem, the 2nd and 3rd eigenvalues $\lambda_{(2)}$ and $\lambda_{(3)}$ were equal. You can still find the first eigenvector, but because of the nonuniqueness, you won't be able to find 2 other eigenvectors in the same way as before.
- What's going on? You know that the other 2 eigenvectors must be orthogonal to the first one. What can you do to complete the new basis set?

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**Class-prep questions from Wednesday (break-out rooms)**

$$(t_{11}-\lambda)\,n_1 + t_{12}n_2 + t_{13}n_3 = 0$$
$$t_{21}n_1 + (t_{22}-\lambda)\,n_2 + t_{23}n_3 = 0$$
$$t_{31}n_1 + t_{32}n_2 + (t_{33}-\lambda)\,n_3 = 0$$

If 2 eigenvalues are the same, either you will get 2 identlcal eigenvector solutions (not good because they must be orthogonal) or else there will be no clear solution. Then what?
Then you can choose ***any*** 2 orthogonal eigenvectors in the plane normal to the first eigenvector.

- What if all 3 eigenvalues are the same?

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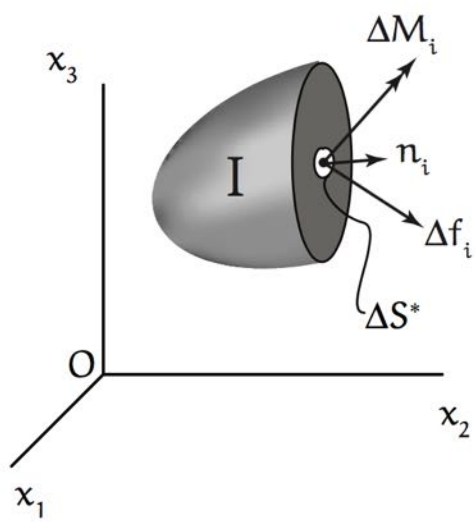
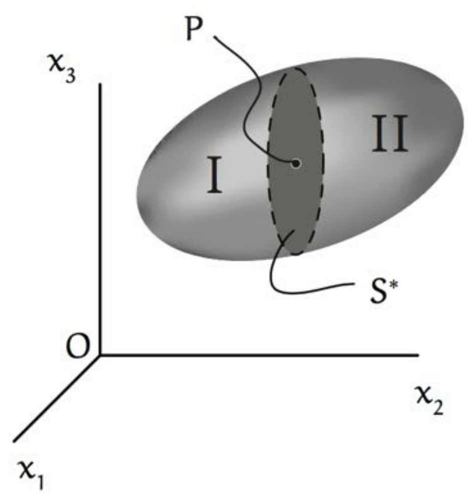
## Class-prep questions for today (break-out rooms)

**Tractions inside volumes**

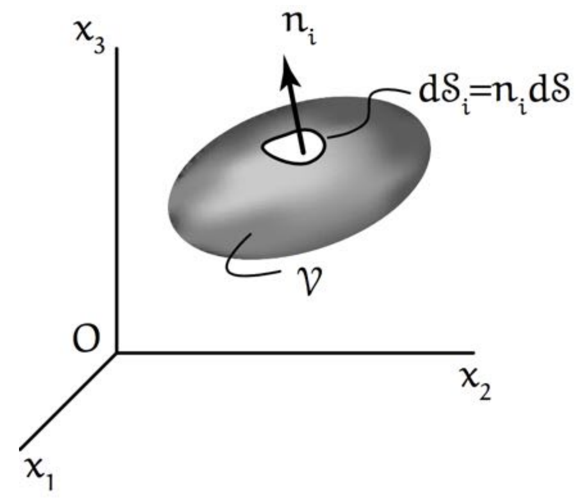
Imagine an arbitrary surface defined by its normal vector $n_i$ inside a continuum under a stress $\sigma_{ij}$. The stress vector or traction vector $t_j^{(n)}$ expresses the force exerted on a unit area of that plane.

1) How would you find the traction vector on plane $n_i$?
2) How can you define the other side (the "back side") of that plane?
3) What might you expect the traction vector to be on that "back side" of the plane $n_i$?
4) What conservation law have you probably invoked in your answer to 3)?



# Integral theorems




We may want to know what is going on inside a body but have access only to its surface (or vice versa)

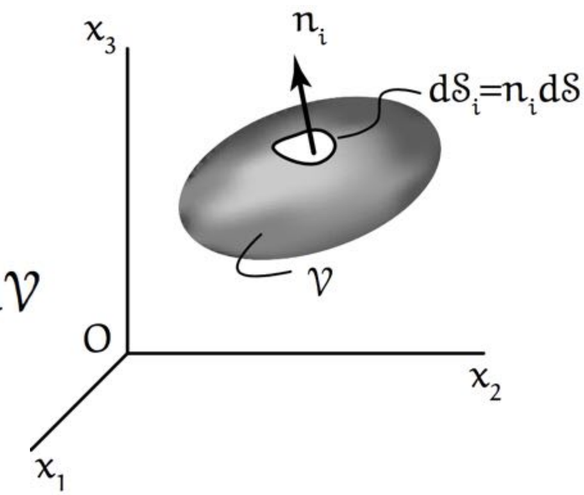
A volume $V$ has surface $S$.

- Each small patch $dS$ on the surface is defined by its normal vector $n_i$.

Divergence theorem $$\int_{\mathcal{S}} t_{ij\ldots k} n_q \, d\mathcal{S} = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$$

The total amount of $t_{ij\ldots k}$ directed out across $S$ is the same as the total amount of spreading (divergence) everywhere inside $V$.

# Special cases




Divergence theorem $\int_{\mathcal{S}} t_{ij\ldots k} n_q \, d\mathcal{S} = \int_{\mathcal{V}} t_{ij\ldots k,q} \, d\mathcal{V}$

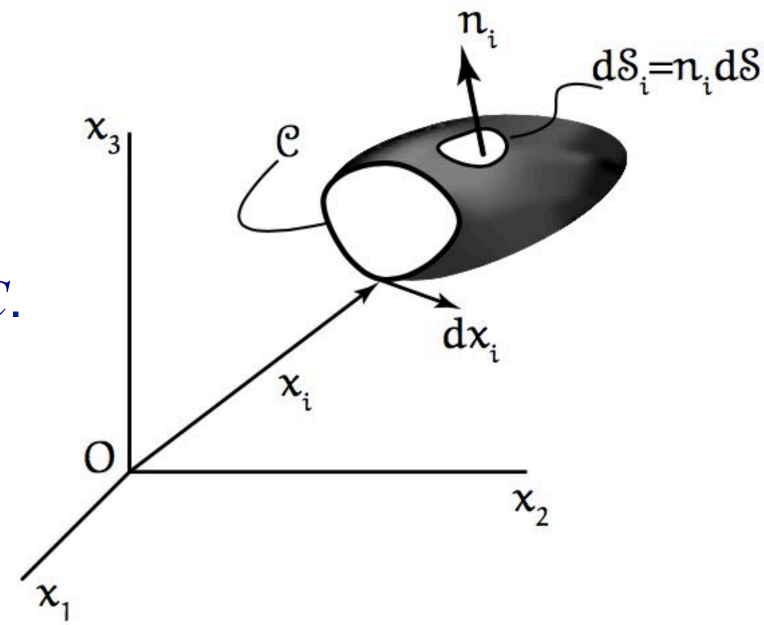
$$\int_{\mathcal{S}} v_q n_q \, d\mathcal{S} = \int_{\mathcal{V}} v_{q,q} \, d\mathcal{V} \quad \text{or} \quad \int_{\mathcal{S}} \boldsymbol{v} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \int_{\mathcal{V}} \operatorname{div} \boldsymbol{v} \, d\mathcal{V}$$

If density $\rho$ is uniform, the total amount of “stuff” flowing out across $S$ with velocity *v* (the flux across $S$) is the same as the total amount of spreading (divergence) of that “stuff” everywhere inside $V$.

# Stokes theorem

$C$ is the perimeter of a cap on an open surface.

- d***x*** is the tangent to the perimeter $C$.
- $\boldsymbol{v}$ is the material velocity.




$$\int_S \varepsilon_{ijk} n_i v_{k,j} \, dS = \int_{\mathcal{C}} v_k \, dx_k \quad \text{or} \quad \int_S \hat{\mathbf{n}} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) \, dS = \int_{\mathcal{C}} \boldsymbol{v} \cdot d\mathbf{x}$$

If density $\rho$ is uniform, the total circulation of "stuff" (curl) within the cap ("churning") is equal to the net flow along the perimeter $C$ ("the racetrack").

($\varepsilon_{ijk} v_{j,k}$ is curl of ***v***)

# Definition of a tensor

In any rectangular coordinate system, a tensor is defined by 9 components that transform according to the rule

$$R'_{ij\ldots k} = a_{iq} a_{jm} \cdots a_{kn} R_{qm\ldots n}$$

and where the basis vectors are related by

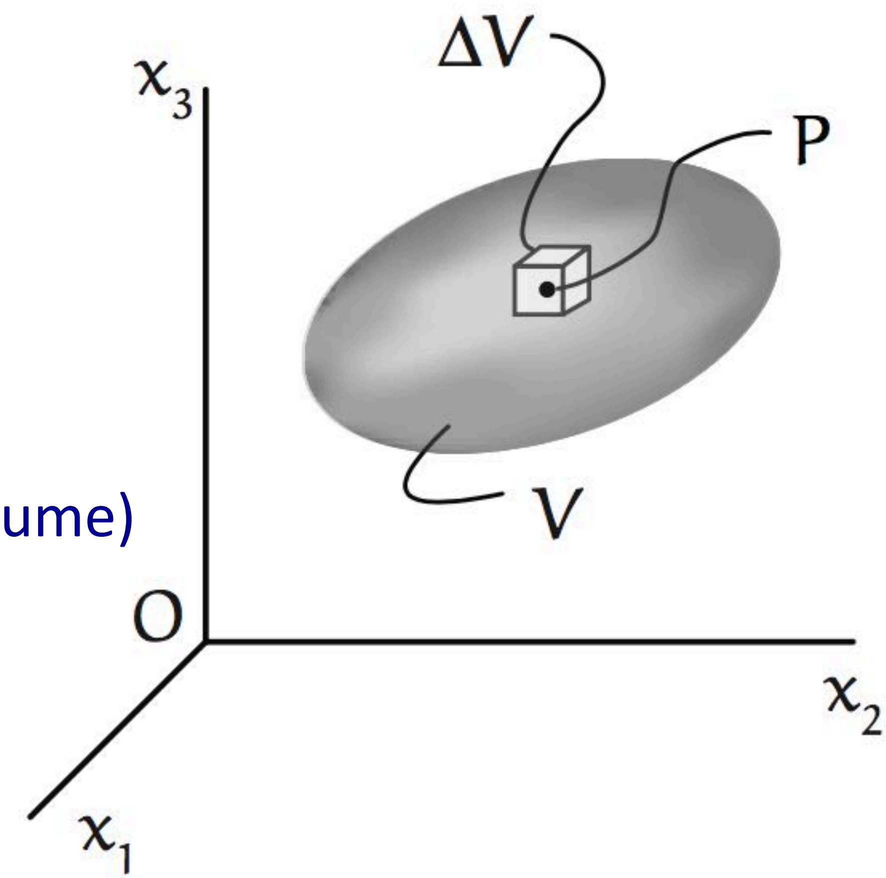
$$\hat{e}'_i = a_{ij} \hat{e}_j$$

# Density $\rho$ in a continuum

$\Delta m$ is the mass in a small volume $\Delta V$ around the point P.

$\rho(\boldsymbol{x})$ = density (mass per unit volume)

$$\rho = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta m}{\Delta \mathcal{V}} = \frac{dm}{d\mathcal{V}}$$



# Forces in a continuum

Body forces $b_i$ force per volume

Surface forces $t^{(n)}_i$ force per area or traction $\sigma_{ji}\, n_j$

(on exterior or interior surfaces)

Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$

In a continuum:

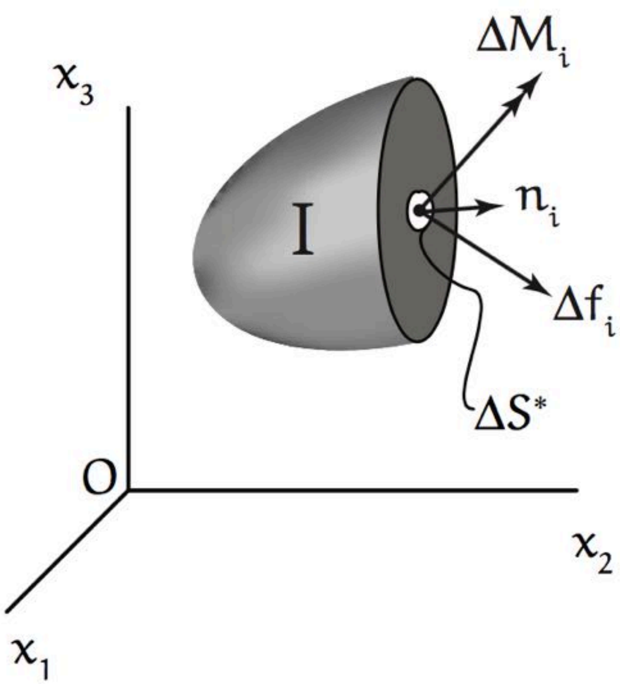
$$\int_V \rho(\vec{x}) b_i dV + \int_S t_i^{(\hat{n})} dS = \frac{d}{dt}\int_V \rho v_i dV$$

# Traction and torque

$f_i$ = force on $\Delta S^*$

$M_i$ = moment or torque on $\Delta S^*$

(force x lever arm from center of $\Delta S^*$)

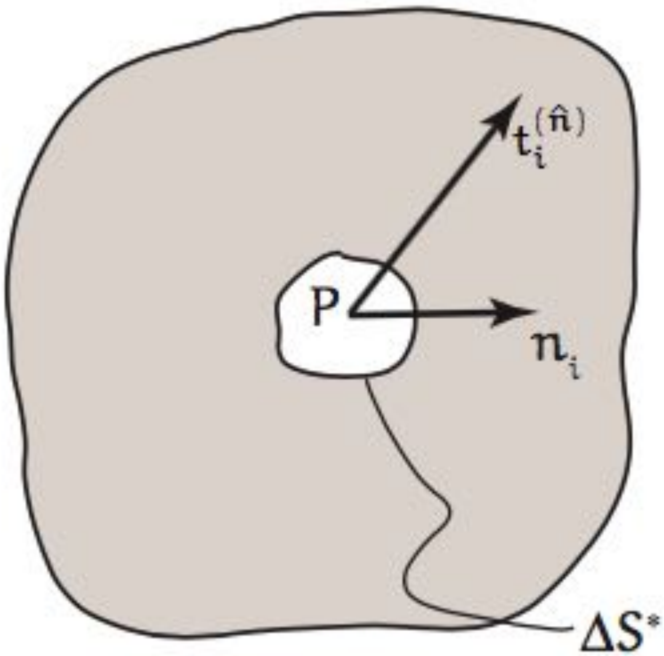



$$\lim_{\Delta S^* \to 0} \frac{\Delta f_i}{\Delta S^*} = \frac{df_i}{dS^*} = t_i^{(\hat{n})}$$

$$\lim_{\Delta S^* \to 0} \frac{\Delta M_i}{\Delta S^*} = 0 \,.$$

# Traction

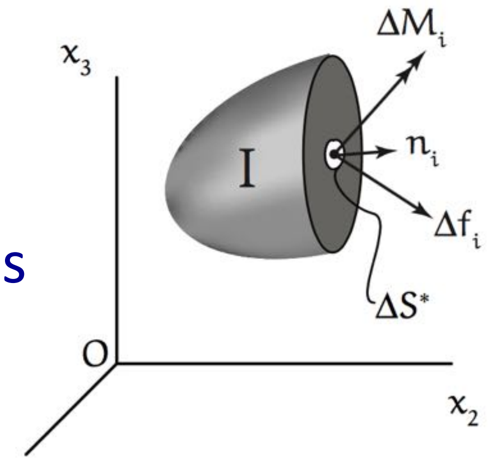
Traction vector acting at point P of plane element $\Delta S^*$ whose normal vector is $n_i$.



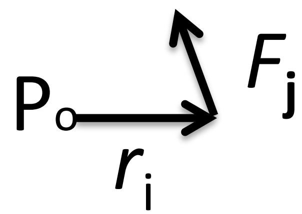
# Torque

Torque causes material to spin

$M_i$ = moment or torque on $\Delta S^*$ (force times lever arm from center of $\Delta S^*$)



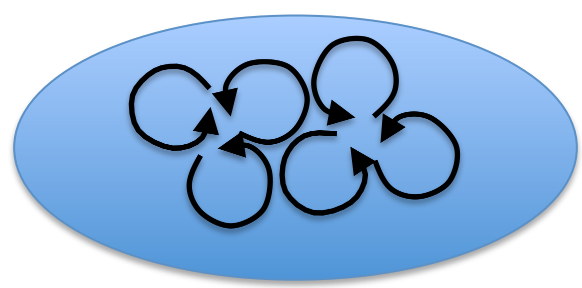

$$M_k = r_i \times F_j$$




$$\lim_{\Delta S^* \to 0} \frac{\Delta M_i}{\Delta S^*} = 0 \, .$$

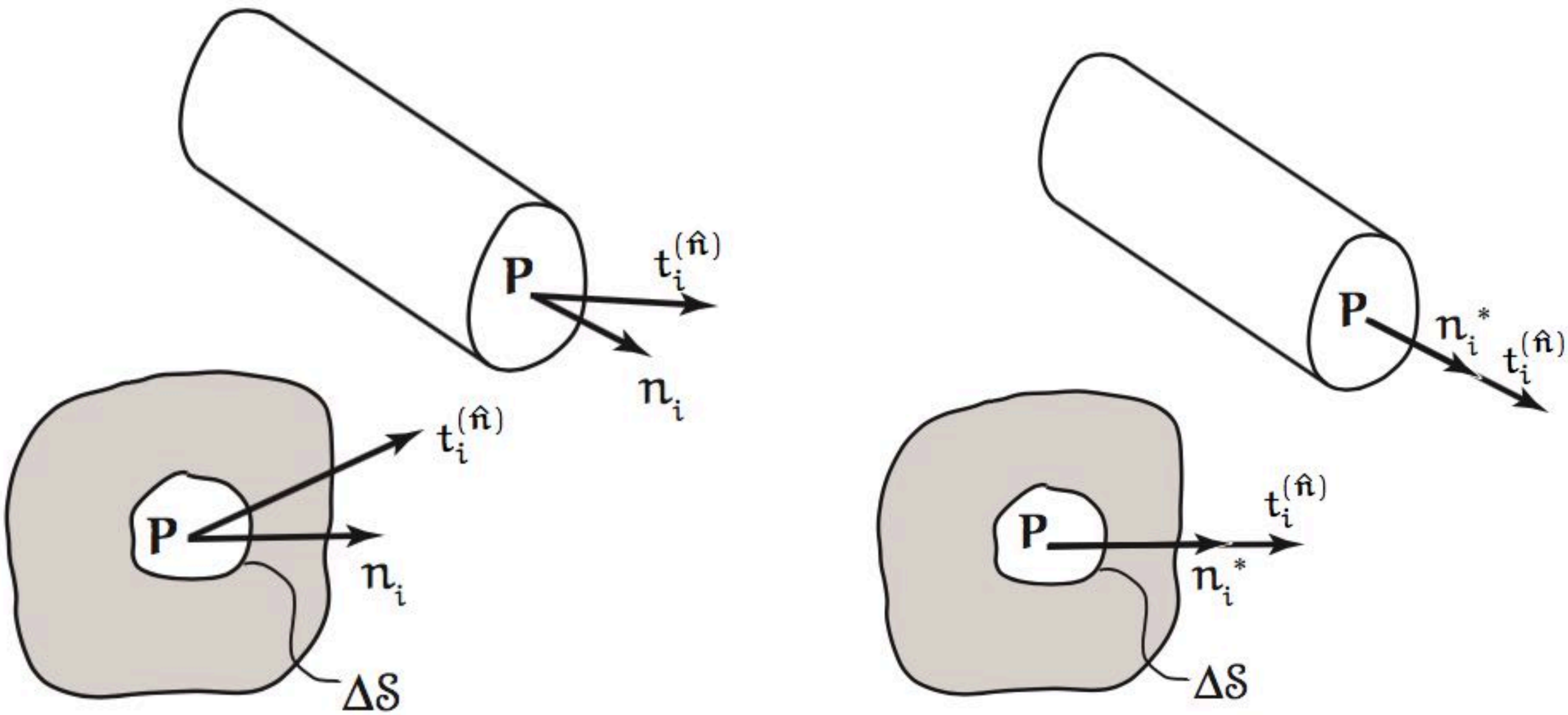
The torque or moment must go to zero because the lever arm must go to zero as $\Delta S^*$ gets smaller.

If it did not, the material would be churned and ripped apart internally …



(This is also why the stress tensor must be symmetric.)

A stress tensor has principal coordinates in which the shear stress vanish, so that the traction vector on the principal planes is parallel to the normal vector



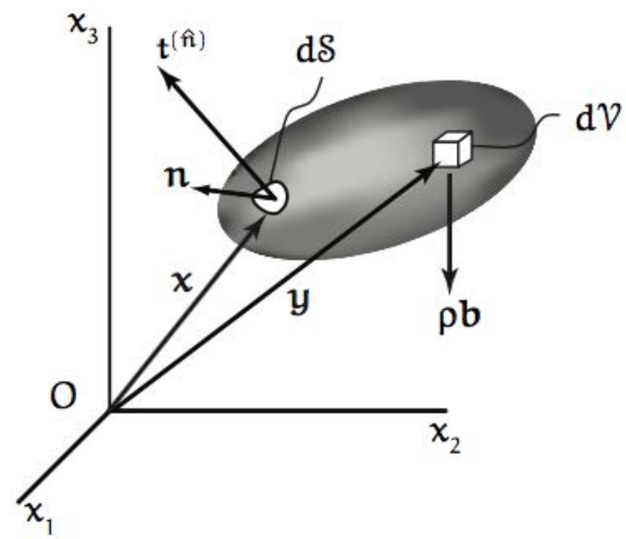

(a) Traction vector at point **P** for an arbitrary plane whose normal is $n_i$.

(b) Traction vector at point **P** for a principal plane whose normal is $n_i^*$.

# Momentum Conservation Equation

Force equilibrium

$$\int_{\mathcal{S}} t_i^{(\hat{n})} d\mathcal{S} + \int_{\mathcal{V}} \rho b_i d\mathcal{V} = 0 \qquad (1)$$




Divergence theorem for first term

$$\int_{\mathcal{S}} t_{ji} n_j d\mathcal{S} = \int_{\mathcal{V}} t_{ji,j} d\mathcal{V}$$

Substitute in (1)

$$\int_{\mathcal{V}} (t_{ji,j} + \rho b_i)\, d\mathcal{V} = 0$$

Volume *V* is arbitrary, so integrand must vanish for any *V*

$$t_{ji,j} + \rho b_i = 0$$