ESS 411/511 Geophysical Continuum Mechanics Class #22

Highlights from Class #21 – Xinyu Wan
Today's highlights on Monday – Barrett Johnson

Kinematics of Deformation and Motion

For Monday, please read MSM Chapter 4.7 and 4.8
- Infinitesimal strain
- Strain compatibility

Also check out 4.11 and 4.12
- Velocity gradient and strain rate
- Material derivatives of lines are, and volumes

Mid-term course evaluation

Due to major changes in teaching and learning resulting from Covid-19, the UW has set up an option for mid-term course evaluations in addition to the traditional end-of-Quarter evaluations.

I hope will all be able to give me feedback. Thanks!

Over half of you have responded so far, which is great. I hope that others among you to will also be able to give me feedback and suggestions.

The evaluation window will close tonight (Friday) at 11:59pm.

Here’s the link (only you who are enrolled in the class can use it).

https://uw.iasystem.org/survey/230856

Indoor Icequakes

I just received this week's edition of *Eos Buzz* from AGU, in which staff reporters publish highlights about papers coming out in AGU journals, such as

- *Journal of Geophysical Research (JGR)*
- *Geophysical Research Letters (GRL)*

https://eos.org/research-spotlights

Researchers at Penn State have reproduced stick/slip failure between ice and bedrock with σ_N = 500 MPa, like under a 500m thick glacier.
The peer-reviewed paper is in *GRL*, and is titled
Application of Constitutive Friction Laws to Glacier Seismicity

Here's the link:
https://doi.org/10.1029/2020GL088964

Problem Set #4

- I'm still working on it ☺

Mid-term

- I'm working on it next …

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Kinematics

- Description without reference to forces

Concept of particle in a continuum

- Just an infinitesimal point in the material, labeled with a vector field X

Displacement

- Vector mapping of an object from initial X to final configuration x

Deformation

- Change of shape described by a displacement field

Rigid-body rotation and translation

- No deformation, but displacement can differ from point to point

Strain or distortion

- Elongation or shear

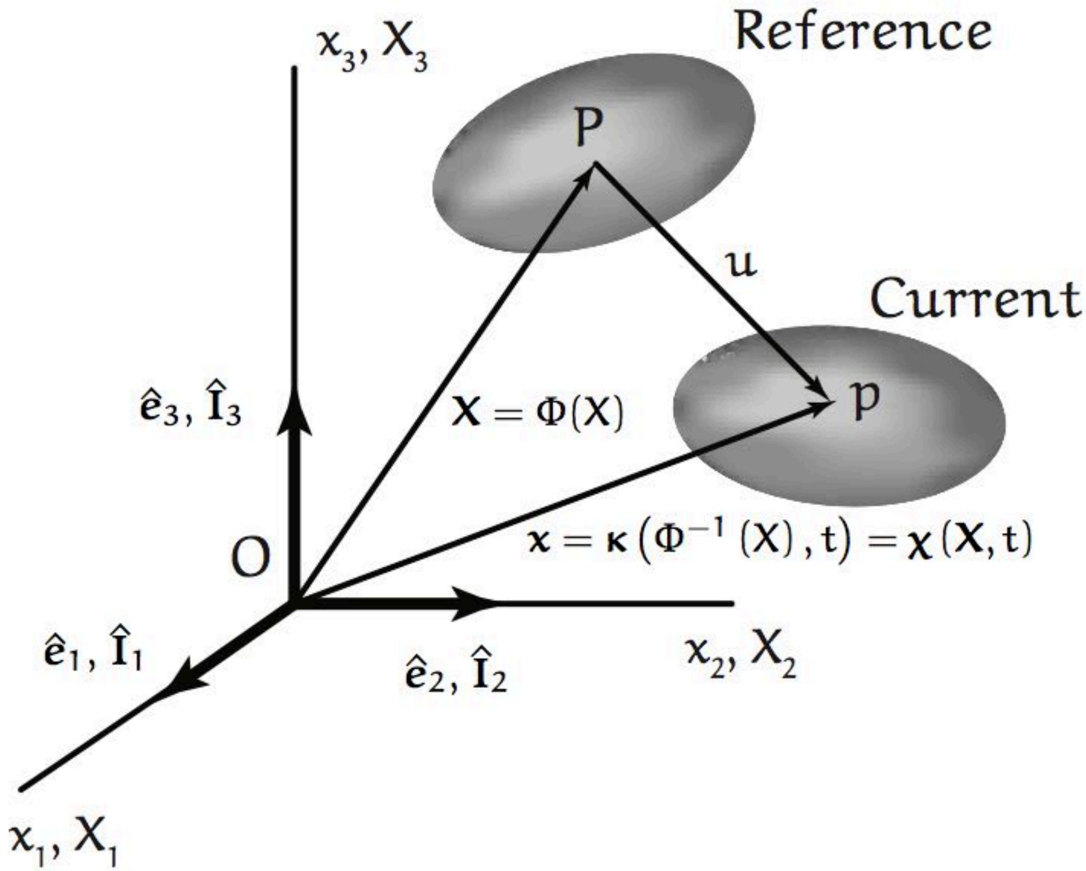
Homogeneous deformation

- Initially straight material lines stay straight

Finite strain

- Material lines can become curved

Initial and Final Configurations



Rates of change in a continuum

When the material is being tracked through time, it is convenient to use two sets of coordinates

Material

The material coordinates X_A are the initial positions of a material particle X in a coordinate system I_A

- Although particle X may move over time, the place X_A where it started from doesn't ever change.
- The coordinates X_A act as a ***label*** identifying particle X, wherever it goes.

Spatial

- The spatial coordinates $x_i(X,t)$ mark the current position of a material particle X in a coordinate system $\hat{e}_i$
  - Conversely, $X(x_i,t)$ indicates which particle X is occupying location x_i at time t.

Temporal Derivatives

As we saw with the traffic on I-5, there are two types of temporal derivatives of some quantity ϕ in a continuum.

- Rate of change of any property $\phi(x_i, t)$ at a fixed point x_i in space, can be written as

  $$\frac{\partial \phi(x_i, t)}{\partial t} \qquad (1)$$

  The partial derivative symbol ∂ indicates that position x_i is held constant.

- Rate of change of $\phi(X_A, t)$ for a particle X_A in the moving material, can be written as

  $$\frac{D\phi(X_A, t)}{Dt} \text{ or } \frac{d\phi(X_A, t)}{dt} \qquad (2)$$

  where "*D*" or "*d*" indicate a "total" or "material-following" derivative.

  The identity *A* of a particle isn't changing through time

  (Calvin and Hobbs transmogrification isn't allowed),

  So (2) is a function of a single variable *t*, and

  $$\frac{d\phi(X_A, t)}{dt} = \frac{\partial \phi(X_A, t)}{\partial t}$$

Material Derivatives

In the material coordinate system I_A, rate of change of ϕ for particle X_A as it moves along its trajectory is relatively simple:

$$\frac{d\phi(X_A,t)}{dt} = \frac{\partial\phi(X_A,t)}{\partial t}$$

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_i$

The rate of change of ϕ for the particle currently at x_i as it moves along its trajectory depends on two things:

1. The rate of change of ϕ seen by an observer at position x_i

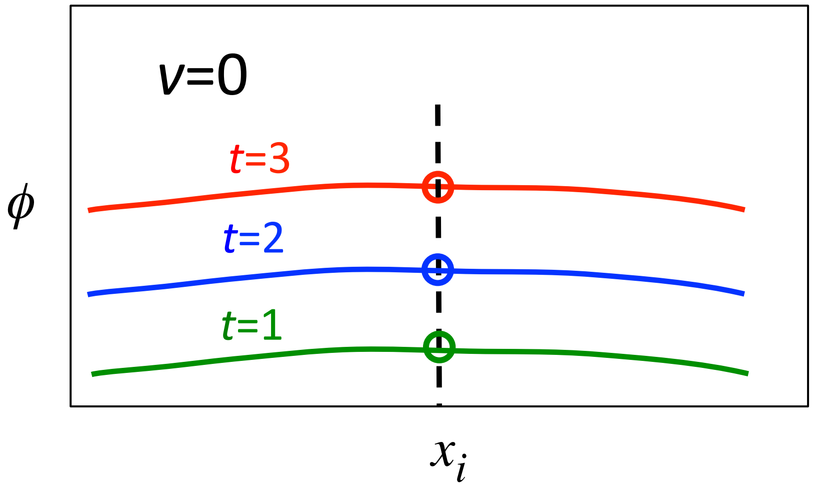
$$\frac{\partial\phi(x_i,t)}{\partial t}$$

2. The rate of change at which the flow *v* carries gradients of ϕ past position x_i, even though ϕ may not be changing on the particles

$$-\frac{\partial\phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}, \quad \frac{\partial x_k}{\partial t} = v_k$$

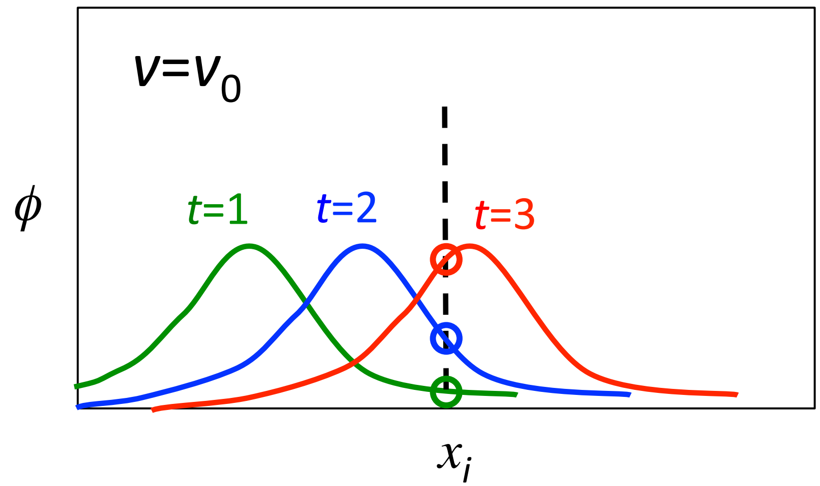
Ways to change ϕ at a point x_i

No motion
e.g. material warming in place



$$\frac{\partial \phi(X_A,t)}{\partial t} = \frac{\partial \phi(x_i,t)}{\partial t}$$

Motion uniform and constant
e.g. a seamount carried by ocean plate



$$\frac{\partial \phi(x_i,t)}{\partial t} = -\frac{\partial \phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}, \quad \frac{\partial x_k}{\partial t} = v_k = v_0$$

Putting it all together

However, it gets uglier if we want to express the material-following derivative in the fixed coordinate system $\hat{e}_i$

In the spatial coordinate system $\hat{e}_i$, rate of change of ϕ for a particle X_A as it passes through x_i :

$$\frac{d\phi(x_i,t)}{dt} = \frac{\partial\phi(x_i,t)}{\partial t} + \frac{\partial\phi(x_i,t)}{\partial x_k}\frac{\partial x_k}{\partial t}$$

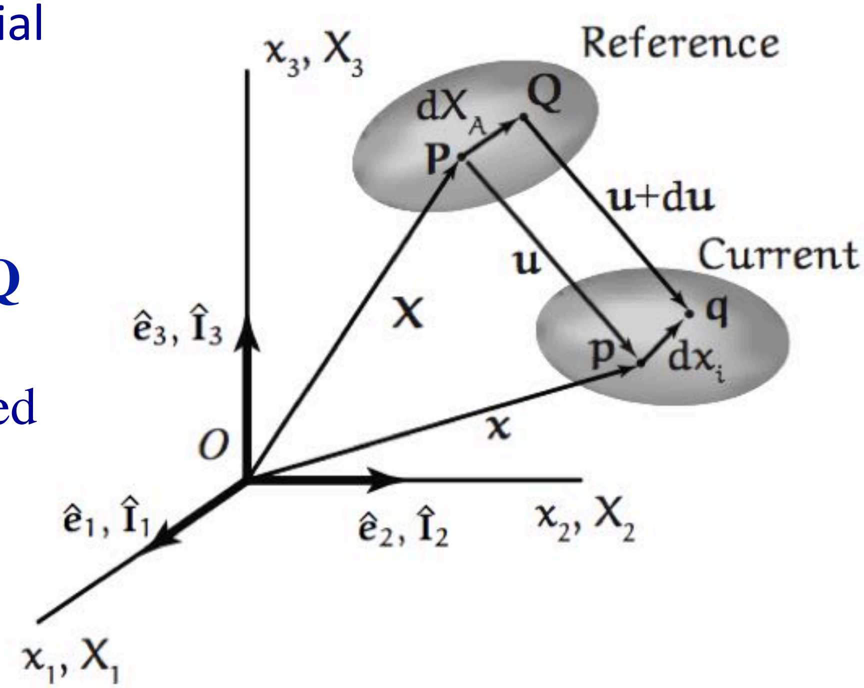
Rate of change seen at x_i

Correction for changes carried in by flow, without ϕ actually changing on the particles

For example, for the seamount, the two terms must cancel each other, because we know that ϕ, the topography of the seamount, is not changing. Other situations can be more complicated … ☺

Displacement and Finite Strain

Any two nearby points **P** and **Q** in the initial configuration are moved to **p** and **q** in the final configuration.
The displacement of point **P** is $\boldsymbol{u}_P = \mathbf{p} - \mathbf{P}$
The displacement of point **Q** is $\boldsymbol{u}_Q = \mathbf{q} - \mathbf{Q}$
Or in general, $\boldsymbol{u} = \boldsymbol{x} - \mathbf{X}$
Because **Q** is close to **P**, $\boldsymbol{u}$ can be expanded as a Taylor series around the point **P**. Here are the first-order terms:



$$u_Q = u_P + \frac{\partial u_i}{\partial X_A} dX_A$$

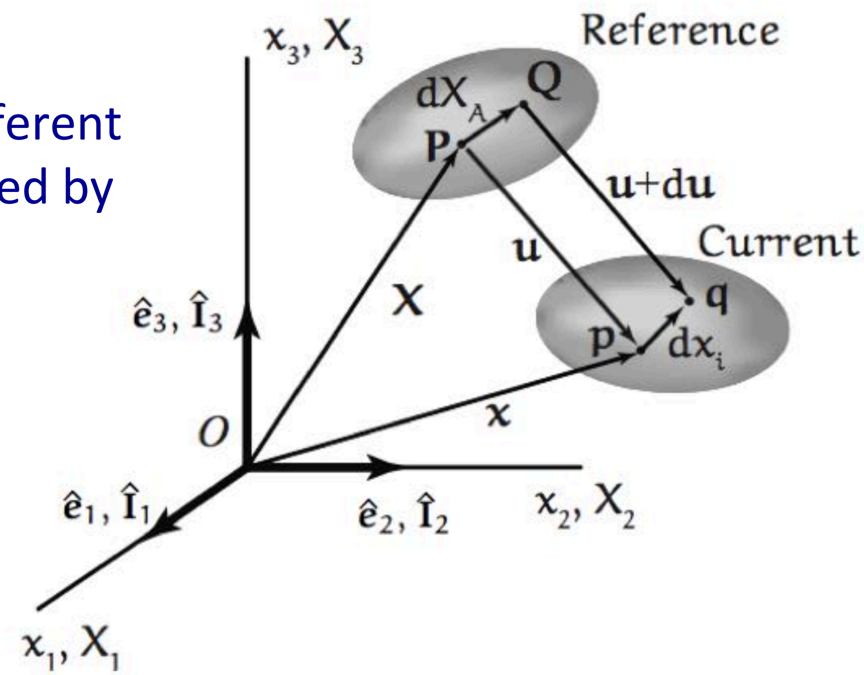
This can be arranged to find d$\boldsymbol{u}$

$$du = \left(u_Q - u_P\right) = \frac{\partial u_i}{\partial X_A} dX_A$$

The small line element dX_A in the initial configuration also gets deformed into a different line element dx_i.

Displacement and Finite Strain

The small line element dX_A in the initial configuration also gets deformed into a different line element dx_i, which can also be expressed by the first-order terms of a Taylor series



$$dx_i = \frac{\partial x_i}{\partial X_A} dX_A$$

The derivatives form the *deformation gradient tensor F_{iA}*

$$F_{iA} = \frac{\partial x_i}{\partial X_A} = x_{i,A}$$

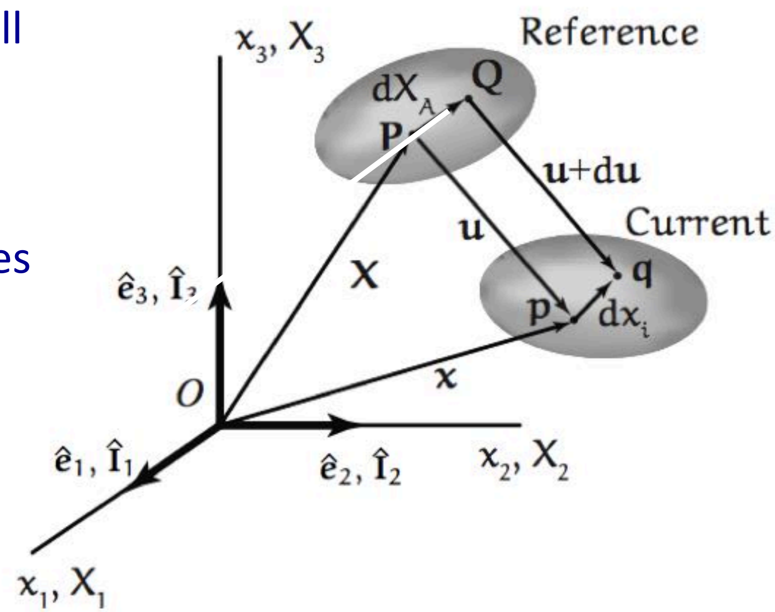
The deformation is reversible, so F_{iA} has an inverse

$$\left(F_{iA}\right)^{-1} = \frac{\partial X_A}{\partial x_i} = X_{A,i}$$

Class-prep: deformation tensor F_{iA} (Break-out rooms)

Figure 4.2 in text MSM shows that an initial small line element dX_A between points **P** and **Q** in a body becomes a small line element dx_i between points **p** and **q** after deformation.
The deformation gradient tensor F_{iA} characterizes the deformation in the vicinity of P and Q by relating dx_i to dX_A, e.g. as expressed in Equation 4.39

$$dx_i = \frac{\partial x_i}{\partial X_A} dX_A = x_{i,A} X_A = F_{iA} dX_A$$



Assignment

Write the general form of the tensor F_{iA} in the equation above as a 3x3 matrix.
Find the 3x3 matrix F_{iA} for the particular deformation field defined by

$x_1 = X_1$
$x_2 = 2\,X_3$
$x_3 = -1/2\,X_2$

Find the vector dx_i resulting from deformation of the column vector

$dX_A = [1,1,1]^T$

and comment on the results in terms of rotation and stretching.

Reference line element dX_A

$$dX_A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Mapping

$$\left\{ \begin{array}{c} x_1 = X_1 \\ x_2 = 2X_3 \\ x_3 = -\frac{1}{2}X_1 \end{array} \right\}$$

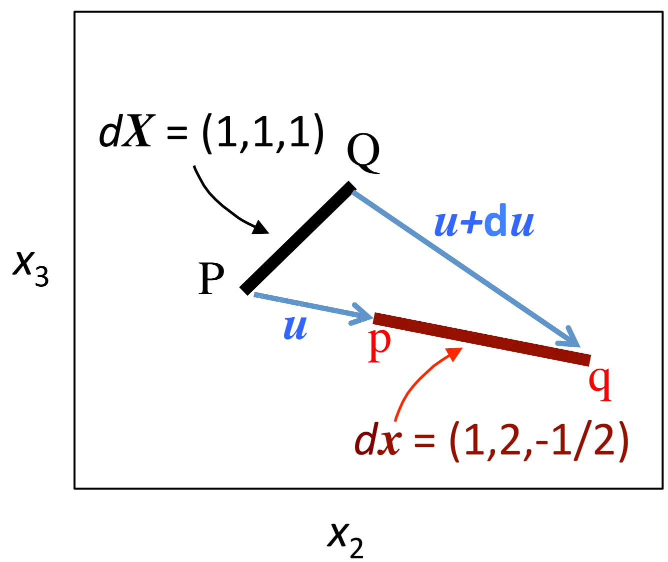
Deformation gradient tensor

$$[F_{iA}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

Finding current line element dx_i

$$F_{iA}dX_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

Luckily for us, there is no change in the X_1 direction ☺



A measure for strain $(dx)^2 - (dX)^2$

$$
\begin{aligned}
(dx)^2 - (dX)^2 &= (x_{i,A}\, dX_A)(x_{i,B}\, dX_B) - \delta_{AB}\, dX_A\, dX_B \\
&= (x_{i,A}x_{i,B} - \delta_{AB})\, dX_A\, dX_B \\
&= (C_{AB} - \delta_{AB})\, dX_A\, dX_B
\end{aligned}
$$

Green's deformation tensor

$$C_{AB} = x_{i,A}x_{i,B} \qquad \text{or} \qquad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

Lagrangian finite strain tensor

$$2E_{AB} = C_{AB} - \delta_{AB} \qquad \text{or} \qquad 2\mathbf{E} = \mathbf{C} - \mathbf{I}$$

A measure for strain $(dx)^2 - (dX)^2$

$$
\begin{aligned}
(dx)^2 - (dX)^2 &= \delta_{ij}\, dx_i\, dx_j - (X_{A,i}\, dx_i)(X_{A,j}\, dx_j) \\
&= (\delta_{ij} - X_{A,i} X_{A,j})\, dx_i\, dx_j \\
&= (\delta_{ij} - c_{ij})\, dx_i\, dx_j
\end{aligned}
$$

Cauchy deformation tensor

$$c_{ij} = X_{A,i} X_{A,j} \qquad \text{or} \qquad \mathbf{c} = (\mathbf{F}^{-1})^{\mathsf{T}} \cdot (\mathbf{F}^{-1})$$

Eulerian finite strain tensor

$$2e_{ij} = (\delta_{ij} - c_{ij}) \qquad \text{or} \qquad 2\mathbf{e} = (\mathbf{I} - \mathbf{c})$$

In terms of displacements $u_i = (x_i - X_i)$

$$2E_{AB} = x_{i,A}x_{i,B} - \delta_{AB} = (u_{i,A} + \delta_{iA})(u_{i,B} + \delta_{iB}) - \delta_{AB}$$

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A}u_{i,B}$$

$$2e_{ij} = \delta_{ij} - X_{A,i}X_{A,j} = \delta_{ij} - (\delta_{Ai} - u_{A,i})(\delta_{Aj} - u_{A,j})$$

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{A,i}u_{A,j}$$