ESS 411/511 Geophysical Continuum Mechanics Class #12

Highlights from Class #11 – Zoe Krauss
Today's highlights on Wednesday – Xinyu Wan

For Wednesday class

- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.8

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

ESS 411/511 Geophysical Continuum Mechanics Class #12

Class-prep questions for today (break-out rooms)

Traction vectors on principal planes, Mohr's circles
The traction vector *tj(n)* expresses the force per unit area of the plane defined by n_i. Section 3.7 explains how to resolve the traction vector into its components σ_N and σ_S normal and parallel to the plane respectively, and shows how to find the maximum and minimum values of σ_N and σ_S.
With respect to faults at km scales, or layered rocks such as shales at the meter describe how you think this might be useful.

Section 3.8 explains how to represent the state of stress at a point with Mohr's circles.
How do you think the Mohr's circle representation of stress t_{ij}in terms of σ_N and σ_S might give potentially more insight than just the stress tensor t_{ij} itself?

Definition of a tensor

In any rectangular coordinate system, a tensor is defined by 9 components that transform according to the rule

$$R'_{ij\ldots k} = a_{iq} a_{jm} \cdots a_{kn} R_{qm\ldots n}$$

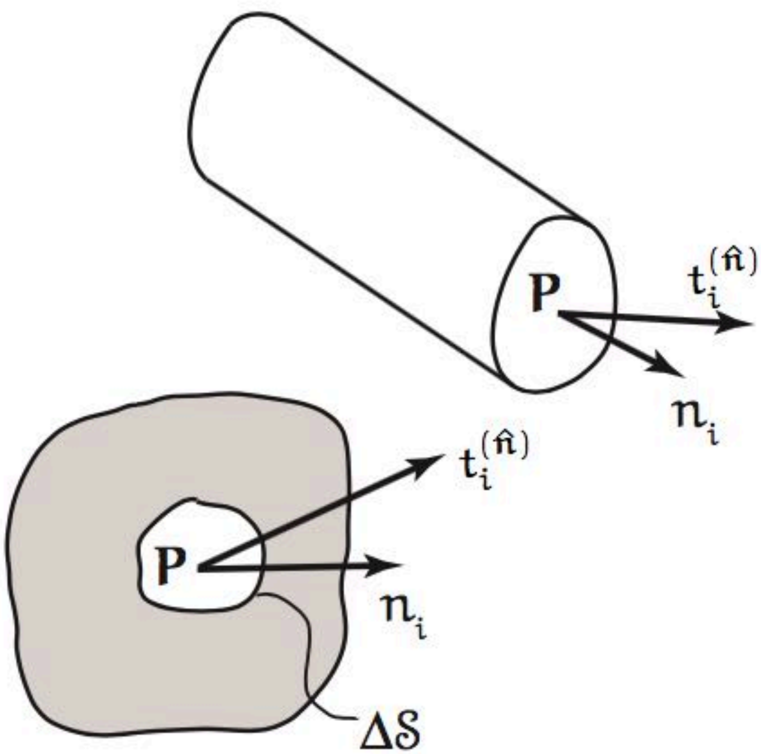
and where the basis vectors are related by

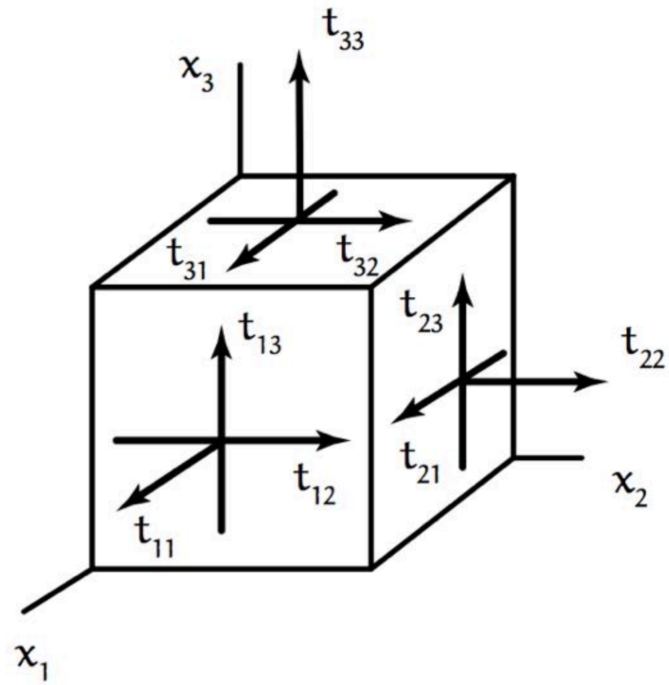
$$\hat{e}'_i = a_{ij} \hat{e}_j$$

In Section 3.3 Equations (3.19) (-3.22) MSM show that stress transforms into new coordinate systems following this rule, and so stress is a tensor.

If a stress tensor ***is not*** expressed in its principal coordinates

- the traction vectors $\hat{t}_i^{(\hat{n})}$ on the coordinate planes may ***not*** be parallel to the coordinate vectors t_{11}^*, and
- off-diagonal elements of t_{ij} may be nonzero.

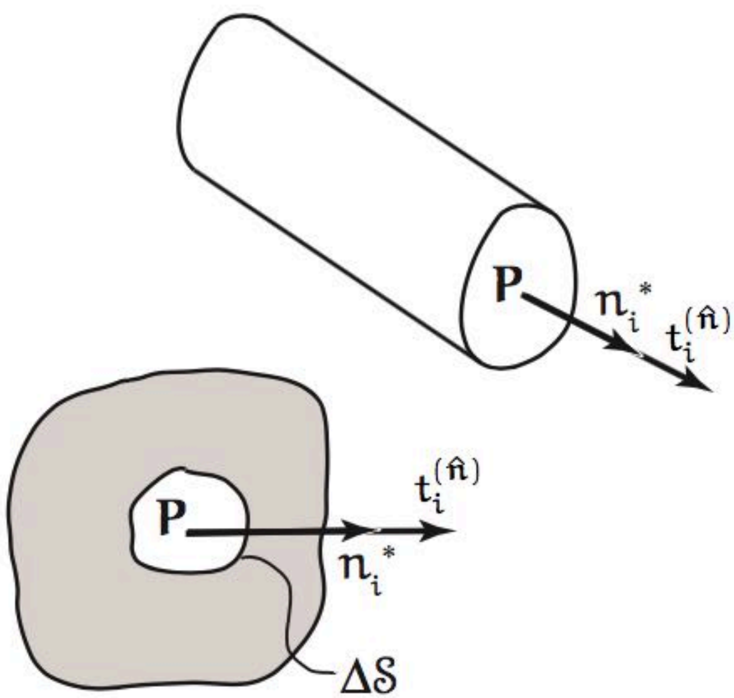


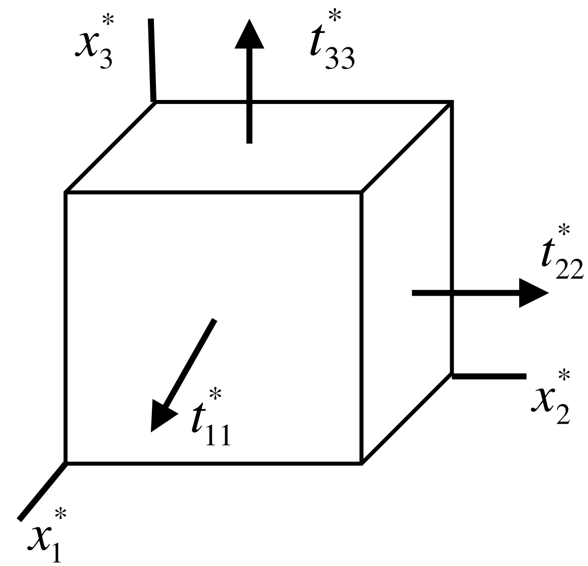


$$\begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

If a stress tensor ***is*** expressed in its principal coordinates

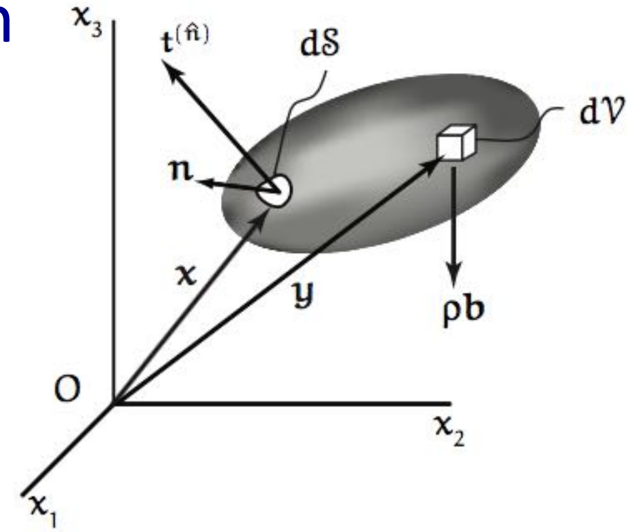
- the traction vectors $\hat{t}_i^{(\hat{n})}$ on the coordinate planes ***must*** be parallel to the coordinate vectors $\hat{n}_i^*$, and
- off-diagonal elements t^*_{ij} must be zero.





$$\begin{bmatrix} t_{11}^* & 0 & 0 \\ 0 & t_{22}^* & 0 \\ 0 & 0 & t_{33}^* \end{bmatrix}$$

Momentum Conservation Equation



Force equilibrium

$$\int_{\mathcal{S}} t_i^{(\hat{n})} d\mathcal{S} + \int_{\mathcal{V}} \rho b_i d\mathcal{V} = 0 \qquad (1)$$

Replace traction vector with stress tensor

$$t_i^{(\hat{n})} = t_{ji} n_j \qquad (2)$$

Apply Divergence Theorem (3) to change surface integral into volume integral

$$\int_{\mathcal{S}} t_{ji} n_j d\mathcal{S} = \int_{\mathcal{V}} t_{ji,j} d\mathcal{V} \qquad (3)$$

Collect volume terms

$$\int_{\mathcal{V}} (t_{ji,j} + \rho b_i) \, d\mathcal{V} = 0$$

Volume *V* is arbitrary, so integrand must vanish for any *V*

$$t_{ji,j} + \rho b_i = 0$$

Symmetry of stress tensor

As discussed in Class 11 on Friday, the stress tensor is symmetric because the moment on an infinitesimal surface element d*S* must go to zero.

The text derives this more rigorously in Section 3.4.

The message to remember is that for stress -

$$t_{ij} = t_{ji}$$

The stress tensor can be reflected across its diagonal without changing.

Transformation laws for stress tensor

Section 3.5 goes over how to express the stress tensor (t_{ij} or σ_{ij}) in different coordinate systems.
This is mainly a repeat of earlier ideas in Chapter 2 about transforming any tensor.

Section 3.6 goes over how to find the principal values (eigenvalues) of the stress tensor, how to find the principal directions (eigenvectors), and how to find the 3 scalar invariants of the stress tensor.
This is also mainly a repeat of earlier ideas in Chapter 2 about transforming any tensor.

Notation

Principal stresses $\sigma_{I} > \sigma_{II} > \sigma_{III}$

$$[t^*_{ij}] = \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & \sigma_{(3)} \end{bmatrix} \quad \text{or} \quad [t^*_{ij}] = \begin{bmatrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

Conventions:

- Compressive stresses are negative
- Principal stresses are numbered from largest (most positive) to smallest
- Other conventions are also used in other texts and in research literature, but this convention is most versatile and correct in all situations

Notation

Scalar Invariants of the stress tensor

These are the coefficients in the cubic characteristic equation when solving for the eigenvalues

$$|t_{ij} - \lambda\delta_{ij}| = 0$$

$$\lambda^3 - I_{\mathbf{T}}\lambda^2 + II_{\mathbf{T}}\lambda - III_{\mathbf{T}} = 0$$

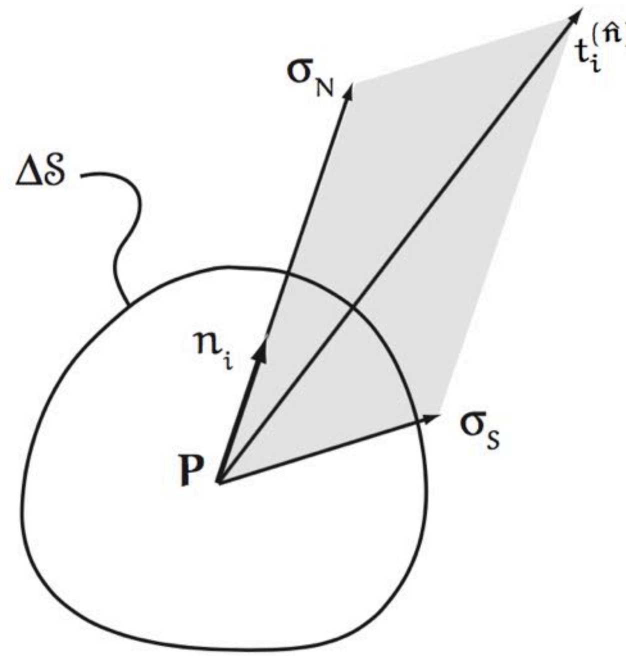
Trace $\quad I_{\mathbf{T}} = t_{ii} = \mathrm{tr}\,\mathbf{T}\,,$

Second invariant $\quad II_{\mathbf{T}} = \frac{1}{2}[t_{ii}t_{jj} - t_{ij}t_{ij}] = \frac{1}{2}\left[(\mathrm{tr}\,\mathbf{T})^2 - \mathrm{tr}\,\mathbf{T}^2\right]$

Determinant $\quad III_{\mathbf{T}} = \varepsilon_{ijk}t_{1i}t_{2j}t_{3k} = \det\mathbf{T}\,.$

Section 3.7 – Normal and shear traction on a plane

On any of the infinite number of plane elements ΔS at P, the traction vector $t_i^{(\hat{n})}$ can be resolved into components σ_N normal to the plane, and σ_S in the plane.
(σ_N and σ_S are just scalar magnitudes.)



Take the dot product to find σ_N as the projection of $t_i^{(\hat{n})}$ onto n_i.

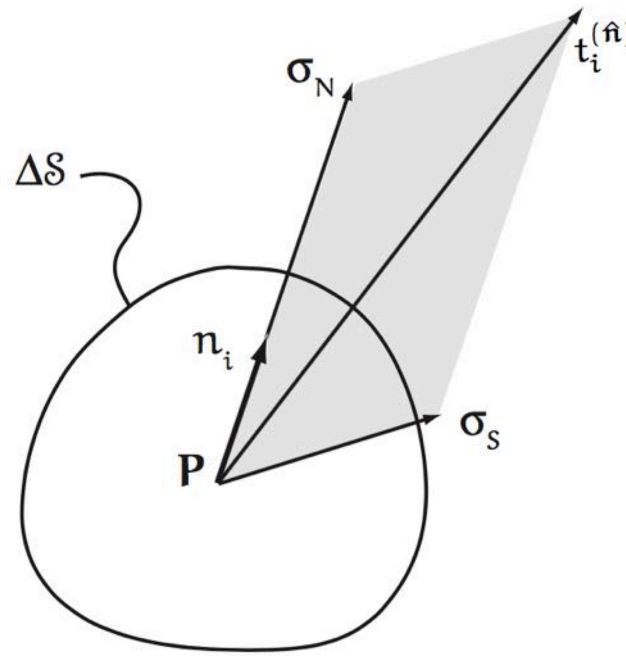
$$\sigma_N = t_i^{(\hat{n})} n_i = t_{ij} n_j n_i$$

Then use the Pythagorean theorem to find σ_S

$$\sigma_S^2 = t_i^{(\hat{n})} t_i^{(\hat{n})} - \sigma_N^2$$

Section 3.7 – Minimum and maximum stress values

What are the largest and smallest values that σ_N and σ_s can take at P when considering all possible planes through P?



Take the dot product to find σ_N

$$\sigma_N = t_i^{(\hat{n})} n_i = t_{ij} n_j n_i$$

Then use the Pythagorean theorem to find σ_S

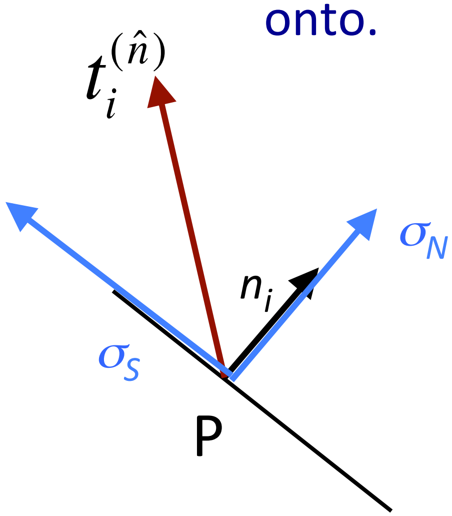
$$\sigma_S^2 = t_i^{(\hat{n})} t_i^{(\hat{n})} - \sigma_N^2$$

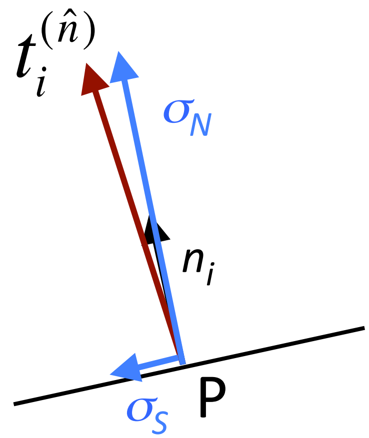
Then find the directions n_i where σ_N and σ_s have extrema

$$\frac{\partial \sigma_N}{\partial n_i} = 0 \quad \text{for } i=1,2,3 \qquad\qquad \frac{\partial \sigma_S}{\partial n_i} = 0 \quad \text{for } i=1,2,3$$

Section 3.7 – Minimum and maximum stress values

σ_N and σ_S can vary dramatically depending on which plane the traction vector $t_i^{(\hat{n})}$ is resolved onto.



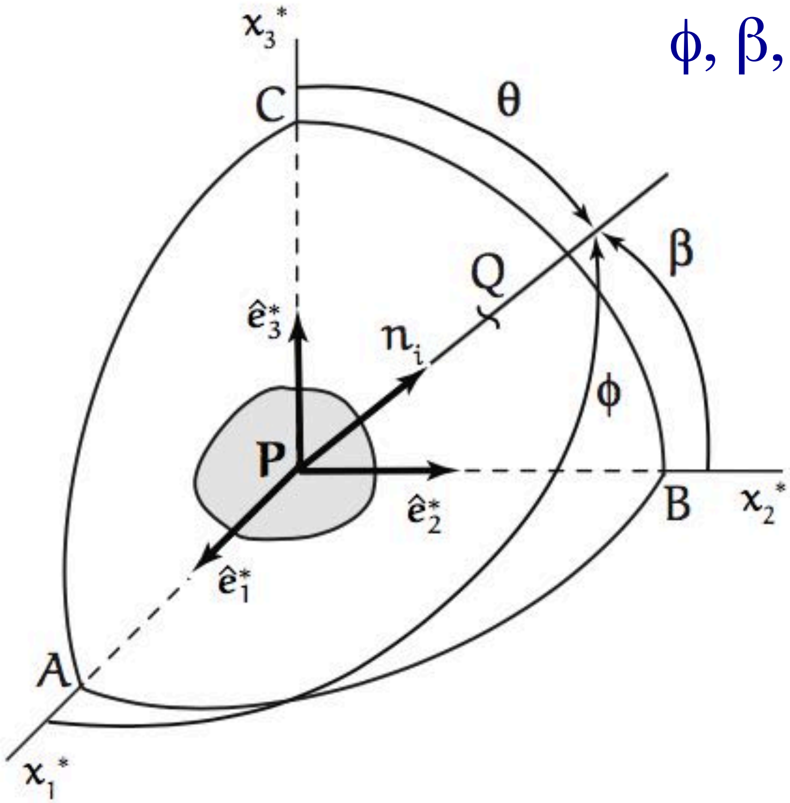


Cartesian Space vs Stress Space

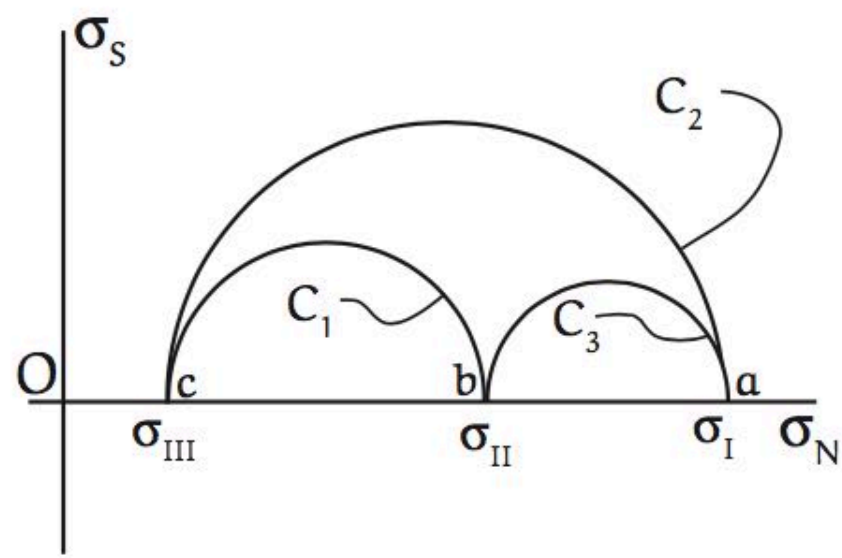
$\hat{e}_i^*$ are principal directions defining principal planes at **P**.
Lower-case letters in stress space correspond to upper-case letters in Cartesian space.

End point **Q** of unit vector n_i can fall anywhere on unit sphere centered at **P**.
ϕ, β, θ relate Q to coordinate axes.

The 3 circles correspond to Q lying in a principal plane.



(a) Octant of small spherical portion of body together with plane at **P** with normal n_i referred to principal axes $Ox_1^*x_2^*x_3^*$.



(b) Mohr's stress semicircle for octant of Fig. 3.14(a).

Cartesian Space *vs* Stress Space

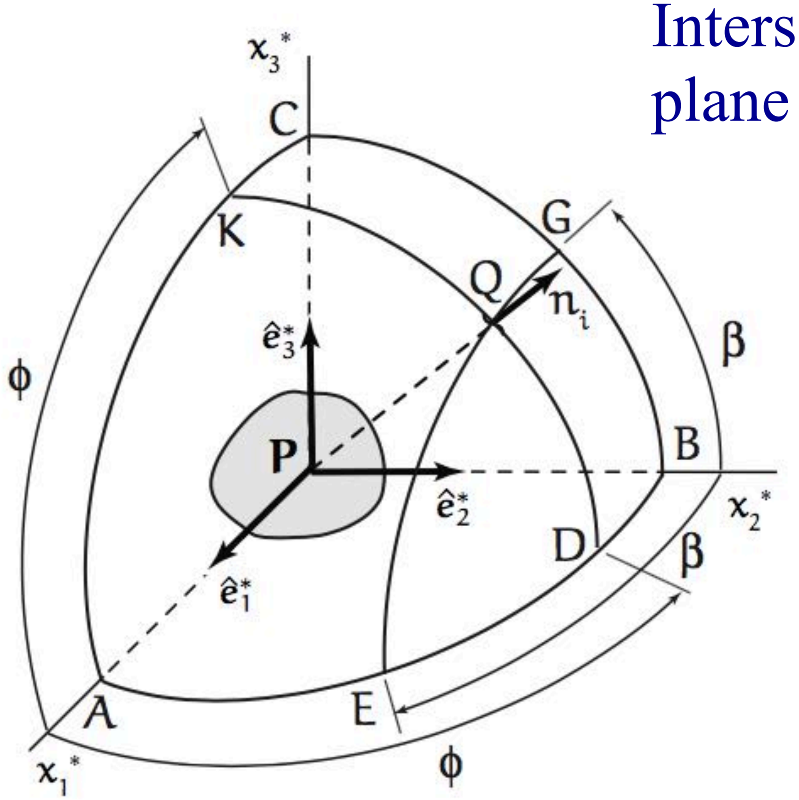
$\hat{e}_i^*$ are principal directions defining principal planes at **P**.

Small circles in Cartesian space (e.g. EQG) map onto circles (e.g. eqg) concentric with primary Mohr's circles (e.g. akc).

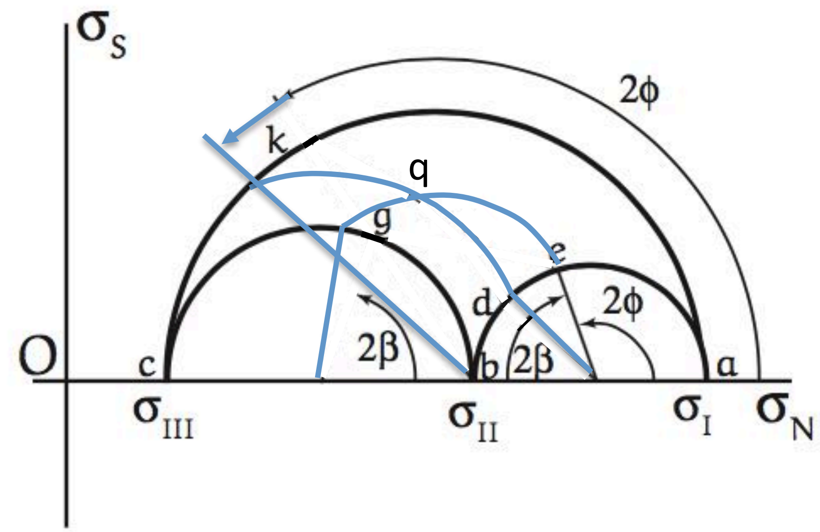
Similarly, KQD maps onto kqd.

Intersection at q shows σ_N and σ_s on plane defined by normal vector n_i at Q.

(I have attempted to (sort of) correct the stress-plane figure below. ☺)



(a) Reference angles ϕ and β for intersection point Q on surface of body octant.



(b) Mohr's stress semicircle for octant of Fig. 3.15(a).