ESS 411/511 Geophysical Continuum Mechanics
Strain Homework

Mase and Mase, Problems 4.5, 4.10, 4.18, and 4.23.
These problems have the same numbers in the 3rd edition Mase, Smelser, and Mase.
Because the answers are given in the text, it is really important that you describe clearly
how you reach your answers before I can give you credit ☺.

1. (MSM 4.5)
The Lagrangian description of a continuum motion is given by

$x_1 = X_1e^{-t} + X_3(e^{-t} - 1)$,

$x_2 = X_2e^{t} - X_3(1 - e^{-t})$,

$x_3 = X_3e^{t}$.

Show that these equations are invertible, and determine the Eulerian description of the
motion.

Answer

$X_1 = x_1e^{t} - x_3(e^{t} - 1)$, $X_2 = x_2e^{-t} + x_3(e^{-2t} - e^{-3t})$, $X_3 = x_3e^{-t}$

2. (MSM 4.10)
A displacement field is given in terms of the spatial variables and time by the equations

$u_1 = x_2t^2,$ $u_2 = x_3t,$ $u_3 = x_1t$.

Using the (spatial) material derivative operator, determine the velocity components.

Answer

$v_1 = (2x_2t + x_3t^2 + x_1t^3)/(1 - t^4)$

$v_2 = (x_3 + x_1t + 2x_2t^3)/(1 - t^4)$

$v_3 = (x_1 + 2x_2t^2 + x_3t^3)/(1 - t^4)$

3. (MSM 4.18)
Given the deformation expressed by

$x_1 = X_1 + AX_2^2,$

$x_2 = X_2,$

$x_3 = X_3 - AX_2^2,$

where *A* is a constant (not necessarily small), determine the finite strain tensors **E** and **e**,
and show that if the displacements are small so that $x \approx X$ and if squares of *A* may be
neglected, both tensors reduce to the infinitesimal strain tensor **ε**.

Answer

$$[\epsilon_{ij}] = \begin{bmatrix} 0 & Ax_2 & 0 \\ Ax_2 & 0 & -Ax_2 \\ 0 & -Ax_2 & 0 \end{bmatrix}$$

4. (MSM 4.23)

Given the displacement field
$$u_1 = AX_2X_3, \quad u_2 = AX_3^2, \quad u_3 = AX_1^2,$$
where A is a very small constant, determine
(a) the components of the infinitesimal strain tensor ε, and the infinitesimal rotation tensor ω.
(b) the principal values of ε, at the point (1, 1, 0).

Answer
(a) $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$, $\varepsilon_{12} = 1/2\ AX_3$, $\varepsilon_{13} = A(X_2+2X_1)$, and $\varepsilon_{23} = AX_3$

   $\omega_{11} = \omega_{22} = \omega_{33} = 0$, $\quad \omega_{12} = -\omega_{21} = AX_3$,

   $\omega_{\square\blacktriangleleft} = -\omega_{\blacktriangleleft\square} = 1/2\ AX_\blacksquare - AX_\square$, and $\omega_{\blacksquare\blacktriangleleft} = -\omega_{\blacktriangleleft\blacksquare} = AX_\blacktriangleleft$
(b) $\varepsilon_{(I)} = 3/2\ A$, $\quad \varepsilon_{(II)} = 0$, $\varepsilon_{(III)} = -3/2\ A$

Note that in M, S, and M there are 2 typos in the answer to 4.23 (a).
In the first line, the correct answer should be $\varepsilon_{12} = 1/2\ AX_3$
and the last line should read $\omega_{\square\blacktriangleleft} = -\omega_{\blacktriangleleft\square} = 1/2\ AX_\blacksquare - AX_\square$ and $\omega_{\blacksquare\blacktriangleleft} = -\omega_{\blacktriangleleft\blacksquare} = AX_\blacktriangleleft$

5. The Material Derivative and Ocean Temperatures
(i) *Moored and Drifting Buoys*
    Imagine a (fictitious) region of the ocean between a north-south trending ice shelf to the west (cold) and a north-south trending spreading ridge to the east (hot). Temperature differences are expressed in Kelvins, where $1K = 1°C$. In this region, temperature $T(x,y,z,t)$ is given by
$$T = T_0 + ax + by + cz + dt,$$
where x-y-z are East-North-Up. In this region, $a = 0.1$ K/km, $b = 0$ K/km, $c = 0.2$ K/km, and $d = 0.05$ K/yr, with the last term due to global warming.
Some oceanographers place thermometers on two neutrally buoyant thingies (technical term) that drift with the current at z_1 and z_2 km depth, and on an anchored deep buoy. For simplicity, assume that the current does not affect $T(x,y,z,t)$, because only the spreading ridge, the ice shelf, and global warming control the temperature (somewhat unphysical).

a) On the anchored buoy at $x = 10$km, $y = 20$km, and $z = -3.597354$ km depth, what rate of temperature change will be measured in units of K/yr?

b) If the current flows 2 km/yr to the east, what rate of temperature change will be measured on the drifting thingies?

c) What if the flow of the current is 5 km/yr to the north?

d) What if the current in the vicinity of the thingies (which are at depth) flows at 3 km/yr 30° north of east and dives 0.5 km/yr downward (deeper)? I hope you will be using a dot product here.