ESS 411/511 Geophysical Continuum Mechanics Class #29

Highlights from Class #28 – Zoe Krauss
Today's highlights on Friday – Chloe Mcburney

Today
- Constitutive Relations, viscous fluids, elastic plane waves

For Friday please read
- Ed's notes on elastic waves
- Ed's notes on kinematic waves

(on right sidebar at)
https://courses.washington.edu/ess511/NOTES/)

Problem Sets

Problem Set #6 has also been graded

- Results are on Canvas
- I will return annotated paper later today

Problem Set #7

- Due on sunday …

For Problems Lab on Thursday

- Study Questions for take-at-home Final exam have been posted.

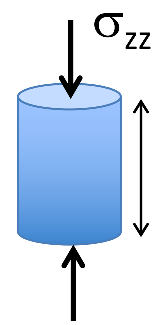
I am preparing some notes about the issues that you encountered on the Mid-term and on Problem Sets #4, #5, and #6

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Elastic solid

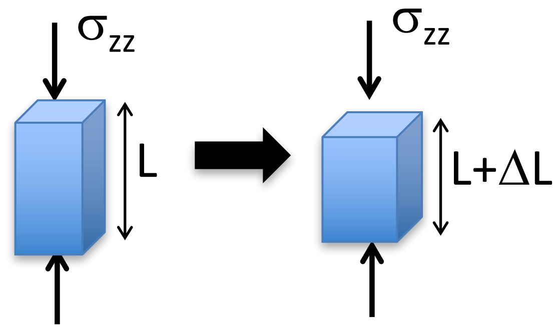


Stress in terms of strain $$\boxed{\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}}$$

Strain in terms of stress $$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu \delta_{ij} \sigma_{kk}\right]$$

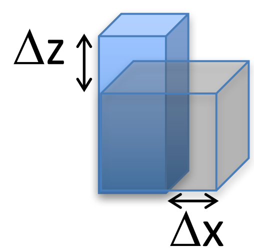
Young's modulus $$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

(Relates normal stresses and strains)



Poisson's ratio $$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

(Ratio of strains in direction of stress and normal to stress)



Other combinations of elastic parameters are possible

$$\boxed{\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}} \qquad \epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu \delta_{ij}\sigma_{kk}\right]$$

Young's modulus $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$

Shear modulus $G = \frac{E}{2(1+\nu)} = \mu$

(Relates shear stress to shear strain)

Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$

Bulk modulus $K = \frac{E}{3(1-2\nu)}$

(Ratio of volumetric stress to volumetric strain)

Warm-up (break-out rooms)

1) What is the value of Poisson's ratio for an incompressible material?
Hint:
- volumetric stress $\sigma_{ii}/3$
- volumetric strain $e_{ii} = 0$

Bulk modulus $K = \frac{E}{3(1-2\nu)}$

(Ratio of volumetric stress to volumetric strain)

2) Can shear stress exist in a fluid?

Classical (Newtonian) Fluids

In a fluid at rest, tractions on any surface are just pressure.

Define pressure as mean compressive normal stress

So with convention compressive stresses are negative

$$p = -\frac{1}{3}\sigma_{ii} \qquad \text{(pressure)}$$

and $t_i^{(n)} = \sigma_{ij} n_j = -p n_i$ (traction)

$$\sigma_{ij} = -p\delta_{ij}$$

When fluid is in motion.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$\tau_{ij} \equiv$ viscous stress tensor.

A viscous fluid can support shear stresses.

Fluid in motion

When fluid is in motion.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$\tau_{ij} \equiv$ viscous stress tensor.

A viscous fluid can support shear stresses.

In general, $\underline{\underline{\tau}}$ can be related to deformation rate tensor $\underline{\underline{D}}$

$$\underline{\underline{\tau}} = f(\underline{\underline{D}})$$

just like $\underline{\underline{\sigma}}$ was related to strain $\underline{\underline{\epsilon}}$ in elasticity. $\underline{\underline{\sigma}} = G(\underline{\underline{\epsilon}})$.

If relation between τ_{ij} and D_{km} is linear, we have a Newtonian fluid.

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_k}\right)$$

$$\tau_{ij} = C^{*}_{ijkm} D_{km}$$

Superscripts $*$ denote viscous analogs of elastic parameters. We can carry out a totally parallel development for fluids.

Class-prep: Constitutive relations

Please read

- Ed's notes on classical fluids
- Ed's notes on elastic waves

Both are available on the class web site at

https://courses.washington.edu/ess511/NOTES/notes.html

In a classical isotropic linear fluid, the deviatoric stress tensor τ_{ij} can be related to the deformation-rate tensor D_{ij} with just 2 scalar coefficients λ^* and μ^* (analogs to the elastic Lamé constants λ and μ)

$$\tau_{ij} = \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij}$$

Assignment

Write out the relation between σ_{ij} and τ_{ij} and describe their difference in prose.

Define ε_{ij} and D_{ij} and describe their difference in prose.

What do the Lamé analog constants λ^* and μ^* represent, and what are their units?

A constitutive relation

We can invoke the same symmetries, of isotropic material, to get.

$$\boxed{\tau_{ij} = \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij}}$$

All information about the fluid is in λ^* and μ^*

or in terms of total stress σ_{ij}:

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij}$$

D_{kk} is the rate of change of volume, so for incompressible fluid, $D_{kk}=0$, and

$$\boxed{\tau_{ij} = 2\mu^* D_{ij} \qquad \sigma_{ij} = -p\delta_{ij} + 2\mu^* D_{ij}}$$

Elastic waves

Equations of motion $$\rho \ddot{u}_l = \sigma_{lm,m} + b_l$$

ρ is density
u_l is displacement
b_l is body force

Stress $$\sigma_{lm} = \lambda\, \delta_{lm}\, \epsilon_{nn} + 2\mu\, \epsilon_{lm}$$

Strain $$\epsilon_{lm} = \frac{1}{2}\left(u_{l,m} + u_{m,l}\right)$$

Putting it together

$$\begin{aligned}
\rho \ddot{u}_l &= \left[\lambda\, \delta_{lm} \epsilon_{nn} + 2\mu\, \epsilon_{lm}\right]_{,m} + b_l \\
&= \left[\lambda\, \delta_{lm} u_{n,n} + \mu\left(u_{l,m} + u_{m,l}\right)\right]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}$$

Elastic waves

Putting it together

$$\begin{aligned}
\rho \ddot{u}_l &= [\lambda\, \delta_{lm} \epsilon_{nn} + 2\mu\, \epsilon_{lm}]_{,m} + b_l \\
&= [\lambda\, \delta_{lm} u_{n,n} + \mu\, (u_{l,m} + u_{m,l})]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}$$

Lets look for plane-wave solutions

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{A} \exp\left(i\, \mathbf{k} \cdot \mathbf{x} - i\, \omega t\right) \qquad (1)$$

$\mathbf{x}$ is the position vector

$\mathbf{A}$ is a polarization vector

direction and magnitude of displacements $\mathbf{u}$ during wave motion

$\mathbf{k}$ is a wavenumber

direction of wave propagation and the spatial periodicity of the waves.

In index notation, (1) is written as

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

Elastic waves

Equations of motion

$$\begin{aligned}
\rho \ddot{u}_l &= [\lambda\, \delta_{lm} \epsilon_{nn} + 2\mu\, \epsilon_{lm}]_{,m} + b_l \\
&= [\lambda\, \delta_{lm} u_{n,n} + \mu\, (u_{l,m} + u_{m,l})]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}$$

In absence of body forces, set b_l=0

Put (2) into equations of motion

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

After a bunch of algebra (in the notes),

$$\rho\, \omega^2 A_l = (\lambda + \mu)\, A_n k_n k_l + \mu\, A_l k_n k_n$$

When ***A*** and ***k*** are parallel, we get compressional waves (p waves)
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

Elastic waves

Equations of motion

$$\begin{aligned}
\rho \ddot{u}_l &= [\lambda\, \delta_{lm} \epsilon_{nn} + 2\mu\, \epsilon_{lm}]_{,m} + b_l \\
&= [\lambda\, \delta_{lm} u_{n,n} + \mu\, (u_{l,m} + u_{m,l})]_{,m} + b_l \\
&= \lambda\, u_{n,nl} + \mu\, u_{l,mm} + \mu\, u_{m,lm} + b_l \\
&= (\lambda + \mu)\, u_{n,nl} + \mu\, u_{l,nn} + b_l
\end{aligned}$$

In absence of body forces, set b_l=0

Put (2) into equations of motion

$$u_l(\mathbf{x}, t) = A_l \exp\left(i\, k_m x_m - i\, \omega t\right) \qquad (2)$$

After a bunch of algebra (in the notes),

$$\rho\, \omega^2 A_l = (\lambda + \mu)\, A_n k_n k_l + \mu\, A_l k_n k_n$$

When ***A*** and ***k*** are parallel, we get compressional waves (p waves)
When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

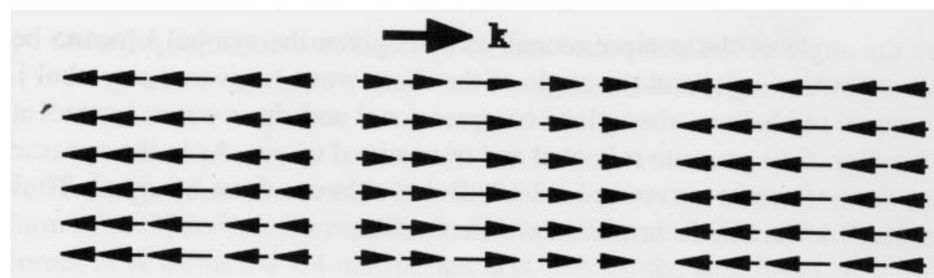
When ***A*** and ***k*** are parallel, we get compressional waves (p waves)

$$\mathbf{A} = c\mathbf{k}$$

$$\rho\,\omega^2 c\,k_l = (\lambda + \mu)\,c\,k_n k_n k_l + \mu\,c\,k_l k_n k_n$$

$$\begin{aligned} \rho\,\omega^2 &= (\lambda + \mu)\,k_n k_n + \mu\,k_n k_n \\ &= (\lambda + \mu)\,k_n k_n \end{aligned}$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \left(\frac{\lambda + 2\mu}{\rho}\right)$$



Phase velocity
of p wave

$$\alpha = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\lambda + 2\mu}{\rho}\right)}$$

When ***A*** and ***k*** are orthogonal, we get shear waves (s waves)

$$A_n k_n = 0$$

$$\rho\,\omega^2 = \mu\, k_n k_n$$

$$\frac{\omega^2}{|\mathbf{k}|^2} = \frac{\mu}{\rho}$$

$$\beta = \frac{\omega}{|\mathbf{k}|} = \sqrt{\left(\frac{\mu}{\rho}\right)}$$