Earth and Space Sciences 411/511
Geophysical Continuum Mechanics

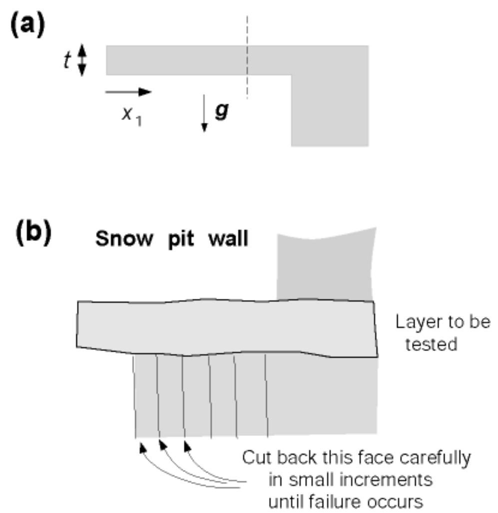
Problem set – Heat, Stress, and Moments in the Earth

1. Cantilevers

A cantilever beam of thickness t and uniform density ρ is acted upon by gravity.

a) Assuming that the normal stress varies linearly with elevation across any vertical section (dashed line in part (a) of figure), calculate the tensile stress in the top of the beam as a function of distance x_1 from its end. (Note that definition of x_1 here differs from class.)

b) This result is often used to get a crude estimate of the tensile strength of deposited snow, by doing the experiment sketched in part (b) of the figure. If the snow slab is 15 cm thick, has a density of 400 kg m^{-3}, and fails when the overhang is 0.75 m, what is its tensile strength?



A similar experiment is done by nature on a larger scale, when wind-drifted snow forms cornices, which can eventually break as they are extended, or as the strength of the snow drops for some reason. One can test the stability of a cornice by standing on its edge, thereby increasing the force moment and the tensile stress; however, this might not be a wise way to conduct the experiment!

2. Subducting Plate

(a) The elastic lithosphere bends at an oceanic trench as in Figure A below. The densities of mantle and water are represented by ρ_m=3300 kg m^{-3} and ρ_w=1000 kg m^{-3} respectively, and D is the flexural rigidity of the plate. The asthenosphere can be treated as a fluid at rest, i.e. it does not support shear stresses.

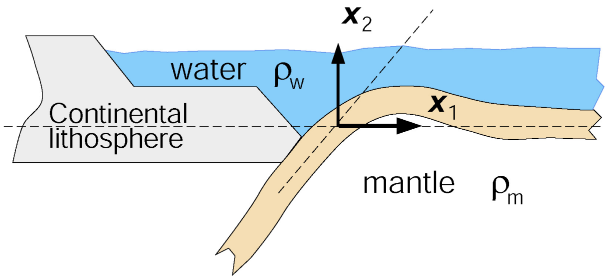
- Starting from the equation for bending plates,

$$D\frac{d^4u_2}{dx_1^4} = F_2(x_1) + P_1\frac{\partial^2 u_2}{\partial x_1^2} \qquad (1)$$

and assuming that the stress in the mantle far from the trench is lithostatic, show that the vertical displacement $u_2(x_1)$ of the warped plate must satisfy

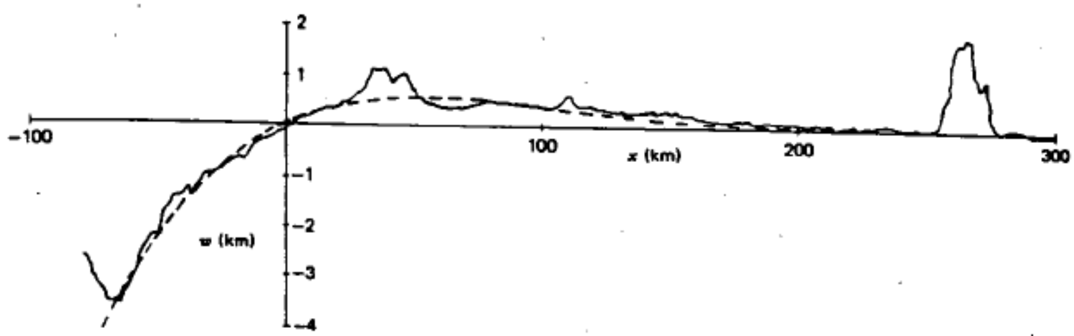
$$D\frac{\mathrm{d}^4 u_2}{\mathrm{d}x_1^4} + (\rho_m - \rho_w) g\, u_2(x_1) = 0 \qquad (2)$$

Figure A



(b) Figure B below from Turcotte and Schubert, *Geodynamics*, shows a bathymetric profile across the Mariana Trench. (The peaks are submarine volcanic edifices, or seamounts.) Note that in T&S, vertical displacements are called $w(x)$.
Referring to our cartoon Figure A, the slope $\partial u_2/\partial x_1$ is 0.03 on the descending limb at Mariana Trench, where the vertical displacement is zero (i.e. $u_2(x_1) = 0$); this is represented by the sloping dashed line at $x_2 = x_1 = 0$ in Figure A.

Figure B



The dashed line in Figure B is a fit using a solution to Equation (1) above.

- Derive a solution to Equation (1) for $x_1 > 0$. (Hint: think about trigonometric functions and exponentials, and show that your proposed solution works.)
- Using measurements from Figure B, calculate an expression for the plate profile oceanward (to the right) from the Mariana Trench in Figure B, and graph your profile.

(c) Assume that flexural rigidity D and thickness t of the subducting plate are related by $D = m\, t^3/12$, where $m \approx 10^{10}$ Pa is an elastic modulus.

- Using the measurements shown in Figure B, make an estimate of the flexural rigidity D of the lithosphere, and its thickness t.

3. Thermal energy and heat diffusion

In a linear viscous continuum, start from an energy equation of the form:

$$\rho\dot{e} - \sigma_{ij}\dot{\varepsilon}_{ij} + q_{i,i} = 0 \tag{3}$$

where σ_{ij} is stress, $\dot{\varepsilon}_{ij}$ is strain rate, q_i is heat flux, and $e(T)$ is internal energy per unit mass, given by

$$e(T) = \int_0^T c(T')dT' \tag{4}$$

In (4), T is temperature, and $c(T)$ is specific heat capacity at constant volume.

a) Assume that thermal conductivity K is uniform and isotropic.

- Derive a conservation-of-energy equation in the spatial coordinates, in terms of temperature T, for a continuum in which Fourier's law of heat conduction applies, i.e. $q_i = -K\,T_{,i}$. Express your equation in both index notation and symbolic notation.

b) Now look at a special case in which the velocity has only an x_1 component, which is uniform in space, and T varies only in x_2.

- Show that T is governed by a linear heat-diffusion equation, i.e.

$$\frac{\partial T}{\partial t} = \left(\frac{K}{\rho c}\right)\frac{\partial^2 T}{\partial x_2^2} \tag{5}$$

c) Suppose that, at time t=0, the initial condition is: $T(x_2, 0) = T_a + T_b \sin(\alpha\, x_2)$, where T_a and T_b are constants. This represents an infinitely tall alternating stack of warm and cold layers.

- Using (5), find the temperature at future times. You can use separation of variables, and assume a solution of the form
  $$T(x_2, t) - T_0 = T_p(t)\sin(\alpha x_2)$$
- Explain in a few words, what is happening in terms of heat diffusion between hot and cold regions, and the effect of using different values of the wavenumber α.
- How is the average temperature of the continuum changing through time?
- How would you expect the average temperature to change through time if velocity $v_1(x_2)$ was not spatially uniform?