ESS 411/511 Geophysical Continuum Mechanics Class #6

Highlights from Class #5 – Barrett Johnson
Today's highlights on Wednesday – Madeline Mamer

Remember we are looking for just 2 or 3 *highlights*, not a summary of the entire class. (What did you think was most important?)

EARTH SCIENCE WEEK

https://www.earthsciweek.org/

Since October 1998, the American Geosciences Institute has organized this national and international event to help the public gain a better understanding and appreciation for the Earth sciences and to encourage stewardship of the Earth. This year's Earth Science Week will be held from October 11 - 17, 2020 and will celebrate the theme "Earth Materials in Our Lives." The coming year's event will focus on the ways that Earth materials impact humans — and the ways human activity impacts these materials — in the 21st century.

ESS 411/511 Geophysical Continuum Mechanics

For Friday class

- Please read Mase, Smelser, and Mase, CH 2 through Section 2.6

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr’s circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Problem Sets

- Problem Set #1 – due in Canvas on Wednesday
- Problem Set #2 – in Lab on Thursday

ESS 411/511 Geophysical Continuum Mechanics Class #6

Warm-up question (break-out) –

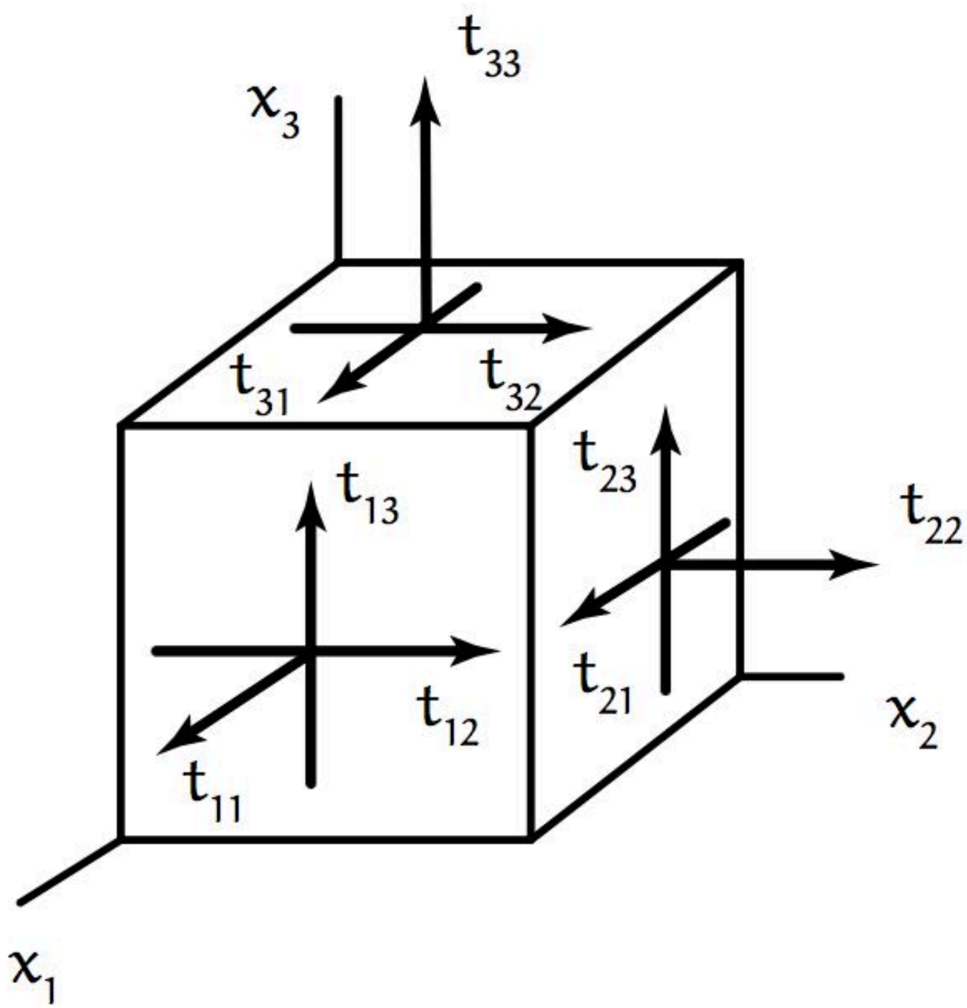
The Summation Convention is your Best Friend in Continuum

- Expand $t_{ii} v_j \hat{e}_j$ according to the summation convention.
- What is the tensor rank of the result?
- How would you interpret the result?

Class-prep answers (break-out)

- Expressing stress as vector? Why or why not?

Why stress has 9 components



Scalars, vectors, and tensors

Scalar (zero-order tensor)

- α, β, γ, θ, *a, b, r,* etc. lower-case italic greek or roman letters.
- Temperature, pressure, density, …

Vector (first-order tensor)

- ***u, v, w, x, y, z*** etc. Bold italic lower-case roman letters.
- Velocity, force, geothermal flux, …

Higher-order tensor

- **A, T, S**, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, E**,** etc. Bold letters, often upper-case roman, but many exceptions.
- Stress, strain rate, anisotropic material properties, …

Scalars, vectors, and tensors

Symbolic notation:

A tensor, or linear transformation **T**, assigns any vector **v** to another vector **T*v*** such that

$\mathbf{T}(\boldsymbol{v} + \boldsymbol{w}) = \mathbf{T}\,\boldsymbol{v} + \mathbf{T}\,\boldsymbol{w}$ (distributive property)

$\mathbf{T}(\alpha\boldsymbol{v}) = \alpha\mathbf{T}\,\boldsymbol{v}$ (associative property)

$(\mathbf{T}+\mathbf{S})\boldsymbol{v} = \mathbf{T}\boldsymbol{v} + \mathbf{S}\boldsymbol{v}$

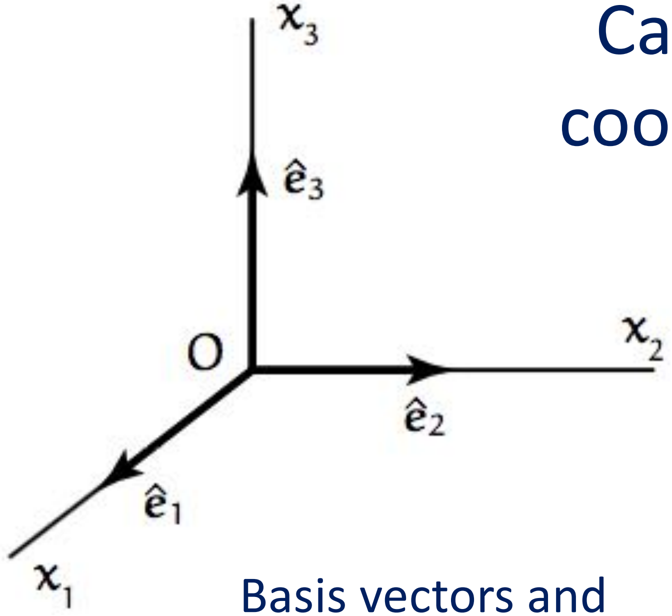
$(\alpha\mathbf{T})\,\boldsymbol{v} = \alpha\,(\mathrm{T}\boldsymbol{v})$

for all ***v*** and ***w***.

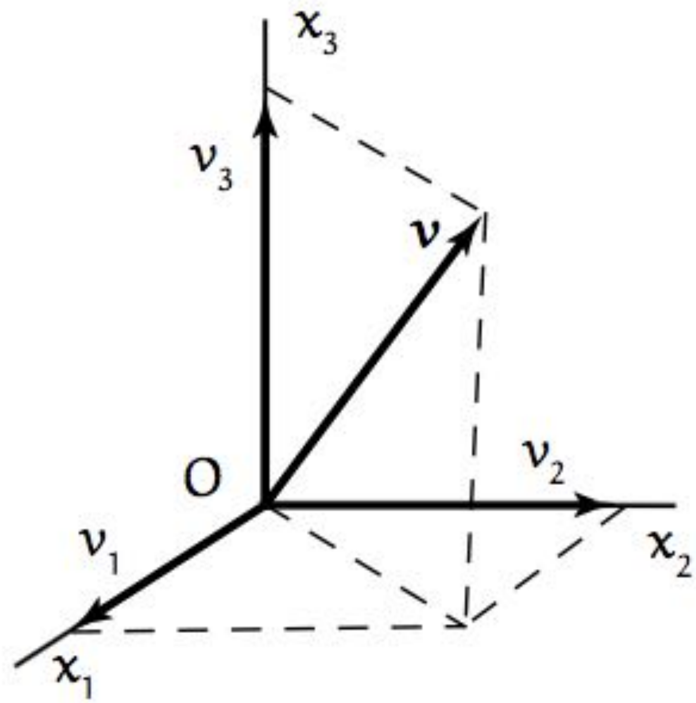
Not necessarily commutative though …

- Things like **T*v*** and ***v*T** get messier.
- May not be equal
- May not even exist …

Cartesian coordinates



Basis vectors and coordinate axes



A vector ***v*** exists independent of a particular coordinate system.

- However, after we choose a coordinate system, ***v*** can be expressed through its components in that coordinate system.

  $\boldsymbol{v} = (v_1, v_2, v_3)$

$$v = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = \Sigma_{i=1}^{3} v_i\hat{e}_i$$

Index notation

- $u_{\mathrm{i}} \Longleftrightarrow (u_1, u_2, u_3)$ vector (3 elements)
- $u_{\mathrm{ij}} \Longleftrightarrow (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33})$
  Second-order tensor (9 elements)
- $u_{\mathrm{ijk}} \Longleftrightarrow (u_{111}, u_{112}, u_{113}, u_{121}, u_{122}, u_{123}, \ldots u_{331}, u_{332}, u_{333},)$.
  Third-order tensor (27 elements)
- $u_{\mathrm{ijkl}} \Longleftrightarrow (u_{1111}, u_{1112}, u_{1113}, \ldots u_{3331}, u_{3332}, u_{3333},)$.
  Fourth-order tensor (81 elements)

Summation Convention

- Whenever an index i is repeated in a term, a summation is implied, in which i takes the values 1, 2 ,3.

$$v_i \hat{e}_i \equiv \Sigma_{i=1}^{3} v_i \hat{e}_i$$

- In an implied summation, index i is a *dummy* index, i.e.

$$v_i \hat{e}_i = v_k \hat{e}_k$$

or

$$u_i v_i = u_k v_k$$

Kronecker delta

Since the base vectors $\hat{\mathbf{e}}_i$ ($i = 1,2,3$) are unit vectors and orthogonal

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{cases} 1 & \text{if numerical value of } i = \text{numerical value of } j \\ 0 & \text{if numerical value of } i \neq \text{numerical value of } j \end{cases}$$

Therefore, if we introduce the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if numerical value of } i = \text{numerical value of } j \\ 0 & \text{if numerical value of } i \neq \text{numerical value of } j \end{cases}$$

we see that

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \qquad (i, j = 1, 2, 3) \ .$$

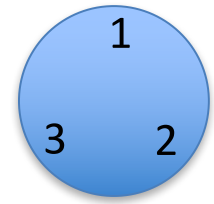
Note that δ_{ij} is like a 3x3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the value of δ_{jj} ?

Permutation symbol

Imagine the numbers 1, 2, 3 are written clock-wise around the circumference of a wheel



$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if the numerical values of } i,j,k \text{ are in clockwise order} \\ -1 & \text{if the numerical values of } i,j,k \text{ are anti-click-wise order} \\ 0 & \text{if the numerical values of } i,j,k \text{ are in any other order} \end{cases}$$

Cross products of basis vectors

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk} \hat{\mathbf{e}}_k \qquad (i, j, k = 1, 2, 3)$$

Switching indices

$$\varepsilon_{ijk} = -\varepsilon_{kji} = \varepsilon_{kij} = -\varepsilon_{ikj}$$

Determinants

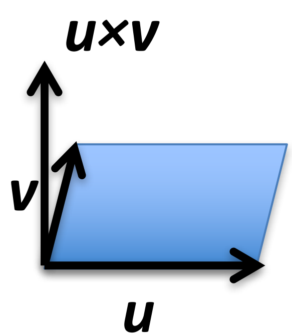
$$\det \boldsymbol{\mathcal{A}} = |\mathcal{A}_{ij}| = \begin{vmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{vmatrix}$$

$$\begin{aligned} \det \boldsymbol{\mathcal{A}} &= \mathcal{A}_{11} \begin{vmatrix} \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{32} & \mathcal{A}_{33} \end{vmatrix} - \mathcal{A}_{12} \begin{vmatrix} \mathcal{A}_{21} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{33} \end{vmatrix} + \mathcal{A}_{13} \begin{vmatrix} \mathcal{A}_{21} & \mathcal{A}_{22} \\ \mathcal{A}_{31} & \mathcal{A}_{32} \end{vmatrix} \\ &= \mathcal{A}_{11} (\mathcal{A}_{22}\mathcal{A}_{33} - \mathcal{A}_{23}\mathcal{A}_{32}) - \mathcal{A}_{12} (\mathcal{A}_{21}\mathcal{A}_{33} - \mathcal{A}_{23}\mathcal{A}_{31}) \\ &\quad + \mathcal{A}_{13} (\mathcal{A}_{21}\mathcal{A}_{32} - \mathcal{A}_{22}\mathcal{A}_{31}) \ . \end{aligned}$$

$$\varepsilon_{qmn} \det \boldsymbol{\mathcal{A}} = \varepsilon_{ijk} \mathcal{A}_{iq} \mathcal{A}_{jm} \mathcal{A}_{kn}$$

Box product

cross (vector) product of two vectors, Eq 2.12:

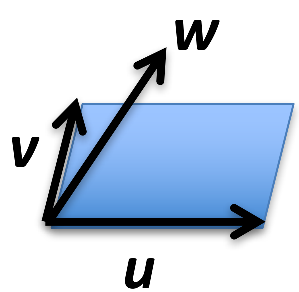


$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = (uv \sin\theta)\, \hat{\mathbf{e}} = \varepsilon_{ijk} u_i v_j \hat{\mathbf{e}}_k$$

What does ***u×v*** measure?

triple scalar product (box product), Eq 2.13:

$$\begin{aligned} [\mathbf{u}, \mathbf{v}, \mathbf{w}] &= u_i \hat{\mathbf{e}}_i \cdot (v_j \hat{\mathbf{e}}_j \times w_k \hat{\mathbf{e}}_k) = u_i \hat{\mathbf{e}}_i \cdot \varepsilon_{jkq} v_j w_k \hat{\mathbf{e}}_q \\ &= \varepsilon_{jkq} u_i v_j w_k \delta_{iq} = \varepsilon_{ijk} u_i v_j w_k \end{aligned}$$



What does **[*u,v,w*]** measure?

Vector algebra