ESS 411/511 Geophysical Continuum Mechanics

## Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

ESS 411/511 Geophysical Continuum Mechanics Class #9

For Wednesday

- Please read Mase, Smelser, and Mase, Ch 3 through Section 3.6

For Friday

- ESS 511 Students will give a 1 minute presentation about their class projects on Friday.

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

Problem Sets

- Problem Set #2 due in Canvas on Wednesday
- Problem Set #3 in Thursday session

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**Warm-up** Finding the eigenvalues and eigenvectors of a 3x3 tensor is complicated and "mathy".

- Explain in words why anyone would want to find the eigenvalues and eigenvectors of a 3x3 tensor. Why bother?

## Class-prep questions

We said earlier this week that stress $s_{ij}$ can't be represented by a vector, it is a second-order tensor, because there are two directions involved in defining stress.

Yet our text (page 55) talks about a "stress vector"! Something strange is going on here!

In math, and in words, what exactly is this vector $t_i^{(\boldsymbol{n})}$?
Hint – what is $\boldsymbol{n}$?

# Principal values and directions (Eigenvalues and eigenvectors)

A 2nd order tensor $s_{ij}$ maps a vector $u_j$ onto another vector $v_i$

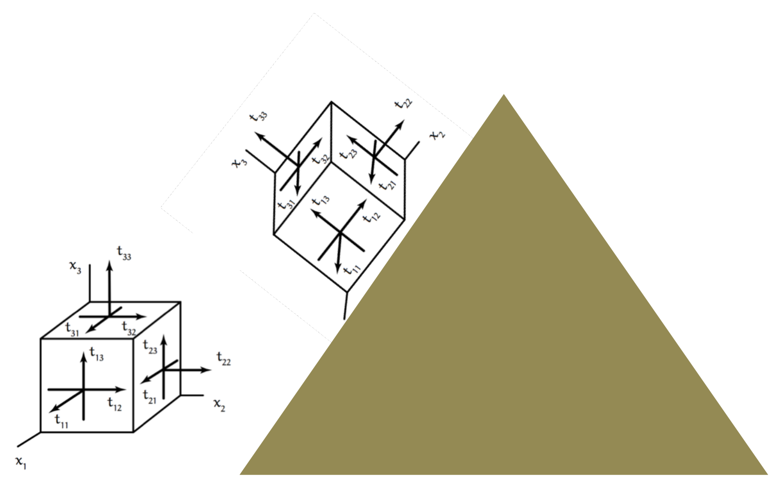
$$s_{ij}u_j = v_i$$

In general $u_j$ and $v_i$ point in different directions.

It would be nice if we could find some special vectors $u_j$ that mapped onto vectors $v_i$ that were parallel to $u_j$.
That could help us to find a coordinate system in which $s_{ij}$ could be expressed more simply.

For example, stress in the rocks on a mountain side.
We know that there is no shear stress on the sloping surface.

- Maybe the stress tensor would be simpler using a coordinate system aligned with the mountain surface.



# Finding eigenvectors

When $t_{ij}$ is symmetric with real components, there will be some vectors $n_j$ that *do* map onto a parallel vector.

$$t_{ij} n_j = \lambda n_i \qquad \text{or} \qquad \mathbf{T} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$$

When $n_j$ is a unit vector, it defines a principal direction or ***eigenvector*** of the tensor $t_{ij}$, and $\lambda$ is called a principal value or ***eigenvalue*** of $t_{ij}$.

$$t_{ij}\, n_j - \lambda\, n_i = 0$$

Since $n_i = \delta_{ij}\, n_j$

$$t_{ij}\, n_j - \lambda\, \delta_{ij}\, n_j = 0$$

or

$$(t_{ij} - \lambda\, \delta_{ij})\, n_j = 0 \quad \text{or in symbolic form,} \quad (\mathbf{T} - \lambda \mathrm{I}) \bullet \boldsymbol{n} = 0$$

# Let's work through an example

**_Example 2.14_**

Determine the principal values and principal directions of the second-order tensor $\mathsf{T}$ whose matrix representation is

$$[t_{ij}] = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

# Let’s work through an example

**Example 2.14**

Determine the principal values and principal directions of the second-order tensor $\mathbf{T}$ whose matrix representation is

$$[t_{ij}] = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

In order for $n_i$ to be an eigenvector, $n_i$ and $t_{ij}\, n_j$ must be parallel

$$t_{ij}\, n_j - \lambda\, n_i = 0$$

Since $n_i = \delta_{ij}\, n_j$

$$t_{ij}\, n_j - \lambda\, \delta_{ij}\, n_j = 0$$

Factor out $n_j$: $(t_{ij} - \lambda\, \delta_{ij})\, n_j = 0$

# Let's work through an example

$$(t_{11}-\lambda)\,n_1 + t_{12}n_2 + t_{13}n_3 = 0$$
$$t_{21}n_1 + (t_{22}-\lambda)\,n_2 + t_{23}n_3 = 0$$
$$t_{31}n_1 + t_{32}n_2 + (t_{33}-\lambda)\,n_3 = 0$$

Obviously these equations are satisfied if $n_1 = n_2 = n_3 = 0$.
But that is no help because we said $n_j$ is a unit vector

Nontrivial solutions can exist
(the equations are not independent)
if the determinant = 0

$$|t_{ij} - \lambda\delta_{ij}| = 0$$

$$\begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

So, expanding on the third row,

$$(3-\lambda)\left(10 - 7\lambda + \lambda^2 - 4\right) = 0$$

which factors into

$$(3-\lambda)(6-\lambda)(1-\lambda) = 0$$

# Let's work through an example

$$(3-\lambda)(6-\lambda)(1-\lambda)=0$$

So, the eigenvalues are: $\lambda_{(1)}=3\quad \lambda_{(2)}=6,\quad \lambda_{(3)}=1$

Now, find the 3 corresponding eigenvectors $n_j$ by solving the equations $(t_{ij}-\lambda\,\delta_{ij})\,n_j=0$ with each value of $\lambda$ in turn:

$$(t_{11}-\lambda)\,n_1+t_{12}n_2+t_{13}n_3=0$$
$$t_{21}n_1+(t_{22}-\lambda)\,n_2+t_{23}n_3=0$$
$$t_{31}n_1+t_{32}n_2+(t_{33}-\lambda)\,n_3=0$$

and

$$[t_{ij}]=\begin{bmatrix}5 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$$

With $\lambda_{(1)}$=3:

$$2n_1+2n_2=0$$
$$2n_1-n_2=0$$

The solution is: $n_1=n_2=0$ and since $n_j$ must be a unit vector, $n_3$ = 1

and the first eigenvector is: (0, 0, ±1), i.e. $\hat{e}'_3=\hat{e}_3$

So the new axes are just the old axes rotated about $\hat{e}_3$

# Let's work through an example

Similarly, using $\lambda_{(2)}$=6,

$$\begin{aligned} -n_1 + 2n_2 &= 0 \\ 2n_1 - 4n_2 &= 0 \\ -3n_3 &= 0 \end{aligned}$$

$n_1 = 2n_2$ and since $n_3 = 0$,

And the unit-vector criterion gives

$$(2n_2)^2 + n_2^2 = 1, \text{ or } n_2 = \pm 1/\sqrt{5} \quad \text{and} \quad n_1 = \pm 2/\sqrt{5}$$

So, the second eigenvector is: $(\pm 2/\sqrt{5}, \pm 1/\sqrt{5}, 0)$

# Let's work through an example

Similarly, using $\lambda_{(3)}$= 1,

$$\begin{aligned} 4n_1 + 2n_2 &= 0 \\ 2n_1 + n_2 &= 0 \end{aligned}$$

$n_1 = 2n_2$ and since $n_3 = 0$,

And the unit-vector criterion gives

$(2n_2)^2 + n_2^2 = 1$, or $n_2 = \pm 1/\sqrt{5}$ and $n_1 = \pm 2/\sqrt{5}$

So, the third eigenvector is: $(\pm 1/\sqrt{5}, \mp 2/\sqrt{5}, 0)$

And the Transformation matrix is:

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm \frac{2}{\sqrt{5}} & \pm \frac{1}{\sqrt{5}} & 0 \\ \pm \frac{1}{\sqrt{5}} & \mp \frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

Each row is an eigenvector

# Let's work through an example

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm\frac{2}{\sqrt{5}} & \pm\frac{1}{\sqrt{5}} & 0 \\ \pm\frac{1}{\sqrt{5}} & \mp\frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

The rows provide 2 sets of eigenvectors, depending on the upper/lower +/- choices.

- How would you decide which set to use?
- Suppose in a different problem, the 2nd and 3rd eigenvalues were equal. There will be some remaining ambiguity you won't be able to find 3 eigenvectors in the same way.
- What's going on? What have you learned about the directions in which $a_{ij}n_j$ points in the same direction as $n_j$?