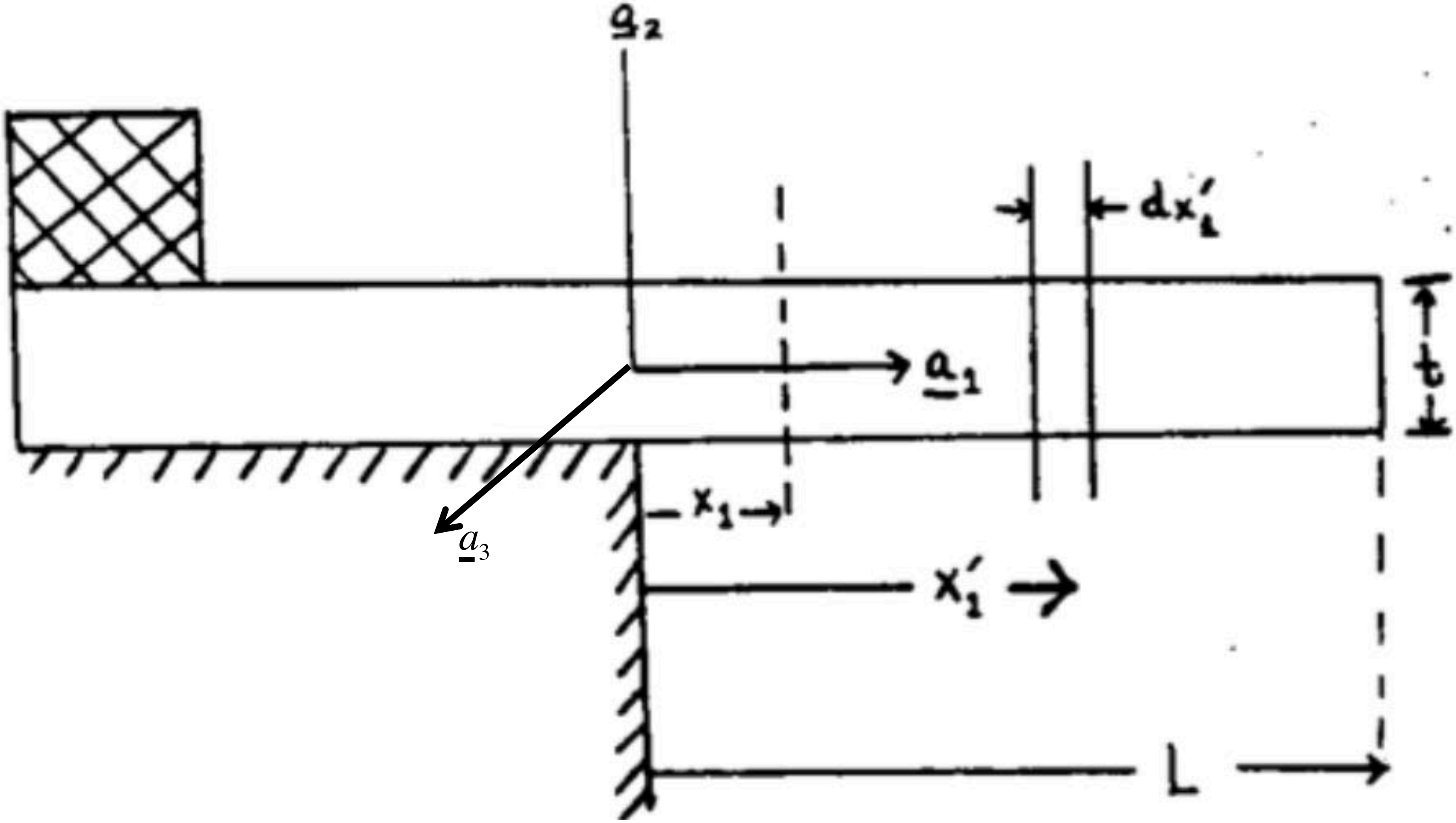
ESS 411/511 Geophysical Continuum Mechanics Class #26

For Problems Lab tomorrow

- Read (*https://courses.washington.edu/ess511/NOTES/*)
  - Raymond notes on stress and moments
  - Turcotte and Schubert Section 3.9

- For Friday class
- Read (*https://courses.washington.edu/ess511/NOTES/*)
  - Ed's note on volume elements
  - Ed's note on conservation laws
  - Ed's note on constitutive relations

A hanging plate



Average stress across the beam (per unit width)

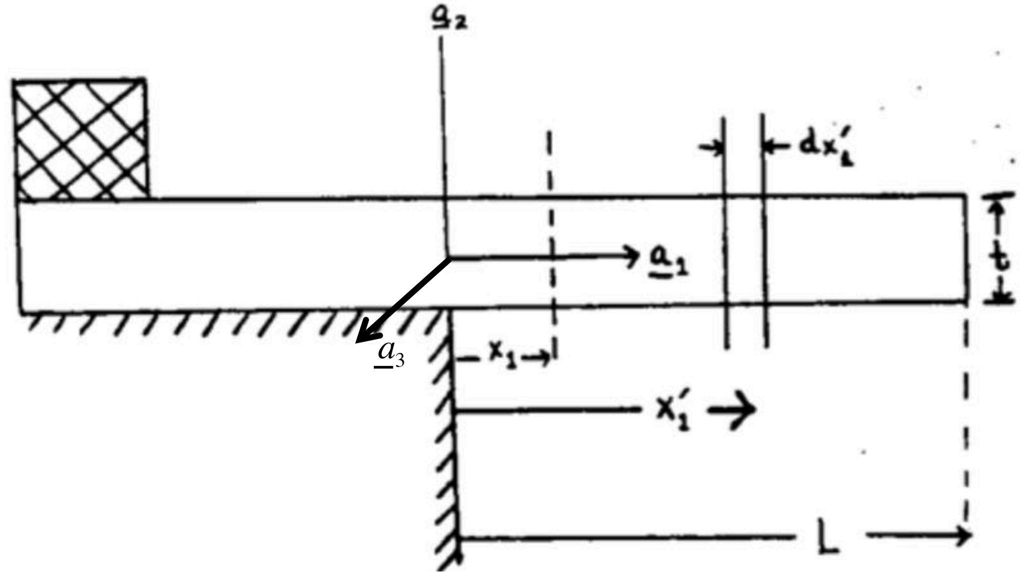
Plane at x_1 must support the weight of all material to the right.

e.g. $\langle\sigma_{12}\rangle t$ is vertically directed force per unit width in x_3

$$\langle\sigma_{11}\rangle t = \rho g_1 (L - x_1)\, t = 0$$

$$\langle\sigma_{12}\rangle t = \rho g_2 (L - x_1)\, t = 0$$

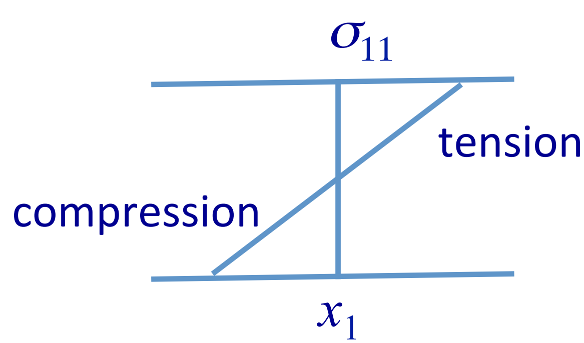
$$\langle\sigma_{13}\rangle t = \rho g_3 (L - x_1)\, t = 0$$



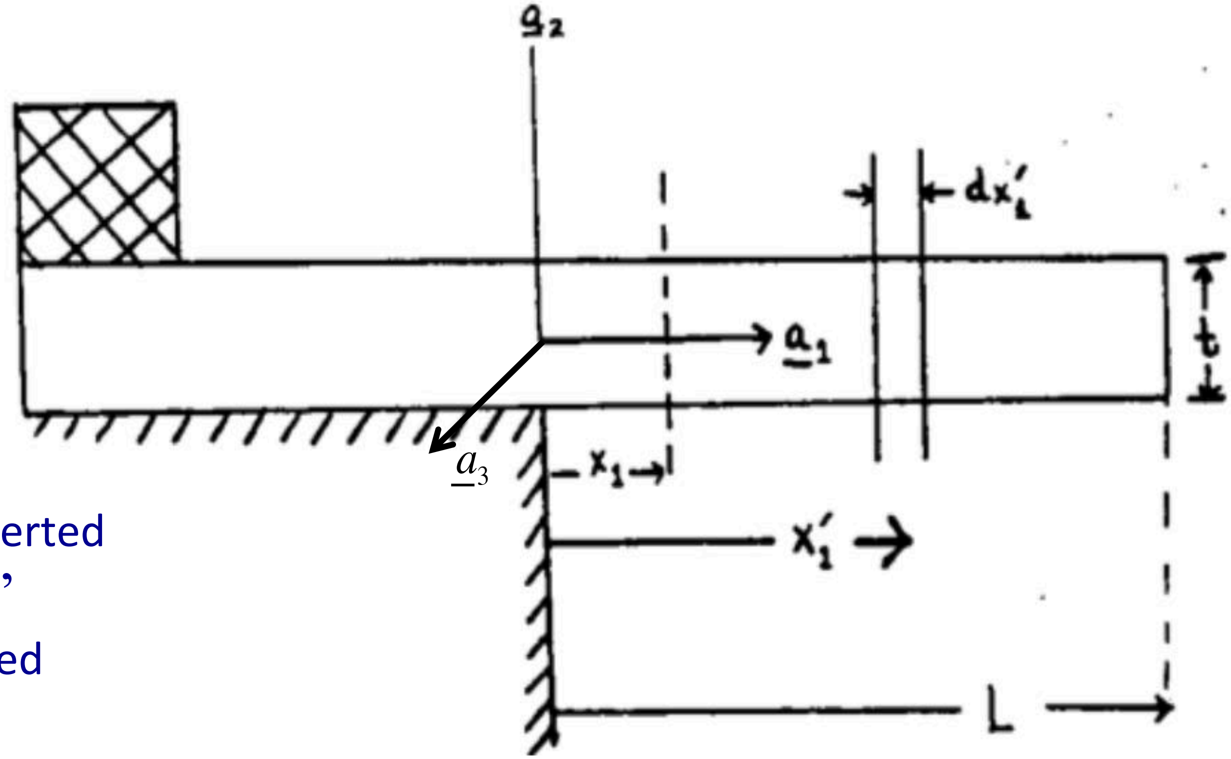
Although $\langle\sigma_{11}\rangle=0$, there must be

- tension ($\sigma_{11}>0$) in the upper part and
- compression ($\sigma_{11}<0$) in the lower part,

in order to prevent the material to the right from falling down.



Incremental moment at x_1 due to outboard weight



Incremental moment at x_1 exerted by thin slice dx_1' of slab at x_1'
e.g. $\langle\sigma_{12}\rangle t$ is vertically directed force per unit width in x_3

$$dM_g = -\rho g t \, dx_1'(x_1' - x_1)$$

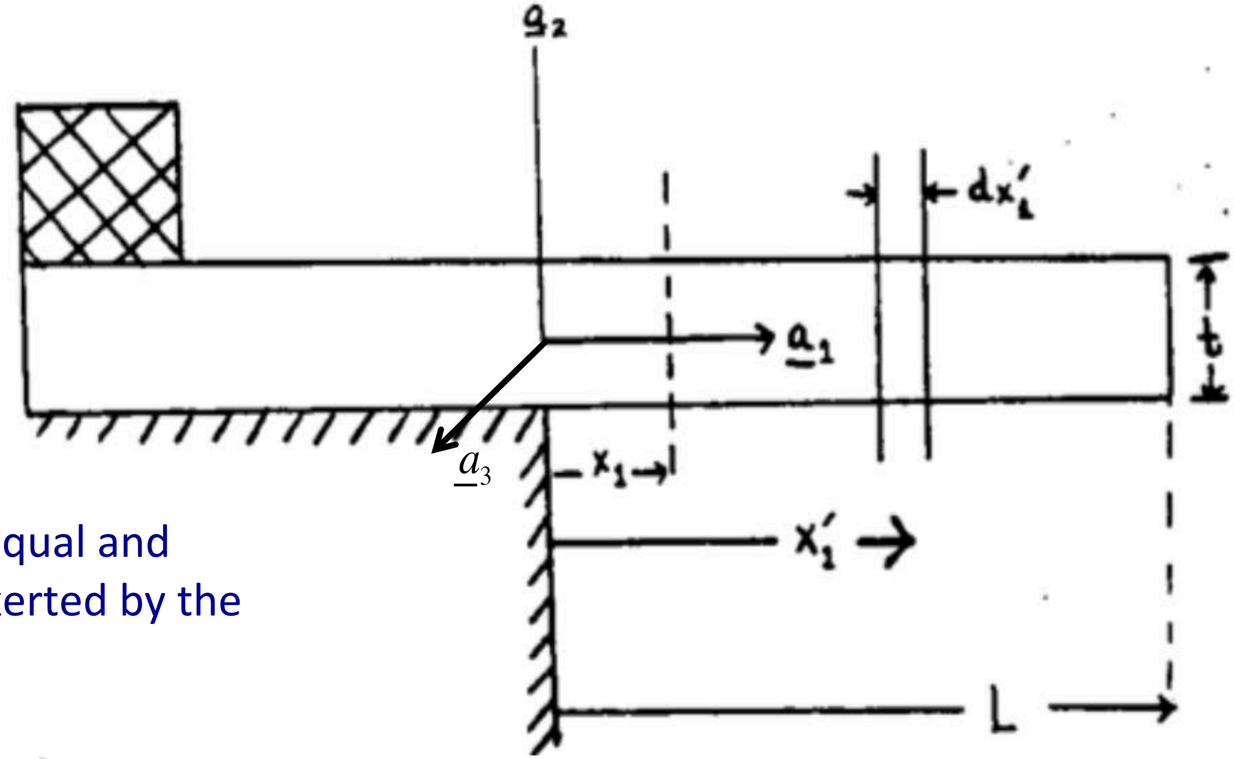
Now add up the incremental moments at x_1 due to *all* thin slices dx_1' to the right of x_1

$$M_g = \int_{x_1}^{L} \rho g t(x_1 - x_1')dx_1' = -\frac{1}{2}\rho g t(L - x_1)^2$$

The force per unit width: $\rho g \, t \, dx_1'$

The lever arm: $(x_1 - x_1')$

Incremental moment at x_1 due to stress in the beam

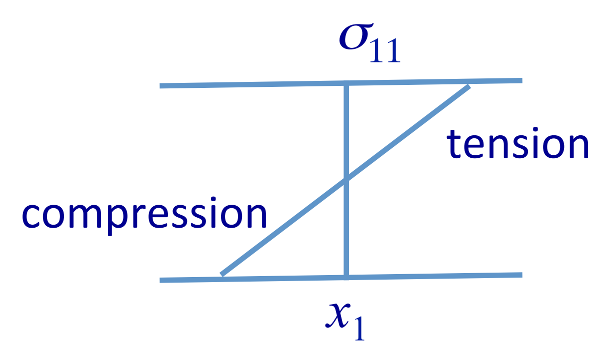


M_g must be balanced by an equal and opposite moment M_t at x_1 exerted by the stress state there.

$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2')\, x_2'\, dx_2'$$

The force per unit width: $\sigma_{11}(x_1, x_2')\, dx_2'$

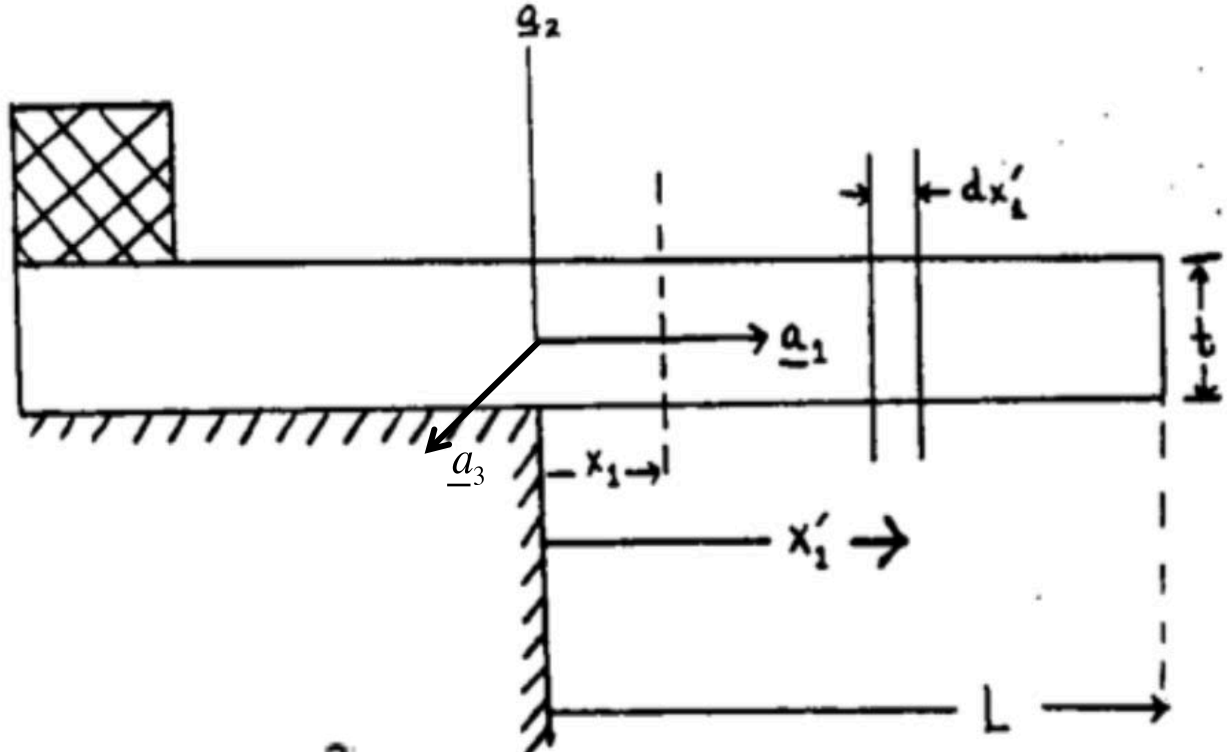
The lever arm: x_2'



Shear stresses at x_1 don't contribute because they have no moment arm around x_1

σ_{11} is assumed to be linear, but it is a very good assumption.

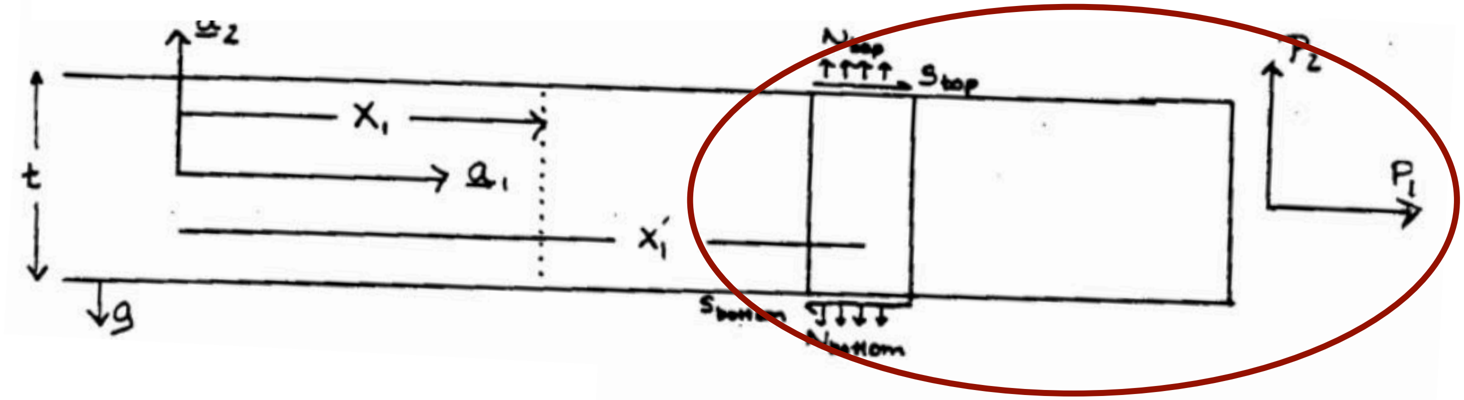
Putting it together -

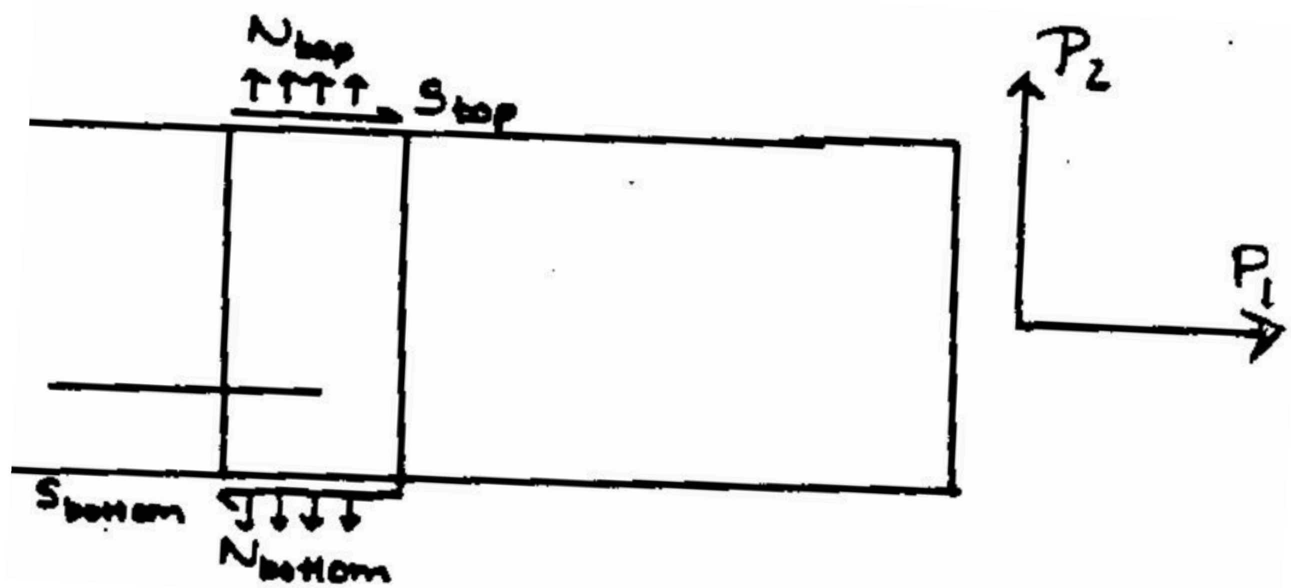


$M_t + M_g = 0$

$$\int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2') x_2' dx_2' = \frac{1}{2}\rho g t \, (L - x_1)^2$$

Now include tractions on the top and bottom





Horizontal forces

$$F_1(x_1') = S_{top} - S_{bottom}$$

Vertical forces

$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

Balanced forces

Zero net force in a_1 direction

$$\langle \sigma_{11} \rangle t + \int_{x_1}^{L} F_1(x_1')dx_1' + P_1 = 0$$

$$\langle \sigma_{11} \rangle = \frac{1}{t}\{P_1 + \int_{x_1}^{L} F_1(x_1')dx_1'\}$$

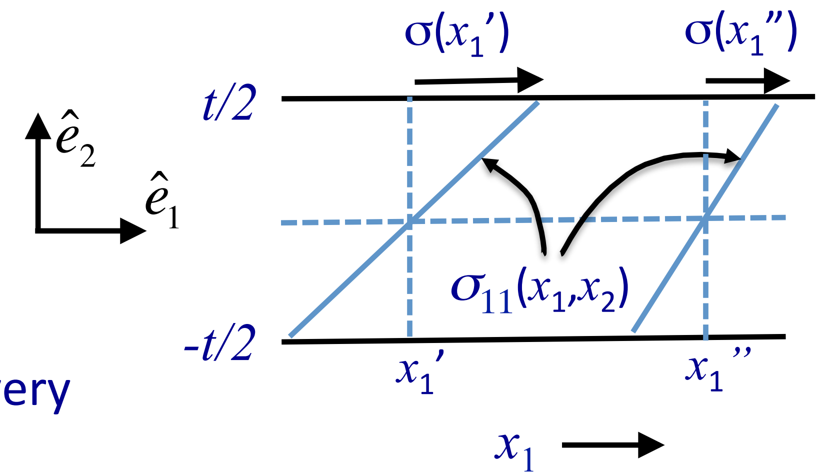
Zero net force in a_2 direction

$$\langle \sigma_{12} \rangle t + \int_{x_1}^{L} F_2(x_1')dx_1' + P_2 = 0$$

$$\langle \sigma_{12} \rangle = \frac{1}{t}\{P_2 + \int_{x_1}^{L} F_2(x_1')dx_1'\}$$

Finding stress σ_{11} in beam

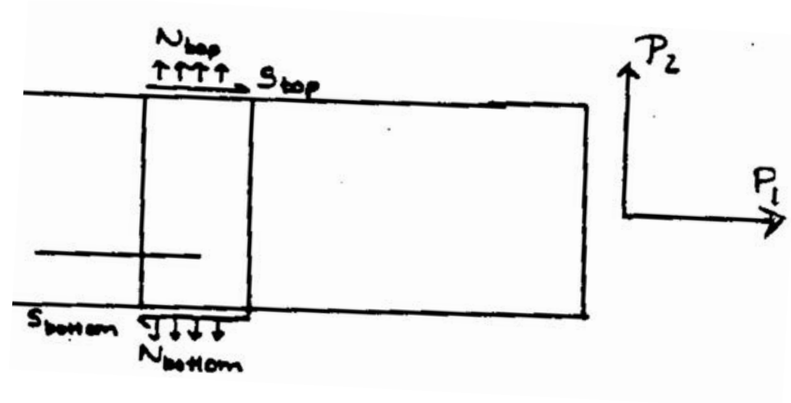
$$\sigma_{11}(x_1,x_2) = 2\frac{\sigma(x_1)x_2}{t}$$



σ_{11} is *assumed* to be linear, but it is a very good assumption.

- $\sigma(x_1)$ gives the magnitude of the stress at the upper surface.
- $x_2/(t/2)$ gives the normalized shape from bottom (x_2= -t/2) to top (x_2= t/2)

Balance of Moments



Moment due to weight outboard from x_1

$$M_L = \int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' + P_2(L - x_1) - P_1(u_2(L) - u_2(x_1))$$

Moment due to stress in plane at x_1

$$M_t = + \int_{-t/2}^{t/2} \sigma_{11}(x_1, x_2')x_2' dx_2'$$

Recall

$$F_2(x_1') = -\rho g t + N_{top} - N_{bottom}$$

$$F_1(x_1') = S_{top} - S_{bottom}$$

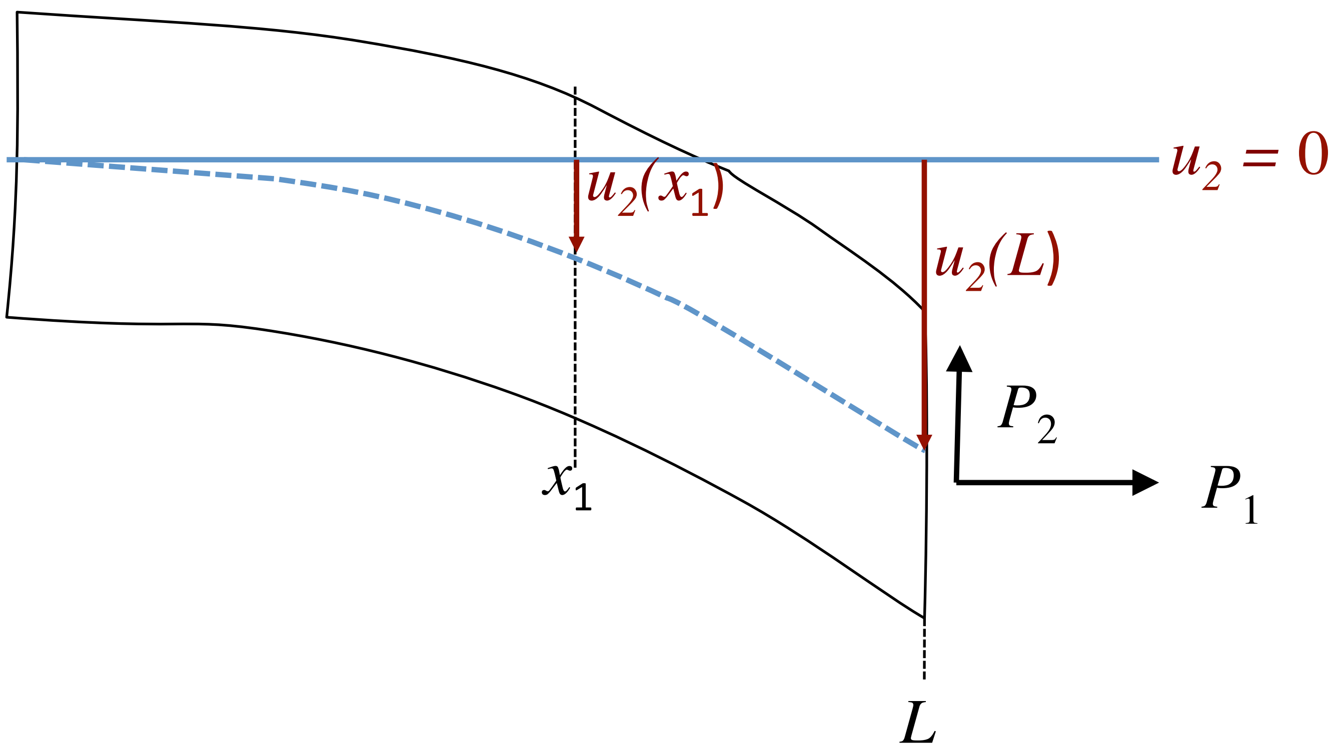
Zero net moment

$$M_t + M_L = 0$$

Calculate $M_t(x_1)$ from known geometry and forces

$$M_t(x_1) = -\int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$

With deflection



Calculate $M_t(x_1)$ from known geometry and forces

$$M_t(x_1) = -\int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$

Integral equations are messy …

$$M_t(x_1) = -\int_{x_1}^{L} F_2(x_1')(x_1' - x_1)dx_1' - P_2(L - x_1) + P_1(u_2(L) - u_2(x_1))$$

So we can differentiate it twice with respect to x_1

$$\frac{d^2 M_t(x_1)}{dx_1^2} = -F_2(x_1) - P_1 \frac{\partial^2 u_2}{\partial x_1^2}$$

Then we use the idea of flexural rigidity *D*, which relates the bending curvature to the force moment.
(D is a property of the geometry and the material.)

$$M_t = -D\frac{\partial^2 u_2}{\partial x_1^2}$$

To get an equation for the shape $\underline{u}_2(x_1)$

$$\frac{\partial^4 u_2}{\partial x_1^4} = \frac{1}{D}\left[F_2(x_1) + P_1\frac{\partial^2 u_2}{\partial x_1^2}\right]$$