ESS 411/511 Geophysical Continuum Mechanics Class #23

Highlights from Class #22 – Barrett Johnson
Today's highlights on Wednesday – Madeleine Lucas

Kinematics of Deformation and Motion

For Wednesday, please read MSM Chapter 4.11 and 4.12

- Velocity gradient and strain rate
- Material derivatives of lines are, and volumes

Problem Set #4

- I'm still working on it ☺

Mid-term

- I'm working on it next …

ESS 411/511 Geophysical Continuum Mechanics

Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Attenuation
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

Kinematics

- Description without reference to forces

Concept of particle in a continuum

- Just an infinitesimal point in the material, labeled with a vector field X

Displacement

- Vector mapping of an object from initial X to final configuration x

Deformation

- Change of shape described by a displacement field

Rigid-body rotation and translation

- No deformation, but displacement can differ from point to point

Strain or distortion

- Elongation or shear

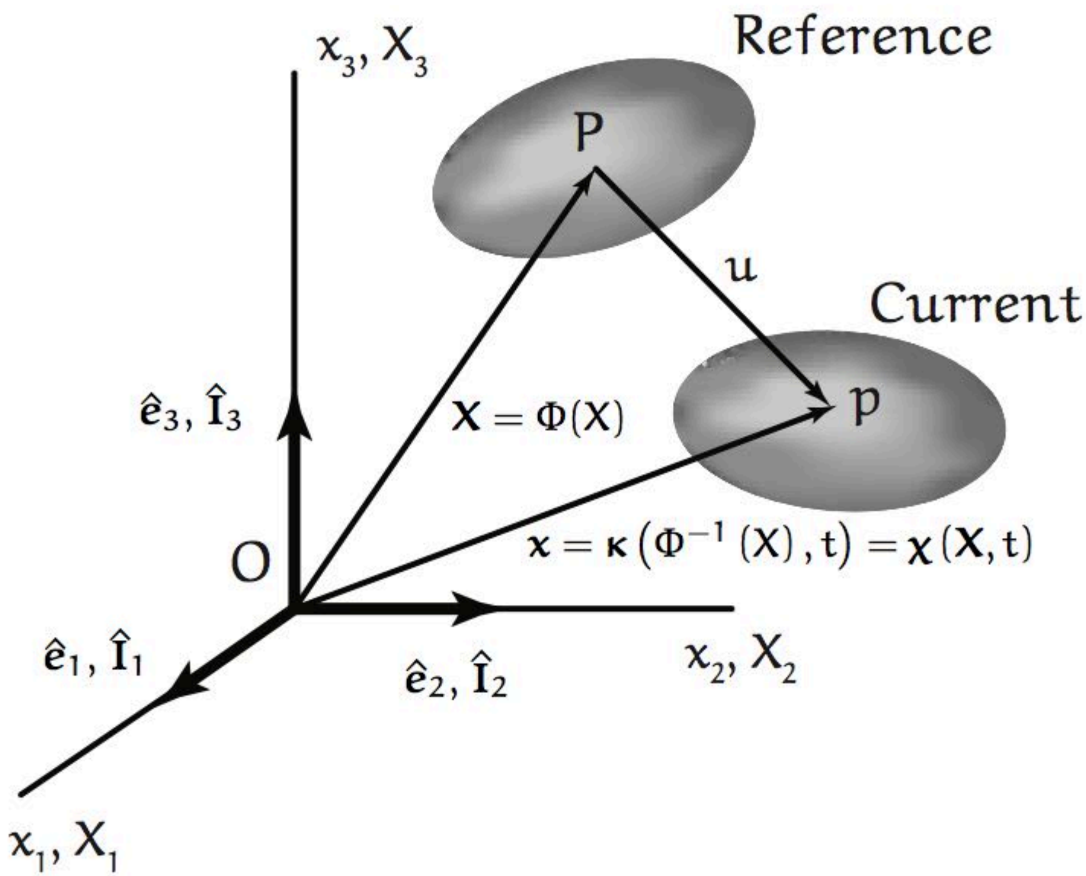
Homogeneous deformation

- Initially straight material lines stay straight

Finite strain

- Material lines can become curved

Initial and Final Configurations



Class-prep: Strain Compatibility (Break-out rooms)

For small strains, the strain is defined (Equation 4.62) as

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial X_A}\delta_{Aj} + \frac{\partial u_j}{\partial X_B}\delta_{Bi} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = u_{i,j} + u_{j,i}$$

where $u_i = x_i - X_A\, d_{iA}$ is displacement, the difference between current and initial positions. (The Kroenecker delta just makes it clear how to relate the "*A*" subscripts in the initial coordinate system to the "*i*" subscripts in the current coordinate system.)

Section 4.8 describes strain compatibility, and the 81 strain-compatibility equations that relate the various second derivatives.

$$\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0$$

Assignment

81 equations– that's a lot of equations!

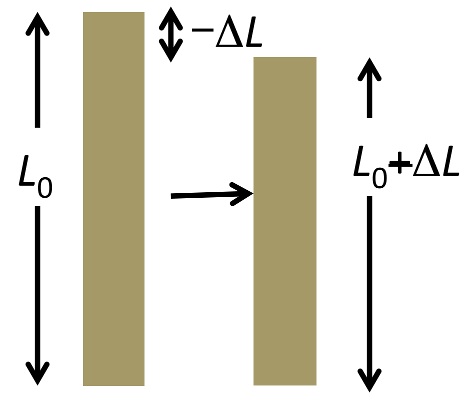
- Why are only 6 of them needed in practice?
- Compatibility with what? What would *strain incompatibility* look like?

On page 129 – "*It may be shown that the compatibility equations, either Eq 4.90 or Eq 4.91, are both necessary and sufficient for a single-valued displacement f*ield of a body occupying a simply connected domain."

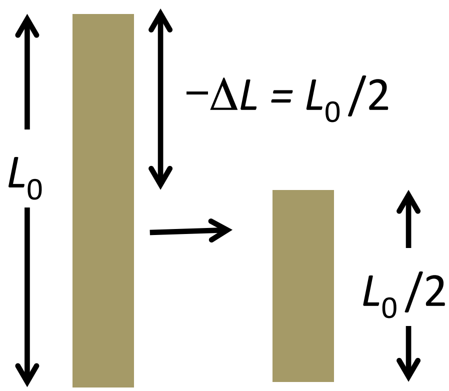
- How would you paraphrase this sentence into simple concepts?

Finite Strain in 1-D

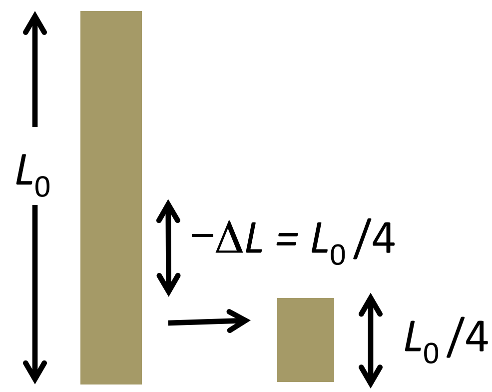
We typically first see strain expressed as $\varepsilon = \dfrac{\Delta L}{L_0}$
and when there is shortening, $\Delta L < 0$



But this approach is not accurate for large strains e.g. when $-\Delta L$ is large fraction of L_0. For example,



$$\varepsilon = \frac{\Delta L}{L_0} = \frac{-L_0/2}{L_0} = -\tfrac{1}{2}$$



$$\varepsilon = \frac{\Delta L}{L_0} = \frac{-L_0/4}{L_0} = -\tfrac{1}{4}$$

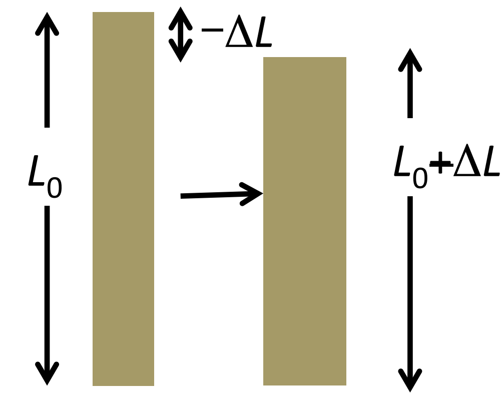
In both steps the height was halved, so the incremental strain should be the same …

Finite Strain in 1-D

We can address large strains by adding up a whole series of small strains e.g. when $-\Delta L$ is large fraction of L_0.

- We have to reset L_0 at each step and call it l

$$\varepsilon = \frac{\Delta L}{L_0}$$



$$\varepsilon = \int_{L_0}^{L} \frac{dl}{l} = \ln(L) - \ln(L_o) = \ln\left(L \big/ L_o \right)$$

?

$$\varepsilon = \ln\left(L \big/ L_o \right) = \ln\left(\frac{L_0 + \Delta L}{L_0} \right) = \ln\left(1 + \frac{\Delta L}{L_0} \right)$$

(remembering that $\Delta L<0$)

Now recall the Taylor series for $\ln(1+x)$ when $-1< x < 1$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$$

With $x= (\Delta L / L_O)$,

$$\varepsilon = \ln\left(1 + \frac{\Delta L}{L_0} \right)$$

$$= \frac{\Delta L}{L_0} - \frac{1}{2}\left(\frac{\Delta L}{L_0} \right)^2 + \text{some small stuff}$$

- The first term looks familiar …
- The second term corrects for the changing l as strain proceeds

A measure for strain $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$

$$
\begin{aligned}
(\mathrm{d}x)^2 - (\mathrm{d}X)^2 &= (x_{i,A}\,\mathrm{d}X_A)(x_{i,B}\,\mathrm{d}X_B) - \delta_{AB}\,\mathrm{d}X_A\,\mathrm{d}X_B \\
&= (x_{i,A}x_{i,B} - \delta_{AB})\,\mathrm{d}X_A\,\mathrm{d}X_B \\
&= (C_{AB} - \delta_{AB})\,\mathrm{d}X_A\,\mathrm{d}X_B
\end{aligned}
$$

Green's deformation tensor

$$C_{AB} = x_{i,A}x_{i,B} \qquad \text{or} \qquad \mathbf{C} = \mathbf{F}^{\mathsf{T}} \cdot \mathbf{F}$$

Lagrangian finite strain tensor

$$2E_{AB} = C_{AB} - \delta_{AB} \qquad \text{or} \qquad 2\mathbf{E} = \mathbf{C} - \mathbf{I}$$

A measure for strain $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$

$$
\begin{aligned}
(\mathrm{d}x)^2 - (\mathrm{d}X)^2 &= \delta_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j - (X_{A,i}\,\mathrm{d}x_i)(X_{A,j}\,\mathrm{d}x_j) \\
&= (\delta_{ij} - X_{A,i}X_{A,j})\,\mathrm{d}x_i\,\mathrm{d}x_j \\
&= (\delta_{ij} - c_{ij})\,\mathrm{d}x_i\,\mathrm{d}x_j
\end{aligned}
$$

Cauchy deformation tensor

$$c_{ij} = X_{A,i}X_{A,j} \qquad \text{or} \qquad \mathbf{c} = (\mathbf{F}^{-1})^{T} \cdot (\mathbf{F}^{-1})$$

Eulerian finite strain tensor

$$2e_{ij} = (\delta_{ij} - c_{ij}) \qquad \text{or} \qquad 2\mathbf{e} = (\mathbf{I} - \mathbf{c})$$

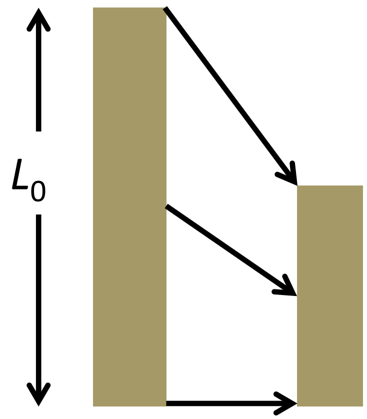
In terms of displacements $u_i = (x_i - X_i)$

$$2E_{AB} = x_{i,A}x_{i,B} - \delta_{AB} = (u_{i,A} + \delta_{iA})(u_{i,B} + \delta_{iB}) - \delta_{AB}$$

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A}u_{i,B}$$

$$2e_{ij} = \delta_{ij} - X_{A,i}X_{A,j} = \delta_{ij} - (\delta_{Ai} - u_{A,i})(\delta_{Aj} - u_{A,j})$$

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{A,i}u_{A,j}$$



A bar is shortened uniformly in the x_1 direction

$x_1 = kX_1$, e.g. $k = 1/2$

$$k = \left({x_1}/{X_1} \right) = \left(\frac{L_0 + \Delta L}{L_0} \right)$$

$$= \left(1 + \frac{\Delta L}{L_0} \right)$$

$$k - 1 = \frac{\Delta L}{L_0}$$

remember $\Delta L<0$

Lagrangian Finite Strain Tensor

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A}u_{i,B} \qquad (**)$$

Let's see how this fancy-pants equation applies to the 1-D case …

The only nonzero terms occur when A=B=i=1

Displacement field
$$u_1 = x_1 - X_1 = kX_1 - X_1$$
$$= (k-1)X_1$$
(particles move down so $u_1<0$)

Displacement gradient
$$u_{1,1} = \frac{\partial u_1}{\partial X_1} = k - 1$$

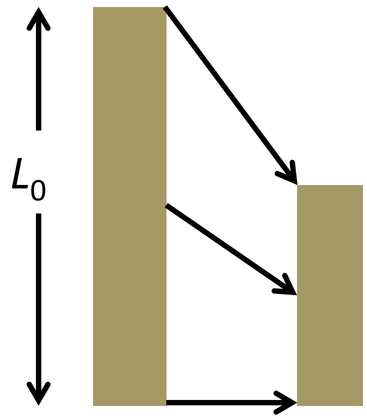
Strain field
$$2E_{AB} = 2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1} \right)^2$$
from (**)

$$= 2(k-1) + (k-1)^2$$
A=B=1

$$2E_{AB} = 2\left(\frac{\Delta L}{L_0} \right) + \left(\frac{\Delta L}{L_0} \right)^2$$
A=B=1

$$\varepsilon = \ln\left(1 + \frac{\Delta L}{L_0}\right)$$

$$= \frac{\Delta L}{L_0} - \frac{1}{2}\left(\frac{\Delta L}{L_0}\right)^2 + \text{some small stuff}$$



A bar is shortened in the x_1 direction

$x_1 = kX_1$, e.g. $k = ½$

Strain is homogeneous since k is a constant

Lagrangian Finite Strain Tensor

$$2E_{AB} = u_{A,B} + u_{B,A} + u_{i,A}u_{i,B}$$

Let's see how this fancy-pants equation applies to the 1-D case …

$$2E_{AB} = 2\left(\frac{\Delta L}{L_0}\right) + \left(\frac{\Delta L}{L_0}\right)^2$$

$$E_{AB} = \left(\frac{\Delta L}{L_0}\right) + \frac{1}{2}\left(\frac{\Delta L}{L_0}\right)^2 \qquad A=B=1$$

Strain tensor

$$E_{AB} = \begin{bmatrix} \left(\left(\frac{\Delta L}{L_0}\right) + \frac{1}{2}\left(\frac{\Delta L}{L_0}\right)^2\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$