ESS 411/511 Geophysical Continuum Mechanics Class #4

Highlights from Class #3 – Andrew Gregovich
Today's highlights on Friday – Madeleine Lucas
Warm-up question (break-out) –

- What does Quality *Q* mean?
- What is it useful for?

Class-prep answers (break-out)

- Discuss your explanations of energy dissipation (or not) in cyclical springs and dash-pots.

ESS 411/511 Geophysical Continuum Mechanics Class #3

## For Friday class

- Please read Mase, Smelser, and Mase, CH 2 through Section 2.3

Your short CR/NC Pre-class prep writing assignment (1 point) in Canvas

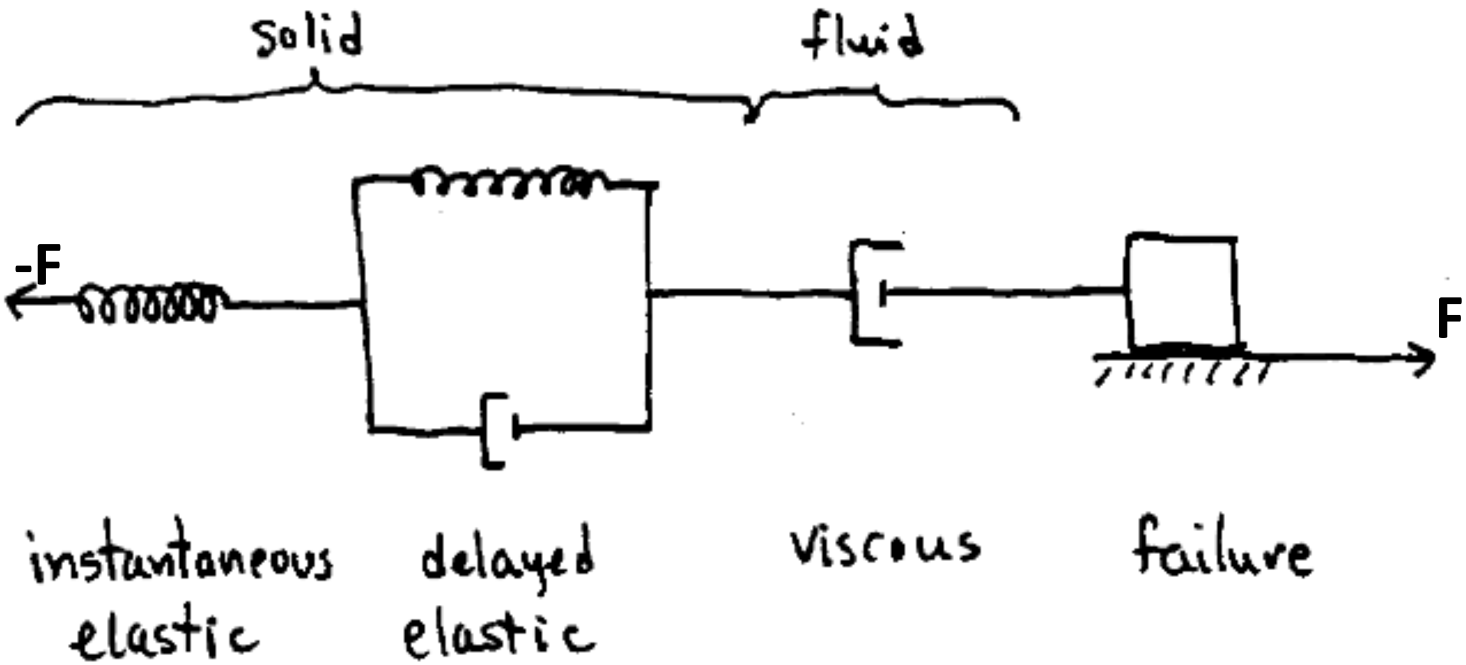
- It will be due in Canvas at the start of class.
- I will send another message when it is posted in Canvas.

ESS 411/511 Geophysical Continuum Mechanics Class #3

## Broad Outline for the Quarter

- Continuum mechanics in 1-D
- 1-D models with springs, dashpots, sliding blocks
- Mathematical tools – vectors, tensors, coordinate changes
- Stress – principal values, Mohr's circles for 3-D stress
- Coulomb failure, pore pressure, crustal strength
- Measuring stress in the Earth
- Strain – Finite strain; infinitesimal strains
- Moments – lithosphere bending; Earthquake moment magnitude
- Conservation laws
- Constitutive relations for elastic and viscous materials
- Elastic waves; kinematic waves

# A model for idealized real materials




Forces are balanced

- Each element feels the same force **F**

# Rheological tests

**Creep tests**

- Apply a constant stress $\sigma$
  e.g. put a weight on top of a sample
- Measure strain *e(t)* or strain rate $\dot{e}(t)$

**Relaxation tests**

- Apply an abrupt strain *e,* then hold it constant
  e.g. abrupt shortening in a vice.
- Measure stress $\sigma(t)$ as sample adjusts.

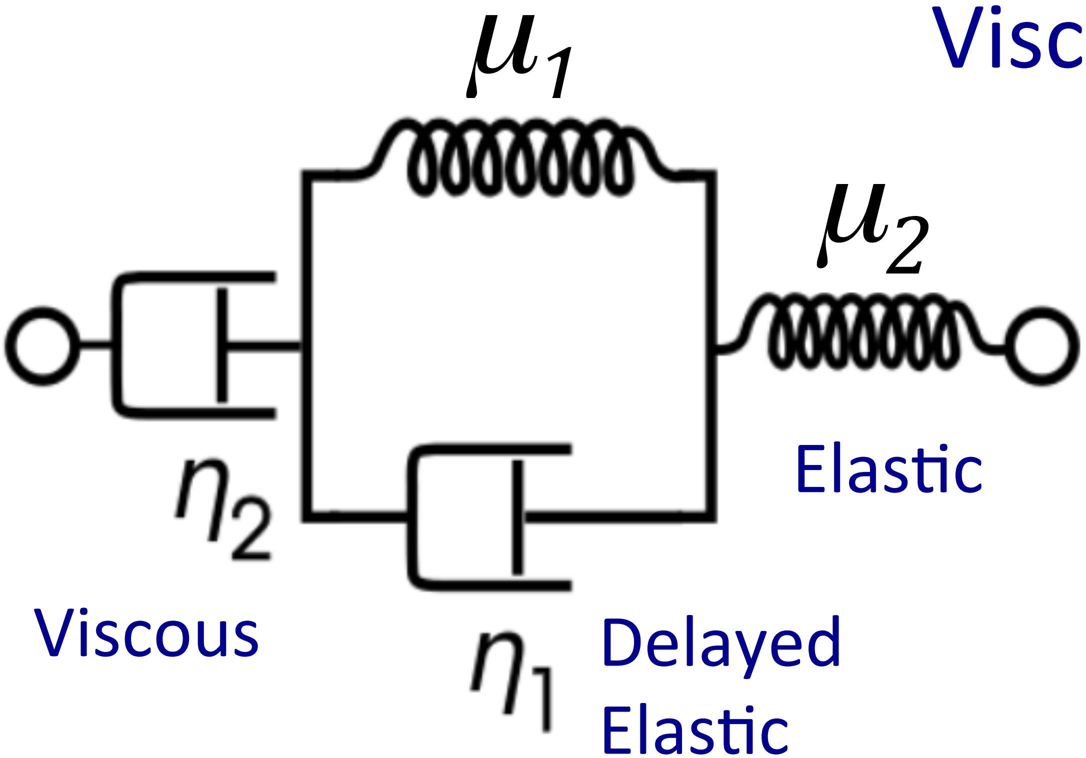
**Constant strain-rate tests**

- Apply a constant strain rate
  e.g. with a motor-driven vice
- Measure stress $\sigma(t)$

# Models for linear solids

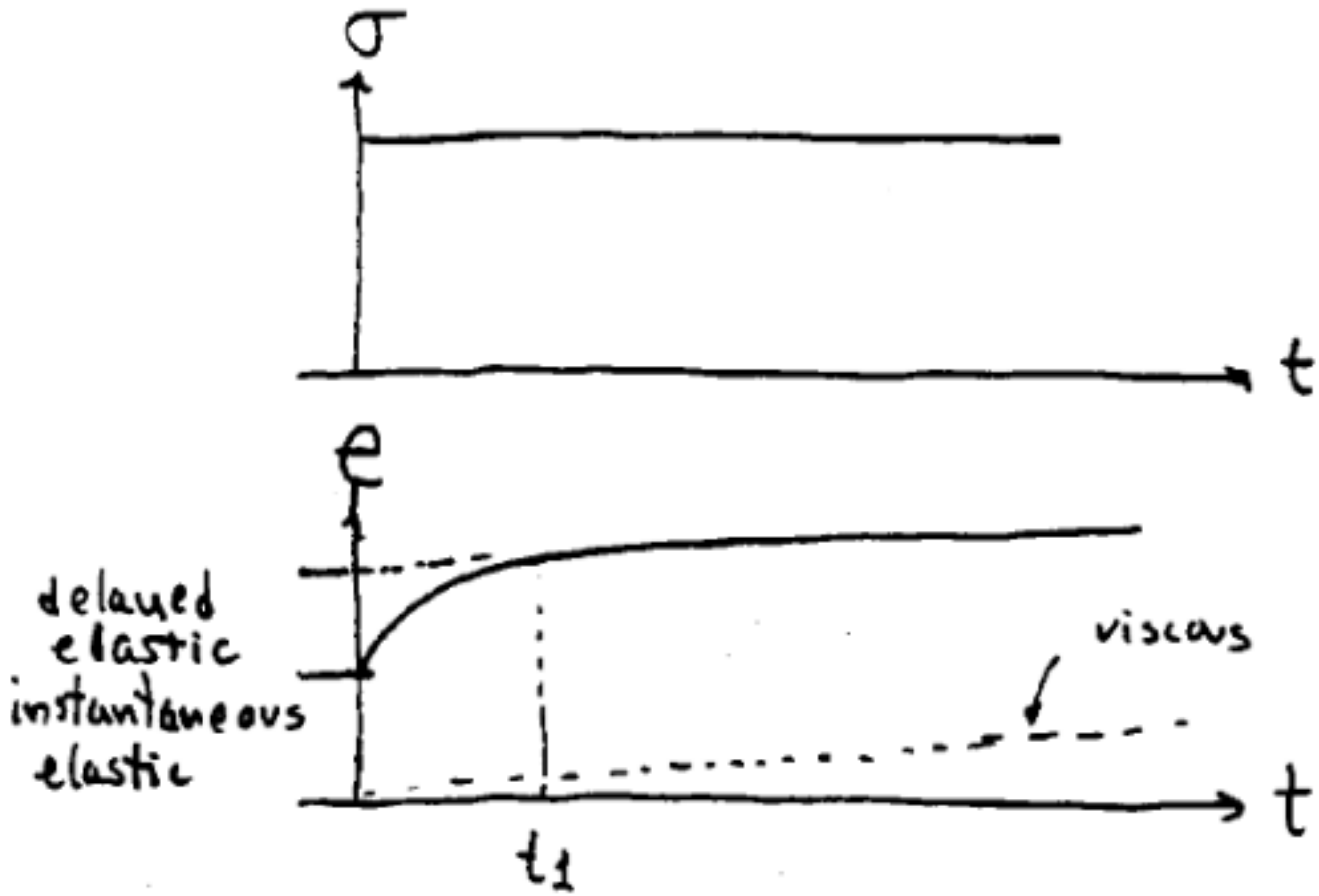
Those springs and dashpots …

# Viscoelastic model

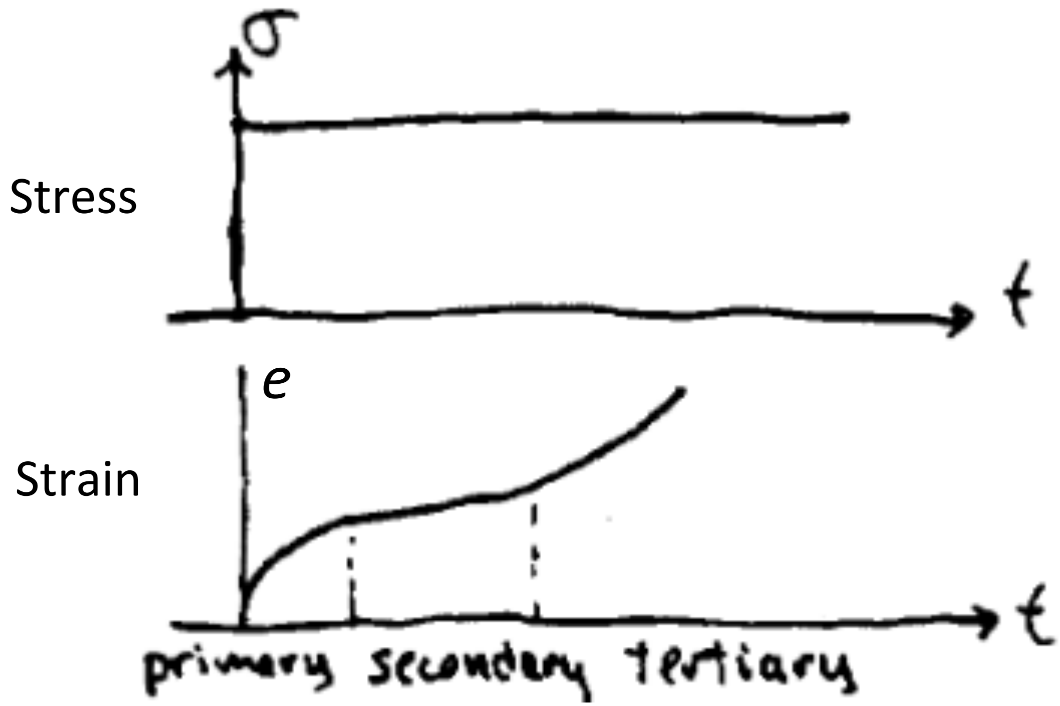



Called *Maxwell Solid*, if $\eta_1 = \infty$, $\mu_1 = \infty$
Called *Kelvin-Voigt Solid*, if $\eta_2 = \infty$, $\mu_2 = \infty$
Called *Standard Linear Solid*, if $\eta_2 = \infty$

# Creep Test with Viscoelastic model



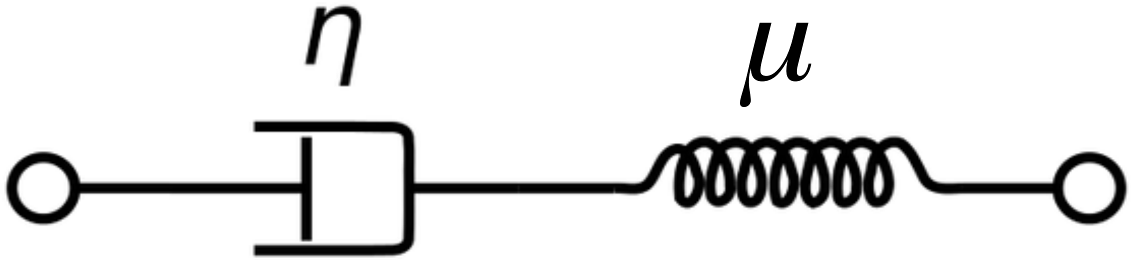
# Viscoelastic behavior in real materials

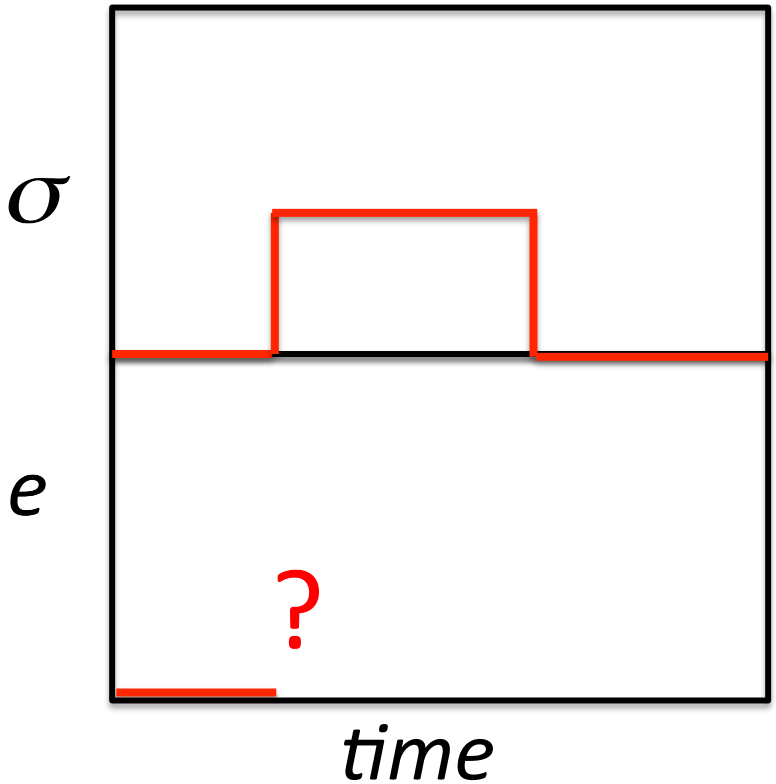



Changes in the microstructure at the crystal level inside the sample can alter the effective viscosity after significant strain as the test progresses.

- e.g. crystal basal planes align for easy glide
- microcracks may develop, allowing internal slip

# Maxwell solid

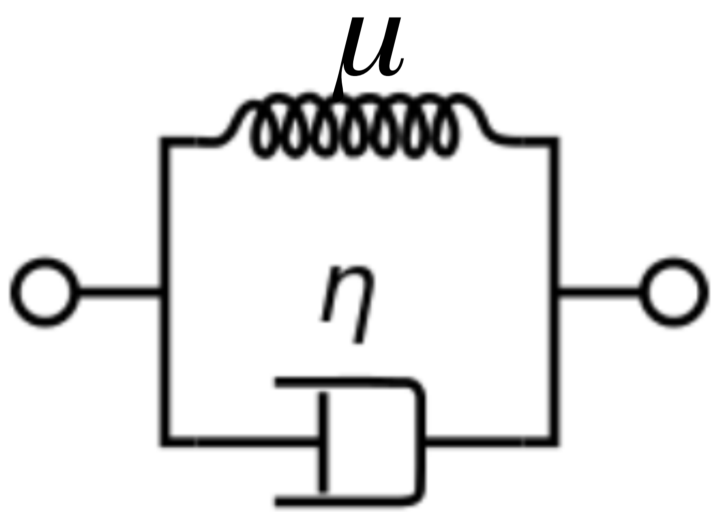


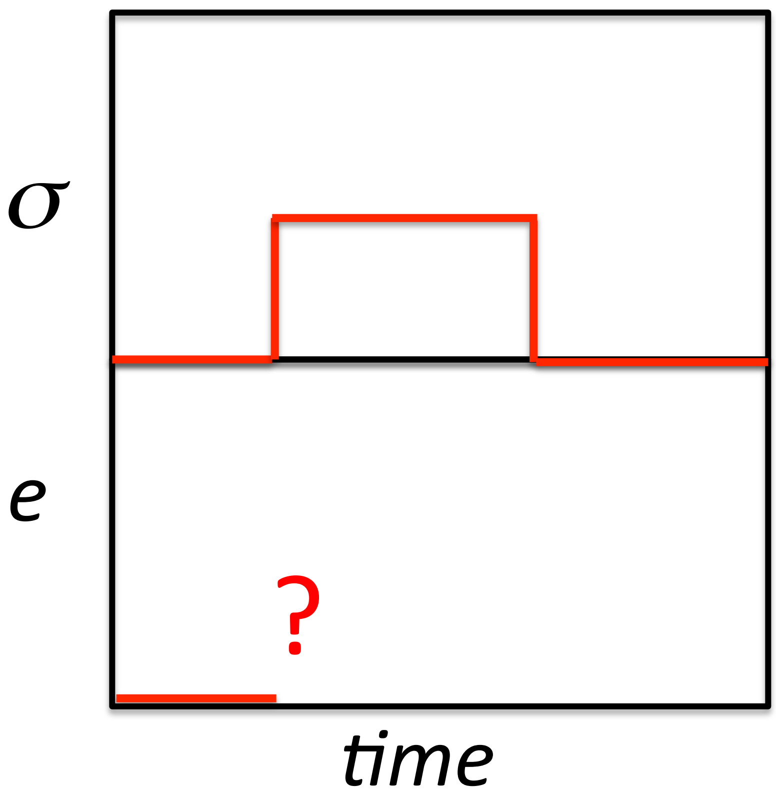




Also Homework set #1

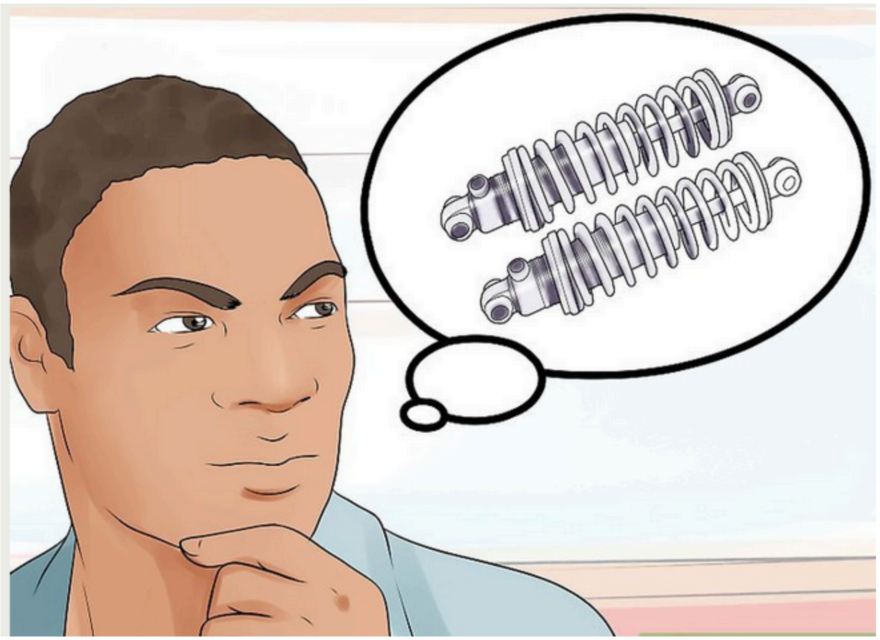
- Is there a characteristic time for the material?
- $\eta/\mu$ ?

# Kelvin-Voigt solid

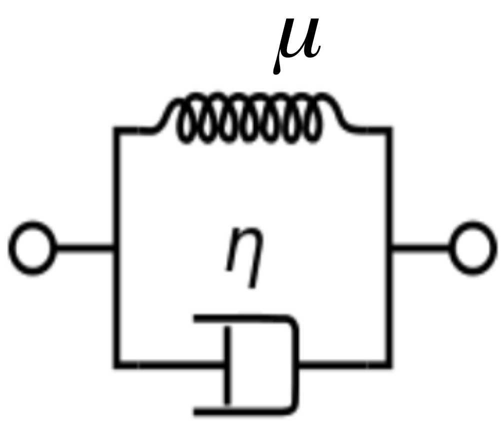


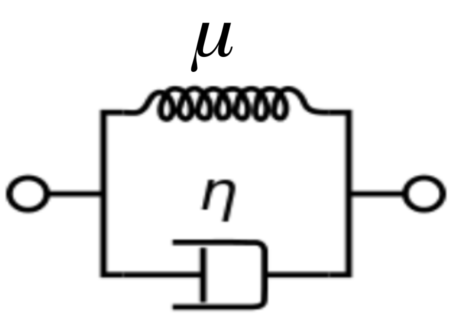

- Is there a characteristic time for the material?
- $\eta/\mu$ ?



A shock absorber can be modeled as a delayed elasticity Kelvin-Voigt solid




# Kelvin-Voigt Response




Spring and dashpot together support stress $\sigma$

$$\sigma(t) = \mu\ e(t) + \eta\ \dot{e}(t)$$

- At $t = 0$, spring hasn't shortened; dashpot supports all the stress $\sigma$, so

  $e(0) = 0$ (*)
- At $t = \infty$, dashpot has stopped; spring supports all the stress $\sigma$, so

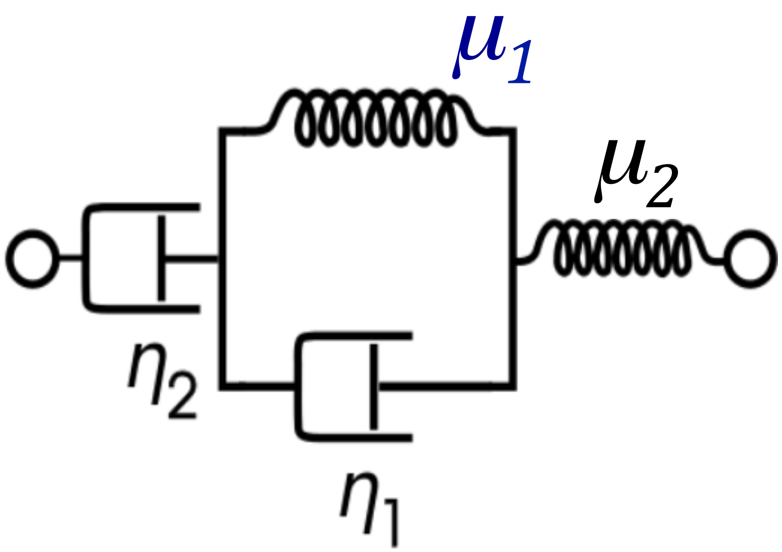
  $e(\infty) = \sigma/\mu$ (**)
- The transition is probably a decaying exponential.
- $\tau=\eta/\mu$ must be the time constant defining the transition.

$$e(t) = \frac{\sigma}{\mu} + \mathrm{A}\exp\left(-\frac{\eta}{\mu}t\right)$$

With the boundary conditions (*) and (**), A can be found, and solution is …

$$e(t) = \frac{\sigma}{\mu}\left[1 - \exp\left(-\frac{\mu}{\eta}t\right)\right]$$

# Generalized linear viscoelastic solid




Response to
constant loading $\sigma$

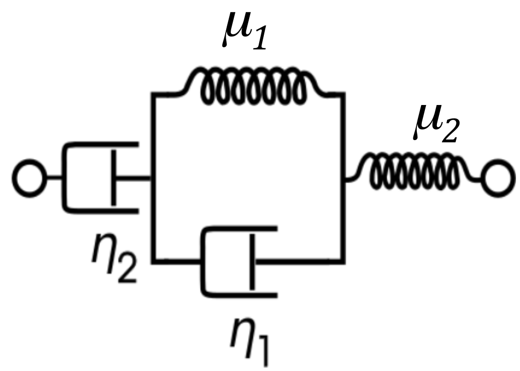
$$e(t) = \frac{\sigma}{\eta_2}t + \frac{\sigma}{\mu_2} + \frac{\sigma}{\mu_1}\left[1 - \exp\left(-\frac{\mu_1}{\eta_1}t\right)\right]$$

Viscous

Elastic

Delayed
Elastic

# How did we get that?!




$$e(t) = \frac{\sigma}{\eta_2}t + \frac{\sigma}{\mu_2} + \frac{\sigma}{\mu_1}\left[1 - \exp\left(-\frac{\mu_1}{\eta_1}t\right)\right]$$

Viscous Elastic Delayed Elastic

Each element feels the same stress $\sigma$,

- We just added up the strains in each element

The Raymond notes also give creep functions and relaxation functions for step changes in stress or strain

Creep test

$$\sigma(t) = \begin{cases} 0 & t=0 \\ 1 & t \geq 0 \end{cases} \qquad \sigma'(t) = \delta(t)$$

Applied stress $\sigma$

$$e(t) = \int_0^t C(t-t')\delta(t')dt' = C(t)$$

$C(t\text{-}t')$ is the creep function

Relaxation test

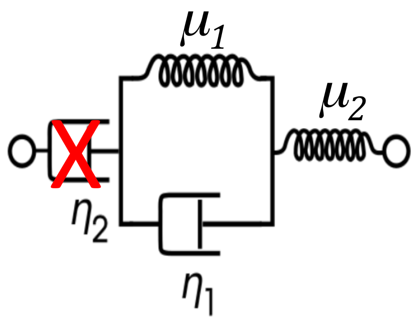
$$e(t) = \begin{cases} 0 & t=0 \\ 1 & t \geq 0 \end{cases} \qquad e'(t) = \delta(t)$$

Applied strain $\sigma$

$$\sigma(t) = \int_0^t k(t-t')\delta(t')dt' = k(t)$$

$k(t\text{-}t')$ is the relaxation function

# Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)




$e \quad = \quad$ constant (after $t$=0)

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1+\mu_2}\left\{1+\frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1+\mu_2}{\eta_1}t\right)\right\} \quad (*)$$

**At $t$=0 :**

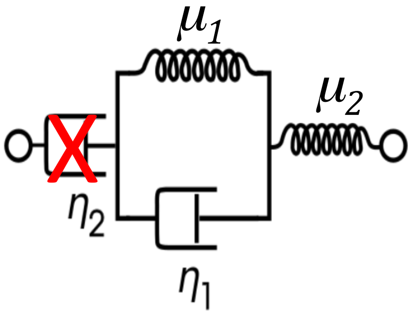
- The spring $\mu_1$ in the K-V element is prevented from deforming, due to $\eta_1$.
- All applied strain $e$ is taken up initially in the spring $\mu_2$. So $\sigma(0) = \mu_2 e$
  (Do you agree that (*) shows this?)
- Stress $\sigma(0)$ also acts on the K-V element, so it also begins to strain.

For a K-V element, $e(t) = \dfrac{\sigma}{\mu}\left[1 - \exp\left(-\dfrac{\mu}{\eta}t\right)\right]$ (strain $e(0) = 0$ ✓).

By differentiating with respect to time $t$, $\dot{e}(t) = \dfrac{\sigma}{\mu}\left[\left(\dfrac{\mu}{\eta}\right)\exp\left(-\dfrac{\mu}{\eta}t\right)\right] = \dfrac{\sigma}{\eta}\exp\left(-\dfrac{\mu}{\eta}t\right)$

At $t$=0, the strain *rate* in the K-V element is $\dot{e}(0) = \dfrac{\sigma}{\eta} = \dfrac{\mu_2 e}{\eta}$

# Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)




$$e = \text{constant}$$

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$

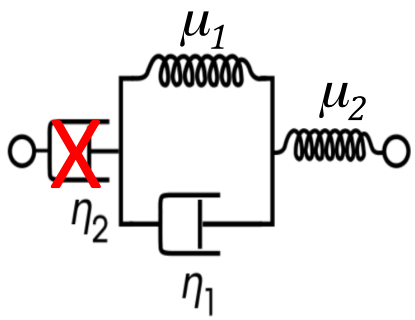
**At t > 0**

- The K-V element is starting to strain at the rate $\dot{e}(0) = \dfrac{\mu_2 e}{\eta}$ ,
- K-V begins to take over some of the strain from the spring $\mu_2$.
- Strain $e_1$ increases in spring $\mu_1$ and $\eta_1$, and strain $e_2$ decreases in spring $\mu_2$

  $$e_1 + e_2 = e$$
- Because strain is decreasing in spring $\mu_2$, stress $\sigma(t)$ must be decreasing.
- Strain $e_1$ in spring $\mu_1$ cannot exceed $\sigma_\infty/\mu_1$
- Dash-pot $\eta_1$ must eventually stop moving.
- This means there is no stress in the dash-pot at $t=t_\infty$
- There will be a time constant $\tau$ that depends on $\mu_{2,}$ $\mu_1$ and $\eta_1$

$$\tau = \left[\frac{\eta_1}{\mu_1 + \mu_2}\right]$$

# Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)




$$e \quad = \quad \text{constant}$$

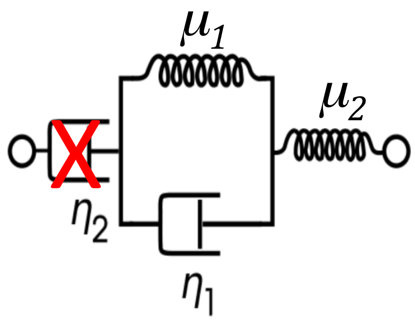
$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$

**At t = ∞**

- Strain $e_1$ in spring $\mu_1$ cannot exceed $\sigma_\infty/\mu_1$
- Dash-pot $\eta_1$ must eventually stop moving.
- This means there is no stress in the dash-pot at $t=t_\infty$
- Both springs $\mu_1$ and $\mu_2$ then support the same stress $\sigma_\infty$, so

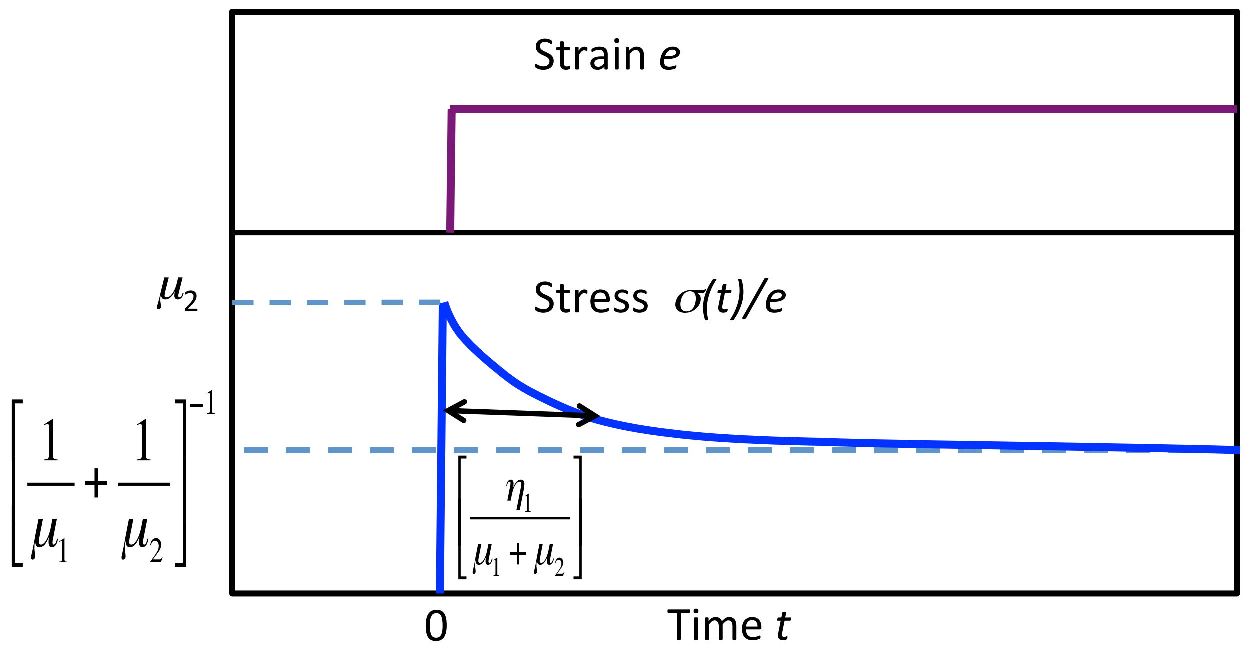
$$e = \frac{\sigma_\infty}{\mu_1} + \frac{\sigma_\infty}{\mu_2}$$ or $$\frac{\sigma_\infty}{e} = \left[\frac{1}{\mu_1} + \frac{1}{\mu_2}\right]^{-1}$$ is the limiting stress at $t=t_\infty$

# Relaxation Function in Standard Linear Solid ($\eta_2 = \infty$)




$$e = \text{constant}$$

$$\sigma(t) = e\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\left\{1 + \frac{\mu_2}{\mu_1}\exp\left(-\frac{\mu_1 + \mu_2}{\eta_1}t\right)\right\} \quad (*)$$



# Energy and Work

Work W is force ***F*** acting through a distance *d*

Work for point particles: W = ***F*** *d*

In Continuum – work done per unit volume:

$$\frac{W}{V} = \frac{Fd}{V} = \left(\frac{F}{A}\right) \bullet \left(\frac{d}{l}\right) = \sigma e$$ *= stress × strain*

*Rate* of doing work per unit volume

$$\frac{d}{dt}\left(\frac{W}{V}\right) = \frac{\dot{W}}{V} = \frac{F\dot{d}}{V} = \left(\frac{F}{A}\right) \bullet \left(\frac{\dot{d}}{l}\right) = \sigma \dot{e}$$ *= stress × strain rate*

(overdots indicate time derivatives)

# Energy and Work

Total energy input between from time 0 to *t*

$$\Delta E(t) = \int_0^t \sigma(t')\dot{e}(t')dt'$$

For elastic material, substitute: $\sigma(t) = \mu e(t)$

$$\Delta E(t) = \int_0^t \mu e(t')\dot{e}(t') \; dt' = \frac{1}{2}\mu e^2(t) = \frac{\sigma^2(t)}{2\mu}$$

$\Delta E(t)$ returns to zero whenever $\sigma$ returns to zero.

- All energy is recovered

# Energy and Work

Total energy input between from time 0 to *t*

$$\Delta E(t) = \int_0^t \sigma(t')\dot{e}(t')dt'$$

For viscous material:

$$\sigma(t) = \eta\dot{e}(t)$$

$$\Delta E(t) = \int_0^t \eta\dot{e}(t)^2 dt'$$

The integrand is always positive.

- $\Delta E(t)$ can never return to zero if strain rate is ever nonzero
- energy is always lost if any strain has occurred.