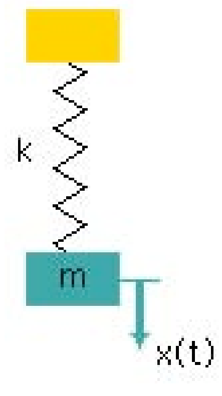
# CHAPTER 4: Some methods for finding natural frequencies.

We have considered two systems each with two degrees-of-freedom. In chapter 2 we considered an axial system comprising two masses and two springs and in chapter 3 a rigid beam mounted on a spring at each end. These two systems will be used to illustrate various methods of finding natural frequencies and mode shapes. The methods can be applied to much more complex systems with many degrees-of-freedom but are more easily understood when applied to the two degree-of-freedom systems considered previously. It is useful at this stage to introduce the concept of principal coordinates as this will confirm the importance of natural frequencies and mode shapes.

## 4.1 Principal Coordinates



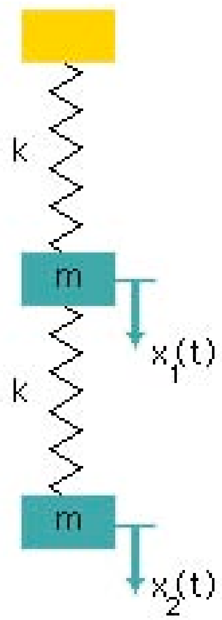

It was shown that for a single degree of freedom system the equation of motion was

$$mx'' + kx = 0 \quad \ldots\ldots\ldots\ldots \quad (1.1)$$

which can be rearranged to give

$$x'' + \omega_n^2 x = 0 \quad \ldots\ldots\ldots\ldots \quad (4.1)$$

where $\omega_n = \sqrt{\dfrac{k}{m}}$ the natural frequency of the system




For the axial two degree-of-freedom system with $m_1$=$m_2$=m and $k_1$=$k_2$=k, the equations of motion were

$$mx_1'' = -kx_1 - k(x_1 - x_2) \quad \ldots\ldots\ldots\ldots \quad (2.1)$$
$$mx_2'' = k(x_1 - x_2) \quad \ldots\ldots\ldots\ldots \quad (2.2)$$

It would be useful if we could write these equations as two single degree-of-freedom system equations similar to (4.1) such that,

$$p_1'' + \omega_{n1}^2 p_1 = 0 \quad \ldots\ldots\ldots\ldots \quad (4.2)$$
$$p_2'' + \omega_{n2}^2 p_2 = 0 \quad \ldots\ldots\ldots\ldots \quad (4.3)$$

as then the maths would be much simpler - we could just use the results from a single degree-of-freedom system. In fact this is possible if we make

$$x_1 = p_1 + p_2 \quad \ldots\ldots\ldots\ldots \quad (4.4)$$
$$x_2 = u_1 p_1 + u_2 p_2 \quad \ldots\ldots\ldots\ldots \quad (4.5)$$

Substitute (4.1)and (4.5) in (2.1) and (2.2)

$$m(p_1'' + p_2'') = -k(p_1 + p_2) - k(p_1 + p_2 - u_1 p_1 - u_2 p_2) \quad \ldots\ldots\ldots\ldots \quad (4.6)$$
$$m(u_1 p_1'' + u_2 p_2'') = k(p_1 + p_2 - u_1 p_1 - u_2 p_2) \quad \ldots\ldots\ldots\ldots \quad (4.7)$$

Now eliminate $p_2$ from the LHS by (4.6)x $u_2$-(4.7)

$$mp_1''(u_2 - u_1) = -kp_1(2u_2 - u_2u_1 + 1 - u_1) - kp_2(u_2 - u_2u_2 + 1) \quad \text{................} \quad (4.8)$$

and also eliminate $p_1$ from the LHS by (4.6)x $u_1$-(4.7)

$$mp_2''(u_1 - u_2) = -kp_1(u_1 - u_1u_1 + 1) - kp_2(2u_1 - u_1u_2 + 1 - u_2) \quad \text{..................} \quad (4.9)$$

To remove $p_2$ from RHS of (4.8) so that (4.8) only involves $p_1$ requires $u_2 - u_2u_2 + 1 = 0$
and to remove $p_1$ from RHS of (4.9) so that (4.9) only involves $p_2$ also requires
$u_1 - u_1u_1 + 1 = 0$

Thus solving $u - u^2 + 1 = 0$ will give the values of $u_1$ and $u_2$ that will achieve our goal.

$$u^2 - u - 1 = 0$$

$$\therefore u = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2} = 1.618 \text{ or } -0.618$$

Thus equations (4.4) and (4.5) become

$$x_1 = p_1 + p_2 \quad \text{.......................................................} \quad (4.10)$$

$$x_2 = 1.618p_1 - 0.618p_2 \quad \text{...............................................} \quad (4.11)$$

It is of interest to note that the ratio of the $p_1$ coefficients is

$$\frac{x_1}{x_2} = \frac{1}{1.618} = 0.618 \quad \text{the first mode shape}$$

and the ratio of the $p_2$ coefficients is

$$\frac{x_1}{x_2} = \frac{1}{-0.618} = -1.618 \quad \text{the second mode shape.}$$

It remains to determine if $\omega_{n1}$ in equation (4.2) and $\omega_{n2}$ in equation (4.3) are the first and second mode natural frequencies.

Substituting from equations (4.10) and (4.11) in equation (4.8)

$$mp_1''(-0.618 - 1.618) = -kp_1(-1.236 + 1 + 1 - 1.618)$$

$$\therefore -2.236mp_1'' = 0.854kp_1$$

$$\therefore p_1'' + 0.382\frac{k}{m}p_1 = 0$$

and hence if $\omega_{n1}^2 = 0.382\frac{k}{m}$ and $\omega_{n1} = 0.618\sqrt{\frac{k}{m}}$ the first natural frequency we may write

$$p_1'' + \omega_{n1}^2 p_1 = 0 \quad \text{..............................................................} \quad (4.2)$$

Substituting from equations (4.10) and (4.11) in equation (4.9)

$$mp_2''(1.618+0.618) = -kp_2(3.236+1+1+0.618)$$
$$2.236mp_2'' = -5.854kp_2$$
$$p_2''+2.618\frac{k}{m}p_2 = 0$$

and hence if $\omega_{n2}^2 = 2.618\frac{k}{m}$ and $\omega_{n2} = 1.618\sqrt{\frac{k}{m}}$ the second natural frequency we may write

$$p_2''+\omega_{n2}^2 p_2 = 0 \quad \text{.......} \quad (4.3)$$

Equations (4.4) and (4.5)

$$x_1 = p_1 + p_2 \quad \text{.......} \quad (4.4)$$
$$x_2 = u_1 p_1 + u_2 p_2 \quad \text{.......} \quad (4.5)$$

may be re-arranged to give

$$p_1 = \frac{x_2 - u_2 x_1}{u_1 - u_2} \quad \text{.......} \quad (4.12)$$
$$p_2 = \frac{u_1 x_1 - x_2}{u_1 - u_2} \quad \text{.......} \quad (4.13)$$

Thus the principal coordinates may be obtained if the mode shapes are known.

Transient vibration: It is possible to use the principal coordinates to determine the transient response resulting from initial conditions with no external excitation. Equations (4.2) and (4.3) are of the form,

$$x'' + \omega_n^2 x = 0$$

for which the solution is

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

Thus

$$p_1(t) = A\cos\omega_{n1} t + B\sin\omega_{n1} t$$
$$\text{and } p_2(t) = C\cos\omega_{n2} t + D\sin\omega_{n2} t$$

If as in chapter 2 the initial conditions are taken to be $x_2'(0) = x_2(0) = x_1'(0) = 0$ and $x_1(0) = X_o$ then substituting in equations (4.12) and (4.13) with $u_1 = 1.618$ and $u_2 = -0.618$ gives,

$$p_1(0) = \frac{0+0.618X_o}{1.618-(-0.618)} = \frac{0.618X_o}{2.236} = 0.276X_o$$
$$p_2(0) = \frac{1.618X_o - 0}{1.618-(-0.618)} = \frac{1.618X_o}{2.236} = 0.724X_o$$
$$\text{and } p_1'(0) = p_2'(0) = 0$$

Thus

$$p_1(t) = 0.276X_o\cos\omega_{n1} t$$

$$\text{and } p_2(t) = 0.724X_o \cos\omega_{n1}t$$

and substituting in (4.4) gives

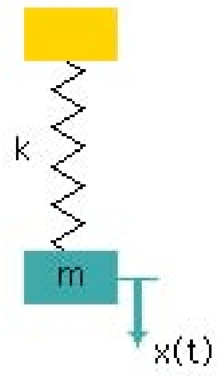
$$x_1 = 0.276X_0 \cos\omega_{n1}t + 0.724X_0 \cos\omega_{n2}t$$ the same as in chapter 2.

Substituting in (4.5) gives

$$x_1 = 0.447X_0 \cos\omega_{n1}t - 0.447X_0 \cos\omega_{n2}t$$ the same as in chapter 2.

**4.2 Energy Methods**

One of the interesting implications of the above is that if the mode shapes are known then it appears that the natural frequencies follow from these shapes. This idea can be used by considering the energy of vibrating systems. First consider a single degree-of-freedom system.




If this system is vibrating sinusoidally, which without any continuing excitation can only be at its natural frequency, then

$$x(t) = A\sin(\omega t + \phi)$$

Consider the maximum energy stored in the spring, which is when the extension is a maximum then

$$V_{max} = \frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2$$

Now consider the maximum kinetic energy of the mass. The velocity of the mass is

$$v = x'(t) = \omega A\cos(\omega t + \phi).$$

So that $v_{max} = \omega A$ and

$$T_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\omega^2 A^2$$

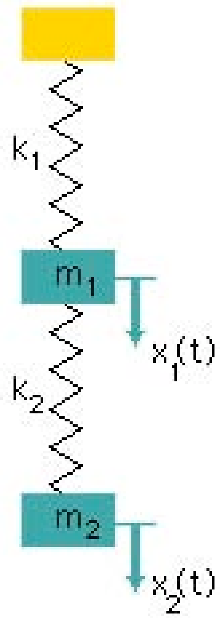
From the principle of conservation of energy $T_{max} = V_{max}$ so that

$$\frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2$$

either A=0 and no motion exists or vibration can exist (at the natural frequency) when

$$\omega = \sqrt{\frac{k}{m}} = \omega_n \text{ the natural frequency of the system}$$

We may use the same method on the two degree-of-freedom system.




If this system is vibrating sinusoidally, which without any continuing excitation can only be at a natural frequency, then

$$x_1(t) = A_1 \sin(\omega t + \phi) \quad \text{and} \quad x_2(t) = A_2 \sin(\omega t + \phi)$$

The maximum energy stored in the springs is

$$V_{max} = \frac{1}{2}k_1 x_{1\,max}^2 + \frac{1}{2}k_2\left(x_{2\,max} - x_{1\,max}\right)^2 = \frac{1}{2}k_1 A_1^2 + \frac{1}{2}k_2\left(A_2 - A_1\right)^2$$

and the maximum kinetic energy is

$$T_{max} = \frac{1}{2}m_1 v_{1\,max}^2 + \frac{1}{2}m_2 v_{2\,max}^2 = \frac{1}{2}m_1\omega^2 A_1^2 + \frac{1}{2}m_2\omega^2 A_2^2$$

From the principle of conservation of energy $T_{max} = V_{max}$ so that

$$\frac{1}{2}m_1\omega^2 A_1^2 + \frac{1}{2}m_2\omega^2 A_2^2 = \frac{1}{2}k_1 A_1^2 + \frac{1}{2}k_2\left(A_2 - A_1\right)^2$$

Thus by rearranging we can find an expression for $\omega$

$$\omega^2 = \frac{k_1 A_1^2 + k_2\left(A_2 - A_1\right)^2}{m_1 A_1^2 + m_2 A_2^2}$$

If we let $\eta = \frac{A_1}{A_2}$ (the mode shape) then $\omega^2 = \frac{k_1\eta^2 + k_2(1-\eta)^2}{m_1\eta^2 + m_2}$.

When $m_1 = m_2 = m$ and $k_1 = k_2 = k$,

$$\omega^2 = \frac{k_1\eta^2 + k_2(1-\eta)^2}{m_1\eta^2 + m_2} = \frac{\eta^2 + (1-\eta)^2}{\eta^2 + 1}\frac{k}{m}$$

When we used the method on the one degree-of-freedom system the mode shape was known as there was a single mass with an amplitude A. For this two degree-of-freedom system we have the mode shapes as an unknowns. If they were known we would get the two natural frequencies.

If $\eta = 0.618$

then $\omega^2 = \frac{0.382 + (1-0.618)^2}{0.382+1}\frac{k}{m} = \frac{0.382 + 0.1459}{1.382}\frac{k}{m} = 0.382\frac{k}{m}$ and $\omega_{n1} = 0.618\sqrt{\frac{k}{m}}$

If $\eta = -1.618$

then $\omega^2 = \frac{2.618 + (1+1.618)^2}{2.618+1}\frac{k}{m} = \frac{2.618 + 6.854}{3.618}\frac{k}{m} = 2.618\frac{k}{m}$ and $\omega_{n2} = 1.618\sqrt{\frac{k}{m}}$

What if they are not known. Then we may make a guess.

If we guess $\eta = 0.5$

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then $\omega^2 = \frac{0.25+(1-0.5)^2}{0.25+1}\frac{k}{m} = \frac{0.25+0.25}{1.25}\frac{k}{m} = 0.4\frac{k}{m}$ and $\omega_{n1} = 0.632\sqrt{\frac{k}{m}}$

This is reasonably close.

It is interesting to see how the prediction for ω varies with the presumed mode shape. This is shown in figure 4.1. The main points of interest are that the first natural frequency is a minimum of the curve. It follows that for any assumed mode shape the predicted natural frequency will not be less than the first natural frequency.

Also for this two degree-of-freedom system the predicted natural frequency will not be higher than the second natural frequency.

For more complex systems these conclusions are also valid.

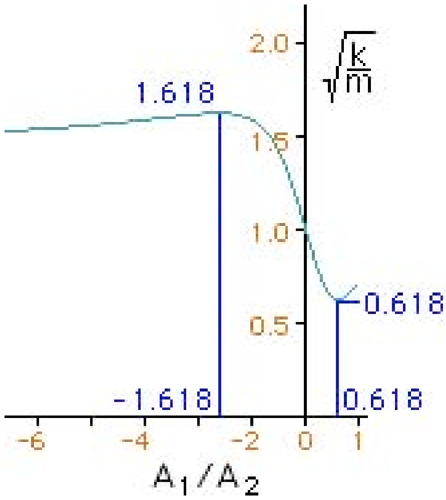
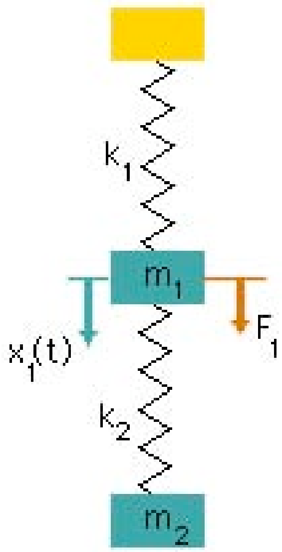



Figure 4.1

## 4.3 Matrix methods

For multi degree-of-freedom vibration it will be useful to use matrix formulations of the equations of motion.

The equations of motion for the axial two degree-of-freedom system without damping are




$$m_1 x_1'' = -k_1 x_1 - k_2(x_1 - x_2) + F_1 \quad \text{......................} \quad (4.1)$$

$$m_2 x_2'' = k_2(x_1 - x_2) \quad \text{................................} \quad (4.2)$$

which may be rearranged as,

$$m_1 x_1'' + x_1(k_1 + k_2) - k_2 x_2 = F_1$$

$$m_2 x_2'' - k_2 x_1 + k_2 x_2 = 0$$

and in matrix form these are

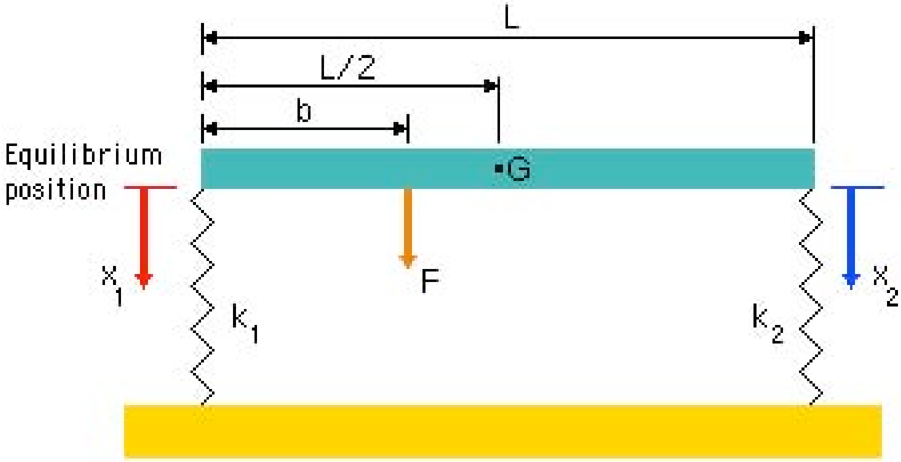
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \quad \text{.......................................} \quad (4.14)$$

which may be written as

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = [\mathbf{F}] \quad \text{..........................................................} \quad (4.15)$$

where $[\mathbf{M}]$ is called the mass matrix and $[\mathbf{K}]$ the stiffness matrix.

Similarly for the beam on springs




The equations of motion for the system without damping are

$$mx_1'' = -4k_1x_1 + 2k_2x_2 + F(4 - 6b/L) \quad \text{......................................} \quad (3.15)$$
$$mx_2'' = 2k_1x_1 - 4k_2x_2 - F(2 - 6b/L) \quad \text{......................................} \quad (3.16)$$

which may be rearranged as,

$$mx_1'' + 4k_1x_1 - 2k_2x_2 = F(4 - 6b/L)$$
$$mx_2'' - 2k_1x_1 + 4k_2x_2 = -F(2 - 6b/L)$$

and in matrix form these are,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} 4k_1 & -2k_2 \\ -2k_1 & 4k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F(4-6b/L) \\ -F(2-6b/L) \end{bmatrix} \quad \text{......................} \quad (4.16)$$

which again may be written as

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = 0 \quad \text{................................................} \quad (4.15)$$

It will be found that for any system we can write the equations of motion in this form and this will apply when there are more than two degrees-of-freedom.

First consider natural frequencies. If it assumed that the system is vibrating sinusoidally so that $[\ddot{\mathbf{X}}] = -\omega^2[\mathbf{X}]$. Then equation (4.15) reduces to,

$$\left[[\mathbf{K}] - \omega^2[\mathbf{M}]\right][\mathbf{X}] = 0 \quad \text{..............................................} \quad (4.16)$$

If we put $\lambda = \omega^2$ and $[\mathbf{X}] = \{u\}$ we obtain the characteristic or eigen value equation of the system:

$$\left[[\mathbf{K}] - \lambda[\mathbf{M}]\right]\{u\} = 0 \quad \text{..................................................} \quad (4.17)$$

For non-zero solutions for {u}

$$\det\left[[\mathbf{K}] - \lambda[\mathbf{M}]\right] = 0 \quad \text{.................................................} \quad (4.18)$$

Solving this equation gives the eigen values $\lambda$ and hence the natural frequencies $\omega = \sqrt{\lambda}$. When each of these values is substituted in (4.17) the associated eigen vector {u} is found and this is the mode shape.

Consider the axial system with $m_1 = m_2 = m$ and $k_1 = k_2 = k$ and no excitation so that equation (4.14) becomes,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Assume that the system is vibrating sinusoidally so that $\left[\ddot{\mathbf{X}}\right] = -\omega^2\left[\mathbf{X}\right]$ ie

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = -\omega^2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

thus

$$-\omega^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

rearranging

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{.............................................} \quad (4.20)$$

for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to be non-zero requires

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = 0$$

$$\text{or} \quad (2k - m\omega^2)(k - m\omega^2) - k^2 = 0$$
$$m^2\omega^4 - 3mk\omega^2 + k^2 = 0$$

This is the natural frequency equation (2.7) found previously. The mode shapes can be found by substituting the natural frequency values into (4.20).

If the system has many degrees-of-freedom the natural frequency equation will be far more difficult to solve. Thus iterative methods are often used (and are the basis for some computer programs).

### 4.3.1 Iterative method

Consider again equation (4.17)

$$\left[[\mathbf{K}] - \lambda[\mathbf{M}]\right]\{u\} = 0 \quad \text{.................................................} \quad (4.17)$$

this can be rearranged as

$$\lambda[\mathbf{M}]\{u\} = [\mathbf{K}]\{u\}$$

and premultiplying by $[\mathbf{M}]^{-1}$ gives,

$$\lambda\{u\} = [\mathbf{M}]^{-1}[\mathbf{K}]\{u\} \quad \text{.......................................................} \quad (4.18)$$

Highest natural frequency.

If we make an initial guess of $\{u\}_1$ then a second estimate $\{u\}_2$ can be made using equation (4.18) so that

$$\lambda^{(1)}\{u\}_2 = [\mathbf{M}]^{-1}[\mathbf{K}]\{u\}_1$$

$\lambda^{(1)}$ is the first estimate of the eigen value if an element of $\{u\}_1$ is made unity and the same element in $\{u\}_2$ is made unity by adjusting $\lambda^{(1)}$. This process is repeated and it will be found that the value converges to the **highest** natural frequency - remember $\lambda = \omega^2$. We will make the first element unity.

For the example above guess the first mode shape to be,

$$\{u\}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(we know the second mode shape will have a sign change in the elements)

Now $[\mathbf{M}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and therefore $[\mathbf{M}]^{-1} = \dfrac{1}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

also $[\mathbf{K}] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

so that $[\mathbf{M}]^{-1}[\mathbf{K}] = \dfrac{k}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \dfrac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

and $[\mathbf{M}]^{-1}[\mathbf{K}]\{u\}_1 = \dfrac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \dfrac{k}{m}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Thus $\lambda^{(1)}\{u\}_2 = \dfrac{3k}{m}\begin{bmatrix} 1 \\ -0.667 \end{bmatrix}$

since the first element in $\{u\}_1$ was unity the first element in $\{u\}_2$ must also be unity and $\{u\}_2 = \begin{bmatrix} 1 \\ -0.667 \end{bmatrix}$.

The first estimate of the eigen value is $\lambda^{(1)} = \dfrac{3k}{m}$ so that $\omega_n{}^{(1)} = \sqrt{\dfrac{3k}{m}} = 1.732\sqrt{\dfrac{k}{m}}$

The second iteration gives

$$\lambda^{(2)}\{u\}_3 = [\mathbf{M}]^{-1}[\mathbf{K}]\{u\}_2$$

$$\lambda^{(2)}\{u\}_3 = \frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.667 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2.667 \\ -1.667 \end{bmatrix} = \frac{2.667k}{m}\begin{bmatrix} 1 \\ -0.625 \end{bmatrix}$$

Thus $\{u\}_3 = \begin{bmatrix} 1 \\ -0.625 \end{bmatrix}$ and $\lambda^{(2)} = \dfrac{2.667k}{m}$ so that $\omega_n{}^{(2)} = \sqrt{\dfrac{2.667k}{m}} = 1.633\sqrt{\dfrac{k}{m}}$

If we remember that the exact values for the highest mode are,

Thus $\{u\} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}$ and $\lambda = \dfrac{2.618k}{m}$ so that $\omega_n = 1.618\sqrt{\dfrac{k}{m}}$ we have got close after 2 iterations.

One more iteration gives,

$$\lambda^{(3)}\{u\}_4 = \frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -0.625 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2.625 \\ -1.625 \end{bmatrix} = \frac{2.625k}{m}\begin{bmatrix} 1 \\ -0.619 \end{bmatrix}$$

Thus $\{u\} = \begin{bmatrix} 1 \\ -0.619 \end{bmatrix}$ and $\lambda = \dfrac{2.625k}{m}$ so that $\omega_n = 1.62\sqrt{\dfrac{k}{m}}$ and we are very close after 3 iterations.

Lowest natural frequency.

Consider again equation (4.17)

$$[[\mathbf{K}] - \lambda[\mathbf{M}]]\{u\} = 0 \quad \text{......................................................} \quad (4.17)$$

this can be rearranged as

$$\frac{1}{\lambda}[\mathbf{K}]\{u\} = [\mathbf{M}]\{u\}$$

and premultiplying by $[\mathbf{K}]^{-1}$ gives,

$$\frac{1}{\lambda}\{u\} = [\mathbf{K}]^{-1}[\mathbf{M}]\{u\} \quad \text{......................................................} \quad (4.19)$$

If we make an initial guess of $\{u\}_1$ then a second estimate $\{u\}_2$ can be made using equation (4.19) so that

$$\frac{1}{\lambda^{(1)}}\{u\}_2 = [\mathbf{K}]^{-1}[\mathbf{M}]\{u\}_1$$

As before $\lambda^{(1)}$ is the first estimate of the eigen value if an element of $\{u\}_1$ is made unity and the same element in $\{u\}_2$ is made unity by adjusting $\lambda^{(1)}$. This process is repeated and it will be found that the value converges to the **lowest** natural frequency - remember $\lambda = \omega^2$. Again making the first element unity.

Guess the first mode shape to be,

$$\{u\}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(we know that for the first mode shape each element has the same sign)

$$[\mathbf{M}] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [\mathbf{K}] = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{so that } [\mathbf{K}]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\text{so that } [\mathbf{K}]^{-1}[\mathbf{M}] = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{m}{k}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\text{and } [\mathbf{K}]^{-1}[\mathbf{M}]\{u\}_1 = \frac{m}{k}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{m}{k}\begin{bmatrix}2\\3\end{bmatrix} = \frac{2m}{k}\begin{bmatrix}1\\1.5\end{bmatrix} = \frac{1}{\lambda^{(1)}}\{u\}_2$$

$$\text{Thus } \{u\}_2 = \begin{bmatrix}1\\1.5\end{bmatrix} \text{ and } \lambda^{(1)} = \frac{k}{2m} \text{ so that } \omega_n{}^{(1)} = \sqrt{\frac{k}{2m}} = 0.707\sqrt{\frac{k}{m}}$$

The second iteration gives

$$[\mathbf{K}]^{-1}[\mathbf{M}]\{u\}_2 = \frac{m}{k}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\1.5\end{bmatrix} = \frac{m}{k}\begin{bmatrix}2.5\\4\end{bmatrix} = \frac{2.5m}{k}\begin{bmatrix}1\\1.6\end{bmatrix} = \frac{1}{\lambda^{(2)}}\{u\}_3$$

$$\text{Thus } \{u\}_3 = \begin{bmatrix}1\\1.6\end{bmatrix} \text{ and } \lambda^{(2)} = \frac{k}{2.5m} \text{ so that } \omega_n{}^{(2)} = \sqrt{\frac{k}{2.5m}} = 0.632\sqrt{\frac{k}{m}}$$

If we remember that the exact values for the lowest mode are,

$$\text{Thus } \{u\} = \begin{bmatrix}1\\1.618\end{bmatrix} \text{ and } \lambda = \frac{k}{2.618m} \text{ so that } \omega_n = 0.618\sqrt{\frac{k}{m}}$$ we have again got close after 2 iterations.

One more iteration gives,

$$[\mathbf{K}]^{-1}[\mathbf{M}]\{u\}_3 = \frac{m}{k}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\1.6\end{bmatrix} = \frac{m}{k}\begin{bmatrix}2.6\\4.2\end{bmatrix} = \frac{2.6m}{k}\begin{bmatrix}1\\1.615\end{bmatrix} = \frac{1}{\lambda^{(3)}}\{u\}_4$$

$$\text{Thus } \{u\} = \begin{bmatrix}1\\1.615\end{bmatrix} \text{ and } \lambda = \frac{k}{2.6m} \text{ so that } \omega_n = 0.62\sqrt{\frac{k}{m}}$$ and we are again very close after 3 iterations.

For the iteration methods, the better the first guess the fewer the number of iterations required to get close to the exact value.

For many methods of solution it is useful to have the mass matrix $[\mathbf{M}]$ and the stiffness matrix $[\mathbf{K}]$ symmetrical. Using Newton's second law to determine the equation of motion does not guarantee symmetrical matrices as can be seen from equation (4.16) which has,

$$[\mathbf{K}] = \begin{bmatrix}4k_1 & -2k_2\\-2k_1 & 4k_2\end{bmatrix}$$

It is possible to obtain the equations using Langrange's equations and have symmetrical matrices.

## 4.4 Lagrange Equations

The conventional form for these equations is,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)+\frac{\partial V}{\partial q_i}=0$$

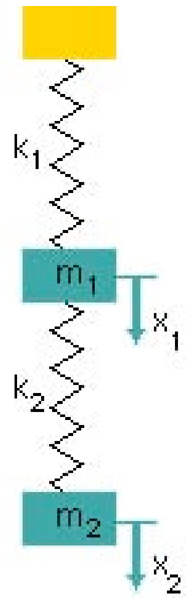
Where,

T is the maximum KE of the system
V is the maximum stored energy in the system
$q_i$ is the ith coordinate.

However a more useful form is

$$\frac{d}{dt^2}\left(\frac{\partial T}{\partial q_i}\right)+\frac{\partial V}{\partial q_i}=0 \quad \text{where} \quad T=\sum\frac{1}{2}m_i x_i^2$$

### 4.4.1 Axial system




For the axial vibration system $q_1 = x_1$ and $q_2 = x_2$ so that

$$T=\frac{1}{2}m_1x_1^2+\frac{1}{2}m_2x_2^2$$

and the stored energy is,

$$V=\frac{1}{2}k_1x_1^2+\frac{1}{2}k_2\left(x_2-x_1\right)^2$$

for the $q_i = x_1$

$$\frac{\partial V}{\partial x_1}=k_1x_1-k_2\left(x_2-x_1\right)$$

$$\text{and} \quad \left(\frac{\partial T}{\partial x_1}\right)=m_1x_1 \quad \text{so that} \quad \frac{d}{dt^2}\left(\frac{\partial T}{\partial x_1}\right)=m_1x_1''$$

Thus $m_1x_1''+k_1x_1-k_2\left(x_2-x_1\right)=0$ as found previously, equation (2.1)

similarly for the $q_i = x_2$

$$\frac{\partial V}{\partial x_2}=k_2\left(x_2-x_1\right)$$

$$\text{and} \quad \left(\frac{\partial T}{\partial x_2}\right)=m_2x_2 \quad \text{so that} \quad \frac{d}{dt^2}\left(\frac{\partial T}{\partial x_2}\right)=m_2x_2''$$

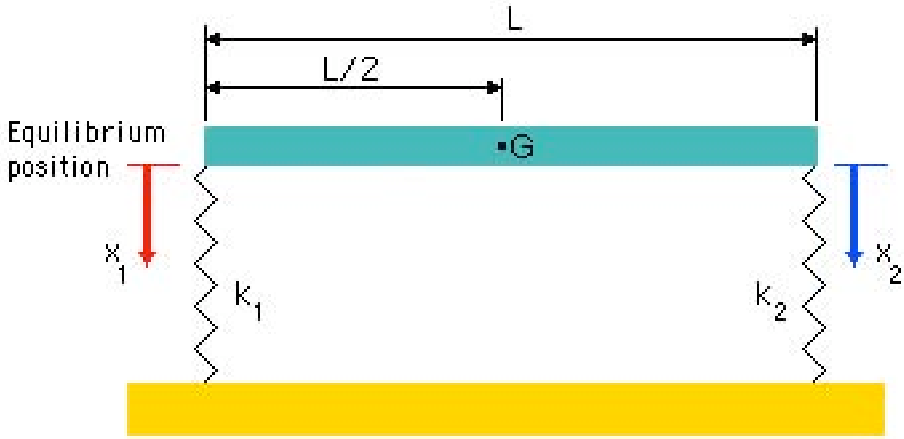
and $m_2x_2''+k_2\left(x_2-x_1\right)=0$ as found previously, equation (2.2)

In matrix form these are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

and both the mass and stiffness matrices are symmetrical.

4.4.2 Beam on springs




$$T = \frac{1}{2} m x_G^2 + \frac{1}{2} I_G \theta^2$$

$$I_G = \frac{mL^2}{12} \text{ (long slender beam)}$$

$$\text{and } V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

We need $x_1$ and $x_2$ in terms of $x_G$ and $\theta$.

From geometry $x_G = (x_1 + x_2)/2$ and $\theta = (x_1 - x_2)/L$ so that ,

$$x_1 = x_G + \frac{\theta L}{2} \text{ and } x_2 = x_G - \frac{\theta L}{2}$$

$$\text{so that } V = \frac{1}{2} k_1 \left( x_G + \frac{\theta L}{2} \right)^2 + \frac{1}{2} k_2 \left( x_G - \frac{\theta L}{2} \right)^2$$

for the $q_i = x_G$

$$\frac{\partial V}{\partial x_G} = k_1 \left( x_G + \frac{\theta L}{2} \right) + k_2 \left( x_G - \frac{\theta L}{2} \right)$$

$$\text{and } \left( \frac{\partial T}{\partial x_G} \right) = m x_G \text{ so that } \frac{d}{dt^2} \left( \frac{\partial T}{\partial x_G} \right) = m x_G''$$

$$m x_G'' + k_1 \left( x_G + \frac{\theta L}{2} \right) + k_2 \left( x_G - \frac{\theta L}{2} \right) = 0$$

for the $q_i = \theta$

$$\frac{\partial V}{\partial \theta} = k_1 \frac{L}{2} \left( x_G + \frac{\theta L}{2} \right) - k_2 \frac{L}{2} \left( x_G - \frac{\theta L}{2} \right)$$

$$\text{and } \left( \frac{\partial T}{\partial \theta} \right) = I_G \theta \text{ so that } \frac{d}{dt^2} \left( \frac{\partial T}{\partial \theta} \right) = I_G \theta''$$

$$I_G\theta'' + k_1\frac{L}{2}\left(x_G + \frac{\theta L}{2}\right) - k_2\frac{L}{2}\left(x_G - \frac{\theta L}{2}\right) = 0$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & I_G \end{bmatrix}\begin{bmatrix} x_G'' \\ \theta'' \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & (k_1 - k_2)L/2 \\ (k_1 - k_2)L/2 & (k_1 + k_2)L^2/4 \end{bmatrix}\begin{bmatrix} x_G \\ \theta \end{bmatrix} = 0$$

The mass and stiffness matrices are symmetrical. We can again write the equations in matrix form as

$$[\mathbf{M}][\ddot{\mathbf{X}}] + [\mathbf{K}][\mathbf{X}] = 0 \quad \text{.........................................................} \quad (4.15)$$

## 4.5 Conclusions.

Various methods have been described that allow the natural frequencies of complex systems to be calculated. Modern computer programs use some of these methods. As long hand calculations are not done for systems with large numbers of degrees of freedom a knowledge of these methods is important to appreciate how natural frequencies are calculated using computers. Often computer programs start from an assumed mode shape and using an iteration technique to find the solution.

The major point of interest is that for all the systems considered thus far, the equations of motion in matrix notation have the same form. Thus it will be useful to have a general approach to matrix solutions. This will be pursued in the next two chapters.

Next chapter

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