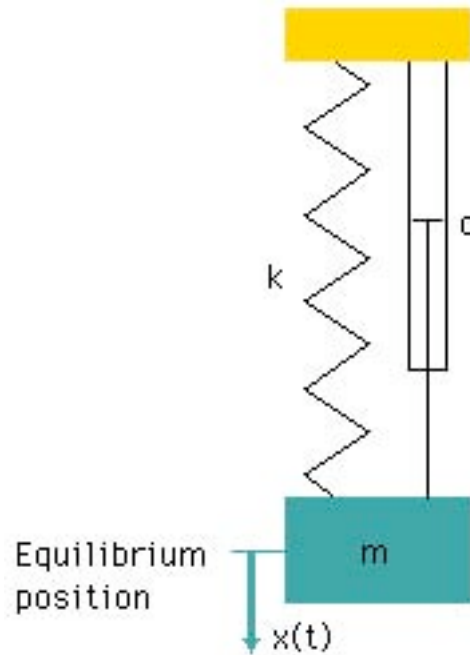


APPENDIX 1: Complex eigenvalues



The equation of motion of the single degree of freedom system shown above is,

$$mx''(t) + cx'(t) + kx(t) = 0 \quad \dots\dots\dots (1.5)$$

Defining the undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ and the damping ratio $\xi = \frac{c}{2\sqrt{mk}}$, equation (1.5) can be rewritten as

$$x'' + 2\xi\omega_n x' + \omega_n^2 x = 0$$

the solution of will be of the form,

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

where the values of A, B, λ_1 and λ_2 may be found from the initial conditions $x(0)$ and $x'(0)$. Thus substituting $x(t) = Ae^{\lambda t}$ in equation (1.5) gives,

$$Ae^{\lambda t}(\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2) = 0 \quad \text{thus} \quad \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

and

$$\lambda = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

and hence if $\xi < 1$ $\lambda_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$ and $\lambda_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$ where $i = \sqrt{-1}$

and therefore

$$x(t) = e^{-\xi\omega_n t} (A_1 e^{i\omega_n\sqrt{1-\xi^2}t} + A_2 e^{-i\omega_n\sqrt{1-\xi^2}t}) \quad \dots\dots\dots (6.73)$$

Now consider the matrix formulation used for more complex systems.

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0 \dots\dots\dots (6.57)$$

As in chapter 6, introducing a velocity vector

$$\{\dot{x}\} = \{y\} \dots\dots\dots (6.58)$$

we obtained the following standard eigenvalue equation:

$$\left(\begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} + \lambda[I] \right) \begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} = 0 \dots\dots\dots (6.60)$$

the eigenvalues are found when,

$$\det \left| \begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} + \lambda[I] \right| = 0$$

If we substitute the single degree of freedom mass and stiffness matrices (1x1 matrices) we obtain,

$$\det \left| \begin{bmatrix} 0 & -1 \\ \frac{k}{m} & \frac{c}{m} \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

expanding

$$\det \left| \begin{bmatrix} \lambda & -1 \\ \frac{k}{m} & \frac{c}{m} + \lambda \end{bmatrix} \right| = 0$$

and

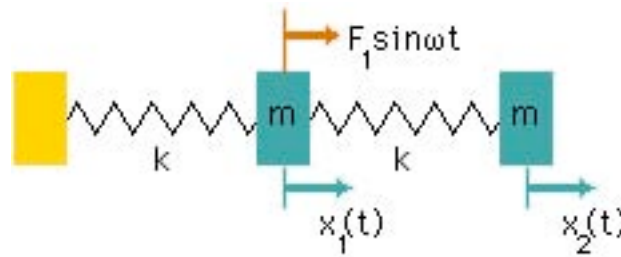
$$\lambda^2 + \lambda \frac{c}{m} + \frac{k}{m} = 0$$

Substituting the undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ and the damping ratio $\xi = \frac{c}{2\sqrt{mk}}$, the equation for λ becomes,

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

and hence as before if $\xi < 0$ $\lambda_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$ and $\lambda_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$ where $i = \sqrt{-1}$

APPENDIX 2: Steady state response of a 2DOF axial spring/mass system, force excitation



For this example let both masses be m and both springs k . The system mass and stiffness matrices are then

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \dots\dots\dots (A2.1)$$

$$[K] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \dots\dots\dots (A2.2)$$

Let the force vector applied to the system be

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \dots\dots\dots (A2.3)$$

This allows us to obtain the response to F_1 alone by putting $F_2 = 0$. However it will also allow the response resulting from F_2 alone to be found by putting $F_1 = 0$.

The displacement vector due to the steady state excitation of this force vector at a frequency ω can be calculated. First the natural frequencies are obtained from the eigenvalue equation of the system:

$$([K] + \lambda^2[M])\{u\} = 0 \dots\dots\dots (6.20)$$

These are given by

$$\det([K] + \lambda^2[M]) = 0 \dots\dots\dots (6.21)$$

For complex systems a computer would be used to produce a solution but for this two degree-of-freedom system we may solve long hand as follows.

$$\det \left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} + \lambda^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} 2k + \lambda^2 m & -k \\ -k & k + \lambda^2 m \end{vmatrix} = 0$$

$$(2k + \lambda^2 m)(k + \lambda^2 m) - (-k)^2 = 0$$

$$m^2 \lambda^4 + 3mk\lambda^2 + k^2 = 0$$

$$\lambda_1 = i0.618\sqrt{\frac{k}{m}}; \lambda_2 = -i0.618\sqrt{\frac{k}{m}}; \lambda_3 = i1.618\sqrt{\frac{k}{m}} \text{ and } \lambda_4 = -i1.618\sqrt{\frac{k}{m}}$$

so that

$$\omega_1 = |\text{Im}(\lambda_1)| = 0.618\sqrt{\frac{k}{m}} \quad \dots\dots\dots (A2.4)$$

$$\omega_2 = |\text{Im}(\lambda_3)| = 1.618\sqrt{\frac{k}{m}} \quad \dots\dots\dots (A2.5)$$

The corresponding eigenvectors are found from equation (6.20). Thus for $\omega_1 = 1.0$, $\lambda^2 = -\omega_1^2 = -1.0$ so that equation (6.20) gives

$$([K] + \lambda^2 [M])\{u\} = 0 \quad \dots\dots\dots (6.20)$$

Thus for $\lambda_1 = 0.382\frac{k}{m}$

$$\left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - 0.382\frac{k}{m} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right) \{u\} = 0$$

$$\begin{pmatrix} 1.618 & -1 \\ -1 & 0.618 \end{pmatrix} \{u\} = 0$$

$$\{u_1\} = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix} \quad \dots\dots\dots (A2.6)$$

(In this example the eigenvectors have not been normalised. The result is the same whether they are normalised or not)

and for $\lambda_2 = 2.618\frac{k}{m}$

$$\left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - 2.618\frac{k}{m} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right) \{u\} = 0$$

$$\begin{pmatrix} -0.618 & -1 \\ -1 & -1.618 \end{pmatrix} \{u\} = 0$$

$$\{u_2\} = \begin{bmatrix} -1.618 \\ 1 \end{bmatrix} \quad \dots\dots\dots (A2.7)$$

It has been shown that

$$\{x\} = \sum_{i=1}^2 \frac{\{u_i\}^T \{F\} \{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t} \quad \dots\dots\dots (6.35)$$

where

$$M_i = \{u_i\}^T [M] \{u_i\} \quad \dots\dots\dots (6.30)$$

Using the values in equations (A2.4 – A2.7)

$$M_1 = \{u_1\}^T [M] \{u_1\} = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0.618 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.618 & 1 \end{bmatrix} \begin{bmatrix} 0.618m \\ m \end{bmatrix} = 1.382m$$

$$M_2 = \{u_2\}^T [M] \{u_2\} = \begin{bmatrix} -1.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} -1.618 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.618 & 1 \end{bmatrix} \begin{bmatrix} -1.618m \\ m \end{bmatrix} = 3.618m$$

Now substituting in equation (6.35)

$$\{x\} = \sum_{i=1}^2 \frac{\{u_i\}^T \{F\} \{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\{u_1\}^T \{F\} \{u_1\}}{M_1(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\{u_2\}^T \{F\} \{u_2\}}{M_2(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and using the values obtained above

$$\{x\} = \frac{\begin{bmatrix} 0.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} -1.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} -1.618 \\ 1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\begin{bmatrix} 0.382F_1 + 0.618F_2 \\ 0.618F_1 + F_2 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} 2.618F_1 - 1.618F_2 \\ -1.618F_1 + F_2 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Thus for $F_2 = 0$

$$\{x\} = \frac{\begin{bmatrix} 0.382F_1 \\ 0.618F_1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} 2.618F_1 \\ -1.618F_1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Since

$$\{x\} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{i\omega t}$$

it follows

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} 0.382F_1 \\ 0.618F_1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} + \frac{\begin{bmatrix} 2.618F_1 \\ -1.618F_1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)}$$

and hence

$$\frac{X_1}{F_1} = \frac{0.276}{m(\omega_1^2 - \omega^2)} + \frac{0.724}{m(\omega_2^2 - \omega^2)}$$

and

$$\frac{X_2}{F_1} = \frac{0.447}{m(\omega_1^2 - \omega^2)} - \frac{0.447}{m(\omega_2^2 - \omega^2)}$$

These equations are usefully rearranged as follows

$$\begin{aligned} \frac{X_2}{F_1} &= \frac{0.276}{m\omega_1^2 \left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.724}{m\omega_2^2 \left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} = \frac{0.276}{m0.382 \frac{k}{m} \left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.724}{m2.618 \frac{k}{m} \left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \\ \frac{kX_1}{F_1} &= \frac{0.724}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.276}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \end{aligned}$$

$$\begin{aligned} \frac{X_2}{F_1} &= \frac{0.447}{m\omega_1^2 \left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.447}{m\omega_2^2 \left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} = \frac{0.447}{m0.382 \frac{k}{m} \left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.447}{m2.618 \frac{k}{m} \left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \\ \frac{kX_2}{F_1} &= \frac{1.17}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \end{aligned}$$

and in a similar manner for $F_1 = 0$

$$\{x\} = \frac{\begin{bmatrix} 0.618F_2 \\ F_2 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} -1.618F_2 \\ F_2 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and hence after some maths

$$\begin{aligned} \frac{kX_1}{F_2} &= \frac{1.17}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \\ \frac{kX_2}{F_2} &= \frac{1.894}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.106}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)} \end{aligned}$$

We can replace the force excitation with abutment excitation as follows

Consider again the axial system with the particular values considered previously ($m_1=m_2=m$ and $k_1=k_2=k$) and with a sinusoidal input on the abutment given by $x_o = X_o \sin \omega t$, as shown in figure 2.7. The equations of motion are,

$$mx_1'' = k(x_o - x_1) - k(x_2 - x_1)$$

$$mx_2'' = -k(x_2 - x_1)$$

For this example let both masses be m and both springs k . These equations can be written in matrix form,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} kX_o \\ 0 \end{bmatrix}$$

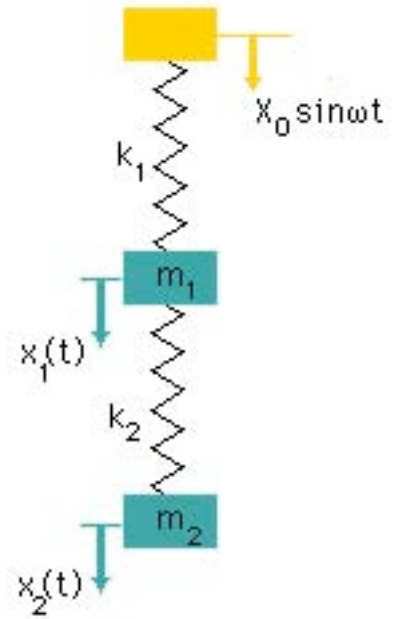


Figure 2.7

Thus by comparison with the previous example,

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} kX_o \\ 0 \end{bmatrix}$$

Thus using the solution above with $F_1 = kX_o$

$$\frac{kX_1}{F_1} = \frac{kX_1}{kX_o} = \frac{X_1}{X_o} = \frac{0.724}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.276}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{kX_2}{F_1} = \frac{kX_2}{kX_o} = \frac{X_2}{X_o} = \frac{1.17}{\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1 - \left(\frac{\omega}{\omega_2}\right)^2\right)}$$

These are the equations presented in [chapter 2](#).