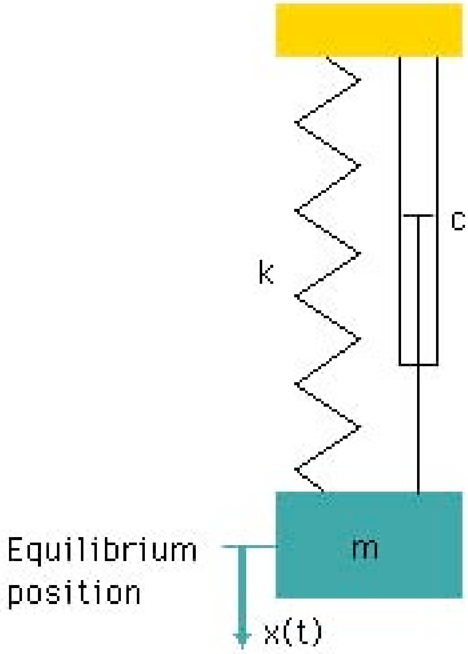
APPENDIX 1: Complex eigenvalues



The equation of motion of the single degree of freedom system shown above is,

$$mx''(t) + cx'(t) + kx(t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.5)$$

Defining the undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ and the damping ratio $\xi = \frac{c}{2\sqrt{mk}}$, equation (1.5) can be rewritten as

$$x''+2\xi\omega_n x'+\omega_n^2 x = 0$$

the solution of will be of the form,

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

where the values of A, B, λ_1 and λ_2 may be found from the initial conditions x(0) and x'(0). Thus substituting $x(t) = Ae^{\lambda t}$ in equation (1.5) gives,

$$Ae^{\lambda t}(\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2) = 0 \quad \text{thus} \quad \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

and

$$\lambda = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

and hence if $\xi < 0$ $\lambda_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$ and $\lambda_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$ where $i = \sqrt{-1}$

and therefore

$$x(t) = e^{-\xi\omega_n t}(A_1 e^{i\omega_n\sqrt{1-\xi^2}t} + A_2 e^{-i\omega_n\sqrt{1-\xi^2}t}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.73)$$

Now consider the matrix formulation used for more complex systems.

$$[M]\{x''\} + [C]\{x'\} + [K]\{x\} = 0 \quad \text{......................................} \quad (6.57)$$

As in chapter 6, introducing a velocity vector

$$\{x'\} = \{y\} \quad \text{..} \quad (6.58)$$

we obtained the following standard eigenvalue equation:

$$\left(\begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} + \lambda[I]\right)\begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} = 0 \quad \text{...........................} \quad (6.60)$$

the eigenvalues are found when,

$$\det\left|\begin{bmatrix} [0] & -[I] \\ [M]^{-1}[K] & [M]^{-1}[C] \end{bmatrix} + \lambda[I]\right| = 0$$

If we substitute the single degree of freedom mass and stiffness matrices (1x1 matrices) we obtain,

$$\det\left|\begin{bmatrix} 0 & -1 \\ \left(\frac{k}{m}\right) & \left(\frac{c}{m}\right) \end{bmatrix} + \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

expanding

$$\det\begin{vmatrix} \lambda & -1 \\ \left(\frac{k}{m}\right) & \left(\frac{c}{m}\right) + \lambda \end{vmatrix} = 0$$

and

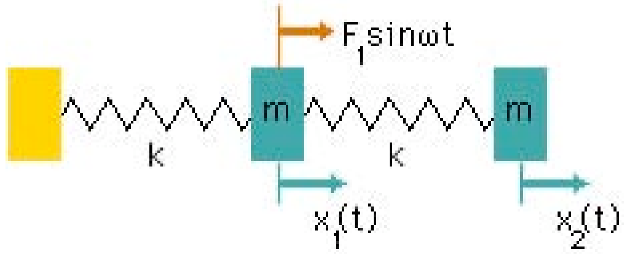
$$\lambda^2 + \lambda\frac{c}{m} + \frac{k}{m} = 0$$

Substituting the undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ and the damping ratio $\xi = \frac{c}{2\sqrt{mk}}$, the equation for λ becomes,

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

and hence as before if $\xi < 0$ $\lambda_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$ and $\lambda_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$ where $i = \sqrt{-1}$

APPENDIX 2: Steady state response of a 2DOF axial spring/mass system, force excitation



For this example let both masses be m and both springs k. The system mass and stiffness matrices are then

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{...} \quad (A2.1)$$

$$[K] = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \quad \text{...} \quad (A2.2)$$

Let the force vector applied to the system be

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{...} \quad (A2.3)$$

This allows us to obtain the response to F_1 alone by putting F_2 =0. However it will also allow the response resulting from F_2 alone to be found by putting $F_1 = 0$.

The displacement vector due to the steady state excitation of this force vector at a frequency ω can be calculated. First the natural frequencies are obtained from the eigenvalue equation of the system:

$$([K] + \lambda^2 [M])\{u\} = 0 \quad \text{...} \quad (6.20)$$

These are given by

$$\det([K] + \lambda^2 [M]) = 0 \quad \text{...} \quad (6.21)$$

For complex systems a computer would be used to produce a solution but for this two degree-of-freedom system we may solve long hand as follows.

$$\det\left(\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} + \lambda^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 2k + \lambda^2 m & -k \\ -k & k + \lambda^2 m \end{vmatrix} = 0$$

$$(2k+\lambda^2 m)(k+\lambda^2 m)-(-k)^2=0$$

$$m^2\lambda^4+3mk\lambda^2+k^2=0$$

$$\lambda_1=i0.618\sqrt{\frac{k}{m}};\lambda_2=-i0.618\sqrt{\frac{k}{m}};\lambda_3=i1.618\sqrt{\frac{k}{m}} \text{ and } \lambda_2=-i1.618\sqrt{\frac{k}{m}}$$

so that

$$\omega_1=|\mathrm{Im}(\lambda_1)|=0.618\sqrt{\frac{k}{m}} \quad \text{..................................} \quad (A2.4)$$

$$\omega_2=|\mathrm{Im}(\lambda_3)|=1.618\sqrt{\frac{k}{m}} \quad \text{....................................} \quad (A2.5)$$

The corresponding eigenvectors are found from equation (6.20). Thus for $\omega_1=1.0$, $\lambda^2=-\omega_1^2=-1.0$ so that equation (6.20) gives

$$([K]+\lambda^2[M])\{u\}=0 \quad \text{...} \quad (6.20)$$

Thus for $\lambda_1=0.382\dfrac{k}{m}$

$$\left(\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}-0.382\frac{k}{m}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\right)\{u\}=0$$

$$\begin{pmatrix}1.618 & -1\\ -1 & 0.618\end{pmatrix}\{u\}=0$$

$$\{u_1\}=\begin{bmatrix}0.618\\ 1\end{bmatrix} \quad \text{......................................} \quad (A2.6)$$

(In this example the eigenvectors have not been normalised. The result is the same whether they are normalised or not)

and for $\lambda_2=2.618\dfrac{k}{m}$

$$\left(\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}-2.618\frac{k}{m}\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\right)\{u\}=0$$

$$\begin{pmatrix}-0.618 & -1\\ -1 & -1.618\end{pmatrix}\{u\}=0$$

$$\{u_2\}=\begin{bmatrix}-1.618\\ 1\end{bmatrix} \quad \text{......................................} \quad (A2.7)$$

It has been shown that

$$\{x\}=\sum_{i=1}^{2}\frac{\{u_i\}^T\{F\}\{u_i\}}{M_i(\omega_i^2-\omega^2)}e^{i\omega t} \quad \text{.......................................} \quad (6.35)$$

where

$$M_i=\{u_i\}^T[M]\{u_i\} \quad \text{...} \quad (6.30)$$

Using the values in equations (A2.4 – A2.7)

$$M_1 = \{u_1\}^T[M]\{u_1\} = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0.618 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.618 & 1 \end{bmatrix} \begin{bmatrix} 0.618m \\ m \end{bmatrix} = 1.382m$$

$$M_2 = \{u_2\}^T[M]\{u_2\} == \begin{bmatrix} -1.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} -1.618 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.618 & 1 \end{bmatrix} \begin{bmatrix} -1.618m \\ m \end{bmatrix} = 3.618m$$

Now substituting in equation (6.35)

$$\{x\} = \sum_{i=1}^{2} \frac{\{u_i\}^T\{F\}\{u_i\}}{M_i(\omega_i^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\{u_1\}^T\{F\}\{u_1\}}{M_1(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\{u_2\}^T\{F\}\{u_2\}}{M_2(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and using the values obtained above

$$\{x\} = \frac{\begin{bmatrix} 0.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} -1.618 \\ 1 \end{bmatrix}^T \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} -1.618 \\ 1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

$$\{x\} = \frac{\begin{bmatrix} 0.382F_1 + 0.618F_2 \\ 0.618F_1 + F_2 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} 2.618F_1 - 1.618F_2 \\ -1.618F_1 + F_2 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Thus for $F_2 = 0$

$$\{x\} = \frac{\begin{bmatrix} 0.382F_1 \\ 0.618F_1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} 2.618F_{12} \\ -1.618F_1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

Since

$$\{x\} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{i\omega t}$$

it follows

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} 0.382F_1 \\ 0.618F_1 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} + \frac{\begin{bmatrix} 2.618F_{12} \\ -1.618F_1 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)}$$

and hence

$$\frac{X_1}{F_1} = \frac{0.276}{m(\omega_1^2 - \omega^2)} + \frac{0.724}{m(\omega_2^2 - \omega^2)}$$

and

$$\frac{X_2}{F_1} = \frac{0.447}{m(\omega_1^2 - \omega^2)} - \frac{0.447}{m(\omega_2^2 - \omega^2)}$$

These equations are usefully rearranged as follows

$$\frac{X_2}{F_1} = \frac{0.276}{m\omega_1^2\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.724}{m\omega_2^2\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)} = \frac{0.276}{m0.382\frac{k}{m}\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.724}{m2.618\frac{k}{m}\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{kX_1}{F_1} = \frac{0.724}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.276}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{X_2}{F_1} = \frac{0.447}{m\omega_1^2\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.447}{m\omega_2^2\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)} = \frac{0.447}{m0.382\frac{k}{m}\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.447}{m2.618\frac{k}{m}\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{kX_2}{F_1} = \frac{1.17}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

and in a similar manner for $F_1 = 0$

$$\{x\} = \frac{\begin{bmatrix} 0.618F_2 \\ F_2 \end{bmatrix}}{1.382m(\omega_1^2 - \omega^2)} e^{i\omega t} + \frac{\begin{bmatrix} -1.618F_2 \\ F_2 \end{bmatrix}}{3.618m(\omega_2^2 - \omega^2)} e^{i\omega t}$$

and hence after some maths

$$\frac{kX_1}{F_2} = \frac{1.17}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{kX_2}{F_2} = \frac{1.894}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.106}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

<u>We can replace the force excitation with abutment excitation as follows</u>

Consider again the axial system with the particular values considered previously (m_1=m_2=m and k_1=k_2=k) and with a sinusoidal input on the abutment given by $x_o = X_o \sin\omega t$, as shown in figure 2.7. The equations of motion are,

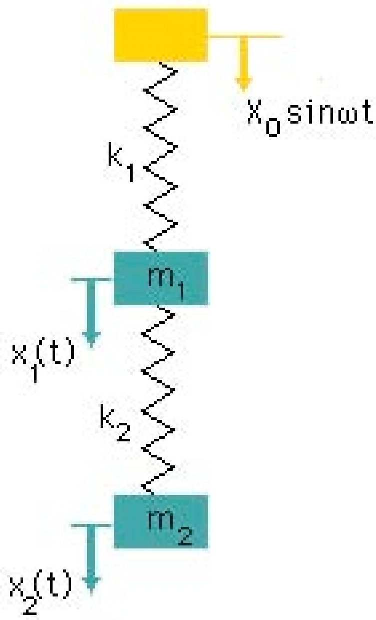


Figure 2.7

$$m{x_1}'' = k(x_o - x_1) - k(x_2 - x_1)$$

$$m{x_2}'' = -k(x_2 - x_1)$$

For this example let both masses be m and both springs k.

These equations can be written in matrix form,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} kX_o \\ 0 \end{bmatrix}$$

Thus by comparison with the previous example,

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} kX_o \\ 0 \end{bmatrix}$$

Thus using the solution above with $F_1 = kX_o$

$$\frac{kX_1}{F_1} = \frac{kX_1}{kX_o} = \frac{X_1}{X_o} = \frac{0.724}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} + \frac{0.276}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

$$\frac{kX_2}{F_1} = \frac{kX_2}{kX_o} = \frac{X_2}{X_o} = \frac{1.17}{\left(1-\left(\frac{\omega}{\omega_1}\right)^2\right)} - \frac{0.17}{\left(1-\left(\frac{\omega}{\omega_2}\right)^2\right)}$$

These are the equations presented in chapter 2.