CHAPTER 7: Continuous Systems.
Axial and Torsional vibration of bars

In the previous chapters we have considered systems made up of a combination of rigid masses, springs and dampers. The number of degrees of freedom, was the sum of the degrees of freedom of each mass. If we remember, that the degrees of freedom equal the number of coordinates needed to specify the position of the mass, then for a continuous system we need an infinite number of degrees of freedom to specify the position. This is because we need to specify the position of each point on the continuous system. In this chapter, we will consider some continuous systems. This will allow us to make some general conclusions about continuous systems. We will consider the axial vibration of a uniform undamped continuous bar, then include damping and also consider tapered bars. It is then a simple matter to extend this to the torsional vibration of such bars.

7.1 Axial vibration of uniform bars

Consider a uniform bar as shown in figure 7.1 and an element of the bar a distance x from one end. When vibrating the forces on the element and the deflection (u) of the element at some time t are as shown

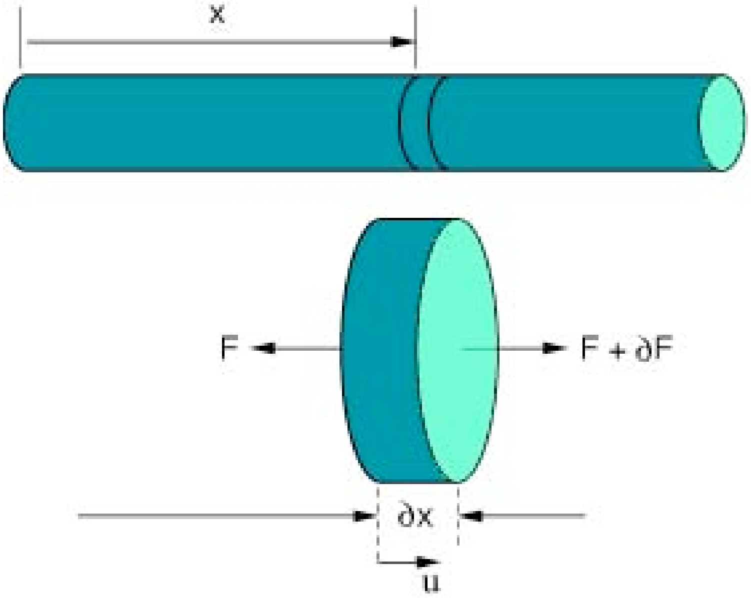

Figure 7.1 Element of bar vibrating axially.

For the element the stress/strain equation gives,

$$F = AE\frac{\partial u}{\partial x} \qquad (7.1)$$

where A is the cross-section area and E is the elastic (Young's) modulus.

Applying Newton II to the element,

$$F + \partial F - F = m\frac{\partial^2 u}{\partial t^2}$$

and if ρ is the density, the mass of the element $m = \rho A \partial x$ so that,

$$\partial F = \rho A \, \partial x \frac{\partial^2 u}{\partial t^2}$$

so that

$$\frac{\partial F}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \quad \text{..} \quad (7.2)$$

and substituting for F from (7.1)

$$\frac{\partial}{\partial x} AE \frac{\partial u}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2}$$

and rearranging

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} \quad \text{..} \quad (7.3)$$

7.1.1 Steady-state sinusoidal motion

Assume that $u(x,t) = U(x)e^{i\omega t}$ then substituting in (7.3)

$$U''(x)e^{i\omega t} = -\frac{\rho\omega^2}{E} U(x)e^{i\omega t}$$

$$\text{thus} \quad U''(x) + \frac{\rho\omega^2}{E} U(x) = 0$$

Let the solution for $U(x) = Be^{\beta_1 x} + Ce^{\beta_2 x}$

so that to find β_1 and β_2 we substitute $U(x) = Be^{\beta x}$ and find two values for β.

Thus

$$\beta^2 Be^{\beta x} + \frac{\rho\omega^2}{E} Be^{\beta x} = 0$$

$$\text{and } \beta^2 = -\frac{\rho\omega^2}{E}$$

Let $\lambda^2 = \dfrac{\rho\omega^2}{E}$ so that

$$\beta^2 = -\lambda^2$$

$$\therefore \beta = \pm i\lambda$$

hence

$$U(x) = Be^{i\lambda x} + Ce^{-i\lambda x} \quad \text{...} \quad (7.4)$$

The constants B and C depend on the end conditions of the bar. If the bar is free/free then there is no force at either end. From equation (1)

$$F = AE \frac{\partial u}{\partial x} \quad \text{so that when there is no force} \quad \frac{\partial u}{\partial x} = 0$$

From (7.4)

$$\frac{\partial u}{\partial x} = i\lambda Be^{i\lambda x} - i\lambda Ce^{-i\lambda x} = 0$$

so that when x=0 $\qquad B - C = 0$ .... (a)

and when x=L $\qquad Be^{i\lambda L} - Ce^{-i\lambda L} = 0$ .... (b)

From (a) $C = B$ and substituting in (b) $\qquad B\left(e^{i\lambda L} - e^{-i\lambda L}\right) = 0$

Thus either B=0 and there is no vibration or B may exist and there will be vibration without any continuing excitation (ie at a natural frequency) if

$$e^{i\lambda L} - e^{-i\lambda L} = 0$$

$$\text{ie when } \cos\lambda L + i\sin\lambda L - \cos\lambda L + i\sin\lambda L = 2i\sin\lambda L = 0$$

The natural frequencies are thus when $\sin\lambda L = 0$ which is when $\lambda L = n\pi$ and $n = 0 \rightarrow \infty$. Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$\omega\sqrt{\frac{\rho}{E}}L = n\pi \quad \text{and hence} \quad \omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}} \qquad n = 0 \rightarrow \infty$$

As a continuous bar has an infinite number of degrees-of-freedom we should not be surprised to find that it has an infinite number of natural frequencies.

To find the mode shapes we return to equation (7.4) with C=B,

$$U(x) = B\left(e^{i\lambda x} + e^{-i\lambda x}\right) = 2B\cos\lambda x$$

And substituting $\lambda L = n\pi$ (ie the natural frequencies)

$$U(x) = A\cos\left(n\pi\frac{x}{L}\right)$$

As the bar is free/free we have a zero frequency mode when n=0 and $U(x) = A$ which is a solid body motion.

The program shows the first three non-zero modes.

<u>For an excited end</u>

From the diagram above it should be noted that at the right hand end when $x = L$ (the length of the bar) the force F is in the same positive direction as u.

Thus for $x = L$ from (7.1) $\dfrac{\partial u(L)}{\partial x} = \dfrac{F}{AE}$

However when $x = 0$ the force F is in the negative u direction

Thus for $x = 0$ from (7.1) $\dfrac{\partial u(0)}{\partial x} = -\dfrac{F}{AE}$

<u>Response when excitation is at x = L</u>

For this case when x=L $F = F_L e^{i\omega t}$ and when x=0 F=0. The response is $u(x) = U(x)e^{i\omega t}$

When x=0 from equation (7.4) $\qquad \dfrac{\partial u}{\partial x} = i\lambda Be^{i\lambda x} - i\lambda Ce^{-i\lambda x} = 0$

so that when x=0 $\qquad B - C = 0$ and hence $B = C$

When x = L, $F = F_L e^{i\omega t}$ and $\dfrac{\partial u(L)}{\partial x} = U'(L)e^{i\omega t} = \dfrac{F_L e^{i\omega t}}{AE}$

therefore
$$i\lambda B e^{i\lambda L} - i\lambda C e^{-i\lambda L} = \frac{F_L}{AE}$$

Put C = B so that
$$B(i\lambda\cos\lambda L - \lambda\sin\lambda L - i\lambda\cos\lambda L - \lambda\sin\lambda L) = \frac{F_L}{AE}$$

Therefore,
$$B = C = \frac{-F_L}{2AE\lambda\sin\lambda L} \qquad (7.5)$$

returning to equation (7.4)
$$U(x) = Be^{i\lambda x} + Ce^{-i\lambda x} \qquad (7.4)$$

and substituting for B and C from (7.5)

$$U(x) = \left(e^{i\lambda x} + e^{-i\lambda x}\right)\frac{-F_L}{2AE\lambda\sin\lambda L}$$

$$\therefore\ U(x) = \left(\cos\lambda x + i\sin\lambda x + \cos\lambda x - i\sin\lambda x\right)\frac{-F_L}{2AE\lambda\sin\lambda L}$$

so that
$$U(x) = -\frac{F_L\cos\lambda x}{AE\lambda\sin\lambda L}$$

and hence the response at a position x along the bar when excited at x=L is
$$\frac{U(x)}{F_L} = -\frac{\cos\lambda x}{AE\lambda\sin\lambda L} \qquad (7.6)$$

for the case when x = L we obtain the response at the excitation position,
$$\frac{U(L)}{F_L} = -\frac{\cos\lambda L}{AE\lambda\sin\lambda L} \qquad (7.7)$$

for the case when x = 0 we obtain the response at the free end,
$$\frac{U(0)}{F_L} = -\frac{1}{AE\lambda\sin\lambda L} \qquad (7.8)$$

<u>Response when excitation is at x = 0</u>

Following similar maths to the above it can be shown that,
$$\frac{U(0)}{F_o} = -\frac{\cos\lambda L}{AE\lambda\sin\lambda L} \qquad (7.9)$$

and as expected from symmetry $\dfrac{U(0)}{F_o} = \dfrac{U(L)}{F_L}$. Compare equations (7.9) and (7.7).

Also it is found that,

$$\frac{U(L)}{F_o} = -\frac{1}{AE\lambda\sin\lambda L} \quad \text{..} \quad (7.10)$$

and as expected from Maxwell's reciprocal theorem $\frac{U(0)}{F_L} = \frac{U(L)}{F_o}$. Compare equations (7.10) and (7.8).

Finally it may be shown that

$$\frac{U(x)}{F_o} = -\frac{\cos\lambda(L-x)}{AE\lambda\sin\lambda L} \quad \text{..} \quad (7.11)$$

The major point to note is that all these responses have $\sin\lambda L$ in the denominator. Thus the response will tend to infinity and resonance will occur when $\sin\lambda L = 0$ which was the natural frequency equation for a free/free bar. The response $\frac{U(L)}{F_L}$ is shown in figure 7.2 for a bar of length L = 2m diameter 0.2m, E = 2.0e11 N/m^2 and ρ =7800.0 kg/m^2 .

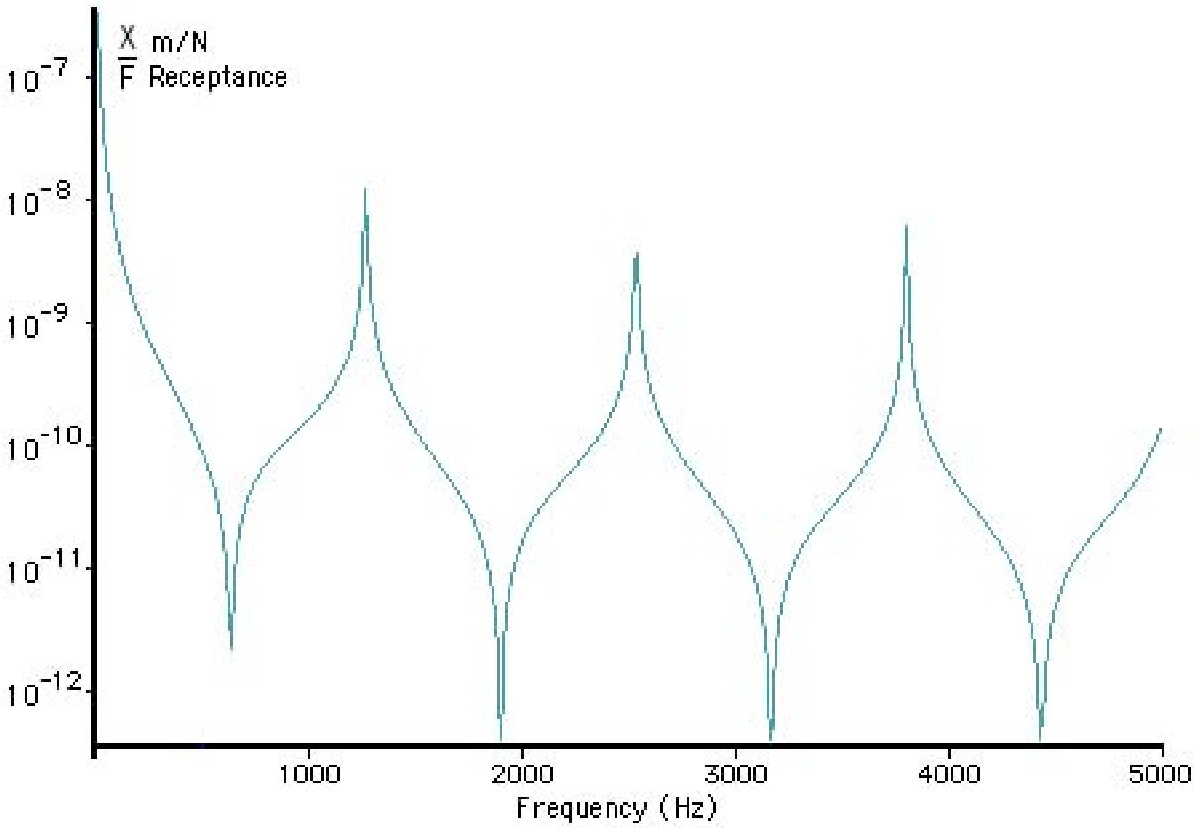

Figure 7.2 Tip response of a free/free bar

It should be noted that there are anti-resonances. If we consider the response at the end of the bar,

$$\frac{u(L)}{F_L} = -\frac{\cos\lambda L}{AE\lambda\sin\lambda L}$$

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it is apparent that which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \to \infty$.

Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

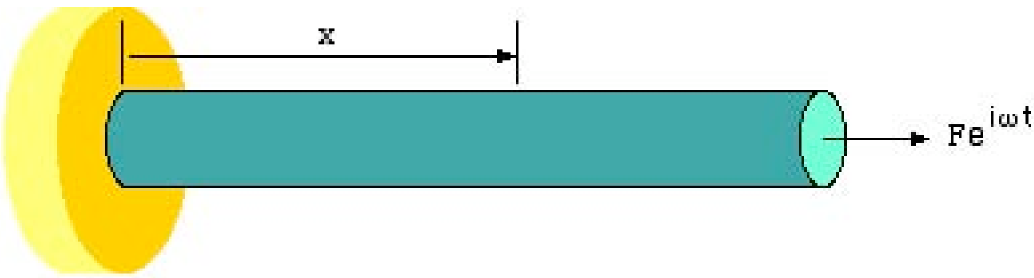
$$\omega\sqrt{\frac{\rho}{E}}L = \frac{2n-1}{2}\pi \quad \text{and hence} \quad \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad n = 1 \to \infty$$

From equation (7.6) we know the response at any position x for an exciting force at the end,

$$\frac{U(x)}{F_L} = -\frac{\cos\lambda x}{AE\lambda\sin\lambda L}$$

we may therefore show the animated deflected shape at any excitation by calculating the response at discrete positions along the bar.

Clamped/free bar



For sinusoidal excitation and response we have shown,

$$U(x) = Be^{i\lambda x} + Ce^{-i\lambda x} \qquad \text{..} \qquad (7.4)$$

Therefore when $x = 0 \quad U(0) = 0 = B + C$ so that $C = -B$

as for the free/free case

When $x = L$, $\quad F = F_L e^{i\omega t}$ and $\frac{\partial u(L)}{\partial x} = U'(L)e^{i\omega t} = \frac{F_L e^{i\omega t}}{AE}$

therefore

$$i\lambda Be^{i\lambda L} - i\lambda Ce^{-i\lambda L} = \frac{F_L}{AE}$$

and substituting $C = -B$

$$i\lambda Be^{i\lambda L} + i\lambda Be^{-i\lambda L} = \frac{F_L}{AE}$$

$$\text{so that} \quad Bi\lambda(\cos\lambda L + i\sin\lambda L + \cos\lambda L - i\sin\lambda L) = \frac{F_L}{AE}$$

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Therefore,

$$B = -C = \frac{F_L}{2GJi\lambda\cos\lambda L} \quad \text{...} \quad (7.12)$$

returning to equation (7.4) and substituting for B and C from (7.12)

$$U(x) = \left(\frac{e^{i\lambda x} - e^{-i\lambda x}}{2i}\right)\frac{F_L}{AE\lambda\cos\lambda L} = \frac{F_L\sin\lambda x}{AE\lambda\cos\lambda L}$$

so that

$$U(x) = \frac{F_L\sin\lambda x}{AE\lambda\cos\lambda L}$$

and hence the response at a position x along the bar when excited at x=L is

$$\frac{U(x)}{F_L} = \frac{\sin\lambda x}{AE\lambda\cos\lambda L} \quad \text{...} \quad (7.13)$$

for the case when x = L we obtain the response at the excitation position,

$$\frac{U(L)}{F_L} = \frac{\sin\lambda L}{AE\lambda\cos\lambda L} \quad \text{...} \quad (7.14)$$

The major point to note is that the responses have $\cos\lambda L$ in the denominator. Thus the response will tend to infinity and resonance will occur when $\cos\lambda L = 0$ which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \to \infty$.

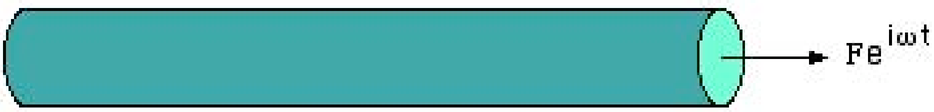
Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{E}}$$

$$\omega\sqrt{\frac{\rho}{E}}L = \frac{2n-1}{2}\pi \quad \text{and hence} \quad \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad n = 1 \to \infty$$

The program shows the first three non-zero modes.

It is of interest to note that these natural frequencies are the frequencies at which anti-resonances occurred for the response of a free/free bar excited at one end.



If the response at the free end (equation (7.14)) is plotted (figure 7.3) then it may be seen by comparing figure 7.3 with 7.2 that all the clamped/free resonances coincide with the anti-resonances for the response of the free/free bar. Note the dimensions of the bar are the same in both cases. This is now a general conclusion for undamped systems. When the response at an end is zero then the frequency at which this occurs is the natural frequency the system would have if clamped at the excitation position.

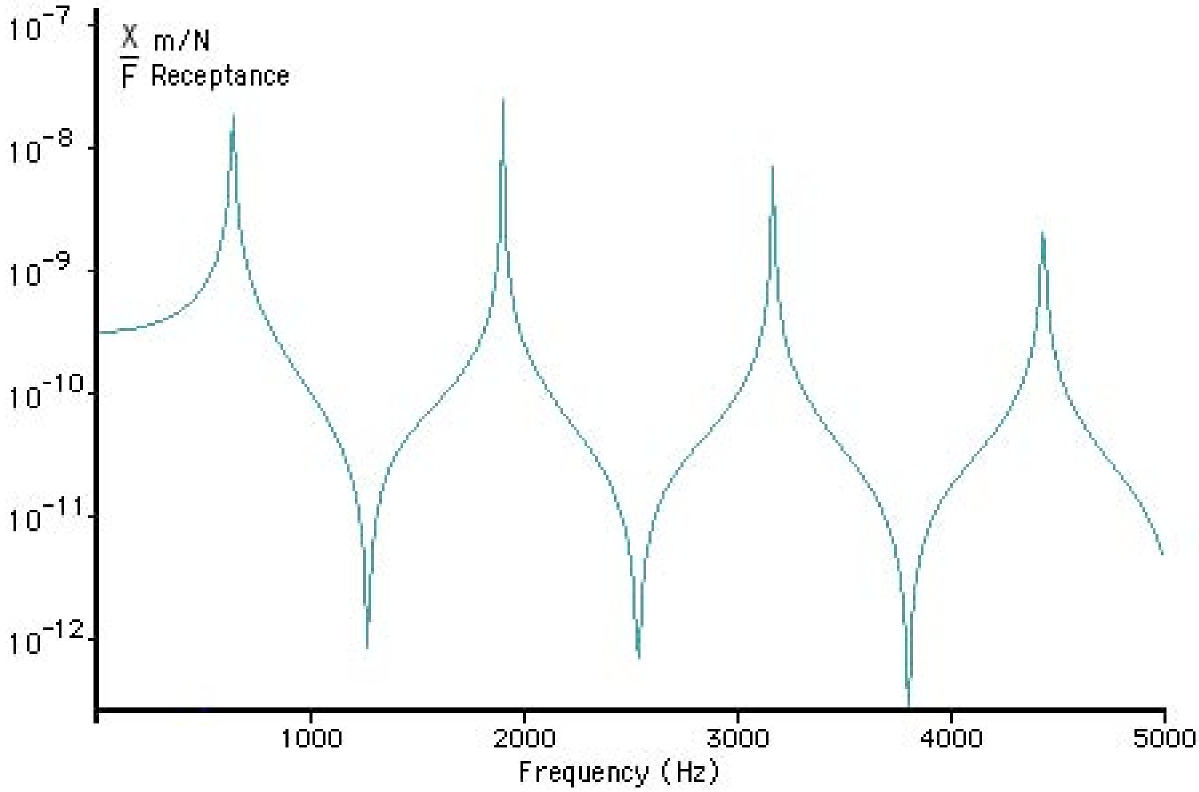


Figure 7.3 Tip response of a clamped/free bar

<u>7.1.2 Damping</u>

In all the previous chapters the damping has been considered to be viscous when it has been added. Thus there has been a damping force proportional to the velocity. For steady state vibration this has given rise to terms of the form $i\omega cX$ where X was the amplitude of vibration and c the viscous damping coefficient. This form of damping has an increasing effect with the frequency. In practice for many systems the damping force does not increase with frequency. Rather there is a constant energy loss each cycle depending on the amplitude but not on the frequency. Such a form of damping is called 'hysteretic' damping and gives rise to damping terms of the form ihX where h is called the hysteretic damping coefficient. When damping is included in the axial vibration of bars both viscous and hysteretic damping should be considered to determine which is the most appropriate.

It is of interest to note that several authors have reported simply taking the equations (7.6) to (7.14) and replacing the elastic modulus by a complex quantity such as $E(1+i\eta))$ with the expectation that this will include hysteretic damping. This does not give the correct response for a damped shaft but simply multiplies the undamped response by a constant complex number $1/(1 + i\eta)$. The response thus clearly still has an infinite value at each of the undamped natural frequencies. It is necessary to return to the basic equations and solve them with a complex modulus. This has been done [1] for torsional vibration and the receptances (ie responses) for axial vibration can be shown to be, for a free/free bar excited at the end x=L,

$$\frac{U(x)}{F_L}=\frac{2\left[\left(a\cosh\lambda Sx\cos\lambda Rx-b\sinh\lambda Sx\sin\lambda Rx\right)-i\left(a\sinh\lambda Sx\sin\lambda Rx+b\cosh\lambda Sx\cos\lambda Rx\right)\right]}{\left(a^2+b^2\right)} \quad \ldots\ldots\ (7.15)$$

where,

$$S=-\frac{\left[(1+\eta^2)^{1/2}-1\right]^{1/2}}{\sqrt{2}(1+\eta^2)^{1/2}} \text{ and } R=\frac{\left[(1+\eta^2)^{1/2}+1\right]^{1/2}}{\sqrt{2}(1+\eta^2)^{1/2}}$$

$$a=2AE\lambda\left[(S+\eta R)\cos(R\lambda L)\sinh(S\lambda L)+(\eta S-R)\sin(R\lambda L)\cosh(S\lambda L)\right]$$
$$b=2AE\lambda\left[(\eta S-R)\cos(R\lambda L)\sinh(S\lambda L)-(S+\eta R)\sin(R\lambda L)\cosh(S\lambda L)\right]$$

If viscous damping is to be modelled then we simply replace η by $\omega\xi$ in the equations above. From equation (7.15) the responses at either end of the bar may be found by putting x = 0 and x = L. The response $\frac{U(L)}{F_L}$ is shown in figure 7.4 for a bar of length l= 2m diameter 0.2m, E = 2.0e11 N/m^2 and ρ =7800.0 kg/m^2 and for various values of η.

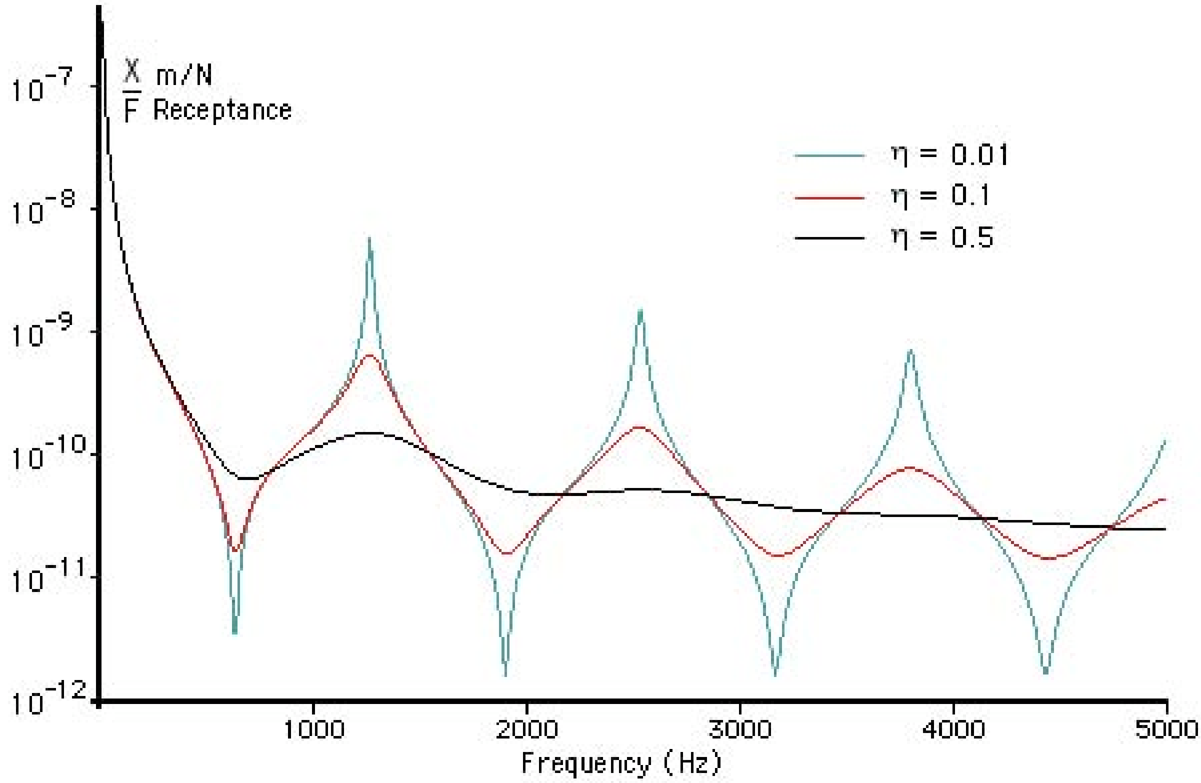


Figure 7.4 Tip response of a damped free/free bar for various hysteretic damping values.

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7.1.3 Tapered bars

Hesterman et al [2] have derived the responses of an undamped tapered bar and their results are given below.

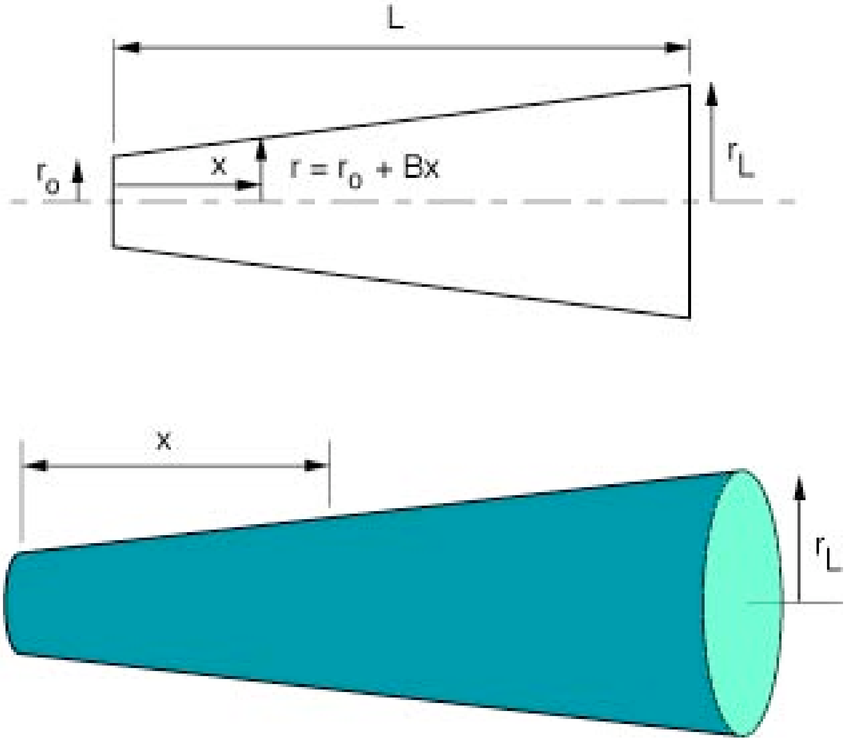

Figure 7.5 Free/free tapered shaft - definition of terms

Defining $z = \lambda\left(x + \frac{r_o}{B}\right)$ where $\lambda = \omega\sqrt{\frac{\rho}{E}}$ as before and $B = \frac{r_L - r_o}{L}$ Hesterman et al [2] showed that,

$$\frac{U(z)}{F_o} = \frac{-1}{E\pi r_o^2 \lambda \Delta z} \cdot \left\{ \left(\frac{\cos z_L}{z_L^2} + \frac{\sin z_L}{z_L} \right) . \sin z + \left(\frac{\cos z_L}{z_L} - \frac{\sin z_L}{z_L^2} \right) . \cos z \right\} \quad \text{................ (7.16)}$$

where

$$\Delta = \left\{ \frac{\cos z_o}{z_o} - \frac{\sin z_o}{z_o^2} \right\} \left\{ \frac{\cos z_L}{z_L^2} + \frac{\sin z_L}{z_L} \right\} - \left\{ \frac{\cos z_o}{z_o^2} + \frac{\sin z_o}{z_o} \right\} \left\{ \frac{\cos z_L}{z_L} - \frac{\sin z_L}{z_L^2} \right\} \quad \text{............ (7.17)}$$

Hence

$$\alpha_{oo} = \frac{U(z_o)}{F_o} = \frac{-1}{E\pi r_o^2 \lambda \Delta z_o} \cdot \left\{ \left(\frac{\cos z_L}{z_L^2} + \frac{\sin z_L}{z_L} \right) . \sin z_o + \left(\frac{\cos z_L}{z_L} - \frac{\sin z_L}{z_L^2} \right) . \cos z_o \right\} \quad \text{.......... (7.18)}$$

and

$$\alpha_{Lo} = \frac{-1}{E\pi r_o^2 \lambda \Delta z_L^2} \quad \text{... (7.19)}$$

Also Hesterman et al [2] showed that,

$$\frac{U(z)}{F_L} = \frac{-1}{E\pi r_L^2 \lambda \Delta z} . \left\{ \left(\frac{\cos z_o}{z_o^2} + \frac{\sin z_o}{z_o} \right) . \sin z + \left(\frac{\cos z_o}{z_o} - \frac{\sin z_o}{z_o^2} \right) . \cos z \right\}$$

Consequently;

$$\alpha_{LL} = \frac{U(z_L)}{F_L} = \frac{-1}{E\pi r_L^2 \lambda \Delta z_L} . \left\{ \left(\frac{\cos z_o}{z_o^2} + \frac{\sin z_o}{z_o} \right) . \sin z_L + \left(\frac{\cos z_o}{z_o} - \frac{\sin z_o}{z_o^2} \right) . \cos z_L \right\} \text{......(7.20)}$$

and

$$\alpha_{oL} = \frac{-1}{E\pi r_L^2 \lambda \Delta z_o^2} \quad \text{... (7.21)}$$

<u>Note</u>: For a linear taper $r_L^2 z_o^2 = r_o^2 z_L^2$. Hence $\alpha_{oL} = \alpha_{Lo}$ .... (7.22)

Figure 7.6 shows α_{oo} and α_{LL} the responses at each end of a bar length L =2m and $D_o =$ 0.05m and = 0.2m. The resonances are the same for each response as the natural frequencies do not depend on the excitation position. However, the anti-resonances are at different frequencies as they correspond to the natural frequencies of a bar clamped at the excitation position.

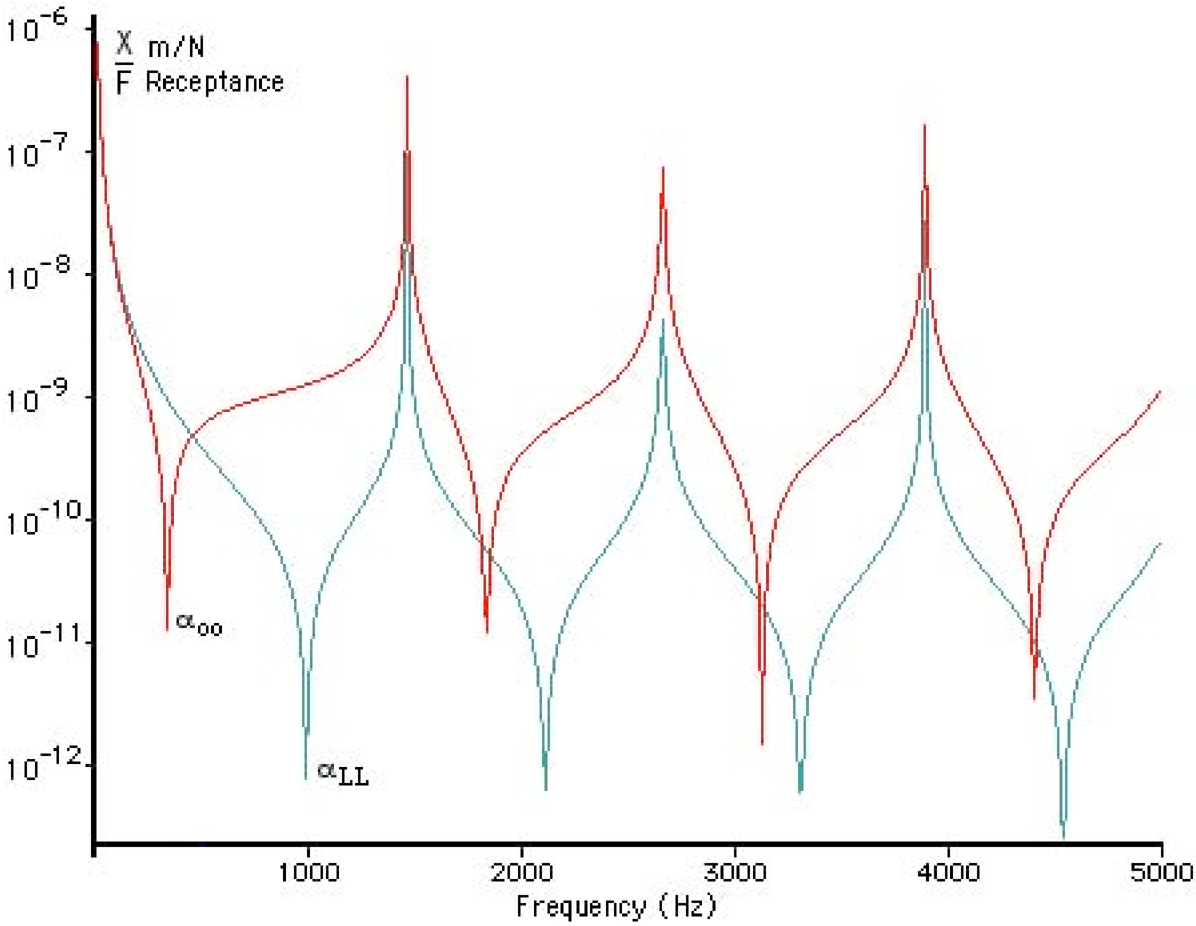

Figure 7.6 Tip responses of a free/free tapered bar.

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7.2 Torsional vibration of uniform bars

For torsional vibration very similar maths applies to that developed for axial vibration. For uniform bars there are only four parameter differences, E is replaced throughout by G; A by J; F by T and u by θ.

The receptances are then the same as for the axial case but with $\lambda^2 = \rho\omega^2 / G$ thus for

7.2.1 Free/free

$$\frac{\Theta(x)}{T_L} = -\frac{\cos\lambda x}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (7.23)$$

$$\frac{\Theta(L)}{T_L} = \frac{\Theta(0)}{T_o} = -\frac{\cos\lambda L}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (7.24)$$

$$\frac{\Theta(0)}{T_L} = \frac{\Theta(L)}{T_o} = -\frac{1}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (7.25)$$

$$\frac{\Theta(x)}{T_o} = -\frac{\cos\lambda(L-x)}{GJ\lambda\sin\lambda L} \quad \text{.......} \quad (7.26)$$

The natural frequencies are thus when $\sin\lambda L = 0$ which is when $\lambda L = n\pi$ and $n = 0 \rightarrow \infty$.
Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{G}}$$

$$\omega\sqrt{\frac{\rho}{G}}L = n\pi \quad \text{and hence} \quad \omega_n = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}} \qquad n = 0 \rightarrow \infty$$

The program shows the first three non-zero modes.

7.2.2 Clamped/free

$$\frac{\Theta(x)}{T_L} = \frac{\sin\lambda x}{GJ\lambda\cos\lambda L} \quad \text{.......} \quad (7.27)$$

$$\frac{\Theta(L)}{T_L} = \frac{\sin\lambda L}{GJ\lambda\cos\lambda L} \quad \text{.......} \quad (7.28)$$

The natural frequencies are thus when $\cos\lambda L = 0$ which is when $\lambda L = \frac{2n-1}{2}\pi$ and $n = 1 \rightarrow \infty$.
Substituting for

$$\lambda = \omega\sqrt{\frac{\rho}{G}}$$

$$\omega\sqrt{\frac{\rho}{G}}L = \frac{2n-1}{2}\pi \quad \text{and hence} \quad \omega = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad n = 1 \rightarrow \infty$$

The program shows the first three modes.

7.2.3 Damping

For a free/free bar excited at the end x=L, again E is replaced throughout by G; A by J; F by T and u by θ.

$$\frac{\Theta(x)}{T_L} = \frac{2\left[\left(a\cosh\lambda Sx\cos\lambda Rx - b\sinh\lambda Sx\sin\lambda Rx\right) - i\left(a\sinh\lambda Sx\sin\lambda Rx + b\cosh\lambda Sx\cos\lambda Rx\right)\right]}{\left(a^2+b^2\right)} \qquad \text{....... (7.29)}$$

where,

$$S = -\frac{\left[(1+\eta^2)^{1/2}-1\right]^{1/2}}{\sqrt{2}(1+\eta^2)^{1/2}} \text{ and } R = \frac{\left[(1+\eta^2)^{1/2}+1\right]^{1/2}}{\sqrt{2}(1+\eta^2)^{1/2}}$$

$$a = 2GJ\lambda\left[(S+\eta R)\cos(R\lambda L)\sinh(S\lambda L) + (\eta S - R)\sin(R\lambda L)\cosh(S\lambda L)\right]$$
$$b = 2GJ\lambda\left[(\eta S - R)\cos(R\lambda L)\sinh(S\lambda L) - (S+\eta R)\sin(R\lambda L)\cosh(S\lambda L)\right]$$

If viscous damping is to be modelled then we simply replace η by $\omega\xi$ in the equations above. From equation (7.29) the responses at either end of the bar may be found by putting x = 0 and x = L.

7.2.4 Tapered bars

For the torsional vibration of tapered bars it not just a simple matter of changing parameters [2]. The solutions are significantly different.

Defining $z = \lambda\left(x + \frac{r_o}{B}\right)$ where $\lambda = \omega\sqrt{\frac{\rho}{G}}$ and $B = \frac{r_L - r_o}{L}$ Hesterman et al [2] showed that,

$$\alpha_{xo} = \frac{\Theta(z)}{T_o} = \frac{2z_o^4}{\Delta G\pi r_o^4\lambda z^2}\cdot\left\{-\left(z_L^2\cos z_L - 3z_L\sin z_L - 3\cos z_L\right)\left(\frac{\sin z}{z} - \cos z\right) + \left(z_L^2\sin z_L + 3z_L\cos z_L - 3\sin z_L\right)\left(\sin z + \frac{\cos z}{z}\right)\right\} \qquad \text{.................. (7.30)}$$

$$\alpha_{xL} = \frac{\Theta(z)}{T_L} = \frac{2z_L^4}{\Delta G\pi r_L^4\lambda z^2}\cdot\left\{-\left(z_o^2\cos z_o - 3z_o\sin z_o - 3\cos z_o\right)\left(\frac{\sin z}{z} - \cos z\right) + \left(z_o^2\sin z_o + 3z_o\cos z_o - 3\sin z_o\right)\left(\sin z + \frac{\cos z}{z}\right)\right\} \qquad \text{.................. (7.31)}$$

where $$\Delta = \left\{z_o^2\sin z_o + 3z_o\cos z_o - 3\sin z_o\right\}\left\{z_L^2\cos z_L - 3z_L\sin z_L - 3\cos z_L\right\} - \left\{z_o^2\cos z_o - 3z_o\sin z_o - 3\cos z_o\right\}\left\{z_L^2\sin z_L + 3z_L\cos z_L - 3\sin z_L\right\} \qquad \text{.................. (7.32)}$$

The tip receptances may then be shown to be,

$$\alpha_{oo} = \frac{\Theta(z_o)}{T_o} = \frac{2z_o^2}{\Delta G\pi r_o^4\lambda}\cdot\left\{-\left(z_L^2\cos z_L - 3z_L\sin z_L - 3\cos z_L\right)\left(\frac{\sin z_o}{z_o} - \cos z_o\right)\right.$$

$$+\left(z_L^2 \sin z_L + 3z_L \cos z_L - 3\sin z_L\right)\left(\sin z_o + \frac{\cos z_o}{z_o}\right)\Bigg\} \quad \text{..................} \quad (7.33)$$

$$\alpha_{LL} = \frac{\Theta(z_L)}{T_L} = \frac{2z_L^2}{\Delta G\pi r_L^4 \lambda} \cdot \Bigg\{-\left(z_o^2 \cos z_o - 3z_o \sin z_o - 3\cos z_o\right)\left(\frac{\sin z_L}{z_L} - \cos z_L\right)$$

$$+\left(z_o^2 \sin z_o + 3z_o \cos z_o - 3\sin z_o\right)\left(\sin z_L + \frac{\cos z_L}{z_L}\right)\Bigg\} \quad \text{..................} \quad (7.34)$$

$$\alpha_{oL} = \alpha_{Lo} = \frac{2z_L^4}{\Delta G\pi r_L^4 \lambda} = \frac{2z_o^4}{\Delta G\pi r_o^4 \lambda} \quad \text{..} \quad (7.35)$$

7.3 Conclusions

In this chapter we have seen that continuous systems have an infinite number of natural frequencies and associated mode shapes.

7.4 References

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