APPENDIX 1: Newton's Second Law

Newton stated that the rate of change of momentum of a body is equal to the resultant force acting on the body. This may be written as,

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$$

It should be noted that **F** and **v** are both vectors, they have magnitude and direction.
If the mass of the body is constant then Newton's second law reduces to,

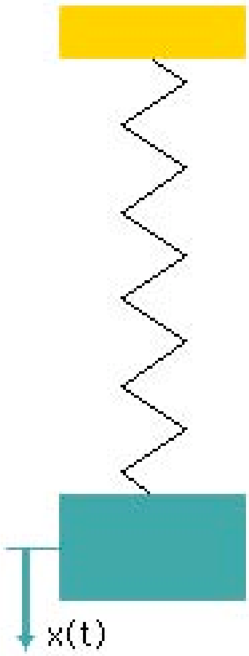
$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

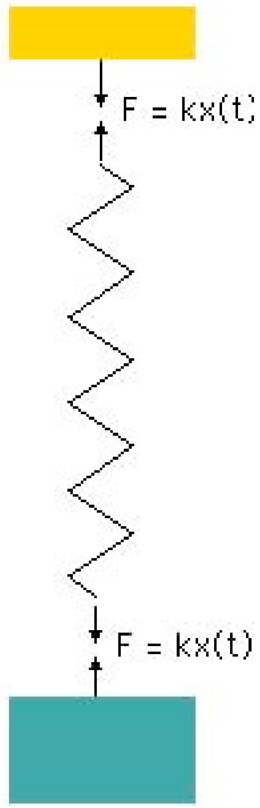
In this case (constant mass) the resultant force equals the mass times its acceleration. Again note that the force and the acceleration are vectors.

APPENDIX 2: Free-body diagrams

In order to use Newton's second law of motion it is necessary to determine the forces acting on a body so that the resultant force (F) can be found. Thus a system of objects such as springs and masses should be divided up into its separate components.

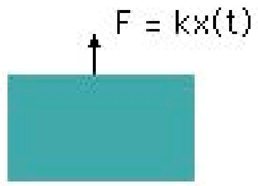
At this stage we will ignore gravity by assuming that the spring/mass system is resting on a frictionless horizontal plane. In the equilibrium position the spring would be unstretched and the mass at rest. When the mass is displaced x(t) from this position the spring is stretched an amount x(t). Thus there are equal and opposite forces (F) at the ends of the spring. The force F is given by F = kx(t).





Newton's third law states that for every force there is an equal and opposite reaction. Thus there is a force F also acting on the mass as shown.

The free body diagram of the mass is thus,



If we now apply Newton's second law we obtain the equation of motion of the mass as,

$$mx''(t) = -kx(t)$$

x"(t) is the acceleration in the positive downwards direction defined by x(t). The force on the mass is upwards and thus -kx(t).

APPENDIX 3: Gravity effects

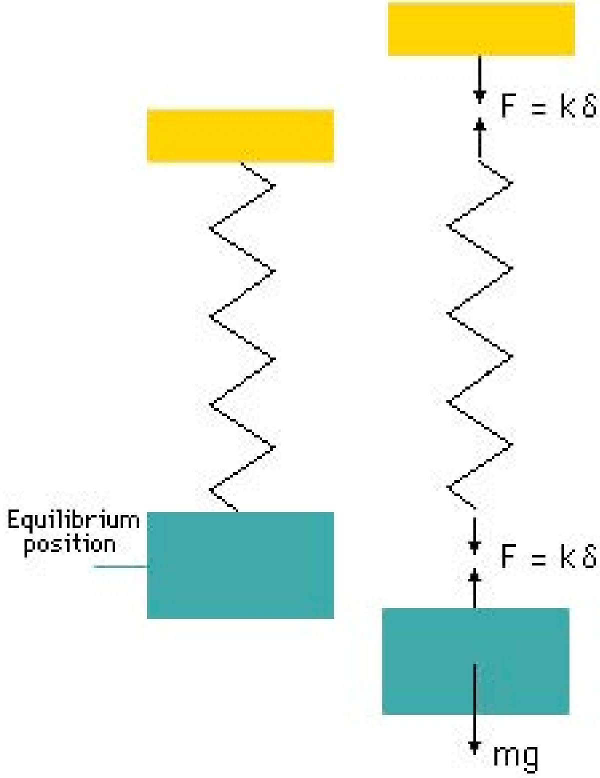
It is appropriate to consider the effect gravity would have if the spring mass system was vertical. In this case we need to draw the free body diagram of the system when in its equilibrium position and also when displaced an amount x(t) from the equilibrium position.

<u>Equilibrium position</u>

Consider the first case when the mass is at the equilibrium position. The spring will be stretched some amount δ because it is supporting the mass.

By definition this is the rest position and the mass has no acceleration, it is not moving. Thus from Newton's second law, remembering that we have defined positive down

$$mg - k\delta = 0$$

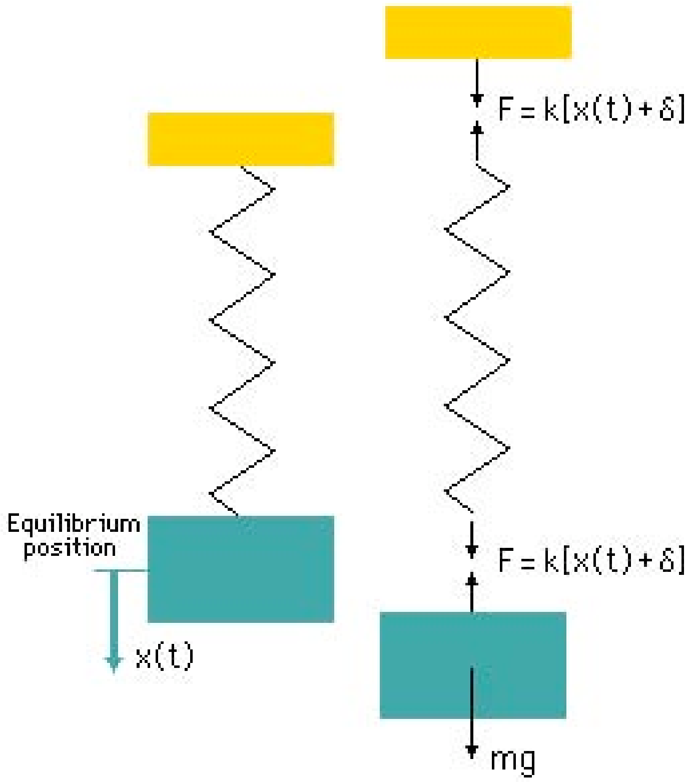


When there is an additional deflection of the spring caused by the mass being displaced an amount x(t) from the equilibrium position, then the total deflection of the spring is x(t) + d.
The equation of motion is thus,

$$mx''(t) = mg - k[x(t) + \delta]$$

The equation of motion may be expanded to

$$mx''(t) = mg - k\delta - kx(t)$$



But we have shown by considering the equilibrium position that

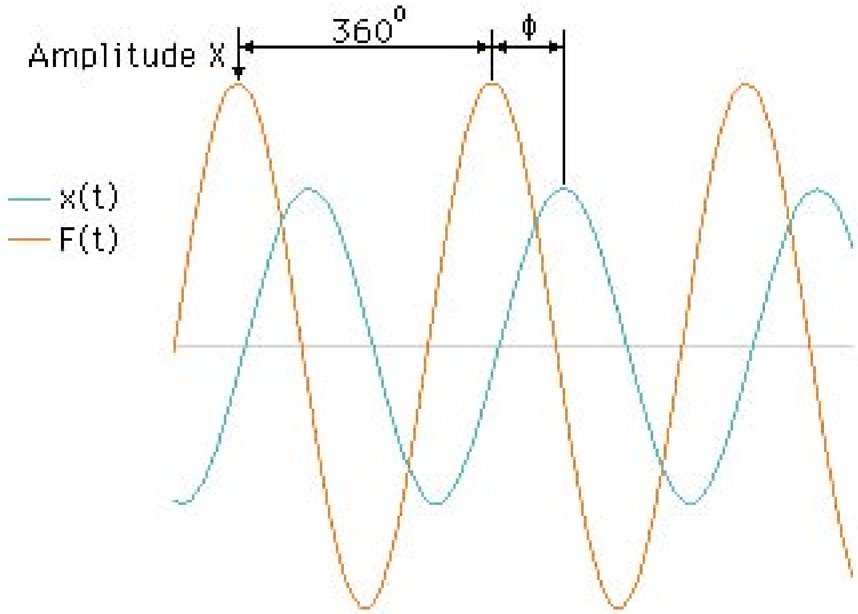
$$mg - k\delta = 0$$

Thus the equation of motion reduces to the form already obtained when gravity effects were ignored.

$$mx''(t) = -kx(t)$$

This is a significant result. It may be shown that if all displacements are measured from the equilibrium position then the equation of motion (for the vast majority of systems) is not dependent on gravitational effects.

APPENDIX 4: Phase angle



The phase angle between two successive peaks is defined as 360°. In the steady state the displacement x(t) lags the excitation f(t) by a phase angle ϕ. Thus

$$x(t) = X\sin(\omega_n t + \phi) \quad \text{and } \phi \text{ is negative.}$$

The animation program will allow you to vary all the parameters and see the connection between rotating vectors and sinusoidal motion.