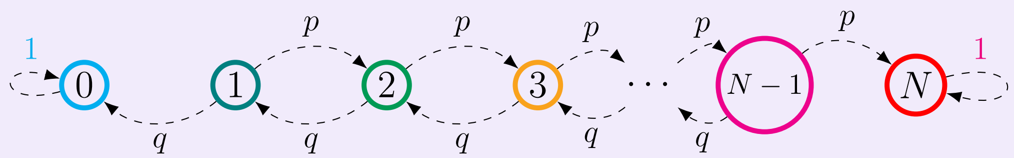
HW 3

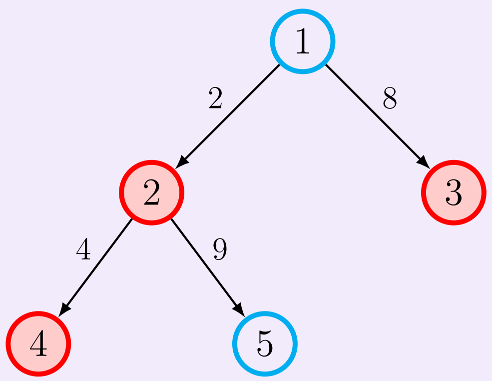
1.

Solution:

Consider the following graph representing the Markov chain of possibilities, where $N = N_1 + N_2$



Using first step decomposition, we can see that



$$
\begin{aligned}
h_{0,0} &= 1 \\
h_{1,0} &= qh_{0,0} + ph_{2,0} \\
h_{2,0} &= qh_{1,0} + ph_{3,0} \\
&\vdots \\
h_{N-1,0} &= qh_{N-2,0} + ph_{N,0} \\
h_{N,0} &= 0
\end{aligned}
$$

Since if you get all the coins and win, you can't lose!

For $1 < n \leqslant N$, we have $h_{n-1,0} = qh_{n-2,0} + ph_{n,0}$

Solving this for $h_{n-1,0}$ yields

$$h_{n,0} = \frac{1}{p}h_{n-1,0} - \frac{q}{p}h_{n-2,0}$$

This is a second order homogeneous linear difference equation with constant coefficients, which can be solved with the auxiliary polynomial.

Let ansatz $cr^n = h_{n,0}$, then

$$cr^n = \frac{1}{p} \cdot cr^{n-1} - \frac{q}{p} \cdot cr^{n-2} \implies r^2 = \frac{1}{p}r - \frac{q}{p}$$

The roots of this polynomial are $r = \frac{1 \pm \sqrt{1-4pq}}{2p}$, and since $q = 1 - p$, we have

$$r = \frac{1 \pm |2p-1|}{2p} = \left\{ \underbrace{\begin{cases} \frac{1}{p} - 1 \text{ if } 0 < p \leqslant \frac{1}{p} \\ 1 \text{ else} \end{cases}}_{+}, \underbrace{\begin{cases} \frac{1}{p} - 1 \text{ if } \frac{1}{2} \leqslant p \leqslant 1 \\ 1 \text{ else} \end{cases}}_{-} \right\}$$

Clearly & luckily, since both roots are the same no matter which case we take, it suffices to say

$$r = \left\{1, \frac{1}{p} - 1\right\} \implies h_{n,0} = c_1 \left(\frac{1}{p} - 1\right)^n + c_2 \cdot 1$$

The boundary conditions are

$$h_{0,0} = 1$$
$$h_{N,0} = 0$$

Thus, we can plug them in to resolve the coefficients c_1, c_2

$$\begin{cases} 1 = c_1 \left(\frac{1}{p} - 1\right)^0 + c_2 = c_1 + c_2 \\ 0 = c_1 \left(\frac{1}{p} - 1\right)^N + c_2 \end{cases}$$

Doing some algebra,

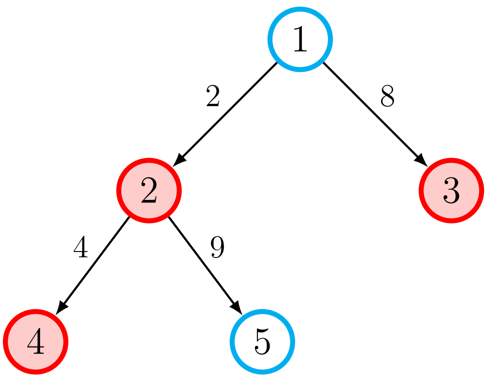
$$c_2 = 1 - c_1 \implies 0 = c_1 \left(\frac{1}{p} - 1\right)^N + 1 - c_1 \implies c_1 = \frac{1}{1 - \left(\frac{1}{p} - 1\right)^N} \implies$$

$$c_2 = 1 - \frac{1}{1 - \left(\frac{1}{p} - 1\right)^N}$$

Thus, we have

$$\begin{aligned} h_{N_1,0} &= \frac{1}{1 - \left(\frac{1}{p} - 1\right)^N} \left(\frac{1}{p} - 1\right)^{N_1} + 1 - \frac{1}{1 - \left(\frac{1}{p} - 1\right)^N} \\ &= \frac{1}{1 - \left(\frac{1}{p} - 1\right)^{N_1+N_2}} \left(\frac{1}{p} - 1\right)^{N_1} + 1 - \frac{1}{1 - \left(\frac{1}{p} - 1\right)^{N_1+N_2}} \end{aligned}$$

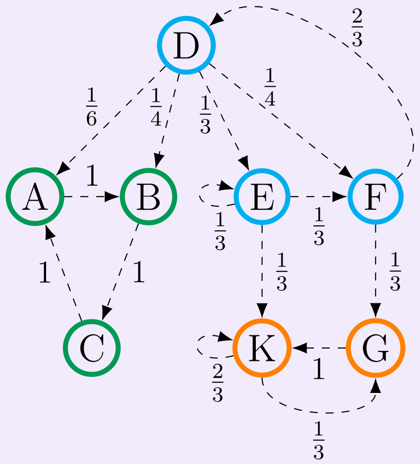
Bruh



2.

Solution:

The graph representation of $(X_n)_{n\geqslant 0}$ is:



a) Then, to calculate $\mathbb{E}[V_E \mid X_0 = D]$, we can use the formula

$$\mathbb{E}[V_E \mid X_0 = D] = \frac{h_{DE}}{1 - f_E}$$

$h_{DE} = \frac{1}{3} + \frac{1}{6}h_{AE} + \frac{1}{4}h_{BE} + \frac{1}{4}h_{FE} = \frac{1}{3} + \frac{1}{4}h_{FE}$ since $A \not\to E$ and $B \not\to E$

$h_{FE} = \frac{2}{3}h_{DE} + \frac{1}{3}h_{GE} = \frac{2}{3}h_{DE}$ since $G \not\to E$

Solving this system yields $h_{DE} = \frac{2}{5}, h_{FE} = \frac{4}{15}$

Calculating f_E can be done similarly;

$$f_E = \frac{1}{3} + \frac{1}{3} + h_{FE} = \frac{1}{3} + \frac{1}{3} \cdot \frac{4}{15} = \frac{19}{45}$$

Then, the answer is simply

$$\mathbb{E}[V_E \mid X_0 = D] = \frac{h_{DE}}{1 - f_E} = \frac{\frac{2}{5}}{1 - \frac{19}{45}} = \frac{9}{13}$$

b)

$$f_G = h_{GG} = \frac{1}{3} = \frac{2}{3}h_{KG}$$
$$h_{KG} = \frac{1}{3} + \frac{2}{3}h_{KG}$$

Solving this system yields that $f_G = h_{KG} = 1$, thus the state G is recurrent as it is guaranteed that G will return to itself once, and since a Markov chain is memoryless, it will continue to do so, thus infinitely often.

c) To check the probability that $D \to G$ first without hitting any other recurrent states along the way (that being $\{A, B, C, K\}$), it suffices to use the same methodology as above but additionally $h_{KG} = 0$ as we are not allowed to visit there.

$$\begin{aligned} h_{DG} &= \frac{1}{3}h_{EG} + \frac{1}{4}h_{FG} \\ h_{EG} &= \frac{1}{3}h_{EG} + \frac{1}{3}h_{FG} + \underbrace{\frac{1}{3}h_{KG}}_{0} \\ h_{FG} &= \frac{1}{3} + \frac{2}{3}h_{DG} \end{aligned}$$

Resolving this linear system yields

$$\begin{aligned} h_{DG} &= \frac{5}{26} \\ h_{EG} &= \frac{3}{13} \\ h_{FG} &= \frac{6}{13} \end{aligned}$$

and so $h_{DG} = \frac{5}{26}$

d) Yes, it does have a stationary distribution. The uniform distribution works just fine, let

$$\pi = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

then

$$\pi P = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{4} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} = \pi$$

Thus, π is stationary on P

3.

Solution:

a) **True**, because $P_{ii} = 0 \implies$ that state i will have to move to some other state $j \neq i$ with probability $1 - P_{ii} = 1$. After that, it will never return to i as $P_{ji} = 0 \, \forall \, j \in S$, so its return probability is certainly not 1.

b) **True**, since this means that all states are reachable from each other, thus they form one big communicating class, which means the chain is irreducible. Suppose that state i was transient, then it must never return to state i ever again. Suppose it ends up at state $j \in S, j \neq i$, which is possible since $P_{ij} > 0$, then the probability of not returning to i is $1 - P_{ji} < 1$

Let $\mu = \max_{j \in S}(1 - P_{ji}) < 1$, then $\mathbb{P}(T_i > k \mid X_0 = i) \leqslant \mu^k$ because if we are at state j, then the probability of $j \not\to i$ i.e. $1 - P_{ji} \leqslant \mu$

Then,

$$\lim_{k\to\infty} \mathbb{P}(T_i > k \mid X_0 = i) = \mathbb{P}(T_i = \infty \mid X_0 = i) \leqslant \lim_{k\to\infty} \mu^k = 0$$

Thus, $\mathbb{P}(T_i < \infty \mid X_0 = i) = 1 - \mathbb{P}(T_i = \infty \mid X_0 = i) = 1$, which means state i is recurrent.

c) **True**, let $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ and $\pi = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Then,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

To show that this is unique, let $\pi = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$, then to satisfy the stationary equation

$$\pi P = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} + \lambda_2 \end{bmatrix}$$

This must equal π, so we have

$$\begin{bmatrix} \frac{\lambda_1}{2} & \frac{\lambda_1}{2} + \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$$

Solving this system yields $\lambda_1 = 0, \lambda_2 \in \mathbb{R}$. However, we have the constraint that $\lambda_1 + \lambda_2 = 1$ in order to be a valid distribution, which locks $\lambda_2 = 1$, thus proving uniqueness.