Disclaimer: These exercises are not provided by the module organiser, and to the best of our knowledge are not present in any course materials. While the aim is to ensure these exercises are appropriate and relevant to each module's content, they may not reflect the difficulty or style of question present in upcoming exam papers - you should first exhaust the module resources provided.

Sets

1. Give the size of each of the following sets:

$$\begin{aligned}
A &= \{y \in \mathbb{Z} \mid 2y \notin \mathbb{Z}, \quad 0 < y < 1000\} \\
B &= \{a + b \mid a, b \in \{3, 4, 7, 8\}\} \\
C &= \{n \in \mathbb{N} \mid 2n^2 - 3n + 1 < 0\} \\
D &= \{a \in \mathbb{Q} \mid a \notin \mathbb{Z}, \quad 5a \in \mathbb{Z}, \quad |2a| \leq 5\}
\end{aligned}$$

Relations

1. Let S be a set such that $|S| = n$. Consider any relation $R \subseteq S \times S$ such that R is neither reflexive, symmetric, nor transitive. For each $n \in \mathbb{N}$, what is the largest that $|R|$ can be?

2. Let $S = \{1, 2, \ldots, 8\}$ and $m \in \mathbb{R}$. We define $\sim_m \subseteq S \times S$ such that

$$a \sim_m b \quad \text{iff} \quad m \cdot a \leq b$$

   (a) For which values of m is $\sim_m$ an equivalence relation?

   (b) For which values of m is $\sim_m$ a partial order?

3. Let $k \in \mathbb{R}$. Define the relation $*_k \subseteq \sim_2 \times \sim_2$ (see question above) such that

$$(a_1, a_2) * (b_1, b_2) \quad \text{iff} \quad (a_1 - a_2)^2 + (b_1 - b_2)^2 \leq k$$

   (a) For which k is $*_k$ an equivalence relation?

   (b) For which k is $*_k$ a partial order?

4. We call a relation $R \subseteq S \times S$ *acyclic* if,

$$\forall a_1, a_2, \cdots, a_k \in S \quad (a_1, a_2), \cdots, (a_{k-1}, a_k), (a_k, a_1) \in R \implies a_1 = \cdots = a_n$$

   (a) Determine whether each of the following relations are acyclic:

      i. $R_1 \subseteq \mathbb{Z} \times \mathbb{Z}$ such that $nR_1k \iff n - k$ is even

      ii. $R_2 \subseteq \mathbb{N} \times \mathbb{N}$ such that $nR_2k \iff n - k$ is even

   (b) Show that all acyclic relations are antisymmetric.