Sets

A

The set of even integers that aren't integers, hence A has size 0.

B

We're looking for all possible *unique* sums of 3, 4, 7 and 8.

$B = \{7, 10, 11, 12, 15\}$, which has size 5.

C

Solving the quadratic yields $n = \frac{1}{2}, 1$. We want integers *between* these two bounds, so C also has 0 elements.

D

If $a \notin \mathbb{Z}$ but $5a \in \mathbb{Z}$, then $a = \frac{k}{5}$, where k is an integer & not a multiple of 5.

$|2a| \leq 5$ tells us that $\frac{-5}{2} \leq a \leq \frac{5}{2}$.

Hence, $D = \{\pm\frac{12}{5}, \pm\frac{11}{5}, \pm\frac{9}{5}, \pm\frac{8}{5}, \pm\frac{7}{5}, \pm\frac{6}{5}, \pm\frac{4}{5}, \pm\frac{3}{5}, \pm\frac{2}{5}, \pm\frac{1}{5}\}$, which has 20 elements.

Relations

1. Base case: $|S| = 1$. No relation is possible (has to be reflexive).

   Let $S = \{1, 2\}$, then $S \times S = \{(1,1), (1,2), (2,1), (2,2)\}$. We only need to remove 2 pairs ((1,1) and (2,1)) to break reflexivity, symmetry and transitivity.

   Now, let $S = \{1, 2, 3\}$, if we remove 3 elements (e.g., (1,1), (1,3) and (2,1)) we can break reflexivity, symmetry and transitivity.
   Note that adding any more elements into S doesn't change the amount of elements we need to remove - just one for each property we want to break.

   Hence, $|R|$ is a function of n, given by:

$$|R| = \begin{cases} 0 & n < 2, \\ 2 & n = 2, \\ n - 3 & n > 2 \end{cases}$$

2. (a) For reflexivity, $m \leq 1$.
For symmetry, $m < 1$.

Does transitivity hold? Check with $m = \frac{1}{2}$.
We can find a counter-example with 5, 2 and 1:
$\frac{1}{2} \times 5 = \frac{5}{2} \Rightarrow 5 \sim_{\frac{1}{2}} 2$
$\frac{1}{2} \times 2 = 1 \Rightarrow 2 \sim_{\frac{1}{2}} 1$
But $\frac{5}{2} > 1 \Rightarrow$ not transitive.

If we repeat the above with $m = \frac{-1}{2}$, we see that the relation is transitive. $m = 0$ would also work here because the smallest number in S is 1, which is greater than 0.

Hence, for $\sim_m$ to be an equivalence relation, we need $m \leq 0$.

(b) For anti-symmetry, $m \geq 1$.

This would also allow for transitivity, as we would have:

$$ma \leq mb < c, \quad a, b, c \in S$$

Intuitively, if $ma \leq b$ & we multiply by some $m \geq 1$, then $ma \leq mb$. If $mb \leq c$, then it follows that $ma \leq c$.

But for reflexivity to hold, we need $m \leq 1$.

Hence, the only value of m for which $\sim_m$ is a partial order is $m = 1$.

3. (a) Here's every possible pair of $S \times S$ under the relation $\sim_2$:

1,2			
1,3			
1,4	2,4		
1,5	2,5		
1,6	2,6	3,6	
1,7	2,7	3,7	
1,8	2,8	3,8	4,8

The largest possible value of $(a_1 - a_2)^2 = (1 - 8)^2 = 49$, so, for reflexivity, $k \geq 98$.
The relation is already symmetric due to the associativity of addition.

For reflexivity to hold, we had to set k to be the maximum possible sum $\Rightarrow$ all other sums are $\leq k \Rightarrow$ transitivity.

Therefore, $*_k$ is an equivalence relation $\forall k \geq 98$.

(b) By associativity of addition, $(a_1, a_2) * (b_1, b_2) \Rightarrow (b_1, b_2) * (a_1, a_2) \Rightarrow$ no possible value of k for which $*_k$ is a partial order.

4. (a) i. No. By counter-example, choose $n = 3$ & $k = 1$:
then $3 - 1 = 2$, which is even, so $3R_1 1$
but $1 - 3 = -2$, which is even, so $1R_1 3$

We have created a cycle $\Rightarrow R_1$ is not acyclic.

ii. No. Again, we have the above counter-example (note there is no requirement that $n - k \in \mathbb{N}$).

(b) Relations can be one of 3 things: symmetric, antisymmetric or neither.

If the relation is symmetric, then nRk & kRn will exist in the relation, so a cycle can be formed $\Rightarrow R$ cannot be symmetric.

What if the relation is neither symmetric nor antisymmetric?

The problem with this is the $\forall$ quantifier in the predicate: we need there to be no cycle for *regardless* of the values we choose (unless they are all the same). Consider this example:

$$1R2 \quad 2R1 \quad 1R3$$

The relation is neither symmetric nor antisymmetric, but a cycle still exists.

If symmetry causes cycles, and being neither symmetric nor antisymmetric can cause a cycle, it follows that any acyclic relation must be antisymmetric.