1st Lecture - Multivariable Controls

2009년 1월 5일 월요일
오전 11:35

David Wang (dwang@uwaterloo.ca) — office E3X-3154

1. Review Single Input Single Output(SISO) design and analysis (380)
2. Performance limitations
3. Advance SISO systems
4. Multi Input Multi Output (MIMO) techniques that build from SISO
5. state Space Design Techniques for MIMO systems
6. Optimal Control methods

Marks

Finals (50%) Projects (30%) 2 midterms (20%)

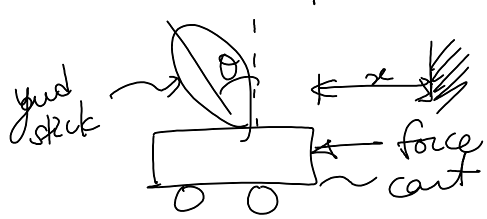
$\underbrace{\text{Finals (50\%) ... 2 midterms (20\%)}}_{\text{Theory based}}$

No textbooks

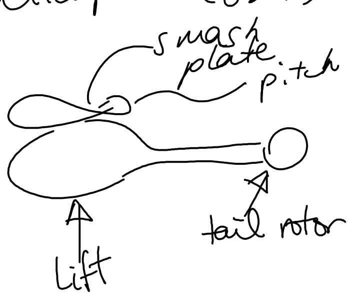
$\boxed{\text{Notes}}$ http://kingcong.uwaterloo.ca/~dwang

Projects
- groups of 4

1. Segway — Inverted pendelum problem

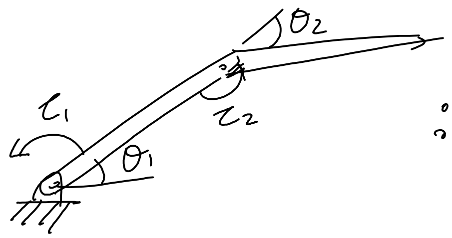


2. Helicopter (6DOF)



→ collective pitch: lift up
cyclic : swash plate in the plane (sideway motion)

3. Planar Elbow Manipulator
(eignoring gravity)



Torque on one changes θ_1 & θ_2

$\therefore$ MIMO

Project presentation: 3 short ppt's showing simulations etc

Marking Scheme /30

<u>5 marks</u>
- finding a model (reference needed)
- Build a simulation in MATLAB/Simulink
- Does the model make sense?
- Power point

<u>20 Marks</u>
- Design a controller
  - → 380 techniques
  - ↳ see performance & stability
  - → Advanced techniques
- Powerpoint

<u>5 Marks</u>
- add complexity to model
  - ↳ eg. add dynamics to actuators
  - → noise to sensors
  - → perturb the model (eg. mass is now 1.2kg instead of 1.3 etc.)
  - → Reevaluate performance
- Powerpoint

By Monday: email dwang@uwaterloo.ca
- Group members (up to 4)
- A brief discussion of what you want to control

Reference Texts:

Reference Texts :

- Modern Control Theory
  - Brogan (state space)
- Control System Design
  - Goodwin, Graebe and Salgado

2nd Lecture

2009년 1월 7일 수요일
오전 11:35

Course Outline 488

1) Review of SISO analysis and design methods (380)
2) Performance limitations
3) Design techniques for SISO systems (Advanced)
4) MIMO design techniques based on SISO methods
5) State Space design methods for MIMO systems
6) Optimal Control Methods

Mark Distribution:
Final 50%
Project 20%
Midterms 30%

No textbook
For notes:
http://kingcong.uwaterloo.ca/~dwang/

TA

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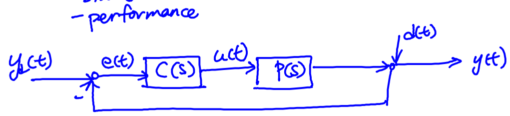
Project:

1. Helicopter
2. Robot
3. Segway
4. CanadArm

Review of SISO analysis and design methods (380)

-In this course, we will study using closed loop feedback techniques in order to modify the behaviour of a system (plant) so that it behaves in a stable manner and can satisfy certain performance criteria

- closed loop feedback
  - stable
  - performance



-Some basic performance criteria are tracking, disturbance rejection and robustness

1) tracking We want $y(t)$ to follow $y_d(t)$

2) disturbance rejection $d(t)$ should not affect $y(t)$ greatly

3) robustness 1) and 2) occur even with uncertainties in $P(s)$

-This section is a quick review of concepts that you learned in 380. This will be very quick and may require you to go back into the notes for more details.

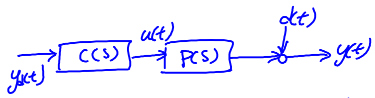
-This course also requires you to become familiar with Matlab. There are several good tutorials.

eg.

http://www.mathworks.com/academia/student_center/tutorials/intropage.html

http://www.mathworks.com/academia/student_center/tutorials/simulink-launchpad.html

-Why is open loop bad?



For perfect tracking, we could let $C(s) = \frac{1}{P(s)}$

Suppose $P(s) = \frac{1}{s+1}$, then $C(s) = s+1$

Probs of OL:

1. We have $y_d(t) = y(t)$, but with $d(t)$, $y(t) = d(t) + y_d(t)$ ∴ No disturbance rejection
2. Recall Sensitivity $= \frac{\partial\{T_{y_dy}\}/T_{y_dy}}{\partial P/P} = \frac{\partial\{PC\}}{\partial P}\cdot\frac{P}{PC} = 1$

   where T_{y_dy} ⇒ transfer function from y_d to y

   ∴ No robustness.

   Small change results in large change in the output.
3. You can't realize $C(s) = s+1$, since it is a pure derivative.
4. If $P(s)$ were unstable (eg $P(s) = \frac{1}{s-1}$) then $C(s) = s-1$ would cause unstable pole-zero cancellation

   i) If you can't cancel exactly, the transfer function is unstable

   ii) Any noise at $u(t)$ can drive the system unstable

-With closed loop, we get $T_{y_dy} = \frac{P(s)C(s)}{1+P(s)C(s)}$

Let $C(s) = K$ and make K "big".

∴ $T_{y_dy} = \lim_{K\to\infty} \frac{PC}{1+PC} = 1$ ← Perfect tracking!!!

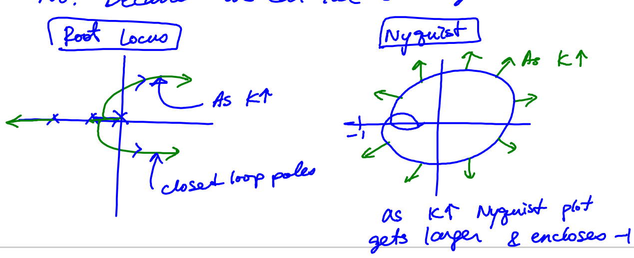
Disturbance: $T_{dy} = \lim_{K\to\infty} \frac{1}{1+PC} = 0$ ← Perfect disturbance rejection

Sensitivity: $\frac{\partial\{T_{y_dy}\}}{\partial P}\cdot\frac{P}{T_{y_dy}} = \frac{1}{1+PC}$ ← As $K\to\infty$ sensitivity $\to 0$

This is called high gain control

-Can we solve all problems with high gain controllers?

No! Because we can lose stability.

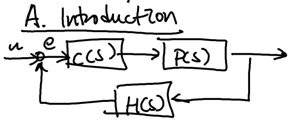


As K↑ noise problems increase

∴ high gain controllers NOT GOOD.

I. Review

A. Introduction



cascade (series) compensator

Assume $P(s)$ is a rational TF.

Linear, time invariant, causal, lumped parameter

ordinary dif. eqn.
NOT PDE.

- If e^{-sT} (delay) exists then this can mess up our system.
- We assume there are no initial conditions in our plant (no stored energy)

B. Stability

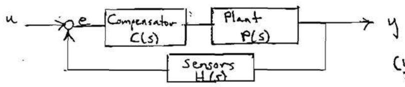
1. Open Loop Stability
   - Definition: Bounded input Bounded output

Review of Analog Control Techniques

2009년 1월 9일 금요일
오전 11:41

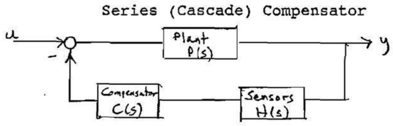
I REVIEW OF ANALOG CONTROL TECHNIQUES

A/ Introduction

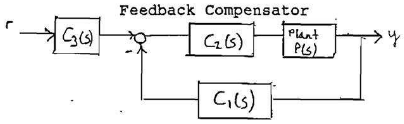


(We assume H=1 in this course)

Series (Cascade) Compensator



Feedback Compensator



Most General Configuration

-It turns out that the techniques to be described can be used for both systems above. Thus, we'll stick to the cascade

-What is $P(s)$? The plant is assumed to be a linear time-invariant and causal (ie does not react until the input is applied) and has a rational transfer function

ie, $\frac{\textit{polynomial in s}}{\textit{polynomial in s}}$

(This happens when our plant is a lumped-parameter system and has no time delays. Most of this will carry over, with minor modifications, to time delays e^{-sT} but we will not consider this at this point)

-We also assume that the initial conditions on the plant are zero. In other words, there is no "energy" initially in the system. Therefore, when we take the Laplace transforms, we can set the initial conditions to 0

eg/ $\dot{y}(t)+y(t)=u(t)$ with initial conditions $y(0)$

Thus, $sY(s)-y(0)+Y(s)=U(s)$

$$Y(s)=\frac{u(s)}{s+1}+\frac{y(0)}{s+1}$$

If the initial condition is zero, then $Y(s)=\frac{1}{s+1}U(s)$

B/Stability

1. Open Loop Stability

-The minimum goal of controllers is always stability

-Def'n- A transfer function with input $u(t)$ and output $y(t)$ is BIBO stable if, whenever $u(t)$ is bounded, then the output $y(t)$ is also bounded (BIBO = Bounded Input Bounded Output)

That is, if $|u(t)| \le M$ for some constant M and for all times t, then $|y(t)| \le N$ for some constant N and for all times t.

eg/ $u(t) \longrightarrow \boxed{K} \longrightarrow y(t)$ This is BIBO stable since
$y = Ku \therefore |y| \le K|u| \le Km$
($\therefore N = Km$)

-If a transfer function is not stable, it is *unstable*. That is, we need to only find one bounded input which gives an unbounded output.

-There are other definitions of stability (eg 482)

-A rational transfer function is proper if the degree of the numerator is less than or equal to the degree of the denominator

-Theorem- A rational transfer function is BIBO stable iff the transfer function is

i) proper
ii) all the poles have negative real parts

Pf/ See E&CE 380 notes

eg/ $\frac{(s+2)(s+3)}{(s+1)}$ is unstable, $\frac{(s+1)}{s(s+3)}$ is unstable
$\frac{(s+1)}{(s+2)(s-3)}$ is unstable, $\frac{(s+1)}{(s^2+2s+2)}$ is stable since poles at $-1 \pm j\sqrt{2}$

-How do we ensure that there is not an unstable zero which cancels an unstable pole? eg $\frac{s-1}{s^2-1}=\frac{1}{s+1}$

-Note that we can only do an unstable pole/zero cancellation for open loop transfer functions. Later, we will look at closed loop transfer functions. YOU CANNOT DO AN UNSTABLE POLE ZERO CANCELLATION IN THE CLOSED LOOP CASE.

-For complicated transfer functions where it is difficult to factor the numerator and denominator, how does one check pole/zero cancellations?

-Euclidean division is a method to find common factors of two polynomials $n(s)$ and $d(s)$ where it is assumed that the order of $n(s)$ is less than or equal to the order of $d(s)$

~ Assumed proper sys.

-Algorithm:

1. Let $r_1=d(s)$, $r_2=n(s)$

   $i=3$ r_1/r_2

2. Let r_i be the remainder of r_{i-2}/r_{i-1}, that is, $r_{i-2}(s)=q_i(s)r_{i-1}(s)+r_i(s)$ where all these are polynomials

3. Eventually, $r_i=0$ for some i_o and the common factor is r_{i_o-1}

eg/ reduce the transfer function

$$\frac{n(s)}{d(s)}=\frac{s^2+4s+3}{s^3+4s^2+4s+1}$$

to its minimal form

sol'n/ Let $r_1(s)=d(s)=s^3+4^2+4s+1$ and $r_2(s)=n(s)=s^2+4s+3$
Can show that $r_2=s$, $r_3=s+1$ and $r_4=0$
Therefore, $i_o=4$ and the common factor is r_3 or $(s+1)$

-If the r_{i_o-1} is a constant, then there are no common factors!

$s \leftarrow q_3$

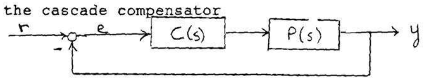
$\underbrace{s^2+4s+3}_{r_2}\ \big)\ s^3+4s^2+4s+1 \leftarrow r_1$
s^3+4s^2+3s
$s+1 \leftarrow r_3$

$s+3 \leftarrow q_4$

$r_3 \rightarrow s+1\ \big)\ s^2+4s+3 \leftarrow r_2$
s^2+s
$3s+3$
$3s+3$
$0 \leftarrow r_4$

2. Closed Loop Stability

-Look at the cascade compensator



-One could just look at the transfer function from r to y, ie $PC/(1+PC)$ to see if the system is stable. As mentioned in the previous section, though, one cannot do an unstable pole/zero cancellation to check for stability. Here, we will see why.

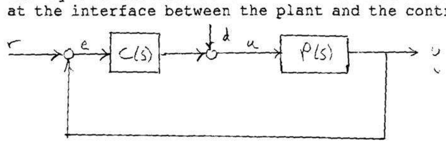
-Suppose $P(s)=\frac{n_p(s)}{d_p(s)}$ where $n_p(s)$, $d_p(s)$ do not have common roots and $C(s)=\frac{n_c(s)}{d_c(s)}$ where $n_c(s)$, $d_c(s)$ do not have common roots. (That is, cancel the common factors from the numerator and denominator)

Thus, $\frac{PC}{1+PC}=\frac{n_pn_c}{d_pd_c+n_pn_c}$

-The stability of this transfer function is given by the roots of the denominator, ie $d_pd_c+n_pn_c$

-Definition- The characteristic polynomial is $\Delta(s)=d_pd_c+n_pn_c$

-What if n_pn_c has an unstable root which cancels an unstable root of $\Delta(s)$? You can't do this in a closed loop system. The reason is that the system becomes nonrobust to disturbances or noise entering at the interface between the plant and the controller, d.



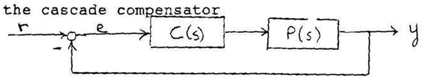
We must have u and y bounded for bounded r and d.

$3s+3$

$0 \leftarrow r_4$

2. Closed Loop Stability

-Look at the cascade compensator



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Thus, $\frac{PC}{1+PC}=\frac{n_p n_c}{d_p d_c+n_p n_c}$

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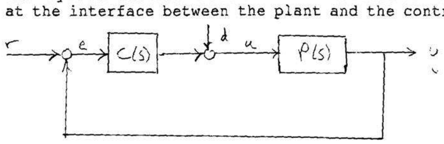


We must have u and y bounded for bounded r and d.

-Definition- A system is closed loop stable if all the transfer functions from r and d to u and y are stable.

-Note that this implies that, if r and d are bounded, then u and y are bounded as well

-Theorem- The system is closed loop stable iff the characteristic polynomial $\Delta(s)$ has all its roots in the open LHP.

Proof/

First note that the transfer function from $[r, d]'$ to $[y, u]'$ is given by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} PC(1+PC)^{-1} & P(1+PC)^{-1} \\ C(1+PC)^{-1} & (1+PC)^{-1} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

$$\therefore \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \frac{n_c n_p}{\Delta} & \frac{n_p d_c}{\Delta} \\ \frac{d_p n_c}{\Delta} & \frac{d_p d_c}{\Delta} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

We need all of these transfer functions to be stable.

$\Leftarrow$ This is obvious as, if Δ has open LHP roots, then all the above transfer functions will be (open loop) stable. Thus, the system is closed loop stable

$\Rightarrow$ If the system is closed loop stable, then all the transfer functions in the above equation must be stable. If Δ has closed RHP roots, then they must be cancelled by corresponding unstable RHP roots in all the numerator terms. However, it is easy to see from the form of the equations that this is not possible. There will be at least be one unstable transfer function or else either P or C has a common factor. ▩

-Therefore, checking closed loop stability only requires checking the roots of $\Delta(s)$

-Note that the above implies there can be NO unstable pole/zero cancellations between P and C. Suppose there is an unstable pole/zero cancellation at λ. Then this would mean that $\Delta(\lambda)=0$. In other words, Δ would have a closed RHP root.

C/ Performance Criteria

1. Time domain

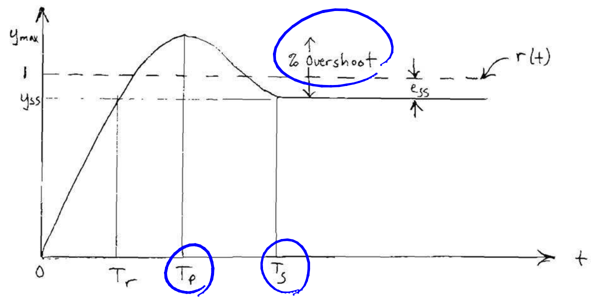
-In addition to stability, we desire other performance criteria

-We have many test signals, eg, step, ramp, impulse, sinusoid

-Easiest to generate is the step. It also contains all frequencies

-Suppose the input to a stable system is a step of unit magnitude

-The output will look like



-We only need to look at a step of unit magnitude since, by linearity, scaling $u(t)$ merely scales $y(t)$ by the same amount.

1. Steady state error

-Let $e(t)=r(t)-y(t)$ and $e_{ss}=\lim_{t\to\infty} e(t)$

-To calculate this, use the final value theorem, that is

$$e_{ss}=\lim_{t\to\infty} e(t)=\lim_{s\to 0} se(s)$$

assuming the limit actually exists. You can check this by examining the poles of $se(s)$

(eg $e(t)=e^t$, $\therefore \lim_{s\to 0} se(s)=0$. However, $e(t)$ is actually unbounded)

Type 0 system = no poles at zero in C(s)P(s)
" 1 " = 1 pole at zero in C(s) P(s)
" 2 " = 2 " " " " "

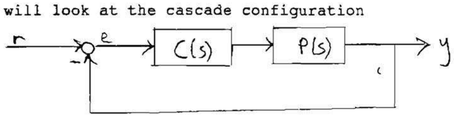
Type	Error to Step	Error to Ramp
0	const.	∞
1	0	const
2	0	0

-Def'n- The system *tracks* the reference input if and only if $e_{ss}=0$

-To track, it is necessary that the system is stable

-If the input $r(t)$ is zero or a constant, we call this a regulator

-We will look at the cascade configuration



-The loop gain is $G=CP$

-Def'n- If $G=\frac{k\prod(s+z_i)}{s^l\prod(s+p_i)}$, then this is a type l system

-Theorem- Let $G=\frac{ka(s)}{s^l b(s)}$. Then the steady state error for various inputs are as follows:

Type	Unit Step	Unit Ramp	Unit Parabola
0	$\frac{1}{1+ka(0)/b(0)}$	∞	∞
1	0	$\frac{1}{(ka(0)/b(0))}$	∞
2	0	0	$\frac{2}{(ka(0)/b(0))}$

2. Percent overshoot: $\%OS=\frac{y_{max}-y_{ss}}{y_{ss}}$

3. Settling time: The time it takes for $y(t)$ to settle to within 2% (5%) of y_{ss}

Type	Error to Step	Error to Ramp
0	const.	∞
1	0	const
2	0	0

-Def'n- The system *tracks* the reference input if and only if $e_{ss}=0$

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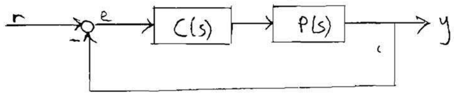

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2. Percent overshoot: $\%OS=\frac{y_{max}-y_{ss}}{y_{ss}}$

3. Settling time: The time it takes for $y(t)$ to settle to within 2% (5%) of y_{ss}

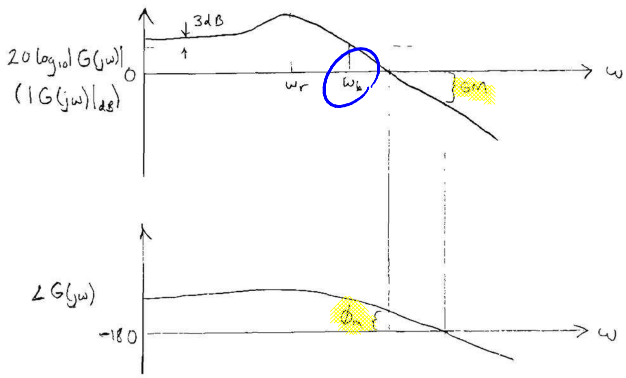
4. Rise Time; i) If there is overshoot, it is the time to first reach y_{ss}

ii) If there is no overshoot, it is the time to reach 90% of y_{ss}

2. Frequency Domain

–Construct a Bode plot by looking at the open loop gain. Apply an input $u(t)=\sin(\omega t)$ and sweep through all from 0 to ∞. The output $y(t)$ will also be sinusoidal at the same frequencies as the input due to linearity. Measure the magnitude of $y(t)$ and the phase difference ϕ between the input and output. Finally, plot $20\log_{10}\left|\frac{y(j\omega)}{u(j\omega)}\right|$ and ϕ vs the frequency ω

–Can show that this is the same as plotting $20\log_{10}|G(j\omega)|$ *vs* ω and $\angle G(j\omega)$ *vs* ω



1. Phase margin ϕ_m- the phase difference from $\angle G(j\omega)$ to -180 when $20\log_{10}|G(j\omega)|=0db$

2. Gain margin *GM*- the difference between 0db and $20\log_{10}|G(j\omega)|$ when $\angle G(j\omega)$ is at -180

3. Bandwidth ω_b- the ω when $|G(j\omega)|$ decreases by 3db from the DC value

4. Resonance peak ω_r- the ω when $20\log_{10}|G(j\omega)|$ is at a maximum

By definition, we will always consider the gain margin to be greater than 1 and the phase margin to be positive

3. Second Order Relationships between Time and Frequency Domain

-Many plants can be represented by the transfer function

$$G(s)=\frac{{}^{2}_{n}}{s(s+2\xi\omega_n)}$$

-In the closed loop, this becomes

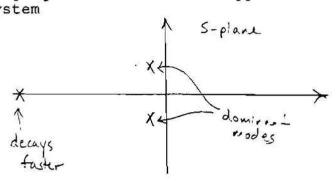
$$\frac{G(s)}{1+G(s)}=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

-The step response of the closed loop system is given by

$$y(t)=1-\frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t+\tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)$$

-This is a crucial relationship that is often used to connect frequency domain and time domain specifications

-Many closed loop systems can also be approximated by a dominant second order system



-Rule of thumb- The real part of the dominant poles should be less than 1/10 the real part of the other poles. As well, there should not be any zeros in the vicinity of the dominant poles (ie may reduce the residues). Otherwise, you must check the residues.

-Eg/

$$\frac{G(s)}{1+G(s)}=\frac{(s+10)}{(s+11)(s+12)(s^2+2s+2)}$$

$$=\frac{1}{s^2+2s+2}\cdot\frac{s+10}{(s+11)(s+12)}$$

$$\approx\frac{10}{132(s^2+2s+2)}$$

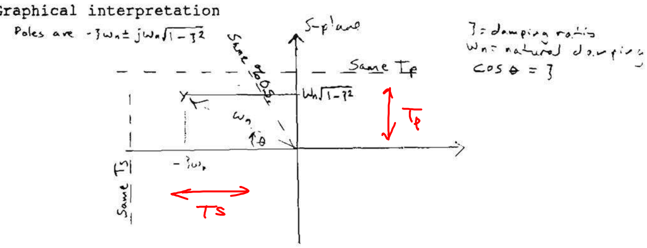
You could also do a partial fraction expansion, and just keep the residues corresponding to the dominant poles. Then recombine these terms.

$$\frac{G(s)}{1+G(s)}=\frac{.001}{s+11}+\frac{.016}{s+12}+\frac{-.0065s+.075}{s^2+2s+2}$$ (Note that 10/132 = .076)

-For a second order system, we have the following relationships between frequency domain and time domain parameters

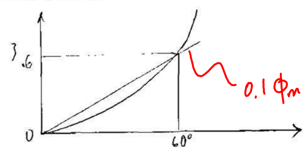
i) $T_s = \dfrac{4}{\xi\omega_n}$ (2%)   ii) $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}}$, $\xi<1$ iii) $\%OS = 100e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$

-Graphical interpretation



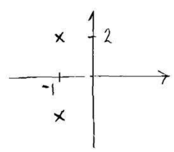
-Another crucial relationship between the frequency domain and time domain is $\xi \approx .01\phi_m$ *for* $\phi_m < 60°$.

This comes from the open loop Bode plot of $G(s)$. One can find the phase margin analytically as $\phi_m = \tan^{-1}\left[2\xi\left(\dfrac{1}{(4\xi^4+1)^{1/2}-2\xi^2}\right)^{1/2}\right]$. If you plot this, one finds the following graphs:



eg/ We want a 2% settling time of 4 seconds with oscillations with a frequency of no more than 2 rad/s. What is the phase margin if the system is approximately 2nd order?

Soln/



$\xi = .44$
$\therefore \phi_m \approx 44°$

January 14th
Wednesday

* PD controller always stabilizes 2nd order sys.
Routh Hurwitz proves this property.

D Stability Tests

$\Delta(s)$ -Recall that our goal is to make stable, ie all roots of $\Delta(s)=0$ in the open left half plane.

-In general, this is very difficult to check if $\Delta(s)$ is a high order polynomial

-We have three main tests to see if $\Delta(s)$ is stable

1. Routh-Hurwitz Criterion

-The proof is hard. Therefore, we will just give the algorithm

must be +ve. If not, then multiply by (−1).

-Suppose $\Delta(s)=a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_o=0$ where $a_n>0$.

1. If any of the coefficients are negative, then this is unstable
2. Set up the table

$$
\begin{array}{l|lll}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_1 & b_2 & b_3 \\
s^{n-3} & c_1 & c_2 & c_3
\end{array}
$$

etc

where $b_1=\dfrac{a_{n-1}a_{n-2}-a_n a_{n-3}}{a_{n-1}}$ $b_2=\dfrac{a_{n-1}a_{n-4}-a_n a_{n-5}}{a_{n-1}}$

$c_1=\dfrac{b_1 a_{n-3}-a_{n-1}b_2}{b_1}$ $c_2=\dfrac{b_1 a_{n-5}-a_{n-1}b_3}{b_1}$ etc

3. $\Delta(s)$ has all open LHP roots iff all the first column elements are positive.

-As well, if there are no zeros in the first element, the number of sign changes are number of unstable roots. When there are zero elements, several possibilities may arise, all of which imply that $\Delta(s)$ include roots that are not in the open LHP (see E&CE 380 notes for more details).

eq/ $\Delta(s)=s^4+3s^3+6s^2+s+2k$

Sol'n/

$$
\begin{array}{l|lll}
s^4 & 1 & 6 & 2K \\
s^3 & 3 & 1 & \\
s^2 & \frac{17}{3} & 2K & \\
s^1 & \alpha & & \\
s^0 & 2K & &
\end{array}
$$

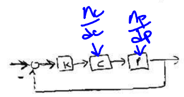
$\therefore K>0$

or

$\frac{17}{3}-6K>0 \therefore K<\frac{17}{18}$

Thus, $0<K<\frac{17}{18}$ for stability

2. Root Locus

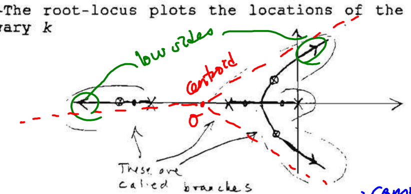


-Look at roots of $\Delta(s)=d_c d_p+kn_c n_p=0$ which are the poles of the closed loop system

-Let $G=n_c n_p/d_c d_p$. Thus, we are really looking the roots of

$$1+kG(s)=0$$

-The root-locus plots the locations of the roots of $\Delta(s)$ as we vary k



Low sides/roots will asymptotically approach the lines from centroid.

Low sides ALWAYS go from poles to zeros. So in this graph we consider there to be 8 zeros in infinity.

-Rewrite as $kG=-1=\|1\|\angle 180$

complex #. ∴ $KG(S)=-1$ $\begin{cases} K\angle G(S) = 180°(2n+1) \\ |KG(S)| = 1 \end{cases}$

$$KG(S) = \frac{(S-z_1)(S-z_2)\cdots(S-z_n)}{(S-P_1)\cdots(S-P_n)}k = 1$$

-Suppose $G(s)=\dfrac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$

$$\angle G(S) = \angle(S-z_1)+\angle(S-z_2)\cdots-\angle(S-P_1)-\angle(S-P_2)\cdots$$

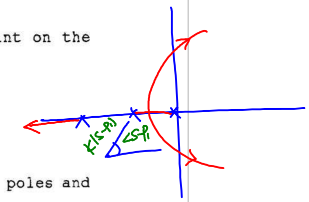
-Since $\|kG\|=1$ and $\angle(kG)=180$, we have the following two conditions:

I. $\dfrac{k\|s-z_1\|\|s-z_2\|\ldots\|s-z_m\|}{\|s-p_1\|\|s-p_2\|\ldots\|s-p_n\|}=1$

2 $\angle(s-z_1)+\angle(s-z_2)+\ldots+\angle(s-z_m)-\angle(s-p_1)-\angle(s-p_2)-\ldots-\angle(s-p_n)=180(2n+1)$

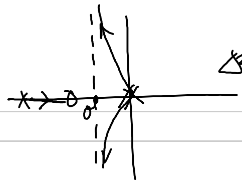
-Use the angle criterion to trace out the loci

-Use the magnitude criterion to find k for a given point on the loci



-Procedure

1. Plot poles and zeros of $G(s)$
2. Root-locus symmetric about the real axis
3. Root-locus on the real axis if the number of poles and zeros of $G(s)$ on the right is odd



$1+KG(S)=0 \Rightarrow \Delta(S)=0$

$G(s) = \frac{s+1}{s^2(s+3)}$

4. The loci goes from the poles to the zeros continuously. There are $n-m$ loci that go to ∞. These are asymptotic to lines centred at

$$\sigma = \frac{\sum \text{poles of } G(s) - \sum \text{zeros of } G(s)}{n-m}$$

and with angles

$$\phi = \frac{(2q+1)\pi}{n-m}$$

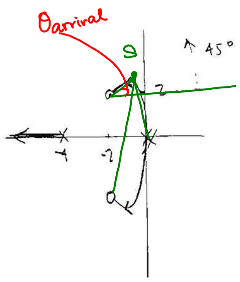
5. Find breakaway points on the real axis, ie, isolate $k=p(s)$, and evaluate $dk/ds=0$.

6. Find the departure angles from the poles by looking at the root-locus very close to the poles

7. Find the arrival angles from the zeros by looking at the root-locus very close to the zeros

-eg/ $KG(s) = K\frac{s^2+4s+8}{s^2(s+4)}$

So the zeros are at $-2 \pm 2j$



# of asymptotes = 1 with angle 180°

$\sigma = 0$

θ_{arr}: (top zero)

$180 = -2(135) - 45 + \theta_{arr} + 90$

$\therefore \theta_{arr} = 45°$

θ_{dep}: (pole at origin)

$180(2n+1) = 2\theta_{dep} - (45-45)$

$\therefore \theta_{dep} = 90, 270$

- Suppose we want to know the gain K for $s=-5$

Use magnitude criteria

$\frac{K|\sqrt{13}||\sqrt{13}|}{5 \times 5} = 1 \quad \therefore K = \frac{25}{13}$

January 16th 2009

3. Nyquist Criterion

-This method handles delays. It is also useful when the only information about the plant is in the form of a Bode plot

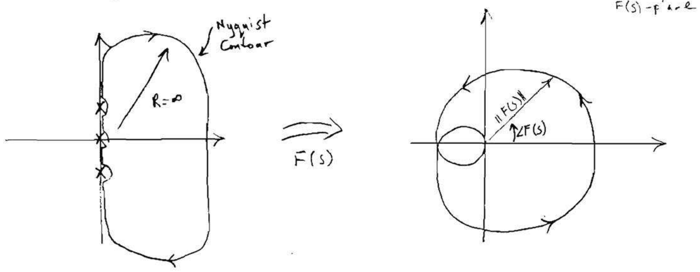
-We would like to examine the number of poles in the right half plane (unstable) for a given rational transfer function

$$F(s) = \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

-Use Cauchy's Principle of the Argument

-Draw the Nyquist contour which encloses the right half plane. Circumvent any imaginary axis poles by going around them on the right

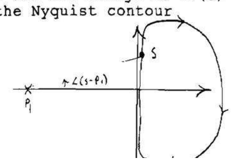
-For each point on the Nyquist contour, calculate $F(s)$ and plot on the "$F(s)$-plane"



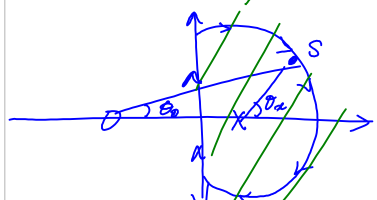
-Now, $\angle F(s) = \angle(s - z_1) + \ldots + \angle(s - z_m) - \angle(s - p_1) - \ldots - \angle(s - p_n)$

-Consider two cases:

i) For poles p_i and zeros z_i outside the Nyquist contour, the net change in $\angle F(s)$ is zero as you go around the Nyquist contour



s-plane, unstable region.

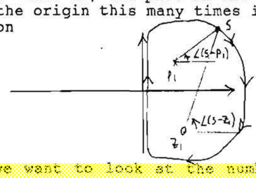


As s travels, θx travels 360°, θ₀ returns back to zero.
Anything in unstable region travels 360°.

Cauchy's Principle of Arg

Net sum of poles/zeros in the unstable region is equal to the # of encirc. of

ii) For poles p_i and zeros z_i inside the Nyquist contour, the number of times $\angle F(s)$ goes around 2π radians clockwise is the number of zeros inside the contour minus the number of poles inside the contour. That is, the plot on the $F(s)$-plane goes around the origin this many times in the clockwise direction



-Now, for stability, we want to look at the number of unstable poles of the closed loop system. Therefore, look for unstable zeros of $F(s)=1+kG(s)=0$

-Note that the "1" simply shifts the origin. Thus, instead of looking at encirclements of the origin for $F(s)=1+kG(s)$, we can look at encirclements of "-1" of the plot of $kG(s)$

-Now, $kG(s)$ simply scales the Nyquist plot larger and smaller

-Thus, we have the following procedure:

1. Map the Nyquist contour to the $kG(s)$-plane (see 380 notes for procedure). We usually plot this for $k=1$ since the gain k simply is a scaling factor

2. Let ν be the clockwise encirclements of -1.

$$\therefore \nu = z_u - p_u$$

where z_u are the unstable zeros of $F(s)=1+kG(s)$ and p_u are the unstable poles of $F(s)=1+kG(s)$ which are also the unstable poles of $G(s)$

-The unstable poles of the closed loop system are $z_u=\nu+p_u$. These must be zero for stability or, equivalently, $-\nu=p_u$.

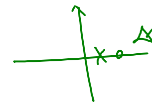
-<u>Nyquist Stability Criterion</u>- The closed loop system is BIBO stable iff

i) the $KG(s)$ plot does not intersect the -1 point (this would imply $j\omega$-axis roots of $\Delta(s)$)

ii) **counterclockwise** encirclements of -1 of the Nyquist plot of $KG(s)$ equal the open RHP poles of $G(s)$

the origin.

eg



In controls, $1+KG(s)=0$.
Hence we care about encirc. of -1.

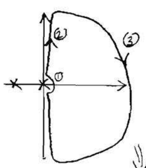
Poles of $G(s)$ are the same as poles of $1+KG(s)$

eg/ $G(s)=\dfrac{10}{s(s+2)}$

Sol'n, $p^u=0$

-Need to plot only the top half since the Nyquist is symmetric

-Divide into 3 regions



① let $s=\varepsilon e^{j\theta}$, θ goes from 0 to $\frac{\pi}{2}$, ε is small

$G(s)=\dfrac{10}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+2)} \approx \dfrac{10}{\varepsilon e^{j\theta}}$

at $\theta=0$, $G(s)\approx\frac{5}{\varepsilon}$

at $\theta=\frac{\pi}{2}$, $G(s)=\frac{5}{\varepsilon}e^{-j\frac{\pi}{2}}$

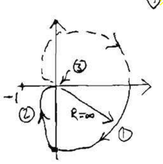
② as $w\to\infty$, $G(jw)=\dfrac{10}{jw(jw+2)}$ approaches $-\dfrac{10}{w^2}$ or $0\angle-180$

$\Downarrow$ $G(s)$

③ let $s=Re^{j\theta}$, θ goes from $\frac{\pi}{2}$ to 0

$G(s)=\dfrac{10}{Re^{j\theta}(Re^{j\theta}+2)} \approx \dfrac{10}{R^2e^{2j\theta}}$

$\therefore$ for $R=\infty$, the magnitude is 0



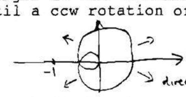
aside: Suppose we were looking at $KG(s)=\dfrac{10K}{s(s+2)}$

We never have a ccw encirclement for any K

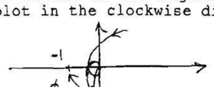
$\therefore$ always stable for any K

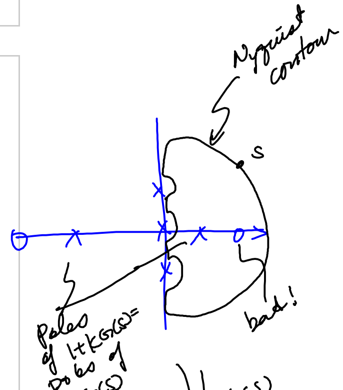
-Can define two quantitative stability "measures" which are related to the frequency domain performance specs.

-Def'n/ The gain margin is the increase in K until instability occurs (that is, until a ccw rotation of -1 occurs).



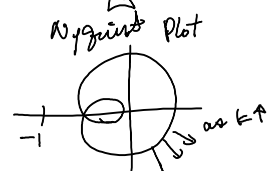
-Def'n/ The phase margin is the decrease in the phase angle until the $|G(s)|=1$ point is forced to the negative real axis (that is, rotate the Nyquist plot in the clockwise direction)





$\Downarrow$ G(s)

Nyquist Plot



let ν be clockwise encirclements of -1

$\therefore \nu = \underbrace{z_u}_{\text{\# of unst. zeros of } 1+KG(s)} - \underbrace{p_u}_{\text{\# of unst. poles of } G(s)}$

eg/ $G(s)=\frac{10}{s(s+2)}$

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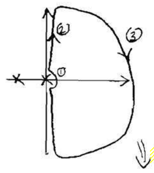
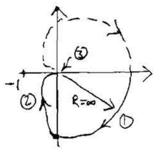


$\Downarrow G(s)$



① let $s=\varepsilon e^{j\theta}$, θ goes from 0 to $\frac{\pi}{2}$, ε is small

$G(s)=\frac{10}{\varepsilon e^{j\theta}(\varepsilon e^{j\theta}+2)}\approx\frac{10}{\varepsilon e^{j\theta}}$

at $\theta=0$, $G(s)\approx\frac{5}{\varepsilon}$

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② as $\omega\to\infty$, $G(j\omega)=\frac{10}{j\omega(j\omega+2)}$ approaches $-\frac{10}{\omega^2}$ or $0\angle-180$

③ let $s=Re^{j\theta}$, θ goes from $\frac{\pi}{2}$ to 0

$G(s)=\frac{10}{Re^{j\theta}(Re^{j\theta}+2)}\approx\frac{10}{R^2e^{2j\theta}}$

$\therefore$ for $R=\infty$, the magnitude is 0

aside: Suppose we were looking at $KG(s)=\frac{10K}{s(s+2)}$

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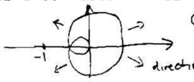
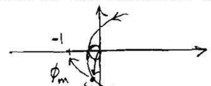


G(s)-plane

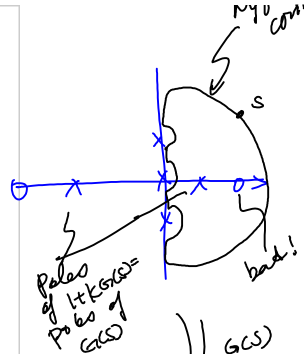
-Def'n/ The phase margin is the decrease in the phase angle until the $|G(s)|=1$ point is forced to the negative real axis (that is, rotate the Nyquist plot in the clockwise direction)



$|G(s)|=1$

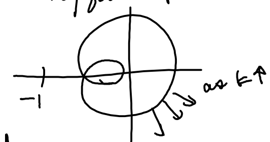
-Note that the Bode and Nyquist plot give exactly the same information in a slightly different form

-The previously mentioned gain and phase margins measured from the Bode plot is exactly the same as the definitions in this section



$\Downarrow G(s)$

Nyquist Plot



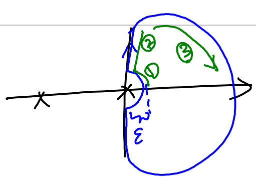
let ν be clockwise encirclements of -1

$\therefore \nu = \underbrace{Z_u}_{\#\text{ of unst. zeros of } 1+KG(s)} - \underbrace{p_u}_{\#\text{ of unst. poles of } G(s)}$

This is zero for stability.

Only the zeros of G(s) is important because they're closed loop poles of G(s).

eg. $G(s)=\frac{10}{s(s+2)}$



① For poles at the origin, Let $s=\varepsilon e^{j\phi}$ ϕ from 0 to 90.

$$\frac{10}{\varepsilon e^{j\phi}(\varepsilon e^{j\phi}+2)}=\frac{(10/\varepsilon)e^{-j\phi}}{\varepsilon e^{j\phi}+2}$$

January 19th 2009

E/ Review of Bode Design Techniques

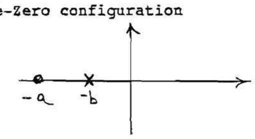
1. Lag Compensator

-Choose $C(s)=k\frac{(1+s/a)}{(1+s/b)}$ where $a>b$

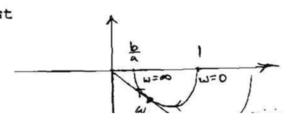
-Ignoring the k, the gain is 1 at $\omega=0$

-Look at plots of $(1+s/a)/(1+s/b)$

-Pole-Zero configuration



-Nyquist



(Nyquist plot is a semi-circle)

E/ Review of Bode Design Techniques

1. Lag Compensator

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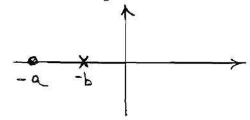
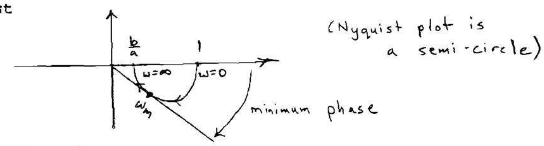
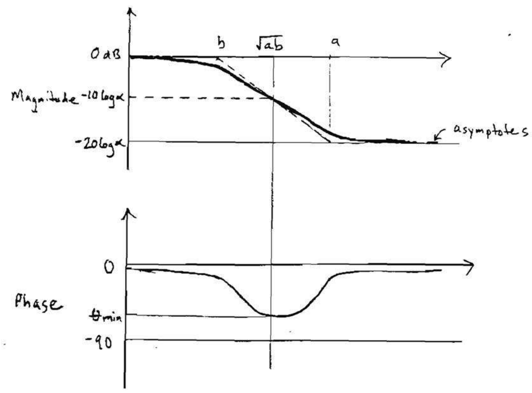


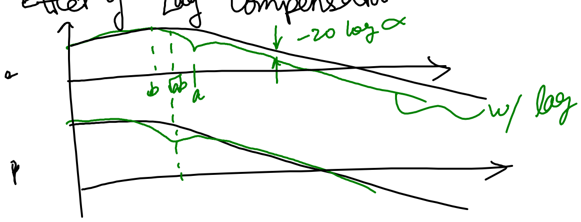
-Nyquist



-Bode (let $\alpha=a/b$)

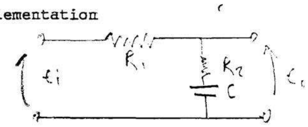


Effect of Lag Compensator



lag looks like PI controller. except PI controller has gain = ∞ at s=0

-Implementation



$E_o(s)=\frac{1+R_2Cs}{1+(R_1+R_2)Cs}E_i(s)$ (Need a gain k)

-Design Goal (Typical. You may have to translate from time domain to frequency domain)

Given e_{ss} less than __ percent, find a C which gives a phase margin ϕ_m of at least __ degrees (Note that e_{ss} is a steady state specification while ϕ_m is a transient specification)

-Usually add a safety margin on ϕ_m of about 5 degrees

-Design Methodology

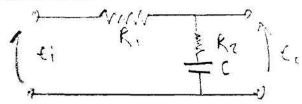
1. Using the final value theorem, find k so that e_{ss} is less than some specified amount (Note that dc gain of C is k)

2. Do a Bode plot with this k

3. Find frequency ω_c where the phase plot is ϕ_m degrees above the -180 degree line

lag looks like PI controller. except PI controller has gain = ∞ at s=0

-Implementation



$$E_o(s) = \frac{1+R_2Cs}{1+(R_1+R_2)Cs}E_i(s)$$ (Need a gain k)

-Design Goal (Typical. You may have to translate from time domain to frequency domain)

Given e_{ss} less than __ percent, find a C which gives a phase margin ϕ_m of at least __ degrees (Note that e_{ss} is a steady state specification while ϕ_m is a transient specification)

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1. Using the final value theorem, find k so that e_{ss} is less than some specified amount (Note that dc gain of C is k)
2. Do a Bode plot with this k
3. Find frequency ω_c where the phase plot is ϕ_m degrees above the -180 degree line
4. Find α such that $20\log\alpha = |G(j\omega_c)|_{db}$. Thus, we have $\alpha = a/b$
5. Choose a arbitrarily below ω_c. Generally, $a=\omega_c/10$.
6. Let $b=a/\alpha$
7. Check the compensated Bode plot and step response

-Effect on closed-loop response

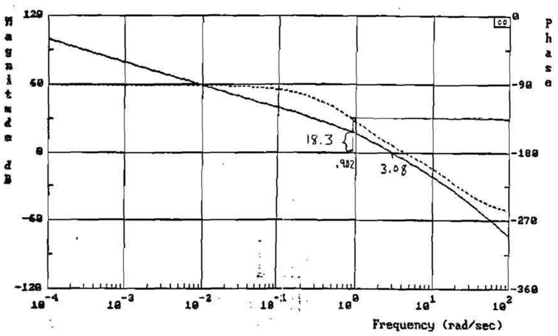
1. Bandwidth reduced. Thus, system is slower
2. Asymptotic tracking remains the same since there are no added poles at zero
3. System is "more stable"
4. Reduced overshoot

-Eg/ $P(s)=\frac{1}{s(s+1)(s+20)}$

Find a $C(s)$ such that
i) e_{ss} to unit ramp is .1
ii) ϕ_m=40°

Solutions/ $e_{ss}=20/k$. $\therefore k$=200.. Design for ϕ_m=45° (ie add a safety margin)

$$\frac{200}{s(s+1)(s+20)} \Rightarrow \frac{10}{j\omega(1+j\omega)(1+j\omega/20)}$$



- initial crossover at $\omega=3.08$
- $\phi_m=45°$ at $\omega=.902$
- Gain at that point =18.3 dB

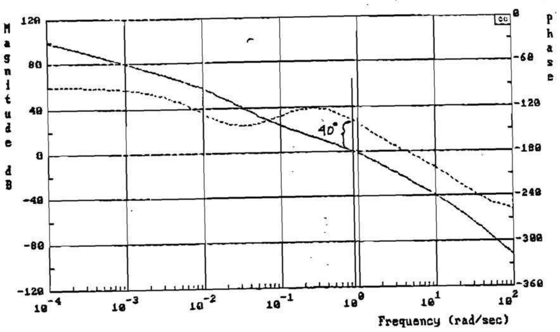
$\therefore 20\log\alpha=18.3$

$\therefore \alpha=8.22$

- Now $a=.902/10=.0902$
- $\therefore b=.0902/8.22=.011$

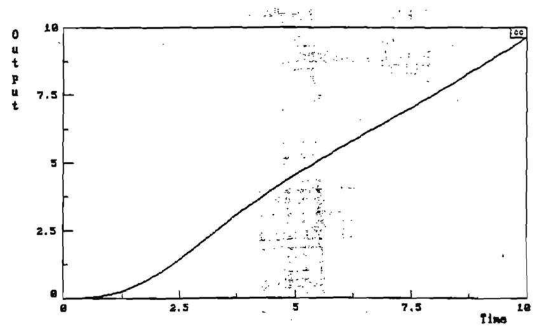
$$\therefore C(s)=200\frac{(1+11.08s)}{(1+90.9s)}$$

- Check compensated Bode Plot



- Can measure ϕ_m to be 40°

- Check ramp response



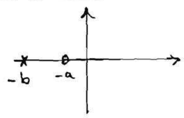
2. Lead Compensator

-Choose $C(s)=k\frac{(1+s/a)}{(1+s/b)}$ where $a<b$

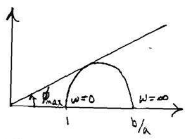
-Ignoring the k, the gain is 1 at $\omega=0$

-Look at plots of $(1+s/a)/(1+s/b)$

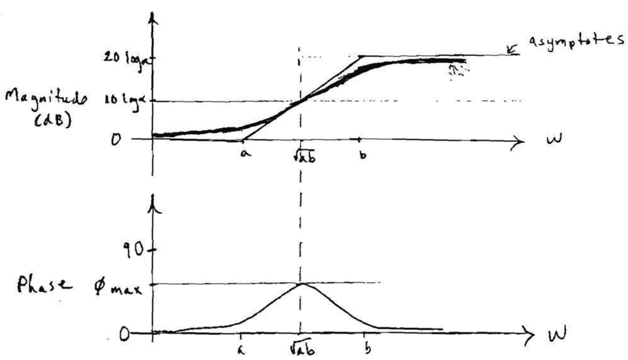
-Pole-Zero configuration



-Nyquist



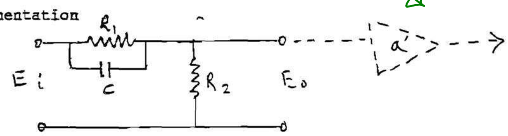
-Bode (let $\alpha=b/a$)



-Thm: $\sin\phi_{max}=\frac{\alpha-1}{\alpha+1}$ and occurs at $\omega_{max}=\sqrt{ab}$

Need an opamp to implement lead compensator because lead requires amplification

-Implementation



$E_o(s) = \frac{1}{a'}\frac{1+Ta's}{1+Ts}E_i(s)$ where $a' = \frac{R_1+R_2}{R_2}$, $T = \frac{R_1C}{a'}$

-Note that this gives a DC gain of $1/a'$. Therfore we need to put in an active gain of a' in addition to the gain k

-Design Goal (Typical. You may have to translate from time domain to frequency domain)

Given e_{ss} less than __ percent, find a C which gives a phase margin ϕ_m of at least __ degrees (Note that e_{ss} is a steady state specification while ϕ_m is a transient specification)

-Usually add a safety margin on ϕ_m of about 5 degrees

-Design Methodology

1. Using the final value theorem, find k so that e_{ss} is less than the specified amount (Note that dc gain of C is k)

2. Do a Bode plot with this k

3. Look at the 0db crossover point of this plot

4. At that frequency, look at the extra phase needed to meet the desired ϕ_m. This will be ϕ'_{max}

5. From ϕ'_{max}, calculate α. This will be our initial guess for ϕ_{max}

6. By trial and error, determine α

i) Calculate $10\log\alpha$. Look at the frequency for which the plot is at $-10\log\alpha$. This determines the new 0db crossover point.
ii) At this frequency, calculate the extra phase margin needed to satisfy the phase criteria. Replace the old ϕ'_{max} with this value. Calculate a new α. Iterate to i) until you find the α such that, at the new 0db crossover, the phase plus the added phase ϕ_{max} from the

lead network is ϕ_m. (It may happen that this iterative technique may not converge. In this case, just start with a small α and increase it until the phase criteria is met.)

7. The new 0db crossover ω_c' is $\sqrt{ab}$. From $\omega_c'=\sqrt{ab}$ and $\alpha=b/a$, we can solve for b and a

-This technique may fail if the phase drops too sharply at the 0db-crossover, or if the required additional phase is too large for the lead network to provide

-Effects on the closed-loop response

1. Bandwidth is increased. Thus, the system is faster

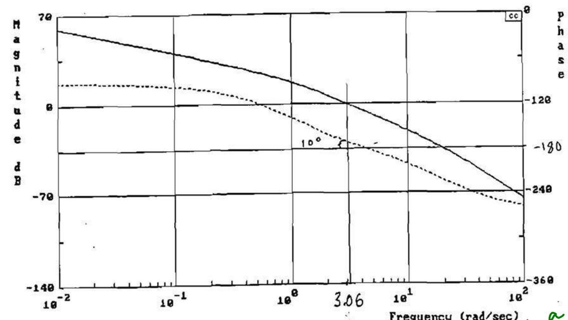
2. More susceptible to noise

-Eg/ $P(s)=\dfrac{1}{s(s+1)(s+20)}$

Find a $C(s)$ such that
i) e_{ss} to unit ramp is .1
ii) ϕ_m=40°

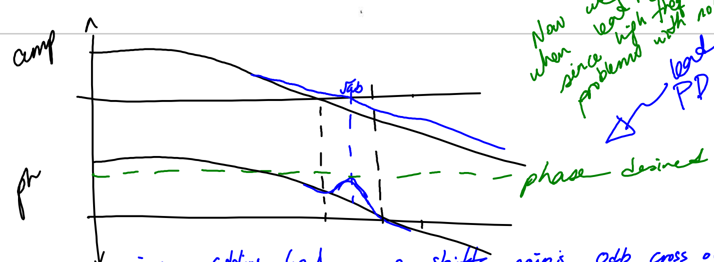
Solutions/ $e_{ss}=20/k$. $\therefore k=200$.. Design for ϕ_m=45° (ie add a safety margin)

Do a Bode plot of $\dfrac{200}{s(s+1)(s+20)}$



Now we opt a faster response when lead comp. added, but since high freq is amplified, problem with noise arises.

lead looks like PD controller



Since adding lead comp shifts gain's 0db cross over freq, which in-turn affects phase margin, we need to iterate.

Initial crossover at 3.06 Phase ≈ -170
$\therefore$ need $\phi_{max}=35$. This gives $\alpha=3.7$
Now $10\log\alpha = 5.6$ dB. The -5.6 dB point is at approximately 4.0.

Iteration 1: The phase at this point is at -178
This requires a $\phi_{max}=43$, $\therefore \alpha=5.3$ and $10\log\alpha = 7.2$
The -7.2 dB point is at approximately $\omega=5$,

Iteration 2: The phase at this point is -182, $\therefore$ we need $\phi_{max}\approx 45°$ again. This gives $\alpha=5.8$

Let's choose this as our final point;
$\omega_c=5$, $\alpha=5.8$

$\therefore \omega_c=\sqrt{ab}=5$, $\alpha=5.8=\dfrac{b}{a}$
$\therefore b^2=145$, $b=12$, $a=2.08$
$\therefore C(s)=200\dfrac{(1+s/2.08)}{(1+s/12)}$

which in turn affects phase margin, we need to iterate.

Initial crossover at 3.06 Phase ≈ -170

$\therefore$ need $\phi_{max} = 35$. This gives $\alpha = 3.7$

Now $10 \log \alpha = 5.6$ dB. The -5.6 dB point is at approximately 4.0.

Iteration 1: The phase at this point is at -178

This requires a $\phi_{max} = 43$, $\therefore \alpha = 5.3$ and $10 \log \alpha = 7.2$

The -7.2 dB point is at approximately $\omega = 5$.

Iteration 2: The phase at this point is -182, $\therefore$ we need $\phi_{max} \approx 45°$ again. This gives $\alpha = 5.8$

Let's choose this as our final point:

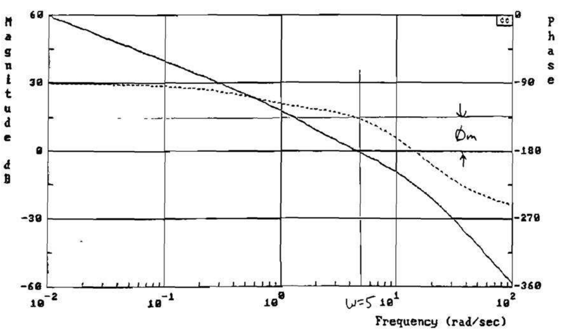
$$\omega_c = 5, \ \alpha = 5.8$$

$$\therefore \omega_c = \sqrt{ab} = 5, \quad \alpha = 5.8 = \frac{b}{a}$$

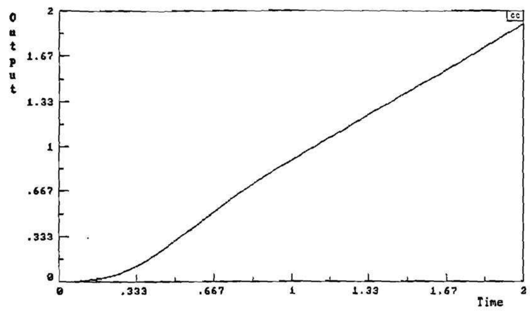
$$\therefore b^2 = 145, \quad b = 12, \quad a = 2.08$$

$$\therefore C(s) = 200 \frac{(1 + s/2.08)}{(1 + s/12)}$$

Look at the compensated plot:



- The ramp response



3. Lead-Lag Compensator

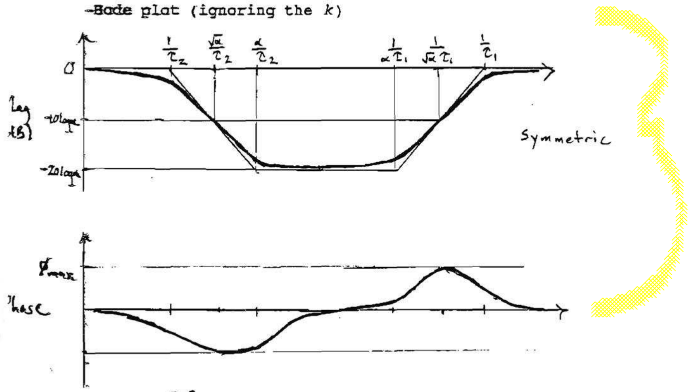
-2 Methods:

1. Divide needed extra phase margin into two parts ie $\phi_{extra}=\phi_1+\phi_2$. Design the lag compensator to satisfy ϕ_1. Consider the plant and lag compensator as the new plant. Design a lead compensator to meet ϕ_2. We can also do this in the reverse order.

2. One step design. The remainder of this section deals with this method.

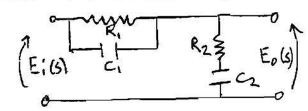
-Assume $C(s)=\dfrac{k(1+\alpha\tau_1 s)(1+\frac{1}{\alpha}\tau_2 s)}{(1+\tau_1 s)(1+\tau_2 s)}$ for $\alpha>1$

-Bode plot (ignoring the k)



- $\sin\phi_{max}=\dfrac{\alpha-1}{\alpha+1}$

--Implementation- RC network (still need k in front)



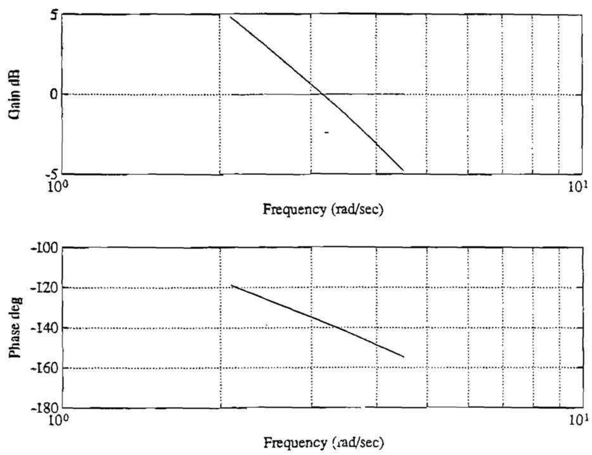
$\alpha\tau_1=R_1C_1$, $\tau_1\tau_2=R_1R_2C_1C_2$, $\tau_1+\tau_2=R_1C_1+R_2C_2+R_1C_2$

Use $\alpha = 2.7$. Since $\omega = 3.14$, $\tau_1 = \frac{1}{\sqrt{\alpha}\,3.14} = .19$

$\therefore \tau_2 = 14.13$

$$\therefore C(s) = 19\,\frac{(1+.513s)}{(1+.19s)}\,\frac{(1+5.23s)}{(1+14.13s)}$$

Check the compensated Bode plot



$\omega_c = 3.17$

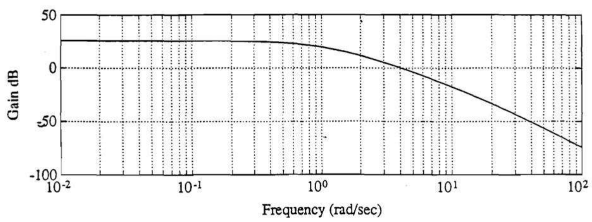
$\Phi_m = 42.6$ $\therefore$ Design works

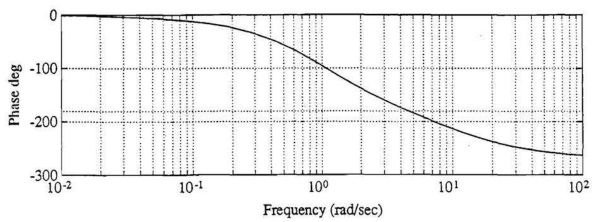
-Example-

$$G(s)=\frac{10}{(s+1)^2(s+10)} \rightarrow G(j\omega)=\frac{1}{(j\omega+1)^2(j\omega.1+1)}$$

Solution/ $e_{ss}=.05=\frac{1}{1+k}$ $\therefore k=19$. Design for $40^\circ+\underbrace{5^\circ}_{\text{safety margin}} = 45^\circ$

Plot $G(j\omega)\times 19$





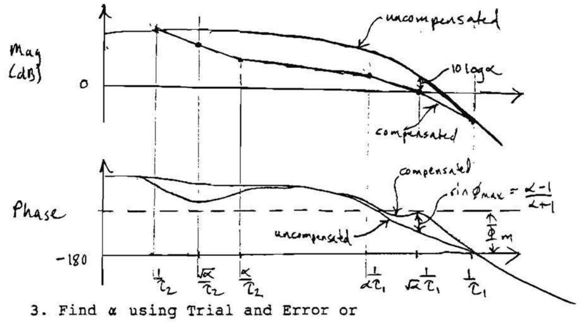
Iterations	0dB crossover	X-tra Phase	α	$10\log\alpha$
0	4	39	4.3	6.4
1	2.8	21	2.1	3.3
2	3.4	30.5	3.1	4.9
3	3.05	25.7	2.53	4.0
4	3.2	28.3	2.8	4.47
5	3.14	27.2	2.7	4.28

-Design goals: $e_{ss}<$ __ and $\phi_m=$ __

-Design methodology (Usually add a safety margin to ϕ_m)

1. Find k such that the steady state error criteria holds

2. Plot $kG(s)$



Trial & Error also required here.

3. Find α using Trial and Error or

   i) start with extra phase margin ϕ_{max} needed at the 0db crossover of $kG(s)$ to meet ϕ_m

   ii) calculate $\alpha=\dfrac{1+\sin\phi_{max}}{1-\sin\phi_{max}}$

   iii) find the frequency where the gain is $10\log\alpha$. This gives a new 0db crossover

   iv) Calculate the new extra phase margin ϕ_{max}

   v) go to ii) until α converges.

4. From the new 0db crossover ω', find τ_1, that is
$\dfrac{1}{\sqrt{\alpha}\tau_1}=\omega'$

5. Choose α/τ_2 to be a little more than one decade below $1/(\alpha\tau_1)$ (arbitrary) and solve for τ_2

6. Check the compensated Bode plot

-Effect on the time response

a) bandwidth decreased
b) less susceptible to noise

ADVANCED SISO CONTROL CONCEPTS

-Recall the concept of gain and phase margins from the Nyquist stability criteria for a stable $G(s)$. There should be no ccw encirclements of the -1 point

-We look at the closed loop system we were looking but fix the gain at 1.

-Can you think a case where you have a particular gain set so that you have essentially unlimited phase margin?

We can have $|G(j\omega)| < 1$

$s = j\omega$

-Is this condition necessary and sufficient? (Iff ?)

If unlimited phase margin, then $|G(j\omega)| < 1$

By contradiction

Suppose there is some range of ω such that $|G(j\omega)| \geq 1$

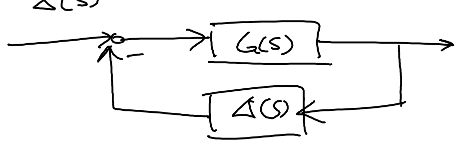
There will be a phase margin.

If we rotate the graph, we will get encirc. of -1 ∴ unstable

will look @

-We can put in a perturbation in the following way. We ~~call this~~ "robust stability"

$\Delta(s)$

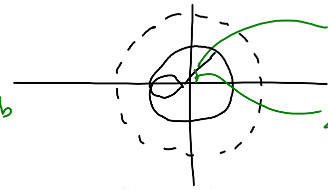


where $|\Delta(j\omega)| < 1 \, \forall \omega$ and $\Delta(s)$ is stable. Note: stability still if no encirclements of -1

(phase can be anything)

-Note that we have essentially replaced $G'(s)$ with $G(s)\Delta(s)$. In the Nyquist plot, we have the magnitude as $|G(j\omega)||\Delta(j\omega)|$ and the phase as $\angle G(j\omega)+\angle\Delta(j\omega)$

$c = ab$
$|c| = |a||b|$
$\angle c = \angle a + \angle b$



-*The Small Gain Theorem* – The closed loop system has robust stability IFF

$$|G(j\omega)| < 1 \;\forall\omega$$

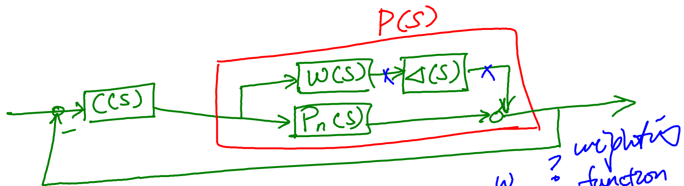
pf. $|G'(j\omega)| = |G(j\omega)||\Delta(j\omega)| < 1$

We want the system to be closed loop stable for all ≤ 1 perturbations $\Delta(s)$. This is called Robust Stability

$\therefore |G(j\omega)| \le 1$

-Any stable $\Delta(s)$ that satisfies $|\Delta(j\omega)| < 1 \;\forall\omega$ is called admissible

-Why is this called robust stability?



W ≗ function
i.e. W(s) tells us "where" the uncertainty is

P_n is the nominal plant. Thus, $P(s) = P_n(s) + W(s)\underbrace{\Delta(s)}_{\text{admissible}}$

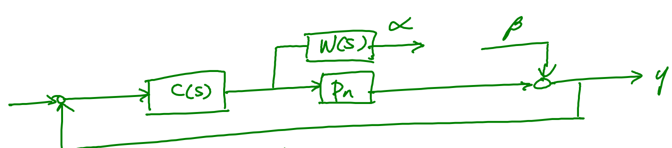
$|\Delta(j\omega)| < 1$
$\angle\Delta(j\omega)$ is arbitrary

This is called additive uncertainty

-Use $W(s)$ to determine the frequency range of uncertainty

- W(s) is low pass: we have low freq. uncertainty
- W(s) is high pass " " high " "
- We want C(s) to stabilize P(s) for all admissible perturb. Δ

-Reformulate this into our previous setup



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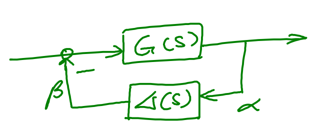
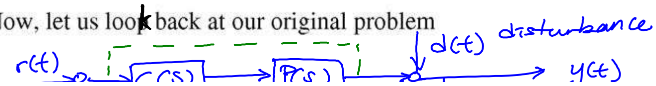


$$G(s) = \frac{C(s)W(s)}{1+P_n(s)C(s)}$$

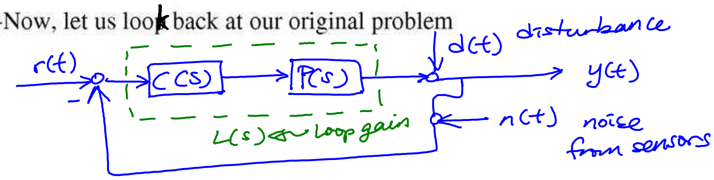
∴ If we design C(s) so that $\left|\frac{C(j\omega)W(j\omega)}{1+P_n(j\omega)C(j\omega)}\right| < 1$, this system is stable for all P(s).

January 26/09

-Now, let us look back at our original problem



-Now, let us look back at our original problem



-Define the loop gain as $L(s) = P(s)C(s)$

we want -Again, we would like ideally like the closed loop transfer functions to be

$$\begin{cases} T_{ry} = \dfrac{PC}{1+PC} = \dfrac{L}{1+L} \\ T_{dy} = \dfrac{1}{1+PC} = \dfrac{1}{1+L} \\ T_{ny} = \dfrac{-PC}{1+PC} = \dfrac{-L}{1+L} \end{cases}$$

high $L(s)$ @ low freq. & low @ high freq.

we want them to be $\Rightarrow$ $\begin{cases} 1 \\ 0 \\ 0 \end{cases}$

@ low freq

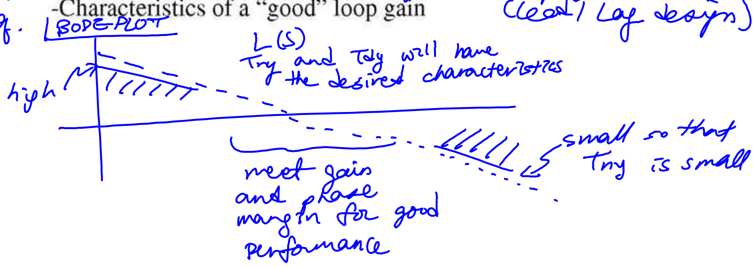
@ " "

But T_{ny} is $-$ve of T_{ry}. @ high freq.

-The key thing is the loop gain

- Recall, Bode plot is of the loop gain $L(s)$! (Lead / Lag design)

-Characteristics of a "good" loop gain



-Thus, we can accomplish these goals if the loop gain has the following frequency domain characteristics

$L(j\omega)$ has to be large at low freq and small at high freq. In loop shaping, you "choose" $L(s)$ to have desired ch. This is your design!

-Once we have the loop gain, we can calculate C(s). Need to approximate sometimes for proper C(s) just like for derivative gain in PD control. This technique is called loopshaping

PD controller is noncausal and not proper.

i.e. Need to approximate $C(s)$ just like in PD control, if $C(s)$ is not proper.

eg. $L(s) = C(s)P(s)$

$\therefore C(s) = \dfrac{L(s)}{P(s)}$

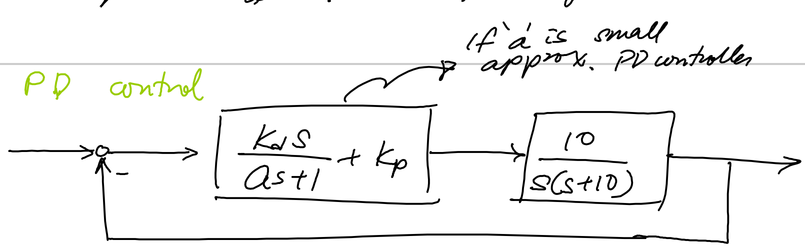
suppose it is improper.

$C(s) = \dfrac{L(s)}{P(s)(as+1)^2}$ choose a small a so that $C(s)$ is proper.

This take a lot of skill!

After you find a $C(s)$ you check it's actual performance in the system, and if the performance is not good, you change $C(s)$ and iterate the design.

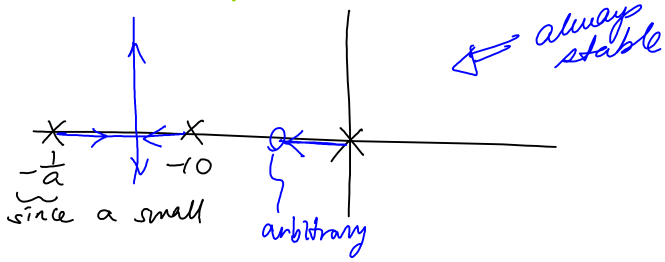
eg. PD control



$L = CP = \dfrac{((K_d + aK_p)s + K_p)\,10}{}$

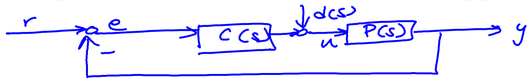
$s(as+1)(s+10)$

stability: always stable



Youla Parametrization

-It is possible to describe the set of all controllers $C(s)$ that stabilize a plant $P(s)$ (*)



$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \frac{PC}{1+PC} & \frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

BIBO stable if all TF stable.

-Assume first that our plant $P(s)$ is stable.

-Theorem-

All stabilizing controllers can be parameterized by

$$C(s) = \frac{Q(s)}{1 - P(s)Q(s)}$$

arbitrary

i.e. all stabilizing controller can be written in this form.

Where $Q(s)$ is a proper, stable transfer function and the denominator is not equal to zero

Both useless controllers

Trivial cases:

$Q(s) = 0$ Then $C(s) = 0$. This is our "0" control strategy.

$Q(s) = k$ Then $C(s) = \frac{k}{1 - P(s)k}$. Let $P = \frac{n}{d}$ where n, d are polynomials.

$\Delta(s) = kd \cdot n + (d - nk)d = d^2 = 0$. ∴ This is the controller that gives you back your open loop polynomials.

closed loop pole pos'n hasn't changed.

-There are ways of "optimal control" which finds the $Q(s)$ that gives the "best" response

-Pf/ If $C(s) = \frac{Q}{1-PQ}$ then $C(s)$ will stabilize the system.

From the defn we need to show that all four TF (*) are stable after substituting $C(s)$.

eg. $\frac{PC}{1+PC} = \frac{\frac{PQ}{1-PQ}}{1 + \frac{PQ}{1-PQ}} = \frac{PQ}{1} \Rightarrow$ stable.

$\frac{1}{1+PC} = \frac{1}{1 + \frac{PQ}{1-PQ}} = 1 - PQ \Rightarrow$ stable

$\frac{P}{1+PC} = \frac{P}{1 + \frac{PQ}{1-PQ}} = P(1-PQ) \Rightarrow$ stable

$\frac{C}{1+PC} = Q \Rightarrow$ stable

Second proof: If $C(s)$ stabilizes, there is a $Q(s)$ in the proper form

i.e. I can write $C(s)$ as $\frac{Q(s)}{1 - P(s)Q(s)}$

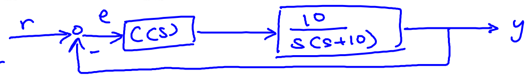
Let $C(s)$ be any stabilizing controller. Then we know that $\frac{C}{1+PC}$ is stable. Let $Q(s) = \frac{C}{1+PC}$ and show this works.

$$\frac{Q(s)}{1 - P(s)Q(s)} = \frac{C(s)}{1+PC} \Big/ \left(1 - \frac{PC}{1+PC}\right) = C!$$

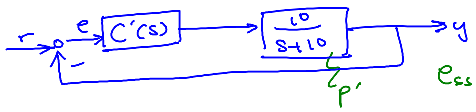
This works!

We force the loop to be a type 1 system so that $C'(s)$ has a pole at the origin.

-Eg- Control of a typical second order system.



P(s) NOT stable. This is a Type 1 system. Thus e_{ss} to a step is 0.



$$e(s) = \frac{1}{1+C'(s)P(s)} r(s)$$

$$e_{ss} = \lim_{s\to 0} s \cdot E(s) = \frac{1}{1+C'(0)P(0)} = 0 = \frac{1}{1+\frac{Q(0)}{1-Q(0)}} = 0 \quad \therefore Q(0) = 1$$

But $C' = \frac{Q}{1-P'Q}$ $\quad \therefore C'(0) = \frac{Q(0)}{1-\underbrace{P'(0)Q(0)}_{1}}$

Jan 30/09

-Eg- Choose $Q(s) = 1$

$$\therefore C'(s) = \frac{1}{1 - 1 \cdot P'(s)} = \frac{s+10}{s} \rightarrow C(s) = s+10 \Rightarrow \text{PD Controller.}$$

-Eg- Choose $Q(s) = \dfrac{100}{s+100}$ $\Rightarrow Q(0) \approx 1$

$$C'(s) = \frac{1}{1 - \frac{100}{s+100} \cdot \frac{10}{s+10}} = \frac{100(s+10)}{s(s+110)}$$

$$C(s) = \underbrace{\frac{100}{s+110}}_{\text{Low pass filter}} \underbrace{(s+10)}_{\text{PD controller}} \Rightarrow \text{This is a PD controller you can implement.}$$

-Theorem in general case where $P(s)$ is not necessarily stable

-Let S be the set of all stable proper rational functions

-Then $P(s)$ can be written as $P(s)=\frac{n_p(s)}{d_p(s)}$ where $n_p(s), d_p(s)$ are in S and satisfy the equality $x_p n_p + y_p d_p = 1$. This means that $n_p(s), d_p(s)$ are coprime, ie, have no common factors in S. There are numerical algorithms to do this which we will re-examine later in the course.

$x_p, y_p \in S$.

S behaves like integers (Ring Theory) as polynomials in that there are concepts of common factors, divisors, primes and coprimes.

-Eg. $P(S) = \frac{1}{S-1}$

Can choose $n_p = \frac{1}{S+1}$, $d_p = \frac{S-1}{S+1}$ both stable

In this simple case, let's assume $x_p = \frac{as+b}{s+1}$ $y_p = \frac{cs+d}{s+1}$. Plug this into $n_p x_p + y_p d_p = 1$

You'll find $x_p = \frac{3s+1}{s+1}$ $y_p = \frac{s}{s+1}$ w/ a=3, b=1, c=1, d=0

-Theorem

The set of all compensators that stabilize $P(s)$ is given by

$$\left\{ C(s)=\frac{x_p + r d_p}{y_p - r n_p}, r \in S, y_p - r n_p \neq 0 \right\}$$

Eg/ If $p(s) = \frac{1}{s-1}$ all controllers are $C(s) = \dfrac{\frac{3s+1}{s+1} + \frac{r(s-1)}{(s+1)}}{\frac{s}{s+1} - r\frac{1}{s+1}}$

Then $C(s) = \frac{3s+1}{s} = 3 + \frac{1}{s} \Leftarrow$ PI controller.

ie. By appropriate choice of 'r' we can generate any type of controller including P, PI, PID, lead, lag etc.

Possible MIDTERM Q.

Eg/ Show that by how the general case can be used to prove the case where $P(s)$ is stable.

Assume $P(s)$ is stable. Then $P(s) = \frac{P(s)}{1} = \frac{n_p(s)}{d_p(s)} \leftarrow$ stable & proper.

We need $x_p P(s) + y_p \cdot 1 = 1$. Choose $x_p = 0$, $y_p = 1$.

$\therefore \left\{ C(s) = \frac{r}{1-rP}, r \in S \right\}$

Waterbed Theory - Handout 4

2009년 2월 2일 월요일
오후 12:13

↳ performance and stability come as tradeoffs.

Push one side & the other side goes up etc.

Topics Covered so far:

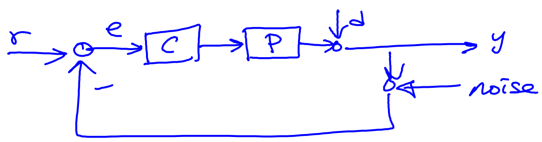
Small gain margin

Loop shaping $L = CP$. → performance in terms of noise rejection. Disturbance rejection etc

Youla parametization

Tradeoffs in Sensitivity and Complementary Sensitivity

-Recall our feedback configuration, where e is the error signal



-Definition- The sensitivity function is $S(s) = \dfrac{1}{1+L(s)} = \dfrac{1}{1+PC}$

- $S(s)$ is the transfer function from r to e $\quad T_{re} = S(s)$

- $S(s)$ is the transfer function from d to y $\quad T_{dy} = S(s)$

- $S(s)$ is the sensitivity of T_{ry} to variations in $P(s)$ $\quad \dfrac{\partial T_{ry} / T_{ry}}{\partial T_P / P} = S(s)$

-The peak of the magnitude of $S(s)$ is a measure of stability robustness

→ Nyquist of OL. ($L(s)$)

distance $= |L(j\omega) - (-1)|$

The minimum distance is $\min_{\omega} |L(j\omega)+1| = \max_{\omega} \left| \dfrac{1}{L(j\omega)+1} \right| = \max_{\omega} S(j\omega)$

∴ We want to minimize $S(j\omega)$.

-In all these cases, we want $S(s)$ to be small at all frequencies!

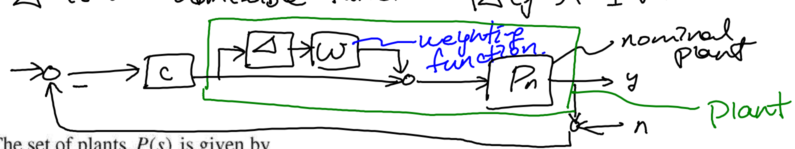
-However, what is the transfer function from the noise n to y?

$$T_{ny} = \frac{CP}{1+CP} = \frac{L}{1+L} = L\,S(s) \Rightarrow$$ Complementary Sensitivity Function

-The complementary sensitivity function is $T = \dfrac{PC}{1+PC}$

-Now, suppose we have a system with a multiplicative uncertainty

Δ is an admissible function $|\Delta(j\omega)| < 1 \ \forall \omega$



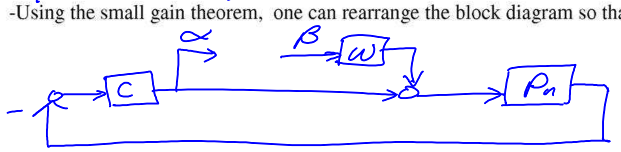
-The set of plants $P(s)$ is given by

$\{P_n(1+\Delta W)\}$

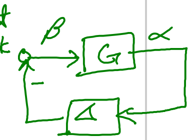
-The weighting function W essentially tells us

If W is low pass we have low freq uncertainty.
If W is high " " " high " uncertainty.

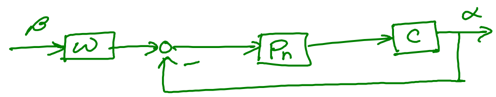
-Using the small gain theorem, one can rearrange the block diagram so that we have



we want it to look like



⇓ Redraw



If T_n satisfies this, my controller will stabilize all the possibilities for $P(s)$

$\therefore G = W \dfrac{P_n C}{1+P_n C} = W T_n$

$\therefore$ small gain $|G(j\omega)| \le 1$ or $|W T_n| \le 1 \ \forall \omega$
or $|T_n(j\omega)| \le \left|\dfrac{1}{W(j\omega)}\right|$

where $G(s) = WT_n$ where $T_n = \dfrac{P_n C}{1+P_n C}$ is again the complementary sensitivity function

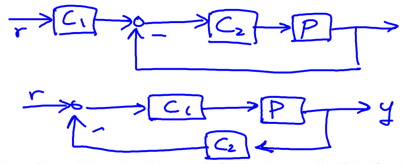
-Thus, ideally, we want the complementary sensitivity function T to also be small at all frequencies which would helpstability to multiplicative uncertainties and reduce noise problems

-PROBLEM: $S+T=1$

-CONCLUSION: We cannot make S and T small for all frequencies!

-Note that one can overcome this limitation by using alternate control topologies that are called two degree-of-freedom topologies as below

doesn't hold



TRY complex number arithmetic

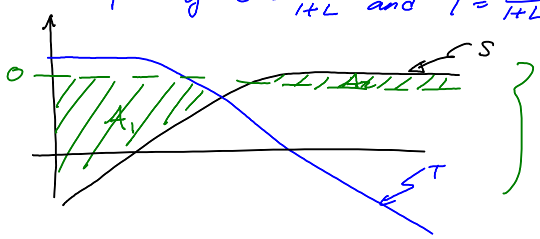
-Note that, since these are complex numbers, it is possible for the magnitude of S and T to be large at the same time. In fact, if one is large, than the other must be as well. Why?

↳ But they have to be out of phase

-Because of these tradeoffs, we often choose

In practice we want S to be small at low frequencies. and T to be small at high frequencies. (Think about how we flipped & got $\left|\frac{1}{\omega G(j\omega)}\right|$)

Bode plot of $S=\frac{1}{1+L}$ and $T=\frac{L}{1+L}$



Ideal shapes of S & T.

Waterbed effect: Tradeoff within S where the area of S below 0dB and area of S above 0dB must cancel.

Review
- Similarity transformation
- Inverse of matrix
- Eigenvalues
- Rank of a matrix.

Peak of mag of $S(s)$

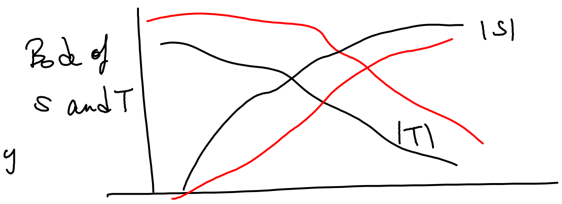
$$\min_{\omega} |L(j\omega)+1| = \left(\max_{\omega} \left|\frac{1}{1+L}\right|\right)^{-1} = \frac{1}{\text{peak } |S|}$$

Tradeoff: S and T should be small

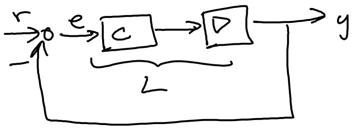
However $S+T=1$

Bode Sensitivity Integral   Midterm Feb. 25^{th} 5:30

2009년 2월 4일 수요일
오전 11:30



Bode Sensitivity integral



-Theorem-(Bode Sensitivity Integral)

{ — C(s) stabilizes system
{ — L(s) has relative dg. at least 2.

Assuming our controller stabilizes the closed loop system above and that the loop gain has a relative degree of at least two (ie order of numerator less than order of denominator by at least two), and suppose N_p denote the number of ORHP poles p_i in the loop gain, then the sensitivity function $S(s) = 1/(1+L(s))$ satisfies

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = \pi\sum_{i=1}^{N_p}\mathrm{Re}(p_i) \ge 0$$

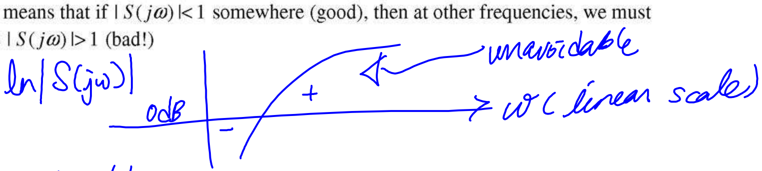
Memorize This for Exam

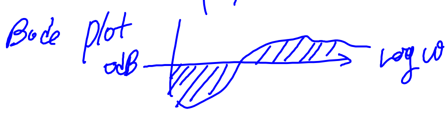
-The proof of this is beyond the scope of this class

When the loop gain L is stable, we have $\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$.

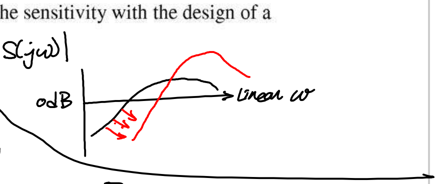
-This has some interesting implications for $S(s)$, even when there are NO unstable poles. This means that if $|S(j\omega)| < 1$ somewhere (good), then at other frequencies, we must have $|S(j\omega)| > 1$ (bad!)

↳ 1 → is 0dB point

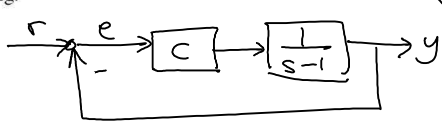




-This is called the waterbed effect since pushing down the sensitivity with the design of a controller will push it up at other points.



-Eg.



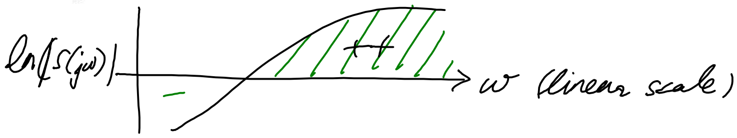
$C_1 = \frac{50}{s+10}$ $C_2 = \frac{200}{s+10}$ $C_3 = 20$

Check stability

Unstable poles in the loop gain

-Suppose there are open right half plane poles in the loop gain

-Now, check out what happens when there are unstable poles using the Bode Sensitivity Integral



-We get automatically worse performance with respect to $S(s)$

- Now, if $P(s)C(s)$ has an ORHP pole at $s = p$, then $S(p) = 0, T(p) = 1$ $\quad \therefore S(p) = \frac{1}{1+P(p)C(p)} = 0.$

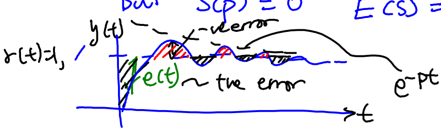
-Theorem- Suppose the closed loop system is stable but the plant $P(s)$ has an ORHP pole at $s = p > 0$. If $r(s) = 1/s$ (ie step input), then $e(t) = r(t) - y(t)$ satisfies

$$\int_0^\infty e(t)e^{-pt}\,dt = 0$$

Pf/ $E(s) = \frac{1}{1+PC} R(s) = S(s)R(s)$

But $S(p) = 0$ $\quad E(s) = \int_0^\infty e(t)\, e^{-st}\, dt$, $\quad E(p) = S(p)\cdot R(p) = 0 = \int_0^\infty e(t) e^{-pt} dt$

QED

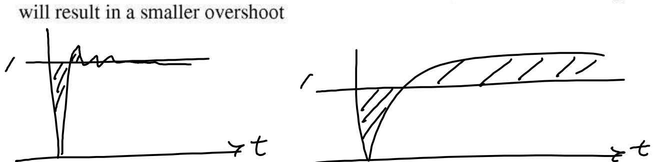


$e = r - y.$

-There must be an overshoot as the areas must cancel

If you have an unstable pole in your plant, you MUST have an overshoot.

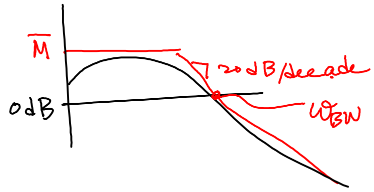
-If we have two controllers with different bandwidths, the one with the higher bandwidth will result in a smaller overshoot



fast bandwidth.

-Frequency domain limitations when there is an ORHP pole

-Suppose we impose the following bounds on the Complementary Sensitivity Function for a stabilizing controller with a "good" characteristic



-Theorem- Suppose there is an ORHP at $s = p$. Then the following is true

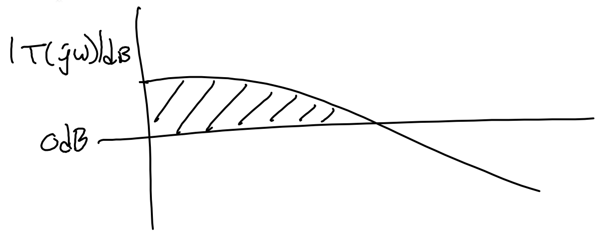
Theorem: If there is an ORHP pole,

-From this, we can use the following rule-of-thumb

$$\underline{\omega_{BW}} > p\frac{\bar{M}}{\bar{M}-1} > \underline{p}$$

- Rule of thumb. Bandwidth > 2p.

-It can also be shown that $\max_{\omega} |T(j\omega)| \geq 1$ when the plant is unstable and the closed loop response is stable



Tutorial

2009년 1월 29일 목요일
오전 11:29

Assignment 1. PD → Lead

PD

$k_p + k_d s$ $K =$ ○ $a =$ ○. $b =$ ○

get $b \to \infty$ and you get PD

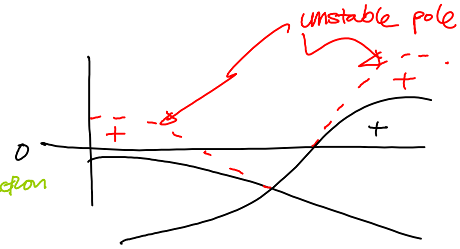
PI → Lag.

$k_p + \frac{k_i}{s} \to \frac{k_p s + k_i}{s}$ $K = \frac{k_i}{b}$ $a \to \frac{k_i}{k_p}$

get $b \to 0$ and get lag equiv. PI.

Lecture
2009년 2월 6일 금요일
오전 11:32

Tradeoffs $\underbrace{S}_{\text{Tracking error, disturbance rejection}} + \underbrace{T}_{\text{noise rejection}} = 1$



Time domain : – ORHP gives overshoot

Unstable Zeros in the Loop Gain

- non-minimum phase system $\frac{s+1}{s+2}$, $\frac{s-1}{s+2}$
- If PC has an ORHP zero at $s=z$ then $T(z)=0$, and $S(z)=1$
- Time delays. e^{-sT_d} where T_d is a time delay — gives you non-min-phase zero.
- Padé approximation $e^{-sT_d} \approx \frac{1-\frac{T_d}{2}s}{1+\frac{T_d}{2}s}$ good approx. at low freq
- Recall $T = \frac{PC}{1+PC}$.

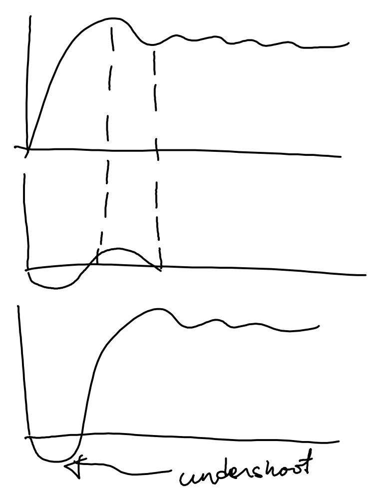
↳ The zeros of the loop gain = zeros of the closed loop $TF_{r\to y}$

- $G(s) = \underbrace{\bar{G}(s)}_{\text{stable}}\underbrace{(-\tau s+1)}_{\text{ORHP zero}}$ pulling out the delays

Intuitively, step response of $\bar{G}$

step response of $-\bar{G}s$

the combined step resp.



Theorem: For the feedback system where the closed loop system is stable and the input is a step and there is an ORHP zero at $s=z$ then

$$\int_0^\infty y(t)\,e^{-zt}\,dt = 0$$

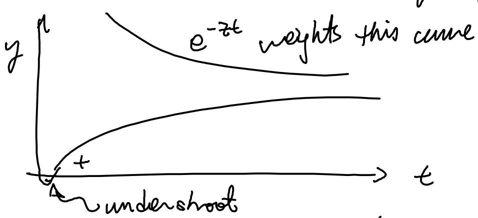
Proof: $\underbrace{Y(s)}_{\text{output}} = \underbrace{T(s)}_{\text{TF}}\underbrace{\frac{1}{s}}_{\text{input}} \rightarrow Y(z) = \cancel{T(z)}^{\,0}\frac{1}{z}$

$Y(s) = \int_0^\infty y(t)\,e^{-st}\,dt \quad \therefore Y(z) = \int_0^\infty y(t)\,e^{-zt}\,dt = 0$

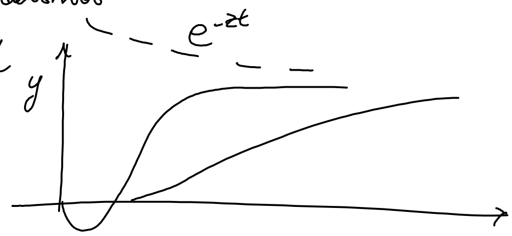
QED

QED

- A real ORHP zero always gives undershoot

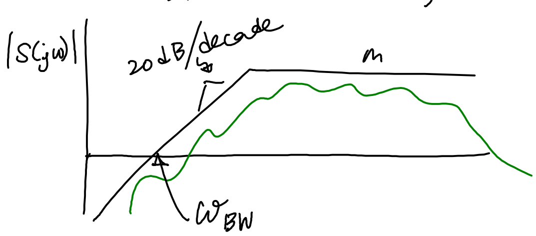


- Bandwidth



faster bandwidth will give us bigger undershoot.
Hence we want to slow down the system

- For an ORHP zero, we have



- Theorem: $\omega_{BW} < z\frac{m-1}{m} < z$
↳ Rule of thumb: Controller bandwidth $< \frac{z}{2}$

- If you have an ORHP pole p and ORHP zero z, the effects combine
For good performance $2p <$ bandwidth $< \frac{z}{2}$

Thus, if $z > 4p$ we might get good performance
but if $z < 4p$ we're out of luck.

Feb. 9/09

Comments about project pres:

Steps: Dynamic eqn. → ODE → state space form
→ Linearization is always about an equil. pt.
$\dot{x} = 0 = f(x_e, u_e)$ equilibrium points
If you start at $x = x_e$, $u = u_e$, then $x(t)$ remains there.

$$x = \delta x + x_e \quad , \quad u = \delta u + u_e$$

Taylor series,

$$\dot{x} = f(x, u)$$

$$\delta \dot{x} + \cancel{\dot{x}_e} = f(\delta x + x_e, \delta u + u_e)$$

$$\approx \cancel{f(x_e, u_e)} + \underbrace{\frac{\partial f}{\partial x}\bigg|_{\substack{x = x_e \\ u = u_e}}}_{A} \delta x + \underbrace{\frac{\partial f}{\partial u}\bigg|_{\substack{x = x_e \\ u = u_e}}}_{B} \delta u + \underbrace{\cdots}_{\text{neglect}}$$

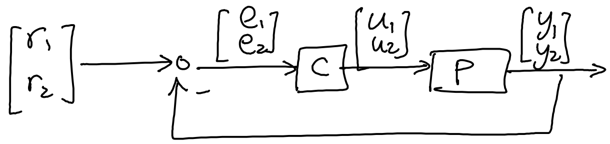
$$\delta \dot{x} = A \delta x + B \delta u$$

Next presentation: Date = 8

40 slides, End of 2nd week of March

Linear Algebra

2009년 2월 9일 월요일
오전 11:43



- Everything is matrices. eg. $\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

- State space

Linear Independence of Vectors

↳ eg. $\mathbb{R}^3$

↳ Vectors are linearly independent if $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$ iff $\alpha_i = 0 \ \forall i$

eg. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix} = 0$

$\therefore$ all $\alpha_i = 0$

→ If this is not true, they're linearly dependent.
This means some of the vectors are combinations of the others

eg. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$-1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0, \quad \alpha_1 = -1 \quad \alpha_2 = 1 \quad \alpha_3 = -1$

Now look at $y = Ax$

$$y = \left[\underset{\substack{\uparrow \\ \text{columns}}}{[A_1]} \, [A_2] \cdots [A_n] \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 [A_1] + x_2 [A_2] + \cdots x_n [A_n]$$

where A_i is the i^{th} column of A. $= \sum_{i=1}^{n} x_i A_i$

Rank → The rank of matrix A is the number of independent columns of A.

Formulas → Rank of A is the l s.t. there is no submatrix of dim. $(l+1)\times(l+1)$ that has a non-zero determinant.

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \leftarrow \text{check 6 times for } \underbrace{[1\times1]}_{\text{Rank will be 1 iff zero matrix.}}, [2\times2], [3\times3].$$

→ The rank of matrix A is the number of indep. rows of A.

→ For a square matrix of dim. $n\times n$, the it is of rank n if $\det A \neq 0$

Span → The span of a set of vectors $\{A_i\}$ is the set of all linear combinations of A,
$Sp\,\{A_i\} = \{\sum_{i=1}^{n} \alpha_i A_i,\ \text{for } \alpha_i \in \mathbb{R}\}$

eg. If $A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ the span is

$\begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_2 \end{bmatrix}$ where α_1, α_2 are any numbers.

eg. $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ There are 2 linearly ind. columns, hence rank is 2

All 3x3 det. are zero. Hence rank=2

Range The range of matrix A is the span of A.

Range is the set of all y's → $y = Ax$ ← x varies over all $\mathbb{R}^n$

$= \sum_{i=1}^{n} A_i x_i = sp\{A_1 A_2, \cdots A_n\}$

∴ The range of A is the span of the linearly indep. columns of A.

eg. $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

Although the choice of linearly independent columns are not unique the spans are the same.

$Sp\{A_1, A_2\} = Sp\{A_2, A_3\} = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}, \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

Theorem

There is a solution to $y = Ax$, where matrix A and vector Y are given.

$$\text{rank}[y \vdots A] = \text{rank}[A] \Leftrightarrow$$

If this doesn't hold there's no soln

Pf/ y has to be a linear combination of columns of A

Null Space The nullspace of a matrix A is the set of all x such that $Ax = 0$.

eg. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Any $\alpha \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$ is an $N(A)$ where α is an arbitrary constant

Elementary row/column matrix do not change the rank of a matrix.

- eg all column operations:

same for rows $\left\{ \begin{array}{l} \text{i) interchange columns} \\ \text{ii) multiply a column by a scalar} \\ \text{iii) " a " by a scalar and add it to another column} \end{array} \right.$

$$\left[\begin{array}{c:c} I & A \\ \hdashline & I \end{array}\right] \Big\}\ \text{row operators} \implies \left[\begin{array}{c:c} R & \Lambda \\ \hdashline & Q \end{array}\right]$$

column operators

$$RAQ = \Lambda$$

eg.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \leftarrow A \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 0 & -3 & -6 \\ 0 & 0 & 1 & 3 & 6 & 9 \end{array}\right]$$

Multiply 1st row by -2 and add to second

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 0 & -3 & -6 \\ -3 & 0 & 1 & 0 & -6 & -12 \end{array}\right] \implies \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 0 & -3 & -6 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{array}\right]$$

Λ (the right-hand block)

$$\text{Rank } \Lambda = 2 = \text{Rank } A$$

$$\text{Check } R \cdot A = \Lambda$$

2009년 2월 11일 수요일
오전 11:31

Matrices A

- The rank is no. of linearly independent columns (rows) of A.
- The span of the columns of A

$$\left\{ \sum_{i=1}^{n} \alpha_i A_i \, , \, \alpha_i \in \mathbb{R} \right\}$$

- This is also the range of A ; $R(A)$
- The Null space of A is all x s.t. $Ax=0$

Find vectors that are lin. ind. and span these spaces

How to find N(A) and R(A)?

R(A) row $\left\{ \begin{matrix} I & A \\ & I \end{matrix} \right.$

elementary column/row operations

Column

$$\begin{matrix} Q & \Lambda \\ & R \end{matrix}$$

$$QAR = \Lambda$$

Fact: Row operations do not affect column dependencies

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix}\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$-1 \times$ col1 + 2col2

Column dependence is the same

The range of A can be described as being the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

It can also be the span of $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Use column operations to find N(A)

$$\left[\begin{array}{c} A \\ \hline I \end{array}\right] \xrightarrow{\text{column ops}} \left[\begin{array}{c|c} \Lambda' & 0 \\ \hline R_1 & R_2 \end{array}\right]$$

$A[R_1 \vdots R_2] = [\Lambda \vdots 0]$ $\quad \therefore A\underbrace{R_2} = 0$

The columns of R_2 satisfy the Null space.

R_2 are Lin. Ind. column vectors that span $N(A)$

eg. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & -6 \\ 3 & 6 & -12 \\ 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ 3 & -6 & -12 \\ 1 & -4 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiply col 1 by -7 and add to col 3

Multiply col 1 by -4 and add to col 2

Mult. col 2 by -2 and add to col 3

$\sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -6 & 0 \\ \hline 1 & -4 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]$ $\quad \therefore N(A) = \text{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Solving $y = Ax$

There is a sol'n when $\text{rank}[y \vdots A] = \text{rank}[A]$

Take $[I][y \vdots A]$ do row operations to get

$[Q]\left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ y' & x & 1 & 0 \\ & x & x & 1 \end{array}\right]$

$y = Ax$, $\quad Qy = QAx$

$$\begin{bmatrix} \\ y' \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Assume A is square)

Eigenvalues

Def'n: $\underbrace{A x}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{scalar}} x$ — eigenvalue

$(A-\lambda I)x = 0, \quad x \neq 0$

If the columns of $(A-\lambda I)$ are all linearly indep. then $x = 0$!

$\therefore$ The columns of $(A-\lambda I)$ are dependent (ie rank$(A) \neq n$)

$\therefore \det(A-\lambda I) = 0$ to have $x \neq 0$.

This gives a polynomial of order n in λ.

ie. $\Delta(\lambda) = 0$ where $\Delta(\lambda)$ is an n^{th} order polynomial.

There are n roots for this. These are the eigenvalues.

How do you find the eigenvectors ($\vec{ev}$)?

Take λ_i, an ev, and plug into $\underbrace{(A-\lambda_i I)}_{\text{matrix}} x = 0$ — $N(A-\lambda_i I)$

I'm looking for n eigenvectors!

Suppose we found n $\vec{ev}$.

$$A x_1 = \lambda_1 x_1$$
$$A x_2 = \lambda_2 x_2$$

$$A \underbrace{[x_1 \ x_2 \ \cdots \ x_n]}_{n \times n \text{ matrix } P} = [\lambda_1 x_1 \ \lambda_2 x_2 \ \cdots \ \lambda_n x_n]$$
$$= \underbrace{[x_1 \ x_2 \ \cdots \ x_n]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{\text{diagonal matrix w/ ev on the diagonal}}$$

$$AP = P\Lambda$$
$$\underbrace{P^{-1}AP}_{\text{similarity transformation}} = \Lambda$$

If P contains the $\vec{ev}$, the matrix becomes diagonalized.

Property
- Any similarity transformation does not change evs.
- $T^{-1}AT$ have the same ev as A.

ie This is coordinate transformation.

Can you always do this? In other words, are there always linearly indep. $\vec{ev}$?

If evs are distinct you can always diagonalize.
If evs are repeated you may not get enough $\vec{ev}$s.

eg. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \det(A-\lambda I)=0 \Rightarrow \lambda^2=0, \quad \lambda=0,0$

Plug $\lambda=0$ into $A-\lambda I$.

$$A-0\cdot I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right]$$

$\vec{ev}$

There is only one $\vec{ev}$!
This cannot be diagonalized!

Midterm Review

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오전 11:31

ev, $\vec{ev}$

$P = [x_1, x_2, \cdots x_n]$

$P^{-1}AP = \Lambda$ diagonalize

you can't always find n linear independent $\vec{ev}$ if you have repeated roots.

The best you can do is a Jordan Block form.

1's above diagonal

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & 1 & 0 & & \\ & & \lambda_2 & 1 & & \\ & & & \lambda_2 & & \\ & & & & \lambda_3 & \\ & & & & & \lambda_3 \end{bmatrix}$$

1 ev for λ_2

- you need generalized eigenvectors
- Beyond scope of the course

Caley Hamilton Theorem

$\det(A - \lambda I) = \Delta(\lambda) = 0$

This gives us the ev.

Theorem: $\Delta(A) = 0$

eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \Delta(\lambda) = \lambda^2 = 0$

$\therefore A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

eg. $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ $\quad \Delta(\lambda) = (4-\lambda)^2 + 9 = 0$

$= \lambda^2 - 8\lambda + 25 = 0$

CH theorem says

$$A^2 - 8A + 25I = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^2 - 8\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

$$A^2 - 8A + 25I = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}^2 - 8\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ & \end{bmatrix} + \begin{bmatrix} -32 & +24 \\ -24 & -32 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

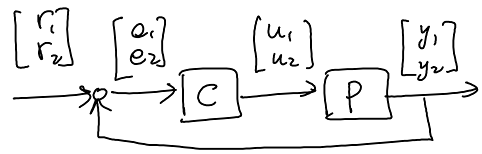
?

Implication: A is 8×8. Look at $A^{100} = \alpha_1 I + \alpha_2 A + \cdots \alpha_8 A^7$

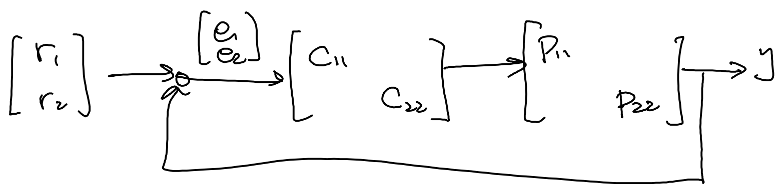
Aside: In our proj.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

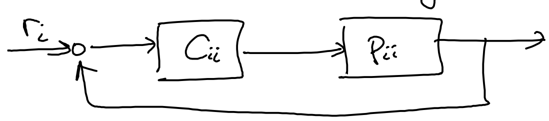
Assume this is not significant i.e. set them equal 0.



Design



These are two SISO designs



Test on the original plant

Choose $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ wisely.

MIMO Control Systems

2009년 2월 13일 금요일
오전 11:55

$$Y(s) = G(s)U(s)$$

where $G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1r}(s) \\ & & \vdots \\ G_{m1}(s) & \cdots & G_{mr}(s) \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}$ $\quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$

This is proper if every element is proper.
The system is square if # inputs = # outputs

Diagonal means the MIMO system is decoupled or decentralized.

- MIMO can also be upper or lower triangular
- off-diagonal terms represent cross-coupling
- you should have a Bode plot for each input-output pair (ie for each element)

MIMO Control Systems
2009년 2월 13일 금요일
오후 12:00

Multi-Input Multi-Output Control Systems

-MIMO transfer function matrix

-This is proper if every element (ie SISO transfer function) is proper

-This is square if there are as many inputs as outputs

-Diagonal means the MIMO system is decoupled or decentralized.

-MIMO systems can also be upper or lower triangular

-Off-diagonal terms represent cross-coupling

-Now, if one does a frequency response of a MIMO system, one needs to generate, in general, one Bode plot for each input-output pairing

-To look at the poles and zeros, there are many definitions. The following is most common

-Definition- The pole of a MIMO system are the roots of the least common denominator of all nonzero minors of all orders of the transfer function matrix

-Eg/ $G(s) = \begin{bmatrix} \frac{2}{s+1} & 5 \\ 6 & \frac{4}{s+1} \end{bmatrix}$

All 1x1 minors: $\frac{2}{s+1}$, 5, 6, $\frac{4}{s+1}$

2x2 minors: $\det(G(s)) = \frac{8}{(s+1)^2} - 30$

$= \frac{8 - 30(s+1)^2}{(s+1)^2}$

lcd $= (s+1)^2$

least common den. $\phi(s) = (s+1)^2$

This $\phi(s)$ is called the pole polynomial

The poles are roots of $\phi(s) = 0 \rightarrow \therefore$ the poles are $-1, -1$.

-Definition- A MIMO System is BIBO stable iff all the elements are BIBO stable

i.e., All the elements have poles in the OLHP.

Another ex. $G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ \frac{-1}{s-1} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$

1x1 minors: $\frac{1}{s+1}$, $\frac{-1}{s-1}$, $\frac{1}{s+2}$, $\frac{s-1}{(s+1)(s+2)}$, $\frac{1}{s+2}$,

2x2 minors (3): $\frac{s-1}{(s+1)(s+2)^2}$, $\frac{2}{(s+1)(s+2)}$, $\frac{1}{(s+1)(s+2)}$

$\phi(s) = (s+1)(s-1)(s+2)^2 = 0$

$\therefore$ the poles are $1, -1, -2, -2$

-There are many, many definitions of zeros. The most common is the following

-Definition- The normal rank of $G(s)$ is the rank for almost all values of s

Eg/ $\begin{bmatrix} \frac{3}{s+1} & \frac{4(s+2)}{(s+1)^2} \\ 9 & \frac{(s+2)}{(s+1)} \end{bmatrix}$

Plug in some values (eg $s=0$) and check the rank. OR calculate the determinant. and see that it is $\neq 0$.

The det. is $\frac{3(s+2)}{(s+1)^2} - \frac{36(s+2)}{s+12} \neq 0$

for almost any values of s.

linearly dependent columns.

eg. $\begin{bmatrix} \frac{3}{s+1} & \frac{6(s+3)}{s+1} \\ \frac{9}{s+2} & \frac{16(s+3)}{(s+2)} \end{bmatrix}$ → plug in $s=0$, rank=1.

- plug in $s=1$, rank=1
- " " $s=-1$, rank=1

Check $\det = ?$ $\det = 0$

$\therefore$ the normal rank = 1.

Feb. 23/09

Def'n of zero:

-Definition- Let r be the normal rank of $G(s)$. Look at all rxr minors adjusted so that the pole polynomial is the denominator for each. The zero polynomial is the greatest common divisor of all the numerators and the zeroes are the roots of the zero polynomial

this is the adj. minor with pole poly. denominator.

Eg/ $G(s) = \begin{bmatrix} \frac{2}{s+1} & 5 \\ 6 & \frac{4}{s+1} \end{bmatrix}$ normal rank = 2

$\therefore$ look at all 2x2 minors $\Rightarrow \frac{8}{(s+1)^2} - 30\frac{(s+1)^2}{(s+1)^2}$

From before, $\phi(s) = (s+1)^2$

$= \frac{-30s^2 - 60s - 22}{(s+1)^2}$

The zeros are the roots of the numerator. They're -1.516, -.434

Eg. $\begin{bmatrix} \frac{1}{s+1} & 0 & \frac{s-1}{(s+1)(s+2)} \\ -\frac{1}{(s-1)} & \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}$ Normal rank = 2

Look at all 2x2 minors: $\frac{1}{(s+1)(s+2)}$, $-\frac{(s-1)}{(s+2)^2(s+1)}$, $\frac{1}{(s+1)(s+2)} - \frac{\cancel{(s-1)}}{(s+1)(s+2)\cancel{(s-1)}} = \frac{2}{(s+1)(s+2)}$

From above, $\phi(s) = (s+1)(s+2)^2(s-1)$

... divisor of numerators is $(s-1)$

Adjusted minors are: $\frac{(s+2)(s-1)}{(s+2)^2(s-1)(s+1)}, \frac{-(s-1)^2}{(s+2)^2(s-1)(s+1)}, \frac{2(s+2)(s-1)}{(s+2)^2(s-1)(s+1)}$ → The greatest common

∴ The zero polynomial is $(s-1)$

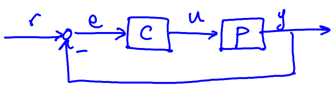
There is a zero at $s=1$.

-Note that, in the MIMO case, poles and zeros can be at the same location without cancelling

-These definitions reduce to our standard definitions when things are SISO

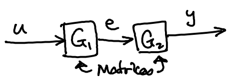
NOTE: In MIMO you could have NO zeros when the greatest common divisor of adjusted minors is '1'.

-prior to this, everything was scalar. Now, the controller and plant have multiple inputs and outputs



-Note that the order of the blocks matter as matrix multiplication does not commute ∴ $PC \neq CP$

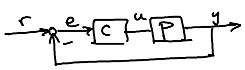
Eg/



$y = G_2 e$, $e = G_1 u$ } $y = G_2 G_1 u$

has to be in the order!

Eg/



$y = Pu$, $u = Ce$, $e = r - y$

$y = PCe = PC(r-y)$

$= PCr - PCy$

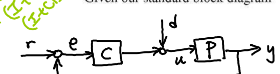
$(I + PC)y = PCr$

$y = (I+PC)^{-1} PCr$

Similarly, $e = (I+PC)^{-1} r$

watch for $(I+PC)^{-1}$ and $(I+CP)^{-1}$

-Given our standard block diagram



$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} (I+PC)^{-1}PC & (I+PC)^{-1}P \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}$$

-We have now input sensitivity and output sensitivity functions. Similarly for complementary sensitivity

Input sensitivity & comp. sensitivity

$L_i = CP$

$S_i = (I + L_i)^{-1}$

$T_i = (I+L_i)^{-1} L_i$

$= L_i (I + L_i)^{-1}$

Output sensitivity & comp. sensitivity

$L_o = PC$

$S_o = (I + L_o)^{-1}$

$T_o = (I + L_o)^{-1} L_o$

$= L_o (I + L_o)^{-1}$

prove these are equal!

$S_i + T_i = I$ ← prove this

$S_o + T_o = I$

- Similar to SISO
- There are also equivalents for MIMO systems of Nyquist Criteria, Bode, Sensitivity Integral, Loop Shaping
- They are difficult to use

MIMO using SISO

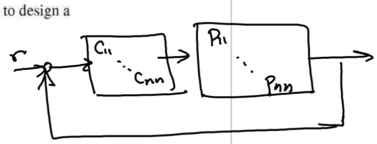
2009년 2월 23일 월요일
오후 12:13

Designing controllers using SISO techniques

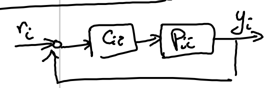
-There are two techniques used for using our previous SISO techniques for "square systems"

-The first is to assume that the plants are decoupled (decentralized) and to design a controller ignoring all the couplings

Assume $P = \begin{bmatrix} P_{11} & & & 0 \\ & P_{22} & & \\ & & P_{33} & \\ 0 & & \cdots & P_{nn} \end{bmatrix}$



This looks like



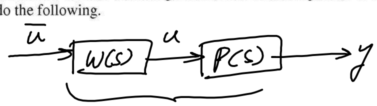
Find C_{ii}

Feb. 25th double check w/ others

-If there are interactions, this can lead to very poor performance or even to instability. This is called Decentralized control

-The second more effective method is decoupling control

-It involves finding a transfer function matrix *W*(*s*) to decouple the system. This is non-trivial (remember, we can't do things like cancel unstable poles). If we can find one, then we can do the following.

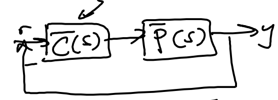


decoupled
O(s) diagonal $\bar{P}(s)$

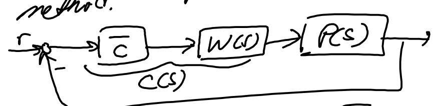
$y = P_{\bar{s}} W(s)\bar{u} = \bar{P}(s) \Rightarrow$ Then

orthogonal using column operations

diagonal



Design is easy since $\bar{P}$ is diagonal. We just use the first method.



$\therefore C(s) = W(s)\bar{C}(s)$

-It is often difficult to find *W*(*s*). One alternate method is to find the low frequency gains and use that instead of *W*(*s*). The idea here is to decouple the system at the critical lower frequencies.

Look at Bode plots. Identify the low freq gains btw the inputs & outputs.

Let $C = \bar{W}\bar{C}$ where $\bar{W}$ is a constant matrix that tries to decouple $P(s)$ at low freq.

-None of these methods are particularly effective or straightforward to design. We will move onto a more systematic approach; State Space Methods

State Space Control Techniques

-Reminder of Matrix properties
- Matrices do not commute
- -det(AB)=det$A$det$B$
- -Block triangular matrices

$(AB)^{-1} = B^{-1}A^{-1}$ formula for transpose

$C = \begin{bmatrix} A & 0 \\ x & B \end{bmatrix}$ eigval C = eig$A$ $\overset{\text{union}}{\cup}$ eig$B$

det C = det$A$ det$B$

-Assume we have state space equations derived from some ordinary differential equations

-Eg.

$\ddot{\theta} + \sin\theta = u$, $y = \theta$ ← eqn for pendulum

Nonlinear

Let $\underbrace{x_1 = \theta, \ x_2 = \dot{\theta}}_{\text{STATE VARIABLES}}$ $\Rightarrow \begin{cases} \dot{x}_1 = \dot{\theta} = x_2 \\ \dot{x}_2 = \ddot{\theta} = u - \sin\theta = u - \sin x_1 \end{cases}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\dot{X} = \begin{bmatrix} x_2 \\ u - \sin x_1 \end{bmatrix}$

I could also choose $x_1' = \dot{\theta}$ $x_2' = \theta$

then $\dot{X}' = \begin{bmatrix} u - \sin x_2' \\ x_1' \end{bmatrix}$, $X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ ⇒ IE. State space eqn are NOT UNIQUE! They depend on our defns for the states

-Assume the system is linearized

-Eg Linearize system $\dot{X} = \begin{bmatrix} x_2 \\ u - \sin x_1 \end{bmatrix} = f(X, u)$

Choose an $\underbrace{\text{equilibrium pt.}}$ Choose $u = 0, \theta = 0, \dot{\theta} = 0$
$(x_1 = 0)$ $(x_2 = 0)$

This gives $\dot{X} = 0$
∴ the states remain unchanged.

Let $\Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$

$\Delta \dot{x} = \frac{\partial f}{\partial X}\Big|_{X_{eq}, u_{eq}} \Delta x + \frac{\partial f}{\partial u}\Big|_{X_{eq}, u_{eq}} \Delta u = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{bmatrix}_{X=0,\, u=0} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{X=0,\, u=0} \Delta u$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$

-Thus, for remainder, we assume

$\dot{x} = Ax + Bu$ — m inputs

$y = Cx + Du$

P outputs, n states

Choose $u = 0, \dot{\theta} = 0, \theta = \pi$ as the eq. pt.

Now we have an inverted pendulum.

Then $\Delta \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$

State space Control

$\dot{x} = Ax + Bu$ where $x \in \mathbb{R}^n$

$y = Cx + Du$ $u \in \mathbb{R}^m$

$X(0) = X_0$ ← Initial cond.

Transfer functions $x(0) = 0$

take Laplace ⇒ $y(s) = [C(sI - A)^{-1}B + D]u(s)$

Feb. 27/09

Let the number of inputs be m, the number of outputs be p and the number of states be n

-If we take the Laplace transform, assuming no initial conditions, we can get the MIMO transfer function

$$\mathcal{L}\{\dot f\} = s\mathcal{L}\{f\} - \underbrace{f(0)}_{\text{I.C.}}$$

$$\therefore sX(s) - \cancel{X(0)} = AX(s) + BU(s)$$

$$\therefore (sI-A)X(s) = BU(s) \longrightarrow X(s) = (sI-A)^{-1}BU(s)$$

$Y(s) = [\underbrace{C(sI-A)^{-1}B+D}_{G(s)}]U(s)$ assuming zero initial conditions

-As we mentioned, these techniques for designing controllers in the MIMO case are not straightforward. We will stay in the state space domain

-The state space representation is not unique. We can do a state space transformation

Let $x' = Px$ where P is some nonsingular matrix (i.e. has an inverse)

eg. $X = \begin{cases} x_1 = \theta \\ x_2 = \dot\theta \end{cases}$ $\quad X' = \begin{cases} x_1' = \dot\theta \\ x_2' = \theta \end{cases}$ $\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\dot X' = P\dot X = P(AX + Bu) = PAX + PBu$$

$$= \underbrace{PAP^{-1}}_{A'}X' + \underbrace{PB}_{B'}u = A'X' + B'u$$

$A' = PAP^{-1}$ is a similarity transformation. The ev do not change

similarly, $Y = CX + Du = \underbrace{CP^{-1}}_{C'}X' + \underbrace{D}_{D'}u$

Look at the transfer function

$$Y(s) = [C'(sI-A')^{-1}B'+D']u(s) = [CP^{-1}(sI-PAP^{-1})^{-1}PB + D]u(s)$$

$$= [C\{P^{-1}(sI-PAP^{-1})P\}^{-1}B + D]u(s) = [C(sI-A)^{-1}B+D]u(s)$$

$\therefore P^{-1}P = I$

$\therefore$ You'll always get the same TF because similarity transform don't change TF

The transfer function is exactly the same

~~-The opposite is not true.~~ IF TWO TRANSFER FUNCTIONS ARE THE SAME, THEY DO NOT HAVE NECESSARILY HAVE THE SAME STATE SPACE REPRESENTATION. IN FACT, THEY MAY BE OF DIFFERENT ORDERS!

-Eg

$$\dot X = Ax + Bu \quad y = Cx + Du \rightarrow Y(s) = [C(sI-A)^{-1}B+D]u(s)$$

$$\dot{\tilde X} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}}_{\tilde A}\tilde x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde B}u \Longrightarrow y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\tilde C}\tilde x + Du$$

Arbitrary square matrix

$\therefore$ -If the order is the ~~same, the~~ systems are related by a similarity transformation (or the dimension of the state)

ie, If the order (ie, the dimension of the state) are the same, all the equivalent ss representations are related by $X = Px$

$$y(s) = [\tilde C(sI-\tilde A)^{-1}\tilde B + \tilde D]u$$

$$= \left[\begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} (sI-A)^{-1} & \\ & (sI-A')^{-1} \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix} + D\right]u$$

$$= [C(sI-A)^{-1}B + D]u(s)\ !$$

NOTE: $(sI-A)^{-1} \overset{\text{scalar}}{\rightsquigarrow} \dfrac{1}{s-a} \rightarrow$ Laplace of e^{at}

$\dot x = x$, $x(0) = 1$ then $sX(s) - 1 = X(s) \rightarrow X(s) = \dfrac{1}{s-1} \rightarrow x(t) = e^t$

-What is the time domain solution to the state space? We need the concept of the matrix exponential. ↳ Analytical soln

In the scalar case:

$$e^{at} = \mathcal{L}^{-1}\{(s-a)^{-1}\}$$

In the matrix case:

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

since $e^{at} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

-Alternate definition

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{i=0}^{\infty}\frac{A^i t^i}{i!}$$

-Properties of the matrix exponential

1. $e^{At} = I$ when $A = 0$
2. $e^{A(t+s)} = e^{At}e^{As}$
3. $e^{(A+B)t} = e^{At}e^{Bt} \Leftrightarrow AB = BA$ ← Only when A and B commute, you can do $e^{(A+B)t} = e^{At}e^{Bt}$. Eg. Diagonal matrices commute. Otherwise you must leave it in $e^{(A+B)t}$ form.
4. $(e^{At})^{-1} = e^{-At}$
5. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Then $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 1^2t^2/2! & 0 \\ 0 & \frac{2^2t^2}{2!} \end{bmatrix} + \cdots$

$$= \begin{bmatrix} 1 + t + 1^2t^2/2! + \cdots & \\ & 1 + 2t + \frac{2^2t^2}{2!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Alternatively: $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} (s-1)^{-1} & 0 \\ 0 & (s-2)^{-1} \end{bmatrix}\right\}$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

-Now, this means the solution to the state space equation is

$$\dot X = AX, \quad X(0) = I$$

we anticipate that the soln would be

$$X = e^{At}$$

Just like how soln to scalar is e^{at}, Matrix gives e^{At}.

eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\right\} \quad \det = \frac{1}{s^2}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ & \end{bmatrix}\right\} = \begin{bmatrix} 1 & t \\ & \end{bmatrix}.$$

$s-1$

-What is the time domain solution to the state space? We need the concept of the matrix exponential. ↳ Analytical soln

In the scalar case:

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In the matrix case:

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

Since $e^{at} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

-Alternate definition

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

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1. $e^{At} = I$ when $A = 0$
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3. $e^{(A+B)t} = e^{At}e^{Bt} \Leftrightarrow AB = BA$
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5. $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$

Only when A and B commute, you can do $e^{(A+B)t} = e^{At}e^{Bt}$.
Eg. Diagonal matrices commute.
Otherwise you must leave it in $e^{(A+B)t}$ form.

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Then $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1t & 0 \\ 0 & 2t \end{bmatrix} + \begin{bmatrix} 1^2t^2/2! & 0 \\ 0 & \frac{2^2t^2}{2!} \end{bmatrix} + \cdots$

$= \begin{bmatrix} 1+t+1^2t^2/2!+\cdots & \\ & 1+2t+\frac{2^2t^2}{2!}+\cdots \end{bmatrix}$

$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$

Same soln

$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} (s-1)^{-1} & 0 \\ 0 & (s-2)^{-1} \end{bmatrix}\right\}$

$= \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$

-Now, this means the solution to the state space equation is

$\dot{X} = AX$, $X(0) = I$

We anticipate that the soln would be

$X = e^{At}$ Just like how soln to scalar is e^{at}, Matrix gives e^{At}.

eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$

$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\right\}$ det $= \frac{1}{s^2}$

$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}\right\} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} t + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{A^2} t^2 + 0 =$

-In general, however, usually numerical integration is used by software packages such as MatLab to calculate the time responses.

Proof of 4. $(e^{At})^{-1} = e^{-At}$

$e^{A(t-t)} = e^{At}e^{-At}$

$e^{A(t-t)} = e^{A\cdot 0} = I \quad \therefore e^{At}e^{-At} = I$

$e^{-At} = (e^{At})^{-1}$ QED

Proof of 5. $\frac{d(e^{At})}{dt} = Ae^{At} = e^{At}A$

$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots$

$\frac{de^{At}}{dt} = 0 + A + \frac{A^2 2t}{2!} + \frac{A^3 3t^2}{3!} + \cdots$

$= A(I + At + A^2\frac{t^2}{2!} + \cdots) = Ae^{At}$

Similarly for $e^{At}A$.

2009년 3월 2일 월요일
오전 11:31

$$e^{At}\begin{cases} L^{-1}\{(sI-A)^{-1}\} \\ I+At+\frac{At^2}{2!}+\cdots \end{cases}$$

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du$$

– The solution

$$sX(s)-X(0)=AX(s)+BU(s)$$
$$Y(s)=CX(s)+DU(s)$$
$$X(s)=(sI-A^{-1})X(0)+(sI-A)^{-1}BU(s)$$
$$Y(s)=C(sI-A)^{-1}X(0)+\underbrace{C(sI-A)^{-1}BU(s)}_{\text{convolution}}+DU(s)$$

$$y(t)=Ce^{At}X(0)+C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau+Du(t)$$

NOTE: convolution $\quad g*u=g(s)\cdot u(s)=\int_0^t g(t)u(t-\tau)d\tau$

In general if the initial time is not 0 but is t_0 we have

$$x(t)=e^{A(t-t_0)}X(t_0)+\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Verify: ◦ the initial conditions $\quad X(t_0)=e^{A\cdot 0}X(t_0)+\cancel{\int_{t_0}^{t_0}}d\tau$
$$=X(t_0)$$

◦ Differential equation

$$\dot{x}=Ae^{A(t-t_0)}X(t_0)+\{e^{A(t-\tau)}Bu(\tau)\}\big|_{\tau=t}+0$$
$$+\int_{t_0}^t Ae^{A(t-\tau)}Bu(\tau)d\tau$$
$$=A\underbrace{\{e^{A(t-t_0)}X(t_0)+\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau\}}_{x(t)}+Bu(t)$$
$$=Ax(t)+Bu(t)$$

∴ satisfies the d.e.

Note Leibniz rule

$$\frac{d}{dt}\int_0^t f(\tau)\,d\tau=f(\tau)\big|_{\tau=t}+0+0=f(t)$$

$$\frac{d}{dt}\int_{\beta(\alpha)}^{\alpha(t)} f(t,\tau)d\tau$$
$$=\dot{\alpha}(t)f(t,\tau)\big|_{\tau=\alpha(t)}$$
$$-\dot{\beta}(t)f(t,\tau)\big|_{\tau=\beta(t)}$$
$$+\int_{\beta(t)}^{\alpha(t)}\frac{\partial f(t,\tau)}{\partial t}d\tau$$

Note: $A=\begin{bmatrix}10{,}000 & \\ & 1\end{bmatrix} \quad e^{At}=\begin{bmatrix}e^{10000t} & \\ & e^{t}\end{bmatrix}$

$t=10{,}000$ sec. → Numerical prob.

-Now, if the eigenvalues of A are in the OLHP, then when we calculate the transfer function, all the denominator terms come from $\det(sI - A)$ and all the poles of each element will also be in the OLHP

-Theorem- If the eigenvalues of A are in the OLHP, then the MIMO system is BIBO stable

-Note that the opposite may not be true
Eg/

-As well, the eigenvalues of A give an indication of the type of response. By looking at e^{At}, we can see that the output are essentially linear combinations of the terms. First, do a similarity transform to get it diagonal (or Jordan block form for repeated eigenvalues). Remember that the transfer function is unchanged

-Thus, the response will look like a linear combination of e^{At} and convolutions of e^{At} with the input.

-Again, the eigenvalues of A determine the response at the output

Note: $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ then $e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

Jordan Block form
$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$ then $e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & te^{\lambda t} \\ & 0 & & & e^{\lambda t} \end{bmatrix}$

Let $\underline{X}' = P\underline{X}$

$$\dot{\underline{X}}' = \underbrace{PAP^{-1}}_{\text{diagonal}}\underline{X}' + \overbrace{PB}^{B'}u, \quad y = \underbrace{CP^{-1}}_{C'}\underline{X} + \underbrace{D}_{D'}u$$

or possible Jordan block form Λ

$$y = C'e^{\Lambda(t-t_0)}\underline{X}'(t_0) + C\int_{t_0}^{t} e^{\Lambda(t-\tau)}Bu(\tau)\,d\tau + Du$$

but $e^{\Lambda(t-t_0)}$ has terms that look like $e^{\lambda t}, te^{\lambda t} \cdots \frac{t^{n-1}}{(n-1)!}e^{\lambda t}$
where λ is an ev.

These terms are called Modes of Response

- The response look like these modes

- The response looks like these modes are convolutions of input with these modes.

eg $A = \begin{bmatrix} -1 & 1 & \\ 0 & -1 & \\ & & 2 \end{bmatrix}$ $u(t)$ is a ramp.

What is the output going to be like?
The ev of A are $-1, -1, -2$
The top left is Jordan Block form already
The modes are e^{-2t}, e^{-t}, te^{-t}
The outputs will be linear combos of
e^{-2t}, e^{-t}, te^{-t} s $\quad t * e^{-2t}, ~~t*e^{-t}~~, ~~t*te^{-t}~~$

March 4th 2009

$\dot{x} = Ax + Bu$
mode of response BIBO stability if the ev of A are in the OLHP

Let $u = -kx + v$ ← new input

$\dot{x} = Ax + B(-kx + v)$
$= (A - Bk)x + Bv$
choose k to modify the behaviour

friday's lecture (next lect)

Today's goal: going from G(S) to ss.rep

-Going from a transfer function to state space. Recall this is not unique. There are a few different "canonical forms"

1. Controllable Canonical Form

-Given $G(s) = \dfrac{b_m s^m + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} ... + a_1 s + a_0} = \dfrac{N(s)}{D(s)} = \dfrac{Y(s)}{U(s)}$

-Introduce auxillary variables $X(s)$

$$\frac{y}{x} = b_m s^m + \dots b_1 s + b_0$$

$$G(s) = \frac{X(s)}{U(s)} \frac{Y(s)}{X(s)} = \frac{b_m s^m + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} ... + a_1 s + a_0}$$

$$\frac{x}{u} = \frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

Therefore, $\dfrac{d^n x}{dt^n} + a_{n-1}\dfrac{d^{n-1}x}{dt^{n-1}} + ... a_1 \dfrac{dx}{dt} + a_0 x = u$, $\quad b_m \dfrac{d^m x}{dt^m} + ... + b_1 \dfrac{dx}{dt} + b_0 x = y$ $\Leftarrow \begin{cases} x(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0) = u \\ y = (b_m s^m + \dots + b_0) x \end{cases}$

Let $x_1 = x, x_2 = \dot{x}$, etc ... $x_n = \dfrac{d^{n-1}x}{dt}$

states

$\therefore u = \dot{x}_n + a_{n-1}x_n + \dots + a_1 x_2 + a_0 x_1 = u$
$y = b_m x_{m-1} + \dots + b_1 x_2 + b_0 x_1 = y$

$\therefore \dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\vdots$
$\dot{x}_n = u - a_0 x_1 - a_1 x_2 - \dots - a_{n-1}$

Then $\dot{x} = Ax + by, y = Cx$ where $A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & . & . & . & -a_{n-1} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix}$

$C = [b_0 \; . \; . \; b_m \; 0 \; . \; . \; 0]$

Similarly, $y = [b_0 \; b_1 \cdots b_m \; 0 \cdots 0]X$

-This is the controllable canonical form

$\dot{X} = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u$

Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \dot{X} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \end{bmatrix}$

-Going from a transfer function to state space. Recall this is not unique. There are a few different "canonical forms"

1. Controllable Canonical Form

-Given $G(s) = \frac{b_m s^m + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} ... + a_1 s + a_0} = \frac{N(s)}{D(s)} = \frac{Y(s)}{U(s)}$

-Introduce auxillary variables $X(s)$

$$\frac{y}{x} = b_m s^m + \dots b_1 s + b_0$$

$$\frac{x}{u} = \frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

$$G(s) = \frac{X(s)}{U(s)}\frac{Y(s)}{X(s)} = \frac{b_m s^m + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} ... + a_1 s + a_0}$$

Therefore, $\begin{cases} \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... a_1 \frac{dx}{dt} + a_0 x = u \\ b_m \frac{d^m x}{dt^m} + ... + b_1 \frac{dx}{dt} + b_0 x = y \end{cases} \Leftarrow \begin{cases} x(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0) = u \\ y = (b_m s^m + \dots + b_0)x \end{cases}$

Let $\boxed{x_1 = x, x_2 = \dot{x}, \text{etc} \dots x_n = \frac{d^{n-1}x}{dt}}$

states

$\therefore \begin{cases} u = \dot{x}_n + a_{n-1}x_n + \dots + a_1 x_2 + a_0 x_1 = u \\ y = b_m x_{m-1} + \dots + b_1 x_2 + b_0 x_1 = y \end{cases}$

Then $\dot{x} = Ax + by,\ y = Cx$ where $A = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & \cdot & \cdot & \cdot & -a_{n-1} \end{bmatrix}, b = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$

$\therefore \dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\vdots$

$\dot{x}_n = u - a_0 x_1 - a_1 x_2 - \dots - a_{n-1}$

$C = [b_0 \ \ . \ \ . \ \ b_m \ \ 0 \ \ . \ \ . \ \ . \ \ 0]$

Similarly, $y = [b_0\ b_1 \cdots b_m\ 0 \cdots 0]X$

-This is the controllable canonical form

Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \dot{X} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ u - a_0 x_1 - a_1 x_2 - \dots - a_{n-1}x_n \end{bmatrix}$

$$\dot{X} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$Y = [b_0\ b_1\ \dots\ b_m\ 0 \cdots 0]\, X$$

$$\frac{Y}{u} = G = \frac{N}{D}$$ Polynomials

-Let's look at the observable canonical form

-Start with $D(s)Y(s) = N(s)U(s)$

$(s^n + a_{n-1}s^{n-1} + \dots + a_0)Y(s) = (b_m s^m + \dots + b_0)U(s)$ let $m = n-1$

$\therefore s^n Y(s) = -a_{n-1}s^{n-1}Y(s) - \dots - a_1 sY(s) - a_0 Y(s) + b_{n-1}s^{n-1}U(s) + \dots + b_1 sU(s) + b_0 U(s)$

This defines my state space eqn

$$Y(s) = \underbrace{\frac{1}{s}[-a_{n-1}Y(s) + b_{n-1}U(s) + \underbrace{\frac{1}{s}[-a_{n-2}Y(s) + b_{n-2}U(s) + \frac{1}{s}[\dots + \underbrace{\frac{1}{s}\underbrace{[-a_0 Y(s) + b_0 U(s)]}_{x_1}\dots]}_{x_2}]]}_{x_{n-1}}}_{x_n}$$

n-brackets

$\therefore y = x_n$

$\dot{x}_n = -a_{n-1}y + b_{n-1}u + x_{n-1}$

$\quad = -a_{n-1}x_n + b_{n-1}u + x_{n-1}$

$\dot{x}_{n-1} = -a_{n-2}x_n + b_{n-2}u + x_{n-2}$

.

.

$\dot{x}_1 = -a_0 x_n + b_0 u$

$$\therefore A = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & & & & -a_1 \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & 1 & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-1} \end{bmatrix}, c = [0 \ \ . \ \ . \ \ 0 \ \ 1]$$

$\therefore y = x_n$

OBSERVABLE CANONICAL FORM

Ex. $Y(s) = \left(\frac{s^3 + 6s^2 + 1}{s^3 + 2s + 1}\right)U(s)$ This is one order too high $\therefore$ we divide

$$\begin{array}{r} 1 \\ s^3 + 2s + 1 \overline{)\, s^3 + 6s^2 + 1} \\ \underline{s^3 + 2s + 1} \\ 6s^2 - 2s \end{array}$$

$b_2 \quad b_1 \quad b_0 = 0$

$y(s) = \Big(\underset{D(s)}{1} + \underbrace{\frac{6s^2 - 2s}{s^3 + 2s + 1}}_{C(s)}\Big)u(s)$

$a_3 = 1 \quad a_2 = 0 \quad a_1 = 2. \quad a_0 = 1$

$$\left\{\begin{array}{l} \therefore \dot{\underline{X}} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}\underline{X} + \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix}u \\ Y = [0 \ 0 \ 1]\underline{X} + 1 \cdot u \end{array}\right.$$

SISO!

C Multivariable Realizations

-Assume $G(s)$ is an $r \times p$ matrix transfer function. Let's first find a state space representation (may not be controllable or observable)

-This is a brute force method. Let $G(s) = \begin{bmatrix} h_{11}(s) & . & . & . & h_{1p}(s) \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ h_{r1}(s) & . & . & . & h_{rp}(s) \end{bmatrix}$

-Suppose we have a scalar representation of each h_{ij} (eg. previous section)

ie $h_{ij}(s) = c_{ij}(sI - A_{ij})^{-1}B_{ij} + D_{ij}$ ← each h_{ij} has C. A. B. D matrix.

-Then a realization is

$$A = \begin{bmatrix} A_{11} & & & & & & & & & & & & \\ & . & & & & & & & & & & & \\ & & . & & & & & & & & & & \\ & & & A_{r1} & & & & & & & & & \\ & & & & A_{12} & & & & & & & & \\ & & & & & . & & & & & & & \\ & & & & & & . & & & & & & \\ & & & & & & & A_{r2} & & & & & \\ & & & & & & & & . & & & & \\ & & & & & & & & & . & & & \\ & & & & & & & & & & A_{1p} & & \\ & & & & & & & & & & & . & \\ & & & & & & & & & & & & . \\ & & & & & & & & & & & & & A_{rp} \end{bmatrix}$$

usually the dim. of this matrix is bigger than we need it to be. (controllability & observability in next lecture)

$$C = \begin{bmatrix} C_{11} & & & & C_{12} & & & & . & . & C_{1p} & & & \\ & C_{21} & & & & C_{22} & & & . & . & & C_{2p} & & \\ & & . & & & & . & & . & . & & & . & \\ & & & . & & & & . & . & . & & & & . \\ & & & C_{r1} & & & & C_{r2} & . & . & & & & C_{rp} \end{bmatrix}$$

when we do $C(sI-A)^{-1}$ using these we can get $G(s) = \begin{bmatrix} h_{11} & \cdots & h_{1p} \\ \vdots & & \vdots \\ h_{r1} & \cdots & h_{rp} \end{bmatrix}$

$$B = \begin{bmatrix} B_{11} & & & \\ \cdot & & & \\ \cdot & & & \\ B_{r1} & & & \\ & B_{12} & & \\ & \cdot & & \\ & \cdot & & \\ & B_{r2} & & \\ & & \cdot & \\ & & & \cdot \\ & & & B_{1p} \\ & & & \cdot \\ & & & \cdot \\ & & & B_{rp} \end{bmatrix} \quad D = \begin{bmatrix} D_{11} & \cdot & \cdot & D_{1p} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ D_{r1} & \cdot & \cdot & D_{rp} \end{bmatrix}$$

Std. form

Eg/
$$y(s) = \begin{bmatrix} \dfrac{2s}{s+5} \\ \dfrac{s+1}{(s+2)^2} \end{bmatrix} u(s) = \begin{bmatrix} 2 - \dfrac{10}{s+5} \\ \dfrac{s+1}{s^2+4s+4} \end{bmatrix} u(s)$$

Use controller canonical form

$$\dot{x} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

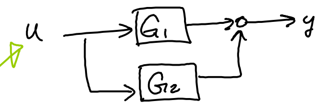
$$y = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

Do this ex.

-Interconnecting state space models

Each G_i is represented by $\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + D_i u_i \end{aligned}$

i) parallel connection



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u = \begin{bmatrix} A_1 x_1 + B_1 u \\ A_2 x_2 + B_2 u \end{bmatrix}$$

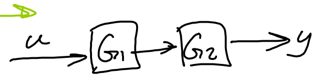
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} E_1 & E_2 \end{bmatrix} u$$

$$G_i \Rightarrow \left.\begin{aligned} \dot{x}_1 &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + E_i u_i \end{aligned}\right\}$$ could use ss2tf and tf2ss to get the ss rep

$$y = y_1 + y_2$$
$$= C_1 x_1 + E_1 u + C_2 x_2 + E_2 u$$
$$= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} E_1 & E_2 \end{bmatrix}}_{\text{Error?}} u$$

$u_1 = u$
$u_2 = u$

ii) series connection



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 E_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} E_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + E_2 E_1 u$$

iii) feedback configuration

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 (I + E_2 E_1)^{-1} E_2 C_1 & -B_1 (I + E_2 E_1)^{-1} C_2 \\ B_2 (I + E_1 E_2)^{-1} C_1 & A_2 - B_2 (I + E_1 E_2)^{-1} E_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 (I + E_2 E_1)^{-1} \\ B_2 (I + E_1 E_2)^{-1} E_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (I + E_1 E_2)^{-1} C_1 & -(I + E_1 E_2)^{-1} E_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (I + E_1 E_2)^{-1} E_1 u$$

-Note: $(I + E_1 E_2)^{-1}$ and $(I + E_2 E_1)^{-1}$ must exist for all t

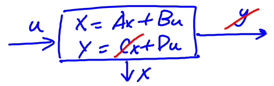
-Eg/

March 30th ← 2nd Quiz
Proj. Pres due April 3rd (20%) (first part 10%)

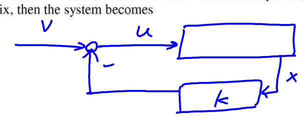
Controllability and State Feedback

-Given the state space description of a plant below $\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}$ where A is nXn

Let's just look at the state equation and not the output equation. First, let us make the huge assumption that the states are *measurable* ← states do not need to be measurable.



-If one lets $u = -kx + v$ where v is a new reference input and k is a state feedback gain matrix, then the system becomes



$$\dot{x} = (A - Bk)x + Bv = \underbrace{(A-Bk)}_{\text{New } A \text{ matrix}}x + Bv$$
$$y = Cx + Du$$

-The system has a new "A" matrix. Can we choose k to arbitrarily move the eigenvalues of the A matrix?

-Goal: Design the state feedback so that the eigenvalues of the controlled state space equations are in the OLHP. Remember that this guarantees that that the system is BIBO stable. The further left of the imaginary axis the poles are, the faster the response but the higher the gains. This higher bandwidth will be susceptible to noise

-Whether there exists a state feedback k is related to the concept of controllability.

-Suppose we have a system that is completely diagonalized.

$\dot{\bar{x}} = \Lambda\bar{x} + \bar{B}u$ where Λ is diagonal

$$\begin{bmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\\ \dot{\bar{x}}_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} + \begin{bmatrix}\bar{b}_1\\ \bar{b}_2\\ \bar{b}_3\end{bmatrix}u \rightarrow \dot{\bar{x}}_i = \lambda_i\bar{x}_i + \bar{b}_i u$$

-Obviously, if one of the b_i is zero, than there is no effect of the input on that particular state.

Suppose $b_2 = 0$

$\dot{\bar{x}}_2 = \lambda_2\bar{x}_2$ $\therefore$ u cannot affect λ_2

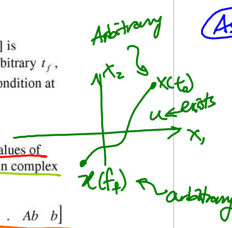
-Look at the following matrix $[\Lambda^2 b \quad \Lambda b \quad b]$. Is the matrix full rank?

$$[\Lambda^2 b \ \Lambda b \ b] = \begin{bmatrix}\lambda_1^2\bar{b}_1 & \lambda_1\bar{b}_1 & \bar{b}_1\\ 0 & 0 & 0\\ \lambda_3^2\bar{b}_3 & \lambda_3\bar{b}_3 & \bar{b}_3\end{bmatrix} \quad \text{rank} = 2 < 3 = n$$

what about
$$[A^2B \quad AB \quad B] = [(T\Lambda T^{-1})^2 T\bar{B} \quad (T\Lambda T^{-1})T\bar{B} \quad T\bar{B}]$$
$$= T[\Lambda^2\bar{B} \quad \Lambda\bar{B} \quad \bar{B}]$$

ASIDE
$\text{rank}(AB) \le \min[\text{rank}(A), \text{rank}(B)]$
$\text{rank}(T) = 3$ since nonsingular
$\text{rank}(\bar{Q}) = 2$
$\text{rank}(Q) \le 2$ (you can show $\text{rank}\,Q = \text{rank}\,\bar{Q}$

-Definition- Controllability- The state space equation is controllable (or $[A,B]$ is controllable) if, for all possible initial conditions and final conditions at an arbitrary t_f, there is an input that takes the system from the initial condition to the final condition at t_f



-Theorem- If the system is controllable, then there is a k such that the eigenvalues of $A - Bk$ can be placed arbitrarily (as long as the complex eigenvalues appear in complex conjugate pairs)

-Theorem- The system is controllable iff the *controllability matrix* $[A^{n-1}b \ \ . \ \ . \ \ Ab \ \ b]$ has rank n ← full rank

-Theorem- The system is controllable iff $[A - \lambda I \quad B]$ for all eigenvalues λ of A — iff $\text{rank}[A - \lambda I \ \vdots \ B] = n$

-In Matlab, the command to find k is "place"

March 9th 2009

-Let's look at the controllable canonical form. Check controllability

$$\dot{x} = \begin{bmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ \underbrace{-a_0 \ \cdots \ -a_{n-1}}_{\text{coefficients of denominator polynomial}}\end{bmatrix}x + \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}u$$

$$b = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix} \quad Ab = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ -a_{n-1}\end{bmatrix} \quad A^2b = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ x\\ x\end{bmatrix} \quad A^3b = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ x\\ x\\ x\end{bmatrix} \quad Q = \begin{bmatrix} & & 1\\ & 1 & 0\\ 1 & & \\ X & & \ddots \\ & & & 1\end{bmatrix}$$

$\therefore \text{rank}(Q) = n$

$\therefore$ Q is always controllable, hence canonical form is always controllable.

-Find the formula for the gain k

$u = -kx + v$
$= [-k_1 \ -k_2 \ \ldots \ -k_n]x + v$

$$\dot{x} = \begin{bmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ -a_0 - k_1 & -a_1 - k_2 & \cdots & -a_{n-1} - k_n\end{bmatrix}x + \begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}v$$

-Look at the following matrix $[\Lambda^2 b \quad \Lambda b \quad b]$. Is the matrix full rank?

$$[\Lambda^2 b \;\; \Lambda b \;\; b] = \begin{bmatrix} \lambda_1^2 \bar{b}_1 & \lambda_1 \bar{b}_1 & \bar{b}_1 \\ 0 & 0 & 0 \\ \lambda_3^2 \bar{b}_3 & \lambda_3 \bar{b}_3 & \bar{b}_3 \end{bmatrix} \quad \text{rank} = 2 < 3 = n$$

What about

$$[A^2 B \;\; AB \;\; B] = [(T\Lambda T^{-1})^2 T\bar{B} \;\; (T\Lambda T^{-1})T\bar{B} \;\; T\bar{B}] = T[\Lambda^2 \bar{B} \;\; \Lambda \bar{B} \;\; \bar{B}]$$

ASIDE

$\text{rank}(AB) \le \min[\text{rank}(A), \text{rank}(B)]$

$\text{rank}(T) = 3$ since nonsingular

$\text{rank}(\bar{Q}) = 2$

$\text{rank}(Q) \le 2$ (you can show $\text{rank}\, Q = \text{rank}\, \bar{Q}$

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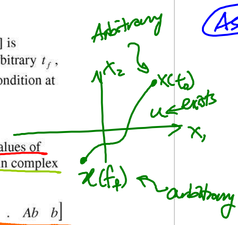


-Theorem- If the system is controllable, then there is a k such that the eigenvalues of $A - Bk$ can be placed arbitrarily (as long as the complex eigenvalues appear in complex conjugate pairs)

-Theorem- The system is controllable iff the *controllability matrix* $[A^{n-1}b \quad . \quad . \quad Ab \quad b]$ has rank n ← full rank

-Theorem- The system is controllable iff $[A - \lambda I \quad B]$ for all eigenvalues λ of A ← iff $\text{rank}[A - \lambda I \; \vdots \; B] = n$

-In Matlab, the command to find k is "place"

March 9th 2009

-Let's look at the controllable canonical form. Check controllability

$$\dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

↳ Coefficients of denominator polynomial

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad Ab = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -a_{n-1} \end{bmatrix} \quad A^2 b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ x \\ x \end{bmatrix} \quad A^3 b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ x \\ x \\ x \end{bmatrix} \quad Q = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & ⋰ & & \\ 1 & & X & \end{bmatrix}$$

$\therefore \text{rank}(Q) = n$

-Find the formula for the gain k

$u = -kx + v$

$= [-k_1 \;\; -k_2 \;\; \ldots \;\; -k_n]\, x + v$

$$\dot{x} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 - k_1 & -a_1 - k_2 & \cdots & -a_{n-1} - k_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v$$

∴ Q is always controllable, hence canonical form is always controllable.

↳ are the 'new' ev determined by the denominator polynomial.

Thus the transfer function denominator polynomial is

$$s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_0 + k_1) = 0$$

Choose k to place ev of A anywhere!

Eg

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

a) what are the ev of this system?

b) design a stable feedback to put all ev of A at $-2, -3, -1$

ANS

a) The denominator polynomial is $s^3 + 3s^2 + 3s + 1 = 0$

These are also the ev of A

$(s+1)^3 = 0 \rightarrow$ ev are at $-1, -1, -1$

b) when $u = [-k_1 \;\; -k_2 \;\; -k_3]\, x + v$ then

$s^3 + (3 + k_3)s^2 + (3 + k_2)s + (1 + k_1) = 0$

Given ev at $-2, -3, -1$ we want $\Delta(s)$ to be

$(s+3)(s+2)(s+1) = (s^2 + 5s + 6)(s+1) = s^3 + \underbrace{6}_{3+k_3} s^2 + \underbrace{11}_{3+k_2} s + \underbrace{6}_{1+k_1}$

$\therefore k = [5 \;\; 8 \;\; 3]$

$u = -kx + v \rightarrow -k = [-5 \;\; -8 \;\; -3]$

-For a SISO system, if the system is controllable, we can always get it into a controllable canonical form

$\dot{x} = Ax + Bu$, $y = Cx + Du$ assume $D = 0$

Find $G(s) = C(sI - A)^{-1}B \Rightarrow G(s) = \dfrac{b_{n-1}s^{n-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$ *

Find $Q = [A^{n-1}B \mid A^{n-2}B \mid \cdots \mid AB \mid B]$

From *, we can write the system in cont. can. form

$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$. Find $\bar{Q} = [\bar{A}^{n-1}\bar{B} \ \ \bar{A}^{n-2}\bar{B} \cdots \bar{A}\bar{B} \ \ \bar{B}]$

-Eg/

Then $T = \bar{Q}^{-1}Q$ Q^{-1} will always exist $\because$ Q has full rank.

i.e. if $\bar{x} = Tx$ this transforms the system into cont. can. form

Now $u = \bar{K}\bar{x} + v$ is easy to design

$= \bar{K}Tx + v$

$\therefore$ K in my original system is $K = \bar{K}T$

- Formulas for MIMO systems are complicated
- simpler problems

eg. $\dot{x} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Find state feedback to place poles at $-1, -3$

a) check controllability

$Q = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}$ rank $= 2$

Let $u = -k_1 x_1 - k_2 x_2 + v$

$A - BK = \begin{bmatrix} 1 & 3 \\ 4 - k_1 & 5 - k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$

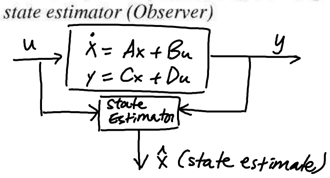
$\det\{sI - (A - BK)\} = (s+1)(s+3)$

You match coefficients.

Observability and State Estimators

-Recall that, when we applied state feedback, we assumed that the states were measurable

-However, in many cases, they are not physical variables. In those cases, we can use a *state estimator (Observer)*



estimator inputs

-The equations of the state estimator are $\dot{\hat{x}} = A\hat{x} + Bu + F(y - \hat{y})$, $\hat{y} = C\hat{x} + Du$ We're looking at $\hat{x}$.

If $D=0$ then $\hat{y} = C\hat{x}$ $\Rightarrow$ $\dot{\hat{x}} = (A-FC)\hat{x} + Bu + Fy$

Awesome midterm question

-If $A-FC$ have eigenvalues in the OLHP, then $e = \hat{x} - x$ converges exponentially to 0 $(e = \hat{x} - x)$

F is what we use to design the estimator.

Pf/

$\dot{e} = \dot{\hat{x}} - \dot{x} = (A-FC)\hat{x} + \cancel{Bu} + Fy - [Ax + \cancel{Bu}]$ ($\to Cx$)

$= (A-FC)\hat{x} - (A-FC)x$

$= (A-FC)e$ since $\dot{x} = Ax$ is $x(t) = x(0)e^{At}$

$\therefore e(t) = e(0)\underbrace{e^{(A-FC)t}}_{\text{goes to zero if the ev lie in the OLHP}}$

Just like how 'controllability' allowed us to arbitrarily locate ev of controller, in estimator we need 'observability' to allow us to arbitrarily locate ev of $(A-FC)$

-Thus, if we know A and C (and also B and D indirectly), then the state estimator will give a "good" estimate of the state that gets better as $t \to \infty$

-Just as in state feedback, one needed controllability to ensure that one could place the eigenvalues of A-BK, in state estimators, we need the concept of observability to place the eigenvalues of $A-FC$ at arbitrary locations

-Theorem- The system $\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}$ is *observable* (or (C,A) is observable) if all initial conditions $x(0)$ can be computed from the knowledge of the input $u(t)$ and output $y(t)$ for all times between 0 and t

critical ones — we really only need these

Intuitive version of theorem

-Note that this t can be arbitrarily small

- Intuitively, if we know $x(0)$ and the s.s. eqn we could generate $x(t)$ exactly.

-Theorem- If the system is observable, then there is a F such that the eigenvalues of $A-FC$ can be placed arbitrarily (as long as the complex eigenvalues appear in complex conjugate pairs)

-Theorem- The system is observable iff the *observability matrix* $R=\begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ CA^{n-1}\end{bmatrix}$ has rank n

$= \begin{bmatrix} B^T \\ (AB)^T \\ \vdots \\ (A^{n-1}B)^T \end{bmatrix} \to P^T = Q$ with columns reversed. Rank doesn't change.

Analogous to controllability check

-Theorem- The system is observable iff $\text{Rank}\begin{bmatrix} C \\ A-\lambda I\end{bmatrix} = n$ for all eigenvalues λ of A

$A-FC$
duality $\to A-BK$

-There is an obvious *duality* here. If we replace A with A^T and C with B^T, then we see that we can use the same techniques and commands to find F (Use the Matlab command "place"). Then, to find F, you use $F=K^T$

IE. the observability results collapse to the controllability results

-The observable canonical form is always observable (exercise)

MIDTERM QUESTION!

-You can also solve for $A-FC$ directly using the observable canonical form

-Similarly to the controllable canonical form, one can always transform a system to an observable canonical form using $T=\bar{R}^{-1}R$

-Eg/ $\dot{x} = \begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1}\end{bmatrix} x + \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1}\end{bmatrix} u \qquad y = [0 \cdots 0 \; 1]x$

coeff of the den. poly. (ie the ev)

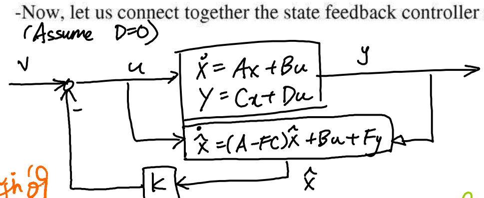
If $F=\begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$, $A-FC = \begin{bmatrix} 0 & \cdots & & -a_0-F_1 \\ 1 & \ddots & & -a_1-F_2 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1}-F_n \end{bmatrix}$

Coeff of the desired characteristic polynomial

$\therefore$ F can be used to place the ev arbitrarily

↳ Useful for systems with things that are not measurable & since observable states can help us.

-Now, let us connect together the state feedback controller and the state estimator
(Assume $D=0$)



Magic !: ev of this system turns out the same as ev of $A-BK$
The system works as though they are completely decoupled.

Morning 8 Gd Midterm Question

Reverse $\boxed{x \to \hat{x} \text{ exponentially}}$

-The state space equations of the two combined are

$\dot{x} = Ax + Bu$, $u = v - K\hat{x}$

$\dot{\hat{x}} = (A-FC)\hat{x} + Fy + Bu$, $y = Cx$

Let $X = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ $\left\{ \begin{array}{l} \dot{X} = \underbrace{\begin{bmatrix} A & -BK \\ FC & A-FC-BK \end{bmatrix}}_{\bar{A}} X + \underbrace{\begin{bmatrix} B \\ B \end{bmatrix}}_{\bar{B}} v \\ y = \underbrace{[C \quad 0]}_{\bar{C}} X \end{array} \right.$

-Transform the system (This is equiv. to using diff. states)

Let $\tilde{X} = TX$ where $T = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$, $T^{-1} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$

$\dot{\tilde{X}} = T\bar{A}T^{-1}\tilde{X} + T\bar{B}v$, $y = \bar{C}T^{-1}\tilde{X}$

(TF doesn't change)

Note: $\tilde{X} = \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix}$ ← error in the estimate

$\dot{\tilde{X}} = \begin{bmatrix} A-BK & -BK \\ 0 & A-FC \end{bmatrix} \tilde{X} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$, $y = [C \quad 0]\tilde{X}$

-The eigenvalues of the system are the eigen values of $\underbrace{(A-BK)}_{\text{state feedback}}$ in union with ev of $\underbrace{(A-FC)}_{\text{state estimator}}$

-Separation principle: The eigenvalues of the combined system are the eigenvalues of the state feedback $A-BK$ in union with the eigenvalues of the state estimator $A-FC$

Make the ev of the estimator at least 2 times faster than the state feedback

-Usually place the poles of the state estimator 2 or more times further to the left (ie faster poles) of the state feedback poles

Because we want our estimation to converge as quickly as possible

-Now, what is the transfer function?

$TF_{v \to y} = G(s) = [C \quad 0] \left[sI - \begin{bmatrix} A-BK & -BK \\ 0 & A-FC \end{bmatrix} \right]^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = [C \quad 0] \begin{bmatrix} sI-(A-BK) & BK \\ 0 & sI-(A-FC) \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = C[sI-(A-BK)]^{-1}B$

because only $sI-(A-BK)$ term remains because of these zeros

same as state feedback TF! ie. It's as if the state estimator doesn't even exist!

NOTE: $\begin{bmatrix} \Lambda_1 & \Lambda_2 \\ 0 & \Lambda_3 \end{bmatrix}^{-1} = \begin{bmatrix} \Lambda_1^{-1} & -\Lambda_1^{-1}\Lambda_2\Lambda_3^{-1} \\ 0 & \Lambda_3^{-1} \end{bmatrix}$

-Thus, the transfer function is as if there were only state feedback

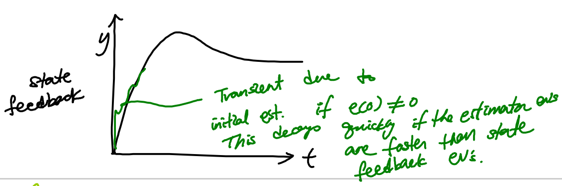
Since we assume zero initial conditions it's as if we have a perfect estimate, hence $\tilde{X} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ ← $(\hat{x} - x)$ system behaves as if

-What is the effect of adding the estimator then?

There will be a transient due to a non-zero initial condition

If $x(0) \neq 0$ (ie $e(0) = \hat{x}(0) - x(0) \neq 0$) then transients occur.

However if $e(0) = 0$ and $x(0) = 0$ then the estimator disappears.



If estimator evs are the same as state feedback evs then the transients will have a lot more effect on the response and

For a step response

$y_{ss} = \lim_{s \to 0} s \cdot G(s) \cdot \frac{1}{s} = G(0) = C(-A+BK)^{-1}B$
$= \text{some number}$

Continued next lecture

Monday 23rd lecture cancelled

-Now, what is the steady state response of the output to a step input?

If V_{ss} = step then $e_{ss} = \lim_{s\to 0} s(Y_{ss} - V_{ss}) = \lim_{s\to 0} s\left[\frac{G(s)}{s} - \frac{1}{s}\right] = G(0) - 1$

$= C(-(A-BK))^{-1}B - 1$

-Note that there can be a huge steady state error!

-One can add an integrator We "augment" the ss eqn

-Consider $\dot{x} = Ax + Bu + Ed$ where d is a disturbance

$y = Cx$ (*)

-Assume there are as many inputs as outputs

-Consider $\underbrace{y_{ref}}_{\text{Constant}}$ and look at $y - y_{ref} = Cx - y_{ref}$

-Design objective:

1. $\lim_{t\to\infty} \dot{x} = 0$ (stability) Note: $x(t)$ must be nonzero if $y = y_{ref}$

2. $\lim_{t\to\infty} (y - y_{ref}) = 0$ (regulator)

-consider d and y_{ref} constant

-differentiate (*)

$\ddot{x} = A\dot{x} + B\dot{u}$ $\dot{y} - \dot{y}_{ref} = C\dot{x}$

-Let $z = \begin{bmatrix} \dot{x} \\ y - y_{ref} \end{bmatrix}$ and $v = \dot{u}$

$\therefore \dot{z} = \begin{bmatrix} \ddot{x} \\ \dot{y} - \dot{y}_{ref} \end{bmatrix} = \begin{bmatrix} A\dot{x} + B\dot{u} \\ C\dot{x} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} v$

-Check to see that this new system is controllable. ie $\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$ has rank n

-Then let $v = -Fz$ and stabilize this system about z=0

-What is the equilibrium point?

If $z = 0$, $\dot{x} = 0$, and $y = y_{ref}$ ie. our design objectives are met!

-Thus, the design equations are met.

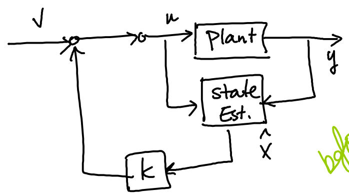
-Now, integrate v to get u and let $F = [F_1 \quad F_2]$

-Thus, the feedback becomes

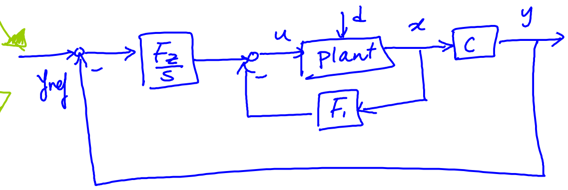
$$V = -Fz = -[F_1 \quad F_2]\begin{bmatrix} \dot{x} \\ y - y_{ref} \end{bmatrix} = -F_1\dot{x} - F_2[y - y_{ref}]$$

$$u = \underbrace{-F_1 x}_{\text{state feedback}} \underbrace{- F_2 \int_0^t (y - y_{ref})\,dt}_{\text{integral control}}$$

-The block diagram is



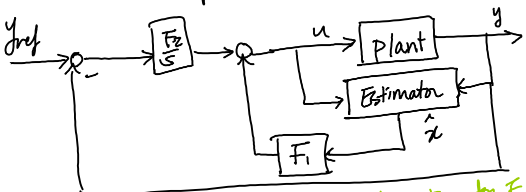
before



-This improves steady state error but can increase transients

-One can show that the separation principle holds here

When we superimpose the block diagrams, And the separation principle still holds!



state estimator F. NOT F in

$$\text{The ev are ev of } (A - FC) \text{ (state estimator) } \cup \text{ the ev of } \underbrace{\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} F}_{\text{Augmented system}} \qquad F \Rightarrow [F_1 \quad F_2]$$

In the project, you need to make the system square

Controllability, Observability and Minimal Realizations

-Recall that, for the multivariable realization, the state space equations you get can be very large in the size of the state

-Note: how to check if state space realizations give the same transfer function

They may not be the same dimension
Even if they are, they're related by $x = T\bar{x}$

$$C(sI-A)^{-1}B + D \overset{?}{=} C'(sI-A')^{-1}B' + D'$$

Note: $(sI-A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$

Pf/ Look at the inverse Laplace and infinite exp. of e^{At}

~~-Eg/~~ $$D + \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \cdots \overset{?}{=} D' + \frac{C'B'}{s} + \frac{C'A'B'}{s^2} + \cdots$$

$\therefore$ to show equality $D = D'$, $CB = C'B'$ $CAB = C'A'B'$

How far do I need to go?
use Cayley Hamilton's theorem so that we don't need to check $CA^{100}B$ etc.
Look at the largest dimension, say n, and check to $CA^{n-1}B = C'A'^{n-1}B'$

since this is controllable & obs. this is irreducible.

Eg. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u$, $y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x$ 2 input 2 output

The transfer function (use ss2tf in Matlab) or

$$C(sI-A)^{-1}B + D \Rightarrow G(s) = \begin{bmatrix} \frac{2s+4}{s^2+3s+2} & \frac{2s+4}{s^2+3s+2} \\ \frac{s+1}{s^2+3s+2} & 0 \end{bmatrix}$$

Here $A \& B$ (A,B) is controllable and A and C (A,C) is observable.
Because $Q = [AB\ B]$, $R = \begin{bmatrix} C \\ CA \end{bmatrix}$ $\underline{\text{rank} = 2}$.

We want a s.s realization of $G(s)$. From notes, we need a realization of each element. Let's use the controllable canonical form.

G_{11}: $A_{11} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C_{11} = [4\ \ 2]$

G_{12}: $A_{12} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C_{12} = [4\ \ 2]$

[illegible]

G_{11}: $A_{11}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$ $B_{11}=\begin{bmatrix}0\\1\end{bmatrix}$ $C_{11}=[4\ \ 2]$

G_{12}: $A_{12}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$ $B_{12}=\begin{bmatrix}0\\1\end{bmatrix}$ $C_{12}=[4\ \ 2]$

G_{21}: $A_{21}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$ $B_{21}=\begin{bmatrix}0\\1\end{bmatrix}$ $C_{21}=[1\ \ 1]$

G_{22}: $A_{22}=0$, $C_{22}=0$, $B_{22}=1$

From notes $A=\begin{bmatrix}A_{11} & & & \\ & A_{21} & & \\ & & A_{12} & \\ & & & A_{22}\end{bmatrix}$ etc

$A^* \quad A=\begin{bmatrix}0 & 1 & & & & & \\ -2 & -3 & & & & & \\ & & 0 & 1 & & & \\ & & -2 & -3 & & & \\ & & & & 0 & 1 & \\ & & & & -2 & -3 & \\ & & & & & & 0\end{bmatrix}$ $B^* \quad B=\begin{bmatrix}0 & \\ 1 & \\ 0 & \\ 1 & \\ & 0\\ & 1\\ & 1\end{bmatrix}$ $C^* \quad C=\begin{bmatrix}4 & 2 & & & 4 & 2 & \\ & & 1 & 1 & & & 0\end{bmatrix}$

This is 7^{th} order.

To check that the calculated ABC give the same TF, check that D^*, $\hat{C}\hat{B}^*$, $\hat{C}\hat{A}\hat{B}^*$ etc.

Look at the cont. matrix Q. Rank Q = 5.

∴ Not controllable.

Observe that obs. matrix P has Rank P = 5.

∴ Not observable

Note: If you system is not controllable/observable then that means mathematically you have inputs that cannot physically affect the output. (Time for redesign of mechanical & electronics)

-As you can see, although the original realization is minimal, the realization we get from the transfer function is not!

-Definition- A realization of $G(s)$ is irreducible or *minimal* iff the associated state space has the smallest possible dimension

-Fact- A state space representation is minimal iff it is controllable and observable
Pf/ Very difficult and messy

- Given a realization, is it possible to isolate the controllable and observable part of the state space equation which gives us the same transfer function? If the unobservable and uncontrollable part can be "removed" and still give us the same transfer function, then this state space realization will be minimal.

-this is the *minreal* command in Matlab

-Theorem- Given a state space realizations an equivalent system (ie $x = T\bar{x}$) system of the form

$$\bar{A} = T^{-1}AT = \begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ 0 & \bar{A}_{22}\end{bmatrix} \qquad \bar{B} = T^{-1}B = \begin{bmatrix}\bar{B}_1\\ 0\end{bmatrix}$$

$$\bar{C} = CT = [\bar{C}_1 \ \ \bar{C}_2] \qquad \bar{D} = D$$

where the system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ is controllable and has the same transfer function as

Observe that obs. matrix P has Rank P = 3.

$\therefore$ Not observable

Note: If you system is not controllable / observable then that means mathematically you have inputs that cannot physically affect the output. (Time for redesign of mechanical & electronics)

-As you can see, although the original realization is minimal, the realization we get from the transfer function is not!

-Definition- A realization of $G(s)$ is irreducible or *minimal* iff the associated state space has the smallest possible dimension

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-this is the *minreal* command in Matlab

-Theorem- Given a state space realizations an equivalent system (ie $x = T\bar{x}$) system of the form

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = T^{-1}B = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CT = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \qquad \bar{D} = D$$

where the system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ is controllable and has the same transfer function as $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$

-Algorithm (This is actually how the proof is constructed)
1. Find the Controllability matrix and calculate the rank q
2. Pick any q linearly independent columns of the Controllability matrix
3. Pick any n-q columns that are linearly independent to the columns in 2. and to each other.
4. Form a matrix T where the first q columns are those in 2. and the rest are the those from 3. This matrix T is nonsingular

Eg/ Continue example.
rank $Q = 5$
Col 1–4 and 6 are linearly independent
Choose randomly $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. T is nonsingular

0s!

$$A = \left[\begin{array}{ccccc|cc} 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ \hline & & & & & -3 & -2 \\ & & 0 & & & 1 & 0 \end{array}\right] \quad A$$

coincidence

$$C = \left[\begin{array}{ccccc|cc} 2 & 2 & -2 & -2 & 2 & 2 & 4 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 \end{array}\right] \quad C$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B$$

-Theorem- Given a state space realizations an equivalent system (ie $\bar{x} = Tx$) system of the form

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = TB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$$

$$\bar{C} = CT^{-1} = [\bar{C}_1 \quad 0] \qquad \bar{D} = D$$

where the system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ is observable and has the same transfer function as $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$

-Algorithm (This is actually how the proof is constructed)
1. Find the Observability matrix and calculate the rank q
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This matrix T is nonsingular

Eg/

-To reduce the system to its minimal realization, we apply these two theorems in succession

Optimal Control

2009년 3월 25일 수요일
오전 11:32

- Introduction
- We need other techniques
- placing poles is tricky
  i) difficult to predict gains
  ii) difficult to predict performance
- General Case
  a) System to be controlled
  $\dot{x} = f(x, u)$ non linear
  b) System constraints
    - The input u may have a bound
    - The state may not exceed certain values
  c) Description of task
    - Initial states $x(t_0) = x_0$
    - You may have final states $x(t_f) = x_f$
  d) Performance Criteria (Cost Function)
    - We try to minimize or maximize some quantity, J, in the system
    eg. $J = t_f - t_0$ ← minimum time prob
    $J = \int_{t_0}^{t_f} u^T u \, dt$ ← minimum effort (fuel consumption)

Problem statement:

The optimal control problem is to find an input u^* which minimizes (or maximizes) the cost function J, subject to any constraints, to do the task.

- Use calculus of variations!
- Simple problem

$$\underbrace{\dot{x} = Ax + Bu \qquad x(0) = x_0 \qquad y = Cx}_{\text{Linear}}$$

Find u^* such that the system is driven in t_f seconds from $x(0) = x_0$ to $x(t_f) = x_f$ such that... the cost function quadratic $\left\{ J = \frac{1}{2} x_f^T S x_f + \frac{1}{2}\int_0^{t_f} (x^T Q x + u^T R u)\, d\tau \right.$

something we choose

is minimized.

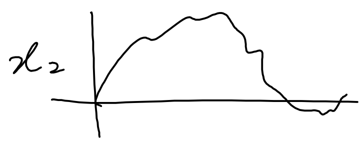
S, Q are positive semi-definite (PSD)
R is positive definite (PD)
Assume S, Q, R are symmetric

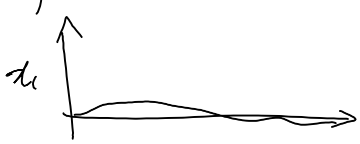
P.S.D $\quad x^T S x \geq 0$ for $x \neq 0$
P.D $\quad x^T R x > 0$ for $x \neq 0$

If S, Q and R are diagonal then
P.S.D if all the elements are ≥ 0
P.D " all the elements are > 0

- This is called an LQR (linear quadratic regulator)
- This problem has a nice solution for $u^*(t)$
- Ignore "S" $\quad J = \frac{1}{2}\int_0^{t_f} (x^T Q x + u^T R u)\, d\tau$

Q, R are diagonal





$$Q = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \qquad Q_1, Q_2 \geq 0 \qquad R = R_1$$

$$J = \frac{1}{2}\int_0^{t_f} (x_1^2 Q_1 + x_2^2 Q_2 + u^2 R_1)\, d\tau$$

$Q_2 \uparrow \; x_2 \downarrow \qquad R_1 \uparrow \; u \downarrow$

Q and R are our new tuning parameters.

- After 'LOTS' of math

$$u^*(t) = -K(t)\, x$$

where $K(t) = -R^{-1} B^T P(t)$

$$-\dot{P}(t) = P(t)A + A^T P(t) - P(t) B R^{-1} B^T P(t) + Q,$$
$$P(t_f) = S$$
Riccati's eqn

we know these because they're our design parameters

Nonlinear differential equation solved backwards in time

- Simplification

$t_f \to \infty \qquad x(t_f) \to 0$

looks identical to state feedback

const. matrix

The solution becomes $\boxed{u = -Kx}$, $K = -R^{-1}B^T P$
where P is a symmetric matrix satisfying algebraic equation:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

Algebraic Riccati equation

Now our cost function is:

$$J = \int_0^\infty (x^T Q x + u^T R u)\, d\tau$$

Fact. If (A, B) is stabilizable and (Q, A) is detectable then the algebraic Riccati eqn has a unique positive definite sol'n P.

- The optimal feedback is $u = -Kx$ and $A - BK$ is stable.
- In Matlab the sol'n to algebraic Riccati eqn is the "ARE" command.

$A, B, Q, R \Rightarrow P$.

- LQR command

gives state feedback!