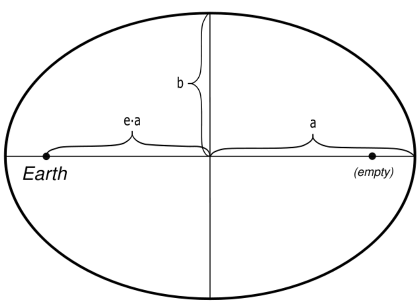
A Very Incomplete Orbital Elements Tutorial

The Keplerian orbital elements for an object are the six parameters that uniquely specify the size, shape and orientation of an orbit and the position of a object along that orbit. The six Keplerian orbital elements are:

a - Semi-major axis	Ω - Longitude of the ascending node
ecc - Eccentricity	ω - Argument of the perihelion
i - Inclination	**M** - Mean anomaly at epoch

Kepler's first law says: *The orbit of every planet is an ellipse with the sun at one focus.* The Semimajor axis (**a**) and the eccentricity (**ecc**) parameterize the size and shape of the ellipse. The units of **a** in our dataset are Astronomical Units (AU), the average distance between the Sun and the Earth.

For a closed elliptical orbit (orbits gravitationally bound to the Sun), $ecc = \sqrt{1 - b^2/a^2}$, where **a** and **b** are the semi-major and semi-minor axes (see drawing below). As you can see from the equation when a = b, ecc = 0 (a circle), and when a >> b, ecc approaches 1. When ecc = 1, the orbit is a parabolic orbit (just bound). When ecc > 1 the orbit is a hyperbolic orbit (unbound).

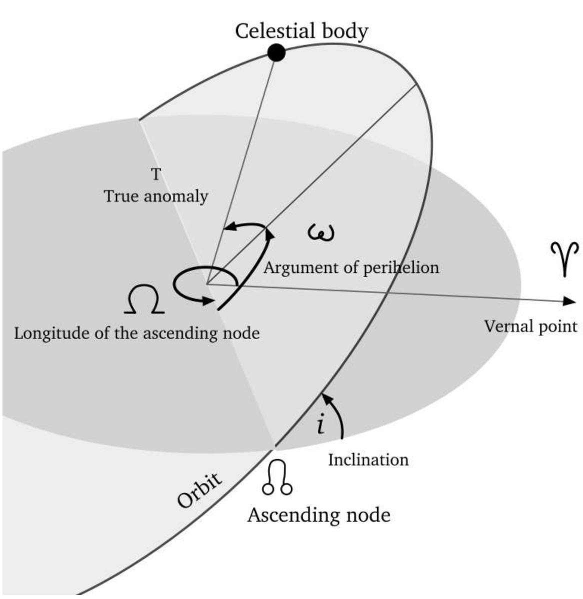


The point in the object's orbit where it is closest to the Sun is called the **perihelion** (periapsis). The farthest point in the orbit is called the **aphelion** (apoapsis). Inspecting the diagram at the right, you can see:

perihelion distance $= a(1 - ecc)$

aphelion distance $= a(1 + ecc)$

The next three elements fix the orientation of the orbit in space (see the diagram below). These three elements are also know as the *Euler angles.* In order to do this we need to define a frame of reference for our coordinate system. For our solar system, the most common frame of reference is to make the X-Y plane correspond to the plane of the ecliptic (the geometric plane containing the mean orbit of the Earth around the Sun). The +X axis points in the direction of the Vernal equinox (the position of the Sun on the first day of spring). It is a "right-handed" coordinate system with the +Z axis pointing in the "northern" direction as seen from the Earth.



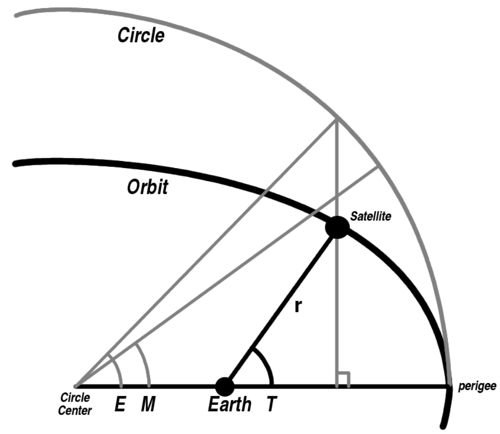
The longitude of the ascending node (Ω, $0° < \Omega < 360°$) orients the ascending node with respect to the vernal point. The ascending node is where the orbit crosses the X-Y plane moving in the +Z direction.

The inclination (**i**, $0° < i < 180°$) orients the orbital plane with respect to the X-Y plane. The equatorial plane is where $i = 0°$.

The argument of perihelion (ω, $0° < \omega < 360°$) orients the semi-major axis with respect to the ascending node.

The Mean anomaly at epoch (**M**) is the fraction of the orbital period that has elapsed since the last perihelion. Think of it as a way to indicate the position of an object along its orbit at a specific time (epoch). In the diagram below, the Mean anomaly is the angle (**M**). The mean anomaly is used to calculate two other angles that have good geometrical meaning: the *eccentric anomaly* (**E**) and the *true anomaly* (**T**). You can see from the drawing that a few easy relationship hold:

For circular orbits ($ecc = 0$): $\mathbf{M} \equiv \mathbf{E} \equiv \mathbf{T}$
For all orbits: At Perihelion, $\mathbf{M} = \mathbf{E} = \mathbf{T} = 0^\circ$ (360°)
For all orbits: At Aphelion, $\mathbf{M} = \mathbf{E} = \mathbf{T} = 180^\circ$



As the names implies, the Mean anomaly at epoch specifies the position of an object *at a specific time.* If we want to know the Mean anomaly at a different time we can find that by the equation:

$$M(t) = M_0 + n(t - t_0)$$

where M_0 is the mean anomaly at time t_0, and n is the mean motion. The mean motion can be thought of as the average daily motion of the object along its orbit:

$$n = \sqrt{\frac{G\mathcal{M}_\odot}{a^3}} \qquad \text{For } \mathbf{a} \text{ in [AU] this becomes :} \qquad n = 0.98560\ a^{-3/2} \left[\frac{degrees}{day}\right]$$

The Eccentric anomaly (E) and the True anomaly (T) are derived from the Mean anomaly (M) and the eccentricity (ecc).

$$E(ecc, M(t)) \qquad\qquad T(ecc, E)$$

The orbital position vector (**r**) is the vector connecting the center-of-mass and the object in orbit (see the drawing above) and is related to the eccentric anomaly (E):

$$r = a\,(1 - ecc \cdot \cos E)$$

With the orbital angles (i, Ω, ω) and the True anomaly (T) we can calculate the position (**X,Y,Z**) of the object in the three-dimensional Cartesian coordinate system, with the origin at the Sun:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \mathbf{r} \cdot \begin{pmatrix} cos(\Omega)\, cos(\omega + T) - sin(\Omega)\, sin(\omega + T)\, cos(i) \\ sin(\Omega)\, cos(\omega + T) + cos(\Omega)\, sin(\omega + T)\, cos(i) \\ sin(\omega + T)\, sin(i) \end{pmatrix}$$

Remember the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the three-dimensional Cartesian coordinate system can be found by:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$