The corresponding formula for dy is

$$dy = \cos u du. \tag{3}$$

It has thus been shown that

$$\frac{\mathbf{d(\sin u)}}{\mathbf{dx}} = \mathbf{\cos u} \frac{\mathbf{du}}{\mathbf{dx}} \tag{4}$$

and

$$\mathbf{d(\sin u) = \cos u du.}$$

Well known properties of the function $y = \sin u$ can be verified by formula (1). Thus $\sin u$ is an increasing function between $u = 0$ and $u = \frac{\pi}{2}$, and between $u = \frac{3\pi}{2}$ and $u = 2\pi$, and decreasing between $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$. The same facts are shown by

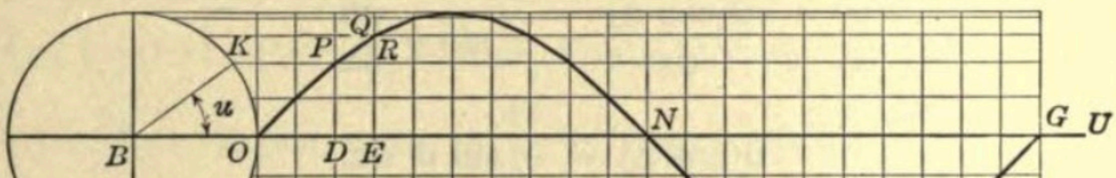

Fig. 53.

the derivative, $\cos u$, which is positive between $u = 0$ and $u = \frac{\pi}{2}$, and between $u = \frac{3\pi}{2}$ and $u = 2\pi$, and negative between $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$. Further, $\sin u$ has maximum and minimum values for $u = \frac{\pi}{2}$ and $u = \frac{3\pi}{2}$, respectively. The same facts are shown by the derivative, $\cos u$, which becomes zero at these points and changes sign at $\frac{\pi}{2}$ from plus to minus, and at $\frac{3\pi}{2}$ from minus to plus.

The slope of the sine curve is approximately the slope of the diagonal PQ of a rectangle in Fig. 53. The greater the number of equal parts into which the circumference of the circle is divided and hence the smaller the subdivisions of the arc, the closer do the slopes of these diagonals approach the slopes of the tangents.

the product of a constant by the cosine of a variable angle, the maximum and minimum values can be found at once. Thus

$$a \cos x + b \sin x = \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{a^2+b^2}} \cos x + \frac{b}{\sqrt{a^2+b^2}} \sin x\right]$$

Now, $\frac{a}{\sqrt{a^2+b^2}}$ and $\frac{b}{\sqrt{a^2+b^2}}$ may be regarded as the cosine and sine, respectively, of an angle α. For if P, Fig. 62, be the point (a, b) and the angle POX be α,

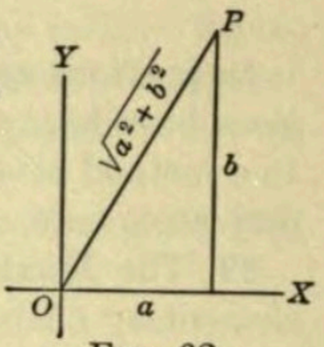

Fig. 62.

$$\cos \alpha = \frac{a}{\sqrt{a^2+b^2}},$$

and

$$\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

Hence

$$a \cos x + b \sin x = \sqrt{a^2+b^2}\,(\cos x \cos \alpha + \sin x \sin \alpha),$$

or

$$a \cos x + b \sin x = \sqrt{a^2+b^2} \cos (x - \alpha). \qquad (1)$$

The quadrant in which α lies will be determined by the signs of a and b.

α is in the first quadrant if a is positive and b is positive.
α is in the second quadrant if a is negative and b is positive.
α is in the third quadrant if a is negative and b is negative.
α is in the fourth quadrant if a is positive and b is negative.

In polar coördinates equation (1) shows that the function $a \cos x + b \sin x$ is represented by a circle passing through the pole, of diameter $\sqrt{a^2+b^2}$, and with its center on the line making an angle α with the polar axis.

The right-hand side of equation (1) shows that the function is represented graphically in rectangular coördinates by a cosine curve of amplitude $\sqrt{a^2+b^2}$. Thus, the maximum value of $a \cos x + b \sin x$ is $\sqrt{a^2+b^2}$ and occurs when $x = \alpha$. The minimum value of the function is $-\sqrt{a^2+b^2}$ and occurs when $x = \alpha + \pi$.

Two examples giving rise to this function are solved below.