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# On the convergence of eigenvalues for mixed for- mulations

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Oxford  
Mathematics

All code to reproduce what  
shown in this slides can be found  
in the following git repository:



<https://github.com/UZerbinati/BrezziBoffiGastaldi>

# On the convergence of eigenvalues for mixed formulations

## Mixed Eigenvalue Problem

- ▶ Solved Bernstein's minimal surface problem together with Enrico Bombieri.
- ▶ Solved 19th Hilbert problem, regarding the regularity of elliptic PDE.
- ▶ He is the father of modern calculus of variation and he is responsible for introducing the notion of  $\Gamma$ -convergence.



**Ennio De Giorgi**  
1928–1996

# The Abstract Problem

## Mixed Eigenvalue Problem

- ▶ Two Hilbert spaces  $\Phi$  and  $\Xi$  are considered.
- ▶ Two **continuous** bilinear forms are also considered,

$$a(\cdot, \cdot) : \Phi \times \Phi \rightarrow \mathbb{R}, \quad b(\cdot, \cdot) : \Phi \times \Xi \rightarrow \mathbb{R}.$$

$$A : \Phi \rightarrow \Phi^*, \quad (A\phi) : \Phi \rightarrow \mathbb{R}, \quad (A\phi)\varphi \mapsto a(\phi, \varphi),$$
$$B : \Phi \rightarrow \Xi^*, \quad (B\phi) : \Xi \rightarrow \mathbb{R}, \quad (B\phi)\xi \mapsto b(\phi, \xi),$$

- ▶ We assume  $a(\cdot, \cdot)$  is **symmetric** and **positive semidefinite**.

# The Abstract Problem

## Mixed Eigenvalue Problem

*Given any  $(f, g) \in \Phi^* \times \Xi^*$  find  $(\psi, \chi) \in \Phi \times \Xi$  such that*

$$\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = \langle f, \varphi \rangle & \forall \varphi \in \Phi \\ b(\psi, \xi) = \langle g, \xi \rangle & \forall \xi \in \Xi \end{cases} \quad (1)$$

*Given any  $(f, g) \in \Phi^* \times \Xi^*$  find  $(\psi, \chi) \in \Phi \times \Xi$  such that*

$$\begin{cases} A\psi + B^*\chi = f \\ B\psi = g \end{cases}$$

### Banach Closed Range **Theorem**

Given a closed linear operator  $B : \Phi \rightarrow \Xi^*$ , the following statements are equivalent:

- ▶  $Range(B)$  is closed in  $\Xi^*$ ,
- ▶  $Range(B) = \left[ Ker(B^T) \right]^0$ , where  $Z^0$  is the the **polar** set of  $Z$ ,  
i.e.  $Z^0 = \left\{ f \in Z^* \text{ s.t } \langle f, z \rangle = 0 \ \forall z \in Z \right\}$ .
- ▶ it exists  $L_B \in \mathcal{L}\left( Range(B), Ker(B)^\perp \right)$  and  $\beta \geq 0$  such that:

$$\beta \|L_B g\|_\Phi \leq \|g\|_{\Xi^*}, \ \forall g \in Range(B).$$

# The Inf-Sup Stability

## A Gentle Introduction to **Brezzi's** Theory

### Brezzi's Theorem

Assuming that the range of  $B$  is  $\Xi^*$  and that  $a(\cdot, \cdot)$  is coercive in the kernel of  $B$ , then it exists one and only one solution to (1).

**How do can one verify that  $B$  is a onto ?**

### Inf-Sup Condition

The operator  $B$  is surjective if and only if it exists  $\beta > 0$  such that

$$\inf_{\xi \in \Xi} \sup_{\varphi \in \Phi} \frac{b(\varphi, \xi)}{\|\varphi\|_{\Phi} \|\xi\|_{\Xi}} \geq \beta$$

# The Discrete Case

## Inf-Sup Stable Finite Element Pairs

- ▶ We introduce two discrete spaces  $\Phi_h \subset \Phi$  and  $\Xi_h \subset \Xi$ .
- ▶ Previous result holds also for the discrete problem,

*Given any  $(f, g) \in \Phi^* \times \Xi^*$  find  $(\psi_h, \chi_h) \in \Phi_h \times \Xi_h$  such that*

$$\begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = \langle f, \varphi_h \rangle & \forall \varphi \in \Phi_h \\ b(\psi_h, \xi_h) = \langle g, \xi_h \rangle & \forall \xi \in \Xi_h \end{cases} \quad (2)$$

- ▶ The pair  $(\Phi_h, \Xi_h)$  is **inf-sup stable** if it exists  $\beta_h$  independent from  $h$  such that:

$$\inf_{\xi_h \in \Xi_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \xi_h)}{\|\varphi_h\|_{\Phi} \|\xi_h\|_{\Xi}} \geq \beta_h$$



We introduce the **solution operator**, i.e.

$$\begin{aligned} S : \Phi^* \times \Xi^* &\rightarrow \Phi \times \Xi \text{ s.t. } S(f, g) = (\psi, \chi) \text{ as in (1),} \\ S_h : \Phi_h^* \times \Xi_h^* &\rightarrow \Phi_h \times \Xi \text{ s.t. } S_h(f, g) = (\psi_h, \chi_h) \text{ as in (2).} \end{aligned}$$

## Proposition Necessity of the Inf-sup

If it exists a constant  $C > 0$  such that for all  $(f, g) \in \Phi^* \times \Xi^*$  and for all  $h > 0$

$$\|S_h(f, g)\|_{\Phi \times \Xi} \leq C \left( \|f\|_{\Phi_h^*} + \|g\|_{\Xi_h^*} \right) \quad (3)$$

then the bilinear form  $a(\cdot, \cdot)$  is elliptic in the kernel of  $B$  and the pair  $(\Phi_h, \Xi_h)$  is **inf-sup** stable.

# On the Necessity of the Inf-Sup For the Source Problem

When dealing with an mixed problem such that  $a(\cdot, \cdot)$  is elliptic in the kernel the **inf-sup stability** condition is not only **sufficient** it is also **necessary**.

## Remark

When proving the existence and uniqueness of solution for 1 and 2, hypothesis can be weaken, i.e. we can require  $A : Ker(B) \rightarrow Ker(B)^*$  to be an isomorphism.

We are now ready to introduce an eigenvalue problem. Given an Hilbert space  $H$  and a selfadjoint compact operator  $T : H \rightarrow H$ , we call eigenvalue of  $T$  the  $\lambda \in \mathbb{R}$  such that

$$\lambda Tu = u \text{ with } u \in H \setminus \{0\}.$$

In particular it is well known that for the above described operator  $T$  it exists a sequence  $\{\lambda_i\}_{i \in \mathbb{N}}$  such that

$$\begin{aligned} \lambda_i Tu_i &= u_i, \\ \lim_{i \rightarrow \infty} \lambda_i &= +\infty \text{ and } \lambda_i \geq 0 \quad \forall i \in \mathbb{N}. \end{aligned} \tag{4}$$

We now consider for all  $h > 0$  the selfadjoint non negative operator  $T_h : H \rightarrow H$ , with finite range  $H_h$ . Let's denote  $N(h)$  is the dimension of  $H_h$ . We are interested in the eigenvalues,

$$\lambda^h T_h u^h = u^h \text{ with } u^h \in H_h \setminus \{0\}.$$

The same characterization of the eigenvalue presented above holds also in the discrete case.

If we assume that the discrete approximation operator  $T_h$  converges to  $T$  with respect to the norm of  $\mathcal{L}(H, H)$ , i.e.

$$\lim_{h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(H, H)} = 0$$

then  $\forall \varepsilon > 0$  and  $\forall n \in \mathbb{N}$  it exists  $h_0 > 0$  such that  $\forall h > h_0$

$$\max_{i=1, \dots, m(N)} |\lambda_i - \lambda_i^h| \leq \varepsilon, \quad (5)$$

$$\delta\left(\oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h\right) \leq \varepsilon. \quad (6)$$

$m(N)$  is the number of eigenvalues corresponding to  $N$  distinct ones,  $E_i = \langle u_i \rangle$  and  $E_i^h = \langle u_i^h \rangle$ . The converse also holds true.

We consider two Hilbert space  $H_\Phi$  and  $H_\Xi$  such that we can identify  $H_\Phi$  with  $H_\Phi^*$ ,  $H_\Xi$  with  $H_\Xi^*$  and

$$\Phi \subset H_\Phi \subset \Phi^*, \quad \Xi \subset H_\Xi \subset \Xi^*.$$

### Proposition Convergence of Discrete Eigenvalue Problem

Assuming that  $a(\cdot, \cdot)$  is elliptic in the kernel of  $B_h$  and the discrete inf-sup condition holds then  $S_h$  converges in  $\mathcal{L}(H_\Phi, H_\Xi)$  to  $S$  if and only if  $S : H_\Phi \times H_\Xi \rightarrow H_\Phi \times H_\Xi$  is compact. The converse holds true.

# The Story Doesn't End Here

## Motivating Example – Stokes Eigenvalue Problem with Q1-P0



```
msh = UnitSquareMesh(10,10,quadrilateral=True)
V = VectorFunctionSpace(msh, "Q", 1)
Q = FunctionSpace(msh, "DG", 0)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
     + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```

# The Story Doesn't End Here

## Motivating Example – Stokes Eigenvalue Problem Q1-P0



```
A = assemble (a)
M = assemble (m)
Asc, Msc = A.M.handle, M.M.handle
E = SLEPc.EPS().create()
E.setType(SLEPc.EPS.Type.ARNOLDI)
E.setProblemType(SLEPc.EPS.ProblemType.GHEP);
E.setOperators(Asc, Msc)
PC = ST.getKSP().getPC();
PC.setType("svd");
E.setST(ST);
E.solve();
```



# The Story Doesn't End Here

## Motivating Example – Stokes Eigenvalue Problem Q1-P0



N	Reference	Q1-P0
1	52.34468	53.56885
2	92.12438	97.57386
3	92.12438	97.57386
4	128.209	97.573867

- ▶ The numerical experiment for the Q1-P0 are obtained using a  $10 \times 10$  uniform square grid, while the reference value are obtained using Hood-Taylor finite element pair on a  $20 \times 20$  square mesh.
- ▶ As  $h \rightarrow 0$  we would see a degraded rate of convergence.

More often the note in practice when we are interested in eigenvalue problem where either  $f$  or  $g$  is zero. For example we call problem of type  $(f \neq 0)$ ,

$$\begin{aligned} &\text{Find } (\psi, \chi) \in \Phi \times \Xi \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ &\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = \lambda \langle \psi, \varphi \rangle & \forall \varphi \in \Phi, \\ b(\psi, \xi) = 0 & \forall \xi \in \Xi. \end{cases} \end{aligned} \quad (7)$$

Which **can not** be cast as an eigenvalue problem of the form of (1).

## Two Type Of Problems

Problem of Type  $(f \ 0)$  and  $(0 \ g)$  Type.

To recast (7) as an eigenvalue problem we need to introduce

$$C_{\Phi} : \Phi^* \rightarrow \Phi^* \times \Xi^*$$

$$f \mapsto (f, 0)$$

$$C_{\Phi}^* : \Phi \times \Xi \rightarrow \Phi$$

$$(\varphi, \xi) \mapsto \varphi$$

then we can study the eigenvalue problem corresponding to

$$T_{\Phi} := C_{\Phi}^* \circ S \circ C_{\Phi} : \Phi^* \rightarrow \Phi \quad (8)$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (7) ?

## Proposition Existence of Solutions

If  $a(\cdot, \cdot)$  is elliptic in the kernel of  $B_h$ , then problem (7) admits at least one solution  $(\psi_h, \chi_h)$ . Moreover  $\psi_h$  is uniquely determined by  $f$  and

$$\|\psi_h\|_{\Phi} \leq C \|f\|_{\Phi_h^*}.$$

Furthermore if it exists  $C > 0$  such that for every  $h > 0$  and for every  $(\psi_h, \chi_h, f) \in \Phi_h \times \Xi_h \times \Phi^*$  the above inequality is verified then the operator  $T_{\Phi}^h$  is defined for all element in  $\Phi$  and  $a$  elliptic in the kernel of  $B_h$ .

## Definition Weak Approximability

Let  $\Xi_0^H$  be the range of  $C_{\Xi}^* \circ S \circ C_{\Phi}$ . We say that  $\Xi_0^H$  verifies the **weak approximability** if for every  $\chi \in \Xi_0^H$

$$\sup_{\varphi_h \in \text{Ker}(B_h)} \frac{b(\varphi_h, \chi)}{\|\varphi\|_{\Phi}} \leq \omega_1(h) \|\chi\|_{\Xi_0^H}, \quad \lim_{h \rightarrow 0} \omega_1(h) = 0.$$

## Remark

The above definition is an approximability condition in fact using the fact that  $b(\varphi_h, \chi') = 0$  for all  $\chi' \in \Xi_h$  to rewrite the weak approximability as for all  $\chi \in \Xi_0^H$   $\inf_{\chi' \in \Xi_h} \|\chi - \chi'\|_{\Xi} \leq \omega_1(h) \|\chi\|_{\Xi_0^H}$ .

### Definition Strong Approximability

Let  $\Phi_0^H$  be the range of  $C_\Phi^* \circ S \circ C_\Phi$ . We say that  $\Phi_0^H$  verifies the **strong approximability** if for every  $\psi \in \Phi_0^H$

$$\inf_{\psi' \in \text{Ker}(B_h)} \|\psi - \psi'\|_\Phi \leq \omega_2(h) \|\psi\|_{\Phi_0^H}, \quad \lim_{h \rightarrow 0} \omega_2(h) = 0.$$

### Proposition Convergence

If  $a(\cdot, \cdot)$  is elliptic in the kernel of  $B_h$  and the weak approximability of  $\Xi_0^H$  and strong approximability of  $\Phi_0^H$  are verified, then for all  $f \in H_\Phi$

$$\left\| T_\Phi f - T_\Phi^h f \right\|_\Phi \leq \omega_3(h), \quad \lim_{h \rightarrow 0} \omega_3(h) = 0. \quad (9)$$

Vice versa if the sequence  $T_\Phi^h$  is bounded in  $\mathcal{L}(\Phi^*, \Phi)$  and converges uniformly to  $T_\Phi$  in  $\mathcal{L}(\Phi^*, \Phi)$  then  $a(\cdot, \cdot)$  is elliptic in the kernel of  $B_h$ , moreover the strong and weak approximability conditions are verified respectively for  $\Phi_0^H$  and  $\Xi_0^H$ .

## An additional Example

### A Connection with Charlie's presentation



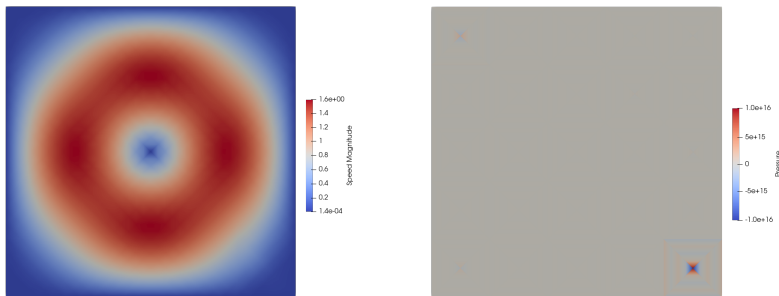
We solve the Stokes eigenvalue problem using Scott-Vogelius(ish) finite element pair and criss-cross. mesh.

```
msh = UnitSquareMesh(5,5,diagonal="crossed")
V = VectorFunctionSpace(msh, "CG", 4)
Q = FunctionSpace(msh, "DG", 3)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
    + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```



# An additional Example

## A Connection with Charlie's presentation



**Figure:** On the LHS the first mode of the Stokes eigenvalue problem, on the RHS the corresponding pressure computed with a SV(ish) method.

## Numerical Example

### The Mixed Laplacian Eigenvalue Problem – Q1-P0



```
msh = SquareMesh(64,64,np.pi,quadrilateral=True)

S = VectorFunctionSpace(msh, "Q", 1)
V = FunctionSpace(msh, "DG", 0)
X = S*V
s,u = TrialFunctions(X); t,v = TestFunctions(X)

a = (inner(s,t)+inner(div(t),u)+inner(div(s),v))*dx
m = -inner(u,v)*dx

bc = DirichletBC (X.sub(0), as_vector([0.0,0.0]),
                  [1,2,3,4])
A = assemble (a,bcs=bc)
M = assemble (m)
```

# Numerical Example

## The Mixed Laplacian Eigenvalue Problem – Q1-P0



N	Ref	Q1-P0
1	1	0.99111
2	1	0.99111
3	2	1.96374
4	4	3.96531
5	4	3.96669
6	5	4.91019
7	5	4.91019
8	8	7.85488
9	9	8.73856
10	9	8.73856
11	10	8.92934

## Two Type Of Problems

### Problem of Type $(0 \quad g)$

What happens when  $f$  is null rather than  $g$  ?

We call problem of this type  $(0 \quad g)$ ,

*Find  $(\psi, \chi) \in \Phi \times \Xi$  and  $\lambda \in \mathbb{R}$  such that*

$$\begin{cases} a(\psi, \varphi) + b(\varphi, \chi) = 0 & \forall \varphi \in \Phi, \\ b(\psi, \xi) = -\lambda \langle \chi, \xi \rangle & \forall \xi \in \Xi. \end{cases} \quad (10)$$

Which once again **can not** be cast as an eigenvalue problem of the form of (1).

## Two Type Of Problems

Problem of Type  $(f \ 0)$  and  $(0 \ g)$  Type.

To recast (10) as an eigenvalue problem we need to introduce

$$\begin{aligned} C_{\Xi} : \Xi^* &\rightarrow \Phi^* \times \Xi^* & C_{\Xi}^* : \Phi \times \Xi &\rightarrow \Xi \\ g &\mapsto (0, g) & (\varphi, \xi) &\mapsto \xi \end{aligned}$$

then we can study the eigenvalue problem corresponding to

$$T_{\Xi} := C_{\Xi}^* \circ S \circ C_{\Xi} : \Xi^* \rightarrow \Xi \quad (11)$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (10) ?

## Proposition Existence of Solutions

If the discrete inf-sup stability condition holds, then problem (10) admits at least one solution  $(\psi_h, \chi_h)$ . Moreover  $\chi_h$  is uniquely determined by  $g$  and

$$\|\chi_h\|_{\Xi} \leq C \|g\|_{\Xi_h^*}.$$

Furthermore if it exists  $C > 0$  such that for every  $h > 0$  and for every  $(\psi_h, \chi_h, g) \in \Phi_h \times \Xi_h \times \Xi^*$  the above inequality is verified then the operator  $T_{\Xi}^h$  is defined for all element in  $\Xi$  and the **discrete** inf-sup condition is verified.

## Definition Weak Approximability

Let  $\Xi_0^H$  be the range of  $C_{\Xi}^* \circ S \circ C_{\Phi}$ . We say that  $\Xi_0^H$  verifies the **weak approximability** if for every  $(\chi, \varphi_h) \in \Xi_0^H \times \text{Ker}(B_h)$

$$b(\varphi_h, \chi) \leq \omega_4(h) \|\chi\|_{\Xi_0^H} \sqrt{a(\varphi_h, \varphi_h)}, \quad \lim_{h \rightarrow 0} \omega_4(h) = 0.$$

- Notice that this is an approximability condition similar to the one presented for the  $(f \quad 0)$  problems.

### Definition Strong Approximability

Let  $\Xi_0^H$  be the range of  $C_{\Xi}^* \circ S \circ C_{\Xi}$ . We say that  $\Xi_0^H$  verifies the **strong approximability** if for every  $\chi \in \Xi_0^H$  it exists  $\chi' \in \Xi_h$  such that,

$$\|\chi - \chi'\|_{\Xi} \leq \omega_5(h) \|\chi\|_{\Xi_H^0}, \lim_{h \rightarrow 0} \omega_5(h) = 0. \quad (12)$$



### Definition Fortin Operator

Given a subspace  $\Phi_\Pi$  of  $\Phi$  we say an operator  $\Pi_h : \Phi_\Pi \rightarrow \Phi_h$  is a **Fortin** operator with respect to the bilinear form  $b(\cdot, \cdot)$  and the subspace  $\Xi_h$  if for all  $\varphi \in \Phi_\Pi$  we have that:

$$b(\varphi - \Pi_h \varphi, \xi_h) = 0, \quad \forall \xi_h \in \Xi_h.$$

### Proposition Sufficient Conditions for Convergence

Assuming that it exists a **bounded Fortin operator**

$\Pi_h : \text{Range}(C_\Phi^* \circ S \circ C_\Xi) \rightarrow \Phi_h$  such that for every  $\phi \in \Phi_H^0$ ,

$$\sqrt{a(\varphi - \Pi_h \varphi, \varphi - \Pi_h \varphi)} \leq \omega_6(h) \|\varphi\|_{\Phi_H^0}, \quad \lim_{h \rightarrow 0} \omega_6(h) = 0. \quad (13)$$

If the weak and strong approximability condition of  $\Xi_H^0$  are verified then

$$\|T_\Xi f - T_\Xi^h f\|_\Xi \leq \omega_7(h) \|g\|_{H_\Xi}, \quad \lim_{h \rightarrow 0} \omega_7(h) = 0,$$

for any  $g \in H_\Xi$ .

### Proposition Necessary Conditions for Convergence

If the sequence of operators  $T_{\Xi}^h$  is bounded in  $\mathcal{L}(\Xi^*, \Xi)$ , it converges to  $T_{\Xi}$  in  $\mathcal{L}(H_{\Xi}, \Xi)$  and the following bounds holds when  $f = 0$ ,

$$\|\varphi_h\|_{\Phi} \leq C \|g\|_{\Xi},$$

then it exists a **bounded Fortin operator** verifying (13), moreover we have that the discrete inf-sup condition is verified together with the weak and strong approximation property of  $\Xi_H^0$ .

- ▶ The inf-sup condition is neither necessary nor sufficient when dealing with eigenvalue problem.
- ▶ For  $(f - 0)$  problem the inf-sup condition is **not necessary**.
- ▶ For  $(0 - g)$  problem the inf-sup condition is **not sufficient**.
- ▶ Why does everything work when dealing with a complex ?  
**Boffi, Acta Numerica (2010).**
- ▶ A more modern approach which also deals with Kolata argument can be found in **Boffi, Acta Numerica (2010).**

# Conclusion

## Is the Approximation Of Mixed Eigenvalue Problem Closed ?

- ▶ Can we create new element pairs specifically to solve the solve  $(f - 0)$  problems ?
- ▶ What happens if we use Babuska version of the inf-sup conditions ?





Boffi, D. (2010).

Finite element approximation of eigenvalue problems.

*Acta Numer.*, 19:1–120.



Boffi, D., Brezzi, F., and Fortin, M. (2013).

*Mixed finite element methods and applications*, volume 44 of  
*Springer Series in Computational Mathematics*.

Springer, Heidelberg.



Boffi, D., Brezzi, F., and Gastaldi, L. (2000).

On the problem of spurious eigenvalues in the approximation  
of linear elliptic problems in mixed form.

*Math. Comp.*, 69(229):121–140.