On the convergence of eigenvalues for mixed formulations

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All code to reproduce what shown in this slides can be found in the following git repository:

https://github.com/UZerbinati/BrezziBoffiGastaldi

- Solved Bernstein's minimal surface problem together with Enrico Bombieri.
- Solved 19th Hilbert problem, regarding the regularity of elliptic PDE.
- He is the father of modern calculus of variation and he is responsible for introducing the notion of Γ-convergence.

Enio De Giorgi
1928–1996

- Two Hilbert spaces Φ and Ξ are considered.
- Two **continuous** bilinear forms are also considered,

$$a(\cdot,\cdot) : \Phi \times \Phi \to \mathbb{R}, \qquad b(\cdot,\cdot) : \Phi \times \Xi \to \mathbb{R}.$$

$$A : \Phi \to \Phi^*, \ (A\phi) : \Phi \to \mathbb{R}, \ (A\phi)\varphi \mapsto a(\phi,\varphi),$$
$$B : \Phi \to \Xi^*, \ (B\phi) : \Xi \to \mathbb{R}, \ (B\phi)\xi \mapsto b(\phi,\xi),$$

- We assume $a(\cdot,\cdot)$ is **symmetric** and **positive semidefinite**.

$$\textit{Given any } (f,g) \in \Phi^* \times \Xi^* \textit{ find } (\psi,\chi) \in \Phi \times \Xi \textit{ such that}$$

$$\begin{cases} a(\psi,\varphi) + b(\varphi,\chi) = \langle f,\varphi\rangle & \forall \varphi \in \Phi \\ b(\psi,\xi) = \langle g,\xi\rangle & \forall \xi \in \Xi \end{cases} \tag{1}$$

$$\textit{Given any } (f,g) \in \Phi^* \times \Xi^* \textit{ find } (\psi,\chi) \in \Phi \times \Xi \textit{ such that}$$

$$\begin{cases} A\psi + B^*\chi = f \\ B\psi = g \end{cases}$$

Banach Closed Range **Theorem**

Given a closed linear operator $B : \Phi \to \Xi^*$, the following statements are equivalent:

- $Range(B)$ is closed in Ξ^*,
- $Range(B) = \Big[Ker(B^T)\Big]^0$, where Z^0 is the the **polar** set of Z, i.e. $Z^0 = \Big\{f \in Z^* \; s.t \; \langle f, z\rangle = 0 \; \forall z \in Z\Big\}$.
- it exists $L_B \in \mathcal{L}\Big(Range(B), Ker(B)^\perp\Big)$ and $\beta \geq 0$ such that:

$$\beta \|L_B g\|_\Phi \leq \|g\|_{\Xi^*}, \; \forall g \in Range(B).$$

Brezzi's **Theorem**

Assuming that the range of B is Ξ^* and that $a(\cdot,\cdot)$ is coercive in the kernel of B, then it exists one and only one solution to (1).

How do can one verify that B is a onto ?

Inf-Sup **Condition**

The operator B is surjective if and only if it exists $\beta > 0$ such that

$$\inf_{\xi\in\Xi}\sup_{\varphi\in\Phi}\frac{b(\varphi,\xi)}{\|\varphi\|_\Phi\|\xi\|_\Xi} \geq \beta$$

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- We introduce two discrete spaces $\Phi_h \subset \Phi$ and $\Xi_h \subset \Xi$.
- Previous result holds also for the discrete problem,

  Given any $(f, g) \in \Phi^* \times \Xi^*$ *find* $(\psi_h, \chi_h) \in \Phi_h \times \Xi_h$ *such that*

$$\begin{cases} a(\psi_h, \varphi_h) + b(\varphi_h, \chi_h) = \langle f, \varphi_h \rangle & \forall \varphi \in \Phi_h \\ b(\psi_h, \xi_h) = \langle g, \xi_h \rangle & \forall \xi \in \Xi_h \end{cases} \tag{2}$$

- The pair (Φ_h, Ξ_h) is **inf-sup stable** if it exists β_h independent from h such that:

$$\inf_{\xi_h \in \Xi_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \xi_h)}{\|\varphi_h\|_\Phi \|\xi_h\|_\Xi} \geq \beta_h$$

We introduce the **solution operator**, i.e.

$$S : \Phi^* \times \Xi^* \to \Phi \times \Xi \ s.t\ S(f,g) = (\psi, \chi) \ as \ in \ (1),$$
$$S_h : \Phi_h^* \times \Xi_h^* \to \Phi_h \times \Xi \ s.t\ S_h(f,g) = (\psi_h, \chi_h) \ as \ in \ (2).$$

Proposition Necessity of the Inf-sup

If it exists a constant $C > 0$ such that for all $(f,g) \in \Phi^* \times \Xi^*$ and for all $h > 0$

$$\|S_h(f,g)\|_{\Phi\times\Xi} \leq C\Big(\|f\|_{\Phi_h^*} + \|g\|_{\Xi_h^*}\Big) \tag{3}$$

then the bilinear form $a(\cdot,\cdot)$ is elliptic in the kernel of B and the pair (Φ_h, Ξ_h) is **inf-sup** stable.

When dealing with an mixed problem such that $a(\cdot,\cdot)$ is elliptic in the kernel the **inf-sup stability** condition is not only **sufficient** it is also **necessary**.

Remark

When proving the existence and uniqueness of solution for 1 and 2, hypothesis can be weaken, i.e. we can require $A : Ker(B) \to Ker(B)^*$ to be an isomorphism.

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We are now ready to introduce an eigenvalue problem. Given an Hilbert space H and a selfadjoint compact operator $T : H \to H$, we call eigenvalue of T the $\lambda \in \mathbb{R}$ such that

$$\lambda Tu = u \textit{ with } u \in H \backslash \{0\}.$$

In particular it is well known that for the above described operator T it exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ such that

$$\begin{gathered} \lambda_i T u_i = u_i, \\ \lim_{i\to\infty} \lambda_i = +\infty \textit{ and } \lambda_i \geq 0 \; \forall i \in \mathbb{N}. \end{gathered} \tag{4}$$

We now consider for all $h > 0$ the selfadjoint non negative operator $T_h : H \to H$, with finite range H_h. Let's denote $N(h)$ is the dimension of H_h. We are interested in the eigenvalues,

$$\lambda^h T u^h = u^h \textit{ with } u^h \in H_h \backslash \{0\}.$$

The same characterization of the eigenvalue presented above holds also in the discrete case.

If we assume that the discrete approximation operator T_h converges to T with respect to the norm of $\mathcal{L}(H,H)$, i.e.

$$\lim_{h\to 0} \| T - T_h \|_{\mathcal{L}(H,H)} = 0$$

then $\forall \varepsilon > 0$ and $\forall n \in \mathbb{N}$ it exists $h_0 > 0$ such that $\forall h > h_0$

$$\max_{i=1,\ldots,m(N)} \left| \lambda_i - \lambda_i^h \right| \leq \varepsilon, \tag{5}$$

$$\delta\left(\oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h \right) \leq \varepsilon. \tag{6}$$

$m(N)$ is the number of eigenvalues corresponding to N distinct ones, $E_i = \langle u_i \rangle$ and $E_i^h = \langle u_i^h \rangle$. The converse also holds true.

We consider two Hilbert space H_Φ and H_Ξ such that we cam identify H_Φ with H_Φ^*, H_Ξ with H_Ξ^* and

$$\Phi \subset H_\Phi \subset \Phi^*, \qquad \Xi \subset H_\Xi \subset \Xi^*.$$

Proposition Convergence of Discrete Eigenvalue Problem

Assuming that $a(\cdot,\cdot)$ is elliptic in the kernel of B_h and the discrete inf-sup condition holds then S_h converges in $\mathcal{L}(H_\Phi, H_\Xi)$ to S if and only if $S : H_\Phi \times H_\Xi \to H_\Phi \times H_\Xi$ is compact. The converse holds true.

```
msh = UnitSquareMesh(10,10,quadrilateral=True)
V = VectorFunctionSpace(msh, "Q", 1)
Q = FunctionSpace(msh, "DG", 0)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
  + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```

Motivating Example – Stokes Eigenvalue Problem Q1-P0

```
A = assemble (a)
M = assemble (m)
Asc, Msc = A.M.handle, M.M.handle
E = SLEPc.EPS().create()
E.setType(SLEPc.EPS.Type.ARNOLDI)
E.setProblemType(SLEPc.EPS.ProblemType.GHEP);
E.setOperators(Asc,Msc)
PC = ST.getKSP().getPC();
PC.setType("svd");
E.setST(ST);
E.solve();
```

Motivating Example – Stokes Eigenvalue Problem Q1-P0

N	Reference	Q1-P0
1	52.34468	53.56885
2	92.12438	97.57386
3	92.12438	97.57386
4	128.209	97.573867

- The numerical experiment for the Q1-P0 are obtained using a 10×10 uniform square grid, while the reference value are obtained using Hood-Taylor finite element pair on a 20×20 square mesh.
- As $h \to 0$ we would see a degraded rate of convergence.

More often the note in practice when we are interested in eigenvalue problem where either f or g is zero. For example we call problem of type $(f \quad 0)$,

$$\textit{Find } (\psi,\chi) \in \Phi \times \Xi \textit{ and } \lambda \in \mathbb{R} \textit{ such that}$$
$$\begin{cases} a(\psi,\varphi) + b(\varphi,\chi) = \lambda\langle\psi,\varphi\rangle & \forall\varphi \in \Phi, \\ b(\psi,\xi) = 0 & \forall\xi \in \Xi. \end{cases} \tag{7}$$

Which **can not** be cast as an eigenvalue problem of the form of (1).

To recast (7) as an eigenvalue problem we need to introduce

$$\begin{aligned} C_\Phi : \Phi^* &\to \Phi^* \times \Xi^* \\ f &\mapsto (f, 0) \end{aligned} \qquad \begin{aligned} C_\Phi^* : \Phi \times \Xi &\to \Phi \\ (\varphi, \xi) &\mapsto \varphi \end{aligned}$$

then we can study the eigenvalue problem corresponding to

$$T_\Phi := C_\Phi^* \circ S \circ C_\phi : \Phi^* \to \Phi \tag{8}$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (7) ?

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Proposition Existence of Solutions

If $a(\cdot,\cdot)$ is elliptic in the kernel of B_h, then problem (7) admits at least one solution (ψ_h, χ_h). Moreover ψ_h in uniquely determined by f and

$$\|\psi_h\|_\Phi \leq C\|f\|_{\Phi_h^*}.$$

Furthermore if it exists $C > 0$ such that for every $h > 0$ and for every $(\psi_h, \chi_h, f) \in \Phi_h \times \Xi_h \times \Phi^*$ the above inequality is verified then the operator T_Φ^h is defined for all element in Φ and a elliptic in the kernel of B_h.

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Definition Weak Approximability

Let Ξ_0^H be the range of $C_\Xi^* \circ S \circ C_\Phi$. We say that Ξ_0^H verifies the **weak approximability** if for every $\chi \in \Xi_0^H$

$$\sup_{\varphi_h \in Ker(B_h)} \frac{b(\varphi_h, \chi)}{\|\varphi\|_\Phi} \leq \omega_1(h)\|\chi\|_{\Xi_0^H}, \ \lim_{h\to 0} \omega_1(h) = 0.$$

Remark

The above definition is an approximability condition in fact using the fact that $b(\varphi_h, \chi^I) = 0$ for all $\chi^I \in \Xi_h$ to rewrite the weak approximability as for all $\chi \in \Xi_0^H$ $\inf_{\chi^I \in \Xi_h} \|\chi - \chi^I\|_\Xi \leq \omega_1(h)\|\chi\|_{\Xi_0^H}$.

Definition Strong Approximability

Let Φ_0^H be the range of $C_\Phi^* \circ S \circ C_\Phi$. We say that Φ_0^H verifies the **strong approximability** if for every $\psi \in \Phi_0^H$

$$\inf_{\psi^I \in Ker(B_h)} \left\| \psi - \psi^I \right\|_\Phi \leq \omega_2(h) \|\psi\|_{\Phi_0^H}, \ \lim_{h \to 0} \omega_2(h) = 0.$$

Proposition Convergence

If $a(\cdot,\cdot)$ is elliptic in the kernel of B_h and the weak approximability of Ξ_0^H and strong approximability of Φ_0^H are verified, then for all $f \in H_\Phi$

$$\left\| T_\Phi f - T_\Phi^h f \right\|_\Phi \le \omega_3(h), \ \lim_{h\to 0} \omega_3(h) = 0. \tag{9}$$

Vice versa if the sequence T_Φ^h is bounded in $\mathcal{L}(\Phi^*, \Phi)$ and converges uniformly to T_Φ in $\mathcal{L}(\Phi^*, \Phi)$ then $a(\cdot,\cdot)$ is elliptic in the kernel of B_h, moreover the strong and weak approximability conditions are verified respectively for Φ_0^H and Ξ_0^H.

We solve the Stokes eigenvalue problem using Scott-Vogelious(**ish**) finite element pair and criss-cross. mesh.

```
msh = UnitSquareMesh(5,5,diagonal="crossed")
V = VectorFunctionSpace(msh, "CG", 4)
Q = FunctionSpace(msh, "DG", 3)
X = V*Q
u,p = TrialFunctions(X)
v,q = TestFunctions(X)
a = (inner(grad(u), grad(v)) - inner(p, div(v))
  + inner(div(u), q))*dx+1e8*inner(u,v)*ds
m = inner(u,v)*dx
sol = Function(X)
```

An additional Example
A Connection with Charlie's presentation

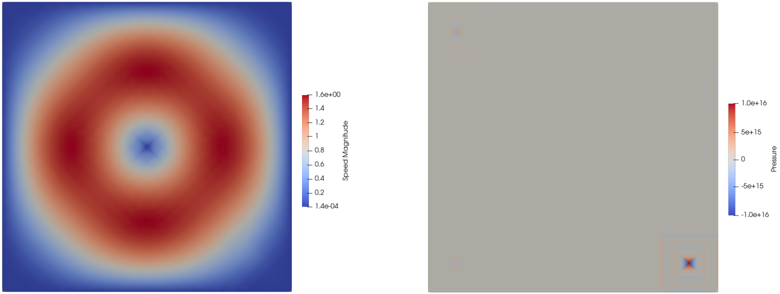

Figure: On the LHS the first mode of the Stokes eigenvalue problem, on the RHS the corresponding pressure computed with a SV(ish) method.

Numerical Example

The Mixed Laplacian Eigenvalue Problem – Q1-P0

```
msh = SquareMesh(64,64,np.pi,quadrilateral=True)

S = VectorFunctionSpace(msh, "Q", 1)
V = FunctionSpace(msh, "DG", 0)
X = S*V
s,u = TrialFunctions(X); t,v = TestFunctions(X)

a = (inner(s,t)+inner(div(t),u)+inner(div(s),v))*dx
m = -inner(u,v)*dx

bc = DirichletBC (X.sub(0), as_vector([0.0,0.0]) ,
     [1,2,3,4])
A = assemble (a,bcs=bc)
M = assemble (m)
```

Numerical Example

The Mixed Laplacian Eigenvalue Problem – Q1-P0

N	Ref	Q1-P0
1	1	0.99111
2	1	0.99111
3	2	1.96374
4	4	3.96531
5	4	3.96669
6	5	4.91019
7	5	4.91019
8	8	7.85488
9	9	8.73856
10	9	8.73856
11	10	8.92934

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What happens when f is null rather then g ?
We call problem of this type $(0 \quad g)$,

$$\textit{Find } (\psi,\chi) \in \Phi \times \Xi \textit{ and } \lambda \in \mathbb{R} \textit{ such that}$$
$$\begin{cases} a(\psi,\varphi) + b(\varphi,\chi) = 0 & \forall \varphi \in \Phi, \\ b(\psi,\xi) = -\lambda\langle \chi,\xi \rangle & \forall \xi \in \Xi. \end{cases} \tag{10}$$

Which once again **can not** be cast as an eigenvalue problem of the form of (1).

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To recast (10) as an eigenvalue problem we need to introduce

$$C_{\Xi} : \Xi^* \to \Phi^* \times \Xi^* \qquad\qquad C_{\Xi}^* : \Phi \times \Xi \to \Xi$$
$$g \mapsto (0, g) \qquad\qquad (\varphi, \xi) \mapsto \xi$$

then we can study the eigenvalue problem corresponding to

$$T_{\Xi} := C_{\Xi}^* \circ S \circ C_{\Xi} : \Xi^* \to \Xi \tag{11}$$

- What are the necessary and sufficient conditions to solve an eigenvalue problem like (10) ?

Proposition Existence of Solutions

If the discrete inf-sup stability condition holds, then problem (10) admits at least one solution (ψ_h, χ_h). Moreover χ_h in uniquely determined by g and

$$\|\chi_h\|_{\Xi} \leq C\|g\|_{\Xi_h^*}.$$

Furthermore if it exists $C > 0$ such that for every $h > 0$ and for every $(\psi_h, \chi_h, g) \in \Phi_h \times \Xi_h \times \Xi^*$ the above inequality is verified then the operator T_{Ξ}^h is defined for all element in Ξ and the **discrete** inf-sup condition is verified.

Definition Weak Approximability

Let Ξ_0^H be the range of $C_{\Xi}^* \circ S \circ C_{\Phi}$. We say that Ξ_0^H verifies the **weak approximability** if for every $(\chi, \varphi_h) \in \Xi_0^H \times Ker(B_h)$

$$b(\varphi_h, \chi) \leq \omega_4(h) \|\chi\|_{\Xi_0^H} \sqrt{a(\varphi_h, \varphi_h)}, \quad \lim_{h \to 0} \omega_4(h) = 0.$$

- Notice that this is an approximability condition similar to the one presented for the $\begin{pmatrix} f & 0 \end{pmatrix}$ problems.

Definition Strong Approximability

Let Ξ_0^H be the range of $C_{\Xi}^* \circ S \circ C_{\Xi}$. We say that Ξ_0^H verifies the **strong approximability** if for every $\chi \in \Xi_0^H$ it exists $\chi^I \in \Xi_h$ such that,

$$\left\|\chi - \chi^I\right\|_{\Xi} \leq \omega_5(h)\|\chi\|_{\Xi_H^0}, \lim_{h\to 0} \omega_5(h) = 0. \tag{12}$$

Definition Fortin Operator

Given a subspace Φ_Π of Φ we say an operator $\Pi_h : \Phi_\Pi \to \Phi_h$ is a **Fortin** operator with respect to the bilinear form $b(\cdot,\cdot)$ and the subspace Ξ_h if for all $\varphi \in \Phi_\Pi$ we have that:

$$b(\varphi - \Pi_h\varphi, \xi_h) = 0, \ \forall \xi_h \in \Xi_h.$$

Proposition Sufficient Conditions for Convergence

Assuming that it exists a **bounded Fortin operator** $\Pi_h : Range(C_\Phi^* \circ S \circ C_\Xi) \to \Phi_h$ such that for every $\phi \in \Phi_H^0$,

$$\sqrt{a(\varphi - \Pi_h\varphi, \varphi - \Pi_h\varphi)} \leq \omega_6(h)\|\varphi\|_{\Phi_H^0}, \ \lim_{h\to 0} \omega_6(h) = 0. \qquad (13)$$

If the weak and strong approximability condition of Ξ_H^0 are verified then

$$\left\| T_\Xi f - T_\Xi^h f \right\|_\Xi \leq \omega_7(h)\|g\|_{H_\Xi}, \ \lim_{h\to 0} \omega_7(h) = 0,$$

for any $g \in H_\Xi$.

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Proposition Necessary Conditions for Convergence

If the sequence of operators T_{Ξ}^{h} is bounded in $\mathcal{L}(\Xi^*, \Xi)$ and it converges to T_{Ξ} in $\mathcal{L}(H_{\Xi}, \Xi)$ and the following bounds holds when $f = 0$,

$$\|\varphi_h\|_{\Phi} \leq C\|g\|_{\Xi},$$

then it exists a **bounded Fortin operator** verifying (13), moreover we have that the discrete inf-sup condition is verified together with the weak approximation property of Ξ_H^0.

- The inf-sup condition is neither necessary nor sufficient when dealing with eigenvalue problem.
- For $(f - 0)$ problem the inf-sup condition is **not necessary**.
- For $(0 - g)$ problem the inf-sup condition is **neither sufficient nor necessary**.

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- Why does everything work when dealing with a complex ? **Thank you Boris !**
- Can we create new element pairs specifically to solve the solve $(f - 0)$ problems ?
- What happens if we use Babuska version of the inf-sup conditions ?