



Mathematical
Institute

Divergence-free discretisations of the Stokes eigenvalue problem

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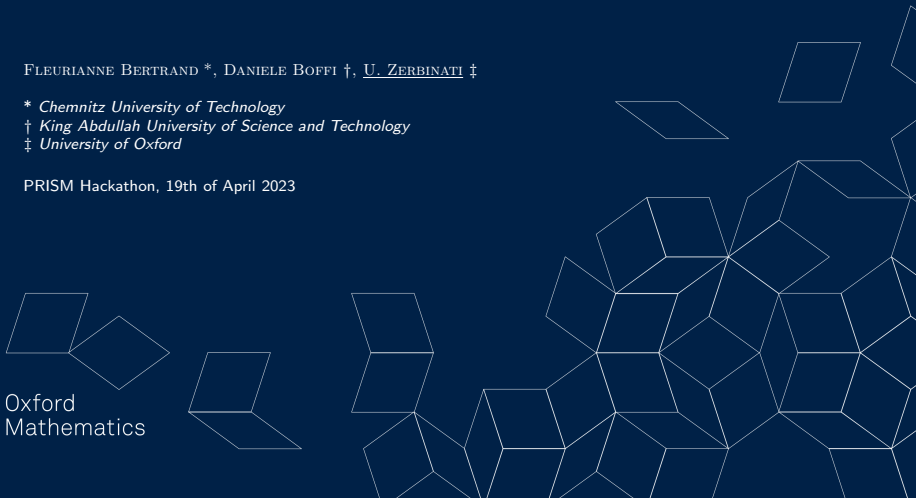
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Oxford
Mathematics



Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$ such that $\forall (\mathbf{v}, q) \in H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega)$,

$$\begin{aligned}\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}, p)_{\mathcal{L}^2(\Omega)} &= \lambda_n (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)}, \\ (\nabla \cdot \mathbf{u}, q)_{\mathcal{L}^2(\Omega)} &= 0,\end{aligned}$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

- Are the eigenvalue of this problem real?
- Do the eigenvalue of this problem diverge?

Stokes eigenvalue problem – Laplace form

We introduce the space $H_{0,0}^1(\Omega) = \left\{ \mathbf{v} \in H_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \right\}$.

Find $\mathbf{u} \in H_{0,0}^1(\Omega)$ such that $\forall \mathbf{v} \in H_{0,0}^1(\Omega)$,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} = \lambda_n (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)},$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

- ▶ The eigenvalue problem is self-adjoint therefore $\lambda_n \in \mathbb{R}$.
- ▶ $H_{0,0}^1(\Omega)$ is compactly embedded in $\mathcal{L}^2(\Omega)$ and therefore operator corresponding to the eigenvalue problem is compact, implying $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

Discrete Stokes eigenvalue problem

Find $(\mathbf{u}^h, p^h) \in V_h \times Q_h$ such that $\forall (\mathbf{v}^h, q^h) \in V_h \times Q_h$,

$$\begin{aligned}\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}^h, p^h)_{\mathcal{L}^2(\Omega)} &= \lambda_n (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)}, \\ (\nabla \cdot \mathbf{u}^h, q^h)_{\mathcal{L}^2(\Omega)} &= 0,\end{aligned}$$

with $\lambda_n^h \in \mathbb{C}$, $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity, and

$$V_h \times Q_h \subset H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega).$$

Under what hypotheses on V_h and Q_h is this eigenvalue problem well-posed ?

The divergence-free constraint

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0$$

Find $\mathbf{u}_h \in \mathbb{K}_h$ such that $\forall \mathbf{v}_h \in \mathbb{K}_h$,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)},$$

with $\lambda_n \in \mathbb{C}$, $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity and

$$\mathbb{K}_h = \left\{ \mathbf{v}^h \in V_h : b(\mathbf{v}^h, q^h) = 0, \forall q^h \in Q_h \right\}.$$

$$\mathbb{K}_h \not\subset H_{0,0}^1(\Omega)$$

$$\nabla \cdot V_h \subset Q_h$$

Under this hypothesis, we have the following result, i.e.

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0 \Leftrightarrow \nabla \cdot \mathbf{v}^h = 0,$$

which implies the functions are point-wise **divergence-free**.

$$\mathbb{K}_h \subset H_{0,0}^1(\Omega)$$

Divergence discretisations eigenvalue problem

Find $\mathbf{u}_h \in \mathbb{K}_h$ such that $\forall \mathbf{v}_h \in \mathbb{K}_h$,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)},$$

with $\nabla \cdot \mathbf{V}_h \subset Q_h$, $\lambda_n \in \mathbb{C}$, $\nu \in \mathbb{R}_{>0}$ is the fluid viscosity.

This problem is well-posed and we can analyse it using Babuska-Osborn theory.

Theorem

For each $n \in \mathbb{N}$, we have

$$\lambda_n \leq \lambda_n^h \leq \lambda_n + C \sup_{\mathbf{u} \in E, \|\mathbf{u}\|=1} \inf_{\mathbf{v}^h \in \mathbb{K}_h} \|\mathbf{u} - \mathbf{v}^h\|_{H^1(\Omega)}^2$$

and there exists $\mathbf{w}_n^h \in \langle \mathbf{u}_n^h, \dots, \mathbf{u}_{n+m-1}^h \rangle$ such that

$$\|\mathbf{u}_n - \mathbf{w}_n^h\| \leq C \sup_{\mathbf{u} \in E, \|\mathbf{u}\|=1} \inf_{\mathbf{v}^h \in \mathbb{K}_h} \|\mathbf{u} - \mathbf{v}^h\|_{H^1(\Omega)}$$

where m , E and \mathbf{u}_n are respectively the multiplicity, eigenspace and eigenvector corresponding to the eigenvalue λ_n .

An example – special mesh

Lemma

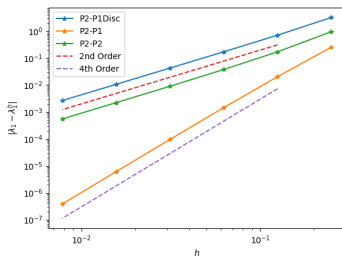
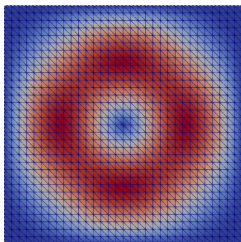
Let $\mathbf{u} \in H^s(\Omega) \cap H_{0,0}^1(\Omega)$, with $s \geq 2$. On a special mesh obtained from a uniform square mesh dividing each cell along one of its diagonals there exists a $\mathbf{u}^h \in [\mathbb{P}^k(\mathcal{T}_h)]^2$ such that

$$\nabla \cdot \mathbf{u}_h = 0,$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} \leq C \|\mathbf{u}\|_{H^s(\Omega)} \cdot \begin{cases} h^{\min(k-1, s-1)}, & k \in \{1, 2, 3\} \\ h^{\min(k, s-1)} k^{(1-s)}, & k \geq 4 \end{cases}$$

An example – special mesh

$$[\mathbb{P}^2(\mathcal{T}_h)]^2 - \mathbb{P}_{disc}^1(\mathcal{T}_h)$$



- We say that Q_h verifies the **weak approximability condition** if there exists $\gamma_1(h)$, such that for every $q \in \mathcal{L}_0^2(\Omega)$

$$\sup_{\mathbf{v}^h \in \mathbb{K}_h} \frac{b(\mathbf{v}^h, q)}{\|\mathbf{v}^h\|_{H^1(\Omega)}} \leq \omega_1(h) \|q\|_{\mathcal{L}^2(\Omega)} \text{ and } \lim_{h \rightarrow 0} \gamma_1(h) = 0.$$

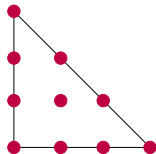
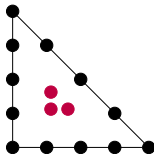
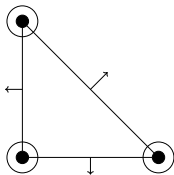
- We say V_h verifies the **strong approximability condition** if there exists $\gamma_2(h)$, such that for every $\mathbf{v} \in H_{0,0}^1(\Omega) \cap H^2(\Omega)$

$$\inf_{\mathbf{v}^h \in \mathbb{K}_h} \|\mathbf{v} - \mathbf{v}^h\|_{H^1(\Omega)} \leq \gamma_2(h) \|\mathbf{v}\|_{H^2(\Omega)} \text{ and } \lim_{h \rightarrow 0} \gamma_2(h) = 0.$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_0^2(\Omega) & \xrightarrow{\nabla \times} & [H_0^1(\Omega)]^2 & \xrightarrow{\nabla \cdot} & \mathcal{L}_0^2(\Omega) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & Q_h \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & S_h & \xrightarrow{\nabla \times} & V_h & \xrightarrow{\nabla \cdot} & \hat{Q}_h \longrightarrow 0
 \end{array}$$

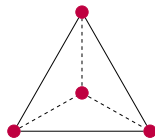
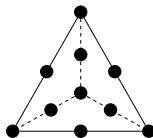
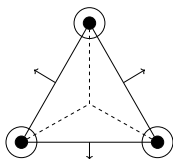
Couple more example

- $[\mathbb{P}^4(\mathcal{T}_h)]^2 - \mathbb{P}_{disc}^3(\mathcal{T}_h)$, will be a converging scheme on a criss-cross mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from the Morgan-Scott-Vogelius complex.

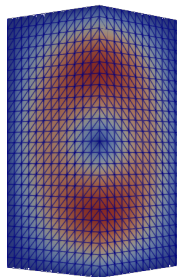


Couple more example

- $[\mathbb{P}^2(\mathcal{T}_h)]^2 - \mathbb{P}^2(\mathcal{T}_h)$, will be a converging scheme on a barycentrically refined mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from Hsieh-Clough-Tocher complex.



- ▶ There is no need to characterise the range of the divergence operator!
This is crucial for three-dimensional problems.
- ▶ A wide variety of finite element space pairs can be used even if they are not **inf-sup** stable.



Thank you for your attention !