Divergence-free discretisations of the Stokes eigenvalue problem

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Oxford Mathematics

Stokes eigenvalue problem

Find $(\mathbf{u}, p) \in H^1_0(\Omega) \times \mathcal{L}^2_0(\Omega)$ such that $\forall (\mathbf{v}, q) \in H^1_0(\Omega) \times \mathcal{L}^2_0(\Omega)$,

$$\begin{cases} \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}, p)_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)}, \\ (\nabla \cdot \mathbf{u}, q)_{\mathcal{L}^2(\Omega)} = 0, \end{cases} \tag{1}$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{\geq 0}$ is the fluid viscosity.

- Are the eigenvalue of this problem real?
- Do the eigenvalue of this problem diverge?

Stokes eigenvalue problem – Laplace form

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We introduce the space $H^1_{0,0}(\Omega) = \left\{ \mathbf{v} \in H^1_0(\Omega) \;:\; \nabla \cdot \mathbf{u} = 0 \right\}$.

Find $\mathbf{u} \in H^1_{0,0}(\Omega)$ such that $\forall \mathbf{v} \in H^1_{0,0}(\Omega)$,

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}, \mathbf{v})_{\mathcal{L}^2(\Omega)}, \tag{2}$$

with $\lambda_n \in \mathbb{C}$, and $\nu \in \mathbb{R}_{\geq 0}$ is the fluid viscosity.

- The eigenvalue problem is self-adjoint therefore $\lambda_n \in \mathbb{R}$.
- $H^1_{0,0}(\Omega)$ is compactly embedded in $\mathcal{L}^2(\Omega)$ and therefore operator corresponding to the eigenvalue problem is compact, implying $\lambda_n \to \infty$ ad $n \to \infty$.

Discrete Stokes eigenvalue problem

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Find $(\mathbf{u}^h, p^h) \in V_h \times Q_h$ such that $\forall (\mathbf{v}^h, q^h) \in V_h \times Q_h$,

$$\begin{cases} \nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} - (\nabla \cdot \mathbf{v}^h, p^h)_{\mathcal{L}^2(\Omega)} = \lambda_n \, (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)}, \\ (\nabla \cdot \mathbf{u}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0, \end{cases} \tag{3}$$

with $\lambda_n^h \in \mathbb{C}$, $\nu \in \mathbb{R}_{\geq 0}$ is the fluid viscosity, and

$$\boxed{V_h \times Q_h \subset H_0^1(\Omega) \times \mathcal{L}_0^2(\Omega).} \tag{4}$$

Under what hypothesis on V_h and Q_h is this eigenvalue problem well-posed ?

The divergence-free constraint

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$$\boxed{b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0} \tag{5}$$

Find $\mathbf{u}_h \in \mathbb{K}_h$ such that $\forall \mathbf{v}_h \in \mathbb{K}_h$,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h \, (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)}, \tag{6}$$

with $\lambda_n \in \mathbb{C}$, $\nu \in \mathbb{R}_{\geq 0}$ is the fluid viscosity and

$$\mathbb{K}_h = \left\{ \mathbf{v}^h \in V_h \ : \ b(\mathbf{v}^h, q^h) = 0 \ \forall q^h \in Q_h \right\}. \tag{7}$$

Divergence-free discretisations

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$$\boxed{\nabla \cdot V_h \subset Q_h} \tag{8}$$

Under this hypothesis, we have the following result, i.e.

$$b(\mathbf{v}^h, q^h) = (\nabla \cdot \mathbf{v}^h, q^h)_{\mathcal{L}^2(\Omega)} = 0 \Leftrightarrow \nabla \cdot \mathbf{v}^h = 0, \tag{9}$$

which implies the functions are really **divergence-free**.

$$\boxed{\mathbb{K}_h \subset H^1_{0,0}(\Omega)} \tag{10}$$

Divergence discretisations eigenvalue problem

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Find $\mathbf{u}_h \in \mathbb{K}_h$ such that $\forall \mathbf{v}_h \in \mathbb{K}_h$,

$$\nu(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h)_{\mathcal{L}^2(\Omega)} = \lambda_n^h (\mathbf{u}^h, \mathbf{v}^h)_{\mathcal{L}^2(\Omega)}, \tag{11}$$

with $\nabla \cdot V_h \subset Q_h$, $\lambda_n \in \mathbb{C}$, $\nu \in \mathbb{R}_{\geq 0}$ is the fluid viscosity.

This problem is well-posed and we can analyse it using Babuska-Osborn theory.

Babuska-Osborn theory

Theorem

For each $n \in \mathbb{N}$, we have

$$\lambda_n \leq \lambda_n^h \leq \lambda_n + C \sup_{\mathbf{u} \in E,\ \|\mathbf{u}\|=1} \inf_{\mathbf{v}^h \in \mathbb{K}_h} \|\mathbf{u} - \mathbf{v}_h\|^2_{H^1(\Omega)} \tag{12}$$

and it exists $\mathbf{w}_n^h \in \langle \mathbf{u}_n^h, \ldots, \mathbf{u}_{n+m-1}^h \rangle$.

$$\|\mathbf{u}_n - \mathbf{w}_n^h\| \leq C \sup_{\mathbf{u} \in E,\ \|\mathbf{u}\|=1} \inf_{\mathbf{v}^h \in \mathbb{K}_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)} \tag{13}$$

where m, E and $\mathbf{u}_n$ are respectively the multiplicity, eigenspace and eigenvector corresponding to the eigenvalue λ_n.

An example – special mesh

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Lemma

Let $\mathbf{u} \in H^s(\Omega) \cap H^1_{0,0}(\Omega)$*, with* $s \geq 2$*. On a special mesh obtained from a uniform square mesh dividing each cell along one of its diagonals there exists a* $\mathbf{u}^h \in [\mathbb{P}^k(\mathcal{T}_h)]^2$ *such that*

$$\nabla \cdot \mathbf{u}_h = 0, \tag{14}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} \leq C\, |\mathbf{u}|_{H^s(\Omega)} \cdot \begin{cases} h^{\min(k-1,s-1)}, & k \in \{1,2,3\} \\ h^{\min(k,s-1)} k^{(1-s)}, & k \geq 4 \end{cases} \tag{15}$$

An example – special mesh

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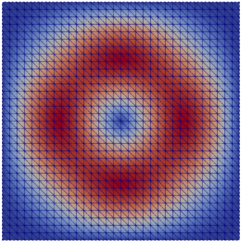
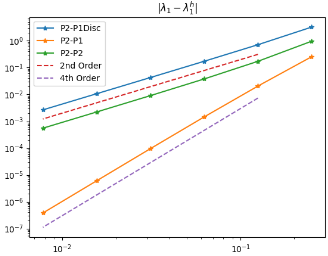
$$\boxed{[\mathbb{P}^2(\mathcal{T}_h)]^2 \;-\; \mathbb{P}^1_{disc}(\mathcal{T}_h)} \tag{16}$$





A systematic approach to the best approximation

Lemma

Let $(V_h,\ \hat{Q}_h)$ *be an* **inf-sup stable** *finite element pair, we assume that the bottom sequence is* **exact and bounded**. *Under this hypothesis given any* $\mathbf{u} \in H^1_{0,0}(\Omega)$, *the following best approximation estimate holds*

$$\inf_{\mathbf{u}^h \in \mathbb{K}_h} \left\| \mathbf{u} - \mathbf{u}^h \right\|_{H^1(\Omega)} \leq C(\beta) \inf_{\mathbf{v}^h \in V_h} \left\| \mathbf{u} - \mathbf{u}^h \right\|_{H^1(\Omega)}, \qquad (17)$$

where β *is the inf-sup constant corresponding to the finite element pair* $(V_h,\ \hat{Q}_h)$.

Boffi-Brezzi-Gastaldi Theory

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- We say that Q_h verifies the **weak approximability condition** if there exists $\gamma_1(h)$, such that for every $q \in \mathcal{L}^2_0(\Omega)$

$$\sup_{\mathbf{v}^h \in \mathbb{K}_h} \frac{b(\mathbf{v}^h, q)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \leq \omega_1(h) \|q\|_{\mathcal{L}^2(\Omega)} \text{ and } \lim_{h \to 0} \gamma_1(h) = 0. \quad (18)$$

- We say V_h verifies the **strong approximability condition** if there exists $\gamma_2(h)$, such that for every $\mathbf{v} \in H^1_{0,0}(\Omega) \cap H^2(\Omega)$

$$\inf_{\mathbf{v}^h \in \mathbb{K}_h} \left\| \mathbf{v} - \mathbf{v}^h \right\|_{H^1(\Omega)} \leq \gamma_2(h) \|\mathbf{v}\|_{H^2(\Omega)} \text{ and } \lim_{h \to 0} \gamma_2(h) = 0. \quad (19)$$

Finite Element Exterior Calculus

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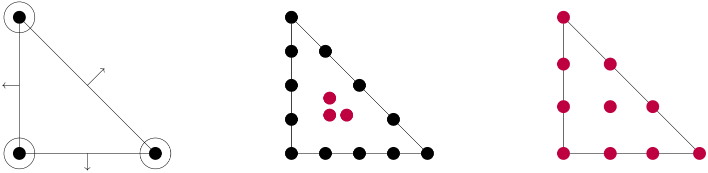
$$\begin{tikzcd}
0 \arrow[r] & H^2_0(\Omega) \arrow[r, "\nabla \times"] \arrow[d] & \left[H^1_0(\Omega)\right]^2 \arrow[r, "\nabla \cdot"] \arrow[d] & \mathcal{L}^2_0(\Omega) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_h \arrow[r, "\nabla \times"] \arrow[d] & V_h \arrow[r, red, "\nabla \cdot"] \arrow[d] & Q_h \arrow[r, red] & 0 \\
0 \arrow[r] & S_h \arrow[r, "\nabla \times"] & V_h \arrow[r, "\nabla \cdot"] & \hat{Q}_h \arrow[r] & 0
\end{tikzcd}$$

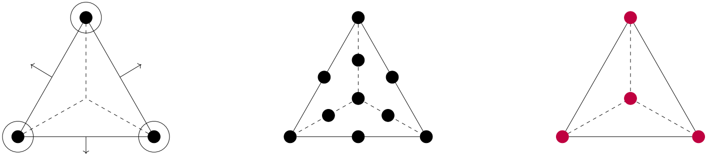
Couple more example

- $[\mathbb{P}^4(\mathcal{T}_h)]^2 \; - \; \mathbb{P}^3_{disc}(\mathcal{T}_h)$, will be a converging scheme on a criss-cross mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from the Morgan-Scott-Vogelious complex.



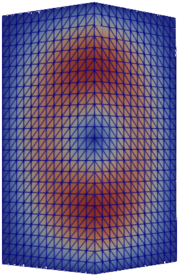
Couple more example

- $[\mathbb{P}^2(\mathcal{T}_h)]^2 \ - \ \mathbb{P}^2(\mathcal{T}_h)$, will be a converging scheme on a barycentrically refined mesh even if this choice of the element is not inf-sup stable. Best approximation estimates can be derived from Hsieh-Clough-Tocher complex.



Conclusions

- There is no need to characterise the range of the divergence operator!
- A wide variety of finite element space pairs can be used even if they are not **inf-sup** stable.



Thank you for your attention !