Quasi-optimality of FEMs for the Helmholtz equation

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The wave equation

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$$
\begin{aligned}
\partial_t^2 p_\delta - c^2 \Delta p_\delta &= f && \text{in} \quad \Omega \times (0, T), \\
p_\delta &= 0 && \text{on } \partial\Omega \times (0, T), \\
p_\delta(\cdot, 0) = p_0(\cdot), &\qquad \partial_t p_\delta(\cdot, 0) = \dot{p}_0(\cdot).
\end{aligned}
$$

Time harmonic solutions of the wave equation

The **time harmonic solutions** of the wave equation are of the form

$$p^{(\delta)}(\cdot, t) = \mathrm{Re}\left\{P(\cdot)e^{-i\omega t}\right\},$$

where $P : \Omega \to \mathbb{C}$ and ω is the **angular frequency**.

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where $P : \Omega \to \mathbb{C}$ and ω is the **angular frequency**.
Substituting this ansatz into the wave equation, we obtain the **Helmholtz equation**

$$\begin{aligned} -\Delta P - k^2 P = F \qquad & \text{in} \quad \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

where $k = \omega/c$ is the **wave number** and $F = \mathrm{Re}\left\{f(\cdot, t)e^{-i\omega t}\right\}$.

The weak formulation

The **weak formulation** of the Helmholtz equation is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla P, \nabla Q)_{L^2} - k^2 (P, Q)_{L^2} = {}_{H^{-1}}\langle F, Q \rangle_{H_0^1} \quad \forall Q \in H_0^1(\Omega),$$

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The weak formulation of the Helmholtz equation is **elliptic**, hence we can make use of **elliptic regularity theory**.

The Helmholtz problem is **ill-posed** if k^2 is an eigenvalue of the Laplacian, in the sense that the solution P is not uniquely determined by the data F.

Finite element discretization

The **finite element discretization** of the Helmholtz equation is to find $u_h \in X_h$ such that

$$(\nabla P_h, \nabla Q_h)_{L^2} - k^2 (P_h, Q_h)_{L^2} = (F, Q_h)_{L^2} \quad \forall Q_h \in X_h,$$

where $X_h \subset H_0^1(\Omega)$ is a finite-dimensional subspace.

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- We discretize the domain Ω into a mesh of elements $\{\mathcal{T}_h\}_{h>0}$.
- We choose as X_h the space of piecewise linear polynomial functions on the mesh, i.e.

$$X_h := \{v \in L^2(\Omega) : v|_\tau \in \mathcal{P}^1(\tau)\ \forall \tau \in T_h\} \cap H_0^1(\Omega) \subset H_0^1(\Omega).$$

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- We obtain a **linear system** of the form $\underline{\underline{\mathcal{A}}}\,\underline{P}_h = \underline{F}$.

A motivating example

$$P(\cdot) = \sin(\pi\omega\cdot), \qquad \omega = 16$$

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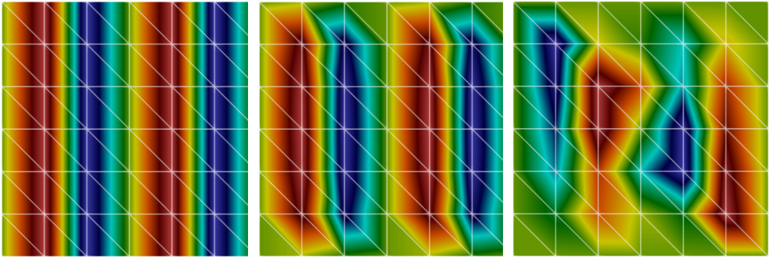

L^2 projection $\qquad k = 0 \qquad k = 256$

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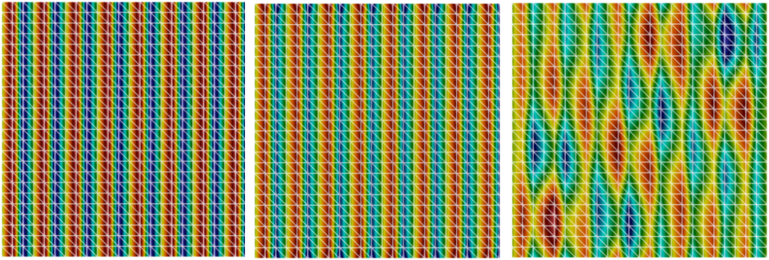

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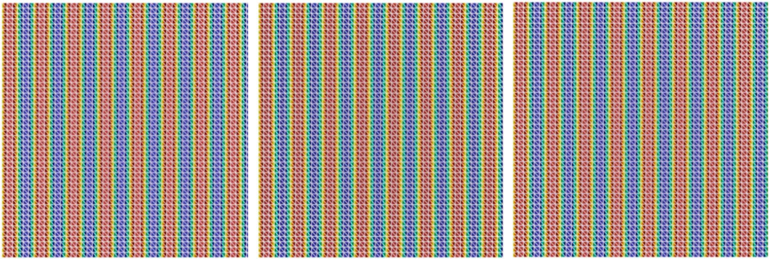

L^2 projection $\qquad$ $k = 0$ $\qquad$ $k = 256$

Coercive elliptic problems

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Definition (Coercivity)

We call a sesquilinear form $\mathcal{A} : X \times X \to \mathbb{C}$ **coercive** on X if there exists a constant $\alpha > 0$ such that

$$\mathcal{A}(P, P) \geq \alpha \|P\|_X^2 \qquad \forall P \in X.$$

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Theorem (Lax-Milgram)

If $\mathcal{A} : X \times X \to \mathbb{C}$ is coercive on X, then the variational problem

$$\text{find } P \in X \text{ such that } \mathcal{A}(P, Q) = {}_{X'}\langle f, Q \rangle_X \quad \forall Q \in X$$

is **well-posed** *for all $f \in X'$.*

Discrete coercive problems

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Cea's lemma

Let $\mathcal{A} : X \times X \to \mathbb{C}$ be coercive on X and let $P \in X$ be the solution of the continuous variational problem. Then the solution $P_h \in X_h$ of the discrete variational problem satisfies

$$\|P - P_h\|_X \leq \frac{M}{\alpha} \inf_{Q_h \in X_h} \|P - Q_h\|_X,$$

where M is the continuity constant of $\mathcal{A}$.

Lack of coercivity of the Helmholtz problem

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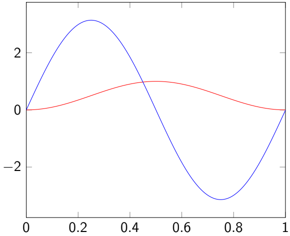
Consider the function $P(x) = \sin(\pi\omega x)$.

$$\|P\|^2_{L^2([0,1])} = \int_0^1 \sin^2(\pi\omega x) = \frac{1}{2}$$

$$\|P'\|^2_{L^2([0,1])} = \int_0^1 2\pi \sin(\pi\omega x)\cos(\pi\omega x)$$



notice that the last integral vanishes for $\omega \in \mathbb{N}$, on the interval $[0,1]$.

T-coercive elliptic problems

Definition (T-coercivity)

We call a sesquilinear form $\mathcal{A}(\cdot,\cdot)$ **T-coercive** on X if there exists a bijective operator $T \in L(X)$ and a constant $\alpha > 0$ s.t.

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Theorem (Ciarlet[1])

If $\mathcal{A}(\cdot,\cdot)$ is T-coercive on X, then the corresponding variational problem is well-posed.

1. see e.g., P. Ciarlet Jr., "T-coercivity: Application to the discretization of Helmholtz-like problems", 2012.

Discrete T-coercivity

- T-coercivity is a **sufficient and necessary** condition for well-posedness of the weak formulation.

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- T-coercivity is **not** automatically inherited onto the discrete level.
- T-coercivity, at the discrete level, is a equivalent to **uniform inf-sup stability**, i.e.

$$\inf_{v_h \in X_h} \sup_{w_h \in X_h} \frac{|a_h(v_h, w_h)|}{\|v_h\|_X \|w_h\|_X} \geq \beta > 0,$$

which guarantees the **well-posedness** of the discrete problem.

Compact Eigenvalue problems

Considering the eigenvalue problem, find $(\lambda, P) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$(\nabla P, \nabla Q)_{L^2} = \lambda (P, Q)_{L^2} \quad \forall Q \in H_0^1(\Omega).$$

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$$(\nabla P, \nabla Q)_{L^2} = \lambda (P, Q)_{L^2} \quad \forall Q \in H_0^1(\Omega).$$

We can rewrite this as an eigenvalue problem associated with an operator $\mathcal{S} : H_0^1(\Omega) \to H_0^1(\Omega)$, i.e. find $(\lambda, P) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$(\nabla \mathcal{S} f, \nabla Q) = \lambda (f, Q) \quad \forall Q \in H_0^1(\Omega).$$

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Elliptic regularity

Via elliptic regularity theory, we know that the operator $\mathcal{S}$ is compact. In fact, by Rellich-Kondrachov we know that $H^2(\Omega) \subset\subset H_0^1(\Omega)$.

Hilbert basis

- Let $(\lambda^{(i)}, \Phi^{(i)})_{i\in\mathbb{N}}$ be the eigenpairs of the operator $\mathcal{S}$.

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- We expand any $\Phi^{(i)} \in H_0^1(\Omega)$ as $P = \sum_{i\in\mathbb{N}} \Phi^{(i)}(P, \Phi^{(i)})_{H^1}$.

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- We expand any $\Phi^{(i)} \in H^1_0(\Omega)$ as $P = \sum_{i\in\mathbb{N}} \Phi^{(i)}(P, \Phi^{(i)})_{H^1}$.
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Spectral decomposition

S is a compact operator on $H^1_0(\Omega)$, hence the eigenvalues $\lambda^{(i)}$ are **discrete** and **tend to zero**. Furthermore, the eigenfunctions $P^{(i)}$ form a **Hilbert basis** of $H^1_0(\Omega)$.

T-coercivity of the Helmholtz problem

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$$
\begin{aligned}
\mathcal{A}(P, P) &= (\nabla P, \nabla P)_{L^2} - k^2 (P, P)_{L^2(\Omega)} \\
&= \sum_{i \in \mathbb{N}} P^{(i)} (\nabla \Phi^{(i)}, \nabla P)_{L^2} - k^2 P^{(i)} (\Phi^{(i)}, P)_{L^2} \\
&= \sum_{i \in \mathbb{N}} \lambda^{(i)} P^{(i)} (\Phi^{(i)}, P)_{L^2} - k^2 P^{(i)} (\Phi^{(i)}, P)_{L^2} \\
&= \sum_{i \in \mathbb{N}} \lambda^{(i)} |P^{(i)}|^2 (\Phi^{(i)}, \Phi^{(i)})_{L^2} - k^2 |P^{(i)}|^2 (\Phi^{(i)}, \Phi^{(i)})_{L^2} \\
&= \sum_{i \in \mathbb{N}} \left(\frac{\lambda^{(i)} - k^2}{1 - \lambda^{(i)}} \right) |P^{(i)}|^2
\end{aligned}
$$

T-coercivity of the Helmholtz problem

Suppose $\exists i_*$ s.t. $\lambda_1 < \cdots < \lambda_{i_*} < k^2 < \lambda_{i_*+1} < \ldots$

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We then construct $W := \operatorname{span}_{0 \le i \le i_*}\{\Phi^{(i)}\}$ and set $T := \mathrm{Id}_X - 2P_W$, i.e.

$$T\Phi^{(i)} = \begin{cases} -\Phi^{(i)} & \text{if } i \le i_*, \\ +\Phi^{(i)} & \text{if } i > i_*. \end{cases}$$

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$$T\Phi^{(i)} = \begin{cases} -\Phi^{(i)} & \text{if } i \leq i_*, \\ +\Phi^{(i)} & \text{if } i > i_*. \end{cases}$$

- T is bijective, since it is self-inverse, i.e. $T^2 = \operatorname{Id}_X$.

We notice that $\mathcal{A}(\cdot,\cdot)$ is T-coercive, provided k^2 not an eigenvalue $\lambda^{(i)}$, since

$$\mathcal{A}(P,TP) = \sum_{i\leq i_*}\left(\frac{k^2-\lambda^{(i)}}{1+\lambda^{(i)}}\right)(\Phi^{(i)})^2 + \sum_{i>i_*}\left(\frac{\lambda^{(i)}-k^2}{1+\lambda^{(i)}}\right)(\Phi^{(i)})^2$$
$$\geq \alpha \sum_{i\in\mathbb{N}} \lambda^{(i)}(\Phi^{(i)})^2 = \alpha\|P\|^2_{H^1},$$

where $\alpha = \min_{i\geq 0}\left\{\left|\frac{\lambda^{(i)}-k^2}{1+\lambda^{(i)}}\right|\right\} > 0$.

Weak T-coercivity

Definition (weak T-coercivity)

A linear operator $A \in L(X)$ is called **weakly T-coercive** if there exists a bijective operator $T \in L(X)$ and $K \in L(X)$ compact s.t. $T^*A + K$ is coercive.

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- A is T-coercive if T^*A is bijective.
- A is weakly T-coercive if $T^*A + K$, B bijective and K compact.

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- A is weakly T-coercive if $T^*A + K$, B bijective and K compact.

Lemma

If A is weakly T-coercive and injective, then A is bijective.

Robin boundary conditions

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Consider the Helmholtz problem with Robin boundary conditions, i.e. find $u \in H^1(\Omega)$ such that $\mathcal{A}(P, Q) = {}_{H^{-1}}\langle f, Q \rangle_{H^1}$ $\forall Q \in X$, where

$$\mathcal{A}(P, Q) := \underbrace{(\nabla P, \nabla Q)_{L^2} - k(P, Q)_{L^2}}_{\mathcal{A}_0(P,Q)} - ik\langle P, Q \rangle_{L^2(\partial\Omega)} = (f, Q)_{L^2}.$$

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Trace theorem

On bounded Lipschitz domains, the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is compact.

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- A is weakly T-coercive (injectivity can also be shown)

Schatz argument

We begin observing that the sesquilinear form $\mathcal{A}_0$ satisfies the *Gårding inequality*, i.e.

$$\mathcal{G}\|P-P_h\|^2_{H^1_k} \leq Re\left\{\mathcal{A}_0(P-P_h, P-Q_h)\right\} + k^2\|P-P_h\|^2_{L^2},$$

where we have introduced the norm $\|P\|^2_{H^1_k} := \|\nabla P\|^2_{L^2} + k^2\|P\|^2_{L^2}$.

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Aubin-Nitsche duality trick

The following bound holds,

$$\|P-P_h\|_{L^2} \leq M^2\psi(X_h)\|P-P_h\|_{H^1_k},$$

where $\psi(X_h) := \sup\limits_{g\in L^2(\Omega)} \inf\limits_{Q_h\in X_h} \dfrac{\|g-Q_h\|_{H^1_k}}{\|g\|_{L^2}}$.

Schatz argument

Combining the Gårding inequality with the Aubin-Nitsche duality trick, we obtain

$$\mathcal{G}\|P - P_h\|^2_{H^1_k} \leq M\|P - Q_h\|_{H^1_k}\|P - P_h\|_{H^1_k} + M^2k^2\psi(X_h)^2\|P - P_h\|^2_{H^1_k}.$$

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Imposing the following condition on $\psi(X_h)$,

$$\psi(X_h) \leq \left(\frac{\mathcal{G}}{2k^2M^2}\right)^{\frac{1}{2}},$$

we obtain the following error bound:

$$\mathcal{G}\|P-P_h\|_{H^1_k} \leq 2M \inf_{Q_h\in X_h}\|P-Q_h\|_{H^1_k}.$$

Schatz argument

Using *Bramble-Hilbert lemma*, we can bound $\psi(X_h)$ as

$$\psi(X_h) \leq C_{\mathcal{I}} h \|Z\|_{H^2(\Omega)} \leq \left(\frac{\mathcal{G}}{2k^2M^2}\right)^{\frac{1}{2}}.$$

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Frequency regularity estimates

The following regularity estimate holds $\|P\|_{H^2} \leq (1+kC_\Omega)\|g\|_{L^2}$, for $P \in H^2(\Omega)$ such that $-\Delta P = g$.

Schatz argument

Using *Bramble-Hilbert lemma*, we can bound $\psi(X_h)$ as

$$\psi(X_h) \leq C_{\mathcal{I}} h \|Z\|_{H^2(\Omega)} \leq \left(\frac{\mathcal{G}}{2k^2M^2}\right)^{\frac{1}{2}}.$$

Frequency regularity estimates

The following regularity estimate holds $\|P\|_{H^2} \leq (1+kC_\Omega)\|g\|_{L^2}$, for $P \in H^2(\Omega)$ such that $-\Delta P = g$.

Combining the previous estimates we obtain,

$$h \lesssim C_{\mathcal{I}} \left(\frac{\mathcal{G}}{2k^2M^2}\right)^{\frac{1}{2}} (1+kC_\Omega)^{-1} \sim k^{-2}.$$

Discrete T-coercivity

Definition

We call a family of sesquilinear forms $(\mathcal{A}_h)_{h>0}$ on X_h **uniformly** T$_h$-coercive, if there exists bijective operators $T_h : X_h \to X_h$ and $\alpha > 0$ independent of h s.t.

$$\mathcal{A}_h(P_h, T_h P_h) \geq \alpha \|P_h\|_X^2 \qquad \forall P_h \in X_h.$$

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- If $\mathcal{A}_h$ is uniformly T_h-coercive, then the discrete problem is stable and quasi-optimal $\|P - P_h\|_X \lesssim \inf_{P_h \in X_h} \|P - Q_h\|_X$. **Provided** it is usually enough to show that

$$\lim_{h\to 0} \|T - T_h\|_X = 0,$$

but this is an asymptotic result not explicit about h.

Discrete weak T-coercivity

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Theorem

Let $A = B + K$, where B is bijective and K compact and suppose that $\ker(A) = \{0\}$. If there exists a family of bijective operators $T_h \in \mathcal{L}(X_h)$ s.t. B is **uniformly $\mathbf{T}_h$-coercive** *on X_h, then there exists $h_0 > 0$ s.t. A is* **uniformly $\mathbf{T}_h$-coercive** *on X_h for $h < h_0$.*

- If A is weakly coercive and injective, and $(T^*)^{-1}B$ is uniformly T_h-coercive, then the discrete problem is stable and quasi-optimal

$$\|P - P_h\|_X \lesssim \inf_{Q_h \in X_h} \|P - Q_h\|_X.$$

Discrete Helmholtz

Find $P_h \in X_h$ such that for any $P_h \in X_h$ the following holds,

$$\mathcal{A}(P_h, Q_h) := \underbrace{(\nabla P_h, \nabla Q_h)_{L^2} - k(P_h, Q_h)_{L^2}}_{=:\mathcal{A}_0(P_h, Q_h)} - ik\langle P_h, Q_h\rangle_{L^2(\partial\Omega)} = (f, Q_h)_{L^2}.$$

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- Only have to show that $\mathcal{A}_0$ is uniformly T_h-coercive.

Define $T_h : X_h \to X_h$ through

$$T_h e_h^{(i)} := \begin{cases} -e_h^{(i)} & \text{if } i \leq i_*, \\ +e_h^{(i)} & \text{if } i > i_*. \end{cases}$$

Discrete T-coercivity of $\mathcal{A}_0$

Following the same steps as before, we can expand P_h in terms of the eigenfunctions of $\mathcal{S} : X_h \to X_h$, i.e. $P_h = \sum_{i\in\mathbb{N}} P_h^{(i)}\Phi_h^{(i)}$.

$$\mathcal{A}_0(P_h, T_h P_h) := \sum_{i\leq i_*}\left(\frac{k-\lambda_h^{(i)}}{1+\lambda_h^{(i)}}\right)(P_h^{(i)})^2 + \sum_{i>i_*}\left(\frac{\lambda_h^{(i)}-k}{1+\lambda_h^{(i)}}\right)(P_h^{(i)})^2.$$

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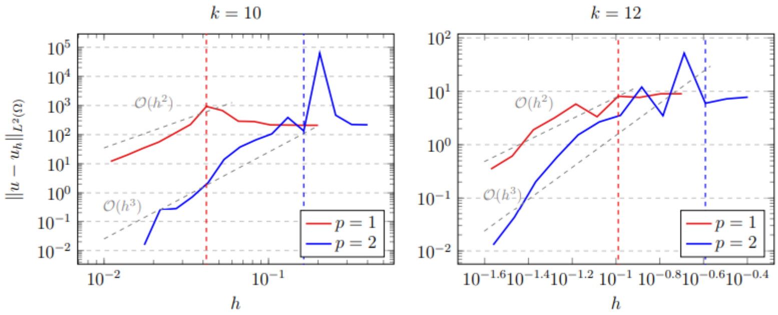
- $\mathcal{A}_0$ is uniformly T_h-coercive if and only if $\lambda_h^{(i_*)} < k^2$.
- This is equivalent to ensure that $\lambda_h^{(i_*)} - \lambda^{(i_*)} < k^2 - \lambda^{(i_*)}$

Discrete T-coercivity of $\mathcal{A}_0$



L^2-error of the approximation of the Helmholtz problem with Dirichlet boundary conditions against a computed reference solution.
The vertical lines indicate when $\lambda_h^{(i_*)} < k^2$.

Discrete T-coercivity of $\mathcal{A}_0$

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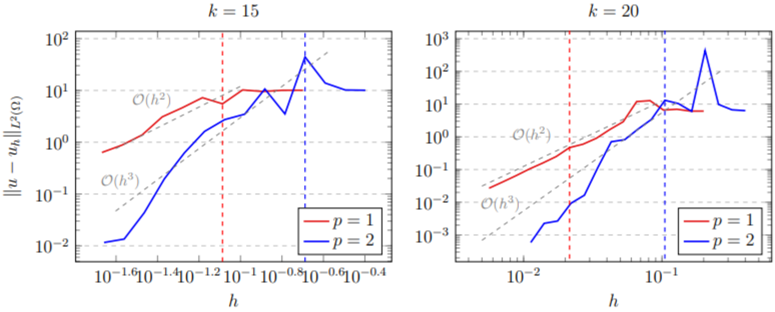

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Eigenvalue estimates

With classical eigenvalue estimates, we get

$$\lambda_h^{(i_*)} - \lambda^{(i_*)} \leq \lambda^{(i_*)} 4\sqrt{i_*} C_{\Omega,X} C_{\mathcal{I}} h^2,$$

where the constant $C_{\Omega,X}$ and $C_{\mathcal{I}}$ are defined as follows:

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So to have uniform T_h-coercivity, we want to ensure that

$$h^2 < \frac{k^2 - \lambda^{(i_*)}}{4\sqrt{i_*}\lambda^{(i_*)} C_{\Omega,X} C_{\mathcal{I}}}.$$

Quasi-optimality of $\mathcal{A}_0$

Theorem

The bilinear form $\mathcal{A}_0$ is uniformly T_h-coercive on X_h, if h is chosen such that

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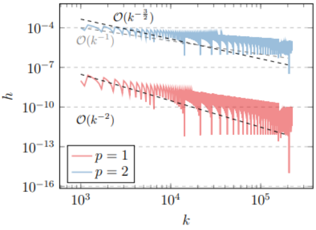

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University of Oxford Mathematical Institute

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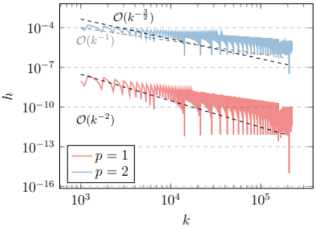

- This guarantees that the discrete problem is quasi-optimal provided h is small enough.

Adaptive scheme

Construct the mesh, with the minimal number of elements, that guarantees the quasi-optimality of the Helmholtz problem:

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- **Solving the Laplace eigenvalue problem adaptively:** Solve the Laplace eigenvalue problem on a sequence of refined meshes and check whether $k^2 - \lambda_h^{(i_*)} < 0$. If yes, we can stop because h is small enough s.t. we have uniform T_h-coercivity. (needs to use the same discretization as for Helmholtz)

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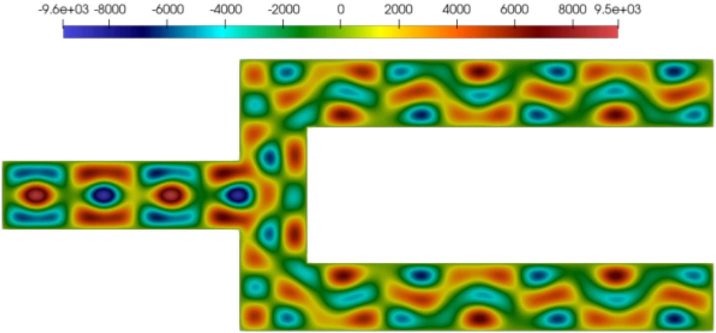
- **Solve the Helmholtz problem.**

Adpative scheme: numerical examples



(a) $Number of DoFs: 21521.$

Adpative scheme: numerical examples

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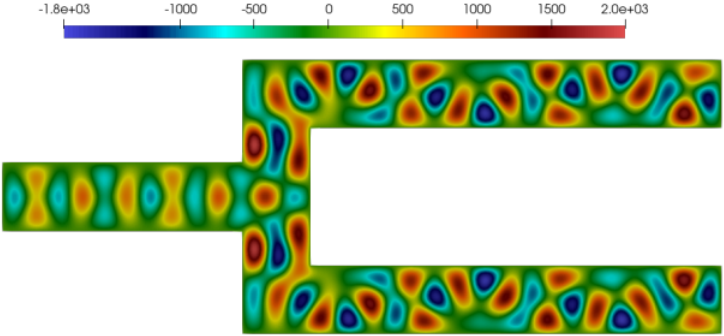
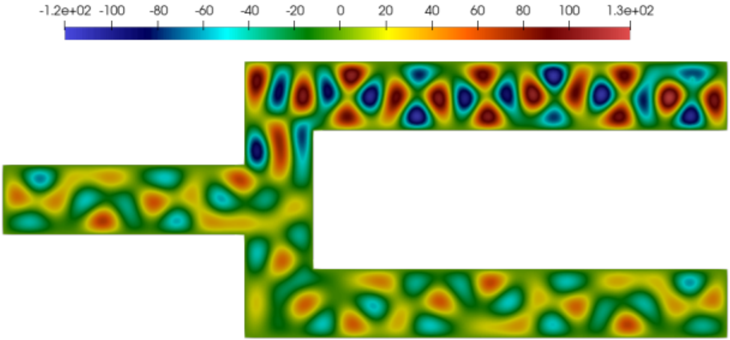


(b) $Number of DoFs : 84769.$

Adpative scheme: numerical examples



(c) $Number of DoFs : 336449.$

Adpative scheme: numerical examples

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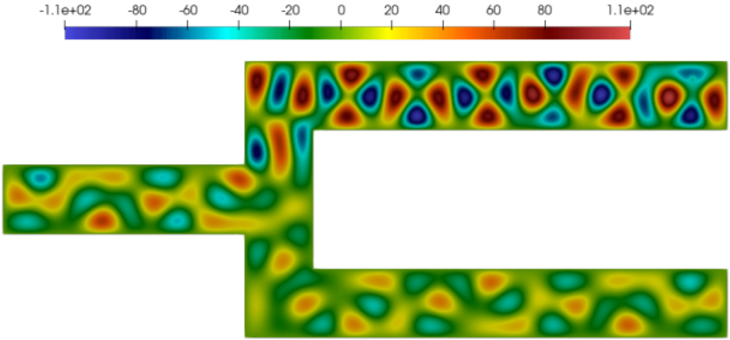


(d) $Number of DoFs : 1340545.$

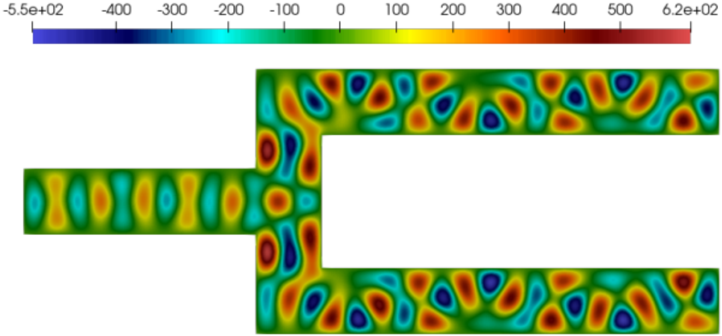
Adaptive^2 scheme

- In the adaptive scheme, we use a **Babuška–Rheinboldt** estimator to adaptively refine the mesh.
- The main idea is to use the **Babuška–Rheinboldt** estimator not one the desired solution or on a specific eigenfunction, but rather on the first $i_* + \ell$ eigenfunctions, i.e.

$$\eta_K = i_*^{-1} \sum_{i=1}^{i_*+\ell} h_K^2 \| \Delta e_h^{(i)} + \lambda_h^{(i)} e_h^{(i)} \|_{L^2(K)}^2 + \frac{h_K}{2} \| \nabla e_h^{(i)} \cdot n \|_{L^2(\partial K \setminus \partial \Omega)}^2 .$$

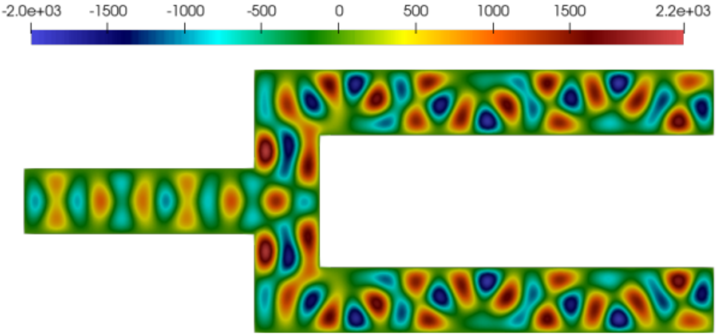
- We then refine the mesh in the elements where the indicator is larger. We can also adapt a **Dörfler marking** strategy.

Adpative^2 scheme: numerical examples



(a) $Number of DoFs : 77615.$

Adpative^2 scheme: numerical examples



(b) $Number of DoFs : 86733.$

Adpative^2 scheme: numerical examples

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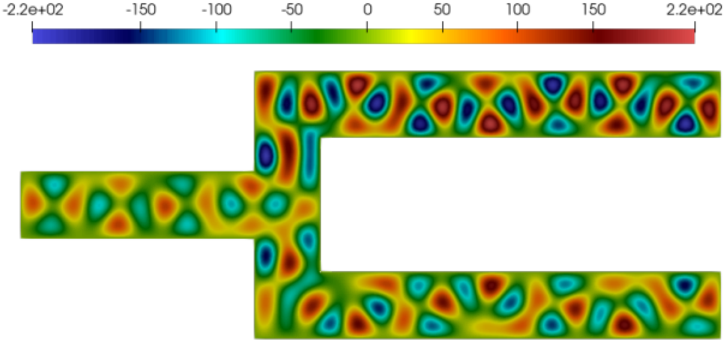
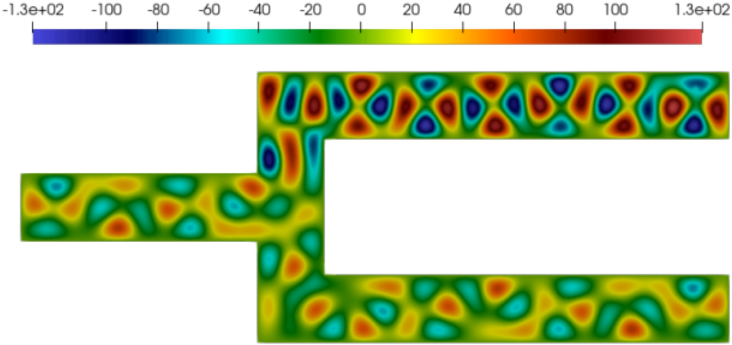


(c) $Number of DoFs$: 161102.

Adpative^2 scheme: numerical examples



(d) $Number of DoFs: 279034.$