Spectra and Pseudo-spectra

We will now focus on the spectral theory of bounded ENDOMORPHISM, i.e. $T \in \mathcal{B}(X,X)$. A fundamental tool for the definition of the is the RESOLVENT, i.e. $R_T(z) := (T - zI)^{-1}$, with $z \in \mathbb{C}$. The resolvent might not be well-defined over the all $\mathbb{C}$. We will drop the notation $\|\cdot\|_X$ in favour of simply $\|\cdot\|$.

Definition Given an operator $T \in \mathcal{B}(X,X)$ we call the resolvent set of T the subset of $\mathbb{C}$ such that $R_T(z)$ is well-defined and belongs to $\mathcal{B}(X,X)$, i.e. $\rho_T = \{ z \in \mathbb{C} : (T - zI) : X \to X \text{ is bijective} \}$.

We call the spectrum of $T \in \mathcal{B}(X,X)$ the complement of the resolvent set, i.e. $\sigma_T = \mathbb{C} \setminus \rho_T$.

Proposition Let $T \in \mathcal{B}(X,X)$, then ρ_T is an open set, and thus σ_T is closed.

Proof Let $z \in \rho_T$ and pick an other $z' \in \mathbb{C}$ such that $|z - z'| < \|R_T(z)\|^{-1}$, the operator $(T - z'I) = (T - zI) + \overbrace{(z - z')I}^{\delta}$ with $\delta = (z - z')I$, is in the hypothesis of the "First stability estimate" since $\|\delta\| = |z - z'| < \|R_T(z)\|^{-1}$ and thus we can apply the "First stability estimate" to state that $(T - z'I)$ is bijective and $R_T(z') \in \mathcal{B}(X,X)$.

Proposition Let $T \in \mathcal{B}(X,X)$ the spectrum of T is never empty, i.e. $\sigma_T \neq \phi$.

Proof Let's assume by contradiction that $\rho_T = \mathbb{C}$, then we have that $R_T(z)$ is analytic on $\mathbb{C}$. If $\sigma_T = \phi$ then it $(T - zI)^{-1}$ exists for all $z \in \mathbb{C}$, and we have that for any $z \in \mathbb{C}$ such that $|z| > \|T\|$ then the Neuman series $(T - zI)^{-1} = z^{-1} \sum_{k=0}^{\infty} (z^{-1} T)^k$ converges, picking $|z| \to \infty$ we see that $\|T - zI\| \to 0$, Hence $\|(z - T)^{-1}\|$ is bounded on the all $\mathbb{C}$ since $T \in \mathcal{B}(X,X)$ implies T bounded also in $B_{\|T\|}(0)$ and therefore we can apply Liouville theorem to say that $\|R_T(z)\| = 0$ on the all $\mathbb{C}$, which is a contradiction for $z = 0$.

Theorem Let $T \in \mathcal{B}(X,X)$ and $z \in \rho_T$ then the following inequality holds $\forall z \in \rho_T$, $\|R_T(z)\| > \gamma(z, \sigma_T)^{-1}$ (*).

Proof We pick $z \in \rho_T$ and observe that in previous theorem we have showed the fact that if $|z - z'| < \|R_T(z)\|^{-1}$ then $z' \in \rho_T$. Negating this steatment we obtain that picked $z \in \rho_T$ then $z' \in \sigma_T$ implies $|z - z'| \geq \|R_T(z)\|^{-1}$. Given that $z' \in \sigma_T$ can be picked arbitrarly we have, $\inf_{z' \in \sigma_T} |z - z'| > \|R_T(z)\|$ hence $\gamma(z, \sigma_T)^{-1} \leq \|R_T(z)\|$.

Proposition If $\cancel{z} \in \sigma_T$ then also $z \notin \sigma_{T+\delta}$, for any $\delta \in \mathcal{B}(X,X)$ such that $\|\delta\| \leq \|R_T(z)\|^{-1}$.

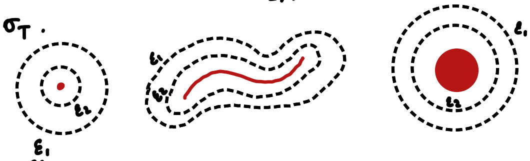
Proof The proof is a consequence of the proof of the fact that ρ_T is an open set.

The previous proposition assert the UPPER SEMI-CONTINUITY of the spectrum, i.e. an infinitesimal perturbation of T can change the spectrum only infinitesimal.

Now we give a definition of PSEUDO-SPECTRA based on the estimate (*) and analogues to the one given by [EMBREE-TREFETHEN].

Definition Let $T \in \mathcal{B}(X,X)$ and fix $\varepsilon > 0$. We denote $\sigma_{\varepsilon,T}$ the ε-PSEUDOSPECTRA of T, i.e.
$\sigma_{\varepsilon,T} := \{ z \in \rho_T : \|R_T(z)\| > 1/\varepsilon \} \cup \sigma_T$.

- SPECTRUM
- LEVEL SET OF ε-PSEUDOSPECTRUM



Example Let us consider an Helmoholtz like equation, i.e. $Tx - zI = y$, for some $z \in \rho_T$.
Notice that by the fact that $z \in \rho_T$ we know that given any $y \in Y$, then $\exists!$ $x \in X$ such that $x = R_T(z) y$. We then notice that, $\|x\| \leq \|R_T(z)\| \|y\|$ unfortunatly the previous proposition only gives us a lower bound on $\|R_T(z)\|$ hence we have no control over the norm of the solution. The situatin is very different when the operator are normal. In fact the lack of a control on the norm of the solution suggests that for non-normal operator the Helmholtz problem can be terribly ill-posed, i.e. for a small perturbation of the data we can obtain a completly different solution from the unperturbed problem. This shuld prompt us to wonder the physical significance of non-normal operators problems. This idea is perfectly captured by the the following quote from Nick Trefethen:

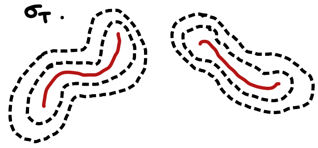
« If your answer is highly sensitive to perturbations, you have probably asked the wrong question. »

In some sense then the complement of the pseudo-spectra is an extremely interesting set, i.e.

$\sigma_{\varepsilon,T}^{\,c} = \{ z \in \rho_T \ : \ \| R_T(z) \| < \varepsilon^{-1} \}$, we will call this set the PSEUDO-RESOLVENT.
Fixed a level of stability we are willing to tollerate the PSEUDO-RESOLVENT $\sigma_{\varepsilon,T}^{c}$ is the set of $z \in \mathbb{C}$ such that we can control how sensitive our problem is to perturbations, i.e. for which the following stability estimate holds, $\|x\| \le \|R_T(z)\| \|y\| \le \varepsilon^{-1} \|y\|$.
This simply means that a small data must provide a small solution, hence a small variation of data can only cause a small variation of the solution. This doesn't mean that a small perturbation of the solution corresponds to a small perturbation of the data.

PROPOSITION Let $T \in B(X,X)$, for any $\varepsilon_1 > \varepsilon_2$ then $\sigma_{\varepsilon_2,T} \subseteq \sigma_{\varepsilon_1,T}$ and $\bigcap_{\varepsilon > 0} \sigma_{\varepsilon,T} = \sigma_T$. Furthermore each connected component of $\sigma_{\varepsilon,T}$ has a non-empty intersection with σ_T. Lastly we notice $\sigma_{\varepsilon,T}$ is closed.

PROOF We begin observing that from the definition it is clear that for $\varepsilon_1 > \varepsilon_2$, $\sigma_{\varepsilon_2,T} \subseteq \sigma_{\varepsilon_1,T}$. Furthermore we also clearly see that the inclusion $\sigma_{\varepsilon,T} \setminus \sigma_T \subsetneq \sigma_{\varepsilon_1,T}$ holds hence $\bigcap_{\varepsilon > 0} \sigma_{\varepsilon,T}$ must be the empty set hence $\bigcap_{\varepsilon>0} \sigma_{\varepsilon,T} = \sigma_T$.
Notice that it $\exists z \in \partial\sigma_T$ such that $\forall\, 0<\delta<\varepsilon$ $B_\delta(z) \cap (\sigma_{\varepsilon,T} \setminus \sigma_T) \neq \emptyset$, this because we know that $\|R_T(z)\| > \gamma(z, \sigma_T)^{-1}$ hence $\emptyset \neq B_\delta(z) \cap \sigma_{\delta,T} \subseteq B_{\varepsilon,T}(z) \cap \sigma_{\varepsilon,T}$.
We observe that $R_T(z)$ is analytic on ρ_T and therefore $\|R_T(z)\|$ is a bounded continuous function is subharmonic. Using the maximum principle of subharmonic function we conclude that each connected component of $\sigma_{\varepsilon,T}$ has a non-empty intersection with σ_T.
By the maximum principle the maximum of $\|R_T(z)\|$ is attained at $\partial \sigma_{\varepsilon,T} \setminus \sigma_T$.



DEFINITION We call $z \in \mathbb{C}$ an eigenvalue of $T \in B(X,X)$ if $N(T - zI) \neq \{0\}$.
We denote the set of eigenvalue of the operator T, Ξ_T, and given $z \in \Xi_T$ we call $N(T-zI)$ the eigenspace corresponding to z.

PROPOSITION Let $T \in B(X,X)$ then $\Xi_T \subseteq \sigma_T$.
PROOF Notice that if $z \in \Xi_T$ then $N(T-zI) \neq \{0\}$ thus the operator is not injective hence $z \in \sigma_T$.
Yet it might happen that $N(T-zI) = \{0\}$ and $R(T-zI) \neq X$, in this case we have that $z \in \sigma_T$ but $z \in \Xi_T$.

EXAMPLE Let $X = \ell^2$ and consider the right-shift operator, i.e. $T: X \to X$ $(u_1, u_2, \cdots) \mapsto (0, u_1, u_2, \cdots)$
Then it is clear that $\forall \lambda \in \mathbb{C}$ $Tu = \lambda u \iff (0, u_1, u_2, \cdots) = (\lambda u_1, \lambda u_2, \cdots) \iff \lambda u_1 = 0, \lambda u_2 = u_1 \iff \underline{u} = \underline{0}$.
Hence Ξ_T is empty. Notice that $\sigma_T \neq \emptyset$ for every T, hence $\Xi_T \subseteq \sigma_T$.
As discussed before the only mode for which $z \in \sigma_T \setminus \Xi_T$ is if $R(T-zI) \neq X$, which by FREDHOLM ALTERNATIVE is impossible if $z \neq 0$.

DEFINITION Given an operator $T \in B(X,X)$ we call the spectral radius of T, $r_T := \sup_{z \in \sigma_T} |z|$.

THEOREM (Gelfand formula) Let $T \in B(X,X)$ then $r_T = \lim_{n \to \infty} \|T^n\|^{1/n}$.

COROLLARY Let $T \in B(H,H)$ if T is a SELF-ADJOINT operator then $r_T = \|T\|$.

PROOF We begin observing that if T is self-adjoint and we pick $\|h\| = 1$ and observe that, $\|Th\|^2 = (Th, Th) = (T^2 h, h) \le \|T^2 h\| \|h\| = \|T^2 h\|$ which implies $\|T^{2n}\| = \|T\|^{2n}$, then we use Gelfand formula $r_T = \lim_{n\to\infty} \|T^{2n}\|^{1/2n} = \lim_{n\to\infty} \|T\| = \|T\|$.

COROLLARY Let $T \in B(H,H)$ be a SELF-ADJOINT operator then if $\sigma_T = \{0\}$, T must vanish.

PROOF The proof of this result can be found as THEOREM 5.11 in [BREZIS].

DEFINITION Let $T \in K(X,X)$ and $z \in \Xi_T \setminus \{0\}$. We define the ASCENT of z the smallest $\alpha \in \mathbb{N}$ such that $N((T-zI)^\alpha) = N((T-zI)^{\alpha+1})$. The dimension of $N((T-zI)^\alpha)$ is the ALGEBRAIC MULTIPLICITY of z and the elements of $N((T-zI)^\alpha)$ are the GENERALISED EIGENVECTORS of T. The dimension of $N(T-zI)$ is known as the GEOMETRIC MULTIPLICITY of z.

THEOREM Let $T \in B(H,H)$ be a SELF-ADJOINT OPERATOR, then $\forall z \in \Xi_T \setminus \{0\}$ the ascent of z is one and the algebraic and geometric multiplicity coincide.

We can generalise the notion of eigenvalues in a similar manner to how we generalised the notion of spectra introducing the PSEUDO SPECTRA.

DEFINITION Let $T \in \mathcal{B}(X,X)$ and fix $\varepsilon > 0$ we call $z \in \mathbb{C}$ an ε-eigenvalue, if it $\exists x \in X$ such that $\|x\| = 1$ and $\|(T - zI)x\| < \varepsilon$. We denote $\Xi_{\varepsilon,T}$ the set of all eigenvalues of T and call x an ε-eigen function associated with T.

PROPOSITION Let $T \in \mathcal{B}(X,X)$ then $\Xi_{\varepsilon,T} \subseteq \sigma_{\varepsilon,T}$.

PROOF Notice that in general $\|R_T(z)\| \geq \|T - zI\|^{-1} = [\sup_{\substack{x \in X \\ \|x\|=1}} \|(T - zI)x\|]^{-1}$. If $z \in \Xi_{\varepsilon,T}$ and $\|(T - zI)x\| < \varepsilon$, it $\exists x \in X$, with $\|x\| = 1$, and for which $\|(T - zI)x\| > \varepsilon$, and therefore $\|T - zI\|^{-1} > \frac{1}{\varepsilon}$, which imply that $z \in \sigma_{\varepsilon,T}$.

We now focus our attention on operators with isolated spectra, of which compact operators are an important class of representatives.

PROPOSITION Let $T \in K(X,X)$, then $0 \in \sigma_T$ and $\sigma_T \setminus \{0\} = \Xi_T \setminus \{0\}$.

PROOF Suppose that $0 \notin \sigma_T$ then T is bijective and $T \circ T^{-1}$ must be compact. Hence the identity operator would be compact on infinite-dimensional Banach space which is a contradiction by the RIESZ UNIT BALL THEOREM.

THEOREM Let $T \in K(X,X)$ and let $\{z_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ be a sequence such that $\lim_{n \to \infty} z_n = z$ and $z_n \in \sigma_T \setminus \{0\}$. Then $z = 0$, hence this implies that $\sigma_T \setminus \{0\}$ is only made of isolated points.

PROOF We know that $z_n \in \Xi_T$ by hypothesis, thus we can consider the eigen functions x_n and denote E_n the space formed by the first n eigen functions.

We begin observing that $E_n \subsetneq E_{n+1}$, i.e. the first n eigenvectors are linearly independent. Thus we assume by contradiction that $e_{n+1} = \sum_{k=1}^{n} \alpha_k e_n$ and therefore, $T e_{n+1} = \sum_{k=1}^{n} z_{k+1} \alpha_k e_k$ then $\alpha_k (z_k - z_{k+1})$ must vanish for all $k = 1, \dots, n$. Thus α_k needs to vanish for $k = 1, \dots, n$ hence $e_{n+1} = 0$ which is a contradiction.

Applying RIESZ LEMMA we can construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\|x_n\| = 1$, $x_n \in E_n$ and $\gamma(x_n, E_{n-1}) \geq \frac{1}{2}$, $\forall n \geq 2$.

Notice now that $\forall n > m > 2$ we have that, since $(T - z_n I) E_n \subseteq E_{n-1}$ and $E_{m-1} \subsetneq E_m \subsetneq E_{n-1} \subsetneq E_n$.

$$(*) \quad \|z_n^{-1} T x_n - z_m^{-1} T x_m\| = \|z_n^{-1} \underbrace{(T x_n - z_n x_n)}_{\in E_{n-1}} - z_m^{-1} \underbrace{(T x_m - z_m x_m)}_{\in E_{m-1} \subsetneq E_{n-1}} + x_n - \underbrace{x_m}_{\in E_m \subseteq E_{n-1}}\| \geq \gamma(x_n, E_{n-1}) \geq \frac{1}{2}.$$

If $z_n \to z$ and $z \neq 0$, this would imply that $\|T x_m - T x_n\| \leq \|z_n x_n - z_m x_m\| \leq |z_n - z_m| \searrow 0$, which would imply $\{T x_n\}$ has a converging subsequence which contradicts $(*)$.

This implies that the spectrum of T is formed by isolated points.

COROLLARY Let $T \in K(X,X)$ then one of the following scenario applies: 1. $\sigma_T = \{0\}$, 2. $\sigma_T \setminus \{0\}$ is finite, 3. $\sigma_T \setminus \{0\}$ is a sequence converging to 0.

PROOF For every integer $n \geq 1$ we notice that the set $\sigma_T \cap \{z \in \mathbb{C} : |z| \geq 1/n\}$ is either empty of finite, if not we would have a subsequence to some z with $|z| \geq 1/n$. This because T is compact. In fact if $\{z_n\}_{n \in \mathbb{N}} \subseteq \{z \in \mathbb{C} : |z| \geq 1/n\}$ then we can consider the sequence of associated eigenvectors $\{x_n\}_{n \in \mathbb{N}}$ and observe the sequence is bounded hence $\{T x_n\}$ has a converging subsequence which imply $\lim_{n \to \infty} z_n = z \geq \frac{1}{n}$ which contradicts the previous theorem.

PROPOSITION Let $T \in \mathcal{B}(X,X)$ be a FINITE-RANK OPERATOR, i.e. $\dim R(T_n) < +\infty$, the $\sigma_T \setminus \{0\}$ has finitely many elements.

PROOF The elements of $\sigma_T \setminus \{0\}$ are the eigenvalues since $\dim R(T_n) < +\infty$, there can only be finite eigenvectors.

DEFINITION Let $\{h_n\}_{n \in \mathbb{N}} \subseteq H$ be a sequence of ortonormal elements of H. We say that $\{e_n\}$ is an Hilbert basis if $\langle \bigcup_{n \geq 1} e_n \rangle$ is dense in H.

THEOREM Every SEPARABLE Hilbert space has an Hilbert basis.

THEOREM Let H be a SEPARABLE Hilbert space and let $T \in K(H,H)$ be SELF-ADJOINT. Then there exist a Hilbert basis composed by the eigenvectors of T.

PROOF Let $\{z_n\}_{n\geq 1}$ be a sequence of all distinct, non zero eigenvalues of T. We will additionaly set $\lambda_0 = 0$, $E_0 = N(T)$ and denote E_n the eigenspace associated to z_n. Notice that $\dim(E_n) < \infty$, as a consequence of FREDHOLM ALTERNATIVE. We will now prove that H is the Hilbert sum of all E_n. We do this in two steps. First we prove that $(E_n)_{n\geq 0}$ are mutually orthogonal. We pick $h \in E_m$ and $h' \in E_n$ with $m \neq n$, then $Th = z_m h$ and $Th' = z_n h'$, thus we have $z_m (h,h') = (Th, h') = (h, Th') = z_n (h,h')$ hence because $z_m \neq z_n$ we have that $(h,h') = 0$. We now set to prove that vector space E spanned by $\bigcup_{n\geq 0} E_n$ is dense in H. First we observe that $TE \subseteq E$, furthermore given $h' \in E^{\perp}$ we have $(Th^{\perp}, h') = (h^{\perp}, Th') = 0$ $\forall h' \in E$ hence $Th' \in E^{\perp}$, thus $TE^{\perp} \subseteq E^{\perp}$. We denote $T^{\perp} : E^{\perp} \to E^{\perp}$ $h^{\perp} \mapsto Th^{\perp}$. We claim that $\sigma(T^{\perp}) = \{0\}$. By contradiction let assume this is not the case and that some $z \neq 0 \in \sigma_{T^{\perp}}$. Since $z \in \Xi_{T^{\perp}}$ then it means that there $\exists x \neq 0$ such that $T^{\perp} h = z h$ hence z is also an eigenvalue of T, hence $h^{\perp} \in E^{\perp} \cap E$ which is a contradiction. Since $\sigma_{T^{\perp}} = \{0\}$ we know that T vanish on $E^{\perp}$, by one of the previous corollary, thus $E^{\perp} \subseteq N(T)$. Furthermore $N(T) \subseteq E_0 \subseteq E$ hence $E^{\perp} \subseteq E$ which implies that $E^{\perp} = \{0\}$, hence E is dense in H, since H separable and $H = E^{\perp} \oplus E$.
Finally we pick in each subspace a Hilbert basis, the existence of such a basis for E_0 is a consequence of the fact that all separable Hilbert space have an Hilbert base and for E_n, with $n>0$ is just a consequence of the fact that E_n is finite-dimensional.

THEOREM The same result as above applies even if $T \in \mathcal{B}(X,X)$ is a normal operator.

THEOREM Let $T \in K(X,X)$ be a normal operator then $\sigma_{\varepsilon,T} = \{ z \in \mathbb{C} \ : \ z_1 + z_2 \in \sigma_T \ , \ z_2 \in B_\varepsilon(\underline{0}) \}$.

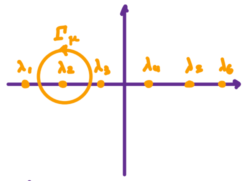
PROOF We begin considering a Hilbert basis for H formed by the eigenvectors of T, i.e. $\{h_n\}_{n\in\mathbb{N}}$.
We then observe that $\|R_T(z)\| = \sup_{n\in\mathbb{N}} \frac{\|R_T(z) h_n\|}{|h_n|} = \sup_{n\in\mathbb{N}} \|(T - zI)^{-1} h_n\| \overset{*}{=} \sup_{n\in\mathbb{N}} \frac{1}{\lambda_n - z} = \frac{1}{\gamma(z, \sigma_T)}$.

$^{*}(T - zI) h_n = Th_n - z h_n = (\lambda_n - z) h_n$

$(T - zI) \frac{1}{\lambda_n - z} h_n = h_n \Rightarrow (T - zI)^{-1} h_n = \frac{1}{\lambda_n - z} h_n$

EXAMPLE We have previously discussed how the complement of the pseudo-spectra tell us for which value of z the Helmholtz problem is not "ill-conditioned" from the physical point of view. Notice that in the case of $T \in \mathcal{B}(X,X)$ being SELF-ADJOINT we also have the upper-bound $\|R_T(z)\| \leq \gamma(z, \sigma_T)$ hence for $z \in \mathbb{C}$ we have that the solution of the Helmholtz problem depends continuosly on the data, i.e. $\|x\| \leq \gamma(z, \sigma_T) \|y\|$. This argument to obtain the continuous dependence of the solution from the data is equivalent to the T-coercivity argument by CIARLET.

THEOREM Let $T \in K(H,H)$ be a NORMAL OPERATOR. Fixed $\mu \in \sigma_T \setminus \{0\}$ we consider a curve $\Gamma_\mu \subseteq \rho_T$ only encapsulating μ and no other element of σ_T, then the operator $\Lambda_{T,\Gamma_\mu} : H \to H$ $\quad x \mapsto \frac{1}{2\pi i} \int_{\Gamma_\mu} R_T(z) x \, dz$ is a projection on the eigenspace associated to μ.



PROOF We consider a Hilbert basis for H composed by the eigenvectors of T, i.e. $\{h_n\}_{n\in\mathbb{N}}$. The action of $R_T(z)$ can be expressed as $R_T(z) h \overset{*}{=} \sum_{n\in\mathbb{N}} (\lambda_n - z)^{-1} (h_n, h) h_n$ and thus we can expressed the action of Λ_μ as $\Lambda_{T,\Gamma_\mu} h = \frac{1}{2\pi i} \int_{\Gamma_\mu} \sum_{n\in\mathbb{N}} \underbrace{(\lambda_n - z)^{-1} (h_n, h) h_n}_{F_n(z)} \, dz$. We begin assuming that all eigenvalues are simple, hence their geometric multiplicity is 1.
Thus we notice by CAUCHY RESIDUE THEOREM that $\Lambda_{T,\Gamma_\mu} h = \frac{1}{2\pi i} 2\pi i \sum_{n\in\mathbb{N}} \mathrm{Res}(F_n(z), \mu)(h_n, h) h_n$ this because μ is the only pole contained in Γ_μ and it is a pole only for F_{n_μ} where n_μ denotes the index of the Hilbert basis associated with μ. Hence only $\mathrm{Res}(F_{n_\mu}(z), \mu)$ is non vanishing and it is equal to 1, thus $\Lambda_{T,\Gamma_\mu} h = (h_n, h) h_n$ which is the orthogonal projection of h onto to the eigenfunction h_n. To generalise the proof to the case when eigenvalues are not simple it

is sufficent to observe that multiple residual will not vanish leaving $\Lambda_{T,\Gamma_\mu} h = \sum_{n \in I_\mu} (h_n, h)\, h_n$ with $I_\mu = \{ n : h_n \in N(T-\mu I) \}$ which is exactly the projection on the eigen space $N(T-\mu I)$.