

## SPECTRA AND PSEUDO-SPECTRA

We will now focus on the spectral theory of bounded ENDOMORPHISM, i.e.  $T \in \mathcal{B}(X, X)$ . A fundamental tool for the definition of the is the RESOLVENT, i.e.  $R_T(z) := (T - zI)^{-1}$ , with  $z \in \mathbb{C}$ . The resolvent might not be well-defined over the all  $\mathbb{C}$ . We will drop the notation  $\| \cdot \|_X$  in favour of simply  $\| \cdot \|$ .

**DEFINITION** Given an operator  $T \in \mathcal{B}(X, X)$  we call the resolvent set of  $T$  the subset of  $\mathbb{C}$  such that  $R_T(z)$  is well-defined and belong to  $\mathcal{B}(X, X)$ , i.e.  $\rho_T = \{z \in \mathbb{C} : (T - zI) : X \rightarrow X \text{ is bijective}\}$ .

We call the spectrum of  $T \in \mathcal{B}(X, X)$  the complement of the resolvent set, i.e.  $\sigma_T = \overline{\mathbb{C}} \setminus \rho_T$ .

**PROPOSITION** Let  $T \in \mathcal{B}(X, X)$ , then  $\rho_T$  is an open set, and thus  $\sigma_T$  is closed.

**PROOF** Let  $z \in \rho_T$  and pick an other  $t' \in \mathbb{C}$  such that  $|z - t'| < \|R_T(z)\|^{-1}$ , the operator  $(T - z'I) = (T - t'I) + (z - t')I$  with  $\delta = (z - t')I$ , is in the hypothesis of the "First stability estimate" since  $\|\delta\| = |z - t'| < \|R_T(z)\|^{-1}$  and thus we can apply the "First stability estimate" to state that  $(T - t'I)$  is bijective and  $R_{T-t'}(t') \in \mathcal{B}(X, X)$ .

**PROPOSITION** Let  $T \in \mathcal{B}(X, X)$  the spectrum of  $T$  is never empty, i.e.  $\sigma_T \neq \emptyset$ .

**PROOF** Let's assume by contradiction that  $\sigma_T = \emptyset$ , then we have that  $R_T(z)$  is analytic on  $\mathbb{C}$ . If  $\sigma_T = \emptyset$  then it  $(T - zI)^{-1}$  exists for all  $z \in \mathbb{C}$ , and we have that for any  $t \in \mathbb{C}$  such that  $|z| > \|T\|$  then the Neuman series  $(T - zI)^{-1} = z^{-1} \sum_{k=0}^{\infty} (z^{-1} T)^k$  converges, picking  $|z| \rightarrow \infty$  we see that  $\|T - zI\| \rightarrow 0$ , Hence  $\|(z - T)^{-1}\|$  is bounded on the all  $\mathbb{C}$  since  $T \in \mathcal{B}(X, X)$  implies  $T$  bounded also in  $B_{\|T\|}(\mathbb{C})$  and therefore we can apply Liouville theorem to say that  $\|R_T(z)\| = 0$  on the all  $\mathbb{C}$ , which is a contradiction for  $z=0$ .

**THEOREM** Let  $T \in \mathcal{B}(X, X)$  and  $z \in \rho_T$  then the following inequality holds  $\forall t \in \rho_T, \|R_T(t)\| \geq \gamma(z, \sigma_T)^{-1}$ .

**PROOF** We pick  $z \in \rho_T$  and observe that in previous theorem we have showed the fact that if  $|z - t| < \|R_T(z)\|^{-1}$  then  $t \in \rho_T$ . Negating this statement we obtain that picked  $t \in \rho_T$  then  $t \in \sigma_T$  implies  $|z - t| \geq \|R_T(z)\|^{-1}$ .

Given that  $z \in \rho_T$  can be picked arbitrarily we have,  $\inf_{z \in \rho_T} |z - t| \geq \|R_T(z)\|$  hence  $\gamma(z, \sigma_T)^{-1} \leq \|R_T(z)\|$ .

**PROPOSITION** If  $z \in \sigma_T$  then also  $z \in \sigma_{T+\delta}$ , for any  $\delta \in \mathcal{B}(X, X)$  such that  $\|\delta\| \leq \|R_T(z)\|^{-1}$ .

**PROOF** The proof is a consequence of the proof of the fact that  $\rho_T$  is an open set.

The previous proposition assert the UPPER SEMI-CONTINUITY of the spectrum, i.e. an infinitesimal perturbation of  $T$  can change the spectrum only infinitesimal.

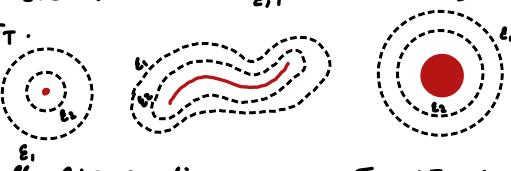
Now we give a definition of PSEUDO-SPECTRA based on the estimate (\*) and analogous to the one given by [ENBREY-TREFETHEN].

**DEFINITION** let  $T \in \mathcal{B}(X, X)$  and fix  $\epsilon > 0$ . We denote  $\sigma_{\epsilon, T}$  the  $\epsilon$ -PSEUDOSPECTRUM of  $T$ , i.e.

$$\sigma_{\epsilon, T} := \{z \in \rho_T : \|R_T(z)\| > 1/\epsilon\} \cup \sigma_T.$$

■ SPECTRUM

■ LEVEL SET OF  $\epsilon$ -PSEUDOSPECTRUM



**EXAMPLE** Let us consider an Helmholtz like equation, i.e.  $Tx - tI = y$ , for some  $t \in \rho_T$ .

Notice that by the fact that  $z \in \rho_T$  we know that given any  $y \in \mathbb{X}$ , there  $\exists! z \in \mathbb{X}$  such that  $x = R_T(z)y$ . We then notice that,  $\|z\| \leq \|R_T(z)\| \|y\|$  unfortunately the previous proposition only gives us a lower bound on  $\|R_T(z)\|$  hence we have no control over the norm of the solution. The situation is very different when the operator are normal. In fact the lack of a control on the norm of the solution suggests that for non-normal operator the Helmholtz problem can be terribly ill-posed, i.e. for a small perturbation of the data we can obtain a completely different solution from the unperturbed problem. This should prompt us to wonder the physical significance of non-normal operators problems. This idea is perfectly captured by the following quote from Nick Trefethen:

"If your answer is highly sensitive to perturbations, you have probably asked the wrong question."

In some sense then the complement of the pseudo-spectra is an extremely interesting set, i.e.

$\sigma_{\epsilon,T}^c = \{ z \in \sigma_T : \|R_T(z)\| < \epsilon^{-1} \}$ , we will call this set the PSEUDO-RESOLVENT.

Fixed a level of stability we are willing to tolerate the PSEUDO-RESOLVENT  $\sigma_{\epsilon,T}^c$  is the set of  $z \in \mathbb{C}$  such that we can control how sensitive our problem is to perturbations, i.e. for which the following stability estimate holds,  $\|z\| \leq \|R_T(z)\| \|y\| \leq \epsilon^{-1} \|y\|$ .

This simply means that a small data must provide a small solution, hence a small variation of data can only cause a small variation of the solution. This doesn't mean that a small perturbation of the solution corresponds to a small perturbation of the data.

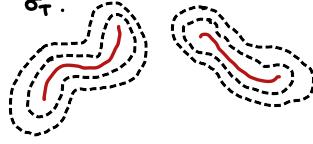
**PROPOSITION** Let  $T \in \mathcal{B}(X,X)$ , for any  $\epsilon_1 > \epsilon_2$  then  $\sigma_{\epsilon_1,T}^c \subseteq \sigma_{\epsilon_2,T}^c$  and  $\bigcap_{\epsilon > 0} \sigma_{\epsilon,T}^c = \sigma_T$ . Furthermore each connected component of  $\sigma_{\epsilon,T}^c$  has a non-empty intersection with  $\sigma_T$ . Lastly we notice  $\sigma_{\epsilon,T}^c$  is closed.

**PROOF** We begin observing that from the definition it is clear that for  $\epsilon_1 > \epsilon_2$ ,  $\sigma_{\epsilon_2,T}^c \subseteq \sigma_{\epsilon_1,T}^c$ . Furthermore we also clearly see that the inclusion  $\sigma_{\epsilon,T}^c \setminus \sigma_T \subseteq \sigma_{\epsilon_1,T}^c$  holds hence  $\bigcap_{\epsilon > 0} \sigma_{\epsilon,T}^c$  must be the empty set hence  $\bigcap_{\epsilon > 0} \sigma_{\epsilon,T}^c = \sigma_T$ .

Notice that if  $\exists z \in \partial \sigma_T$  such that  $\forall \epsilon < \epsilon_0 \quad B_\epsilon(z) \cap (\sigma_{\epsilon,T}^c \setminus \sigma_T) \neq \emptyset$ , this because we know that  $\|R_T(z)\| > \gamma(z, \sigma_T)^{-1}$  hence  $\emptyset \neq B_\epsilon(z) \cap \sigma_{\epsilon,T}^c \subseteq B_\epsilon(z) \cap \sigma_T$ .

We observe that  $R_T(z)$  is analytic on  $\sigma_T$  and therefore  $\|R_T(z)\|$  is a bounded continuous function is subharmonic. Using the maximum principle of subharmonic function we conclude that each connected component of  $\sigma_{\epsilon,T}^c$  has a non-empty intersection with  $\sigma_T$ .

By the maximum principle the maximum of  $\|R_T(z)\|$  is attained at  $\partial \sigma_{\epsilon,T}^c \setminus \sigma_T$ .



**DEFINITION** We call  $z \in \mathbb{C}$  an eigenvalue of  $T \in \mathcal{B}(X,X)$  if  $N(T-zI) \neq \{0\}$ .

We denote the set of eigenvalues of the operator  $T$ ,  $\sigma_T$ , and given

$z \in \sigma_T$  we call  $N(T-zI)$  the eigenspace corresponding to  $z$ .

**PROPOSITION** Let  $T \in \mathcal{B}(X,X)$  then  $\sigma_T \subseteq \sigma_T$ .

**PROOF** Notice that if  $z \in \sigma_T$  then  $N(T-zI) \neq \{0\}$  thus the operator is not injective hence  $z \in \sigma_T$ .

Yet it might happen that  $N(T-zI) = \{0\}$  and  $R(T-zI) \neq X$ , in this case we have that  $z \in \sigma_T$  but  $z \notin \sigma_T$ .

**EXAMPLE** Let  $X = \ell^2$  and consider the right-shift operator, i.e.  $T: X \rightarrow X \quad (u_1, u_2, \dots) \mapsto (0, u_1, u_2, \dots)$

Then it is clear that  $\forall \lambda \in \mathbb{C} \quad Tu = \lambda u \Leftrightarrow (0, u_1, u_2, \dots) = (\lambda u_1, \lambda u_2, \dots) \Leftrightarrow \lambda u_1 = 0, \lambda u_2 = u_1 \Leftrightarrow \lambda = 0$ .

Hence  $\sigma_T$  is empty. Notice that  $\sigma_T \neq \emptyset$  for every  $T$ , hence  $\sigma_T \subseteq \sigma_T$ .

As discussed before the only mode for which  $z \in \sigma_T \setminus \sigma_T$  is if  $R(T-zI) \neq X$ , which by FREDHOLM ALTERNATIVE is impossible if  $z \neq 0$ .

**DEFINITION** Given an operator  $T \in \mathcal{B}(X,X)$  we call the spectral radius of  $T$ ,  $r_T := \sup_{z \in \sigma_T} |z|$ .

**THEOREM (Gelfand formula)** Let  $T \in \mathcal{B}(X,X)$  then  $r_T = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ .

**COROLLARY** Let  $T \in \mathcal{B}(H,H)$  if  $T$  is a SELF-ADJOINT operator then  $r_T = \|T\|$ .

**PROOF** We begin observing that if  $T$  is self-adjoint and we pick  $\|h\| = 1$  and observe that,

$\|Th\|^2 = (Th, Th) = (T^2 h, h) \leq \|T^2 h\| \|h\| = \|T^2 h\|$  which implies  $\|T^{2n}\| = \|T\|^{2n}$ , then we use Gelfand formula  $r_T = \lim_{n \rightarrow \infty} \|T^{2n}\|^{1/2n} = \lim_{n \rightarrow \infty} \|T\| = \|T\|$ .

**COROLLARY** Let  $T \in \mathcal{B}(H,H)$  be a SELF-ADJOINT operator then if  $\sigma_T = \{0\}$ ,  $T$  must vanish.

**PROOF** The proof of this result can be found as THEOREM 5.11 in [BREZIS].

**DEFINITION** Let  $T \in \mathcal{B}(X,X)$  and  $z \in \sigma_T \setminus \{0\}$ . We define the ASCENT of  $z$  the smallest  $\alpha \in \mathbb{N}$  such that  $N((T-zI)^\alpha) = N((T-zI)^{\alpha+1})$ . The dimension of  $N((T-zI)^\alpha)$  is the ALGEBRAIC MULTIPLICITY of  $z$  and the elements of  $N((T-zI)^\alpha)$  are the GENERALISED EIGENVECTORS of  $T$ . The dimension of  $N(T-zI)$  is known as the GEOMETRIC MULTIPLICITY of  $z$ .

**THEOREM** Let  $T \in \mathcal{B}(H,H)$  be a SELF-ADJOINT OPERATOR, then  $\forall z \in \sigma_T \setminus \{0\}$  the ascent of  $z$  is one and the algebraic and geometric multiplicity coincide.

We can generalize the notion of eigenvalues in a similar manner to how we generalized the notion of spectra introducing the PSEUDO SPECTRA.

**DEFINITION** Let  $T \in \mathcal{B}(X, X)$  and fix  $\varepsilon > 0$  we call  $\lambda \in \mathbb{C}$  an  $\varepsilon$ -eigenvalue, if it  $\exists z \in X$  such that  $\|z\| = 1$  and  $\|(T - \lambda I)z\| < \varepsilon$ . We denote  $\Sigma_{\varepsilon, T}$  the set of all eigenvalues of  $T$  and call  $z$  an  $\varepsilon$ -eigenfunction associated with  $T$ .

**PROPOSITION** Let  $T \in \mathcal{B}(X, X)$  then  $\Sigma_{\varepsilon, T} \subseteq \sigma_{\varepsilon, T}$ .

**PROOF** Notice that in general  $\|R_T(z)\| \geq \|T - zI\|^{-1} = [\sup_{x \in X} \|(T - zI)x\|]^{-1}$ . If  $z \in \Sigma_{\varepsilon, T}$  and  $\|(T - zI)z\| < \varepsilon$ , it  $\exists x \in X$ , with  $\|x\|=1$ , and for which  $\|(T - zI)x\| > \varepsilon$ , and  $\|x\|=1$  therefore  $\|T - zI\|^{-1} > \frac{1}{\varepsilon}$ , which imply that  $z \in \sigma_{\varepsilon, T}$ .

We now focus our attention on operators with isolated spectra, of which compact operators are an important class of representatives.

**PROPOSITION** Let  $T \in K(X, X)$ , then  $0 \in \sigma_T$  and  $\sigma_T \setminus \{0\} = \Sigma_T \setminus \{0\}$ .

**PROOF** Suppose that  $0 \notin \sigma_T$  then  $T$  is bijective and  $T^{-1}$  must be compact. Hence the identity operator would be compact on infinite-dimensional Banach space which is a contradiction by the RIESZ UNIT BALL THEOREM.

**THEOREM** Let  $T \in K(X, X)$  and let  $\{\varepsilon_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$  be a sequence such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $\varepsilon_n \in \sigma_T \setminus \{0\}$ . Then  $z=0$ , hence this implies that  $\sigma_T \setminus \{0\}$  is only made of isolated points.

**PROOF** We know that  $\varepsilon_n \in \Sigma_T$  by hypothesis, thus we can consider the eigenfunctions  $x_n$  and denote  $E_n$  the space formed by the first  $n$  eigenfunctions.

We begin observing that  $E_n \subseteq E_{n+1}$ , i.e. the first  $n$  eigenvectors are linearly independent. Thus we assume by contradiction that  $\varepsilon_{n+1} = \sum_{k=1}^n \alpha_k \varepsilon_k$  and therefore,  $T\varepsilon_{n+1} = \sum_{k=1}^{n+1} \varepsilon_{k+1} \alpha_k e_k$  then  $\alpha_k (\varepsilon_k - \varepsilon_{k+1})$  must vanish for all  $k=1, \dots, n$ . Thus  $\alpha_k$  needs to vanish for  $k=1, \dots, n$  hence  $\varepsilon_{n+1} = 0$  which is a contradiction.

Applying RIESZ LEMMA we can construct a sequence  $\{x_n\}_{n \in \mathbb{N}}$  such that  $\|x_n\|=1$ ,  $x_n \in E_n$  and  $\gamma(x_n, E_{n-1}) \geq \frac{1}{2}$ ,  $\forall n \geq 2$ .

Notice now that  $\forall n > m > 2$  we have that, since  $(T - \varepsilon_n I)E_n \subseteq E_{n-1}$  and  $E_{m-1} \subseteq E_m \subseteq E_{m-1} \subseteq E_n$ .

$$(*) \quad \|\varepsilon_n^{-1} T x_n - \varepsilon_m^{-1} T x_m\| = \left\| \varepsilon_n^{-1} \left( \underbrace{T x_n - \varepsilon_n x_n}_{\in E_{n-1}} \right) - \varepsilon_m^{-1} \left( \underbrace{T x_m - \varepsilon_m x_m}_{\in E_{m-1} \subseteq E_{n-1}} \right) + x_n - x_m \right\| \geq \gamma(x_n, E_{n-1}) \geq \frac{1}{2}.$$

If  $x_n \rightarrow z$  and  $z \neq 0$ , this

would imply that  $\|T z - T x_n\| \leq \|\varepsilon_n x_n - \varepsilon_m x_m\| \leq |\varepsilon_n - \varepsilon_m| \xrightarrow{n \rightarrow \infty} 0$ , which would imply  $\{T x_n\}$  has a converging subsequence which contradicts (\*).

This implies that the spectrum of  $T$  is formed by isolated points.

**COROLLARY** Let  $T \in K(X, X)$  then one of the following scenario applies: 1.  $\sigma_T = \{0\}$ , 2.  $\sigma_T \setminus \{0\}$  is finite, 3.  $\sigma_T \setminus \{0\}$  is a sequence converging to 0.

**PROOF** For every integer  $n \geq 1$  we notice that the set  $\sigma_{T^n} \setminus \{z \in \mathbb{C} : |z| \geq 1/n\}$

is either empty or finite, if not we would have a subsequence to some  $z$  with  $|z| \geq 1/n$ . This because  $T$  is compact. In fact if  $\{z_n\}_{n \in \mathbb{N}} \setminus \{z \in \mathbb{C} : |z| \geq 1/n\}$  then we can consider the sequence of associated eigenvectors  $\{x_n\}_{n \in \mathbb{N}}$  and observe the sequence is bounded hence  $\{T x_n\}$  has a converging subsequence which imply  $\lim_{n \rightarrow \infty} z_n = z \geq \frac{1}{n}$  which contradicts the previous theorem.

**PROPOSITION** Let  $T \in \mathcal{B}(X, X)$  be a FINITE-RANK OPERATOR, i.e.  $\dim R(T_n) < \infty$ , then  $\sigma_T \setminus \{0\}$  has finitely many elements.

**PROOF** The elements of  $\sigma_T \setminus \{0\}$  are the eigenvalues since  $\dim R(T_n) < \infty$ , there can only be finite eigenvectors.

**DEFINITION** Let  $\{x_n\}_{n \in \mathbb{N}} \subseteq H$  be a sequence of orthonormal elements of  $H$ . We say that  $\{x_n\}_{n \in \mathbb{N}}$  is an Hilbert basis if  $\langle \sum_{n \geq 1} c_n x_n \rangle$  is dense in  $H$ .

**THEOREM** Every SEPARABLE Hilbert space has an Hilbert basis.

**THEOREM** Let  $H$  be a SEPARABLE Hilbert space and let  $T \in K(H, H)$  be SELF-ADJOINT. Then there exist a Hilbert basis composed by the eigenvectors of  $T$ .

**PROOF** Let  $\{\lambda_n\}_{n \geq 0}$  be a sequence of all distinct, non zero eigenvalues of  $T$ . We will additionally set  $\lambda_0 = 0$ ,  $E_0 = N(T)$  and denote  $E_n$  the eigenspace associated to  $\lambda_n$ . Notice that  $\dim(E_n) < \infty$ , as a consequence of FREDHOLM ALTERNATIVE. We will now prove that  $H$  is the Hilbert sum of all  $E_n$ . We do this in two steps. First we prove that  $(E_n)_{n \geq 0}$  are mutually orthogonal.

We pick  $h \in E_m$  and  $h' \in E_n$  with  $m \neq n$ , then  $Th = z_m h$  and  $Th' = z_n h'$ , thus we have  $z_m(h, h') = (Th, h') = (h, Th') = z_n(h, h')$  hence because  $z_m \neq z_n$  we have that  $(h, h') = 0$ . We now set to prove that vector space  $E$  spanned by  $\bigcup_{n \geq 0} E_n$  is dense in  $H$ . First we observe that  $TE \subseteq E$ , furthermore given  $h^* \in E^\perp$  we have  $(Th^*, h^*) = (h^*, Th^*) = 0 \quad \forall h^* \in E$  hence  $Th^* \in E^\perp$ , thus  $TE^\perp \subseteq E^\perp$ . We denote  $T^\perp : E^\perp \rightarrow E^\perp \quad h^* \mapsto Th^*$ . We claim that  $\sigma(T^\perp) = \{0\}$ . By contradiction let assume this is not the case and that some  $z \neq 0 \in \sigma_{T^\perp}$ . Since  $z \in \sigma_{T^\perp}$  then it means that there  $\exists x \neq 0$  such that  $T^\perp x = zx$  hence  $z$  is also an eigenvalue of  $T$ , hence  $h^* \in E^\perp \cap E$  which is a contradiction. Since  $\sigma_{T^\perp} = \{0\}$  we know that  $T$  vanish on  $E^\perp$ , by one of the previous corollary, thus  $E^\perp \subseteq N(T)$ . Furthermore  $N(T) \subseteq E_0 \subseteq E$  hence  $E^\perp \subseteq E$  which implies that  $E^\perp = \{0\}$ , hence  $E$  is dense in  $H$ , since  $H$  separable and  $H = E^\perp \oplus E$ .

Finally we pick in each subspace a Hilbert basis, the existence of such a basis for  $E_0$  is a consequence of the fact that all separable Hilbert space have an Hilbert basis and for  $E_n$ , with  $n > 0$  is just a consequence of the fact that  $E_n$  is finite-dimensional.

**THEOREM** The same result as above applies even if  $T \in \mathcal{B}(X, X)$  is a normal operator.

**THEOREM** Let  $T \in K(X, X)$  be a normal operator then  $\sigma_{E,T} = \{z \in \mathbb{C} : z_1 + z_2 \in \sigma_T, z_2 \in \mathcal{B}_E(\Omega)\}$ .

**PROOF** We begin considering a Hilbert basis for  $H$  formed by the eigenvectors of  $T$ , i.e.  $\{h_n\}_{n \in \mathbb{N}}$ .

We then observe that  $\|R_T(z)\| = \sup_{n \in \mathbb{N}} \frac{\|R_T(z)h_n\|}{\|h_n\|} = \sup_{n \in \mathbb{N}} \|(T-zI)^{-1}h_n\| \stackrel{(1)}{=} \sup_{n \in \mathbb{N}} \frac{1}{|\lambda_n - z|} = \frac{1}{\gamma(z, \sigma_T)}$ .

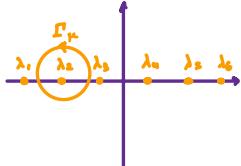
$$(1) (T-zI)h_n = Th_n - zh_n = (\lambda_n - z)h_n$$

**EXAMPLE** we have previously discussed how the complement of the  $(T-zI) \frac{1}{\lambda_n - z} h_n = h_n \Rightarrow (T-zI)h_n = \frac{1}{\lambda_n - z} h_n$  pseudo-spectra tell us for which value of  $z$  the Helmholtz problem is not "ill-conditioned" from the physical point of view. Notice that in the case of  $T \in \mathcal{B}(X, X)$  being SELF-ADJOINT we also have the upper-bound  $\|R_T(z)\| \leq \gamma(z, \sigma_T)$  hence for  $z \in \mathbb{C}$  we have that the solution of the Helmholtz problem depends continuously on the data, i.e.  $\|x\| \leq \gamma(z, \sigma_T) \|y\|$ . This argument to obtain the continuous dependence of the solution from the data is equivalent to the T-coercivity argument by CSECT.

**THEOREM** Let  $T \in K(H, H)$  be a NORMAL OPERATOR. Fixed  $\mu \in \sigma_T \setminus \{0\}$  we consider

a curve  $\Gamma_\mu \subseteq \sigma_T$  only encapsulating  $\mu$  and no other element of  $\sigma_T$ , then

the operator  $\Delta_{T, \Gamma_\mu} : H \rightarrow H \quad x \mapsto \frac{1}{2\pi i} \int_{\Gamma_\mu} R_T(z)x dz$  is a projection on the eigenspace associated to  $\mu$ .



**PROOF** We consider a Hilbert basis for  $H$  composed by the eigenvectors of  $T$ , i.e.  $\{h_n\}_{n \in \mathbb{N}}$ . The action of  $R_T(z)$  can be expressed as  $R_T(z)h = \sum_{n \in \mathbb{N}} (\lambda_n - z)^{-1}(h_n, h)h_n$  and thus we can express the action of  $\Delta_\mu$  as  $\Delta_{T, \Gamma_\mu}h = \frac{1}{2\pi i} \int_{\Gamma_\mu} \sum_{n \in \mathbb{N}} (\lambda_n - z)^{-1}(h_n, h)h_n dz$ .

We begin assuming that all eigenvalues are simple, hence their geometric multiplicity is 1.

Then we notice by CAUCHY RESIDUE THEOREM that  $\Delta_{T, \Gamma_\mu}h = \frac{1}{2\pi i} \sum_{n \in \mathbb{N}} \text{Res}(F_{n, \mu}(z), \mu)(h_n, h)h_n$  this because  $\mu$  is the only pole contained in  $\Gamma_\mu$  and it is a pole only for  $F_{n, \mu}$  where  $n_\mu$  denotes the index of the Hilbert basis associated with  $\mu$ . Hence only  $\text{Res}(F_{n, \mu}(z), \mu)$  is non vanishing and it is equal to 1, thus  $\Delta_{T, \Gamma_\mu}h = (h_n, h)h_n$  which is the orthogonal projection of  $h$  onto to the eigenfunction  $h_n$ . To generalise the proof to the case when eigenvalues are not simple it

is sufficient to observe that multiple residual will not vanish proving  $\Delta_{T, \mathbb{I}_\mu} h = \sum_{n \in \mathbb{I}_\mu} (h_n, h) h_n$  with  $\mathbb{I}_\mu = \{n : h_n \in N(T - \mu I)\}$  which is exactly the projection on the eigen space  $N(T - \mu I)$ .