

Discretisation of the Helmholtz-Korteweg equation

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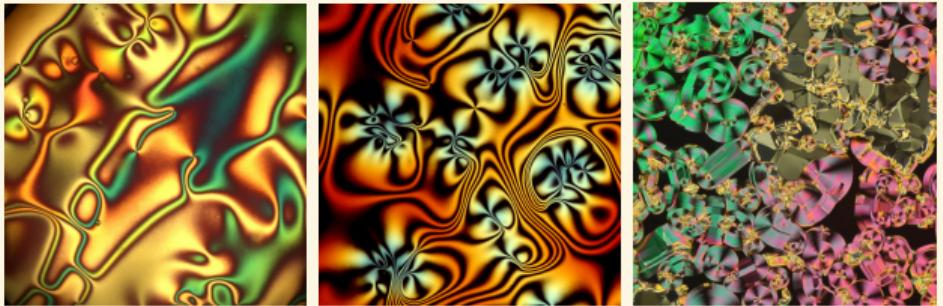
LIQUID CRYSTALS



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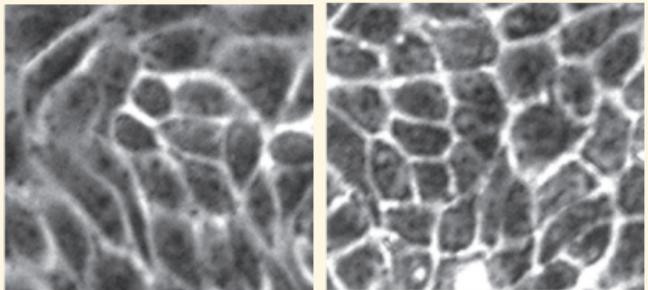
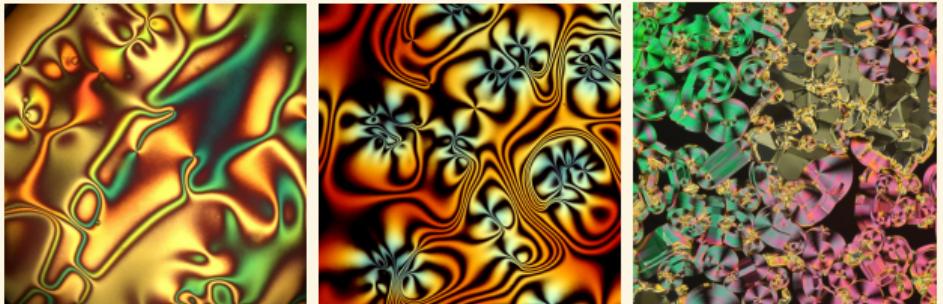


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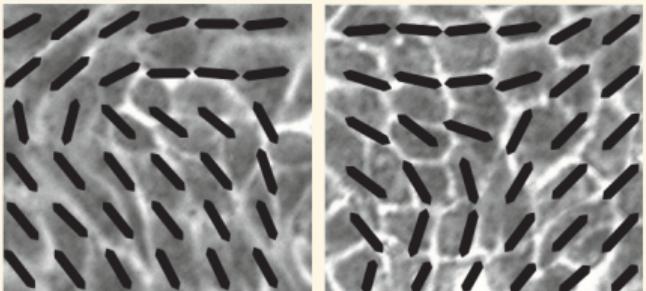
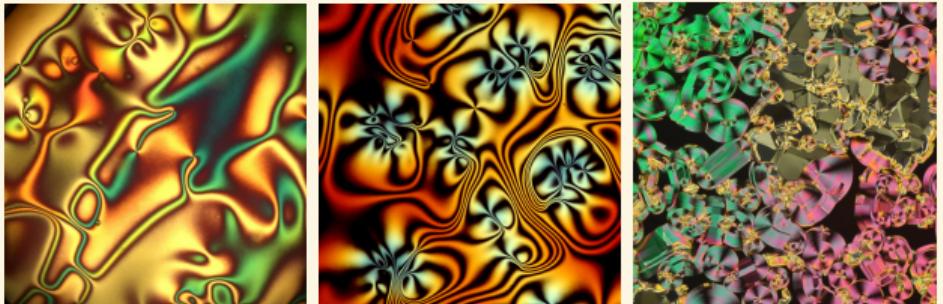


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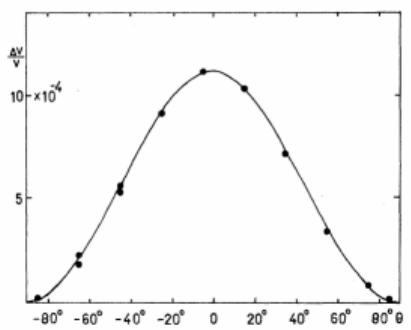


Figure: Angular dependence of sound velocity. $T = 21\text{ C}$, $v = 10\text{ MHz}$, and $H = 5\text{ kOe}$. θ is the angle between the field direction and propagation direction. Solid line is $12.5 \cdot 10^{-4} \cos(\theta)^2$.

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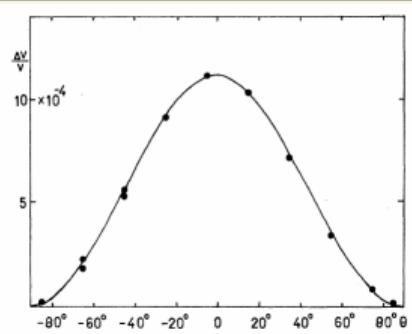


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- ▶ We here assume the aligning torque acting on the nematic director field is of elastic nature, rather than of a dissipative viscous one. This idea was already proposed, and validated experimentally, by Mullen, Lüthi, and Stephen.



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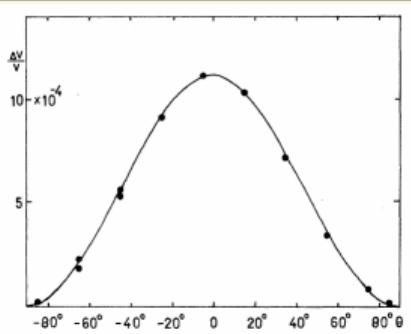


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THE NEMATIC HELMHOLTZ–KORTEWEG EQUATION



Farrell, P.E. and , 2025. Time-harmonic waves in Korteweg and nematic-Korteweg fluids. *Physical Review E*, 111(3), p.035413.

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Under this hypothesis the **nematic Helmholtz–Korteweg equation** can be derived, i.e.

$$-\omega^2 S(\mathbf{x}) - c_0^2 \Delta S(\mathbf{x}) + \rho_0^2 \alpha \Delta^2 S(\mathbf{x}) + \rho_0^2 u_2 \nabla \cdot \nabla [\mathbf{n} \cdot \underline{\mathcal{H}} \underline{S} \mathbf{n}] = 0.$$

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- ▶ It can be proven that this PDE is compatible with anhyperelastic formulation, i.e. it can be derived from the free energy functional

$$W(\rho, \nabla \rho, \mathbf{n}) = c_0^2 \rho^2 + \frac{1}{2} \alpha \|\nabla \rho\|^2 + \frac{1}{2} \beta (\nabla \rho \cdot \mathbf{n})^2.$$

BOUNDARY CONDITIONS

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Sound-soft boundary conditions

Sound-soft boundary conditions impose that the excess-pressure defined as

$$c_0^2 \rho_0 S(\mathbf{x}) - \rho_0^3 \alpha \Delta S(\mathbf{x}) - u_2 \rho_0^3 (\mathbf{n} \cdot \underline{\mathcal{H}} \underline{\mathcal{S}} \mathbf{n}) = 0.$$

vanish along the boundary. Sound-soft boundary conditions thus correspond to imposing homogeneous Dirichlet boundary conditions on $S(\mathbf{x})$ and

$$\Delta S(\mathbf{x}) = -\frac{u_2}{\alpha} (\mathbf{n} \cdot \underline{\mathcal{H}} \underline{\mathcal{S}} \mathbf{n}).$$

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Sound-hard boundary conditions

Sound-hard boundary conditions also change since the normal derivative of the fluid velocity $\partial_\nu \mathbf{v}$ now satisfies the equation

$$i\omega\rho_0(\mathbf{n} \cdot \boldsymbol{\nu}) = c_0^2\partial_\nu S(\mathbf{x}) - \rho_0^2\alpha\partial_\nu\Delta S(\mathbf{x}) - \rho_0^2u_2\partial_\nu (\mathbf{n} \cdot \underline{\mathcal{H}}\underline{S}\mathbf{n}).$$

Sound-hard boundary conditions thus correspond to imposing homogeneous Neumann boundary conditions on $S(\mathbf{x})$ and

$$\partial_\nu\Delta S(\mathbf{x}) = -\frac{u_2}{\alpha}\partial_\nu (\mathbf{n} \cdot \underline{\mathcal{H}}\underline{S}\mathbf{n}).$$

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Impedance boundary conditions

Some computation shows that the impedance boundary conditions for the nematic Helmholtz–Korteweg equation are equivalent to imposing Robin boundary conditions on $S(x)$ and

$$\partial_\nu \Delta S(x) = i\zeta \Delta S(x) + i\zeta \frac{u_2}{\alpha} (\mathbf{n} \cdot \underline{\mathcal{H}} \underline{S} \mathbf{n}) - \frac{u_2}{\alpha} \partial_\nu (\mathbf{n} \cdot \underline{\mathcal{H}} \underline{S} \mathbf{n}),$$

where ζ is the impedance of the boundary.

WEAK FORMULATION

 Farrell, P.E., van Beeck, T. and Zerbinati, U., 2025. Analysis and numerical analysis of the Helmholtz-Korteweg equation. arXiv preprint arXiv:2503.10771.

We want to find $u \in X$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in X,$$

where

$$a(u, v) := \underbrace{\alpha(\Delta u, \Delta v)_{L^2(\Omega)} + \beta(\mathbf{n}^T(\mathcal{H}u)\mathbf{n}, \Delta v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}}_{=:e(u,v)} - k^2(u, v)_{L^2(\Omega)}$$

We only consider sound-soft boundary conditions for which $X = H_0^2(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ for simplicity and impose the boundary conditions using Nitsch's trick at the discrete level.

C^1 -DISCRETIZATION OF THE HELMHOLTZ–KORTEWEG EQUATION

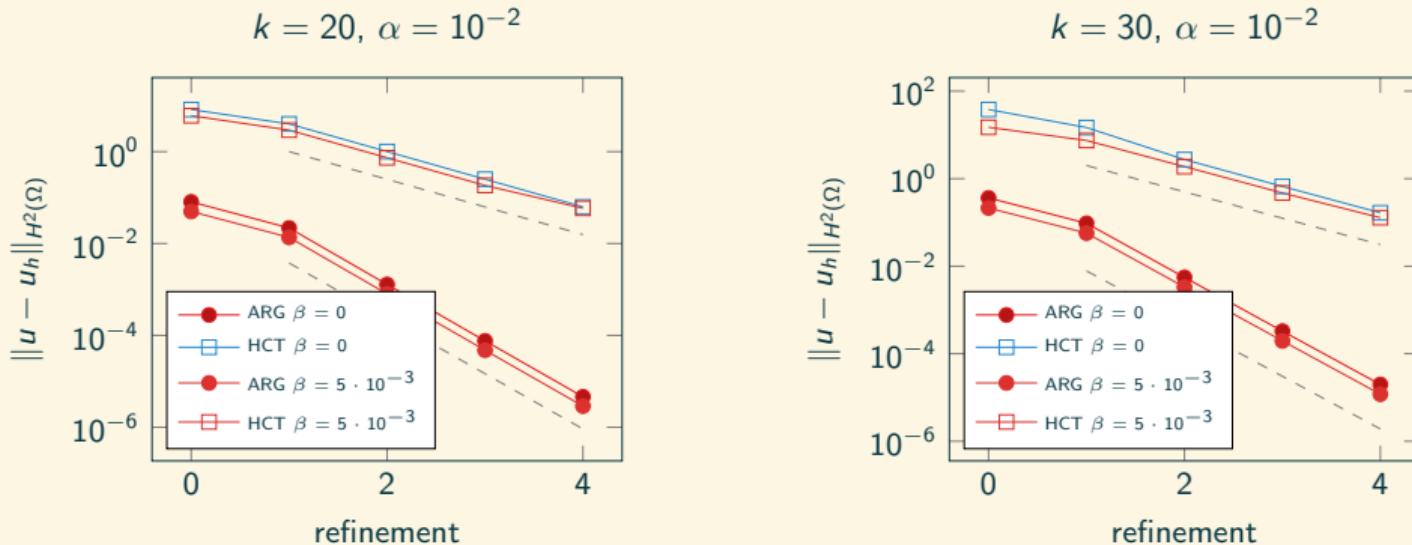
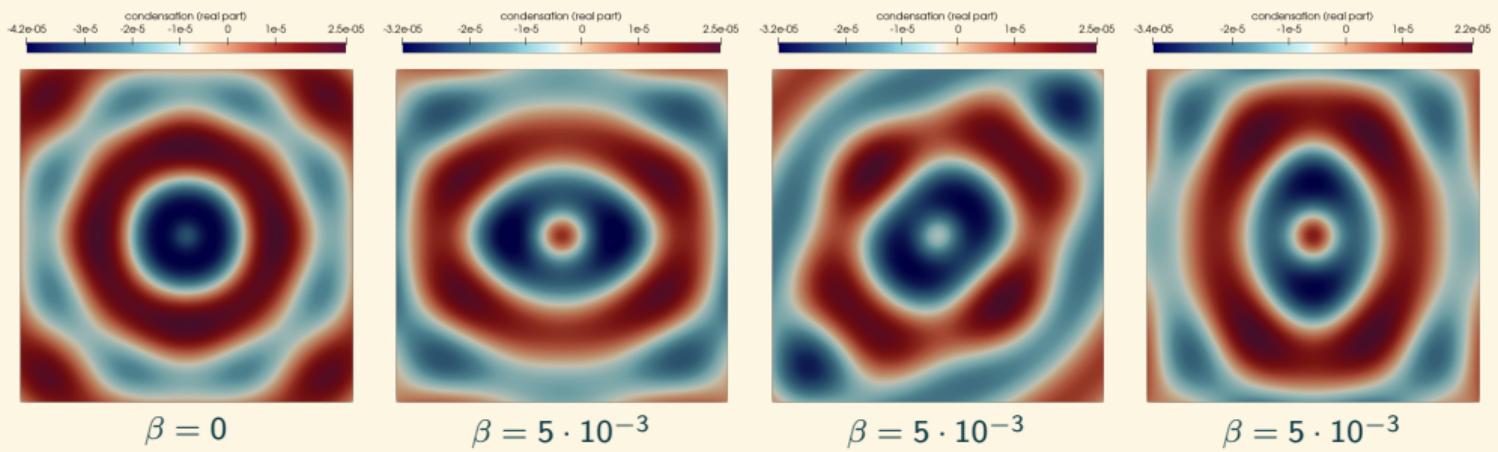


Figure: The convergence of the H^2 -norm of the error for the Helmholtz–Korteweg equation for different values of k (top row) and the corresponding manufactured solution (bottom row).

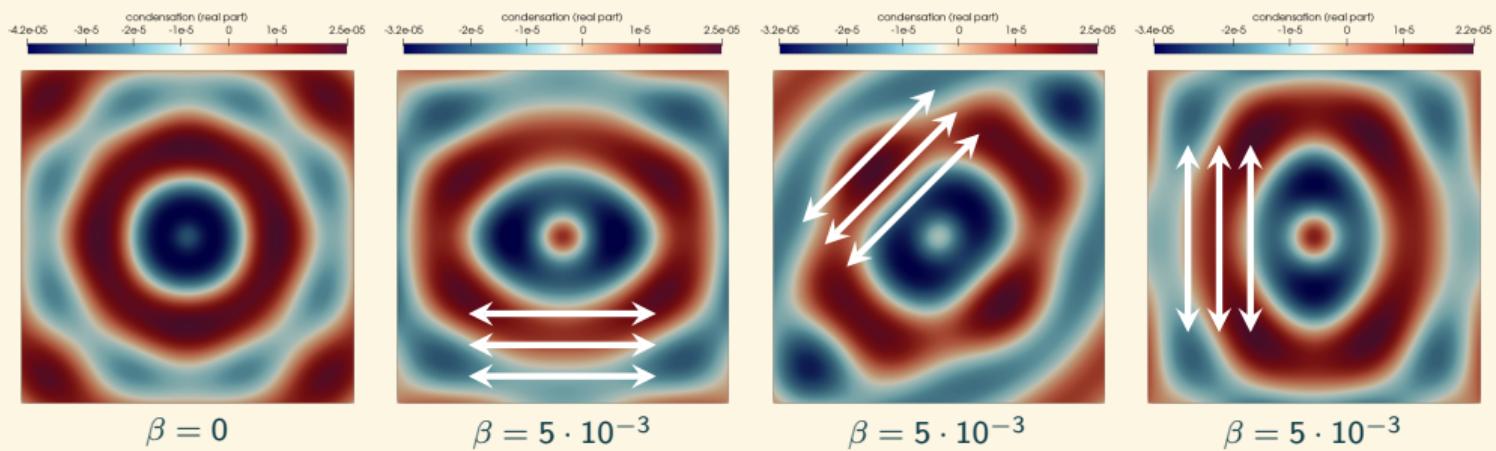
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We demonstrate the anisotropic speed of sound considering as right-hand side asymmetric Gaussian pulse in $(0, 0)$, impedance BCs, $k = 40$, $\alpha = 10^{-2}$



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TOTAL INTERNAL REFLECTION

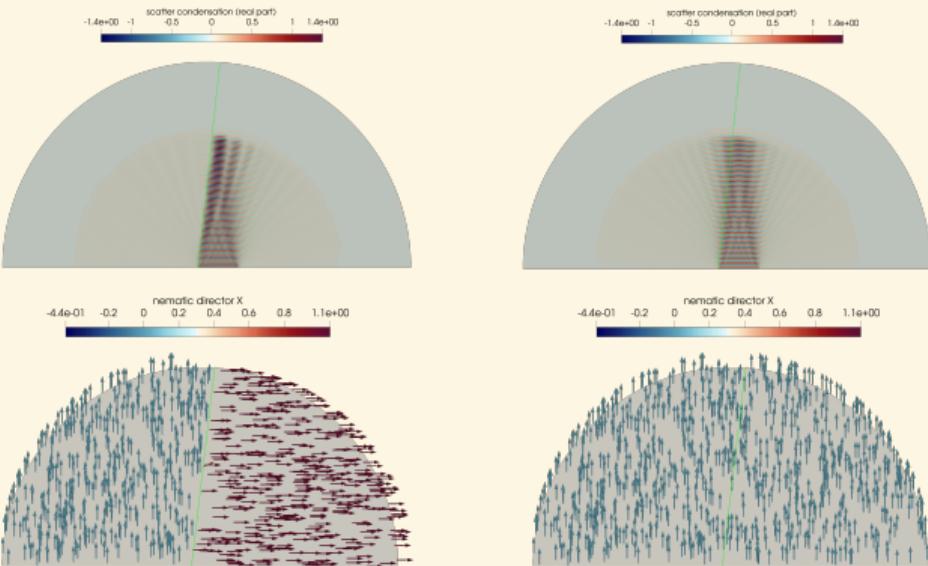


Figure: An acoustic reflection phenomenon in a nematic Korteweg fluid can be caused by a discontinuity in the nematic director field. We consider a Gaussian beam travelling upwards in a semicircular domain, with two different nematic director fields.

SCATTERING BY A CIRCULAR OBSTACLE

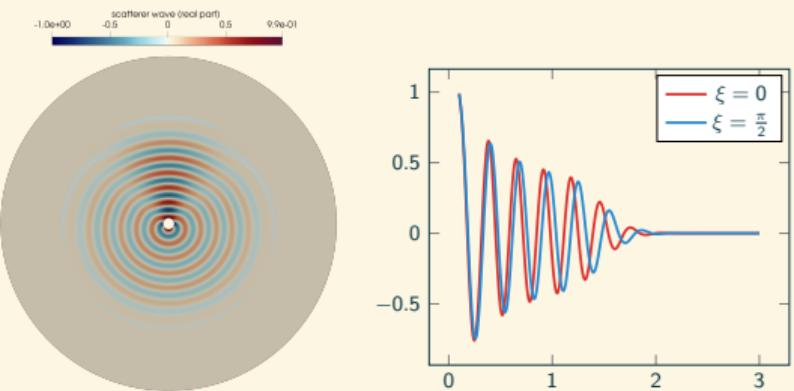


Figure: The scattered wave produced by a circular obstacle in a nematic Korteweg fluid with $\alpha = 10^{-3}$ and $u_2 = 5 \cdot 10^{-4}$, has a greater amplitude when the incoming plane wave is orthogonal to the nematic director field. Recall that ξ is the angle between \mathbf{d} and \mathbf{n} . We simulated a plane wave propagating parallel to the y -axis and impinging on a circular obstacle, centered at the origin (left). The amplitude of the scattered wave, for different values of ξ , is measured along the y -axis (right). An adiabatic layer has been used to implement the Sommerfeld radiation condition on the outer boundary.

THANK YOU!

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<https://doi.org/10.1103/PhysRevE.111.035413>

<https://doi.org/10.48550/arXiv.2502.17626>

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