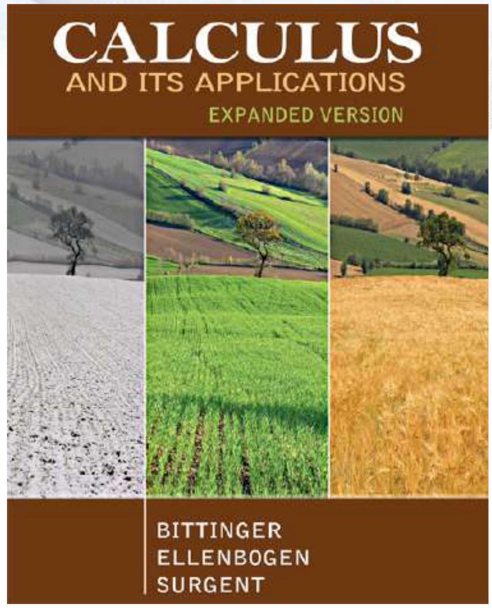
Section 2.3



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Graph Sketching: Asymptotes and Rational Functions

2.3

OBJECTIVE

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

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2.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A **rational function** is a function f that can be described by

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x) \neq 0$.

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DEFINITION:

The line $x = a$ is a **vertical asymptote** if any of the following limit statements are true:

$$\lim_{x \to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty \quad \text{or}$$

$$\lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$

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DEFINITION (continued):

The graph of a rational function *never* crosses a vertical asymptote. If the expression that defines the rational function f is simplified, meaning that it has no common factor other that –1 or 1, then if a is an input that makes the denominator 0, the line $x = a$ is a vertical asymptote.

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Example 1: Determine the vertical asymptotes of the function given by

$$f(x) = \frac{P(x)}{Q(x)}$$

$$f(x) = \frac{x(x-2)}{x(x-1)(x+1)}$$

$$f(x) = \frac{(x-2)}{(x-1)(x+1)}$$

Since $x = 1$ and $x = -1$ make the denominator 0, $x = 1$ and $x = -1$ are vertical asymptotes.

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2.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 1

Determine the vertical asymptotes: $f(x)=\dfrac{1}{x\left(x^2-16\right)}$

$$f(x)=\frac{1}{x\left(x^2-16\right)}$$

$$f(x)=\frac{1}{x(x+4)(x-4)}$$

After factoring out the denominator, we see that $x = 0$, $x = 4$, and $x = –4$ make the denominator 0. Thus, there are vertical asymptotes at $x = 0$, $x = 4$, and $x = –4$.

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DEFINITION:

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements are true:

$$\lim_{x\to-\infty} f\left(x\right)=b \qquad \text{or} \qquad \lim_{x\to\infty} f\left(x\right)=b.$$

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DEFINITION (continued):

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator. (The degree of a polynomial in one variable is the highest power of that variable.)

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Example 2: Determine the horizontal asymptote of the function given by

$$f(x) = \frac{3x^2 + 2x - 4}{2x^2 - x + 1}.$$

First, divide the numerator and denominator by x^2.

$$f(x) = \frac{3 + \frac{2}{x} - \frac{4}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}}$$

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Example 2 (continued):

Second, find the limit as $|x|$ gets larger and larger.

$$\lim_{x\to-\infty} \frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}} = \frac{3}{2} \quad \text{and} \quad \lim_{x\to\infty} \frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}} = \frac{3}{2}$$

Thus, the line $y = \frac{3}{2}$ is a horizontal asymptote.

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Quick Check 2

Determine the horizontal asymptote of the function given by

$$f(x) = \frac{(2x-1)(x+1)}{(3x+2)(5x+6)}.$$

First we should multiply both the numerator and denominator out:

$$f(x) = \frac{(2x-1)(x+1)}{(3x+2)(5x+6)} = \frac{2x^2+x-1}{15x^2+28x+12}$$

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Quick Check 2 Concluded

Since both the numerator and denominator have the same power of x, we can divide both by that power:

$$f(x) = \frac{2x^2 + x - 1}{15x^2 + 28x + 12} = \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{15 + \frac{28}{x} + \frac{12}{x^2}}$$

Now we can see that as $|x|$ gets very large, the numerator approaches 2 and the denominator approaches 15. Therefore the value of the function gets very close to $\frac{2}{15}$. Thus, $\lim_{x\to-\infty} f(x) = \frac{2}{15}$ and $\lim_{x\to\infty} f(x) = \frac{2}{15}$.

Therefore there is a horizontal asymptote at $y = \frac{2}{15}$.

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DEFINITION:

A linear asymptote that is neither vertical nor horizontal is called a **slant**, or **oblique**, **asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

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2.3 Graph Sketching: Asymptotes and Rational Functions

Example 3: Find the slant asymptote:

$$f(x)=\frac{x^2-4}{x-1}$$

First, divide the numerator by the denominator.

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \quad -4 \\ & \underline{x^2-x} \\ & \quad x-4 \\ & \quad \underline{x-1} \\ & \quad -3 \end{array}$$

$$\Rightarrow \quad f(x)=\frac{x^2-4}{x-1}=(x+1)+\frac{-3}{x-1}$$

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Example 3 (concluded):

Second, now we can see that as $|x|$ gets very large, $-3/(x-1)$ approaches 0. Thus, for very large $|x|$, the expression $x + 1$ is the dominant part of

$$(x+1)+\frac{-3}{x-1}$$

thus $y = x + 1$ is the slant asymptote.

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Quick Check 3

Find the slant asymptote: $g(x)=\dfrac{2x^2+x-1}{x-3}$

Use polynomial division to solve for this:

$$\begin{array}{r}
2x\ +7 \\
x-3\overline{)\,2x^2+x-1} \\
-(2x^2-6x)\downarrow \\
\hline
7x\ -1 \\
-(\ 7x-21\ ) \\
\hline
20
\end{array}$$

Since we have a remainder of 20, we can see that as $|x|$ gets very large, the remainder approaches 0. Thus the dominant part of $2x+7+\dfrac{20}{x-3}$ is $2x = 7$.

Therefore, there is slant asymptote at $y = 2x + 7$

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2.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs:

a) *Intercepts*. Find the x-intercept(s) and the y-intercept of the graph.

b) *Asymptotes*. Find any vertical, horizontal, or slant asymptotes.

c) *Derivatives and Domain*. Find $f'(x)$ and $f''(x)$. Find the domain of f.

d) *Critical Values of f*. Find any inputs for which $f'(x)$ is not defined or for which $f'(x) = 0$.

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Strategy for Sketching Graphs (continued):

e) *Increasing and/or decreasing; relative extrema*. Substitute each critical value, x_0, from step (d) into $f''(x)$, and apply the Second Derivative Test. If no critical value exists, use f' and test values to find where f is increasing or decreasing.

f) *Inflection Points*. Determine candidates for inflection points by finding x-values for which $f''(x)$ does not exist or for which $f''(x) = 0$. Find the function values at these points.

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Strategy for Sketching Graphs (concluded):

g) *Concavity*. Use the values c from step (f) as endpoints of intervals. Determine the concavity by checking to see where f' is increasing – that is, $f''(x) > 0$ – and where f' is decreasing – that is, $f''(x) < 0$. Do this by selecting test points and substituting into $f''(x)$. Use the results of step (d).

h) *Sketch the graph*. Use the information from steps (a) – (g) to sketch the graph, plotting extra points as needed.

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2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4: Sketch the graph of $f(x) = \dfrac{8}{x^2 - 4}$.

a) *Intercepts*. The x-intercepts occur at values for which the numerator equals 0. Since $8 \neq 0$, there are no x-intercepts. To find the y-intercept, we find $f(0)$.

$$f(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2$$

Thus, we have the point (0, –2).

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2.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 (continued):

b) *Asymptotes*.

Vertical:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \end{aligned}$$

So, $x = 2$ and $x = -2$ are vertical asymptotes.

Horizontal: The degree of the numerator is less than the degree of the denominator. So, the x-axis, $y = 0$ is the horizontal asymptote.

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Example 4 (continued):

Slant: There is no slant asymptote since the degree of the numerator is not 1 more than the degree of the denominator.

c) *Derivatives and Domain*. Using the Quotient Rule, we get

$$f'(x) = \frac{-16x}{(x^2-4)^2} \quad \text{and} \quad f''(x) = \frac{16(3x^2+4)}{(x^2-4)^3}.$$

The domain of f is all real numbers, $x \neq 2$ and $x \neq -2$.

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Example 4 (continued):

d) *Critical Values of f*. $f'(x)$ equals 0 where the numerator equals 0 and does not exist where the denominator equals 0.

$$\begin{aligned} -16x &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} (x^2-4)^2 &= 0 \\ x^2-4 &= 0 \\ (x-2)(x+2) &= 0 \end{aligned}$$

$$x = 2 \quad or \quad x = -2$$

However, since f does not exist at $x = 2$ or $x = -2$, $x = 0$ is the only critical value.

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Example 4 (continued):

e) *Increasing and/or decreasing; relative extrema.*

$$f''(0) = \frac{16(3\cdot 0^2 + 4)}{(0^2 - 4)^3} = \frac{64}{-64} = -1 < 0$$

Thus, $x = 0$ is a relative maximum and f is increasing on (–2, 0) and decreasing on (0, 2).

Since f'' does not exist at $x = 2$ and $x = –2$, we use f' and test values to see if f is increasing or decreasing on (–∞, 2) and (2, ∞).

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Example 4 (continued):

$$f'(-3) = \frac{-16(-3)}{\left((-3)^2 - 4\right)^2} = \frac{48}{25} > 0$$

So, f is increasing on (–∞, 2).

$$f'(3) = \frac{-16(3)}{\left((3)^2 - 4\right)^2} = \frac{-48}{25} < 0$$

So, f is decreasing on (2, ∞).

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Example 4 (continued):

f) *Inflection points*. f'' does not exist $x = 2$ and $x = –2$. However, neither does f. Thus we consider where f'' equals 0.

$$16(3x^2 + 4) = 0$$

Note that $16(3x^2 + 4) > 0$ for all real numbers x, so there are no points of inflection.

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Example 4 (continued):

g) *Concavity*. Since there are no points of inflection, the only places where f could change concavity would be on either side of the vertical asymptotes.

Note that we already know from step (e) that f is concave down at $x = 0$. So we need only test a point in $(–\infty, 2)$ and a point in $(2, \infty)$.

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Example 4 (continued):

$$f''(-3)=\frac{16(3\cdot(-3)^2+4)}{((-3)^2-4)^3}=\frac{496}{125}>0$$

Thus, f is concave up on $(-\infty, 2)$.

$$f''(3)=\frac{16(3\cdot(3)^2+4)}{((3)^2-4)^3}=\frac{496}{125}>0$$

Thus, f is concave up on $(2, \infty)$.

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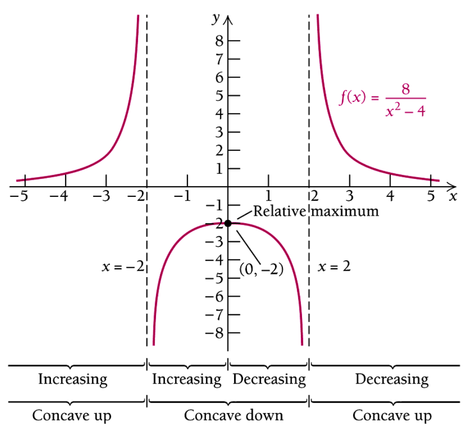
Example 4 (continued):

h) *Sketch the graph*. Using the information in steps (a) – (g), the graph follows.

x	$f(x)$ approximately
−5	0.38
−4	0.67
−3	1.6
−1	−2.67
0	−2
1	−2.67
3	1.6
4	0.67
5	0.38



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2.3 Graph Sketching: Asymptotes and Rational Functions

Section Summary

- A line $x = a$ is a *vertical asymptote* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$
- A line $y = b$ is a *horizontal asymptote* if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$
- A graph may cross a horizontal asymptote but never a vertical asymptote.
- A *slant asymptote* occurs when the degree of the numerator is 1 greater than the degree of the denominator. Long division of polynomials can be used to determine the equation of the slant asymptote.
- Vertical, horizontal, and slant asymptotes can be used as guides for accurate curve sketching. Asymptotes are not a part of a graph but are visual guides only.

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Gamma, Beta Functions, Differentiation Under the Integral Sign

21.1 GAMMA FUNCTION

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that $\Gamma 1 = 1$*

Solution. $$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

Put $n = 1$, $$\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$$ **Proved**

Example 2. *Prove that*

(*i*) $\overline{\mid n+1} = n \overline{\mid n}$ (*ii*) $\overline{\mid n+1} = \underline{\mid n}$ (*Reduction formula*)

Solution.

(*i*) $$\overline{\mid n} = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad \ldots(1)$$

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \ldots + \frac{x^n}{\lfloor n} + \ldots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$\therefore$ $$\Gamma n = (n-1)\,\overline{\mid n-1} \qquad \ldots(2)$$

$$\overline{\mid n+1} = n\,\Gamma n$$ Replacing n by $(n + 1)$ **Proved**

(*ii*) Replace n by $n - 1$ in (2), we get

$$\overline{|n-1} = (n-2)\,\overline{|n-2}$$

Putting the value $\lceil n-1$ in (2), we get

$$\lceil n = (n-1)(n-2)\overline{|n-2}$$

Similarly $$\lceil n = (n-1)(n-2)\ldots 3.2.1\lceil 1 \qquad \ldots (3)$$

Putting the value of $\overline{|1}$ in (3), we have

$$\lceil n = (n-1)(n-2)\ldots 3.2.1.1$$

$$\lceil n = \underline{|n-1}$$

Replacing n by $n+1$, we have

$$\overline{|n+1} = \underline{|n} \qquad \textbf{Proved}$$

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx \qquad \ldots(1)$

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2}\, e^{-t}\, 2\,t\,dt = 2\int_0^\infty t^{3/2}\, e^{-t}\, dt$$

$$= 2\left\lceil \frac{5}{2} \right. \qquad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\left\lceil\frac{3}{2}\right. = 2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{3}{2}\sqrt{\pi} \qquad \textbf{Ans.}$$

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx \qquad \ldots(1)$

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3\,t^2\,dt$ in (1) we get

$$I = \int_0^\infty t^{3/2}\, e^{-t}\, 3\,t^2\,dt = 3\int_0^\infty t^{7/2}\, e^{-t}\, dt = 3\left\lceil\frac{9}{2}\right. = 3\,.\,\frac{7}{2}\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{315}{16}\sqrt{\pi} \qquad \textbf{Ans.}$$

Example 5. *Evaluate* $\int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx.$

Solution. Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2}\, dx \qquad \ldots(1)$

Putting $t = h^2x^2$ or $x = \dfrac{\sqrt{t}}{h}$ or $dx = \dfrac{dt}{2\,h\sqrt{t}}$,

(1) becomes $$I = \int_o^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\,\frac{dt}{2\,h\,\sqrt{t}}$$

$$= \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-1}{2}}\, e^{-t}\,\frac{dt}{\sqrt{t}} = \frac{1}{2\,h^n}\int_0^\infty t^{\frac{n-2}{2}}\, e^{-t}\, dt$$

$$= \frac{1}{2\,h^n}\left\lceil\frac{n}{2}\right. \qquad \textbf{Ans.}$$

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma\left(-\frac{1}{2}\right)$ (*ii*) $\Gamma\left(\frac{-3}{2}\right)$ (*iii*) $\Gamma\left(\frac{-15}{2}\right)$ (*iv*) $\Gamma\left(\frac{7}{2}\right)$ (*v*) $\Gamma 0$

Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\frac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\, e^{-x}\, dx$ **Ans.** $\Gamma\left(\frac{3}{2}\right)$

3. $\int_0^\infty x^4 e^{-x^2}\, dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}\, dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

21.3 BETA FUNCTION

$$\int_0^\infty x^{l-1}(1-x)^{m-1}\, dx$$

is called the Beta function of l, m. It is also written as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx.$$

21.4 EVALUATION OF BETA FUNCTION

$$\beta(l, m) = \frac{\Gamma l\, \Gamma m}{\Gamma(l+m)}$$

21.5 A PROPERTY OF BETA FUNCTION

$$\beta(l, m) = \beta(m, l)$$

Solution. We have

Example 8. *Evaluate* $\int_0^1 x^4 (1 - \sqrt{x})^5 \, dx$

Solution. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\, tdt$

$$\int_0^1 x^4 (1 - \sqrt{x})^5 \, dx = \int_0^1 (t^2)^4 (1 - t)^5 (2\, tdt)$$

$$= 2 \int_0^1 t^9 (1 - t)^5 \, dt = 2\, \beta(10, 6) = 2 \frac{\Gamma 10 \, \Gamma 6}{\Gamma 16} = 2 \frac{\lfloor 9 \, \lfloor 5}{\lfloor 15}$$

$$= 2 \, . \, \frac{\lfloor 5}{10 \times 11 \times 12 \times 13 \times 14 \times 15} = \frac{2 \times 1 \times 2 \times 3 \times 4 \times 5}{10 \times 11 \times 12 \times 13 \times 14 \times 15}$$

$$= \frac{1}{11 \times 13 \times 7 \times 15} = \frac{1}{15015}$$

Example 9. *Evaluate* $\int_0^1 (1 - x^3)^{-\frac{1}{2}} \, dx$

Solution. Let $x^3 = y$ or $x = y^{1/3}$ or $dx = \frac{1}{3} y^{-\frac{2}{3}} dy$

$$\int_0^1 (1 - x^3)^{-\frac{1}{2}} \, dx = \int_0^1 (1 - y)^{-\frac{1}{2}} \left(\frac{1}{3} y^{-\frac{2}{3}} dy \right)$$

$$= \frac{1}{3} \int_0^1 y^{-\frac{2}{3}} (1 - y)^{-\frac{1}{2}} dy = \frac{1}{3} \beta\left(\frac{1}{3}, \frac{1}{2} \right) = \frac{1}{3} \frac{\Gamma\frac{1}{3} \, \Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$

The Reduction Formulas:

$$\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx \;(n \neq 1)$$

$$\int \cot^n x\,dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx \;(n \neq 1)$$

$$\int \sec^n x\,dx = \frac{\sec^{n-1} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \quad (n \neq 1)$$

$$\int \csc^n x\,dx = \frac{-\csc^{n-1} x\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx \quad (n \neq 1)$$

5.10 WORK DONE AS A SCALAR PRODUCT

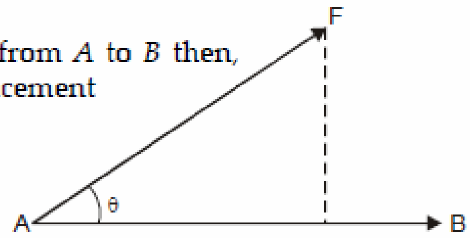
If a constant force F acting on a particle displaces it from A to B then,

Work done = (component of F along AB). Displacement

$= F \cos \theta . AB$

$= \vec{F} . \vec{AB}$

$$\boxed{\textbf{Work done} = \textbf{Force . Displacement}}$$



5.13 AREA OF PARALLELOGRAM

Example 3. *Find the area of a parallelogram whose adjacent sides are $i - 2j + 3k$ and $2i + j - 4k$.*

Solution. Vector area of || gm $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}$

$$= (8 - 3)\hat{i} - (-4 - 6)\hat{j} + (1 + 4)\hat{k} = 5\hat{i} + 10\hat{j} + 5\hat{k}$$

Area of parallelogram $= \sqrt{(5)^2 + (10)^2 + (5)^2} = 5\sqrt{6}$ **Ans.**

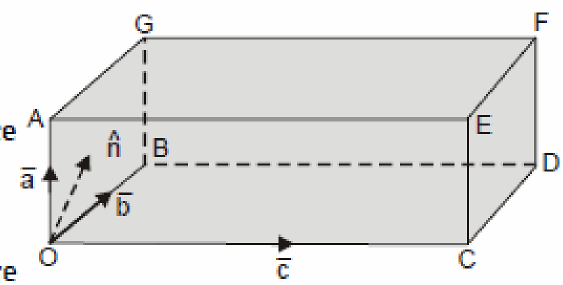
5.17 GEOMETRICAL INTERPRETATION

The scalar triple product $\vec{a} . (\vec{b} \times \vec{c})$ represents the volume of the parallelopiped having $\vec{a}$, $\vec{b}$, $\vec{c}$ as its co-terminous edges.

$\vec{a} . (\vec{b} \times \vec{c}) = \vec{a}$. Area of || gm $OBDC\ \hat{\eta}$

= Area of || gm $OBDC$ × perpendicular distance between the parallel faces $OBDC$ and $AEFG$.

= Volume of the parallelopiped



Note. (1) If $\vec{a} . (\vec{b} \times \vec{c}) = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

(2) Volume of tetrahedron $\frac{1}{6}(\vec{a}\ \vec{b}\ \vec{c})$.

Example 4. *Find the volume of parallelopiped if*

$$\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}, \quad \vec{b} = -3\hat{i} + 7\hat{j} - 3\hat{k}, \text{ and } \vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$$

are the three co-terminous edges of the parallelopiped.

Solution.

$$\text{Volume} = \vec{a} . (\vec{b} \times \vec{c})$$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3(-21 - 15) - 7(9 + 21) + 5(15 - 49)$$

$$= 108 - 210 - 170 = -272$$

Volume = 272 cube units. **Ans.**

$a \times (a \times b)$ ...

Example 14. *Find the angle between the surface* $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$. (*M.D.U. Dec. 2009*)

Solution. Here, we have

$$x^2 + y^2 + z^2 = 9 \qquad ...(1)$$

$$z = x^2 + y^2 - 3 \qquad ...(2)$$

Normal to (1) $\eta_1 = \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal to (1) at $(2, -1, 2)$, $\eta_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(3)

Normal to (2), $\eta_2 = \nabla(z - x^2 - y^2 + 3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(z - x^2 - y^2 + 3) = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

Normal to (2) at $(2, -1, 2)$, $\eta_2 = -4\hat{i} + 2\hat{j} + \hat{k}$ ...(4)

$$\eta_1 \cdot \eta_2 = |\eta_1||\eta_2|\cos\theta$$

$$\cos\theta = \frac{\eta_1 \cdot \eta_2}{|\eta_1||\eta_2|} = \frac{(4\hat{i} - 2\hat{j} + 4\hat{k}).(-4\hat{i} + 2\hat{j} + \hat{k})}{|4\hat{i} - 2\hat{j} + 4\hat{k}||-4\hat{i} + 2\hat{j} + \hat{k}|} = \frac{-16 - 4 + 4}{\sqrt{16 + 4 + 16}\sqrt{16 + 4 + 1}}$$

$$= \frac{-16}{6\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-8}{3\sqrt{21}}\right)$$

Hence the angle between (1) and (2) $\cos^{-1}\left(\dfrac{-8}{3\sqrt{21}}\right)$ **Ans**

Example 23. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (*Nagpur University, Summer 2002*)

Solution. Normal on the surface $(x^2 + y^2 + z^2 - 9 = 0)$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = (2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k})$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$ ...(1)

Normal on the surface $(z = x^2 + y^2 - 3) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z - 3)$

$$= 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

Normal at the point $(2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$ ...(2)

Let θ be the angle between normals (1) and (2).

$$(4\hat{i} - 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k}) = \sqrt{16 + 4 + 16}\,\sqrt{16 + 4 + 1}\cos\theta$$

$$16 + 4 - 4 = 6\sqrt{21}\cos\theta \qquad \Rightarrow \qquad 16 = 6\sqrt{21}\cos\theta$$

$\left(\sqrt{9+4} \right) \quad \sqrt{13}$

Example 26. *Find the directional derivative of* $\phi\,(x, y, z) = x^2 y z + 4 x z^2$ *at* $(1, -2, 1)$ *in the direction of* $2\hat{i} - \hat{j} - 2\hat{k}$. *Find the greatest rate of increase of* ϕ.

(*Uttarakhand, I Semester, Dec. 2006*)

Solution. Here, $\phi\,(x, y, z) = x^2 y z + 4xz^2$

Now,
$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$
$$= (2xyz + 4z^2)\,\hat{i} + (x^2z)\,\hat{j} + (x^2y + 8xz)\,\hat{k}$$
$$\nabla\phi \text{ at } (1, -2, 1) = \{2(1)(-2)(1) + 4(1)^2\}\,\hat{i} + (1\times 1)\,\hat{j} + \{1(-2) + 8(1)(1)\}\,\hat{k}$$
$$= (-4 + 4)\,\hat{i} + \hat{j} + (-2 + 8)\,\hat{k} = \hat{j} + 6\hat{k}$$

Let
$$\hat{a} = \text{unit vector} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

So, the required directional derivative at $(1, -2, 1)$
$$= \nabla\phi.\hat{a} = (\hat{j} + 6\hat{k}).\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{1}{3}(-1-12) = \frac{-13}{3}$$

Greatest rate of increase of $\phi = \left|\hat{j} + 6\hat{k}\right| = \sqrt{1+36}$
$$= \sqrt{37}$$
Ans.

Example 27. *Find the directional derivative of the function* $\phi = x^2 - y^2 + 2z^2$ *at the point P* $(1, 2, 3)$ *in the direction of the line PQ where Q is the point* $(5, 0, 4)$.

(*AMIETE, Dec. 20010, Nagpur University, Summer 2008, U.P., I Sem., Winter 2000*)

Solution. Directional derivative $= \overline{\nabla}\phi$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\,\hat{i} - 2y\,\hat{j} + 4z\,\hat{k}$$

Directional Derivative at the point $P\,(1, 2, 3) = 2\hat{i} - 4\hat{j} + 12\hat{k}$ ...(1)

$$\overline{PQ} = \overline{Q} - \overline{P} = (5, 0, 4) - (1, 2, 3) = (4, -2, 1) \qquad ...(2)$$

Directional Derivative along $PQ = (2\hat{i} - 4\hat{j} + 12\hat{k}).\dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$ [From (1) and (2)]

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$
Ans.

Example 39. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.*

Solution. $\text{div}(\vec{u}) = \nabla.\vec{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}) = 4x^3 + 4y^3 + 4z^3$$

Outer normal of the sphere $= \nabla(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Outer normal of the sphere at (1, 2, 2) $= 2\hat{i} + 4\hat{j} + 4\hat{k}$ ...(1)

Directional derivative $= \vec{\nabla}(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^3 + 4y^3 + 4z^3) = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$$

Directional derivative at (1, 2, 2) $= 12\hat{i} + 48\hat{j} + 48\hat{k}$ ...(2)

Directional derivative along the outer normal $= (12\hat{i} + 48\hat{j} + 48\hat{k}).\dfrac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}}$ [From (1), (2)]

$$= \frac{24 + 192 + 192}{6} = 68$$ **Ans.**

Example 41. *Find the divergence and curl of* $\vec{v} = (x\,y\,z)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$ *at* *(2, −1, 1)* *(Nagpur University, Summer 2003)*

Solution. Here, we have

$$\vec{v} = (x\,y\,z)\hat{i} + (3x^2 y)\hat{j} + (xz^2 - y^2 z)\hat{k}$$

$$\text{Div. } \vec{v} = \nabla\phi$$

$$\text{Div } \vec{v} = \frac{\partial}{\partial x}(x\,y\,z) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 3x^2 + 2x\,z - y^2 \qquad = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{Curl } \overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2 y & xz^2 - y^2 z \end{vmatrix} \qquad = -2yz\,\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k}$$

$$= -2yz\,\hat{i} + (xy - z^2)\hat{j} + (6xy - xz)\hat{k}$$

Curl at (2, −1, 1)

$$= -2(-1)(1)\,\hat{i} + \{(2)(-1) - 1\}\hat{j} + \{6(2)(-1) - 2(1)\}\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k}$$ **Ans.**

Example 43. *Prove that* $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ *is both solenoidal and irrotational.* (*U.P., I Sem, Dec. 2008*)

Solution. Let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

For solenoidal, we have to prove $\vec{\nabla}.\vec{F} = 0$.

Now, $$\vec{\nabla}.\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].\left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$$
$$= -2 + 2x - 2x + 2 = 0$$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\overline{F} = 0$.

Now, $$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$
$$= (3z + 2y - 2y + 3z)\hat{i} - (-2z + 3y - 3y + 2z)\hat{j} + (3z + 2y - 2y - 3z)\hat{k}$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

Thus, $\vec{F}$ is irrotational.

Hence, $\vec{F}$ is both solenoidal and irrotational. **Proved.**

Example 44. *Determine the constants a and b such that the curl of vector*

$\overline{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ *is zero.*

Example 65. *If a force* $\vec{F} = 2x^2 y\hat{i} + 3\,xy\hat{j}$ *displaces a particle in the xy-plane from (0, 0) to (1, 4) along a curve* $y = 4\,x^2$. *Find the work done.*

Solution. Work done $= \int_c \vec{F}\,.\,\vec{dr}$ $\left[\begin{array}{l} \vec{r} = x\hat{i} + y\hat{j} \\ \vec{dr} = dx\hat{i} + dy\,\hat{j} \end{array}\right]$

$$= \int_c (2\,x^2 y\,\hat{i} + 3\,xy\,\hat{j})\,.\,(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_c (2\,x^2 y\,dx + 3\,xy\,dy)$$

Putting the values of y and dy, we get $\left(\begin{array}{l} y = 4\,x^2 \\ dy = 8\,x\,dx \end{array}\right)$

$$= \int_0^1 [2\,x^2\,(4\,x^2)\,dx + 3\,x\,(4\,x^2)\,8\,x\,dx]$$

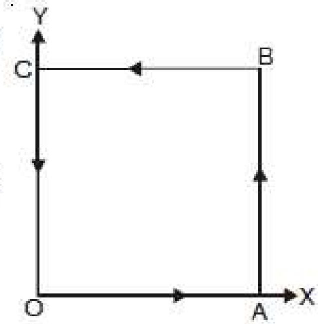
$$= 104\int_0^1 x^4\,dx = 104\left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$ **Ans.**

Example 66. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *where* $\vec{F} = x^2\hat{i} + xy\hat{j}$ *and C is the boundary of the square in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = a.*

(*Nagpur University, Summer 2001*)

Solution. $\int_C \vec{F}\cdot\vec{d}r = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr}$

Here $\vec{r} = x\hat{i} + y\hat{j},\quad \vec{d}r = dx\hat{i} + dy\hat{j},\quad \vec{F} = x^2\hat{i} + xy\,\hat{j}$



$$\vec{F}\cdot\vec{dr} = x^2dx + xydy \qquad \text{...(1)}$$

On OA, $y = 0$ $\quad \therefore\ \vec{F}\cdot\vec{dr} = x^2dx$

$$\int_{OA} \vec{F}\cdot\vec{dr} = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad \text{...(2)}$$

On AB, $x = a$ $\quad \therefore\ dx = 0$

(1) becomes

$$\therefore\ \vec{F}\cdot\vec{dr} = aydy$$

$$\int_{Ab} \vec{F}\cdot\vec{dr} = \int_0^a aydy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2} \qquad \text{...(3)}$$

On BC, $y = a$ $\quad \therefore\ dy = 0$

$\Rightarrow$ (1) becomes $\quad \vec{F}\cdot\vec{dr} = x^2dx$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3} \qquad \text{...(4)}$$

On CO, $x = 0$, $\quad \therefore\ \vec{F}\cdot\vec{dr} = 0$

(1) becomes

$$\int_{CO} \vec{F}\cdot\vec{dr} = 0 \qquad \text{...(5)}$$

On adding (2), (3), (4) and (5), we get $\int_C \vec{F}\cdot\vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0 = \frac{a^3}{2}$ **Ans.**

Example 67. *A vector field is given by*

$\vec{F} = (2y+3)\,\hat{i} + xz\hat{j} + (yz - x)\,\hat{k}$. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *along the path c is* $x = 2t$, $y = t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$. (*Nagpur University, Winter 2003*)

Solution. $\int_C \vec{F}\cdot\vec{dr} = \int_C (2y+3)\,dx + (xz)\,dy + (yz - x)\,dz$

$$\left[\begin{array}{lll} \text{Since } x = 2t & y = t & z = t^3 \\ \therefore \dfrac{dx}{dt} = 2 & \dfrac{dy}{dt} = 1 & \dfrac{dz}{dt} = 3t^2 \end{array}\right]$$

$$= \int_0^1 (2t+3)\,(2\,dt) + (2t)\,(t^3)\,dt + (t^4 - 2t)\,(3t^2\,dt) = \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1 = \left[2t^2 + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{3}{2}t^4\right]_0^1$$

$$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = 7.32857.$$ **Ans.**

Example 69. *If* $\vec{A} = (3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\,\hat{k}$, *evaluate the line integral* $\oint \vec{A}.\vec{dr}$ *from* (0, 0, 0) to (1, 1, 1) *along the curve C.*

$x = t$, $y = t^2$, $z = t^3$. (*Uttarakhand, I Semester, Dec. 2006*)

Solution. We have,

$$\int_C \vec{A}.\vec{dr} = \int_C [\,(3x^2 + 6y)\,\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\,].[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz]$$

$$= \int_C [\,(3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz]$$

If $x = t$, $y = t^2$, $z = t^3$, then points (0, 0, 0) and (1, 1, 1) correspond to $t = 0$ and $t = 1$ respectively.

Now, $\int_C \vec{A}.\vec{dr} = \int_{t=0}^{t=1} [\,(3t^2 + 6t^2)\,d\,(t) - 14t^2\,t^3\,d\,(t^2) + 20t\,(t^3)^2\,d\,(t^3)]$

$$= \int_{t=0}^{t=1} [\,9t^2\,dt - 14t^5\,.\,2t\,dt + 20\,t^7\,.\,3\,t^2\,dt] = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

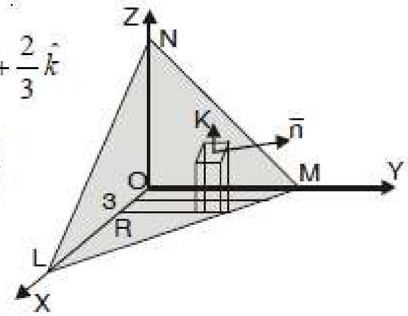
$$= \left[9\left(\frac{t^3}{3}\right) - 28\left(\frac{t^7}{7}\right) + 60\left(\frac{t^{10}}{10}\right)\right]_0^1 \qquad = 3 - 4 + 6 = 5$$ **Ans.**

Example 70. *Evaluate* $\iint_S \vec{A}.\hat{n}\,ds$ *where* $\vec{A} = (x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.* (*Nagpur University, Summer 2000*)

Solution. A vector normal to the surface "S" is given by

$$\nabla(2x + y + 2z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

And $\hat{n}$ = a unit vector normal to surface S



$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{k}.\hat{n} = \hat{k}.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \qquad \iint_S \overline{A}.\hat{n}\,ds = \iint_R \overline{A}.\hat{n}\frac{dx\,dy}{\hat{k}.\overline{n}}$$

Where R is the projection of S.

Now, $\vec{A}.\hat{n} = [(x + y^2)\,\hat{i} - 2x\hat{j} + 2yz\hat{k}].\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

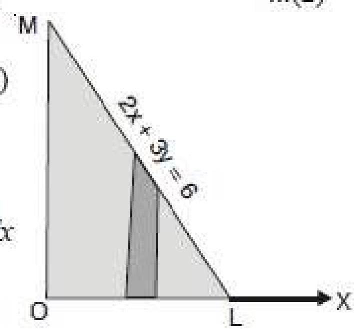
$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz \qquad \text{...(1)}$$

Putting the value of z in (1), we get

$$\vec{A}.\hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right) \qquad \left(\because \text{ on the plane } 2x + y + 2z = 6,\; z = \frac{(6-2x-y)}{2}\right)$$

$$\vec{A}.\hat{n} = \frac{2}{3}y(y + 6 - 2x - y) = \frac{4}{3}y(3 - x) \qquad \text{...(2)}$$

Hence, $$\iint_S \vec{A}.\hat{n}\,ds = \iint_R \overline{A}.\overline{n}\frac{dx\,dy}{|\hat{k}.\overline{n}|} \qquad \text{...(3)}$$



Putting the value of $\vec{A}.\hat{n}$ from (2) in (3), we get

$$\iint_S \vec{A}.\hat{n}\,ds = \iint_R \frac{4}{3}y(3 - x).\frac{3}{2}dx\,dy = \int_0^3\int_0^{6-2x} 2y(3 - x)\,dy\,dx$$

$$= \int_0^3 2(3 - x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3 - x)(6 - 2x)^2\,dx = 4\int_0^3 (3 - x)^3\,dx$$

$$= 4.\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = -(0 - 81) = 81$$ **Ans.**

Example 75. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\,dS$, *where* $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ *and S is the part of the plane* $2x + 3y + 6z = 12$ *included in the first octant.* (*Uttarakhand, I semester, Dec. 2006*)

Solution. Here, $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

Given surface $f(x, y, z) = 2x + 3y + 6z - 12$

Normal vector $= \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$

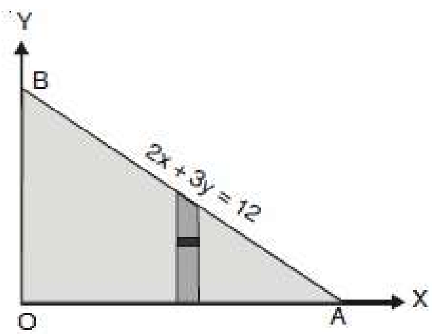
$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$dS = \frac{dx\,dy}{\hat{n}\,.\,\hat{k}} = \frac{dx\,dy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\,.\,\hat{k}} = \frac{dx\,dy}{\frac{6}{7}} = \frac{7}{6}\,dx\,dy$$

Now, $\iint \vec{A}\,.\,\hat{n}\,dS = \iint (18z\hat{i} - 12\hat{j} + 3y\hat{k})\,.\,\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\frac{7}{6}\,dx\,dy$

$$= \iint (36z - 36 + 18y)\frac{dx\,dy}{6} \qquad = \iint (6z - 6 + 3y)\,dx\,dy$$

Putting the value of $6z = 12 - 2x - 3y$, we get



$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$$

$$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)\,dx\,dy$$

$$= \int_0^6 (6 - 2x)\,dx \int_0^{\frac{1}{3}(12-2x)} dy$$

$$= \int_0^6 (6 - 2x)\,dx\,(y)_0^{\frac{1}{3}(12-2x)}$$

$$= \int_0^6 (6 - 2x)\frac{1}{3}(12 - 2x)\,dx = \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx$$

$$= \frac{1}{3}\left[\frac{4x^3}{3} - 18x^2 + 72x\right]_0^6 = \frac{1}{3}[4 \times 36 \times 2 - 18 \times 36 + 72 \times 6] = \frac{72}{3}[4 - 9 + 6] = 24$$ **Ans.**

Input interpretation

4^{th} roots of $-3-3i$

Results

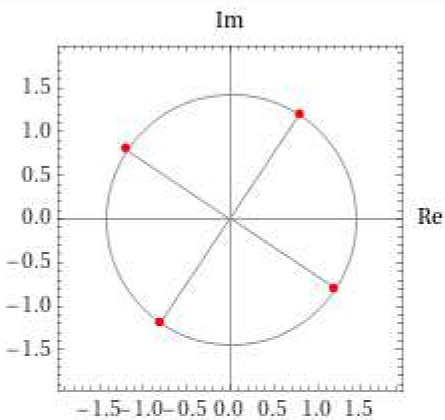
$\sqrt[8]{2}\ \sqrt[4]{3}\ e^{-(3 i \pi)/16} \approx 1.1933 - 0.7973\, i$

$\sqrt[8]{2}\ \sqrt[4]{3}\ e^{(5 i \pi)/16} \approx 0.7973 + 1.1933\, i$ (principal root)

$\sqrt[8]{2}\ \sqrt[4]{3}\ e^{(13 i \pi)/16} \approx -1.1933 + 0.7973\, i$

$\sqrt[8]{2}\ \sqrt[4]{3}\ e^{-(11 i \pi)/16} \approx -0.7973 - 1.1933\, i$

Plot



Find $\sqrt[4]{81i}$.

SOLUTION

The polar form of $81i$ is $81\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$ (for steps, see polar form calculator).

According to the De Moivre's Formula, all n-th roots of a complex number $r\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)$ are given by $r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$, $k=\overline{0..n-1}$.

We have that $r=81$, $\theta=\frac{\pi}{2}$, and $n=4$.

- $k=0$: $\sqrt[4]{81}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot0}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot0}{4}\right)\right)=3\left(\cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right)\right)=3\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}+3i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$
- $k=1$: $\sqrt[4]{81}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot1}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot1}{4}\right)\right)=3\left(\cos\left(\frac{5\pi}{8}\right)+i\sin\left(\frac{5\pi}{8}\right)\right)=-3\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}+3i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}$
- $k=2$: $\sqrt[4]{81}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot2}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot2}{4}\right)\right)=3\left(\cos\left(\frac{9\pi}{8}\right)+i\sin\left(\frac{9\pi}{8}\right)\right)=-3\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}-3i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$
- $k=3$: $\sqrt[4]{81}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot3}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot3}{4}\right)\right)=3\left(\cos\left(\frac{13\pi}{8}\right)+i\sin\left(\frac{13\pi}{8}\right)\right)=3\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}-3i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}$

ANSWER

$\sqrt[4]{81i}=3\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}+3i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}\approx 2.77163859753386+1.148050297095269i$ A

$\sqrt[4]{81i}=-3\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}+3i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}\approx -1.148050297095269+2.77163859753386i$ A

$\sqrt[4]{81i}=-3\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}-3i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}\approx -2.77163859753386-1.148050297095269i$ A

$\sqrt[4]{81i}=3\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}-3i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}\approx 1.148050297095269-2.77163859753386i$ A

Find $\sqrt[4]{16i}$.

SOLUTION

The polar form of $16i$ is $16\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$ (for steps, see polar form calculator).

According to the De Moivre's Formula, all n-th roots of a complex number $r\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)$ are given by $r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$, $k=\overline{0..n-1}$.

We have that $r=16$, $\theta=\frac{\pi}{2}$, and $n=4$.

- $k=0$: $\sqrt[4]{16}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 0}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 0}{4}\right)\right)=2\left(\cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right)\right)=2\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}+2i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$
- $k=1$: $\sqrt[4]{16}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 1}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 1}{4}\right)\right)=2\left(\cos\left(\frac{5\pi}{8}\right)+i\sin\left(\frac{5\pi}{8}\right)\right)=-2\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}+2i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}$
- $k=2$: $\sqrt[4]{16}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 2}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 2}{4}\right)\right)=2\left(\cos\left(\frac{9\pi}{8}\right)+i\sin\left(\frac{9\pi}{8}\right)\right)=-2\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}-2i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$
- $k=3$: $\sqrt[4]{16}\left(\cos\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 3}{4}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\cdot\pi\cdot 3}{4}\right)\right)=2\left(\cos\left(\frac{13\pi}{8}\right)+i\sin\left(\frac{13\pi}{8}\right)\right)=2\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}-2i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}$

ANSWER

$\sqrt[4]{16i}=2\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}+2i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}\approx 1.847759065022574+0.76536686473018i$ A

$\sqrt[4]{16i}=-2\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}+2i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}\approx -0.76536686473018+1.847759065022574i$ A

$\sqrt[4]{16i}=-2\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}-2i\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}\approx -1.847759065022574-0.76536686473018i$ A

$\sqrt[4]{16i}=2\sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}-2i\sqrt{\frac{\sqrt{2}}{4}+\frac{1}{2}}\approx 0.76536686473018-1.847759065022574i$ A

Find $\sqrt[4]{1+i}$.

SOLUTION

The polar form of $1+i$ is $\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$ (for steps, see polar form calculator).

According to the De Moivre's Formula, all n-th roots of a complex number $r\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)$ are given by $r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$, $k=\overline{0..n-1}$.

We have that $r=\sqrt{2}$, $\theta=\frac{\pi}{4}$, and $n=4$.

- $k=0$: $\sqrt[4]{\sqrt{2}}\left(\cos\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 0}{4}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 0}{4}\right)\right)=\sqrt[8]{2}\left(\cos\left(\frac{\pi}{16}\right)+i\sin\left(\frac{\pi}{16}\right)\right)=\sqrt[8]{2}\cos\left(\frac{\pi}{16}\right)+\sqrt[8]{2}i\sin\left(\frac{\pi}{16}\right)$
- $k=1$: $\sqrt[4]{\sqrt{2}}\left(\cos\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 1}{4}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 1}{4}\right)\right)=\sqrt[8]{2}\left(\cos\left(\frac{9\pi}{16}\right)+i\sin\left(\frac{9\pi}{16}\right)\right)=-\sqrt[8]{2}\cos\left(\frac{7\pi}{16}\right)+\sqrt[8]{2}i\sin\left(\frac{7\pi}{16}\right)$
- $k=2$: $\sqrt[4]{\sqrt{2}}\left(\cos\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 2}{4}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 2}{4}\right)\right)=\sqrt[8]{2}\left(\cos\left(\frac{17\pi}{16}\right)+i\sin\left(\frac{17\pi}{16}\right)\right)=-\sqrt[8]{2}\cos\left(\frac{\pi}{16}\right)-\sqrt[8]{2}i\sin\left(\frac{\pi}{16}\right)$
- $k=3$: $\sqrt[4]{\sqrt{2}}\left(\cos\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 3}{4}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\cdot\pi\cdot 3}{4}\right)\right)=\sqrt[8]{2}\left(\cos\left(\frac{25\pi}{16}\right)+i\sin\left(\frac{25\pi}{16}\right)\right)=\sqrt[8]{2}\cos\left(\frac{7\pi}{16}\right)-\sqrt[8]{2}i\sin\left(\frac{7\pi}{16}\right)$

ANSWER

$\sqrt[4]{1+i} = \sqrt[8]{2}\cos\left(\frac{\pi}{16}\right) + \sqrt[8]{2}i\sin\left(\frac{\pi}{16}\right) \approx 1.069553932363986 + 0.212747504726743i$ A

$\sqrt[4]{1+i} = -\sqrt[8]{2}\cos\left(\frac{7\pi}{16}\right) + \sqrt[8]{2}i\sin\left(\frac{7\pi}{16}\right) \approx -0.212747504726743 + 1.069553932363986i$ A

$\sqrt[4]{1+i} = -\sqrt[8]{2}\cos\left(\frac{\pi}{16}\right) - \sqrt[8]{2}i\sin\left(\frac{\pi}{16}\right) \approx -1.069553932363986 - 0.212747504726743i$ A

$\sqrt[4]{1+i} = \sqrt[8]{2}\cos\left(\frac{7\pi}{16}\right) - \sqrt[8]{2}i\sin\left(\frac{7\pi}{16}\right) \approx 0.212747504726743 - 1.069553932363986i$ A

Find $\sqrt[5]{32}$.

SOLUTION

The polar form of 32 is $32\left(\cos\left(0\right)+i\sin\left(0\right)\right)$ (for steps, see polar form calculator).

According to the De Moivre's Formula, all n-th roots of a complex number $r\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)$ are given by $r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$, $k=\overline{0..n-1}$.

We have that $r=32$, $\theta=0$, and $n=5$.

- $k=0$: $\sqrt[5]{32}\left(\cos\left(\frac{0+2\cdot\pi\cdot 0}{5}\right)+i\sin\left(\frac{0+2\cdot\pi\cdot 0}{5}\right)\right)=2\left(\cos\left(0\right)+i\sin\left(0\right)\right)=2$
- $k=1$: $\sqrt[5]{32}\left(\cos\left(\frac{0+2\cdot\pi\cdot 1}{5}\right)+i\sin\left(\frac{0+2\cdot\pi\cdot 1}{5}\right)\right)=2\left(\cos\left(\frac{2\pi}{5}\right)+i\sin\left(\frac{2\pi}{5}\right)\right)=-\frac{1}{2}+\frac{\sqrt{5}}{2}+2i\sqrt{\frac{\sqrt{5}}{8}+\frac{5}{8}}$
- $k=2$: $\sqrt[5]{32}\left(\cos\left(\frac{0+2\cdot\pi\cdot 2}{5}\right)+i\sin\left(\frac{0+2\cdot\pi\cdot 2}{5}\right)\right)=2\left(\cos\left(\frac{4\pi}{5}\right)+i\sin\left(\frac{4\pi}{5}\right)\right)=-\frac{\sqrt{5}}{2}-\frac{1}{2}+2i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$
- $k=3$: $\sqrt[5]{32}\left(\cos\left(\frac{0+2\cdot\pi\cdot 3}{5}\right)+i\sin\left(\frac{0+2\cdot\pi\cdot 3}{5}\right)\right)=2\left(\cos\left(\frac{6\pi}{5}\right)+i\sin\left(\frac{6\pi}{5}\right)\right)=-\frac{\sqrt{5}}{2}-\frac{1}{2}-2i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$
- $k=4$: $\sqrt[5]{32}\left(\cos\left(\frac{0+2\cdot\pi\cdot 4}{5}\right)+i\sin\left(\frac{0+2\cdot\pi\cdot 4}{5}\right)\right)=2\left(\cos\left(\frac{8\pi}{5}\right)+i\sin\left(\frac{8\pi}{5}\right)\right)=-\frac{1}{2}+\frac{\sqrt{5}}{2}-2i\sqrt{\frac{\sqrt{5}}{8}+\frac{5}{8}}$

ANSWER

$\sqrt[5]{32}=2$ A

$\sqrt[5]{32}=-\frac{1}{2}+\frac{\sqrt{5}}{2}+2i\sqrt{\frac{\sqrt{5}}{8}+\frac{5}{8}}\approx 0.618033988749895+1.902113032590307i$ A

$\sqrt[5]{32}=-\frac{\sqrt{5}}{2}-\frac{1}{2}+2i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\approx -1.618033988749895+1.175570504584946i$ A

$\sqrt[5]{32}=-\frac{\sqrt{5}}{2}-\frac{1}{2}-2i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\approx -1.618033988749895-1.175570504584946i$ A

$\sqrt[5]{32}=-\frac{1}{2}+\frac{\sqrt{5}}{2}-2i\sqrt{\frac{\sqrt{5}}{8}+\frac{5}{8}}\approx 0.618033988749895-1.902113032590307i$ A

GAMMA AND BETA FUNCTIONS

Useful Definitions and Formulas

1. $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad n > 0$

2. $\Gamma(n+1) = n\Gamma(n) = n!$

3. $\Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad m,n > 0$

5. $B(m,n) = B(n,m)$

6. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

7. $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

8. $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

9. $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad 0 < n < 1$

10. $\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = B(m,n)$

A. Evaluate in terms of gamma function

1. $\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}} dx$ **2.** $\int_0^b y^5\sqrt{b^2-y^2}\,dy$ **3.** $\int_0^\infty e^{-x^2} dx$

4. $\int_0^\infty x^5 e^{-4x} dx$ **5.** $\int_0^\infty x^6 e^{-3x} dx$ **6.** $\int_0^\infty x^5 e^{-x^2} dx$

7. $\int_0^\infty x^9 e^{-x^2} dx$ **8.** $\int_0^\infty \sqrt{x} e^{-x^2} dx$ **9.** $\int_0^1 \frac{x^3}{\sqrt{1-x^3}} dx$

10. $\int_0^1 \frac{1}{\sqrt{\ln\left(\frac{1}{x}\right)}} dx$ **11.** $\int_0^1 \frac{1}{\sqrt{x\ln\left(\frac{1}{x}\right)}} dx$ **12.** $\int_0^1 \left(1-\frac{1}{x}\right)^{\frac{1}{3}} dx$

B. Evaluate in terms of beta function

1. $\int_0^1 \frac{x^2}{\sqrt{1-x}} dx$ **2.** $\int_0^1 x^7(1-x)^3 dx$ **3.** $\int_0^1 \frac{1}{\sqrt{1-x^3}} dx$

4. $\int_0^1 x^3\sqrt{1-x}\,dx$ **5.** $\int_0^1 x^{\frac{5}{2}}(1-x)^{\frac{3}{2}} dx$ **7.** $\int_0^a y^7\sqrt{a^4-y^4}\,dy$

7. $\int_0^4 y^3\sqrt{64-y^3}\,dx$ **8.** $\int_0^1 x^2(1-x^3)^{\frac{3}{2}} dx$ **9.** $\int_0^\infty \frac{1}{1+x^4} dx$

C. Evaluate the following integrals

1. $\int_0^\pi \sin^5\theta\cos^4\theta\,d\theta$ **2.** $\int_0^\pi \sin^6\theta\cos^7\theta\,d\theta$ **3.** $\int_0^{\frac{\pi}{6}} \sin^2 6\theta\cos^4 3\theta\,d\theta$

4. $\int_0^{\frac{\pi}{4}} \sin^2 4\theta\cos^3 2\theta\,d\theta$ **5.** $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta\,d\theta$ **6.** $\int_0^{\frac{\pi}{8}} \sin^2 8\theta\cos^4 4\theta\,d\theta$

Double Integral Worksheet

Useful Properties of double integrals.

1. $\iint_D [f(x,y) + g(x,y)]\ dA = \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA$
2. $\iint_D c\, f(x,y)\ dA = c \iint_D f(x,y)\ dA$
3. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$ then $\iint_D f(x,y)\ dA \geq \iint_D g(x,y)\ dA$
4. $\iint_D 1\ dA = A(D)$
5. If $D = D_1 \cup D_2$ for non-overlapping "nice" regions, then $\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA$
6. If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ then $mA(D) \leq \iint_D f(x,y)\ dA \leq MA(D)$

Double Integrals over general regions in x, y coordinates

Sketch regions too

1. $\int_0^4 \int_0^{4-x} xy\, dy\, dx$
2. $\iint_D (x+y)\ dA$ where D is the triangle with vertices $(0,0), (0,2), (1,2)$
3. $\iint_D 48xy\ dA$ where D is the region bounded by $y = x^3$ and $y = \sqrt{x}$

Reverse order of integration.

1. $\int_0^1 \int_x^{2x} e^{y-x}\ dy\, dx$
2. $\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} 1\ dx\, dy$
3. $\int_0^7 \int_{x^2-6x}^{x} f(x,y)\ dy\, dx$
4. $\int_1^2 \int_x^{x^3} f(x,y)\ dy\, dx + \int_2^8 \int_x^8 f(x,y)\ dy\, dx$

Find Volume of solid

1. Tetrahedron in first octant bounded by coordinate planes and $z = 7 - 3x - 2y$.
2. Solid inside both the sphere $x^2 + y^2 + z^2 = 3$ and paraboloid $2z = x^2 + y^2$.

CURVATURE AND RADIUS OF CURVATURE

5.1 Introduction:

Curvature is a numerical measure of bending of the curve. At a particular point on the curve , a tangent can be drawn. Let this line makes an angle Ψ with positive x- axis. Then curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length s.

∴ Curvature at P = $\left|\frac{d\Psi}{ds}\right|$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature.

Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

5.2

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+y_1^2\right)^{3/2}}{|y_2|} \text{ (When tangent is parallel to x – axis)}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \text{ (When tangent is parallel to y – axis)}$$

- **Radius of curvature of parametric curve:**
  $\mathbf{x = f(t), y = g(t)}$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|}, \quad \text{where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 1 Find the radius of curvature at any pt of the cycloid

$$x = a(\theta + \text{Sin}\,\theta), \quad y = a(1 - \cos\theta)$$

Solution: $x' = \frac{dx}{d\theta} = a(1+\cos\theta)$ and $y' = \frac{dy}{d\theta} = a\sin\theta$

$x'' = \frac{d^2x}{d\theta^2} = -\text{ a}\sin\theta$ and $y'' = \frac{d^2y}{d\theta^2} = \text{ a}\cos\theta$

Now $\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} = \frac{\{a^2(1+Cos\ \theta)^2 + a^2 sin^2\theta\}^{3/2}}{a^2(1+Cos\ \theta)\ Cos\ \theta + a^2\ sin^2\theta}$

$= \frac{a\left(1+cos^2\theta+2cos\theta+sin^2\theta\right)^{3/2}}{cos\theta+cos^2\theta+sin^2\theta}$

$= \frac{a(2+2cos\theta)^{3/2}}{1+cos\theta}$

$= 2\sqrt{2}\text{ a}\sqrt{1+cos\theta}$

$=2\sqrt{2}\text{ a}\sqrt{2\frac{\cos^2\theta}{2}} = 4\text{a}\cos\frac{\theta}{2}$

Example 2 Show that the radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ (x = a $\cos^3\theta$, y = a $\sin^3\theta$) is equal to three times the lenth of the perpendicular from the origin to the tangent.

Solution : $\frac{dx}{d\theta} = -\text{ 3a } cos^2\theta\ sin\theta = x'$

$\frac{dy}{d\theta} = -\text{ 3a } sin^2\theta\ cos\theta = y'$

$x'' = \frac{d^2x}{d\theta^2} = \frac{d}{d\theta}(-\text{ 3a }\cos^2\theta\sin\theta)$

$= -\text{3a }[-2\cos\theta\sin^2\theta+\cos^3\theta]$

$= 6\text{ a}\cos\theta\sin^2\theta - 3\text{a}\cos^3\theta$

$y'' = \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(3\text{a}\sin^2\theta\cos\theta)$

$= 3a(2\sin\theta\ cos^2\theta - sin^3\theta)$

$= 6a\sin\theta\ cos^2\theta - 3a\ sin^3\theta$

Now $\rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|}$

$= \frac{\left(9a^2cos^4\theta sin^2\theta+9a^2sin^4\theta cos^2\theta\right)^{3/2}}{|(-3acos^2\theta\ sin\theta)(6a\ sin\theta cos^2\theta-3asin^3\theta)-3asin^2\theta\cos\theta(6a\ cos\theta\ sin^2\theta-3a\ cos^3\theta)|}$

$$= \frac{\left[9a^2cos^2sin^2\theta\left(cos^2\theta + sin^2\theta\right)\right]^{3/2}}{\left|-18a^2\, sin^2\,\theta cos^4\,\theta + 9a^2\,cos^2\theta\,sin^4\theta - 18a^2\,sin^4\,\theta cos^2\theta + 9a^2\,sin^2\,\theta cos^4\theta\right|}$$

$$= \frac{9^{3/2}(a\,cos\theta\,sin\theta)^3}{\left|-9a^2sin^2\theta cos^4\theta - 9a^2\,cos^2\theta\,sin^4\theta\right|}$$

$$= \frac{(9)^{3/2}(a\,cos\theta\,\sin\theta)^3}{9a^2\,cos^2\theta sin^2\theta\,(cos^2\theta + sin^2\theta)}$$

$\Longrightarrow\ \rho$ = 3a sin$\theta cos\theta$ …….(1)

The equation of the tangent at any point on the curve is

y – a sin^3 θ = – tan θ (x – a cos$^3\theta$)

$\Longrightarrow$ x sinθ + y cos θ – a sin θ cos θ = 0 ……..(2)

∴ The length of the perpendicular from the origin to the tangent (2) is

$$p = \frac{\left|0.sin\theta + 0.cos\theta - a\,sin\theta\,cos\theta\right|}{\sqrt{sin^2\theta + cos^2\theta}}$$

= a sinθ cosθ ……..(3)

Hence from (1) & (3), ρ = 3p

Example 3 If ρ & ρ ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove that

$$\left(\rho^{2/3} + \rho'^{2/3}\right)(ab)^{2/3} = a^2 + b^2$$

Solution: Parametric equation of the ellipse is

x = a cos θ, y=b sin θ

x'= – a sin θ, y'= b cos θ

x''= – a cos θ, y'' = – b sin θ

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{\left|x'y'' - y'x''\right|} = \frac{\left(a^2sin^2\theta + b^2\,cos^2\,b\right)^{3/2}}{\left|(-a\,sin\theta)\,(-bsin\theta) - (bcos\theta)\,(-acos\theta)\right|}$$

$$= \frac{\left(a^2 sin^2\theta + b^2\, cos^2\theta\right)^{3/2}}{ab} \qquad \ldots\ldots(1)$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by replacing θ by $\theta + \frac{\pi}{2}$ in (1).
Let it be denoted by ρ'. Then

$$\therefore\ \rho' = \frac{\left(a^2 sin^2\theta + b^2 sin^2\,\theta\right)^{3/2}}{ab}$$

$$\therefore \rho^{2/3} + \rho'^{2/3} = \frac{a^2 sin^2\theta + b^2\, cos^2\theta}{(ab)^{2/3}} + \frac{a^2 cos^2\theta + b^2\, cos^2\,\theta}{(ab)^{2/3}}$$

$$= \frac{a^2 + b^2}{(ab)^{2/3}}$$

$$\Longrightarrow (ab)^{2/3}\left(\rho^{2/3} + \rho'^{2/3}\right) = a^2 + b^2$$

Example 4 Find the points on the parabola y^2 = 8x at which the radius of curvature is $\frac{125}{16}$.

Solution: y = $2\sqrt{2}\,\sqrt{x}$

$$y_1 = \frac{\sqrt{2}}{\sqrt{x}} \qquad , \qquad y_2 = \frac{-1}{\sqrt{2}x^{3/2}}$$

$$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{|y_2|} = (1 + \frac{2}{x})^{3/2}.\sqrt{2}\, x^{3/2} = \sqrt{2}\,(x+2)^{3/2}$$

Given $\rho = \frac{12.5}{16}$ $\therefore (x + 2)^{3/2} = \frac{125}{16\sqrt{2}} = \left(\frac{5}{2\sqrt{2}}\right)^3$

$$\therefore (x + 2)^{3/2} = \frac{5}{2\sqrt{2}}$$

$\Longrightarrow$ x + 2 = $\frac{25}{8}$ $\Longrightarrow$ x = $\frac{9}{8}$

$\Longrightarrow$ y^2 = 8 $\left(\frac{9}{8}\right)$ i.e. y = 3,-3

Hence the points at which the radius of curvature is $\frac{125}{16}$ are (9,$\pm$ 3).

Example 5 Find the radius of curvature at any point of the curve y = C cos h (x/c)

Solution: $y_1 = \frac{c}{c}$ Sin h $\frac{x}{c}$ = Sin h $\left(\frac{x}{c}\right)$

$y_2 = \frac{1}{c}$ cos h $\frac{x}{c}$

Now, $\rho = \frac{\left(1+y_1{}^2\right)^{3/2}}{y_2}$

$$= \frac{\left(1+Sin\ h^2\left(\frac{x}{c}\right)\right)^{3/2}}{\frac{1}{c}\cos h\,\frac{x}{c}}$$

$$= \text{C cos h}^2\left(\frac{x}{c}\right)$$

$\Rightarrow \rho = \frac{1}{c}y^2$

Example 6 For the curve y = $\frac{ax}{a+x}$, prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

where ρ is the radius of curvature of the curve at its point (x, y)

Solution: Here y = $\frac{ax}{a+x}$

$\Rightarrow$ $y_1 = \frac{(a+x)a - ax\,(1)}{(a+x)^2}$

$= \frac{a^2}{(a+x)^2}$

$\therefore$ $y_2 = \frac{-2a^2}{(a+x)^3}$

Now, $\rho = \frac{\left(1+y^{1^2}\right)^{3/2}}{y_2}$

$$= \left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{(-2a^2)}$$

$$\therefore \rho^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \frac{(a+x)^2}{(-2)^{2/3}\,a^{4/3}}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[1+\frac{a^4}{(a+x)^4}\right] \quad \frac{(a+x)^2}{2^{2/3}\,a^{4/3}} \times \frac{2^{2/3}}{a^{2/3}}$$

$$= \frac{1}{a^2}\left[1+\frac{a^4}{(a+1)^4}\right](a+x)^2$$

$$= \frac{1}{a^2}\left[(a+x)^2+\frac{a^4}{(a+x)^2}\right]$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

Example 7 Find the curvature of x = 4 cost, y = 3 sint. At what point on this ellipse does the curvature have the greatest & the least values? What are the magnitudes?

Solution: $\rho = \frac{(x'^2+y'^2)^{3/2}}{|x'y''-y'x''|}$

Now, $x' = -\ 4 \sin t \qquad \Rightarrow \ x'' = -\ 4 \cos t$

$y' = -3\cos t \qquad \Rightarrow \ x'' = -\ 3 \sin t$

$$\therefore\ \rho = \frac{(16 sin^2\, t + 9\, cost^2\, t)^{3/2}}{-4\sin t\,(-3 sint) - 3\, Cost\,(-4 Cost)}$$

$$= \frac{1}{12}(9\, cost^2\, t + 16 sin^2\, t)^{3/2}$$

$\Rightarrow (\rho.12)^{2/3} = 9\ \cos t^2 t + 16 \sin^2 t$

Now, curvature is the reciprocal of radius of curvature. Curvature is maximum & minimum when ρ is minimum and maximum respectively . For maximum and minimum values;

$$\frac{d}{dt}(16 \sin^2 t + 9 \cos^2 t) = 0$$

$\Rightarrow$ 32 sint cost + 18 cost (–sint) = 0

$\Rightarrow \qquad 4 \sin t \cos t = 0$

$\Rightarrow \quad t = 0 \ \& \ \frac{\pi}{2}$

At $t = 0$ ie at (4,0)
$(12\,\rho)^{2/3} = 9$
$\Rightarrow \ 12\,\rho = 9^{3/2}$

$\Rightarrow \rho = \frac{9}{4} \quad \therefore \frac{1}{\rho} = \frac{4}{9}$
Similarly, at $t = \frac{\pi}{2}$ ie at (0,3)
$(12\,\rho)^{2/3} = 16$
$12\rho = 4^3$
$\rho = 16/3 \quad \therefore \frac{1}{\rho} = \frac{3}{16}$

Hence, the least value is $\frac{3}{16}$ and the greatest value is $\frac{4}{9}$

Example 8 Find the radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the coordinate axes.
Solution: On differentiating the given , we get

$$\frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}}\ \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{by}{ax}} \qquad \ldots\ldots.(1)$$

The curve touches the x-axis if $\frac{dy}{dx} = 0$ or $y = 0$
When $y = 0$, we have $x = a$ (from the given eq^n)

$\Rightarrow$ given curve touches x – axis at (a,0)

The curve touches y – axis if $\frac{dx}{dy} = 0$ or $x = 0$
When $x = 0$, we have $y = b$

$\Rightarrow$ Given curve touches y-axis at (o, b)

$$\frac{d^2y}{dx} = \sqrt{\frac{b}{a}} \left\{ \sqrt{\frac{b}{a}} \cdot \frac{1}{2x} - \frac{1}{2}\sqrt{\frac{y}{x}} \right\} \qquad \{\text{from (1)}\}$$

At (a,0), $\frac{d^2y}{dx^2} = \frac{1}{2a}\frac{b}{a} = \frac{b}{2a^2}$

$\therefore$ At (a,o), $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = (1+\ 0)^{3/2}\ \frac{2a^2}{b} = \frac{2a^2}{b}$

At (o,b), $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{2b^2}{a}$

5.3 Radius of curvature of Polar curves r = f (θ):

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{2r_1^2+r^2-rr_2} \qquad \left(where\ r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}\right)$$

Example 9 Prove that for the cardioide r = a (1 + cos θ),

$\frac{\rho^2}{r}$ is const.

Solution: Here r = a (1+ cos θ)

$\Rightarrow r_1 = -$ a Sin θ and $r_2 = -$ a cos θ

$$\therefore\ r^2 + r_1^2 = a^2[(1+\cos\theta)^2 + sin^2\theta] = 2a^2(1+\cos\theta)$$

$$r^2 + 2r_1^2 - rr^2 = a^2[(1+\cos\theta)^2 + 2sin^2\theta + \cos\theta(1+\cos\theta)]$$

$$= 3a^2(1+\cos\theta)$$

$$\therefore\ \rho^2 = \frac{(r^2+r_1^2)^3}{(r^2+2r_1^2-rr_2)^2} = \frac{8a^6(1+cos\theta)^3}{9a^4(1+\cos\theta)^2} = \frac{8}{9}a^2(1+\cos\theta)$$

$\Rightarrow\ \rho^2 = \frac{8a}{9}$ r

$\therefore\ \frac{\rho^2}{r} = \frac{8a}{9}$ which is a constant.

Example 10 Show that at the point of intersection of the curves r = a θ and r θ = a, the curvatures are in the ratio 3:1 ($0 < \theta < 2\pi$)

Solution: The points of intersection of curves r = a θ & r θ = a are given by a θ^2 = a or $\theta = \pm 1$

Now for the curve r=a θ we have r_1 = a and r_2 = 0

$$\therefore \text{At } \theta = \pm 1,\ \rho = \left[\frac{(r^2+r_1^2)^{3/2}}{2a^2+a^2\theta^2-0}\right]_{\theta=\pm 1} = \frac{a\,(2\sqrt{2})}{3} = \rho_1$$

For the curve r θ = a,

$$r_1 = \frac{-a}{\theta^2} \quad \text{and} \quad r_2 = \frac{2a}{\theta^3}$$

$$\text{At } \theta = \pm 1,\ \rho = \left[\frac{\left(\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right)^{3/2}}{\frac{2a^2}{\theta^4}+\frac{a^2}{\theta^2}-\frac{2a^2}{\theta^4}}\right]_{\theta=\pm 1,} = \left[a\,\frac{(1+\theta^2)^{3/2}}{\theta^4}\right]_{\theta=\pm 1}$$

$$= 2a\sqrt{2} = \rho_2$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{2a\sqrt{2}}{2a\sqrt{2/3}} = \frac{3}{1}$$

$$\therefore \rho_2 : \rho_1 = 3 : 1$$

Example 11 Find the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\theta$

Solution: $r^m = a^m \cos m\theta$

$$\Rightarrow m\log r = m\log a + \log\cos m\theta$$

$$\Rightarrow \frac{m}{r} r_1 = -m\,\frac{\sin m\theta}{\cos m\theta} \quad \text{(on differentiating w.r.t. } \theta)$$

$$\Rightarrow r_1 = -r\tan m\theta \qquad \ldots\ldots.(1)$$

Now $r_2 = -(r_1 \tan m\theta + rm\sec^2 m\theta)$

$$= r\tan^2 m\theta - rm\sec^2 m\theta \quad \text{(from (1))}$$

$$\therefore \rho = \frac{\left(r^2+r^2\tan^2 m\theta\right)^{3/2}}{r^2+2r^2\tan^2 m\theta - r^2\tan^2 m\theta + r^2 m\sec^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta + r^2 m\sec^2 m\theta} \qquad = \frac{r}{m+1}\sec m\theta$$

Example 12 Show that the radius of curvature at the point (r, θ)

of the curve $r^2 \cos 2\theta = a^2$ is $\frac{r^3}{a^2}$

Solution: $r^2 = a^2 \sec 2\theta$

$$\Rightarrow 2rr_1 = 2a^2 \sec 2\theta \tan 2\theta$$
$$\Rightarrow \quad r_1 = r \tan 2\theta$$

and $r_2 = 2r\, sec^2\theta + r_1 tan 2\theta$

$= 2r \sec^2 2\theta + r \tan^2 2\theta$ ($\because$ r = r tan 2 θ)

Now $\rho = \frac{(r^1+r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} \Rightarrow \rho = \frac{\left((r^2+r^2\tan^2 2\theta)\right)^{3/2}}{2r^2 \tan^2 2\theta + r^2 - r^2 (2\sec^2 2\theta + \tan^2 2\theta)}$

$$= \frac{(r^2\sec^2 2\theta)^{3/2}}{r^2(2\tan^2 2\theta + 1 - 2\sec^2 2\theta - \tan^2 2\theta)}$$

$$= \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta}$$

$$= r \sec 2\theta$$

$$= r \cdot \frac{r^2}{a^2} = \frac{r^3}{a^2}$$

5.4 Radius of curvature at the origin by Newton's method

It is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent there.

When x-axis is the tangent, then

$$\rho = \lim_{x\to 0} \frac{x^2}{2y}$$

When y- axis is the tangent, then

$$\rho = \lim_{x \to 0} \frac{y^2}{2x}$$

Example13 Find the radius of curvature at the origin of the curve

$$x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$$

Solution: Tangent is x = 0 ie y–axis,

$$\rho = \lim_{y \to 0} \frac{y^2}{2x}$$

Dividing the given equation by 2x, we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{+xy}{2x} \ \frac{-y^2}{2x} + \frac{2x}{2x} = 0$$

$$x^3\left(\frac{y}{2x}\right) - xy\left(\frac{y^2}{2x}\right) + xy + x\left(\frac{y}{2x}\right) - \left(\frac{y^2}{2x}\right) + 1 = 0$$

Taking limit $y \to 0$ on both the sides , we get $\rho = 1$

Exercise 5A

1. Find the radius of curvatures at any point the curve

   y = 4 sin x – sin2x at $x = \frac{\pi}{2}$

   Ans $\rho = \frac{1}{4}(5)^{3/2}$

2. If ρ_1 ,ρ_2 are the radii of curvature at the extremes of any chord of the cardioide r = a (1 + cos θ) which passes through the pole, then

   $\rho_1^1 + \rho_2^2 = \frac{16a^2}{9}$

3 Find the radius of curvature of $y^2 = x^2$ (a+x) (a –x) at the origin

Ans. a$\sqrt{2}$

4. Find the radius of curvature at any point 't' of the curve

   x = a (cost + log tan t/2), y = a sin$t$

   Ans. a cos$t$

5. Find the radius of curvature at the origin, for the curve
$2x^3 - 3x^2y + 4y^3 + y^2 - 3x = 0$

Ans. ρ =3/2

6. Find the radius of curvature of $y^2 = \frac{4a^2(2a-x)}{x}$ at a point where the curve meets x – axis

Ans. ρ = a

7. Prove the if ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus rectum is l then
$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}$

8. Find the radius of curvature to the curve r = a (1+ cos θ) at the point where the tangent is parallel to the initial line.

Ans. $\rho = \frac{2}{\sqrt{3}}$. a

9. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho = \frac{a^2b^2}{p^3}$ where p is the perpendicular distance from the centre on the tangent at (x,y).