Design and Analysis of Algorithms

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Complexity Analysis

Want to achieve platform-independence

- Use an abstract machine that uses *steps* of time and *units* of memory, instead of seconds or bytes
  - each elementary operation takes 1 step
  - each elementary instance occupies 1 unit of memory

Analysing an Algorithm

- **Simple statement sequence**

  $s_1;\ s_2;\ \ldots\ ;\ s_k$

  - *O(1)* as long as *k* is constant

- **Simple loops**

  ```
  for(i=0; i<n; i++) { s; }
  ```

  where `s` is *O(1)*

  - Time complexity is *O(n)*

- **Nested loops**

  ```
  for(i=0; i<n; i++)
      for(j=0; j<n; j++) { s; }
  ```

  - Complexity is $O(n^2)$

This part is *O(n)*

Analysing an Algorithm

- **Loop index doesn't vary linearly**

```
h = 1;
while ( h <= n ) {
        s;
        h = 2 * h;
        }
```

  - `h` takes values $1, 2, 4, \ldots$ until it exceeds n
  - There are $1 + \log_2 n$ iterations
  - Complexity $O(\log n)$

Analysing an Algorithm

- **Loop index depends on outer loop index**

```
for(j=0;j<=n;j++)
    for(k=0;k<j;k++){
        s;
        }
```

  - **Inner loop executed**
    - **1, 2, 3, …., n times**

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$\therefore$ **Complexity** $O(n^2)$

Distinguish this case - where the iteration count increases (decreases) by a factor $\leftarrow O(n^k)$
from the previous one - where it changes by a factor $\leftarrow O(\log n)$

Analysing an Algorithm

// Input: int A[N], array of N integers
// Output: Sum of all numbers in array A

```
int Sum(int A[], int N)
{
  int s=0;
  for (int i=0; i< N; i++)
    s = s + A[i];
  return s;
}
```

How should we analyse this?

Analysing an Algorithm

```
// Input: int A[N], array of N integers
// Output: Sum of all numbers in array A

int Sum(int A[], int N){
    int s=0;  1

    for (int i=0; i< N; i++)
    2                  3        4
        s = s + A[i];
    5           6       7
    return s;
}               8
```

1,2,8: Once

3,4,5,6,7: Once per each iteration of for loop, N iteration

Total: 5N + 3

The *complexity function* of the algorithm is : *f(N) = 5N +3*

Analysing an Algorithm

Growth of 5n+3

Estimated running time for different values of N:

N = 10	**=> 53 steps**
N = 100	**=> 503 steps**
N = 1,000	**=> 5003 steps**
N = 1,000,000	**=> 5,000,003 steps**

As N grows, the number of steps grow in *linear* proportion to N for this function *"Sum"*

What Dominates in Previous Example?

What about the +3 and 5 in 5N+3?

- **As N gets large, the +3 becomes insignificant**
- **5 is inaccurate, as different operations require varying amounts of time and also does not have any significant importance**

What is fundamental is that the time is *linear* in N.
<u>Asymptotic Complexity</u>: As N gets large, concentrate on the highest order term:

- **Drop lower order terms such as +3**
- **Drop the constant coefficient of the highest order term i.e. N**

Asymptotic Complexity

- **The 5N+3 time bound is said to "grow asymptotically" like N**
- **This gives us an approximation of the complexity of the algorithm**
- **Ignores lots of (machine dependent) details, concentrate on the bigger picture**

Comparing Functions: Asymptotic Notation

- **Big Oh Notation: Upper bound**
- **Omega Notation: Lower bound**
- **Theta Notation: Tighter bound**

Big Oh Notation

If f(N) and g(N) are two complexity functions, we say

$$f(N) = O(g(N))$$

(read "f(N) as order g(N)", or "f(N) is big-O of g(N)")

if there are constants c and N_0 such that for N > N_0,

$$f(N) \leq c * g(N)$$

for all sufficiently large N.

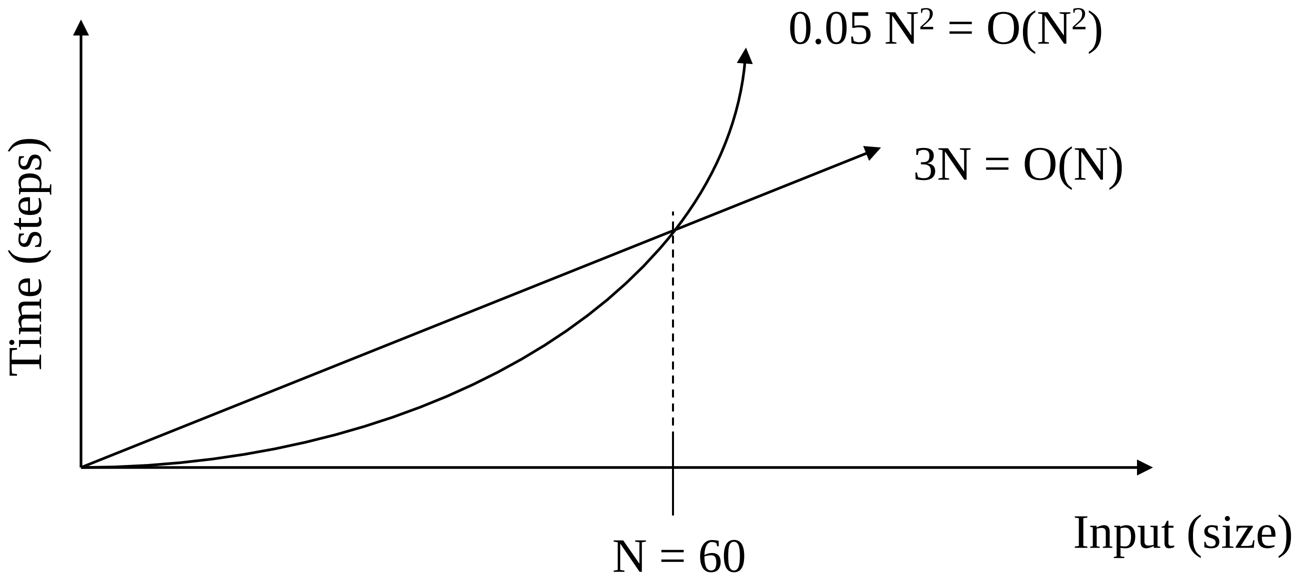
Polynomial and Intractable Algorithms

- **Polynomial Time complexity**
  - **An algorithm is said to be polynomial if it is $O(n^d)$ for some integer d**
  - **Polynomial algorithms are said to be efficient**
    - **They solve problems in reasonable times!**

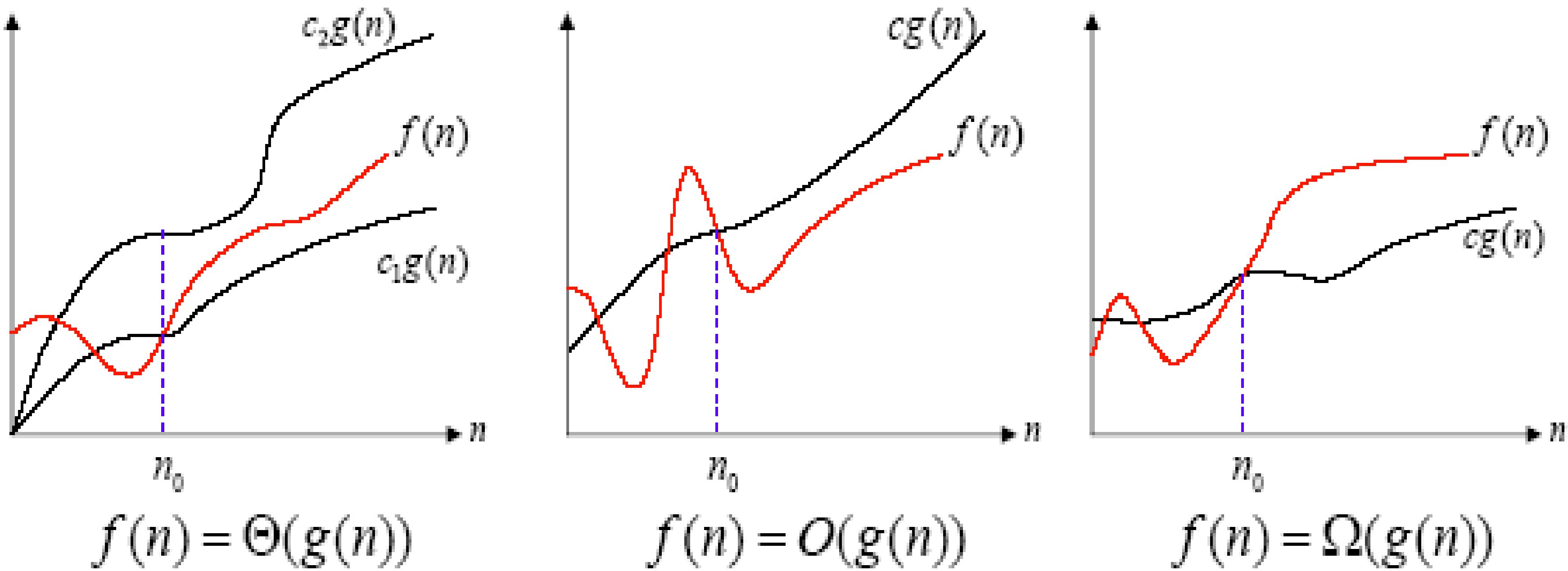
- **Intractable algorithms**
  - **Algorithms for which there is no *known* polynomial time algorithm**
  - ***We will come back to this important class later***

Comparing Functions

- **As inputs get larger, any algorithm of a smaller order will be more efficient than an algorithm of a larger order**



Asymptotic notation



- **Example:**
  - **$f(n) = 3n^5+n^4 = \Theta(n^5)$**

Performance Classification

f(n)	Classification
1	*Constant*: run time is fixed, and does not depend upon n. Most instructions are executed once, or only a few times, regardless of the amount of information being processed
`log` n	*Logarithmic*: when *n* increases, so does run time, but much slower. When *n* doubles, `log` *n* increases by a constant, but does not double until *n* increases to n^2. Common in programs which solve large problems by transforming them into smaller problems.
n	*Linear*: run time varies directly with *n*. Typically, a small amount of processing is done on each element.
n `log` n	When *n* doubles, run time slightly more than doubles. Common in programs which break a problem down into smaller sub-problems, solves them independently, then combines solutions
n^2	*Quadratic*: when *n* doubles, runtime increases fourfold. Practical only for small problems; typically the program processes all pairs of input (e.g. in a double nested loop).
n^3	*Cubic*: when n doubles, runtime increases eightfold
2^n	*Exponential*: when n doubles, run time squares. This is often the result of a natural, “brute force” solution.

Size does matter

What happens if we double the input size N?

N	$\log_2 N$	N	$N \log_2 N$	N^2	2^N
8	3	40	24	64	256
16	4	80	64	256	65536
32	5	160	160	1024	$\sim 10^9$
64	6	320	384	4096	$\sim 10^{19}$
128	7	640	896	16384	$\sim 10^{38}$
256	8	1280	2048	65536	$\sim 10^{76}$

Review of Three Common Sets

g(n) = O(f(n)) means $c \times f(n)$ is an *Upper Bound* on g(n)

g(n) = Ω(f(n)) means $c \times f(n)$ is a *Lower Bound* on g(n)

g(n) = Θ(f(n)) means $c_1 \times f(n)$ is an *Upper Bound* on g(n)
and $c_2 \times f(n)$ is a *Lower Bound* on g(n)

These bounds hold for all inputs beyond some threshold n_0.

Standard Analysis Techniques

- **Constant time statements**
- **Analyzing Loops**
- **Analyzing Nested Loops**
- **Analyzing Sequence of Statements**
- **Analyzing Conditional Statements**

Constant time statements

- Simplest case: O(1) time statements
- Assignment statements of simple data types

  int x = y;
- Arithmetic operations:

  x = 5 * y + 4 - z;
- Array referencing:

  A[j] = 5;
- Most conditional tests:

  if (x < 12) ...

Analyzing Loops

- **Any loop has two parts:**
  - **How many iterations are performed?**
  - **How many steps per iteration?**

```
int sum = 0,j;
for (j=0; j < N; j++)
  sum = sum +j;
```

  - **Loop executes N times (0..N-1)**
  - **O(1) steps per iteration**
- **Total time is N * O(1) = O(N*1) = O(N)**

Analyzing Loops

- **What about this for loop?**

```
int sum =0, j;
for (j=0; j < 100; j++)
   sum = sum + j;
```

- **Loop executes 100 times**
- **O(1) steps per iteration**
- **Total time is 100 * O(1) = O(100 * 1) = O(100) = O(1)**

Analyzing Nested Loops

- **Treat just like a single loop and evaluate each level of nesting as needed:**

```
int j,k;
for (j=0; j<N; j++)
   for (k=N; k>0; k--)
      sum += k+j;
```

- **Start with outer loop:**
  - **How many iterations? N**
  - **How much time per iteration? Need to evaluate inner loop**
- **Inner loop uses O(N) time**
- **Total time is N * O(N) = O(N*N) = O(N^2)**

Analyzing Nested Loops

- **What if the number of iterations of one loop depends on the counter of the other?**

```
int j,k;
for (j=0; j < N; j++)
   for (k=0; k < j; k++)
      sum += k+j;
```

- **Analyze inner and outer loop together:**
- **Number of iterations of the inner loop is:**
- $\mathbf{0 + 1 + 2 + ... + (N-1) = O(N^2)}$

Analyzing Sequence of Statements

- **For a sequence of statements, compute their complexity functions individually and add them up**

```
for (j=0; j < N; j++)
   for (k =0; k < j; k++)
      sum = sum + j*k;
```

$O(N^2)$

```
for (l=0; l < N; l++)
   sum = sum - l;
```

$O(N)$

```
cout<<"Sum="<<sum;
```

$O(1)$

Total cost is $O(N^2) + O(N) + O(1) = O(N^2)$

SUM RULE

Analyzing Conditional Statements

What about conditional statements such as

```
if (condition)
    statement1;
else
    statement2;
```

where statement1 runs in O(N) time and statement2 runs in O(N^2) time?

We use "worst case" complexity: among all inputs of size N, that is the maximum running time?

The analysis for the example above is O(N^2)

Properties of the O notation

- **Constant factors may be ignored**
  - $\forall\ k > 0$, kf is $O(f)$
- **Higher powers grow faster**
  - n^r is $O(n^s)$ if $0 \le r \le s$

← **Fastest growing term dominates a sum**

- If f is $O(g)$, then $f + g$ is $O(g)$

  $eg\ \ an^4 + bn^3$ is $O(n^4)$

← **Polynomial's growth rate is determined by leading term**

- If f is a polynomial of degree d,

  then f is $O(n^d)$

Properties of the *O* notation

- f is $O(g)$ is transitive
  - If f is $O(g)$ and g is $O(h)$ then f is $O(h)$
- Product of upper bounds is upper bound for the product
  - If f is $O(g)$ and h is $O(r)$ then fh is $O(gr)$
- Exponential functions grow faster than powers
  - n^k is $O(b^n)$ $\forall$ $b > 1$ and $k \geq 0$
    e.g. n^{20} is $O(1.05^n)$
- Logarithms grow more slowly than powers
  - $\log_b n$ is $O(n^k)$ $\forall$ $b > 1$ and $k > 0$
    e.g. $\log_2 n$ is $O(n^{0.5})$

Important!

Properties of the O notation

- All logarithms grow at the same rate
  - $\log_b n$ is $O(\log_d n)$ $\forall$ $b, d > 1$
- Sum of first n r^{th} powers grows as the $(r+1)^{th}$ power
  - $\sum_{k=1}^{n} k^r$ is $\Theta(n^{r+1})$

$$e.g. \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{is } \Theta(n^2)$$