MATHEMATICAL INDUCTION FOR
DIVISIBILITY PROBLEMS
INEQUALITY PROBLEMS

DIVISIBILITY:

Let n and d be integers and $d \neq 0$. Then n is divisible by d or d divides n written $d|n$. iff $n = d \cdot k$ for some integer k.

Alternatively, we say that
n is a multiple of d
d is a divisor of n
d is a factor of n

Thus $d|n \Leftrightarrow \exists$ an integer k such that $n = d \cdot k$

EXERCISE:

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

SOLUTION:

1. Basis Step:

For n = 1

$$n^3 - n = 1^3 - 1 = 1 - 1 = 0$$

which is clearly divisible by 3, since $0 = 0 \cdot 3$

Therefore, the given statement is true for n = 1.

2.Inductive Step:

Suppose that the statement is true for n = k, i.e., k^3-k is divisible by 3 for all $n \in Z+$

Then

$$k^3-k = 3 \cdot q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

for some $q \in Z$

We need to prove that $(k+1)^3 - (k+1)$ is divisible by 3.

Now

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1)\\ &= k^3 + 3k^2 + 2k\\ &= (k^3-k) + 3k^2 + 2k + k\\ &= (k^3 - k) + 3k^2 + 3k\\ &= 3 \cdot q + 3 \cdot (k^2 + k) \qquad \text{using(1)}\\ &= 3[q+k^2 + k]\end{aligned}$$

$\Rightarrow \quad (k+1)^3 - (k+1)$ is divisible by 3.

Hence by mathematical induction $n^3- n$ is divisible by 3, whenever n is a positive integer.

EXAMPLE:

Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n}-1$ is divisible by 3.

SOLUTION:

Let P(n): $2^{2n} -1$ is divisible by 3.

1.Basis Step:

P(1) is true
Now P(1): $2^{2(1)}- 1$ is divisible by 3.
Since $2^{2(1)}- 1= 4 - 1 = 3$
which is divisible by 3.
Hence P(1) is true.

2.Inductive Step:

Suppose that P(k) is true. That is $2^{2k}-1$ is divisible by 3. Then, there exists an integer q such that

$$2^{2k} - 1 = 3\cdot q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To prove P(k+1) is true, that is $2^{2(k+1)}- 1$ is divisible by 3.
Now consider

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 2^{2k}\, 2^2 - 1 \\ &= 2^{2k}4 - 1 \\ &= 2^{2k}(3+1) - 1 \\ &= 2^{2k}\cdot 3+(2^{2k} - 1) \\ &= 2^{2k}\cdot 3+3\cdot q \qquad \text{[by using (1)]} \\ &=3(2^{2k} + q) \end{aligned}$$

$\Rightarrow 2^{2(k+1)} - 1$ is divisible by 3.
Accordingly, by mathematical induction. $2^{2n}- 1$ is divisible by 3, for all integers $n \geq 1$.

<u>EXERCISE:</u>

Use mathematical induction to show that the product of any two consecutive positive integers is divisible by 2.

<u>SOLUTION:</u>

Let n and n + 1 be two consecutive integers. We need to prove that n(n+1) is divisible by 2.

1. Basis Step:

For n = 1
$n (n+1) = 1\cdot(1+1) = 1\cdot 2 = 2$
which is clearly divisible by 2.

2. Inductive Step:

Suppose the given statement is true for n = k. That is
k (k+1) is divisible by 2, for some $k \in Z+$
Then $k (k+1) = 2\cdot q$ ………………….(1) $q \in Z+$
We must show that

(k+1)(k+1+1) is divisible by 2.

Consider

$$\begin{aligned} (k+1)(k+1+1) &= (k+1)(k+2) \\ &= (k+1)k + (k+1)2 \\ &= 2q + 2 (k+1) \quad \text{using (1)} \\ &= 2(q+k+1) \end{aligned}$$

Hence (k+1) (k+1+1) is also divisible by 2.
Accordingly, by mathematical induction, the product of any two consecutive positive integers is divisible by 2.

EXERCISE:

Prove by mathematical induction $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

SOLUTION:

1.Basis Step:

For $n = 2$

$n^3 - n = 2^3 - 2 = 8 - 2 = 6$

which is clearly divisible by 6, since $6 = 1 \cdot 6$

Therefore, the given statement is true for $n = 2$.

2.Inductive Step:

Suppose that the statement is true for $n = k$, i.e., $k^3 - k$ is divisible by 6, for all integers $k \geq 2$.

Then

$$k^3 - k = 6 \cdot q \ldots\ldots\ldots\ldots\ldots(1) \quad \text{for some } q \in Z.$$

We need to prove that

$(k+1)^3 - (k+1)$ is divisible by 6

Now
$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^3 + 3k + 1) - (k+1) \\ &= k^3 + 3k^3 + 2k \\ &= (k^3 - k) + (3k^3 + 2k + k) \\ &= (k^3 - k) + 3k^3 + 3k \qquad \text{Using (1)} \\ &= 6 \cdot q + 3k\,(k+1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

Since k is an integer, so k(k+1) being the product of two consecutive integers is an even number.

Let $\quad k(k+1) = 2r \qquad r \in Z$

Now equation (2) can be rewritten as:

$$\begin{aligned}(k+1)^3 - (k+1) &= 6 \cdot q + 3 \cdot 2\,r \\ &= 6q + 6r \\ &= 6\,(q+r) \qquad q, r \in Z\end{aligned}$$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 6.

Hence, by mathematical induction, $n^3 - n$ is divisible by 6, for each integer $n \geq 2$.

EXERCISE:

Prove by mathematical induction. For any integer $n \geq 1$, $x^n - y^n$ is divisible by $x - y$, where x and y are any two integers with $x \neq y$.

SOLUTION:

1.Basis Step:

For $n = 1$

$x^n - y^n = x^1 - y^1 = x - y$

which is clearly divisible by $x - y$. So, the statement is true for $n = 1$.

2.Inductive Step:

Suppose the statement is true for $n = k$, i.e.,

$x^k - y^k$ is divisible by $x - y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

We need to prove that $x^{k+1} - y^{k+1}$ is divisible by $x - y$

Now

$$x^{k+1} - y^{k+1} = x^k \cdot x - y^k \cdot y$$

$= x^k \cdot x - x \cdot y^k + x \cdot y^k - y^k \cdot y$ (introducing $x.y^k$)

$= (x^k - y^k) \cdot x + y^k \cdot (x-y)$

The first term on R.H.S=$(x^k - y^k)$ is divisible by x - y by inductive hypothesis (1).

The second term contains a factor (x-y) so is also divisible by x - y.

Thus $x^{k+1} - y^{k+1}$ is divisible by x - y. Hence, by mathematical induction $x^n - y^n$ is divisible by x - y for any integer $n \geq 1$.

PROVING AN INEQUALITY:

Use mathematical induction to prove that for all integers $n \geq 3$.

$$2n + 1 < 2^n$$

SOLUTION:

1.Basis Step:

For n = 3

L.H.S= 2(3) + 1 = 6 + 1 = 7

R.H.S = 2^3 = 8

Since 7 < 8, so the statement is true for n = 3.

2.Inductive Step:

Suppose the statement is true for n =k, i.e.,

$2k + 1 < 2^k$…………………(1) $k \geq 3$

We need to show that the statement is true for n = k+1,

i.e.;

$2(k+1) + 1 < 2^{k+1}$…………..(2)

Consider L.H.S of (2)

$= 2(k+1) + 1$

$= 2k + 2 + 1$

$= (2k + 1) + 2$

$< 2^k + 2$ using (1)

$< 2^k + 2^k$ (since $2 < 2^k$ for $k \geq 3$)

$< 2 \cdot 2^k = 2^{k+1}$

Thus $2(k+1)+1 < 2k+1$ (proved)

EXERCISE:

Show by mathematical induction

$$1 + n x \leq (1+x)^n$$

for all real numbers x > - 1 and integers $n \geq 2$

SOLUTION:

1. Basis Step:

For n = 2

L.H.S = 1 + (2) x= 1 + 2x

RHS $= (1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ $(x^2 > 0)$

$\Rightarrow$ statement is true for n = 2.

2.Inductive Step:

Suppose the statement is true for n = k.

That is, for $k \geq 2$, $1 + k x \leq (1 + x)^k$……………….(1)

We want to show that the statement is also true for n = k + 1 i.e.,

$$1 + (k + 1)x \le (1 + x)^{k+1}$$

Since $x > -1$, therefore $1 + x > 0$.

Multiplying both sides of (1) by $(1+x)$ we get

$$\begin{aligned}(1+x)(1+x)^k &\ge (1 + x)(1 + kx)\\ &= 1 + kx + x + kx^2\\ &= 1 + (k + 1)x + kx^2\end{aligned}$$

but

$$\left[\begin{array}{l} x > -1, \quad \text{so } x^2 \ge 0 \\ \& k \ge 2, \quad \text{so } kx^2 \ge 0 \end{array}\right.$$

so

$$(1+x)(1+x)^k \ge 1 + (k + 1)x$$

Thus $1 + (k+1)x \le (1+x)^{k+1}$. Hence by mathematical induction, the inequality is true.

PROVING A PROPERTY OF A SEQUENCE:

Define a sequence $a_1, a_2, a_3, \ldots$ as follows:

$a_1 = 2$

$a_k = 5a_k - 1$ for all integers $k \ge 2$ **……………**(1)

Use mathematical induction to show that the terms of the sequence satisfy the formula.

$a_n = 2 \cdot 5^{n-1}$ for all integers $n \ge 1$

SOLUTION:

1.Basis Step:

For $n = 1$, the formula gives

$a_1 = 2\cdot5^{1-1} = 2\cdot5^0 = 2\cdot1 = 2$

which confirms the definition of the sequence. Hence, the formula is true for $n = 1$.

2.Inductive Step:

Suppose, that the formula is true for $n = k$, i.e.,

$a_k = 2\cdot5^{k-1}$ for some integer $k \ge 1$

We show that the statement is also true for $n = k + 1$. i.e.,

$a_{k+1} = 2\cdot5^{k+1-1} = 2\cdot5^{k}$

Now

$$\begin{aligned} a_{k+1} &= 5\cdot a_{k+1-1} && \text{[by definition of } a_1, a_2, a_3 \ldots \text{ or by putting k+1 in (1)]}\\ &= 5\cdot a_k \\ &= 5\cdot(2\cdot5^{k-1}) && \text{by inductive hypothesis}\\ &= 2\cdot(5\cdot5^{k-1})\\ &= 2\cdot5^{k+1-1}\\ &= 2\cdot5^k \end{aligned}$$

which was required.

EXERCISE:

A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$ for all integers $k \ge 2$. Show that $d_n = \dfrac{2}{n!}$ for all integers $n \ge 1$, using mathematical induction.

SOLUTION:

1.Basis Step:

For n = 1, the formula $d_n = \frac{2}{n!}$; n ≥1 gives

$$d_1 = \frac{2}{1!} = \frac{2}{1} = 2$$

which agrees with the definition of the sequence.

2.Inductive Step:

Suppose, the formula is true for n=k. i.e.,

$d_k = \frac{2}{k!}$ for some integer k ≥ 1…………..(1)

We must show that

$$d_{k+1} = \frac{2}{(k+1)!}$$

Now, by the definition of the sequence.

$$d_{k+1} = \frac{d_{(k+1)-1}}{(k+1)} = \frac{1}{(k+1)} d_k \qquad \text{using } d_k = \frac{d_{k-1}}{k}$$

$$= \frac{1}{(k+1)} \frac{2}{k!} \qquad \text{using (1)}$$

$$= \frac{2}{(k+1)!}$$

Hence the formula is also true for n = k + 1. Accordingly, the given formula defines all the terms of the sequence recursively.

EXERCISE:

Prove by mathematical induction that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Whenever n is a positive integer greater than 1.

SOLUTION:

1. Basis Step: for n = 2

L.H.S $= 1 + \frac{1}{4} = \frac{5}{4} = 1.25$

R.H.S $= 2 - \frac{1}{2} = \frac{3}{2} = 1.5$

Clearly LHS < RHS

Hence the statement is true for n = 2.

2.Inductive Step:

Suppose that the statement is true for some integers k > 1, i.e.;

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}<2-\frac{1}{k} \qquad (1)$$

We need to show that the statement is true for n = k + 1. That is

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}<2-\frac{1}{k+1} \qquad (2)$$

Consider the LHS of (2)

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(k+1)^2}=1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$<\left(2-\frac{1}{k}\right)+\frac{1}{\left(k+1\right)^2}$$

$$=2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)$$

We need to prove that

$$2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq 2-\frac{1}{k+1}$$

or $$-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)\leq -\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{(k+1)^2}\geq\frac{1}{k+1}$$

or $$\frac{1}{k}-\frac{1}{k+1}\geq\frac{1}{(k+1)^2}$$

Now $$\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{k(k+1)}$$

$$=\frac{1}{k(k+1)}>\frac{1}{(k+1)^2}$$