Non Exact Diff Eq

A diff eq of the form $Mdx + Ndy = 0$ is said to be non-exact iff $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now if this diff eq is multiplied by a suitable function, then the resulting eq is Exact Diff^l Eq. This suitable fn is called Integrating Factor (I.F)

Note The number of Integrating Factors may be infinite.

Some Rules to Find Integrating Factors

If $Mdx + Ndy = 0$ is not exact then find Integrating Factor using

1) $\frac{M_y - N_x}{N} = P$ then $I.F = e^{\int P dx}$ where P is fn of x alone.

2) $\frac{N_x - M_y}{M} = Q$ then $I.F = e^{\int Q dy}$ where Q is fn of y alone

3) If $Mdx + Ndy = 0$ is Homogeneous then $I.F = \frac{1}{xM + yN}$ where $xM + yN \neq 0$

4) If diff eq is the form $y.f(xy)dx + x.g(xy)dy = 0$ then $I.F = \frac{1}{xM - yN}$ where $xM - yN \neq 0$

Note In some cases I.F can be found only after properly regrouping the terms of a diff eq and then recognising each group as an Exact differential of known function.

1) $xdy + ydx = d(xy)$

2) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

3) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

4) $xdx + ydy = d\left(\frac{x^2 + y^2}{2}\right)$

5) $\frac{xdy + ydx}{xy} = d(\log(xy))$

6) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

7) $\frac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

8) $\frac{xdy + ydx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$

(42)

Ex 9.5

Solve by finding an I.F

① $(xy^2+y)dx - x\,dy = 0$ — ①

$M = xy^2+y \qquad N = -x$

$M_y = 2xy+1 \qquad N_x = -1$

$\because M_y \neq N_x \quad \therefore$ Non Exact

$\dfrac{M_y - N_x}{N} = \dfrac{2xy+1+1}{-x}$ Not fn of x alone.

$\dfrac{N_x - M_y}{M} = \dfrac{-1-2xy-1}{xy^2+y}$

$= -2\dfrac{(1+xy)}{y(xy+1)} = -\dfrac{2}{y}$ Fn of 'y' alone

$\therefore I.F = e^{-\int \frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \dfrac{1}{y^2}$

Multiply both sides of eq① by $IF = \dfrac{1}{y^2}$

$\dfrac{1}{y^2}(xy^2+y)dx - \dfrac{x}{y^2}dy = 0$

$\left(x+\dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ — ⑪

Now $M = x+\dfrac{1}{y} \qquad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2} \qquad N_x = -\dfrac{1}{y^2}$

$\because M_y = N_x \quad \therefore$ ⑪ is Exact Diff Eq

So $\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int\left(x+\dfrac{1}{y}\right)dx + \text{Nil} = C$

$\dfrac{x^2}{2} + \dfrac{x}{y} = C$

③ $(x^2+x-y)dx + x\,dy = 0$ — ①

$M = x^2+x-y \qquad N = x$

$M_y = -1 \qquad N_x = 1$

$M_y \neq N_x \quad \therefore$ Non Exact Diff Eq

$\dfrac{M_y - N_x}{N} = \dfrac{-1-1}{x} = -\dfrac{2}{x}$ fn of x alone.

$\therefore I.F = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \dfrac{1}{x^2}$

Multiply both sides of eq① by $I.F = \dfrac{1}{x^2}$

$\dfrac{1}{x^2}(x^2+x-y)dx + \dfrac{x}{x^2}dy = 0$

$\left(1+\dfrac{1}{x}-\dfrac{y}{x^2}\right)dx + \dfrac{1}{x}dy = 0$ — ⑪

Now $M = 1+\dfrac{1}{x}-\dfrac{y}{x^2} \qquad N = \dfrac{1}{x}$

$M_y = -\dfrac{1}{x^2} \qquad N_x = -\dfrac{1}{x^2}$

$M_y = N_x \quad \therefore$ ⑪ is Exact Diff Eq.

So $\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int\left(1+\dfrac{1}{x}-\dfrac{y}{x^2}\right)dx + \text{Nil} = C$

$x + \ln x + \dfrac{y}{x} = C$

9 - مشق نمبر

(4) $dy + \left(\dfrac{y-\sin x}{x}\right)dx = 0$ ——(i)

$M = \dfrac{y-\sin x}{x}$ $\quad N = 1$

$M_y = \dfrac{1}{x} - 0$ $\quad N_x = 0$

$M_y \neq N_x \therefore$ (i) is Non Exact Diff Eq

Now $\dfrac{M_y - N_x}{N} = \dfrac{\frac{1}{x} - 0}{1} = \dfrac{1}{x}$ fn of x alone

$I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x}$

Multiplying both sides of eq (i) by I.F $= x$

$x\,dy + \cancel{x}\left(\dfrac{y-\sin x}{\cancel{x}}\right)dx = 0$ ——(ii)

$M = y - \sin x$ $\quad N = x$

$M_y = 1$ $\quad N_x = 1$

$M_y = N_x \quad \therefore$ (ii) is Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int (y - \sin x)\,dx = c$

$xy + \cos x = c$

(5) $y(2xy + e^x)dx - e^x dy = 0$

$(2xy^2 + e^x y)dx - e^x dy = 0$ ——(i)

$M = 2xy^2 + e^x y$ $\quad N = -e^x$

$M_y = 4xy + e^x$ $\quad N_x = -e^x$

$M_y \neq N_x \quad \therefore$ (i) is Non Exact Diff Eq.

$\dfrac{M_y - N_x}{N} = \dfrac{4xy + e^x + e^x}{-e^x}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{-e^x - 4xy - e^x}{2xy^2 + ye^x} = \dfrac{-2e^x - 4xy}{y(2xy + e^x)}$

$= \dfrac{-2\cancel{(e^x + 2xy)}}{y\cancel{(2xy + e^x)}} = \dfrac{-2}{y}$ fn of y alone

$I.F = e^{\int -\frac{2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = \boxed{\dfrac{1}{y^2}}$

Multiply both sides of (i) by I.F $= \dfrac{1}{y^2}$

$\dfrac{1}{y^2}(2xy^2 + e^x y)dx - \dfrac{1}{y^2}e^x dy = 0$

$\left(2x + \dfrac{e^x}{y}\right)dx - \dfrac{e^x}{y^2}dy = 0$ ——(ii)

$M = 2x + \dfrac{e^x}{y}$ $\quad N = -\dfrac{e^x}{y^2}$

$M_y = 0 + \left(-\dfrac{e^x}{y^2}\right)$ $\quad N_x = -\dfrac{e^x}{y^2}$

$M_y = N_x \quad \therefore$ (ii) is Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int \left(2x + \dfrac{e^x}{y}\right)dx + \text{Nil} = c$

$x^2 + \dfrac{e^x}{y} = c$ Ans.

(6) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ —(i)

$M = y^4 + 2y$ $\quad N = xy^3 + 2y^4 - 4x$

$M_y = 4y^3 + 2$ $\quad N_x = y^3 - 4$

$M_y \neq N_x \quad \therefore$ (i) is Non Exact Diff Eq.

$\dfrac{N_x - M_y}{M} = \dfrac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3y^3 - 6}{y(y^3 + 2)} = -\dfrac{3}{y}$

$I.F = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3} = \boxed{\dfrac{1}{y^3}}$

$\times$ by $\frac{1}{y^3}$

$\dfrac{1}{y^3}(y^4 + 2y)dx + \dfrac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$

$\left(y + \dfrac{2}{y^2}\right)dx + \left(x + 2y - \dfrac{4x}{y^3}\right)dy = 0$ (ii)

Now $M = y + \dfrac{2}{y^2}$ $\quad N = x + 2y - \dfrac{4x}{y^3}$

$M_y = 1 - \dfrac{4}{y^3}$ $\quad N_x = 1 + 0 - \dfrac{4}{y^3}$

$\because M_y = N_x \quad \therefore$ (ii) is Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)\,dy = c$

$\int\left(y + \dfrac{2}{y^2}\right)dx + \int 2y\,dy = c$

$xy + \dfrac{2x}{y^2} + \dfrac{2y^2}{2} = c$

$xy + \dfrac{2x}{y^2} + y^2 = c$

(44)

⑦ $(x^2+y^2+2x)dx + 2y\,dy = 0$ —①

$M = x^2+y^2+2x$ $\quad N = 2y$

$M_y = 2y$ $\quad N_x = 0$

$M_y \neq N_x$ ∴ ① is Non Exact Diff Eq

$\frac{N_x - M_y}{M} = \frac{0-2y}{x^2+y^2+2x}$ Not fn of y only

$\frac{M_y - N_x}{N} = \frac{2y-0}{2y} = 1 = x^0$ fn of x only

$I.F = e^{\int 1.dx} = e^x$

Multiply both sides of eq ① by $I.F = \boxed{e^x}$

$e^x(x^2+y^2+2x)dx + e^x(2y)dy = 0$ —⑪

$M = e^x(x^2+y^2+2x)$ $\quad N = e^x 2y$

$M_y = e^x 2y$ $\quad N_x = e^x 2y$

$M_y = N_x$ ∴ ⑪ is Exact Diff Eq.

∴ $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int e^x(x^2+y^2+2x)dx + \text{Nil} = c$

$\int e^x x^2 dx + \int e^x y^2 dx + \int e^x 2x\,dx = c$

$x^2 e^x - \cancel{\int 2x e^x dx} + e^x y^2 + \cancel{\int e^x 2x\,dx} = c$

$(x^2+y^2)e^x = c$

⑩ $(4x+3y^2)dx + 2xy\,dy = 0$ —①

$M = 4x+3y^2$ $\quad N = 2xy$

$M_y = 0+6y$ $\quad N_x = 2y$

$M_y \neq N_x$ ∴ ① is Non Exact Diff Eq

$\frac{N_x - M_y}{M} = \frac{2y-6y}{4x+3y^2}$ not fn of y alone

$\frac{M_y - N_x}{N} = \frac{6y-2y}{2xy} = \frac{4y}{2xy} = \frac{2}{x}$ fn of x alone.

(i.e. Easy 2nd Method on Page 41)

⑧ $(x^2+y^2)dx - 2xy\,dy = 0$ —①

$M = x^2+y^2$ $\quad N = -2xy$

$M_y = 2y$ $\quad N_x = -2y$

$M_y \neq N_x$ ∴ ① is Non Exact Diff Eq

$\frac{N_x - M_y}{M} = \frac{-2y-2y}{x^2+y^2}$ Not fn of y only

$\frac{M_y - N_x}{N} = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x}$ fn of x only

$I.F = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \boxed{\frac{1}{x^2}}$

Multiply both sides of eq ① by $I.F = \frac{1}{x^2}$

$\frac{1}{x^2}(x^2+y^2)dx - \frac{1}{x^2}(2xy)dy = 0$

$(1+\frac{y^2}{x^2})dx - \frac{2y}{x}dy = 0$ —⑪

$M = 1+\frac{y^2}{x^2}$ $\quad N = -\frac{2y}{x}$

$M_y = \frac{2y}{x^2}$ $\quad N_x = +\frac{2y}{x^2}$

$M_y = N_x$ ∴ ⑪ is Exact Diff Eq.

∴ $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(1+\frac{y^2}{x^2})dx + \text{Nil} = c$

$x - \frac{y^2}{x} = c$ Ans.

Note ⑧ can be done by $I.F = \frac{1}{xM+yN}$ i.e. Homogeneous Method.

∴ $I.F = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply both sides of ① by $I.F = \boxed{x^2}$

$(4x^3+3y^2x^2)dx + (2x^3y)dy = 0$ —⑪

$M_y = 6yx^2$ $\quad N_x = 6x^2y$

$M_y = N_x$ ∴ ⑪ is Exact Diff Eq.

∴ $\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(4x^3+3y^2x^2)dx + \text{Nil} = c$

$\cancel{4}\frac{x^4}{\cancel{4}} + \cancel{3}y^2\frac{x^3}{\cancel{3}} = C$

$x^4 + y^2x^3 = C$ Ans

45

(12) $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$ —(i)

$M=3x^2y^4+2xy \qquad N=2x^3y^3-x^2$

$M_y=12x^2y^3+2x \qquad N_x=6x^2y^3-2x$

$M_y\neq N_x \quad \therefore$ (i) is Non Exact Diff Eq

$$\frac{M_y-N_x}{N}=\frac{12x^2y^3+\cancel{2x}-6x^2y^3-\cancel{2x}}{2x^3y^3-x^2}=\frac{6\cancel{x^2}y^3}{\cancel{x^2}(2xy^3-1)}$$

$$\frac{N_x-M_y}{M}=\frac{6x^2y^3-2x-12x^2y^3-2x}{3x^2y^4+2xy}=\frac{-6x^2y^3-4x}{xy(3xy^3+2)}$$

$$=\frac{-2x\cancel{(3xy^3+2)}}{xy\cancel{(3xy^3+2)}}=-\frac{2}{y} \text{ fn of } y \text{ alone.}$$

$$\text{So I.F}=e^{\int-\frac{2}{y}dy}=e^{-2\ln y}=e^{\ln y^{-2}}=y^{-2}=\boxed{\frac{1}{y^2}}$$

Multiply both sides of eq (i) by $I.F=\frac{1}{y^2}$

$$\frac{1}{y^2}(3x^2y^4+2xy)dx+\frac{1}{y^2}(2x^3y^3-x^2)dy=0$$

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0 \text{ —(ii)}$$

$M=3x^2y^2+\frac{2x}{y} \qquad N=2x^3y-\frac{x^2}{y^2}$

$M_y=6x^2y-\frac{2x}{y^2} \qquad N_x=6x^2y-\frac{2x}{y^2}$

$M_y=N_x \quad \therefore$ (ii) is Exact Diff Eq.

$\therefore \int Mdx+\int(\text{terms of } N \text{ free from } x)dy=C$

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx+\text{Nil}=C$$

$$\cancel{3}\frac{x^3}{\cancel{3}}y^2+\cancel{2}\frac{x^2}{\cancel{2}y}=C$$

$$x^3y^2+\frac{x^2}{y}=C \quad \text{Ans}$$

(14) $\frac{dy}{dx}=e^{2x}+y-1$

$dy=(e^{2x}+y-1)dx$

$(e^{2x}+y-1)dx-dy=0$ —(i)

$M=e^{2x}+y-1 \qquad N=-1$

$M_y=1 \qquad N_x=0$

$M_y\neq N_x \quad \therefore$ (i) is Non Exact Diff Eq

$\frac{N_x-M_y}{M}=\frac{0-1}{e^{2x}+y-1}$ Not fn of y alone

$\frac{M_y-N_x}{N}=\frac{1-0}{-1}=-1=-x^0$ fn of x alone.

$$\therefore I.F=e^{\int-1dx}=\boxed{e^{-x}}$$

Multiply both sides of eq (i) by e^{-x}

$$e^{-x}(e^{2x}+y-1)dx-e^{-x}dy=0$$

$$(e^x+e^{-x}y-e^{-x})dx-e^{-x}dy=0 \text{ —(ii)}$$

$M=e^x+e^{-x}y-e^{-x} \qquad N=-e^{-x}$

$M_y=e^{-x} \qquad N_x=+e^{-x}$

$M_y=N_x \therefore$ (ii) is Exact Diff Eq

So $\int Mdx+\int(\text{terms of } N \text{ free from } x)dy=c$

$$\int(e^x+e^{-x}y-e^{-x})dx+\text{Nil}=c$$

$$e^x-e^{-x}y+e^{-x}=c$$

2nd Method of Q15 See on Page 46 & 3rd Method on 49

Q15 $(y^2+xy)dx-x^2dy=0$

$\frac{dy}{dx}=\frac{y^2+xy}{x^2}$ HDE —(i)

Put $y=vx$ —(ii)

$\frac{dy}{dx}=v+x\frac{dv}{dx}$ —(iii)

$$v+x\frac{dv}{dx}=\frac{v^2x^2+xvx}{x^2}$$

$$v+x\frac{dv}{dx}=\frac{\cancel{x^2}(v^2+v)}{\cancel{x^2}}$$

$$x\frac{dv}{dx}=v^2+\cancel{v}-\cancel{v}$$

$$\frac{dv}{v^2}=\frac{dx}{x} \quad \text{separable Eqs}$$

$$\int v^{-2}dv=\int\frac{dx}{x}$$

$$-\frac{1}{v}=\ln x+c$$

$$0=\ln x+\frac{1}{v}+c$$

$$=\ln x+\frac{x}{y}+c \quad \text{Ans}$$

$\because y=vx$, $\frac{y}{x}=v$

(46)

Easy Method ...

(15) $(y^2+xy)dx - x^2 dy = 0$ —— (1)

$M = y^2+xy \qquad N = -x^2$

$M_y = 2y+x \qquad N_x = -2x$

$M_y \neq N_x \quad \therefore$ (1) is Non Exact Diff Eq

$\dfrac{M_y - N_x}{N} = \dfrac{2y+x+2x}{-x^2}$ Not fun of x only

$\dfrac{N_x - M_y}{M} = \dfrac{-2x-2y-x}{y^2+xy}$ Not fun of y only

(1) is Homogeneous diff eq of degree 2.

$\therefore xM + yN = xy^2 + \cancel{x^2y} + (y)(-x^2)$

$I.F = \dfrac{1}{xM+yN} = \boxed{\dfrac{1}{xy^2}}$

Multiply both sides of (1) by $I.F = \dfrac{1}{xy^2}$

$\dfrac{1}{xy^2}(y^2+xy)dx - \dfrac{1}{xy^2}x^2 dy = 0$

$\left(\dfrac{1}{x}+\dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ —— (11)

$M = \dfrac{1}{x}+\dfrac{1}{y} \qquad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2} \qquad N_x = -\dfrac{1}{y^2}$

$M_y = N_x \quad \therefore$ (11) is Exact Diff Eq

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{1}{x}+\dfrac{1}{y}\right)dx + \text{Nil} = c$

$\ln x + \dfrac{x}{y} = c$ Ans.

(16) $(3xy+y^2)dx + (x^2+xy)dy = 0$ —— (1)

$M = 3xy+y^2 \qquad N = x^2+xy$

$M_y = 3x+2y \qquad N_x = 2x+y$

$M_y \neq N_x \quad \therefore$ (1) is Non Exact Diff Eq

$\dfrac{N_x - M_y}{M} = \dfrac{2x+y-3x+2y}{3xy+y^2} = \dfrac{-x+3y}{y(3x+y)}$ Not fun of y only

$\dfrac{M_y - N_x}{N} = \dfrac{3x+2y-2x-y}{x^2+xy} = \dfrac{(x+y)}{x(x+y)} = \dfrac{1}{x}$ fun of x only

$I.F = e^{\int\frac{1}{x}dx} = e^{\ln x} = \boxed{x}$

Multiply both sides of eq (1) by $I.F = x$

$(3x^2y+xy^2)dx + (x^3+x^2y)dy = 0$ —— (11)

$M = 3x^2y+xy^2 \qquad N = x^3+x^2y$

$M_y = 3x^2+2xy \qquad N_x = 3x^2+2xy$

$M_y = N_x \quad \therefore$ (11) is Exact Diff Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(3x^2y+xy^2)dx + \text{Nil} = c$

$3\dfrac{x^3}{3}y + \dfrac{x^2y^2}{2} = c$

$x^3y + \dfrac{x^2y^2}{2} = c$ Ans.

Q(23) $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

$y - x = \dfrac{dy}{dx}(x+y)$

$\dfrac{dy}{dx} = \dfrac{y-x}{x+y}$ H.D.Eq —— (1)

$y = vx$ —— (11)

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ —— (111)

$v + x\dfrac{dv}{dx} = \dfrac{vx-x}{x+vx}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{x}(v-1)}{\cancel{x}(1+v)} - v$

$x\dfrac{dv}{dx} = \dfrac{v-1-v(1+v)}{1+v}$

$x\dfrac{dv}{dx} = \dfrac{\cancel{v}-1-\cancel{v}-v^2}{1+v}$

$\int\dfrac{(1+v)}{1+v^2}dv = -\int\dfrac{dx}{x}$

$\int\dfrac{dv}{1+v^2} + \int\dfrac{v\,dv}{1+v^2} = -\int\dfrac{dx}{x}$

$\tan^{-1}v + \dfrac{1}{2}\int\dfrac{2v\,dv}{1+v^2} = -\ln x$

$\tan^{-1}v + \dfrac{1}{2}\ln(1+v^2) + \ln x = c$

$\tan^{-1}v + \ln(1+v^2)^{\frac{1}{2}} + \ln x = c$

$\tan^{-1}\dfrac{y}{x} + \ln\left(1+\dfrac{y^2}{x^2}\right)^{\frac{1}{2}} + \ln x = c$

$\tan^{-1}\dfrac{y}{x} + \ln\dfrac{\sqrt{x^2+y^2}}{\cancel{x}}(\cancel{x}) = c$

$\tan^{-1}\dfrac{y}{x} + \ln\sqrt{x^2+y^2} = c$ Ans

(47)

(17) $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$ —(i)

$M=3x^2y+2xy+y^3 \qquad N=x^2+y^2$

$M_y=3x^2+2x+3y^2 \qquad N_x=2x$

$M_y\neq N_x \quad \therefore$ (i) is Non Exact Diff Eq.

$\dfrac{M_y-N_x}{N}=\dfrac{3x^2+\cancel{2x}+3y^2-\cancel{2x}}{x^2+y^2}$

$=\dfrac{3(x^2+y^2)}{(x^2+y^2)}=3=3x^0$ fn of x only.

$I.F=e^{\int 3\,dx}=\boxed{e^{3x}}$

Multiply both sides of eq (i) by $I.F=e^{3x}$

$(3x^2ye^{3x}+2xye^{3x}+y^3e^{3x})dx+(e^{3x}x^2+e^{3x}y^2)dy=0$ —(ii)

$M=3x^2ye^{3x}+2xye^{3x}+y^3e^{3x} \qquad N=e^{3x}x^2+e^{3x}y^2$

$M_y=3x^2e^{3x}+2xe^{3x}+3y^2e^{3x} \qquad N_x=3e^{3x}x^2+e^{3x}2x+e^{3x}3y^2$

$M_y=N_x \quad \therefore$ (ii) is Exact Diff Eq.

$\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\int(3x^2ye^{3x}+2xye^{3x}+y^3e^{3x})dx+\text{Nil}=c$

$3y\int x^2e^{3x}dx+2y\int xe^{3x}dx+y^3\int e^{3x}dx=c$

$3y\left[x^2\dfrac{e^{3x}}{3}-\int 2x\dfrac{e^{3x}}{3}dx\right]+2y\int xe^{3x}+y^3\dfrac{e^{3x}}{3}=c$

$\Rightarrow x^2ye^{3x}-2y\int \cancel{xe^{3x}dx}+2y\int\cancel{xe^{3x}}+y^3\dfrac{e^{3x}}{3}=c$

$\Rightarrow x^2ye^{3x}+y^3\dfrac{e^{3x}}{3}=c$

(18) $y\,dx+(2xy-e^{-2y})dy=0$ —(i)

$M=y \qquad N=2xy-e^{-2y}$

$M_y=1 \qquad N_x=2y$

$M_y\neq N_x$ (i) is Non Exact Diff Eq

$\dfrac{M_y-N_x}{N}=\dfrac{1-2y}{2xy-e^{-2y}}$ Not fn of x alone

$\dfrac{N_x-M_y}{M}=\dfrac{2y-1}{y}=2-\dfrac{1}{y}$ fn of y alone

$I.F=e^{\int(2-\frac{1}{y})dy}=e^{2y-\ln y}$

$=e^{2y+\ln y^{-1}}=e^{2y}e^{\ln(\frac{1}{y})}=\boxed{e^{2y}\cdot\frac{1}{y}}$

Multiply (i) by $I.F=e^{2y}\cdot\frac{1}{y}$

$e^{2y}\frac{1}{y}\,y\,dx+e^{2y}\cdot\frac{1}{y}(2xy-e^{-2y})dy=0$

$e^{2y}dx+(e^{2y}2x-\frac{1}{y})dy=0$ —(ii)

$M=e^{2y} \qquad N=e^{2y}2x-\frac{1}{y}$

$M_y=2e^{2y} \qquad N_x=2e^{2y}-0$

$M_y=N_x \quad \therefore$ (ii) is Exact Diff Eq

$\therefore\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\Rightarrow\int e^{2y}dx+\int-\frac{1}{y}dy=c$

$\Rightarrow xe^{2y}-\ln y=c$

(8) $(x^2-y^2)dx-2xy\,dy=0$

$\dfrac{dy}{dx}=\dfrac{x^2+y^2}{2xy}$ —(i)

Put $y=vx$ —(ii)

$\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ —(iii)

Put (ii) & (iii) in (i) $\quad v+x\dfrac{dv}{dx}=\dfrac{x^2+v^2x^2}{2x\,vx}$

$v+x\dfrac{dv}{dx}=\dfrac{x^2(1+v^2)}{x^2\,2v}$

$\therefore x\dfrac{dv}{dx}=\dfrac{1+v^2}{2v}-v$

$=\dfrac{1+v^2-2v^2}{2v}$

$x\dfrac{dv}{dx}=\dfrac{1-v^2}{2v}$

$\int\dfrac{2v\,dv}{1-v^2}=\int\dfrac{dx}{x}$

x by (-1) $\quad -\int\dfrac{-2v}{1-v^2}dv=-\int\dfrac{dx}{x}$

$\ln(1-v^2)=-\ln x+\ln c$

$\ln(1-v^2)=\ln(\frac{c}{x})$

$1-v^2=\frac{c}{x}$

$1-\frac{y^2}{x^2}=\frac{c}{x}$

$\dfrac{x^2-y^2}{x^2}=\dfrac{c}{x}$

$x^2-y^2=\frac{c}{x}x^2$

$x^2-y^2=cx$

(48)

(19) $e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)dy = 0$ — (I)

$M = e^x \qquad N = e^x \cot y + 2y \operatorname{cosec} y$

$M_y = 0 \qquad N_x = e^x \cot y$

$M_y \neq N_x \therefore$ (I) is Non Exact Diff Eq

$\dfrac{M_y - N_x}{N} = \dfrac{0 - e^x \cot y}{e^x \cot y + 2y \operatorname{cosec} y}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{e^x \cot y - 0}{e^x} = \cot y$ fn of y alone

$\therefore I.F = e^{\int \cot y\, dy} = e^{\ln \sin y} = \boxed{\sin y}$

Multiply both sides of (I) by $I.F = \sin y$

$\sin y\, e^x dx + (\sin y\, e^x \cot y + 2y \sin y \operatorname{cosec} y)dy = 0$

$\sin y\, e^x dx + (e^x \cos y + 2y)dy = 0$ (II)

$M = \sin y\, e^x \qquad N = e^x \cos y + 2y$

$M_y = \cos y\, e^x \qquad N_x = e^x \cos y + 0$

$M_y = N_x \therefore$ (II) is Exact Diff Eq

$\therefore \int M dx + \int (\text{terms of } N \text{ free from } x)\, dy = c$

$\int e^x \sin y\, dx + \int 2y\, dy = c$

$e^x \sin y + \dfrac{2y^2}{2} = c$

$e^x \sin y + y^2 = c$

(20) $(x+2)\sin y\, dx + x\cos y\, dy = 0$ (I)

$M = (x+2)\sin y \qquad N = x\cos y$

$M_y = (x+2)\cos y \qquad N_x = \cos y$

$M_y \neq N_x \therefore$ (I) is Non Exact

$\dfrac{N_x - M_y}{M} = \dfrac{\cos y - (x+2)\cos y}{(x+2)\sin y}$ Not fn of y

$\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x\cos y}$

$= \dfrac{((x+2) - 1)\cancel{\cos y}}{x \cancel{\cos y}} = \dfrac{x+1}{x}$

$\dfrac{M_y - N_x}{N} = 1 + \dfrac{1}{x}$ fn of x alone

$I.F = e^{\int (1+\frac{1}{x})dx} = e^{x + \ln x} = e^x e^{\ln x}$

$= \boxed{e^x x}$

Multiply by xe^x on both sides of (I)

$xe^x (x+2)\sin y\, dx + xe^x x\cos y\, dy = 0$ (II)

$M = xe^x (x+2)\sin y \quad , N = x^2 e^x \cos y$

$M = (x^2 e^x + 2xe^x)\sin y, N_x = (2xe^x + x^2 e^x)\cos y$

$M_y = (x^2 e^x + 2xe^x)\cos y$

$M_y = N_x \therefore$ (II) is Exact Diff Eq.

$\therefore \int M dx + \int (\text{terms of } N \text{ free from } x)\, dy = c$

$\int (\underset{I}{x^2} \underset{II}{e^x} \sin y + 2xe^x \sin y)dx + \text{Nil} = c$

$\int \underset{I}{x^2} \underset{II}{e^x} \sin y\, dx + \int 2xe^x \sin y\, dx = c$

$x^2 e^x \sin y - \cancel{\int 2xe^x \sin y\, dx} + \cancel{\int 2xe^x \sin y\, dx} = c$

$x^2 e^x \sin y = c$

Easy Method on Page 46.

3) $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$

$x - y + (x+y)\frac{dy}{dx} = 0$

$(x-y)dx + (x+y)dy = 0$ ——— (I)

$M = x - y \qquad N = x+y$

$M_y = 0-1 \qquad N_x = 1+0$

$\frac{M_y - N_x}{N} = \frac{-1-1}{x+y}$ Not fn of x alone

$\frac{N_x - M_y}{M} = \frac{1+1}{x-y}$ Not fn of y alone

$\because$ (I) is Homogeneous So $I.F = \frac{1}{xM+yN}$

$I.F = \frac{1}{x(x-y)+y(x+y)} = \frac{1}{x^2 - \cancel{xy} + \cancel{xy} + y^2}$

$= \frac{1}{x^2+y^2}$

Multiply (I) by $I.F = \frac{1}{x^2+y^2}$

$\frac{(x-y)}{x^2+y^2}dx + \frac{(x+y)}{x^2+y^2}dy$ ——— (II)

$M_y = \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}$, $N_x = \frac{(x^2+y^2).1-(x+y)2x}{(x^2+y^2)^2}$

$M_y = \frac{-[x^2+y^2-2xy+2y^2]}{(x^2+y^2)^2}$, $N_x = \frac{x^2+y^2-2x^2-2xy}{(x^2+y^2)^2}$

$M_y = \frac{+y^2-x^2-2xy}{(x^2+y^2)^2} \qquad N_x = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$

$M_y = N_x \therefore$ (II) is Exact Diffl Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\frac{x-y}{x^2+y^2}\right)dx + Nil = c$

$\int\frac{x\,dx}{x^2+y^2} - \int\frac{y\,dx}{x^2+y^2} = c$

$\frac{1}{2}\int\frac{2x\,dx}{x^2+y^2} - \int\frac{d}{dx}\tan^{-1}\frac{x}{y} = c$ $\left[\begin{array}{l}\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2} \\ \therefore \int\frac{a}{a^2+x^2}dx = \tan^{-1}\frac{x}{a}\end{array}\right]$

$\frac{1}{2}\ln(x^2+y^2) - \tan^{-1}\frac{x}{y} = c$

(49) (16) $(3xy+y^2)dx + (x^2+xy)dy = 0$

$\frac{dy}{dx} = -\frac{(3xy+y^2)}{x^2+xy}$ ——— (I)

Put $y = vx$ ——— (II)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ——— (III)

Put (II) & (III) in (I)

$v + x\frac{dv}{dx} = -\frac{(3x^2v+v^2x^2)}{x^2+x^2v}$

$x\frac{dv}{dx} = -\frac{\cancel{x^2}(3v+v^2)}{\cancel{x^2}(1+v)} - v$

$= -\frac{3v - v^2 - v(1+v)}{1+v}$ LCM

$= \frac{-3v-v^2-v-v^2}{1+v}$

$x\frac{dv}{dx} = -\frac{4v-2v^2}{1+v}$

$x\frac{dv}{dx} = -\frac{2(2v+v^2)}{1+v}$

$\int\frac{1+v}{2v+v^2}dv = -2\int\frac{dx}{x}$ separating Variables

$\frac{1}{2}\int\frac{(2+2v)}{2v+v^2}dv = -2\int\frac{dx}{x}$

$\frac{1}{2}\ln(2v+v^2) = -2\ln x + \ln c$

$\ln(2v+v^2)^{\frac{1}{2}} = \ln x^{-2} + \ln c$

$\ln\sqrt{2v+v^2} = \ln(cx^{-2})$

$\sqrt{2v+v^2} = \frac{c}{x^2}$

Squaring $2v+v^2 = \frac{c^2}{x^4}$

$v(2+v) = \frac{c^2}{x^4}$

$\frac{y}{x}\left(2+\frac{y}{x}\right) = \frac{c^2}{x^4}$

$\frac{y}{x}\left(\frac{2x+y}{x}\right) = \frac{c^2}{x^4}$

$x^4\,\frac{y(2x+y)}{x^2} = c^2$

$x^2(2xy+y^2) = c^2$

$2x^3y + x^2y^2 = c^2$ Ans

(49)

Easy Method and general

(3) $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$

$x - y + (x+y)\dfrac{dy}{dx} = 0$

$(x-y)dx + (x+y)dy = 0$ —— (1)

$M = x - y \qquad N = x + y$

$M_y = 0 - 1 \qquad N_x = 1 + 0$

$\dfrac{M_y - N_x}{N} = \dfrac{-1-1}{x+y}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{1+1}{x-y}$ Not fn of y alone

$\because$ (1) is Homogeneous So $I.F = \dfrac{1}{xM + yN}$

$I.F = \dfrac{1}{x(x-y)+y(x+y)} = \dfrac{1}{x^2 - \cancel{xy} + \cancel{xy} + y^2}$

$= \dfrac{1}{x^2+y^2}$

Multiply (1) by $I.F = \dfrac{1}{x^2+y^2}$

$\dfrac{(x-y)}{x^2+y^2}dx + \dfrac{(x+y)}{x^2+y^2}dy$ —— (11)

$M_y = \dfrac{(x^2+y^2)(-1) - (x-y)(2y)}{(x^2+y^2)^2}, \quad N_x = \dfrac{(x^2+y^2)\cdot 1 - (x+y)2x}{(x^2+y^2)^2}$

$M_y = \dfrac{-x^2 - y^2 - 2xy + 2y^2}{(x^2+y^2)^2}, \quad N_x = \dfrac{x^2 + y^2 - 2x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = \dfrac{+y^2 - x^2 - 2xy}{(x^2+y^2)^2} \qquad N_x = \dfrac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$

$M_y = N_x \therefore$ (11) is Exact Diff Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{x-y}{x^2+y^2}\right)dx + Nil = c$

$\int \dfrac{x\,dx}{x^2+y^2} - \int \dfrac{y\,dx}{x^2+y^2} = c$

$\dfrac{1}{2}\int \dfrac{2x\,dx}{x^2+y^2} - \int \dfrac{d}{dx}\tan^{-1}\dfrac{x}{y} = c \qquad \left(\because \dfrac{d}{dx}\tan^{-1}\dfrac{x}{a} = \dfrac{a}{a^2+x^2}\right)$

$\dfrac{1}{2}\ln(x^2+y^2) - \tan^{-1}\dfrac{x}{y} = c$

(15) $(y^2 + xy)dx - x^2 dy = 0$ —— (1)

$M = y^2 + xy \qquad N = -x^2$

$M_y = 2y + x \qquad N_x = -2x$

$M_y \neq N_x \therefore$ (1) is not Exact Diff Eq.

$\dfrac{M_y - N_x}{N} = \dfrac{2y + x + 2x}{-x^2}$ Not fn of x alone

$\dfrac{N_x - M_y}{M} = \dfrac{-2x - 2y - x}{y^2 + xy}$ Not fn of y alone.

(1) is Homogeneous diff eq of degree (2)

$\therefore I.F = \dfrac{1}{xM + yN} = \dfrac{1}{xy^2 + \cancel{x^2y} + y(\cancel{-x^2})} = \boxed{\dfrac{1}{xy^2}}$

Multiply both sides of (1) by $I.F = \dfrac{1}{xy^2}$

$\dfrac{1}{xy^2}(y^2 + xy)dx - \dfrac{1}{xy^2}x^2 dy = 0$

$\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx - \dfrac{x}{y^2}dy = 0$ —— (11)

$M = \dfrac{1}{x} + \dfrac{1}{y} \qquad N = -\dfrac{x}{y^2}$

$M_y = -\dfrac{1}{y^2} \qquad N_x = -\dfrac{1}{y^2}$

$\because M_y = N_x \therefore$ (11) is Exact Diff Eq.

$\therefore \int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int\left(\dfrac{1}{x} + \dfrac{1}{y}\right)dx + Nil = c$

$\ln x + \dfrac{x}{y} = c$

(50)

Q9 $(3y+4xy^2)dx + (2x+3x^2y)dy = 0$ ——(i)

$M_y = 3+8xy \qquad N_x = 2+6xy$

$M_y \neq N_x$ ∴ (i) is Non Exact.

$$\frac{M_y - N_x}{N} = \frac{3+8xy-2-6xy}{2x+3x^2y} = \frac{1+2xy}{x(2+3xy)}$$

$$\frac{N_x - M_y}{M} = \frac{2+6xy-3-8xy}{3y+4xy^2} = \frac{-1-2xy}{y(3+4xy)}$$

Eq (i) is not Homogeneous

Eq (i) is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$

∴ (i) is $y(3+4xy)dx + x(2+3xy)dy = 0$

So I.F $= \dfrac{1}{xM - yN} = \dfrac{1}{3xy+4x^2y^2-2xy-3x^2y^2} = \dfrac{1}{xy+x^2y^2}$

Multiply both sides of eq (i) by I.F $= \dfrac{1}{xy+x^2y^2}$

$$\therefore \frac{(3y+4xy^2)}{(xy+x^2y^2)}dx + \frac{(2x+3x^2y)}{(xy+x^2y^2)}dy = 0$$

$$\frac{\cancel{y}}{\cancel{y}}\left[\frac{3+4xy}{x+x^2y}\right]dx + \frac{\cancel{x}}{\cancel{x}}\left[\frac{2+3xy}{y+xy^2}\right]dy = 0 \quad \text{——(ii)}$$

$$M_y = \frac{(x+x^2y)(4x)-(3+4xy)(x^2)}{(x+x^2y)^2}$$

$$= \frac{4x^2+\cancel{4x^3y}-3x^2-\cancel{4x^3y}}{(x+x^2y)^2} = \frac{x^2}{x^2(1+xy)^2}$$

$$= \frac{1}{(1+xy)^2}$$

$$N_x = \frac{(y+xy^2)(3y)-(2+3xy)(y^2)}{(y+xy^2)^2}$$

$$= \frac{3y^2+\cancel{3xy^3}-2y^2-\cancel{3xy^3}}{(y+xy^2)^2} = \frac{y^2}{y^2(1+xy)^2}$$

$$= \frac{1}{(1+xy)^2}$$

$M_y = N_x$ ∴ (ii) is Exact Diff Eq.

see above

∴ $\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$$\int\left(\frac{3+4xy}{x+x^2y}\right)dx + \int\frac{2}{y}dy = c$$

$$\int\frac{(3+3xy+xy)}{x(1+xy)}dx + \int\frac{2}{y}dy = c$$

$$3\int\frac{\cancel{(1+xy)}}{x\cancel{(1+xy)}}dx + \int\frac{\cancel{x}y\,dx}{\cancel{x}(1+xy)} + \int\frac{2\,dy}{y} = c$$

$$3\int\frac{dx}{x} + \int\frac{y}{1+xy}dx + \int\frac{2}{y}dy = c$$

$3\ln x + \ln(1+xy) + 2\ln y = c$

$3\ln x + \ln(1+xy) + \ln y^2 = c\ln e$

$\ln x^3(1+xy)y^2 = \ln e^c$

$\ln x^3(1+xy)y^2 = \ln e^c$

Antilog $x^3(1+xy)y^2 = e^c$

$x^3(1+xy)y^2 = c'$

x————x

$$\because N = \frac{2+3xy}{y(1+xy)}$$

$$= \frac{2+2xy+xy}{y(1+xy)}$$

$$= \frac{2\cancel{(1+xy)}}{y\cancel{(1+xy)}} + \frac{x\cancel{y}}{\cancel{y}(1+xy)}$$

$$= \underbrace{\frac{2}{y}}_{\text{free from } x} + \frac{x}{1+xy}$$

51

(11) $(y - xy^2)\,dx + (x + x^2y^2)\,dy = 0$

$M = y - xy^2 \qquad N = x + x^2y^2$

$M_y = 1 - 2xy \qquad N_x = 1 + 2xy^2$

$\dfrac{M_y - N_x}{N} = \dfrac{1 - 2xy - 1 - 2xy^2}{x + x^2y^2} = \dfrac{-2x(y+y^2)}{x(1+xy^2)}$ Not fn of x

$\dfrac{N_x - M_y}{M} = \dfrac{1 + 2xy^2 - 1 + 2xy}{y - xy^2} = \dfrac{2xy(y+1)}{y(1-xy)}$ Not fn of y

Rearranging $y\,dx - xy^2\,dx + x\,dy + x^2y^2\,dy = 0$

$y\,dx + x\,dy - xy^2\,dx + x^2y^2\,dy = 0$

x & ÷ by x $\quad y\,dx + x\,dy - x^2y^2\left(\dfrac{dx}{x}\right) + x^2y^2\,dy = 0$

$y\,dx + x\,dy - x^2y^2\left(\dfrac{dx}{x} - dy\right) = 0$

÷ by x^2y^2 on both sides $\quad \dfrac{y\,dx + x\,dy}{x^2y^2} - \dfrac{\cancel{x^2y^2}}{\cancel{x^2y^2}}\left(\dfrac{dx}{x} - dy\right) = 0$

$d\left(-\dfrac{1}{xy}\right) - \dfrac{dx}{x} + dy = 0$

Integrating $\quad -\dfrac{1}{xy} - \ln|x| + y = c$

(2) $x\,dy - y\,dx = (x^2+y^2)\,dx$ ——(1)

$(x^2 + y^2 + y)\,dx - x\,dy = 0$

$M_y = 2y + 1 \qquad N_x = -1$

$\because M_y \neq N_x$ Hence (1) is Non Exact

$\dfrac{N_x - M_y}{M} = \dfrac{-1 - 2y - 1}{x^2 + y^2 + y}$ Not fn of y

$\dfrac{M_y - N_x}{N} = \dfrac{2y + 1 + 1}{-x}$ Not fn of x

from (1) $\quad x\,dy - y\,dx = (x^2 + y^2)\,dx$

$\displaystyle\int \dfrac{x\,dy - y\,dx}{x^2 + y^2} = \int dx$

$\tan^{-1}\left(\dfrac{y}{x}\right) = x + c$

$\left(\dfrac{y}{x}\right) = \tan(x + c)$

$y = x\tan(x + c)$ Ans