PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1.Basis Step:

The proposition P(1) is true.

2.Inductive Step:

If P(k) is true then P(k + 1) is true for all integers k ≥ 1.

i.e. $\forall$ k p(k) $\to$ P(k + 1)

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$ for all integers n ≥1

SOLUTION:

Let $$P(n):1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers k ≥ 1.

$$1+2+3+\cdots+k=\frac{k(k+1)}{2} \qquad (1)$$

To prove P(k + 1) is true. That is,

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2} \qquad (2)$$

Consider L.H.S. of (2)

$$
\begin{aligned}
1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)} \\
&= (k+1)\left[\frac{k}{2}+1\right] \\
&= (k+1)\left[\frac{k+2}{2}\right] \\
&= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}
\end{aligned}
$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that

$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let P(n) be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step**:

P(1) is true

For n = 1, L.H.S of P(1) = 1and

R.H.S = 12 = 1

Hence the equation is true for n = 1

2. **Inductive Step**:

Suppose P(k) is true for some integer $k \geq 1$. That is,

$1 + 3 + 5 + \ldots + (2k - 1) = k^2$ …………………(1)

To prove P(k+1) is true; i.e.,

$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2$ ………….……(2)

Consider L.H.S. of (2)

$$
\begin{aligned}
1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1) \\
&= 1+3+5+\cdots+(2k-1)+(2k+1) \\
&= k^2+(2k+1) \qquad \text{using (1)} \\
&= (k+1)^2 \\
&= \text{R.H.S. of (2)}
\end{aligned}
$$

Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers n≥1.

EXERCISE:

Use mathematical induction to prove that

1+2+22 + … + 2n = 2n+1 - 1 for all integers n ≥0

SOLUTION:

Let P(n): 1 + 2 + 22 + … + 2n = 2n+1 - 1

1. **Basis Step:**

P(0) is true.
For n = 0
L.H.S of P(0) = 1
R.H.S of P(0) = 20+1 - 1 = 2 - 1 = 1
Hence P(0) is true.

2. **Inductive Step:**

Suppose P(k) is true for some integer k ≥ 0; i.e.,
1+2+22+…+2k = 2k+1 – 1……………………(1)

To prove P(k+1) is true, i.e.,

1+2+22+…+2k+1 = 2k+1+1 – 1…………………(2)

Consider LHS of equation (2)

1+2+22+…+2k+1= (1+2+22+…+2k) + 2k+1
= (2k+1 - 1) +2k+1
= 2·2k+1 - 1
= 2k+1+1 - 1 = R.H.S of (2)

Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers n≥0.

EXERCISE:

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$ for all integers n ≥1.

SOLUTION:

Let P(n) denotes the given equation

1. **Basis step:**

P(1) is true
For n = 1
L.H.S of P(1) = 12 = 1

R.H.S of P(1) $= \frac{1(1+1)(2(1)+1)}{6}$

$= \frac{(1)(2)(3)}{6} = \frac{6}{6} = 1$

So L.H.S = R.H.S of P(1).Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true for some integer k ≥1;

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \text{.........(1)}$$

To prove P(k+1) is true; i.e.;

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \qquad \text{...(2)}$$

Consider LHS of above equation (2)

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 \quad &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right] \\
&= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right] \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}
\end{aligned}$$

<u>EXERCISE:</u>

Prove by mathematical induction

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \qquad \text{for all integers } n \geq 1$$

SOLUTION:

Let P(n) be the given equation.

1.Basis Step:

P(1) is true

For n = 1

L.H.S of P(1) = $\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) = $\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true, for some integer k≥1. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1} \qquad \ldots\ldots(2)$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+2)}$$

= RHS of (2)

Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i2^i = n \cdot 2^{n+2} + 2, \qquad \text{for all integers } n \geq 0$$

SOLUTION:

1.Basis Step:

To prove the formula for n = 0, we need to show that

$$\sum_{i=1}^{0+1} i.2^i = 0 \cdot 2^{0+2} + 2$$

Now, L.H.S = $\sum_{i=1}^{1} i \cdot 2^i = (1)2^1 = 2$

R.H.S = $0 \cdot 2^2 + 2 = 0 + 2 = 2$

Hence the formula is true for n = 0

2.Inductive Step:

Suppose for some integer n=k ≥0

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+1+2} + 2 \qquad \ldots\ldots\ldots..(2)$$

Consider LHS of (2)

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2} \\
&= (k \cdot 2^{k+2} + 2) + (k+2) \cdot 2^{k+2} \\
&= (k + k + 2)2^{k+2} + 2 \\
&= (2k+2) \cdot 2^{k+2} + 2 \\
&= (k+1)2 \cdot 2^{k+2} + 2 \\
&= (k+1) \cdot 2^{k+1+2} + 2 \\
&= \text{RHS of equation (2)}
\end{aligned}$$

Hence the inductive step is proved as well. Accordingly by mathematical induction the given formula is true for all integers n≥0.

EXERCISE:

Use mathematical induction to prove that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \quad \text{for all integers n} \geq 2$$

SOLUTION:

1. **Basis Step**:

For n = 2

$$\text{L.H.S } = 1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$$

$$\text{R.H.S } = \frac{2+1}{2(2)}=\frac{3}{4}$$

Hence the given formula is true for n = 2

2. **Inductive Step**:

Suppose for some integer k ≥2

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)} \ldots\ldots\ldots\ldots..(2)$$

Consider L.H.S of (2)

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left[\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)}\right)$$

$$= \frac{k^2+2k}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)}$$

$$= \frac{k+1+1}{2(k+1)} = \quad \text{RHS of (2)}$$

Hence by mathematical induction the given equation is true

EXERCISE:

Prove by mathematical induction

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1 \qquad \text{for all integers } n \geq 1$$

SOLUTION:

1.Basis step:

For n = 1

L.H.S $= \sum_{i=1}^{n} i(i!) = (1)(1!) = 1$

R.H.,S = (1+1)! - 1 = 2! - 1

= 2 -1 = 1

Hence

$\sum_{i=1}^{1} i(i!) = (1+1)! - 1$

which proves the basis step.

2.Inductive Step:

Suppose for any integer k ≥1

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1 \qquad \text{..............................(1)}$$

We need to prove that

$$\sum_{i=1}^{k+1} i(i!) = (k+1+1)! - 1 \qquad \text{..........................(2)}$$

Consider LHS of (2)

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)! \qquad \text{Using (1)}$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= [1 + (k+1)](k+1)! - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

= RHS of (2)

Hence the inductive step is also true.
Accordingly, by mathematical induction, the given formula is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove the generalization of the following DeMorgan's Law:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

where $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and n≥2.

SOLUTION:

Let P(n) be the given propositional function

1.Basis Step:

P(2) is true.

$$\text{L.H.S of P(2)} = \overline{\bigcap_{j=1}^{2} A_j} = \overline{A_1 \bigcap A_2} \qquad \text{By DeMorgan's Law}$$
$$= \overline{A_1 \bigcup A_2}$$
$$= \bigcup_{i=1}^{2} \overline{A_j} = \text{RHS of } P(2)$$

2.Inductive Step:

Assume that P(k) is true for some integer k ≥2; i.e.,

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. If A_{k+1} is another set of U, then we need to show that

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Consider L.H.S of (2)

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$
$$= \left(\overline{\bigcap_{j=1}^{k} A_j}\right) \cup \overline{A_{k+1}} \qquad \text{By DeMorgan's Law}$$
$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}}$$
$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$
$$= \text{R.H.S of (2)}$$

Hence by mathematical induction, the given generalization of DeMorgan's Law holds.