SZABIST Shaheed Zulfikar Ali Bhutto Institute of Science & Technology

Total Marks: 04

Obtained Marks: ______

Discrete mathematics

Assignment #04

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Question no.1

Prove by mathematical induction for any. For any induction. For any integer $n \geq 1$, $x^n - y^n$ is divisible by $x-y$ where x and y are two integers with $x \neq y$

Solution:

1. Base step:

For $n=1$

$$x^n - y^n = x^1 - y^1 = x-y$$

Which is clearly divisible by $x-y$, so the statement is true for $n=1$

2. Inductive step:

Suppose the statement is true for $n=k$ i.e,

$x^k - y^k$ is divisible by $x-y$ .... ①

we need to prove that $x^{k+1} - y^{k-1}$ is divisible by $x-y$

$$x^{k+1} - y^{k+1} = x^k . x - y^k . y$$

$$= x^k . x - xy^k + xy^k - y^k . y \quad \text{(introducing } xy^k\text{)}$$

$$= (x^k - y^k) . x + y^k (x-y)$$

The first term on R.H.S is $(x^k - y^k)$ is divisible by $x-y$ by inductive hypothesis — ①

The second term contains a factor $(x-y)$ so is also divisible by $x-y$.

Thus $x^{k+1} - y^{k+1}$ is divisible by $x-y$, Hence by mathematical induction $x^n - y^n$ is divisible by $x-y$ for any integer $n \geq 1$.

Question no.2

Prove by mathematical induction $n^3 - n$ is divisible by 6, for each integer $n \geq 2$

Solution:

1. Base step:

For $n=2$

$$n^3 - n = 2^3 - 2 = 8 - 2 = 6$$

which is clearly divisible by 6. since $6 = 1 \cdot 6$

Therefore, the given statement is true for $n=2$

2. Inductive step:

Suppose that the statement is true for $n=k$ i.e,

$k^3 - k$ is divisible by 6,

Then

$$k^3 - k = 6 \cdot q \quad \ldots \text{①} \text{ for some } q \in \mathbb{Z}$$

we need to prove that

$(k+1)^3 - (k+1)$ is divisible by 6.

Now $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$

$$= k^3 + 3k^3 + 2k$$

$$= (k^3 - k) + (3k^3 + 2k + k)$$

$$= (k^3 - k) + 3k^3 + 3k$$

$$= (k^3 - k) + 3(k^3 + k)$$

$$= 6.q + 3k(k^2 + k) \quad \ldots \text{②}$$

Since k is an integer, so $k(k+1)$ being a product of two consecutive integers is an even number.

let

$$k(k+1) = 2r \qquad r \in Z$$

Now eq ② can be written as

$$(k+1)^3 - (k+1) = 6.q + 3(2r)$$

$$= 6q + 6r$$

$$= 6(q + r) \qquad q, r \in Z$$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 6.

Hence, by mathematical induction, $n^3 - n$ is divisible by 6, for each integer $n \geq 1$.

Question no. 3

Use mathematical induction to prove that $n^3 - n$ is divisible by 3, whenever n is a positive integer.

Solution:

1. Base step:

For $n = 1$

$n^3 - n = (1)^3 - 1 = 0$

which is divisible by 3, since $0 = 0 \cdot 3$

Therefore the given statement is true for $n = 1$.

2. Inductive step

Suppose that the statement is true for $n = k$ ie,

$k^3 - k$ is divisible by 3. for all $n \in \mathbb{Z}^+$

Then

$$k^3 - k = 3q \quad \dots \text{①}$$

we need to prove that $(k+1)^3 - (k+1)$ is divisible by 3

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 2k + k \\ &= 3q + 3(k^2 + k) \\ &= 3(q + k^2 + k)\end{aligned}$$

$(k+1)^3 - (k+1)$ is divisible by 3.

Hence by mathematical induction $n^3 - n$ is divisible by 3, whenever n is a positive integer.

Question no. 4

Use mathematical induction to prove that for all integers $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Solution:

(5)

let $P(n): 2^{2n}-1$ is divisible by 3.

1. Base step:

$P(1)$ is true

Now $P(1): 2^{2(1)}-1 = 4-1 = 3$

which is divisible by 3.

Hence $P(1)$ is true.

2. Inductive step:

Suppose that $P(k)$ is true. That is $2^{2k}-1$ is divisible by 3. There there exists an integer q such that

$$2^{2k}-1 = 3.q \quad \ldots\ldots \quad (1)$$

To prove $P(k+1)$ is true, that is $2^{2(k+1)}-1$ is divisible by 3.

Now consider

$$2^{2(k+1)}-1 = 2^{2k.2}-1$$

$$= 2^{2k}.2^{2}-1$$

$$= 2^{2k}.4-1$$

$$= 2^{2k}(3+1)-1$$

$$= 2^{2k}.3+(2^{2k}-1)$$

$$= 2^{2k}.3+3q$$

$= 3(2^{2k} + q)$

$2^{2(k+1)} - 1$ is divisible by 3.

Accordingly, by mathematical induction. $2^{2n} - 1$ is divisible by 3, for all integers $n \geq 1$.

Question no. 5

Use mathemathical induction to show that the product of any two positive consecutive positive integers is divisible by 2.

Solution:

let n and $n+1$ are two consecutive positive integers we need to prove that $n(n+1)$ is divisible by 2.

1. Base step:

For $n = 1$

$n(n+1) = 1(1+1) = 1(2) = 2$

which is divisible by 2

2. Inductive step:

Suppose the given statement is true for $n = k$ that is $k(k+1)$ is divisible by 2, for some $k \in Z+$

we must show that

$$(k+1)(k+1+1) = (k+1)(k+2)$$
$$= k(k+1) + 2(k+1)$$
$$= 2q + 2(k+1)$$
$$= 2(q + k + 1)$$

Hence $(k+1)(k+2)$ is divisible by 2

Accordingly, by mathematical Induction, the product of any two positive integers is divisible by 2.

Question no. 6

Use mathematical induction to prove that for all integers $n \geq 3$, $2n+1 < 2^n$

Solution:

1. Base step:

For $n=3$

$L.H.S = 2(3) + 1 = 6+1 = 7$

$R.H.S = 2^3 = 8$

Since $7 < 8$, so the statement is true for $n=3$

2. Inductive step:

Suppose that the statement is true for $n=k$ i.e,

$$2k+1 < 2^k \quad \ldots \text{①} \qquad k \geq 3$$

We need to show that the statement is true for $n=k+1$ i.e

$$2(k+1)+1 < 2^{(k+1)} \quad \ldots \text{②}$$

Consider the L.H.S of eq ②

$= 2(k+1)+1$

$= 2k+2+1$

$= (2k+1)+2$

$< 2^k + 2$

$< 2^k + 2^k$

$< 2 \cdot 2^k = 2^{k+1}$

Thuse, $2(k+1)+1 < 2^{k+1}$ proved

Question no. 8

Solve all examples, discussed in online class

Exercise 1

Compute each of the following:

a, $\frac{7!}{5!}$ b, $(-2)!$ c, $\frac{(n+1)!}{n!}$ d, $\frac{(n-1)!}{(n+1)!}$

Solution:

a, $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$

b, $(-2)!$ is not defined

c, $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} = n+1$

d, $\frac{(n-1)!}{(n+1)!} = \frac{(n-1)!}{(n+1) \cdot n \cdot (n-1)!} = \frac{1}{n(n+1)} = \frac{1}{n^2+n}$

Exercise 2

Write in terms of factorials

a, $25.24.23.22$

b, $n(n-1)(n-2)\ldots(n-r+1)$

c, $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots(r-1)r}$

Solution:

a, $25.24.23.22$

$$= \frac{25.24.23.22.21!}{21!} = \frac{25!}{21!}$$

b, $n(n-1)(n-2)\ldots(n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

c, $= \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots(r-1)r}$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

Exercise 3

How many possible outcomes are there when a fair coin is tossed three times.

Solution:

Each time a coin is tossed it's outcome is either a head (H) or a tail (T).

Hence in successive tosses, H and T are repeated, Also the order in wich they appear is important. Accordingly, the problem 3 samples from a set of two elements H and T $[k=3, n=2]$

Hence number of samples

$$n^k$$

$$= 2^3 = 8$$

These 8 samples may be listed as

HHH, HHT, HTH, H,TT, T,HH, THT, TTH, TTT

Exercise 4

Suppose repetition of digits is premitted

Part A:

How many three digits can be formed from the six digits 2, 3, 4, 5, 7 and 9

Given distinct elements $= n = 6$

Digits can be chosen $= k = 3$

While forming numbers, order of digits is important. Also digits may be repeated. Hence this is a case of three samples from 6 elements.

Number of 3 digit numbers $= n^k = 6^3 = 216$

Part B:

How many of these numbers are less than 400?

From a given 6 digits 2, 3, 4, 5, 7, 9, a three digit number would be less than 400 if and only if its first digit is either 2 or 3.

The next two digits positions may be filled with anyone of the six digits.

Hence by product rule there are

$2 \cdot 6 \cdot 6 = 72$

three digit numbers less than 400.

Part C:

How many are even?

A number is an even if its right most digit is even. Thus a three digit number formed by the digits 2, 3, 4, 5, 7 and 9 is even its last digit is 2 or 4. Thus the right / last digit position may be filled in 2 ways only while each of the first two position may be filled in 6 ways.

Hence there are

$6 \cdot 6 \cdot 2 = 72$

3-digit even numbers

Part D:

How many are odd?

A number is odd if its right most digit is odd. Thus a three digit number is formed by the digits 2, 3, 4, 5, 7 and 9 is odd if its last digit is one of 3, 5, 7 an 9, Those, the last digit is filled by 4 way, while each of the first two positions may be filled in 6 ways.

Hence there are.

$6 \cdot 6 \cdot 4 = 144$

3 digit odd numbers.

Part E:

How many are multiples of 5?

A number is a mutiple of 5 if its right most digit is either 0 or 5. Thus a digit number is formed by the digits 2, 3, 4, 5, 7 and 9, is a multiple of 5 if its last digit is 5, thus the position of last digit is filled in only one way, while each of the first two positions may be filled in 6 ways.

Hence there are

$6 \cdot 6 \cdot 1 = 36$

3-digit numbers that are multiple of 5.

Exercise 5

A box contain 10 different coloured light bulbs
Find the number of ordered sample of size 3
with replacement.

Solution:

Numbers of light bulb $= n = 10$

Bulbs can be drawn $= k = 3$

Since, bulbs can be drawn with replacement, so repetition is allowed. Also while drawing a sample order of elements in the sample is important.

Hence number of sample size $= 3 = n^k$

$\Rightarrow 10^3 = 1000$

Exercise 6

A multiple choice contains 10 questions;
There are four possible answer for each question

Part A:

How many way a student answers the question on the test if every question is answered?.

Each question can be answered in 4 ways

Suppose answers are labeled as A, B, C, D.

Since lable A may be used as the answer of more then one question. So repetation is allowed. Also the order in which A, B, C, D are choosen as answer for 10 question is important. Hence this is one of the k-sample in ~~with~~ which

$n =$ no. of distint labels $= 4$

$k =$ no. of labels selecting from answering $= 10$

No. of way to answer 10 question $= n^k$

$\Rightarrow 4^{10} = 1048576$

Part B:

How many way a student can answer the question on the test if the student can leave answers blank?

If the student can leave answer blank, then in addition, to the four answers, a fifth option to leave answer blank is possible. Hence, in such case

$n = 5$

and $k = 10$ (as before)

No. of possible answers $= n^k$

$\Rightarrow 5^{10} = 9765625$

15

Exercise 7

How many 2-permutation are there of $\{w, x, y, z\}$? Write them all.

Solution:

Number of 2-permutation of 4 elements is

$$P(4,2) = {}^4P_2 = \frac{4!}{(4-2)!}$$

$$= \frac{4.3.2!}{2!} = 4.3 = 12$$

These 12 permutation are

wx, wy, wz
xw, xy, xz
yw, y,x, yz
zw, z,x, z,y

Exercise 8

Find

a, $P(8,3)$ b, $P(8,8)$ c, $P(8,1)$ d, $P(6,8)$

Solution:

$$P(8,3) = \frac{8!}{(8-3)!} = 8.7.6 = 336$$

$$P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$$

as $0! = 1$

$P(8,1) = \frac{8!}{(8-1)!} = \frac{8 \cdot 7!}{7!} = 8$ (16

$P(6,8) \rightarrow$ is not defined, since the second integer cannot exceeds the first integer

Exercise 9

Find n if

Part A

$$P(n,2) = 72$$

Given $P(n,2) = 72$.

$n.(n-1) = 72$ by using definition of permutation

$n^2 - n = 72$

$n^2 - n - 72 = 0$

$n = 9$; $n = -8$

Since n must be positive, so the only acceptable value of $n = 9$.

Part B

$$P(n,4) = 42\ P(n,2)$$

$\Rightarrow n(n-1)(n-2)(n-3) = 42\ n(n-1)$ ∴ defination of permutation

$\Rightarrow (n-2)(n-3) = 42$ if $n \neq 0$; $n \neq 1$

$\Rightarrow n^2 - 5n + 6 = 42$

$= n^2 - 5n - 36 = 0$

$= (n-9)(n+4) = 0$

$\Rightarrow n = 9$; $n = -4$

Since n must be positive numbers

So $n = 9$;

Exercise 10

Prove that for all integers $n \geq 3$

$$P(n+1, 3) - P(n, 3) = 3P(n, 2)$$

Solution:

Suppose n is an integer greater than or equal to 3.

Now $L.H.S = P(n+1, 3) - P(n, 3)$

$\Rightarrow (n+1)(n)(n-1) - n(n-1)(n-2)$

$\Rightarrow n(n-1)[(n+1) - (n-2)]$

$\Rightarrow n(n-1)[n+1-n+2]$

$\Rightarrow 3n(n-1)$

$R.H.S = 3P(n, 2)$

$\Rightarrow 3n(n-1)$

Thus $L.H.S = R.H.S$ Hence the result

Exercise 11

Part A

How many way can 5 of the letters of the word ALGORITHM be selected and written in a row?

The answer equals the number of 5-permutation of a set of 9 elements and

$$P(9,5) = \frac{9!}{(9-5)!} = 9.8.7.6.5 = 15120$$

Part B:

How many ways can 5 of letter of the word ALGORITHM be selected and written in a row if the first two letter must be TH?

Since of the first two letter must be TH, Hence we need to choose the remaining three letter of of the left 9-2 = 7 alphabets.

Hence, the answer is the number of 3-permutation of a set of 7 elements. which is

$$P(7,3) = \frac{7!}{(7-3)!} = 7.6.5 = 210$$

Exercise 12

Find the number of ways that a party of seven person can arrange themselfies in a row of seven chairs.

Solution:

The seven persons can arrange themselves in a row in P(7,7) ways.

$$P(7,7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 7!$$

Exercise 13

A debating team consists of three boys and two girls. Find the numbers of ways they can sit in a row if the boys and girls are each sit together.

Solution:

There are two ways to distribute them according to sex: BBBGG or GGBBB

In each case, the boy can sit in a row in

$P(3,3) = 3! = 6$ ways

the girl can sit in $P(2,2) = 2! = 2$ ways

Every row consist of boys and girls which is $2! = 2$, Thus.

total number of ways $= n = 2 \cdot 3! \cdot 2!$

$2 \cdot 6 \cdot 2 = 24$

Exercise 14

Find the number n of ways that five large books, four medium sized book and three small books can be placed on the shelf so that all books of the same size are together.

Solution:

In each case, the large books can be arranged among themselves in

$P(5,5) = 5!$ ways, the medium size books
$P(4,4) = 4!$ ways, the small books
$P(3,3) = 3!$ ways.

These are three block of books can be arranged on the shelf in $P(3,3) = 3!$ ways

thus

$$n = 3! \cdot 5! \cdot 4! \cdot 3!$$

$$= 103680$$

Example

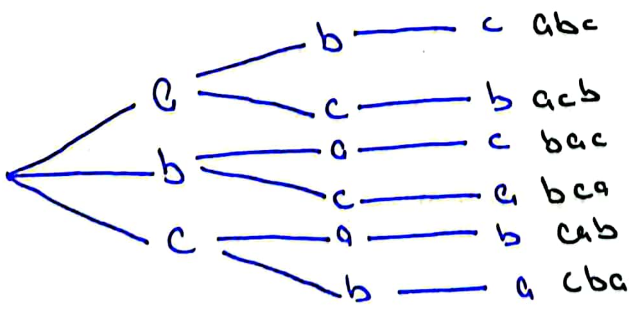
Find the permutation of $\{a, b, c\}$

Solution:

The number of permutation of 3 number are

$$P(3,3) = \frac{3!}{(3-3)!} = 3! = 6$$

We find the six permutation by constructing the appropriate tree diagram. The six premutation are listed on the right of the diagram



Exercise 15

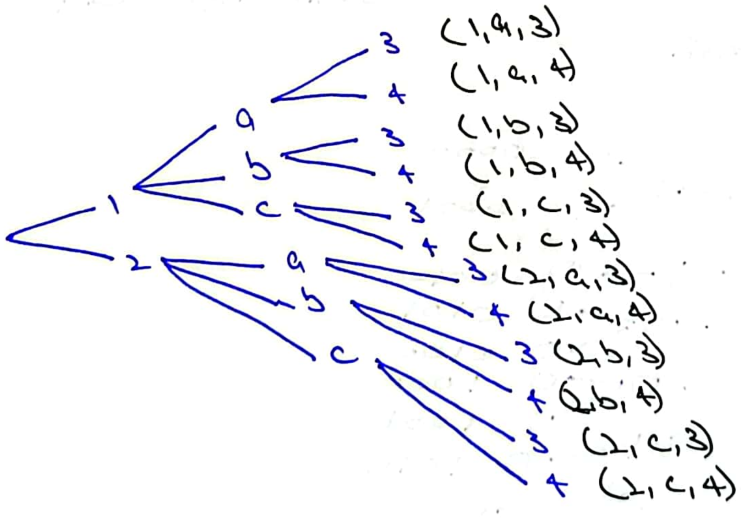
Find the product set A X B X C , where

$A = \{1, 2\}$, $B = \{a, b, c\}$ $C = \{3, 4\}$

by constructing the appropriate tree diagram

Solution:

The required diagram is shown next, Each path is begining of the tree to the end point designates an element of A X B X C which is listed to the right of the tree.

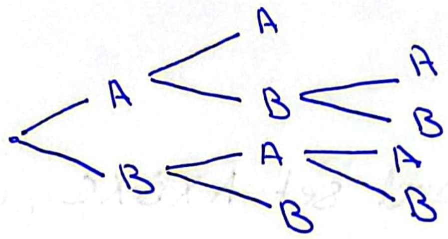


Exercise 16

Team A and team B play in a tournment, the first team win two games the tournment. Find the numbers of possible ~~names~~ way in which the tournment can occurs

Solution:

We construct the approprate tree diagram



The tournment can occurs in 6 ways

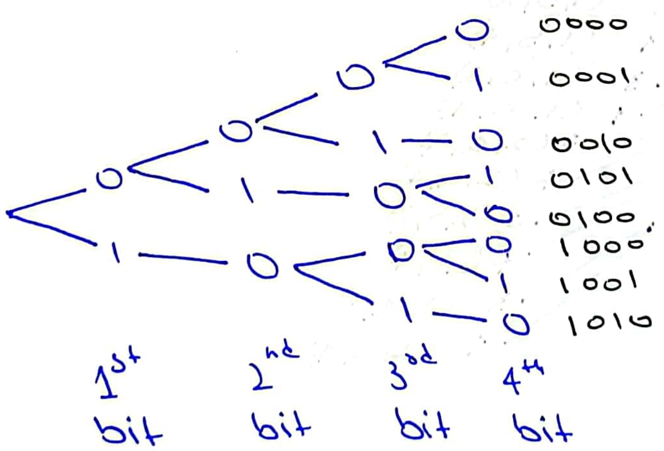
AA, ABA, ABB, BAA, BAB, BB

Exercise 17

How many bit string of length four do not have two consecutive 1's

Solution:

The following tree diagrams display all the bits string of length four without two consecutive 1's, Clearly there are 8 bit strings

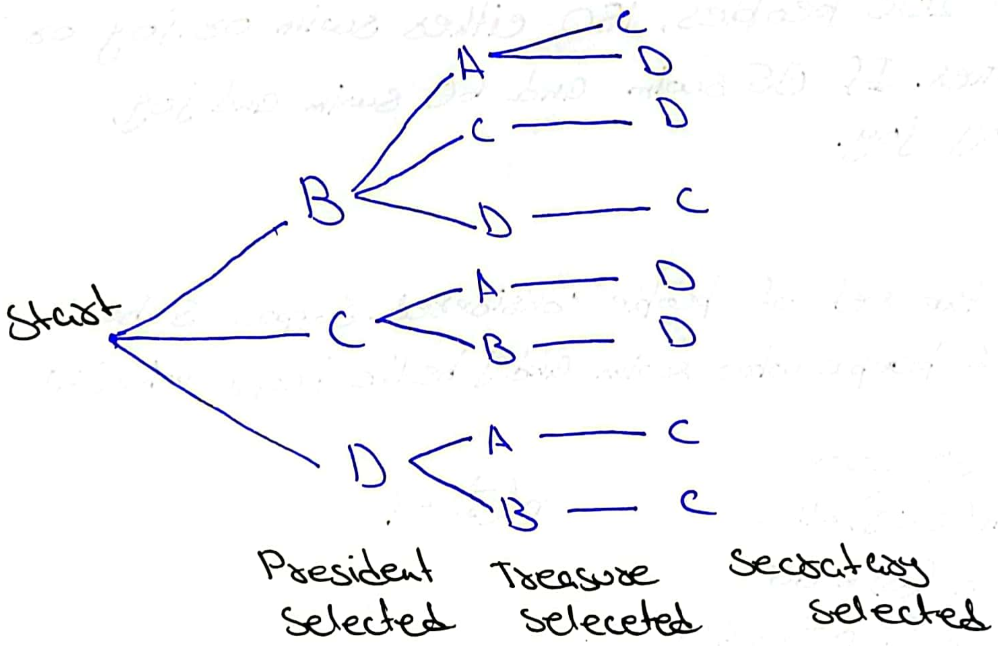


Exercise 18

Three officers, a president, a treasurer and a secretary are to be choosen from amoung four possible: A, B, C and D. Suppose A cannot be president and either C or D must be secretary. How many ways can the officer can be choosen.

Solution:

We construct the possibility tree to see all the possible choices. From the tree given below, see that there are only 8 ways possible to choosen the offices under the given condition



Example

There are 15 girls students and 25 boys students in a class. How many student are there in total.

Solution:

let G be the girls student and B be the boys student

Then

$n(G) = 15$

$n(B) = 25$

$n(G \cup B) = ?$

Since, the set of boys and girls students are disjoint

here total number of students are

$$n(G \cup B) = n(G) + n(B)$$
$$= 15 + 25$$
$$= 40$$

Exercise 19

Among 200 peoples, 150 either swim or jog or both together. If 85 swim and 60 swim and jog, how many jog.

Solution:

let U be the set of people considered. Suppose S be the set of people who swim and J be the people who jog

Then

$n(U) = 200$

$n(S \cup J) = 150$

$n(S) = 85$

$n(S \cap J) = 60$

$n(J) = ?$

By inclusion - exclusion principle,

$$n(S \cup J) = n(S) + n(J) - n(S \cap J)$$
$$150 = 85 + n(J) - 60$$
$$n(J) = 150 - 85 + 60$$
$$n(J) = 125$$

Hence 125 people jog.

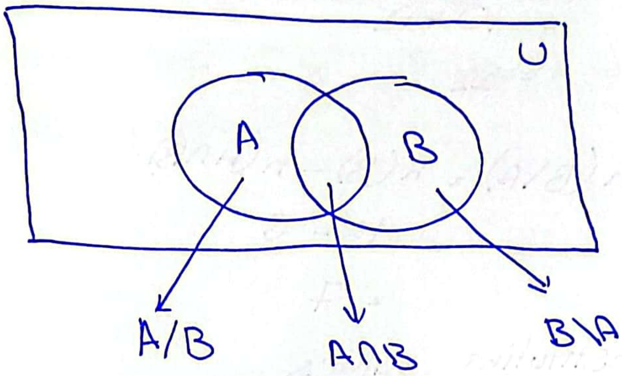
Exercise 20

Suppose A and B are finite sets, show that

$$n(A/B) = n(A) - n(A \cap B)$$

Solution:

Set A may be written as the union of two disjoint set A/B and A∩B



i.e $A = (A\setminus B) \cup (A \cap B)$

Hence by using inclusion and exclusion principle

$n(A) = n(A\setminus B) + n(A\cap B)$

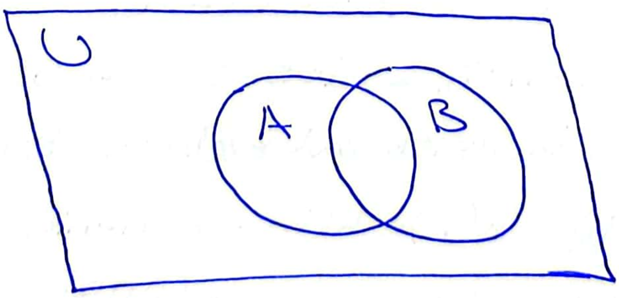
$n(A\setminus B) = n(A) - n(A\cap B)$

Exercise 21

let A and B are subset of U with $n(A) = 10$, $n(B) = 15$, $n(A') = 12$ and $n(A\cap B) = 8$

find $n(A \cup B')$

Solution:



From the diagram $(A \cup B') = U \setminus (B \setminus A)$

$$n(A \cup B') = n(U \setminus (B \setminus A))$$
$$= n(U) - n(B \setminus A)$$

Now $U = A \cup A'$

$$n(U) = n(A) + n(A')$$
$$= 10 + 12$$
$$= 22$$

Also
$$n(B \setminus A) = n(B) - n(A \cap B)$$
$$= 15 - 8$$
$$= 7$$

Substituting values
$$n(A \cup B') = n(U) - n(B \setminus A)$$
$$= 22 - 7$$
$$= 15$$

Exercise 22

let A and B are subset of U with $n(U) = 100$;
$n(A) = 50$; $n(B) = 60$; $n((A \cup B)') = 20$
find
$n(A \cap B)$

Solution:

Since $(A \cup B)' = U \setminus (A \cup B)$
$$n((A \cup B)') = n(U) - n(A \cup B)$$
$$20 = 100 - n(A \cup B)$$
$$n(A \cup B) = 100 - 20 = 80$$

Now by inclusion and exclusion principle
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$80 = 50 + 60 - n(A \cap B)$$
$$n(A \cap B) = 50 + 60 - 80 = 30$$

Exercise 23

Q7

Suppose 18 people read english newspaper (E) or Urdu newspaper (U) or both. Given 5 people read only English newspaper and 7 read both. find the number of "r" of people who read only Urdu newspaper.

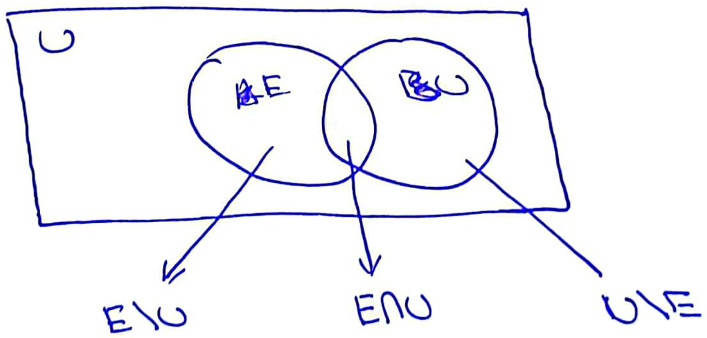
Solution:

Given

$n(E \cup U) = 18 \qquad n(E \setminus U) = 5 \quad n(E \cap U) = 7$

$r = n(U \setminus E) = ?$

From the diagram



$E \cup U = (E \setminus U) \cup (E \cap U) \cup (U \setminus E)$

and the union is disjoint therefore,

$n(E \cup U) = n(E \setminus U) + n(E \cap U) + n(U \setminus E)$

$18 = 5 + 7 + r$

$r = 18 - 5 - 7 = 6$