(CS1.301) Advanced Algorithm Design (Monsoon 2023)

Practice Problem Set 1

Instructions:

- Discussions amongst the students are not discouraged.
- Referring sources other than the lecture notes and textbooks is discouraged as the corresponding solutions available on the internet need not be accurate.
- Please attend tutorials to ensure that you understand the problem correctly.

Question 1 Consider the following functions.

$$T_1(n) = a \cdot T_1\left(\frac{n}{b}\right) + b \cdot n$$
$$T_2(n) = b \cdot T_2\left(\frac{n}{a}\right) + a \cdot n$$

If $a \geqslant b$ and $T_1(1) = T_2(1) = 1$, how do these functions compare as n grows large.

Question 2 A binary tree is a rooted tree in which each node has at most two children. Show by any means possible that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.

Question 3 We have a connected graph $G = (V, E)$, and a specific vertex $u \in V$. Suppose we compute a depth-first search rooted at u, and obtain a tree T that includes all nodes of G. Suppose we then compute a breadth-first search tree rooted at u, and obtain the same tree T. Prove that $G = T$. That is, if T is both a depth-first tree and breadth-first tree rooted at u, then G cannot contain any more edges than those in T.

Question 4 Prove or disprove the following claim. Let G be a graph on n nodes, where n is an even number. If every node of G has degree at least $n/2$, then G is connected.

Question 5 There is a natural intuition that two nodes that are far apart in a communication network separated by many hops have a more tenuous connection than two nodes that are close together. There are a number of algorithmic results that are based to some extent on different ways of making this notion precise. Here's one that involves the susceptibility of paths to the deletion of nodes.

Suppose that an n node undirected graph $G = (V, E)$ contains two nodes s and t such that the distance between s and t is strictly greater than $n/2$. Show that there must exist some node v, not equal to either s or t, such that deleting v from G destroys all s to t paths. (In other words, the graph obtained from G by deleting v contains no path from s to t.) Give an algorithm with running time $O(m + n)$ to find such a node v.

Question 6 A vertex u in a connected (undirected) graph $G = (V, E)$ is called an articulation point if removal of u from the vertex set, and removal of edges incident on the vertex u disconnects the graph. Give an algorithm to find articulation point(s) in a given graph.

Question 7 A directed graph $G = (V, E)$ is strongly connected if and only if every pair of vertices is strongly connected. Equivalently, a strong component of G is a maximal strongly connected subgraph of G. A directed graph G is strongly connected if and only if G has exactly one strong component; at the other extreme, G is a DAG if and only if every strong component of G consists of a single vertex.

Give an algorithm that computes all strong components of a graph in time at most $O(V \cdot E)$. Bonus: Can this be improved to $O(V + E)$?

Question 8 Recall that a directed graph G is strongly connected if, for any two vertices u and v, there is a path in G from u to v and a path in G from v to u.

Describe an algorithm to determine, given an undirected graph G as input, whether it is possible to direct each edge of G so that the resulting directed graph is strongly connected.