(CS1.301) Advanced Algorithm Design (Monsoon 2023)

Practice Problem Set 5

Instructions:

- Discussions amongst the students are not discouraged.
- Referring sources other than the lecture notes and textbooks is discouraged as the corresponding solutions available on the internet need not be accurate.
- Please attend tutorials to ensure that you understand the problem correctly.

Question 1

Prove or disprove the following. Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a minimum s-t cut with respect to these capacities $c_e : e \in E$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum s-t cut with respect to these new capacities $\{1 + c_e : e \in E\}$.

Question 2

Suppose you are given a directed graph $G = (V, E)$, with a positive integer capacity c_e on each edge e, a source $s \in V$, and a sink $t \in V$. You are also given a maximum s-t flow in G, defined by a flow value f_e on each edge e. The flow f is acyclic: There is no cycle in G on which all edges carry positive flow. The flow f is also integer-valued. Now suppose we pick a specific edge $e' \in E$ and reduce its capacity by 1 unit.

Show how to find a maximum flow in the resulting capacitated graph in time $O(m + n)$, where m is the number of edges in G and n is the number of nodes.

Question 3

A packaging company operates within a city, which is represented by a directed graph with n nodes and m edges. Each edge in the graph has a specified weight limit or capacity, denoted by c_i. The company's task is to transport goods from node 1 to node n along a simple path.

The company possesses x trucks, and each truck is capable of carrying any possible weight. However, it is necessary that all trucks must carry the same weight. Additionally, all trucks are dispatched from node 1 simultaneously, meaning no truck waits for others to traverse an edge. Therefore, it is crucial to ensure that the total weight transported across any edge does not exceed the weight limit assigned to that edge.

Provide an $O(mx\log(c_{\max}))$ algorithm to determine the maximum total weight of goods that can be delivered by these x trucks. $c_{\max}$ is the maximum weight limit across all the edges. You can assume that the weights are integral.

Question 4

Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and non-negative edge capacities c_e. Give a polynomial-time algorithm to decide whether G has a unique minimum s-t cut (i.e., an s-t cut of capacity strictly less than that of all other s-t cuts).

Question 5

Consider a round-robin scenario in league matches. The league comprises N teams. At a certain stage during the season, team i has accumulated $w[i]$ wins and has $g[i][j]$ games left to play against team j. A team faces elimination if there is no possibility for the team to secure first place, with the exception that two teams may be tied for first place, in which case neither of the two teams is eliminated.

Design an algorithm that, given team i at a specific point in the season, can determine whether the team is subject to elimination or not. It is assumed that no matches result in ties and that the final rankings are solely based on the number of matches won.

Question 6

In a battlefield, depicted as an $n \times m$ grid, certain cells are occupied by mines. The objective is to clear these mines, but doing so incurs a cost. There are two methods for removing the mines:

1. Select an individual cell containing a mine and eliminate it at a cost x.

2. Choose two neighboring cells (having a common edge), each having a mine, and clear both of them at a cost y.

Provide an algorithm to determine the minimum cost necessary to completely eliminate all the mines.

Question 7

Give an algorithm for computing the max-flow with the following additional constraints. (It is okay to give the solutions to both the parts separately)

1. Each edge *e* has a lower bound *l(e)* on the flow through it.
2. There are multiple sources and sinks, and the value of the flow is computed as the total flow out of all the sources (equivalent to the total flow into all of the sinks).

Question 8

In an image, for each pixel i in the image, we have a likelihood a_i that it belongs to the foreground and a likelihood b_i that it belongs to the background.

For each pixel, we aim to label pixel i as belonging to the foreground if $a_i > b_i$ and to the background otherwise. However, if many of a pixel's neighbors are labeled as background, we would be inclined to label that pixel as background as well. Therefore, for each pair of pixels (i, j) that are neighbors, there's a separation penalty $p_{ij} \geqslant 0$ for placing one in the background and the other in the foreground.

Design an algorithm to determine if each pixel belongs to the foreground or background.

Hint: Think in terms of what you'd want to maximize to find the optimal segmentation.

Question 9

Let's define an algorithm for finding the maximum flow which searches for s-t paths in a graph G'_f consisting only of edges e for which $f(e) < c_e$, and terminates when there is no augmenting path consisting entirely of such edges. We'll call this the Forward-Edge-Only Algorithm.

Prove or disprove: There is an absolute constant $b > 1$ (independent of the particular input flow network), so that on every instance of the Maximum-Flow Problem, the Forward-Edge-Only Algorithm is guaranteed to find a flow of value at least $\frac{1}{b}$ times the maximum flow value (regardless of how it chooses its forward-edge paths).