Random Walks MFTP

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Master Equation

Master Equation for Evolution of Probability

$$P_{ij}(t+1) = \sum_{l=1}^{N} A_{lj} w_{lj} P_{il}(t),$$

$P_{ij}(t+1)$ = Probability that RW starting at node i is at node j at time (t+1)

Follows Preferential Local Transition Probability in case of Biased Random Walks

$$w_{lj} = \frac{k_j^{\alpha}}{\sum_{m=1}^{k_l} k_m^{\alpha}},$$

Master Equation - Mean Field Approximation

To solve the master equation for an ensemble of networks, we need to use mean field approximation:

$$P_j^{\infty} \simeq \sum_{l=1}^{N} \langle A_{lj} \rangle \langle w_{lj} \rangle P_l^{\infty} .$$

For Uncorrelated Networks: We use Annealed Field Approximation for Adjacency Matrix (Different for Assortative and Dissortative, uses Poissonian Nature):

$$\langle A_{lj} \rangle = \frac{k_l k_j}{\langle k \rangle N} .$$

Master Equation - MFA (cont.)

For Uncorrelated networks, degree distribution of neighbours is independent of distribution for a node. So avg. transition prob $\langle w_{lj} \rangle = \frac{\langle k \rangle}{\langle k^{\alpha+1} \rangle k_l} k_j^{\alpha}.$ preferential attachment:

Therefore, the final expression obtained from MFA is:

$$P_j^{\infty} = \frac{k_j^{\alpha+1}}{N \langle k^{\alpha+1} \rangle}.$$

Depends on:

Total Number of Nodes (N)

Alpha moment of degree distribution

Node Degree

MFPT

MFPT is described by the equation:

$$P_{ij}(t) = \delta_{t0}\delta_{ij} + \sum_{\tau=0}^{t} P_{jj}(t-\tau)F_{ij}(\tau).$$

Solving using Laplace Transform :

$$T_{ij} = \sum_{t=0}^{\infty} tF_{ij}(t) = -\widetilde{F}'_{ij}(0) = \begin{cases} 1/P_j^{\infty}, & \text{for } j = i \\ [R_{jj}^{(0)} - R_{ij}^{(0)}]/P_j^{\infty}, & \text{for } j \neq i \end{cases}.$$

Where relaxation time is introduced during analysis:

$$R_{ij}^{(n)} = \sum_{t=0}^{\infty} t^n [P_{ij}(t) - P_j^{\infty}]$$

MFPT (cont.)

Theoretical Prediction:

$$T_{ii} = \frac{N\langle k^{\alpha+1}\rangle}{k_i^{\alpha+1}},$$

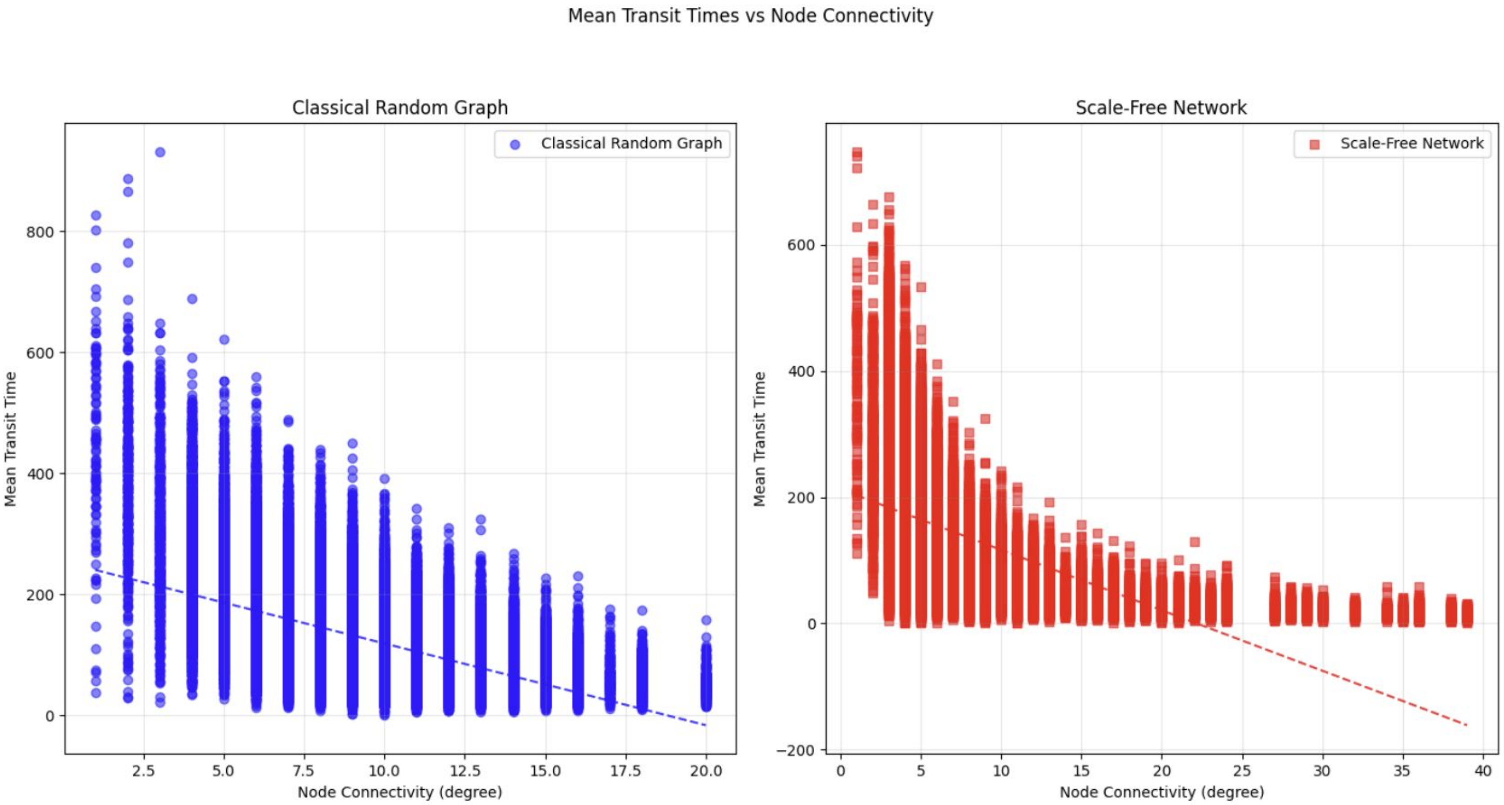
Assuming fast convergence to Stationary Distribution:

$$\begin{aligned} T_{ij} &\simeq \left(\sum_{t=0}^{2} [P_{jj}(t) - P_j^{\infty}] - \sum_{t=0}^{0} [P_{ij}(t) - P_j^{\infty}] \right) / P_j^{\infty} \\ &= \frac{N\langle k^{\alpha+1}\rangle}{k_j^{\alpha+1}} + \frac{N\langle k\rangle^2}{\langle k^2\rangle} \frac{1}{k_j} - 2, \end{aligned}$$

Numerical Simulation

1. Create graphs (erdos renyi or BA)
2. Simulate ensemble of random walkers on graphs
3. Measure the relevant statistics
4. Compare vs Theoretical Prediction

Numerical Simulation - Transit Time vs Connectivity



References

1. Fronczak and Fronczak
2. Amritkar
3. Naoki, Porter, Lambiotte