Homework 1

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For loops

Exercise 2

(a) The first 29 elements of the sequence of ratios of the form f_n/f_{n-1} is given by:

1, 2, 1.5, 1.6666667, 1.6, 1.625, 1.6153846, 1.6190476, 1.6176471, 1.6181818, 1.6179775, 1.6180556, 1.6180258, 1.6180371, 1.6180328, 1.6180344, 1.6180338, 1.6180341, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034, 1.618034

Note that the sequence seems to be converging to about 1.618034.

(b) The golden ratio is $\frac{1+\sqrt{5}}{2} \approx 1.618034$. This does seem to be the value that the sequence in part (a) is converging to. We will prove that this is the case.

Let $n > 4$. Then

$$\frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-2} + f_{n-3}} = 1 + \frac{1}{\frac{f_{n-2}+f_{n-3}}{f_{n-2}}}.$$

Now note that we can rewrite the denominator of the final fraction, $\frac{f_{n-2}+f_{n-3}}{f_{n-2}}$, the same way as the second fraction, $\frac{f_{n-1}+f_{n-2}}{f_{n-1}}$, and as long as n is big enough we can continue doing this. Thus if we are looking at the limit for n to ∞, we can express the ratio as an "infinite fraction".

$$\frac{f_n}{f_{n-1}} = \frac{1}{1 + \frac{1}{1+\frac{1}{\cdots}}}$$

Now all that remains to be show is that this infinite fraction is equal to the golden ratio. To do this, let us first assign the value x to it, such that $x = \frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}$. However, then $1 + \frac{1}{x} = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}} = x$. Now we can simply compute x from this quadratic equation.

$$1 + \frac{1}{x} = x \implies x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{1 - 4 * 1 * -1}}{2 * 1} = \frac{1 \pm \sqrt{5}}{2}.$$

Noting that x should be positive, finally proves that indeed $\frac{f_n}{f_{n-1}} \to \frac{1+\sqrt{5}}{2}$, as $n \to \infty$.