ETH Zurich, IDSC

Car model Equations

Edo Jelavic

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1 Four Wheeled Car Model

Four wheeled car model is built upon the schematic depicted in 1.1.

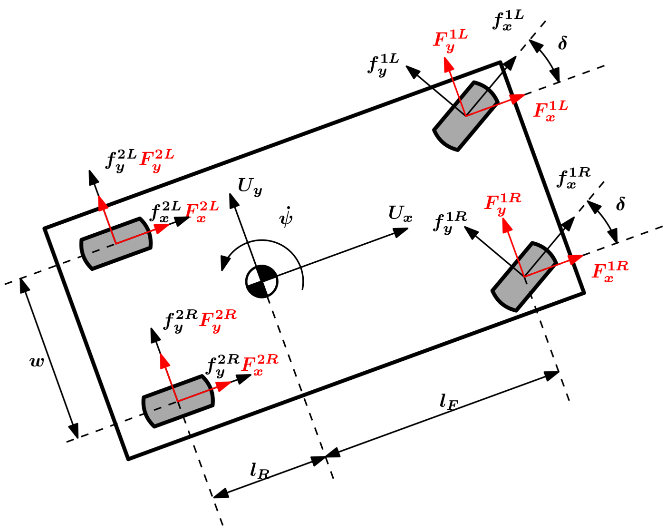

Figure 1.1: 4 wheeled car model

In Figure 1.1, superscript $1L$ refers to the front left wheel, $1R$ refers to front right wheel and similarly $2L$ and $2R$ refer to rear left and right wheel, respectively. Subscripts x and y refer to longitudinal or lateral forces. Capital letter F denotes a force that is tied to the car coordinate frame (red frames and forces painted red in Figure 1.1) while the lower case letters f denote forces in the tire coordinate frame. Four wheeled model is described by following equations:

$$\dot{U}_x = \frac{1}{m}\left(F_x^{1L} + F_x^{1R} + F_x^{2L} + F_x^{2R}\right) + U_y r \tag{1.1}$$

$$\dot{U}_y = \frac{1}{m}\left(F_y^{1L} + F_y^{1R} + F_y^{2L} + F_y^{2R}\right) - U_x r \tag{1.2}$$

$$\dot{r} = \frac{1}{I_z}\left(l_F\left(F_y^{1L} + F_y^{1R}\right) - l_R\left(F_y^{2L} + F_y^{2R}\right) + \frac{w}{2}\left(F_x^{1R} + F_x^{2R} - F_x^{1L} - F_x^{2L}\right)\right) \tag{1.3}$$

$$\dot{\psi} = r \tag{1.4}$$

$$\dot{x} = U_x \cos\psi - U_y \sin\psi \tag{1.5}$$

$$\dot{y} = U_x \sin\psi + U_y \cos\psi \tag{1.6}$$

$$\dot{\omega}^{1L} = \frac{1}{I_\omega}\left(\hat{T}^{1L} - f_x^{1L} R\right) \tag{1.7}$$

$$\dot{\omega}^{1R} = \frac{1}{I_\omega}\left(\hat{T}^{1R} - f_x^{1R} R\right) \tag{1.8}$$

$$\dot{\omega}^{2L} = \frac{1}{I_\omega}\left(\hat{T}^{2L} - f_x^{2L} R\right) \tag{1.9}$$

$$\dot{\omega}^{2R} = \frac{1}{I_\omega}\left(\hat{T}^{2R} - f_x^{2R} R\right) \tag{1.10}$$

$$\hat{T}^j = \begin{cases} T^j, & \omega^j > 0 \\ f_x^j R, & \omega^j = 0 \end{cases} \quad \text{, where} \quad j \in \{1L, 1R, 2L, 2R\} \tag{1.11}$$

In equations 1.1 - 1.10 w denotes width of the car. Superscript 1 denotes the front wheel and superscript 2 denotes the rear wheel. Subscript x denotes longitudinal direction and subscript y denotes lateral direction. For example, f_x^1 is a force on the front tire in the longitudinal direction (in the tire frame). U_x and U_y denote longitudinal and lateral velocities of the car, respectively. Yaw rate is denoted with r and ψ is the yaw angle (or heading angle). m is a mass of the car and I_z is a moment of inertia around the z-axis, l_F is a distance between Center of Gravity (COG) and the front axle, l_R is the distance between COG and the rear axle. w is the distance from left to the right wheel. R is the radius of the wheels. The rotational velocity of the wheels is denoted with ω and μ is a friction coefficient between the tire and the surface. Steering angle is denoted with δ and T denotes torque applied to the wheels (those are inputs into the system). Lateral and longitudinal forces in the tire frame are calculated as follows:

$$\mu^j = D\sin(C\arctan(Bs^j)) \tag{1.12}$$

$$\mu_i^j = -\frac{s_i^j}{s^j}\mu^j \tag{1.13}$$

$$f_i^j = \mu f_z^j \mu_i^j \tag{1.14}$$

$$s_x^j = \frac{v_x^j - \omega^j R}{\omega^j R} \tag{1.15}$$

$$s_y^j = \left(1 + s_x^j\right)\frac{v_y^j}{v_x^j} \tag{1.16}$$

$$s^j = \sqrt{(s_x^j)^2 + (s_y^j)^2} \tag{1.17}$$

$$where \quad i \in \left\{x, y\right\}, j \in \{1L, 1R, 2L, 2R\} \tag{1.18}$$

Relationship between forces in the car frame (red frame in Figure 1.1) and forces in the tire frame is given by:

$$\begin{bmatrix} F_x^j \\ F_y^j \end{bmatrix} = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} f_x^j \\ f_y^j \end{bmatrix} \text{where} \quad j \in \{1L, 1R\} \tag{1.19}$$

$$\begin{bmatrix} F_x^j \\ F_y^j \end{bmatrix} = \begin{bmatrix} f_x^j \\ f_y^j \end{bmatrix} \text{where} \quad j \in \{2L, 2R\} \tag{1.20}$$

$$\tag{1.21}$$

Similar relations hold for velocities as well:

$$\begin{bmatrix} v_x^{1L} \\ v_y^{1L} \end{bmatrix} = \begin{bmatrix} cos\delta & sin\delta \\ -sin\delta & cos\delta \end{bmatrix} \begin{bmatrix} U_x - r\frac{w}{2} \\ U_y + rl_F \end{bmatrix} \tag{1.22}$$

$$\begin{bmatrix} v_x^{1R} \\ v_y^{1R} \end{bmatrix} = \begin{bmatrix} cos\delta & sin\delta \\ -sin\delta & cos\delta \end{bmatrix} \begin{bmatrix} U_x + r\frac{w}{2} \\ U_y + rl_F \end{bmatrix} \tag{1.23}$$

$$\begin{bmatrix} v_x^{2L} \\ v_y^{2L} \end{bmatrix} = \begin{bmatrix} U_x - r\frac{w}{2} \\ U_y - rl_R \end{bmatrix} \tag{1.24}$$

$$\begin{bmatrix} v_x^{2R} \\ v_y^{2R} \end{bmatrix} = \begin{bmatrix} U_x + r\frac{w}{2} \\ U_y - rl_R \end{bmatrix} \tag{1.25}$$

Still missing, are the expressions for computing vertical forces F_z^j (note that $F_z^j = f_z^j$). To obtain expressions for these forces a weight transfer model has to be derived. It all starts by looking at the car confined to the $y-z$ plane. Rear view of the car in the $y-z$ plane is shown in Figure 1.2.
Looking at the Figure 1.2 one obtains the first equation of four equations in total. The first equation is expressed in the equation 1.26; it is merely a balance of torques around the center of gravity. h denotes the distance between COG and the ground.

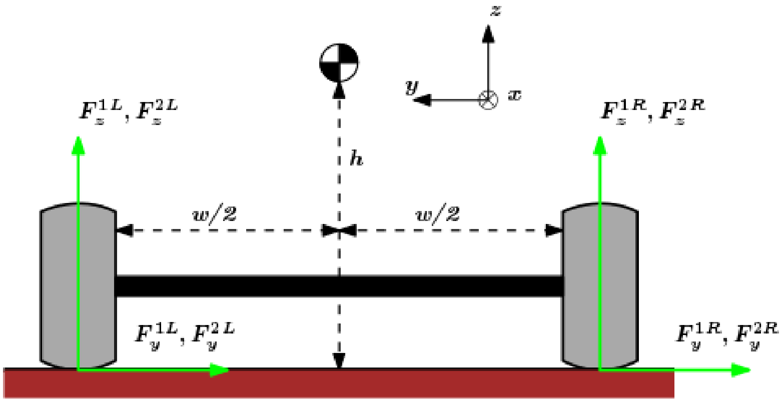

Figure 1.2: 4 wheeled car model, back view

$$\left(F_z^{2R}+F_z^{2L}\right) l_R + h\left(F_x^{1L}+F_x^{1R}+F_x^{2L}+F_x^{2R}\right) = \left(F_z^{1R}+F_z^{1L}\right) l_F \tag{1.26}$$

Second equation is obtained by looking at the torque balance in $x-z$ plane. Side view of the car is shown in Figure 1.3.

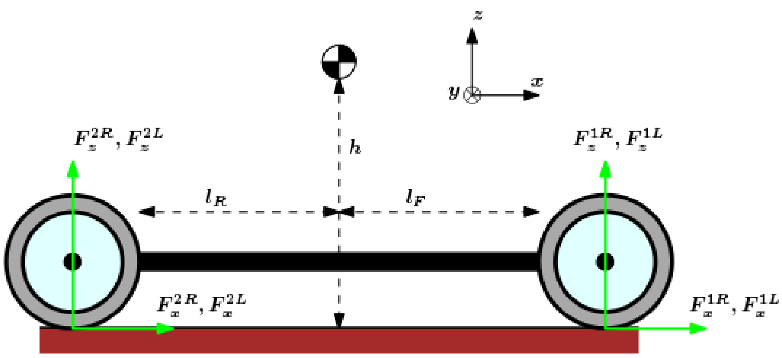

Figure 1.3: 4 wheeled car model, side view

Equation of torque balance in $x-z$ plane is given by:

$$\left(F_z^{1L}+F_z^{2L}\right)\frac{w}{2} + h\left(F_y^{1L}+F_y^{1R}+F_y^{2L}+F_y^{2R}\right) = \left(F_z^{1R}+F_z^{2R}\right)\frac{w}{2} \tag{1.27}$$

The third equation is given by:

$$F_z^{1L}+F_z^{1R}+F_y^{2L}+F_y^{2R} = mg \tag{1.28}$$

Where m is mass and g is gravitational acceleration. Finally, the last equation arises from geometrical considerations. In Figure 1.4, a deflected $x-y$ plane is shown. It is assumed that all the rotations happen around the COG.
From Figure 1.4 it can be seen that deflection on the wheel $2R$ has to be equal to the deflection at the diagonally opposite wheel $1L$. Hence, for the deflections z, it has to hold:

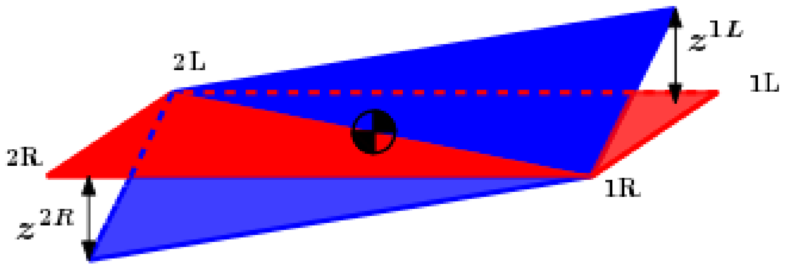

Figure 1.4: 4 wheeled car model, deflected $x-y$ plane amid weight transfer

$$z^{1L} + z^{2R} = 0 \tag{1.29}$$
$$z^{1R} + z^{2L} = 0 \tag{1.30}$$
$$\tag{1.31}$$

If one sums up the equations 1.29 and 1.30, one obtains:

$$z^{1L} + z^{2R} = z^{1R} + z^{2L} \tag{1.32}$$

It is assumed that there is a spring on each wheel. Therefore, the following equation holds:

$$F_z^j = c^j z^j, where \quad j \in \{1L, 1R, 2L, 2R\} \tag{1.33}$$

In equation 1.33, z are deflections and c are spring constants. Combining equations 1.33 and 1.32 and using a simplification $c^i = c^j, \forall i, j \in \{1L, 1R, 2L, 2R\}$, one gets the fourth equation:

$$F_z^{1L} + F_z^{2R} = F_z^{1R} + F_z^{2L} \tag{1.34}$$

By plugging in the expressions for lateral and longitudinal forces (F_x, F_y) from equation 1.14 and equations 1.19 - 1.20 into equations 1.26, 1.27, 1.28 and 1.34, one obtains expressions for vertical forces F_z. These are given by the following set of equations:

$$F_z^{1L} = mg\frac{BG - CF + BH - DF - CH + DG}{den} \tag{1.35}$$
$$F_z^{1R} = mg\frac{AG - EC + AH - ED + CH - DG}{den} \tag{1.36}$$
$$F_z^{2L} = mg\frac{AF - EB + AH - ED + BH - DF}{den} \tag{1.37}$$
$$F_z^{1R} = mg\frac{AF - EB - AG + EC - BG + CF}{den} \tag{1.38}$$
$$\tag{1.39}$$

where:

$$den = 2(AF - EB - AG + EC + BH - DF - CH + DG) \tag{1.40}$$
$$A = -\frac{w}{2} - C_1 - C_2 \tag{1.41}$$
$$B = \frac{w}{2} - C_3 - C_4 \tag{1.42}$$
$$C = -\frac{w}{2} - C_5 \tag{1.43}$$
$$D = \frac{w}{2} - C_6 \tag{1.44}$$
$$E = K_1 - K2 - l_F \tag{1.45}$$
$$F = K_3 - K4 - l_F \tag{1.46}$$
$$G = K_5 + l_R \tag{1.47}$$
$$H = K_6 + l_R \tag{1.48}$$
$$C_1 = -\mu\mu_x^{1L} h \sin(\delta) \tag{1.49}$$
$$C_2 = -\mu\mu_y^{1L} h \cos(\delta) \tag{1.50}$$
$$C_3 = -\mu\mu_x^{1R} h \sin(\delta) \tag{1.51}$$
$$C_4 = -\mu\mu_y^{1R} h \cos(\delta) \tag{1.52}$$
$$C_5 = -\mu\mu_y^{2L} h \tag{1.53}$$
$$C_6 = -\mu\mu_y^{2R} h \tag{1.54}$$
$$K_1 = \mu\mu_x^{1L} h \cos(\delta) \tag{1.55}$$
$$K_2 = \mu\mu_y^{1L} h \sin(\delta) \tag{1.56}$$
$$K_3 = \mu\mu_x^{1R} h \cos(\delta) \tag{1.57}$$
$$K_4 = \mu\mu_y^{1R} h \sin(\delta) \tag{1.58}$$
$$K_5 = \mu\mu_x^{2L} h \tag{1.59}$$
$$K_6 = \mu\mu_x^{2R} h \tag{1.60}$$
$$\tag{1.61}$$

This concludes the derivation of the four-wheeled car model.
Should any errors or inconsistencies be found, feel free to correct them ot make the author aware of them.