

Car model Equations

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1 FOUR WHEELED CAR MODEL

Four wheeled car model is built upon the schematic depicted in 1.1.

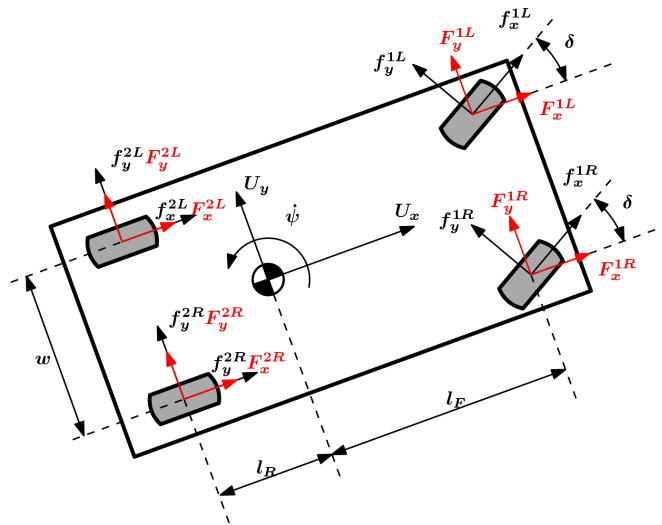


Figure 1.1: 4 wheeled car model

In Figure 1.1, superscript $1L$ refers to the front left wheel, $1R$ refers to front right wheel and similarly $2L$ and $2R$ refer to rear left and right wheel, respectively. Subscripts x and y refer to longitudinal or lateral forces. Capital letter F denotes a force that is tied to the car coordinate frame (red frames and forces painted red in Figure 1.1) while the lower case letters f denote forces in the tire coordinate frame. Four wheeled model is described by following equations:

$$\dot{U}_x = \frac{1}{m} (F_x^{1L} + F_x^{1R} + F_x^{2L} + F_x^{2R}) + U_y r \quad (1.1)$$

$$\dot{U}_y = \frac{1}{m} (F_y^{1L} + F_y^{1R} + F_y^{2L} + F_y^{2R}) - U_x r \quad (1.2)$$

$$\dot{r} = \frac{1}{I_z} (l_F (F_y^{1L} + F_y^{1R}) - l_R (F_y^{2L} + F_y^{2R})) + \frac{w}{2} (F_x^{1R} + F_x^{2R} - F_x^{1L} - F_x^{2L}) \quad (1.3)$$

$$\dot{\psi} = r \quad (1.4)$$

$$\dot{x} = U_x \cos \psi - U_y \sin \psi \quad (1.5)$$

$$\dot{y} = U_x \sin \psi + U_y \cos \psi \quad (1.6)$$

$$\dot{\omega}^{1L} = \frac{1}{I_\omega} (\hat{T}^{1L} - f_x^{1L} R) \quad (1.7)$$

$$\dot{\omega}^{1R} = \frac{1}{I_\omega} (\hat{T}^{1R} - f_x^{1R} R) \quad (1.8)$$

$$\dot{\omega}^{2L} = \frac{1}{I_\omega} (\hat{T}^{2L} - f_x^{2L} R) \quad (1.9)$$

$$\dot{\omega}^{2R} = \frac{1}{I_\omega} (\hat{T}^{2R} - f_x^{2R} R) \quad (1.10)$$

$$\hat{T}^j = \begin{cases} T^j, & \omega^j > 0 \\ f_x^j R, & \omega^j = 0 \end{cases}, \text{ where } j \in \{1L, 1R, 2L, 2R\} \quad (1.11)$$

In equations 1.1 - 1.10 w denotes width of the car. Superscript 1 denotes the front wheel and superscript 2 denotes the rear wheel. Subscript x denotes longitudinal direction and subscript y denotes lateral direction. For example, f_x^1 is a force on the front tire in the longitudinal direction (in the tire frame). U_x and U_y denote longitudinal and lateral velocities of the car, respectively. Yaw rate is denoted with r and ψ is the yaw angle (or heading angle). m is a mass of the car and I_z is a moment of inertia around the z-axis, l_F is a distance between Center of Gravity (COG) and the front axle, l_R is the distance between COG and the rear axle. w is the distance from left to the right wheel. R is the radius of the wheels. The rotational velocity of the wheels is denoted with ω and μ is a friction coefficient between the tire and the surface. Steering angle is denoted with δ and T denotes torque applied to the wheels (those are inputs into the system). Lateral and longitudinal forces in the tire frame are calculated as follows:

$$\mu^j = D \sin(\text{Carctan}(Bs^j)) \quad (1.12)$$

$$\mu_i^j = -\frac{s_i^j}{s^j} \mu^j \quad (1.13)$$

$$f_i^j = \mu f_z^j \mu_i^j \quad (1.14)$$

$$s_x^j = \frac{v_x^j - \omega^j R}{\omega^j R} \quad (1.15)$$

$$s_y^j = \left(1 + s_x^j\right) \frac{v_y^j}{v_x^j} \quad (1.16)$$

$$s^j = \sqrt{(s_x^j)^2 + (s_y^j)^2} \quad (1.17)$$

$$\text{where } i \in \{x, y\}, j \in \{1L, 1R, 2L, 2R\} \quad (1.18)$$

Relationship between forces in the car frame (red frame in Figure 1.1) and forces in the tire frame is given by:

$$\begin{bmatrix} F_x^j \\ F_y^j \end{bmatrix} = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} f_x^j \\ f_y^j \end{bmatrix} \text{ where } j \in \{1L, 1R\} \quad (1.19)$$

$$\begin{bmatrix} F_x^j \\ F_y^j \end{bmatrix} = \begin{bmatrix} f_x^j \\ f_y^j \end{bmatrix} \text{ where } j \in \{2L, 2R\} \quad (1.20)$$

$$(1.21)$$

Similar relations hold for velocities as well:

$$\begin{bmatrix} v_x^{1L} \\ v_y^{1L} \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} U_x - r \frac{w}{2} \\ U_y + r l_F \end{bmatrix} \quad (1.22)$$

$$\begin{bmatrix} v_x^{1R} \\ v_y^{1R} \end{bmatrix} = \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} U_x + r \frac{w}{2} \\ U_y + r l_F \end{bmatrix} \quad (1.23)$$

$$\begin{bmatrix} v_x^{2L} \\ v_y^{2L} \end{bmatrix} = \begin{bmatrix} U_x - r \frac{w}{2} \\ U_y - r l_R \end{bmatrix} \quad (1.24)$$

$$\begin{bmatrix} v_x^{2R} \\ v_y^{2R} \end{bmatrix} = \begin{bmatrix} U_x + r \frac{w}{2} \\ U_y - r l_R \end{bmatrix} \quad (1.25)$$

Still missing, are the expressions for computing vertical forces F_z^j (note that $F_z^j = f_z^j$). To obtain expressions for these forces a weight transfer model has to be derived. It all starts by looking at the car confined to the $y-z$ plane. Rear view of the car in the $y-z$ plane is shown in Figure 1.2.

Looking at the Figure 1.2 one obtains the first equation of four equations in total. The first equation is expressed in the equation 1.26; it is merely a balance of torques around the center of gravity. h denotes the distance between COG and the ground.

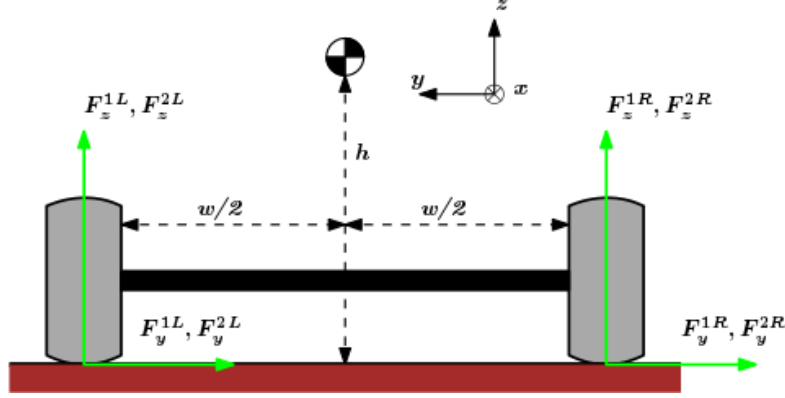


Figure 1.2: 4 wheeled car model, back view

$$(F_z^{2R} + F_z^{2L}) l_R + h(F_x^{1L} + F_x^{1R} + F_x^{2L} + F_x^{2R}) = (F_z^{1R} + F_z^{1L}) l_F \quad (1.26)$$

Second equation is obtained by looking at the torque balance in $x - z$ plane. Side view of the car is shown in Figure 1.3.

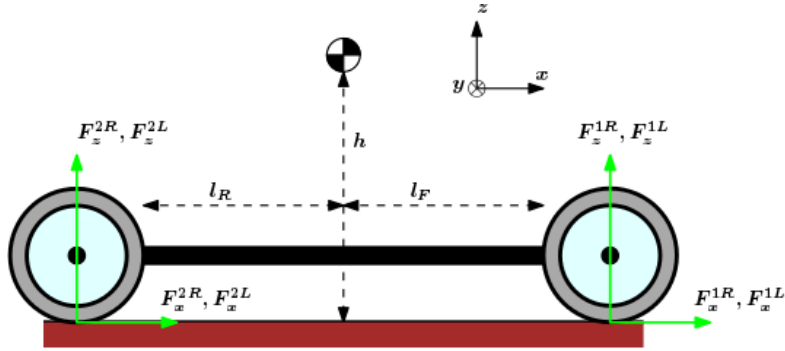


Figure 1.3: 4 wheeled car model, side view

Equation of torque balance in $x - z$ plane is given by:

$$(F_z^{1L} + F_z^{2L}) \frac{w}{2} + h(F_y^{1L} + F_y^{1R} + F_y^{2L} + F_y^{2R}) = (F_z^{1R} + F_z^{2R}) \frac{w}{2} \quad (1.27)$$

The third equation is given by:

$$F_z^{1L} + F_z^{1R} + F_y^{2L} + F_y^{2R} = mg \quad (1.28)$$

Where m is mass and g is gravitational acceleration. Finally, the last equation arises from geometrical considerations. In Figure 1.4, a deflected $x - y$ plane is shown. It is assumed that all the rotations happen around the COG.

From Figure 1.4 it can be seen that deflection on the wheel 2R has to be equal to the deflection at the diagonally opposite wheel 1L. Hence, for the deflections z , it has to hold:

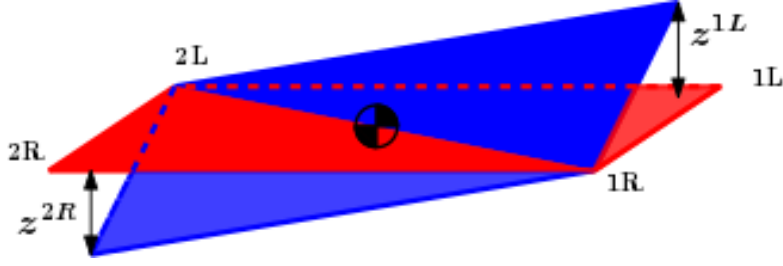


Figure 1.4: 4 wheeled car model, deflected $x - y$ plane amid weight transfer

$$z^{1L} + z^{2R} = 0 \quad (1.29)$$

$$z^{1R} + z^{2L} = 0 \quad (1.30)$$

$$(1.31)$$

If one sums up the equations 1.29 and 1.30, one obtains:

$$z^{1L} + z^{2R} = z^{1R} + z^{2L} \quad (1.32)$$

It is assumed that there is a spring on each wheel. Therefore, the following equation holds:

$$F_z^j = c^j z^j, \text{ where } j \in \{1L, 1R, 2L, 2R\} \quad (1.33)$$

In equation 1.33, z are deflections and c are spring constants. Combining equations 1.33 and 1.32 and using a simplification $c^i = c^j, \forall i, j \in \{1L, 1R, 2L, 2R\}$, one gets the fourth equation:

$$F_z^{1L} + F_z^{2R} = F_z^{1R} + F_z^{2L} \quad (1.34)$$

By plugging in the expressions for lateral and longitudinal forces (F_x, F_y) from equation 1.14 and equations 1.19 - 1.20 into equations 1.26, 1.27, 1.28 and 1.34, one obtains expressions for vertical forces F_z . These are given by the following set of equations:

$$F_z^{1L} = mg \frac{BG - CF + BH - DF - CH + DG}{den} \quad (1.35)$$

$$F_z^{1R} = mg \frac{AG - EC + AH - ED + CH - DG}{den} \quad (1.36)$$

$$F_z^{2L} = mg \frac{AF - EB + AH - ED + BH - DF}{den} \quad (1.37)$$

$$F_z^{1R} = mg \frac{AF - EB - AG + EC - BG + CF}{den} \quad (1.38)$$

$$(1.39)$$

where:

$$den = 2(AF - EB - AG + EC + BH - DF - CH + DG) \quad (1.40)$$

$$A = -\frac{w}{2} - C_1 - C_2 \quad (1.41)$$

$$B = \frac{w}{2} - C_3 - C_4 \quad (1.42)$$

$$C = -\frac{w}{2} - C_5 \quad (1.43)$$

$$D = \frac{w}{2} - C_6 \quad (1.44)$$

$$E = K_1 - K_2 - l_F \quad (1.45)$$

$$F = K_3 - K_4 - l_F \quad (1.46)$$

$$G = K_5 + l_R \quad (1.47)$$

$$H = K_6 + l_R \quad (1.48)$$

$$C_1 = -\mu\mu_x^{1L} h \sin(\delta) \quad (1.49)$$

$$C_2 = -\mu\mu_y^{1L} h \cos(\delta) \quad (1.50)$$

$$C_3 = -\mu\mu_x^{1R} h \sin(\delta) \quad (1.51)$$

$$C_4 = -\mu\mu_y^{1R} h \cos(\delta) \quad (1.52)$$

$$C_5 = -\mu\mu_y^{2L} h \quad (1.53)$$

$$C_6 = -\mu\mu_y^{2R} h \quad (1.54)$$

$$K_1 = \mu\mu_x^{1L} h \cos(\delta) \quad (1.55)$$

$$K_2 = \mu\mu_y^{1L} h \sin(\delta) \quad (1.56)$$

$$K_3 = \mu\mu_x^{1R} h \cos(\delta) \quad (1.57)$$

$$K_4 = \mu\mu_y^{1R} h \sin(\delta) \quad (1.58)$$

$$K_5 = \mu\mu_x^{2L} h \quad (1.59)$$

$$K_6 = \mu\mu_x^{2R} h \quad (1.60)$$

$$(1.61)$$

This concludes the derivation of the four-wheeled car model.

Should any errors or inconsistencies be found, feel free to correct them or make the author aware of them.