PINNs for Navier Stokes equations

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Table of Contents

Table of Contents

Introduction to Navier-Stokes equations

- Incompressible General Form:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

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- 2D Steady Incompressible Navier-Stokes eqn. reduces to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

- **Nonlinearity:** The equations involve nonlinear terms $(u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y})$, making them challenging to solve analytically or numerically.

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- Additional Transport equations (Ex: Energy conservation)

Table of Contents

Frame Title

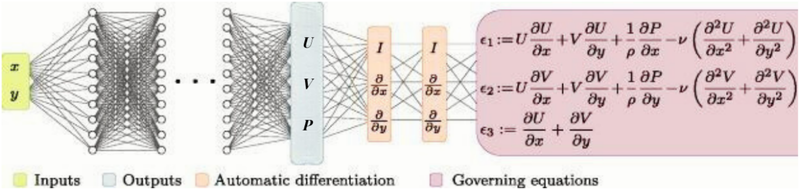

Figure: PINNs for NSE2d

Let, $Y_{pred}(\theta) = [U, V, P]_{pred}$, where θ represents the parameters of the network and $Y_{true} = [U, V, P]$ be the ground truth.

$$\text{residual loss, } L_r = w_1\epsilon_1 + w_2\epsilon_2 + w_3\epsilon_3 \tag{6}$$

$$\text{Total Loss, } L = L_r + \beta L_b \tag{7}$$

Optimization problem reads to,

$$\min_{\theta} L\left(Y_{pred}(\theta), Y_{true}\right)$$

Table of Contents

Kovasznay flow

We will solve Kovasznay flow equation on $\Omega = [0,1]^2$. The governing eqns. are 3, 4 and 5 with Dirichlet Boundary Conditions $u(x,y) = 0, (x,y) \in \partial\Omega$. The reference solution is $u = 1 - e^{\zeta x} cos(2\pi y)$, $v = \frac{\zeta}{2\pi} e^{2\pi x}$ and $p = \frac{1}{2}(1 - e^{1-e^{2\zeta x}})$, where $\zeta = \frac{1}{2\nu} - \sqrt{\frac{1}{4\nu^2} + 4\pi^2}$