3.14 RADIATION FROM HALF-WAVE DIPOLE

Radiated power by half-wave dipole, $P_T = 73.0 I_{eff}^2$

Radiation resistance of half-wave dipole, $R_r = 73\,\Omega$.

Proof   Proof consists of the following steps:

- Write expressions for the assumed current distribution in the element.
- Obtain expression for vector magnetic potential, $\mathbf{A}$.
- Obtain $\mathbf{H}$ from $\mathbf{A}$.
- Obtain $\mathbf{E}$ from $\left(\dfrac{E}{H}\right) = \eta_0$.
- Obtain average radiated power P_{av}.
- Obtain total power radiated.
- Obtain the value of radiation resistance.

The sinusoidal current distribution is represented by Fig. 3.7.

$$I = I_m \sin \beta (H - Z) \quad \text{for } z > 0$$
$$= I_m \sin \beta (H + Z) \quad \text{for } z < 0$$
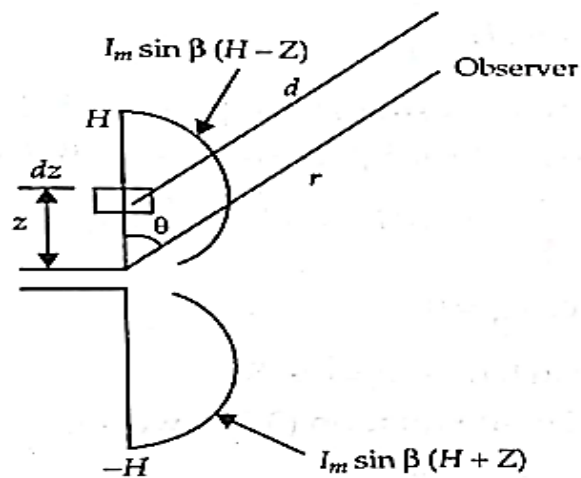
Here I_m = current maximum.

Fig. 3.7 *Dipole*

The vector potential at a point P due to the current element $I\,dz$ is given by,

$$d\mathbf{A} = dA_z\,\mathbf{a}_z = \frac{\mu_0\,Ie^{-j\beta d}\,dz}{4\pi d}\,\mathbf{a}_z \qquad\qquad ...(3.32)$$

Here d is the distance from the current element to the point P. The total vector potential at P due to all current elements is given by

$$A_z = \frac{\mu_0}{4\pi}\int_{-H}^{H}\frac{Ie^{-j\beta d}}{d}\,dz \qquad\qquad ...(3.33)$$

$$= \frac{\mu_0}{4\pi}\int_{-H}^{0}\frac{I_m \sin\beta(H+z)}{d}\,e^{-j\beta d}\,dz + \frac{\mu}{4\pi}\int_{0}^{H}\frac{I_m \sin\beta(H-z)}{d}\,e^{-j\beta d}\,dz$$

$$...(3.34)$$

It is of interest here to consider radiation fields. d in the denominator can be approximated to r. But in the numerator, d is in the phase term and it is given by

$$d = r - z \cos \theta$$

Now Equation (3.34) becomes

$$A_z = \frac{\mu_0}{4\pi} \int\limits_{-H}^{0} \frac{I_m \sin \beta (H + z)}{r} e^{-j\beta (r - z \cos \theta)} \, dz$$

$$+ \frac{\mu_0}{4\pi} \int\limits_{0}^{H} \frac{I_m \sin \beta (H - z)}{r} e^{-j\beta (r - z \cos \theta)} \, dz$$

$$= \frac{\mu_0 I_m e^{-j\beta r}}{4\pi r} \left[\int\limits_{-H}^{0} \sin \beta (H + z) e^{j\beta z \cos \theta} \, dz + \int\limits_{0}^{H} \sin \beta (H - z) e^{j\beta z \cos \theta} \, dz \right]$$

$$...(3.35)$$

For a half-wave dipole, $H = \dfrac{\lambda}{4}$

But,

$$\sin \beta (H + z) = \sin \beta H \, \cos \beta z + \cos \beta H \, \sin \beta z$$

$$\sin \beta (H - z) = \sin \beta H \, \cos \beta z - \cos \beta H \, \sin \beta z$$

As

$$\beta = \dfrac{2\pi}{\lambda}, \quad \sin \beta H = \sin \dfrac{2\pi}{\lambda} \times \dfrac{\lambda}{4} = 1,$$

$$\cos \beta H = \cos \dfrac{\pi}{2} = 0$$

So $\quad \sin \beta (H + z) = \sin \beta (H - z) = \cos \beta z$ ...(3.36)

Putting Equation (3.36) in Equation (3.35), we get

$$A_z = \dfrac{\mu_0 \, I_m \, e^{-j\beta r}}{4\pi r} \left[\int\limits_{-H}^{0} \cos \beta \, z \, e^{+j\beta z \cos \theta} + \int\limits_{0}^{H} \cos \beta \, z \, e^{+j\beta z \cos \theta} \, dz \right]$$

...(3.37)

$$\text{But} \int_{-H}^{0} \cos \beta z e^{+j\beta z \cos \theta} \, dz = \int_{0}^{H} \cos \beta z e^{-j\beta z \cos \theta} \, dz$$

$$A_z = \frac{I_m \mu_0}{4\pi r} e^{-j\beta r} \left[\int_{0}^{\lambda/4} \cos \beta z \, (e^{j\beta z \cos \theta} + e^{-j\beta z \cos \theta}) \, dz \right]$$

$$= \frac{I_m \mu_0}{4\pi r} e^{-j\beta r} \left[\int_{0}^{\lambda/4} \cos \{\beta z (1 + \cos \theta)\} + \cos \{\beta z (1 - \cos \theta)\} \right] dz$$

$$= \frac{I_m \mu_0}{4\pi r} e^{-j\beta r} \left[\int_0^{\lambda/4} \cos\{\beta z (1 + \cos\theta)\} + \cos\{\beta z (1 - \cos\theta)\} \right] dz$$

$$= \frac{I_m \mu_0}{4\pi r} e^{-j\beta r} \left[\frac{\sin\{\beta z (1 + \cos\theta)\}}{\beta (1 + \cos\theta)} + \frac{\sin\{\beta z (1 - \cos\theta)\}}{\beta (1 - \cos\theta)} \right]_0^{\lambda/4}$$

$$= \frac{\mu_0 I_m}{4\pi\beta r} e^{-j\beta r} \left[\frac{(1 - \cos\theta) \cos\left(\frac{\pi}{2} \cos\theta\right) + (1 + \cos\theta) \cos\left(\frac{\pi}{2} \cos\theta\right)}{\sin^2\theta} \right]$$

$$A_z = \frac{\mu_0 I_m}{2\pi\beta r} e^{-j\beta r} \left[\frac{\cos\frac{\pi}{2} \cos\theta}{\sin^2\theta} \right] \qquad \ldots(3.3$$

$$A_z = \frac{\mu_0 I_m}{2\pi\beta r} e^{-j\beta r} \left[\frac{\cos\frac{\pi}{2}\cos\theta}{\sin^2\theta} \right]$$

But we have

$$\mu_0 H_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r \right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r} r(-A_z \sin\theta) - \frac{\partial}{\partial\theta}(A_z \cos\theta) \right]$$

$$\mu_0 H_\phi = - \sin \theta \frac{\partial A_z}{\partial r} \qquad \qquad ...(3.39)$$

From Equations (3.38) and (3.39), we have

$$\mu_0 H_\phi = - \frac{\partial}{\partial r} \left(\frac{j \mu_0 I_m e^{-j\beta r}}{2\pi\beta r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \right) \sin\theta$$

$$H_\phi = \frac{j I_m e^{-j\beta r}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \qquad \qquad ...(3.40)$$

We also know that $E_\theta = \eta_0 H_\phi$, $\quad \eta_0 = 120\pi\Omega$

$$E_\theta = \frac{j120\pi I_m e^{j\beta r}}{2\pi r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right]$$

$$= \frac{j60 I_m e^{-j\beta r}}{r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \qquad ...(3.41)$$

The magnitude of E for the radiation field is

$$E_\theta = \frac{60 I_m}{r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \text{ V/m} \qquad \text{...(3.42)}$$

E_θ and H_ϕ are in time phase. Hence the maximum value of Poynting vector is

$$P_{max} = (E_\theta)_{max} \ (H_\phi)_{max}$$

$$= \frac{60 I_m}{r} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right] \times \frac{I_m}{2\pi r} \left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right)$$

$$= \frac{30 I_m^2}{\pi r^2} \left[\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \right] \qquad \text{...(3.43)}$$

The average value of Poynting vector is one half of the peak value.

So
$$P_{av} = \frac{15 I_m^2}{\pi r^2} \left[\frac{\cos^2 \left(\frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta} \right]$$

or
$$P_{av} = \frac{\eta_0 I_m^2}{8 \pi^2 r^2} \left[\frac{\cos^2 \left(\frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta} \right] \qquad ...(3.44)$$

Therefore, total power radiated through a spherical surface by half wave dipole is

$$P_T = \oint P_{av} ds = \frac{\eta_0 I_m^2}{8 \pi r^2} \int_0^\pi \frac{\cos^2 \left(\frac{\pi}{2} \cos \theta \right)}{\sin^2 \theta} 2 \pi r^2 \sin \theta \, d\theta$$

$$= \frac{\eta_0 I_m^2}{4 \pi} \int_0^\pi \frac{\cos^2 \left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \, d\theta \qquad ...(3.45)$$

But the numerical evaluation of the integral $\int_0^{} \dfrac{\left(\right)}{\sin \theta}\, d\theta$ by Simpson's or the Trapezoidal rule gives a value of 1.218.

So
$$P_T = \frac{\eta_0\, I_m^2}{4\pi}$$

$$= \frac{120\pi I_m^2}{4\pi} \times 1.218 = 36.54 I_m^2 \qquad \text{...(}$$

As $I_m = \sqrt{2}\, I_{eff}$, Equation (3.46) becomes

$$P_T = 36.54 \times 2 \times I_{eff}^2$$

or
$$P_T = 73.08\Omega\, I_{eff}^2, \text{ watts} \qquad \text{...(}$$

The coefficient of I_{eff}^2 is the radiation resistance. That is,

$$R_r = 73.08\Omega. \qquad \text{...(}$$

$$P_{av} = \frac{\eta_0 I_m^2}{8\pi^2 r^2} \left[\frac{\cos^2 \frac{\pi}{2} \cos \theta}{\sin^2 \theta} \right] \qquad \qquad ...(3.49)$$
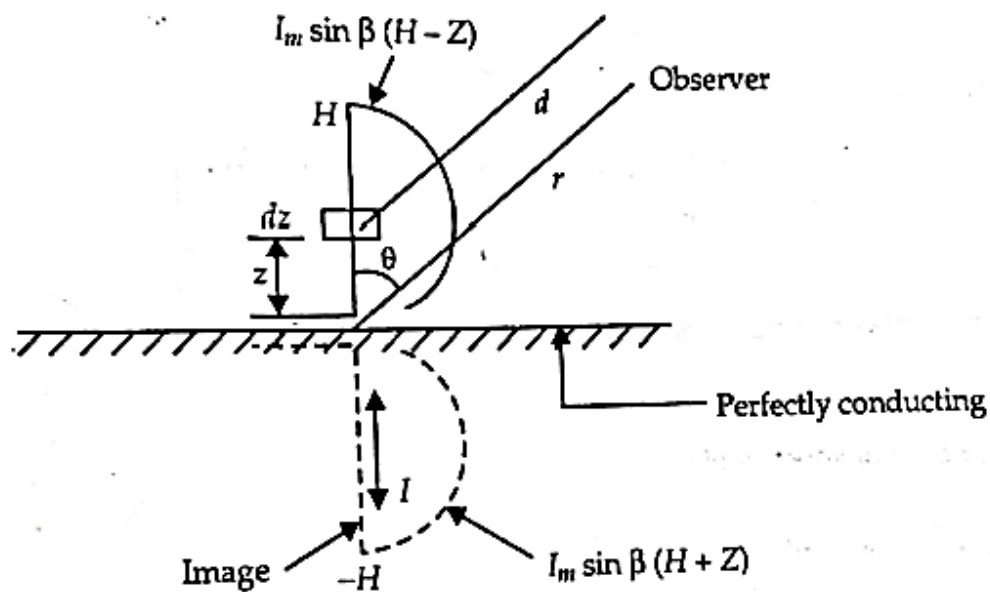

Fig. 3.8 *Monopole with current distribution*

As the monopole is fed with a perfectly conducting plane at one end, it radiates only through a hemi-spherical surface. Therefore, the total radiated power is

$$P_T = \oint P_{av} \, ds$$

$$= \frac{\eta_0 I_m^2}{8\pi r^2} \int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \, 2\pi r^2 \sin\theta \; d\theta$$

$$= \frac{\eta_0 I_m^2}{4\pi} \int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \, d\theta$$

Numerical evaluation of the integral $\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta$ by Simpson's or the Trapezoidal rule gives a value of 0.609.

So $$P_T = \frac{\eta_0 I_m^2}{4\pi} \times 0.609$$

$$= 18.27 I_m^2$$

As $$I_m = \sqrt{2}\, I_{eff}$$

$$P_T = 36.54 I_{eff}^2 \text{ watts} \qquad \qquad \text{...(3.50)}$$

The Radiation resistance, $R_r = 36.54\,\Omega$ \qquad \qquad ...(3.51)

$$\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1 + \cos(\pi\cos\theta)}{\sin\theta}\,d\theta$$

Let

$$u = \cos\theta$$

$$du = -\sin\theta\,d\theta$$

$$\frac{d\theta}{\sin\theta} = \frac{-du}{\sin^2\theta} = -\frac{du}{1-u^2}$$

$$\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta = -\frac{1}{2}\int_1^0 \frac{(1+\cos\pi u)}{1-u^2}\,du$$

$$= \frac{1}{4}\int_0^1 (1+\cos\pi u)\left(\frac{1}{1+u} + \frac{1}{1-u}\right)du$$

$$= \frac{1}{4}\int_{-1}^{+1} \frac{1+\cos\pi u}{1+u}\,du$$

Let

$$v = \pi(1+u)$$

$$dv = \pi\,du$$

$$dv = \pi\, du$$

$$\frac{dv}{v} = \frac{du}{1+u}$$

$$\pi u = v - \pi$$

$$\cos \pi u = \cos v \cos \pi + \sin v \sin \pi = -\cos v$$

Therefore
$$\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos v}{v}\, dv$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\frac{v^1}{2!} - \frac{v^3}{4!} + \frac{v^5}{6!} - \frac{v^7}{8!} + \ldots\right) dv$$

$$= \frac{1}{4}\left(\frac{v^2}{2\cdot 2!} - \frac{v^4}{4\cdot 4!} + \frac{v^6}{6\cdot 6!} - \frac{v^8}{8\cdot 8!} + \ldots\right)_0^{2\pi} \qquad \text{(10-6}$$

The series of eq. (10-62) can be evaluated by substitution. It do
not converge rapidly and so a number of terms must be used. Th
evaluation is shown below.

$$v = 2\pi = 6.2832 \qquad\qquad \log_{10} v = 0.79818$$

$$2\cdot 2! = 4 \qquad\qquad\qquad \log_{10} v^2 = 1.59636$$

$$\log_{10} 2 \cdot 2! = 0.60206$$

$$\log_{10} \frac{v^2}{2 \cdot 2!} = .99430 \qquad \frac{v^2}{2 \cdot 2!} = 9.870$$

The other terms are found in a similar manner. Using eight terms, the sum of the positive terms is 26.878 and the sum of the negative terms is 24.441. Therefore

$$\int_0^{\pi/2} \frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta = 0.6093 \qquad\qquad (10\text{-}63)$$

It is also possible to integrate such a function as

$$\int_0^{\pi/2} \frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta$$

graphically or by Simpson's or the trapezoidal rule. For example by the trapezoidal rule if θ is taken in increments of 5°, then the following table is constructed:

θ in degrees	0	5	10	15	20	25	30	35	40	45
$\dfrac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}$	0	0	.003	.011	.028	.050	.086	.138	.201	.280

θ in degrees	50	55	60	65	70	75	80	85	90
$\dfrac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}$	.369	.468	.578	.688	.788	.875	.942	.980	1.00

Now

$$\int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta = \frac{\pi}{2} \times \frac{1}{18}\left[\frac{1.000 + 0}{2} + \sum_{\theta=5^\circ}^{\theta=85^\circ} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$= \frac{\pi}{36} \times 6.987$$

$$= 0.609$$

the center to zero at the ends [Fig. 10-7(a)]. For the same current (at the terminals) the (short) practical dipole of length l will radiate only one-quarter as much power as the current element of the same length, which has the current I throughout its entire length. (The field strengths at every point are reduced to one-half, and the power density will be reduced to one-quarter.) Therefore, the radiation resistance of a practical short dipole is one-quarter that of the current element of the same length. That is

$$R_{rad} \text{ (short dipole)} = 20\pi^2 \left(\frac{l}{\lambda}\right)^2$$

$$\approx 200 \left(\frac{l}{\lambda}\right)^2 \qquad \text{ohms} \qquad (10\text{-}54a)$$
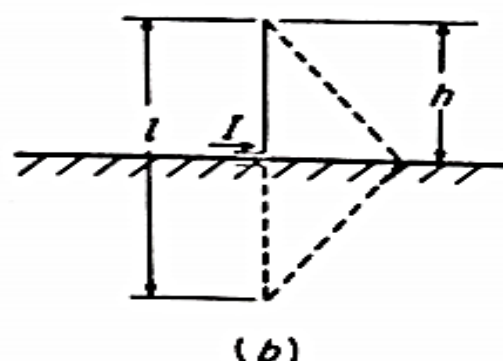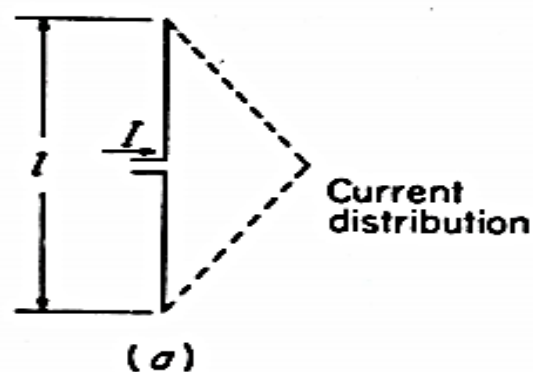
(a)

Current distribution

(b)

Figure 10-7. Current distribution on short antennas: (a) short dipole; (b) short monopole.

The monopole of height h [Fig. 10-7(b)], or short vertical antenna mounted on a reflecting plane, produces the same field strengths above the plane as does the dipole of length $l = 2h$ when both are fed with the same current. However, the short vertical antenna radiates only through the hemispherical surface above the plane, so its radiated power is only one-half that of the corresponding dipole. Therefore the radiation resistance of the monopole of height $h = l/2$ is

$$R_{rad} \text{ (monopole)} = 10\pi^2 \left(\frac{l}{\lambda}\right)^2$$

$$= 40\pi^2 \left(\frac{h}{\lambda}\right)^2$$

$$\approx 400 \left(\frac{h}{\lambda}\right)^2$$

ohms   (10-54b)

Directivity of Current Element

So
$$E = \frac{60 I\, dl \sin \theta}{\lambda r}$$

Maximum radiation occurs at
$$\theta = \frac{\pi}{2}$$

or
$$E_{max} = \frac{60 I\, dl}{\lambda r}$$

The radiated power of current element is
$$P_r = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 I^2 \text{ watts}$$

If P_r is assumed to be 1 watt, then
$$I = \frac{\lambda}{\sqrt{80}\, \pi\, dl} \text{ amp}$$

From Equations (3.54) and (3.55), we get
$$E_{(max)} = \frac{60}{r \sqrt{80}} \text{ V/m}$$

The maximum radiation intensity is given by

$$RI = \frac{r E^2_{(max)}}{\eta_0}$$

$$= \frac{r^2}{120\pi} \frac{60^2}{r^2 \times 80}$$

$$RI = \frac{3}{8\pi}$$

The maximum directive gain, g_d (max)

$$g_d \text{ (max)} = \frac{4\pi \, (RI)}{P_r}$$

As $P_r = 1$ watt

$$= \frac{4\pi \, (RI)}{P_r}$$

$$= 4\pi \times \frac{3}{8\pi}$$

$$= \frac{3}{2} = 1.5$$

The magnitude of E for the radiation field is

$$E_\theta = \frac{60 I_m}{r} \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta\right)}{\sin\theta} \right] \text{ V/m}$$

Problem 3.3   Find the directivity of a half-wave dipole.

Solution   For a half-wave dipole, from Equation (3.42)

$$E_{(max)} = \frac{60 I}{r}$$

But
$$P_r = 73 I^2 \text{ watts}$$

For
$$P_r = 1 \text{ w}$$

$$I = \frac{1}{\sqrt{73}}$$

'r
$$E_{(max)} = \frac{60}{r} \times \frac{1}{\sqrt{73}}$$

$$g_{d\,(max)} = \frac{4\pi (RI)}{P_r}$$

$$= 4\pi (RI) \qquad\qquad\qquad\qquad\qquad \text{[as } P_r = 1 \text{ watt]}$$

$$= 4\pi \times \frac{r^2 E^2}{\eta_0} \qquad\qquad\qquad\qquad\qquad \left[\text{as } RI = r^2 \frac{E^2}{\eta_0} \right]$$

$$= 4\pi \, (RI)$$

<div align="right">[as $P_r = 1$ watt]</div>

$$= 4\pi \times \frac{r^2 E^2}{\eta_0}$$

<div align="right">$\left[\text{as } RI = r^2 \dfrac{E^2}{\eta_0} \right]$</div>

$$= \frac{4\pi \times r^2}{\eta_0} \, \frac{(60)^2}{r^2} \, \frac{1}{73}$$

$$= \frac{4\pi \times (60) \times (60)}{120\pi} \, \frac{1}{73}$$

$$= \frac{120}{73} = 1.644$$

So $\qquad z_d \, (\text{max}) = D = 1.644.$