a relatively narrow frequency band in the neighbourhood of f_0.

→ Problems :

Prove that ① $E[n_c^2(t)] = E[n_s^2(t)]$

② $E[n_c(t)\, n_s(t)] = 0$.

Ans] ① $E[n_c^2(t) = E[n_s^2(t)]$

$n_c(t)$ & $n_s(t)$ are stationary random processes &

we know that $E[a_k^2] = E[b_k^2]$ → equal variance

$E[a_k b_k] = 0$ → uncorrelated to each other.

$$n_c(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} \left[a_k \cos 2\pi (k-K)\Delta f t + b_k \sin 2\pi (k-K)\Delta f t \right]$$

& $$n_s(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} \left[a_k \sin 2\pi (k-K)\Delta f t - b_k \cos 2\pi (k-K)\Delta f t \right]$$

$$\therefore E[n_c^2(t)] = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\left[a_k^2 \cos^2 2\pi (k-K)\Delta f t + b_k^2 \sin^2 2\pi (k-K)\Delta f t + 2 a_k b_k \cos 2\pi (k-K)\Delta f t \sin 2\pi (k-K)\Delta f t \right]$$

$$= \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\left[a_k^2 (\cos^2 2\pi (k-K)\Delta f t + \sin^2 2\pi (k-K)\Delta f t \right] + 0$$

$$\left\{ \because E[a_k^2] = E[b_k^2] \text{ & } E[a_k b_k] = 0 \right\}$$

$$= \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E[a_k^2] \qquad \left\{ \because \cos^2\theta + \sin^2\theta = 1 \right\}$$

$$E[n_s^2(t)] = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\left[a_k^2 \sin^2 2\pi (k-K)\Delta f t + b_k^2 \cos^2 2\pi (k \right.$$

$$\left. - 2 a_k b_k \cos 2\pi (k-K)\Delta f t \sin 2\pi (k- \right.$$

$$= \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\left[a_k^2 \left(\sin^2 2\pi (k-K)\Delta f t + \cos^2 2\pi [k-K] \right. \right.$$

$$+ 0$$

$$\Rightarrow E[n_c^2(t)] = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E[a_k^2]$$

$$\therefore E[n_c^2(t)] = E[n_c^2(t)]$$

(2) $E[n_c(t)\, n_s(t)]$

$$\Rightarrow \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\Big[a_k \cos 2\pi(k-K)\Delta f t + b_k \sin 2\pi(k-K)\Delta f t \Big]$$
$$\Big[a_k \sin 2\pi(k-K)\Delta f t - b_k \cos 2\pi(k-K)\Delta f t \Big]$$

$$\Rightarrow \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\Big[a_k^2 \cos 2\pi(k-K)\Delta f t \sin 2\pi(k-K)\Delta f t + b_k a_k \sin^2 2\pi(k-K)\Delta f t - a_k b_k \cos^2 2\pi(k-K)\Delta f t - b_k^2 \sin 2\pi(k-K)\Delta f t \cos 2\pi(k-K)\Delta f t \Big]$$

we know that for a stationary random process

$E[a_k b_k] = 0 \longrightarrow$ Uncorrelated to each other

$E[a_k^2] = E[b_k^2] \rightarrow$ Equal variance.

$\therefore$ we have

$$E[n_c(t)\, n_s(t)] = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\Big[a_k^2 \cos 2\pi(k-K)\Delta f t \sin 2\pi(k-K)\Delta f t + 0 - 0 - b_k^2 \sin 2\pi(k-K)\Delta f t \cos 2\pi(k-K)\Delta f t \Big]$$

$$= \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} E\Big[a_k^2 \cos 2\pi(k-K)\Delta f t \sin 2\pi(k-K)\Delta f t - a_k^2 \sin 2\pi(k-K)\Delta f t \cos 2\pi(k-K)\Delta f t \Big]$$

$= 0 \quad \therefore E[n_c(t)\, n_s(t)] = 0$

Problem :-

→ Find the value of $E[\dot{n}_c(t)\, n_s(t)]$ & $E[n_c(t)\, \dot{n}_s(t)]$

where $\dot{n}_c(t) = \frac{d}{dt} n_c(t)$ &

$\dot{n}_s(t) = \frac{d}{dt} n_s(t)$

Answer :- 0

→ Power Spectral Density of Quadrature Components

(or) [PSD of $n_d(t)$, $n_c(t)$]

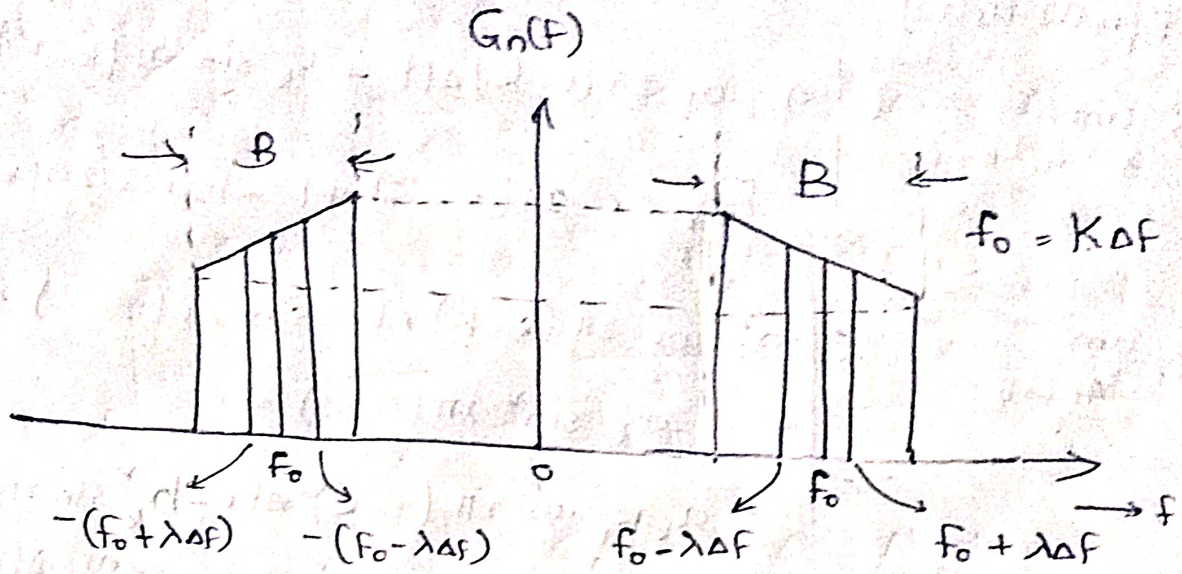


fig :- Power Spectrum of bandlimited noise

Consider non-white noise is having non-uniform noise PSD and f_0 need not be located at its centre. To determine the PSD of $n_c(t)$,

① select the spectral components of noise $n(t)$ from the value of $k = (K-\lambda)$ to $(K+\lambda)$ where λ is an integer.

Since $K\Delta f = f_0$, the selected components correspond to frequencies $(K-\lambda)\Delta f$ to $(K+\lambda)\Delta f$ i.e., from $(f_0-\lambda\Delta f)$ to $(f_0+\lambda\Delta f)$. We know $n_c(t)$ as

$$n_c(t) = \lim_{\Delta f \to 0} \sum_{k=1}^{\infty} \left[a_k \cos 2\pi(k-K)\Delta f t + b_k \sin 2\pi(k-K)\Delta f \right.$$

Find $\Delta n_c(t)$ within this interval i.e,

$$\boxed{\Delta n_c(t) = n_c(t)\Big|_{k=K-\lambda} + n_c(t)\Big|_{k=K+\lambda}} \longrightarrow (1)$$

$$\Rightarrow \Delta n_c(t) = a_{K-\lambda} \cos 2\pi (K-\lambda-K)\Delta f t + b_{K-\lambda} \sin 2\pi(-\lambda)\Delta f t$$

$$+ a_{K+\lambda} \cos 2\pi (K+\lambda-K)\Delta f t + b_{K+\lambda} \sin 2\pi(\lambda)\Delta f t$$

$$\Rightarrow \Delta n_c(t) = a_{K-\lambda} \cos 2\pi\lambda\Delta f t - b_{K-\lambda} \sin 2\pi\lambda\Delta f t +$$

$$a_{K+\lambda} \cos 2\pi\lambda\Delta f t + b_{K+\lambda} \sin 2\pi\lambda\Delta f t$$

These 4 terms are of same freq $\lambda\Delta f$ and are uncorrelated random processes.

(2) These random processes are stationary, consider at time $t = t_1$; $\cos 2\pi\lambda\Delta f t_1 = 1$ & $\sin 2\pi\lambda\Delta f t_1 = 0$ where $\lambda\Delta f t_1$ is integer

then

$$\Delta n_c(t_1) = a_{K-\lambda}(1) - 0 + a_{K+\lambda}(1) + 0$$

$$\Rightarrow \boxed{\Delta n_c(t_1) = a_{K-\lambda} + a_{K+\lambda}} \longrightarrow (2)$$

(3) Find the power P_λ associated with $\Delta n_c(t_1)$ and interrelate P_λ with PSD $G_{nc}(f)$ & $G_n(f)$

$$\therefore P_\lambda = E\left[\Delta n_c(t_1)^2\right]$$

$$= E\left[(a_{K-\lambda} + a_{K+\lambda})^2\right]$$

$$= E\left(a_{K-\lambda}^2\right) + E\left[a_{K+\lambda}^2\right] + \underbrace{2E\left[a_{K-\lambda} a_{K+\lambda}\right]}_{\hookrightarrow 0 \text{ : uncorrelated to each other}}$$

$$\Rightarrow \boxed{P_\lambda = E\left[a_{K-\lambda}^2\right] + E\left[a_{K+\lambda}^2\right]} \longrightarrow (3)$$

We know that $P_k = 2\,G_n(k\Delta f)\,\Delta f$

$\therefore P_\lambda = 2\,G_{nc}(\lambda\Delta f)\,\Delta f$

$F[a_{k-\lambda}^2] = P_{k-\lambda} = 2\,G_n((k-\lambda)\,\Delta f)\,\Delta f$

$F[a_{k+\lambda}^2] = P_{k+\lambda} = 2\,G_n((k+\lambda)\,\Delta f)\,\Delta f$

$\therefore$ eq ③ $\Rightarrow P_\lambda = F[a_{k-\lambda}^2] + F[a_{k+\lambda}^2]$

$\Rightarrow P_\lambda = \overline{a_{k-\lambda}^2} + \overline{a_{k+\lambda}^2}$

$\Rightarrow 2\,G_{nc}(\lambda\Delta f)\,\Delta f = 2\,G_n((k-\lambda)\,\Delta f)\,\Delta f + 2\,G_n((k+\lambda)\,\Delta f)\,\Delta f$

$\Rightarrow$ $\boxed{G_{nc}(\lambda\Delta f) = G_n[(k-\lambda)\,\Delta f) + G_n((k+\lambda)\,\Delta f)}$ → ④

We know that $f_0 = K\Delta f$, replace $\lambda\Delta f$ by a continuous freq. variable f, $\therefore$ eq ④ becomes

$$\boxed{G_{nc}(f) = G_n(f_0 - f) + G_n(f + f_0)}$$

Similarly the same procedure can be adapted to find the PSD of $n_s(t)$ & a similar expression can be obtained for $n_s(t)$

$$\boxed{G_{ns}(f) = G_{nc}(f) = G_n(f_0 - f) + G_n(f_0 + f)}$$

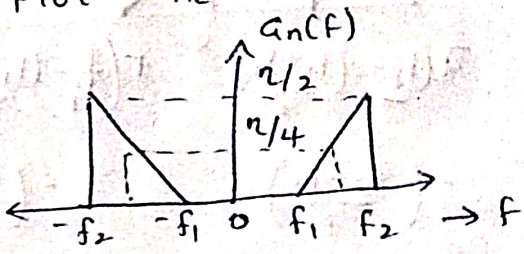
for $n_s(t)$:- $P_\lambda = \overline{b_{k-\lambda}^2} + \overline{b_{k+\lambda}^2}$ we know that $\overline{a_k^2} = \overline{b_k^2}$

$= \overline{a_{k-\lambda}^2} + \overline{a_{k+\lambda}^2}$ & remaining is same.

To get the PSD of either $n_c(t)$ or $n_s(t)$ graphically displace +ve & -ve portions of $G_n(f)$ towards left & right sides respectively by an amount f_0, so that the portion of the plot originally located at f_0 now coincide with the ordinate and by adding these two plots, the plot for either $n_c(t)$ or $n_s(t)$ can be obtained

→ Problem :-

The noise PSD for noise n(t) is shown in fig. Plot the PSD for $n_c(t)$ or $n_s(t)$ i.e, plot $G_{nc}(f)$ or $G_{ns}(f)$



① $f_0 = f_1$

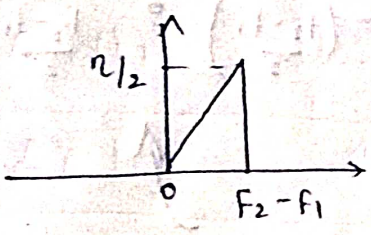
② $f_0 = f_2$

③ $f_0 = \frac{f_1 + f_2}{2}$

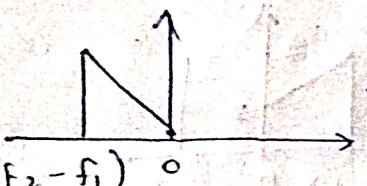
Ans] ① $f_0 = f_1$ $\quad G_{nc}(f) = G_n(f_0 - f) + G_n(f_0 + f)$

$= G_{ns}(f)$

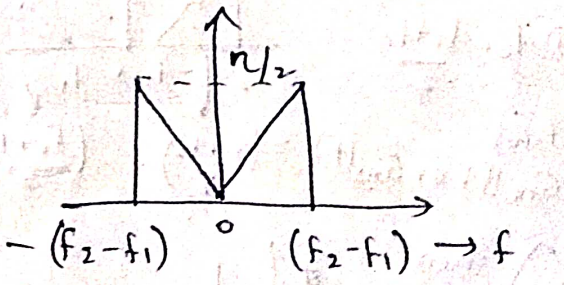
$G_n(f_0 - f)$



$G_n(f_0 + f)$

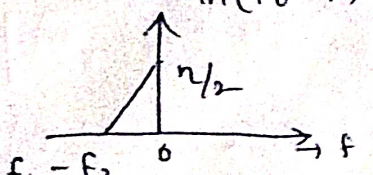


$G_{nc}(f)$ or $G_{ns}(f)$

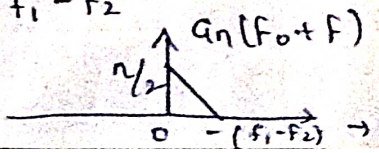


② $f_0 = f_2$

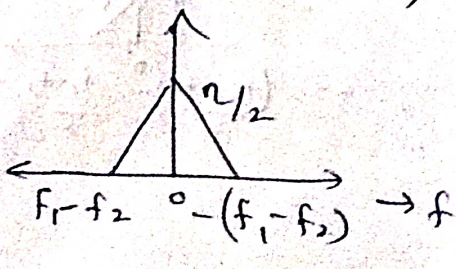
$G_n(f_0 - f)$



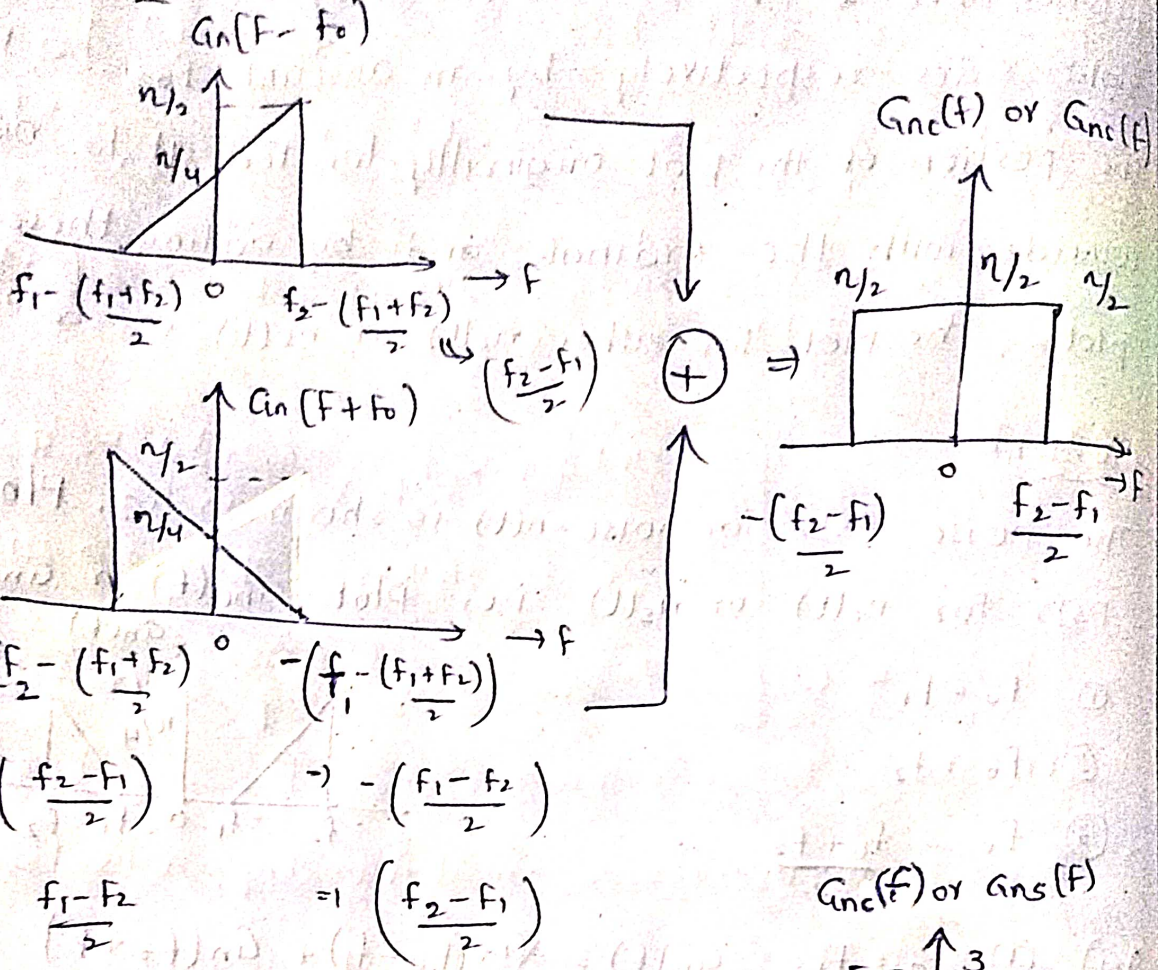
$G_n(f_0 + f)$



$G_{nc}(f)$ or $G_{ns}(f)$



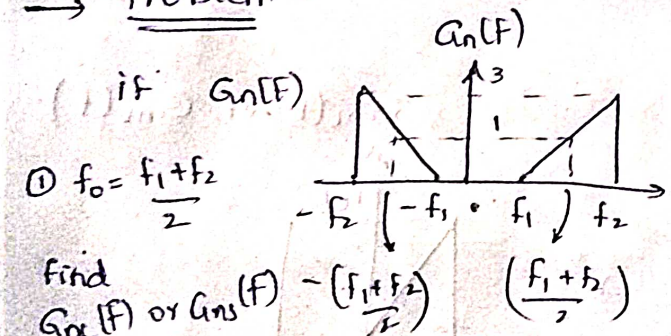
③ $f_0 = \frac{f_1 + f_2}{2}$



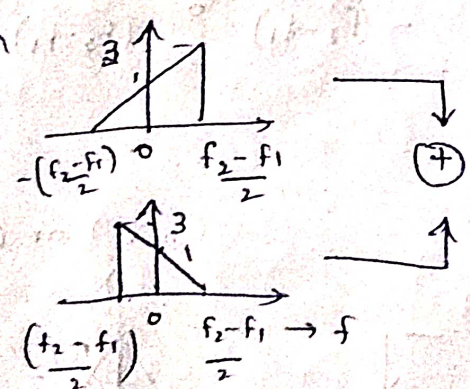
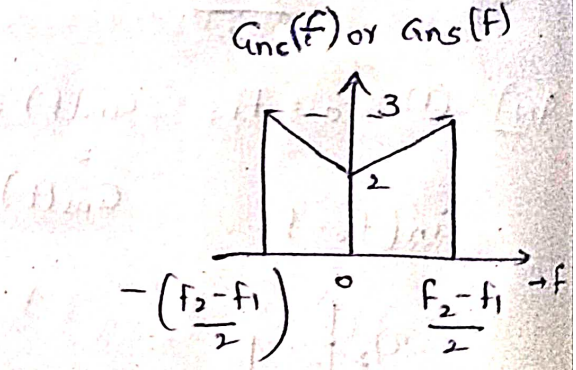
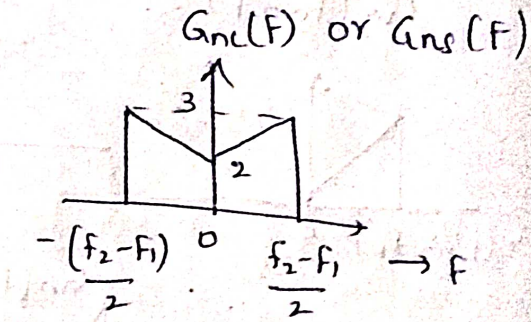
$\Rightarrow -\left(\frac{f_2-f_1}{2}\right) \quad -\left(\frac{f_1-f_2}{2}\right)$

$\Rightarrow \frac{f_1-f_2}{2} = 1\left(\frac{f_2-f_1}{2}\right)$

→ Problem :-



① $f_0 = \frac{f_1+f_2}{2}$

Find $G_{nc}(F)$ or $G_{ns}(f)$ $-\left(\frac{f_1+f_2}{2}\right)$ $\left(\frac{f_1+f_2}{2}\right)$

Ans then