UNIT – I

LINEAR WAVE SHAPING

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High pass, low pass RC circuits-response to sinusoidal, step, pulse, square and ramp inputs, The High pass RC circuit as a differentiator and the Low pass RC circuit as an integrator, Attenuators.

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A linear network is a network made up of linear elements only. A linear network can be described by linear differential equations. The principle of superposition and the principle of homogeneity hold good for linear networks. In pulse circuitry, there are a number of waveforms, which appear very frequently. The most important of these are sinusoidal, step, pulse, square wave, ramp, and exponential waveforms. The response of *RC*, *RL*, and *RLC* circuits to these signals is described in this chapter. Out of these signals, the sinusoidal signal has a unique characteristic that it preserves its shape when it is transmitted through a linear network, i.e. under steady state, the output will be a precise reproduction of the input sinusoidal signal. There will only be a change in the amplitude of the signal and there may be a phase shift between the input and the output waveforms. The influence of the circuit on the signal may then be completely specified by the ratio of the output to the input amplitude and by the phase angle between the output and the input. No other periodic waveform preserves its shape precisely when transmitted through a linear network, and in many cases the output signal may bear very little resemblance to the input signal.

The process whereby the form of a non-sinusoidal signal is altered by transmission through a linear network is called linear wave shaping.

Definitions:

Linear elements: Resistor, capacitors and inductors are called linear elements because the current passing to the elements is proportional to the applied voltage, there is a linear relation between current and voltage.

Linear network: Circuit designed with linear elements Resistors, Capacitor and Inductors is called linear network.

When a sinusoidal signal applied to linear network the output also sinusoidal in nature but a non-sinusoidal signal response is different.

Linear wave shaping: A Signal can also be called as a **Wave**. If a circuit is designed with components like R, L and C then it is called linear circuit. When sinusoidal signal is applied, the shape of the signal is preserved at the output with or without change in the amplitude and shape. But a non-sinusoidal signal alters the output when it is transmitted through a linear circuit.

The process whereby the form of non-sinusoidal signals such as step, pulse, square wave, ramp and exponential is altered by transmission through a linear network is called linear wave shaping.

1.1 High-Pass RC circuit:

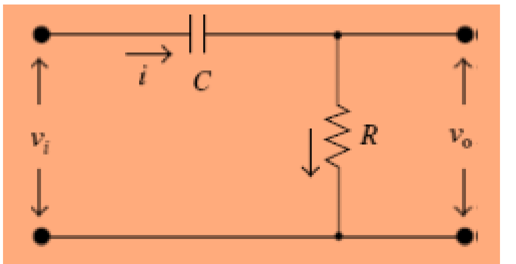


Figure 1.1 High Pass RC Circuit

Consider high pass RC circuit as shown in Figure1.1. At zero frequency the reactance of the capacitor is infinity and so it blocks the input and hence the output is zero. Hence, this capacitor is called the *blocking capacitor* and this circuit, also called the *capacitive coupling circuit*, is used to provide dc isolation between the input and the output. As the frequency increases, the reactance of the capacitor decreases and hence the output and gain increase.

At very high frequencies, the capacitive reactance is very small so a very small voltage appears, across *C* and, so the output is almost equal to the input and the gain is equal to 1. Since this circuit attenuates low-frequency signals and allows transmission of high-frequency signals with little or no attenuation, it is called a high-pass circuit.

The capacitor offers high reactance at low frequency and low reactance at high frequency. Hence low frequency components are not transmitted, but high frequencies are with less attenuation. Therefore, the output is large and the circuit is called a high pass circuit. Let us see now is, what will be the response if different types of inputs, such as, sinusoidal, step, pulse, square wave, exponential and ramp are applied to a high pass circuit. Consider the high-pass *RC* circuit shown in Figure1.1. The capacitor offers a low reactance ($X_c = 1/j\omega C$) as the frequency increases; hence, the output is large. Consequently, high- frequency signals are passed to the output with negligible attenuation whereas, at low frequencies, due to the large reactance offered by the condenser, the output signal is small. Thus, in the high-pass circuit of Figure 1.1, *C* appears as a series element. The time constant τ is given by: $\tau = RC$.

1.2 High-Pass RC Response for a Sinusoidal Input:

Figure 1.2 (a) shows the Laplace transformed high-pass *RC* circuit. The gain versus frequency curve of a high-pass circuit excited by a sinusoidal input is shown in Figure 1.2(b). For a sinusoidal inputv_i, the output signal v_o increases in amplitude with increasing frequency. The frequency at which the gain is $\frac{1}{\sqrt{2}}$ of its maximum value is called the lower cut-off or lower 3-dB frequency. For a high-pass circuit, there is no upper cut-off frequency because all high frequency signals are transmitted with zero attenuation. Therefore $f_2 = \infty$. Hence bandwidth = B.W. = $f_2 - f_1 = \infty$.

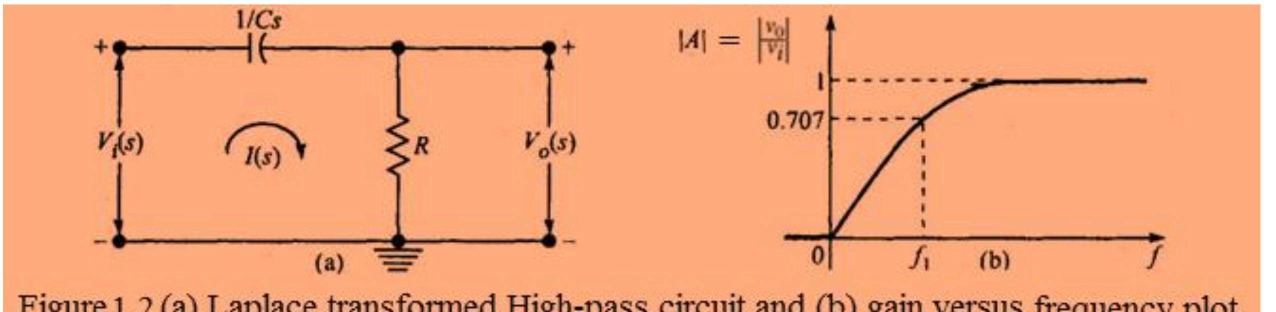


Figure 1.2 (a) Laplace transformed High-pass circuit and (b) gain versus frequency plot.

Expression for the lower cut-off frequency:

For the high-pass *RC* circuit shown in Figure 1.2 (a), the magnitude of the steady-state gain *A*, and the angle θ by which the output leads the input are given by

The output voltage is $V_o(s) = V_i(s)\frac{R}{R+1/j\omega\, C\, s}$

$$\therefore A = \frac{V_o(s)}{V_i(s)} = \frac{R}{R+\frac{1}{Cs}} = \frac{1}{1+\frac{1}{RCs}}$$

$$Putting\ \ s = j\omega,\ A = \frac{R}{R - j\frac{1}{\omega\, R\, C}} = \frac{1}{1 - j\frac{1}{2\,\pi\, f\, R\, C}}$$

$$\therefore |A| = \frac{1}{\sqrt{1+\left(\frac{1}{2\,\pi\, f\, R\, C}\right)^2}} \text{ and } \theta = -\,tan^{-1}\frac{1}{2\,\pi\, f\, R\, C}$$

At the lower cut-off frequency f_1, $|A| = \frac{1}{\sqrt{2}}$, $|A| = \left|\frac{V_o}{V_i}\right| = \frac{1}{\sqrt{2}} = 0.707$

$$\therefore \frac{1}{\sqrt{1+\left(\frac{1}{2\,\pi\, f_1\, R\, C}\right)^2}} = \frac{1}{\sqrt{2}}$$

Squaring and equating the denominators,

$$\therefore \frac{1}{2\,\pi\, f_1\, R\, C} = 1$$

$\Rightarrow f_1 = \frac{1}{2\,\pi\, R\, C}$ --1.1

This is the expression for the lower cut-off frequency of a high-pass circuit.

Therefore at $\omega = \omega_1$ *or* $f = f_1$, the gain drops to a value equal to 0.707 times its maximum value. i.e. Gain = 0.707 |*A*|.

Here, f_1 is the lower cut-off frequency of the high pass circuit. The signal undergoes a phase change and the phase angle, *θ*, is given by: *θ* = tan−1 (*ω*₁/*ω*) = tan−1 (*T/τ*). A typical frequency response curve for a sinusoidal input is shown in Figure 1.2(b).

Relation Between f_1 and Tilt:

The lower cut-off frequency of a high-pass circuit is /i = *\/2nRC*. The lower cut-off frequency produces a tilt. For a 10% change in capacitor voltage, the time or pulse width involved is

$$t = 0.1\, R\, C = PW$$

$$\therefore \frac{PW}{RC} = 0.1 = Fractional\ tilt$$

$\therefore\ Fractional\ tilt = \frac{PW}{RC} = 2\,\pi\, f_1 \times PW$ --1.2

This equation applies only when the tilt is 10% or less. When the tilt exceeds 10%, the voltage should be treated as exponential instead of linear and the equation $V_o = V_f - (V_f - V_i)\, e^{-t/RC}$ should be applied.

1.3 High-Pass RC Response for a Step Input:

A step is a sudden change in voltage, say at an instant t = 0, from say zero to V, in which case, it is called a positive step. The voltage change could also be from zero to -V, in which case it is called a negative step. This is an important signal in pulse and digital circuits.

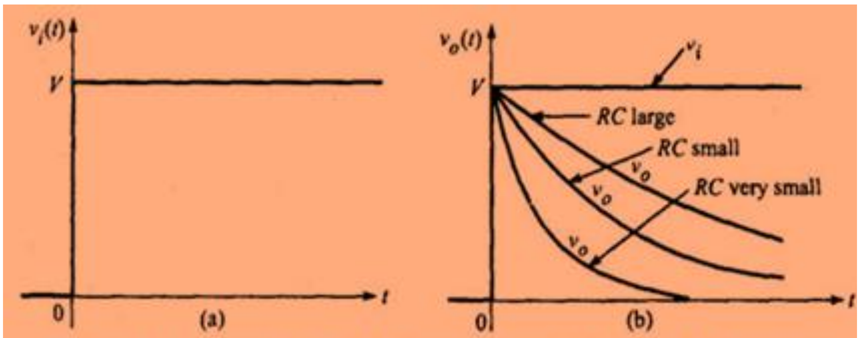


Figure 1.3 (a) Step input and (b) step response for different time constants.

$$\left.\begin{aligned} v_i &= 0 \quad for \quad t<0 \\ v_i &= V \quad for \quad t\geq 0 \end{aligned}\right\}$$

When a step signal of amplitude V volts shown in Figure1.3(a) is applied to the high-pass RC circuit of Figure1.1, since the voltage across the capacitor cannot change instantaneously the output will be just equal to the input at t = 0 (for t < 0, v_i = 0 and v_o = 0). Later when the capacitor charges exponentially, the output reduces exponentially with the same time constant RC. The expression for the output voltage for t > 0 is given by

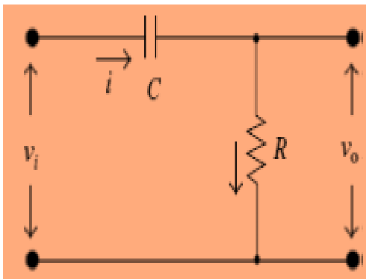


Figure 1.1 High Pass RC Circuit

A Step voltage as shown in Figure 1.3 (a), is defined as,

$$\left.\begin{aligned} v_i &= 0 \quad for \quad t<0 \\ v_i &= V \quad for \quad t\geq 0 \end{aligned}\right\} \text{------------------------------------}1.3$$

Considering the high pass RC circuit as shown in Figure1.1

For a step input, let the output voltage be of the form

$$v_o = B_1 + B_2 e^{-t/\tau} \text{------------------------------------}1.4$$

Where, $\tau = RC$ is the time constant of the circuit.

B_1 is the steady-state value of the output voltage because as $t \longrightarrow \infty$, $v_o \longrightarrow B_1$.

Let the final value of this output voltage be called v_f. Then

$$v_f = B_1 \text{------------------------------------}1.5$$

B_2 is determined by the initial output voltage. At t = 0, when the step voltage is applied, the change at the output is the same as the change at the input, because a capacitor is connected between the input and the output. Hence, $v_o = v_i$

$$v_i = v_O = B_1 + B_2$$

$$\therefore \quad B_2 = v_i - B_1$$

Using Eq. 1.5 we get,

$$B_2 = v_i - v_f \text{------------------------------------}1.6$$

Substituting the values of B_1 and B_2 from Eqs. (1.5) and (1.6) respectively in Eq. (1.4), the general solution for a wave changing exponentially is given by the relation

$v_o = v_f + (v_i - v_f)e^{-t/\tau}$---1.7

For step response of a high-pass *RC* circuit, let us calculate v_i and v_f. As the capacitor blocks the dc component of the input, $v_f = 0$. Since the capacitor does not allow sudden voltage changes, a change in the voltage of the input signal is necessarily accompanied by a corresponding change in the voltage of the output signal. Hence, at $t = 0^+$ when the input abruptly rises to V, the output also changes by V. Therefore, $v_i = V$.

Substituting the values of v_f and v_i in Eq. (1.7), we get

$v_o = Ve^{-t/\tau}$--1.8

At $t = 0$, when a step voltage V is applied as input to the high-pass circuit, as the capacitor will not allow any sudden changes in voltage, it behaves as a short circuit. Hence, the input voltage V appears at the output. As the input remains constant, the charge on the capacitor discharges exponentially with the time constant τ. The response of the step signal for high pass RC circuit is shown in Figure 1.4.

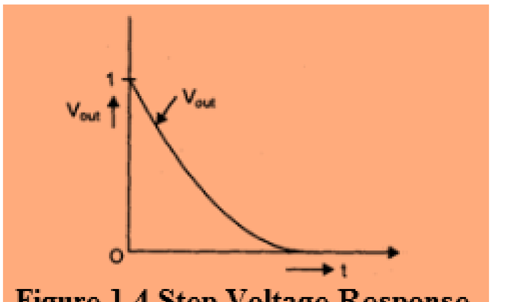


Figure 1.4 Step Voltage Response of High-Pass RC Circuit

$V_o = V_f - (V_f - V_i)\, e^{-t/RC} \;=\; 0 - (0 - V)\; e^{-t/RC} \;=\; V\, e^{-t/RC}$----------------------------1.9

Figure 1.3(b) shows the response of the circuit for large, small, and very small-time constants. For t > 5r, the output will reach more than 99% of its final value. Hence although the steady state is approached asymptotically, for most applications we may assume that the final value has been reached after 5f. If the initial slope of the exponential is maintained, the output falls to zero in a time t = T. The voltage across a capacitor can change instantaneously only when an infinite current pass through it, because for any finite current *i(t)* through the capacitor, the instantaneous change in voltage across the capacitor is given by $\frac{1}{C}\int_0^0 i(t)dt = 0$

1.4 High-Pass RC Response for a Pulse Input:

A positive pulse is mathematically represented as the combination of a positive step followed by a delayed negative step i.e., $v_i(t) = Vu(t) - Vu(t - t_p)$ where, t_p is the duration of the pulse and it is shown in Figure1.5.

A pulse of amplitude V and duration t_p shown in Figure1.5(c) is nothing but the sum of a positive step of amplitude V starting at $t = 0$ shown in Figure1.5(a) and a negative step of amplitude V starting at t_p as shown in Figure1.5(b). Considering the high pass RC circuit as shown in Figure1.1. At $t = 0$, v_i abruptly rises to V. As a capacitor is connected between the input and output, the output also changes abruptly by the same amount. As the input remains constant, the output decays

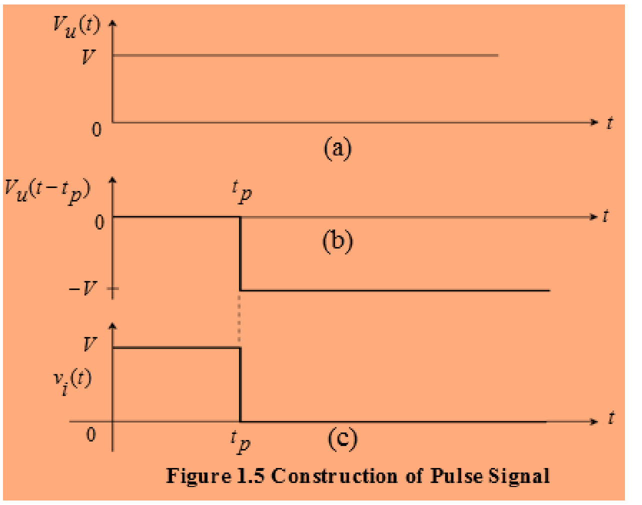


Figure 1.5 Construction of Pulse Signal

exponentially to V_1 at $t = t_p$. Therefore,

$$V_o = V_f - (V_f - V_i)\, e^{-t/RC}\, .Here\ V_f = 0, V_i = V$$
$$\therefore V_o = 0 - (0 - V)\ e^{-t/RC} = V\, e^{-t/RC}.\quad At\ \ t = t_p, \quad V_o = V_1\ \ and\ \ RC = \tau$$

$V_1 = V\, e^{-t_p/\tau}$---1.10

At $t = t_p$, the input abruptly falls by V, v_o also falls by the same amount. In other words, $v_o = V_2 = V_1 - V$. Since V_1 is less than V, v_o is negative and its value is V_2 and this decays to zero exponentially. For $t > t_p$,

$$V_o = V_f - (V_f - V_i)\, e^{(-t-t_p)/RC}\, .Here\ V_f = 0, V_i = V_2$$
$$\therefore V_o = 0 - (0 - V_2)\ e^{(-t-t_p)/RC} = V_2\, e^{(-t-t_p)/RC}.\quad At\ \ t = t_{p+}, \quad V_o = V_1 - V$$

$v_o = (V_1 - V)e^{-(t-t_p)/\tau}$---1.11

Substituting Eq. (1.10) in Eq. (1.11), we get

$$v_o = (V\, e^{-t_p/\tau} - V)e^{-(t-t_p)/\tau}$$

$v_o = V(\, e^{-t_p/\tau} - 1)e^{-(t-t_p)/\tau}$--1.12

The response of pulse input for high-pass circuit is plotted in Figure1.6.

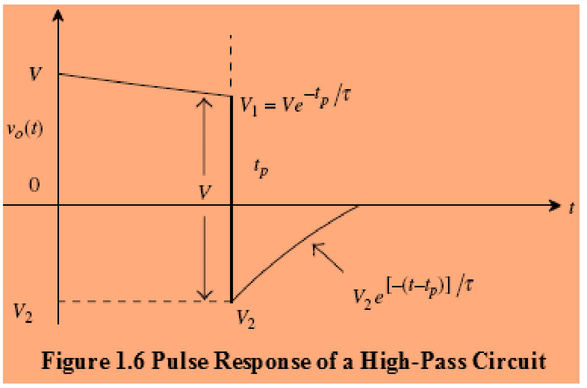


Figure 1.6 Pulse Response of a High-Pass Circuit

The response of high-pass circuits with different values of τ to pulse input is plotted in Figure1.7.

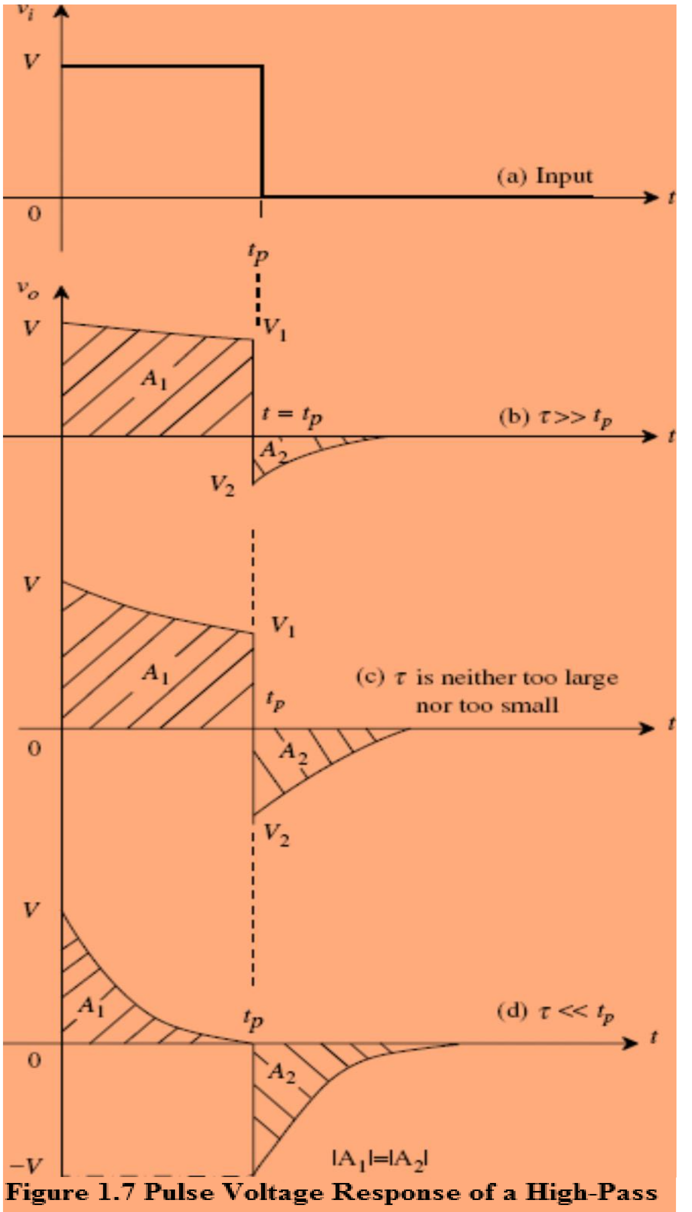


Figure 1.7 Pulse Voltage Response of a High-Pass RC Circuit for Different Time Constants

1.4.1 Verification of $|A_1| = |A_2|$:

The area (A_1) above the reference level is the same as the area (A_2) below the reference level.

Let us verify this using Figure1.8.

Area $A1$:

$For \;\; 0 < t < t_p \;\; the \; output \; voltage \; is \; given \; by \; v_o = Ve^{-t/\tau}$.

So, the area A_1 under the output wave form during $0 < t < t_p$ is given by

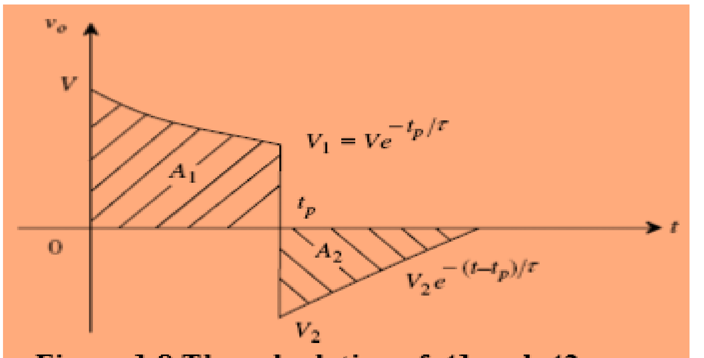


Figure 1.8 The calculation of *A*1 and *A*2

$$A_1 = \int_0^{t_p} Ve^{-t/\tau}\,dt = \left[-V\tau\, e^{-t/\tau}\right]_0^{t_p}$$

$$A_1 = \left[-V\,\tau\, e^{--t_p/\tau} + V\tau\right] = V\tau(1 - e^{-t_p/\tau}) \text{ --1.13}$$

Similarly, the output wave form when $t > t_p$ is $v_o = V(\,e^{-t_p/\tau} - 1)e^{-(t-t_p)/\tau}$

The area under the output wave form when $t > t_p$

$$A_2 = \int_{t_p}^{\infty} V(\,e^{-t_p/\tau} - 1)e^{-(t-t_p)/\tau}dt = \int_{t_p}^{\infty}\{Ve^{-t_p/\tau} - Ve^{-(t-t_p)/\tau}\}dt$$

$$= \left[\frac{Ve^{-t_p/\tau}}{\frac{-1}{\tau}}\right]_{t_p}^{\infty} - \left[\frac{Ve^{-(t-t_p)/\tau}}{\frac{-1}{\tau}}\right]_{t_p}^{\infty}$$

$$A_2 = \left[V\tau\, e^{-t_p/\tau} - V\tau\right] = -V\tau(1 - e^{-t_p/\tau}) \text{ --1.14}$$

From the above Equations (1.13) and (1.14) it is evident that

$$|A_1| = |A_2| \text{--1.15}$$

Example Problem 1: A pulse of amplitude 10 V and duration 10 μs is applied to a high-pass RC circuit. Sketch the output waveform indicating the voltage levels for (i) $RC = t_p$, (ii) $RC = 0.5t_p$ and (iii) $RC = 2t_p$.

Solution:

(i) When $RC = t_p = \tau$, at $t = t_p$

$V_1 = 10\, e^{-(10\times10-6)/(10\times10-6)} = 10-1 = 3.678$ V

$v_{o1} = 10\, e^{-t/(10\times10-6)}$ for $t < t_p$

$v_o\,(t > t_p) = (V_1 - 10)e^{-(t-10\times10-6)/(10\times10-6)} = -6.322\, e^{-(t-10\times10-6)/(10\times10-6)}$

(ii) When $RC = \tau = 0.5\, t_p$, at $t = t_p$

$V_1 = 10\ e^{-(10 \times 10-6)/(0.5 \times 10 \times 10-6)} = 10e^{-2} = 1.35$ V

$v_{o1} = 10\ e^{-t/(0.5 \times 10 \times 10-6)}$ for $t < t_p$

$v_o\ (t > t_p) = -8.65\ e^{-(t-10 \times 10-6)/(0.5 \times 10 \times 10-6)}$

(iii) When $RC = \tau = 2\ t_p$, at $t = t_p$

$V_1 = 10\ e^{-(10 \times 10-6)/(2 \times 10 \times 10-6)} = 10e^{-0.5} = 6.05$ V

$v_{o1} = 10\ e^{-t/(2 \times 10 \times 10-6)}$ for $t < t_p$

$v_o(t > t_p) = -3.935\ e^{-(t-10 \times 10-6)/(2 \times 10 \times 10-6)}$

Based on these results, the output waveforms are sketched as in Figure1.9 (a), (b) and (c) corresponding to cases (i), (ii) and (iii), respectively.

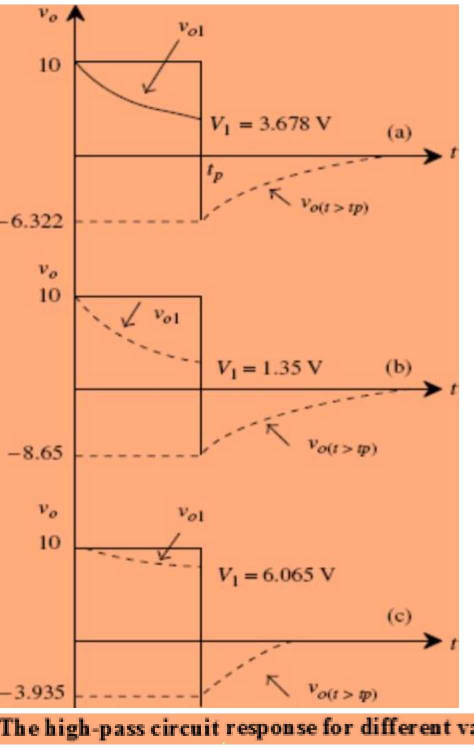


Figure 1.9 The high-pass circuit response for different values of τ

1.5 High-Pass RC Response for a Square-Wave Input:

A waveform that has a constant amplitude, say, V' for a time T_1 and has another constant amplitude, V'' for a time T_2, and which is repetitive with a time $T = (T_1 + T_2)$, is called a square wave. In a symmetric square wave, $T_1 = T_2 = T/2$. Figure 1.10 shows typical input–output waveforms of the high-pass circuit when a square wave is applied as the input signal. As the capacitor blocks the DC, the DC component in the output is zero. Thus, as expected, even if the signal at the input is referenced to an arbitrary dc

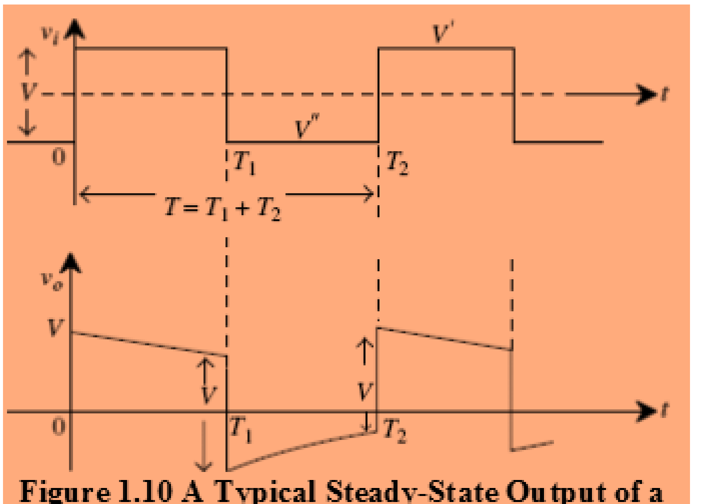


Figure 1.10 A Typical Steady-State Output of a High-Pass Circuit with a Square Wave as Input

level, the output is always referenced to the zero level. It can be proved that whatever the dc component associated with a periodic input waveform, the dc level of the steady-state output signal for the high-pass circuit is always zero.

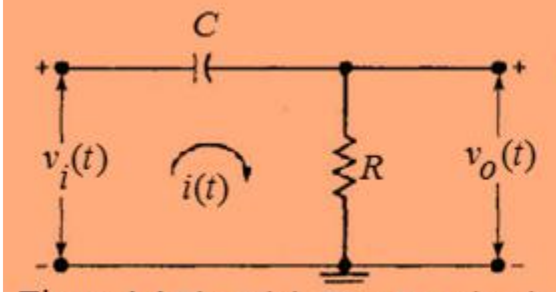


Figure 1.1 The High-Pass RC circuit

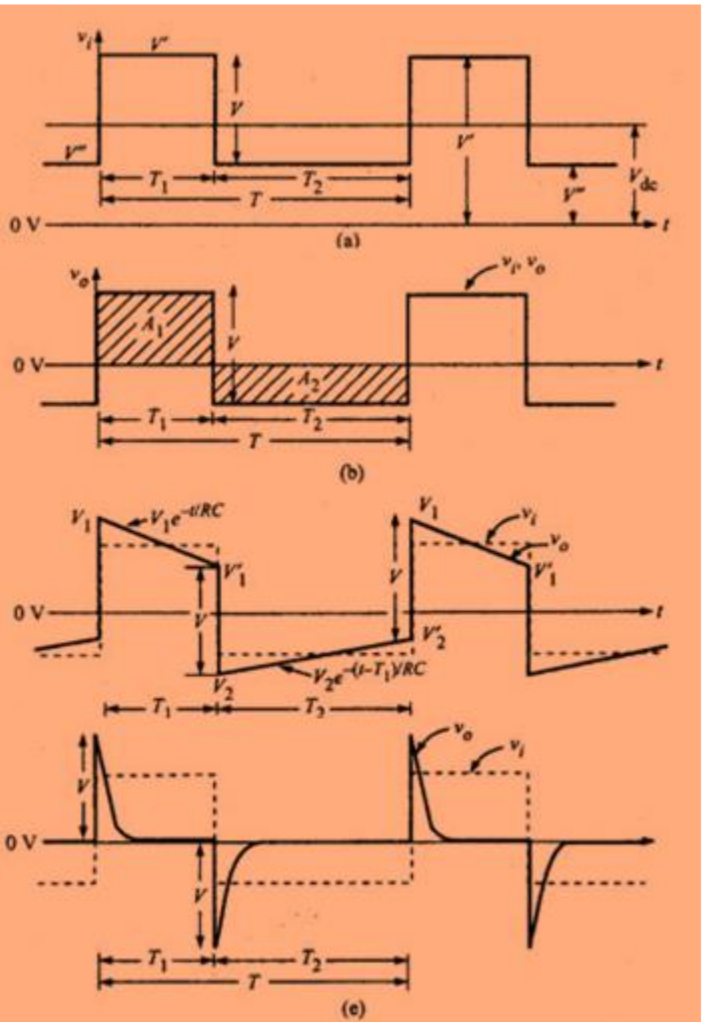


Figure 1.11 (a) A square wave input, (b) output when RC is arbitrarily large, (c) output when $RC > T$, (d) output when RC is comparable to T, and (e) output when $RC << T$.

To verify this statement, we write the KVL equation for the high-pass RC circuit as shown in Figure 1.1.

$$v_i = \frac{q}{C} + v_o \quad \text{------ 1.16}$$

Where, q is the charge on the capacitor. Differentiating with respect to t,

$$\frac{dv_i}{dt} = \frac{1}{C}\frac{dq}{dt} + \frac{dv_o}{dt} \quad \text{------ 1.17}$$

But, $i = \frac{dq}{dt}$

Substituting this condition in Eq. (1.17), we get

$$\frac{dv_i}{dt} = \frac{i}{C} + \frac{dv_o}{dt}$$

Since $v_o = iR$, $i = v_o/R$ and $RC = \tau$.

$\therefore \quad \frac{dv_i}{dt} = \frac{v_o}{\tau} + \frac{dv_o}{dt}$ --1.18

Multiplying by dt and integrating over the time period T we get

$\int_0^T dv_i = \int_0^T \frac{v_o}{\tau} dt + \int_0^T dv_o$ ---1.19

$[v_i(T) - v_i(0)] = \frac{1}{\tau}\int_0^T v_o \, dt + [v_o(T) - v_o(T)]$ ---1.20

Under steady-state conditions, the output and the input waveforms are repetitive with a time period T. Therefore, $v_i(T) = v_o(T)$ and $v_i(0) = v_o(0)$. Hence, from Eq. (1.20)

$\frac{1}{\tau}\int_0^T v_o \, dt = 0$--1.21

Since this integral represents the area under the output waveform over one cycle, we can say that the average level of the steady-state output signal is always zero. As the area under the output waveform over one cycle represents the DC component. In the output, from Eq. (1.21) it is evident that the DC component in the steady-state is always zero.

This waveform is therefore periodic with a fundamental period T but without a dc component. With respect to the high-pass circuit of Figure 1.1, we can say that:

1. The average level of the output signal is always zero, independently of the average level of the input. The output must consequently extend in both negative and positive directions with respect to the zero-voltage axis and the area of the part of the waveform above the zero axis must equal the area which is below the zero axis.

2. When the input changes abruptly by an amount V, the output also changes abruptly by an equal amount and in the same direction.

3. During any finite time interval when the input maintains a constant level, the output decays exponentially towards zero voltage.

Now let us consider the response of the high-pass RC circuit for a square-wave input for different values of the time constant τ, as shown in Figure 1.11. As is evident from the waveform in Figure 1.11 (b), there is no appreciable distortion in the output if τ is large. The output is almost the same as the input except for the fact that there is no DC component in the output. As τ decreases, as in Figure 1.11 (c), there is a tilt in the positive duration (amplitude decreases from V_1 to V_1' during the period 0 to T_1) and there is also a tilt in the negative duration (amplitude increases from V_2 to V_2' during the period T_1 to T_2). A further decrease in the value of τ [see Figure1.11 (d)] gives rise to positive and negative spikes. However, this condition is imposed on high-pass circuits to derive spikes.

Under steady-state conditions, the capacitor charges and discharges to the same voltage levels in each cycle. So, the shape of the output waveform is fixed.

$For\ 0 < t < T_1, the\ out\ put\ is\ given\ by\ v_{o1} = V_1\, e^{-t/RC}$

$$At\ t = T_1\,, v_{o1} = V_1^{|} = V_1\, e^{-T_1/RC}$$

$For\ T_1 < t < T_1 + T_2\,, the\ out\ put\ is\ given\ by\ \ v_{o2} = V_2\, e^{-(t-T_1)/RC}$

$At\ t = T_1 + T_2\,, v_{o2} = V_2^{|} = V_2\, e^{-T_2/RC}$

$Also\ V_1^{|} - V_2 = V\ \ and\ \ V_1 - V_2^{|} = V$

$From\ these\ relations\ V_1, V_1^{|}\,, V_2\ and\ V_2^{|}\ can\ be\ computed.$

1.5.1. Percentage of Tilt:

Let us consider the typical response of the high-pass circuit for a square-wave input shown in Figure 1.12.

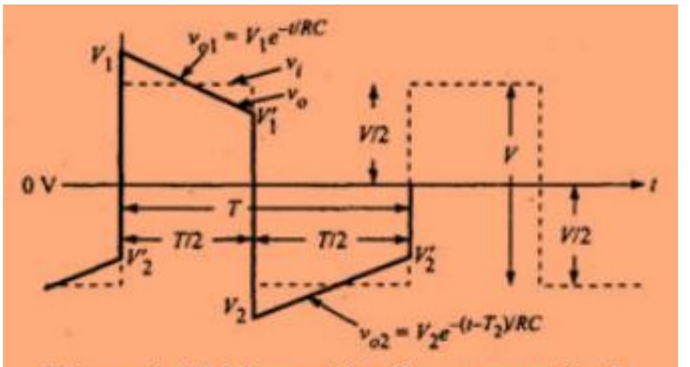


Figure 1.12 Linear tilt of a symmetrical square wave when $RC >> T$

We will derive an expression for the percentage tilt when the time constant RC of the circuit is very large compared to the period of the input waveform, i.e. $RC >> T$ for a symmetrical square wave with zero average value. $RC = \tau$

From Figure1.12 and using Eq. (1.7) we can write

$V_1^{|} = V_1 e^{-T_1/\tau}$ *and* $V_1^{|} - V_2 = V$, $V_2^{|} = V_2 e^{-T_2/\tau}$ and $V_1 - V_2^{|} = V$ ------------------------1.22

For a symmetric square wave $T_1 = T_2 = T/2$. And, because of symmetry:

$V_1 = -V_2,\ i.e.\ V_1 = |V_2|\,, V_1^{|} = -V_2^{|}, i.e.\ V_1^{|} = \left|V_2^{|}\right|\ , and\ T_1 = T_2 = \frac{T}{2}$ ----------------------------1.23

The output waveform for RC>>T is shown in Figure 1.12. Here

$$V_1^{|} = V_1 e^{-T/2RC}\ \ and\ \ V_2^{|} = V_2 e^{-T/2RC}$$
$$V_1 - V_2^{|} = V.\ And\ substituting\ the\ value\ of\ V_2^{|}\ ,we\ get$$
$$i.e.\ V_1 - V_2 e^{-T/2RC} = V.\ \ \ Here\ -V_2 = V_1$$
$$V_1 + V_1 e^{-T/2RC} = V$$
$$\therefore\ \ V_1 = \frac{V}{(1 + e^{-T/2RC})}\qquad or\ \ V = V_1\left(1 + e^{-T/2RC}\right)$$

% tilt $P = \frac{V_1 - V_1^{|}}{\frac{V}{2}} \times 100\% \ = \ \frac{V_1 - V_1 e^{-T/2RC}}{V_1\left(1+e^{-T/2RC}\right)} \times 200\% \ = \ \frac{1-e^{-T/2RC}}{1+e^{-T/2RC}} \times 200\%$

If $T/2\tau << 1$ or $T/2RC << 1$,

$e^{-T/2\tau} = 1 - \frac{T}{2\tau}$ --1.24

Therefore,

$$P = \frac{1 - (1 - \frac{T}{2\tau})}{1 + (1 - \frac{T}{2\tau})} \times 200\% = \frac{\frac{T}{2\tau}}{2 - \frac{T}{2\tau}} \times 200\% \cong \frac{T}{2\tau} \times 100\%\ \ \text{since}\ \frac{T}{2\tau} << 1$$

Thus, for a symmetrical square wave:

$P = \frac{T}{2\tau} \times 100\%$ ---1.25

$P = \frac{T\pi}{2\pi\ R\ C} \times 100\% \quad = \frac{\pi\ f_1}{f} \times 100\% \qquad \text{since } T = \frac{1}{f}$

$P = \frac{T\pi}{2\pi\ R\ C} \times 100\% \quad = \frac{\pi\ f_1}{f} \times 100\% \qquad \text{since } T = \frac{1}{f} \text{ and } f_1 = \frac{1}{2\pi\ R\ C}$

$P = \pi f_1 T \times 100\%$ --1.26

Eq. (1.25) tells us that the smaller the value of τ when compared to the half-period of the square wave ($T/2$), the larger is the value of P. In other words, distortion is large with small τ and is small with large τ. The lower half-power frequency, $f_1 = 1/2\pi RC$.

Example Problem 2: A 10 Hz square wave whose peak-to-peak amplitude is 2 V is fed to an amplifier. Calculate and plot the output waveform if the lower 3-dB frequency is 0.3 Hz.

Solution: Let C be the condenser through which the signal is connected to the amplifier, having an input resistance R, as shown in Figure 1.13 (a).

The lower 3-dB frequency f_1 = 0.3 Hz Input frequency is f = 10 Hz

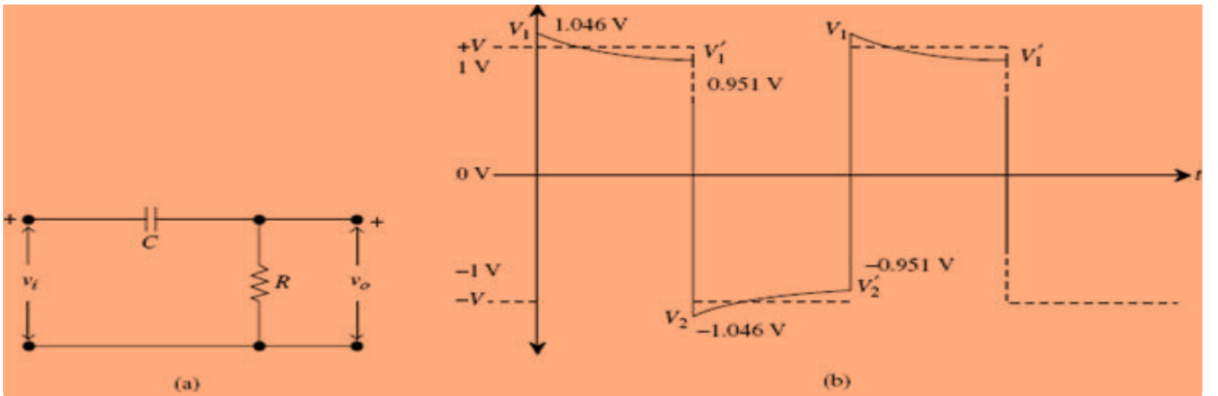


Figure 1.13 (a) The Coupling Network and (b) The Response of the Circuit

$$\tau = RC = \frac{1}{2\pi f_1} = \frac{1}{2\pi(0.3)} = 0.53 \text{ s}$$

$$T = \frac{1}{f} = \frac{1}{10} = 0.1 \text{ s}$$

Therefore,

$$\frac{T}{2} = 0.05 \text{ s}$$

$$V_1 = \frac{V}{1 + e^{-T/2\tau}} = \frac{2}{1 + e^{-0.05/0.53}} = 1.046 \text{ V} \qquad V_1' = V_1 e^{-T/2\tau} = 1.046e^{-0.05/0.53} = 0.951 \text{ V}$$

$$V_1 = -V_2 \quad \text{and} \quad V_1' = -V_2'$$

$$V_1 = |V_2| = 1.046 \text{ V} \qquad V_1' = |V_2'| = 0.951 \text{ V}$$

The response of the circuit is shown in Figure1.13 (b).

Example Problem 3: A 20-Hz symmetrical square wave, shown in Figure 1.14 (a), with peak-to-peak amplitude of 2 V is impressed on a high-pass circuit shown in Figure1.13 (a) whose lower 3-dB frequency is 10 Hz. Calculate and sketch the output waveform. What is the peak-to-peak output amplitude of the above waveform?

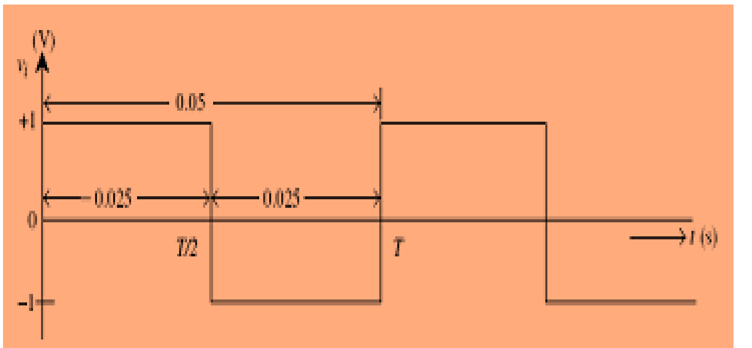


Figure 1.14 (a) Input to the High-Pass Circuit

Solution: The lower 3-dB frequency:

$$f_1 = \frac{1}{2\pi RC} = 10\,\text{Hz} \qquad RC = \tau = \frac{1}{2\pi f_1} = \frac{1}{2\pi \times 10} = 0.0159\,\text{s}$$

Input signal frequency $f = 20$ Hz

Time period of the input,

$$T = \frac{1}{f} = \frac{1}{20} = 0.05\,\text{s}$$
$$\frac{T}{2} = \frac{0.05}{2} = 0.025\,\text{s}$$

Therefore, τ is small compared to $T/2$; So, the capacitor charges and discharges appreciably in each half-cycle. Since the input is a symmetrical square wave, $V_1 = -V_2$, i.e., $|V_1| = |V_2|$, i.e., The peak-to-peak input = 2 V. Hence,

$$V_1 = \frac{V}{1 + e^{(-T/2)/\tau}} = \frac{2}{1 + e^{-0.025/0.0159}} = 1.656\,\text{V} \qquad V_2 = -V_1 = -1.656\,\text{V}$$

Peak-to-peak value of output = $V_1 - V_2 = 3.312$ V.

$$V_1' = V_1 e^{(-T/2)/\tau} = 1.656\, e^{-(0.025/0.0159)} = 0.344\ \text{V}$$

$$V_1' = -V_2' = 0.344\,\text{V}$$

The output is plotted in Figure1.14 (b).

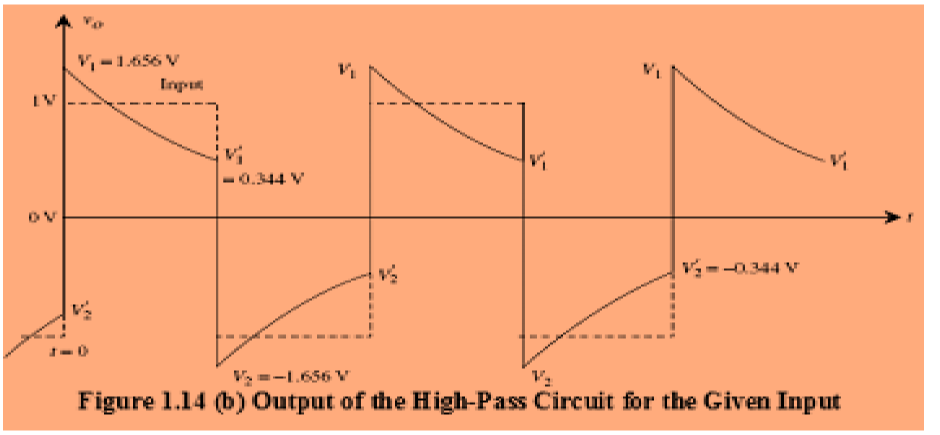


Figure 1.14 (b) Output of the High-Pass Circuit for the Given Input

1.6. High-Pass RC Response for a Ramp Input:

Ramp is a waveform in which the voltage increases linearly with time, for $t > 0$, and is zero for $t < 0$. It is used to move the spot in a CRO linearly with time along the x-axis. This type of waveform is generated by sweep circuits which we shall study later. However, if a ramp is applied as an input to a high-pass circuit, there could be deviation from linearity in the output. We can calculate and plot the output for different values of τ to understand how it influences the output. Let the input to the high-pass circuit be $V_i = \alpha t$, where, α is the slope, as shown in Figure1.15.

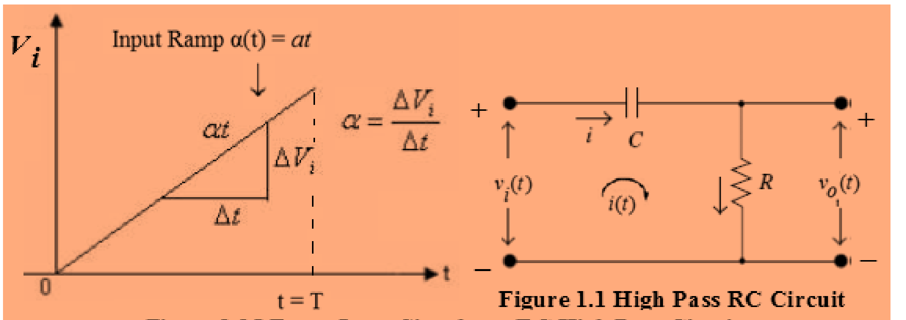


Figure 1.15 Ramp Input Signal to a RC High Pass Circuit

By applying KVL to the High Pass RC circuit as shown in Figure1.1,

$$v_i(t) = \frac{1}{C}\int i(t)dt + v_o(t)$$

$$= \frac{1}{RC}\int v_o(t)\, dt + v_o(t) \quad \left(\because \;\; i(t) = \frac{v_o(t)}{R}\right)$$

$$\therefore \;\; \alpha\, t \;\; = \;\; \frac{1}{RC}\int v_o(t)\, dt + v_o(t) \text{ --1.27}$$

Taking Laplace transforms

$$\frac{\alpha}{s^2} \;\; = \;\; \frac{1}{s\,RC} v_o(s) \;\; + \; v_o(s) \;\; = \;\; v_o(s) \; \left(1 + \frac{1}{s\,RC}\right) \text{--------------------------------1.28}$$

Multiplying throughout by s

$$\frac{\alpha}{s} \;\; = \;\; v_o(s) \; \left(s + \frac{1}{RC}\right) \text{--1.29}$$

Therefore, $v_o(s) \;\; = \;\; \frac{\alpha}{s} \frac{1}{\left(s+\frac{1}{RC}\right)} \;\; = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{RC}\right)}$

From which, $A = \alpha\, RC$ and $B = -\alpha\, RC$

$$v_o(s) \;\; = \;\; \frac{\alpha\, RC}{s} - \frac{\alpha\, RC}{\left(s + \frac{1}{RC}\right)} = \alpha\, RC \left[\frac{1}{s} - \frac{1}{\left(s + \frac{1}{RC}\right)}\right]$$

Taking inverse Laplace transforms

$$v_o(t) \;\; = \;\; \alpha\, RC \left[1 - e^{-t/RC}\right] \text{--1.30}$$

If $t/RC << 1$, then $e^{-t/RC} = 1 - \frac{t}{RC} + \frac{t^2}{2(RC)^2}$

Therefore, $v_o(t) \;\; = \;\; \alpha\, RC \left[1 - 1 - \frac{t}{RC} + \frac{t^2}{2(RC)^2}\right]$

$$v_o(t) \;\; = \;\; \alpha\, t \; \left[1 - \frac{t}{2RC}\right] \text{---1.31}$$

The output falls away from the input, as shown in Figure 1.16. From the waveforms in Figure 1.16, we see that for the output to be the same as the input, $\tau >> T$ (the duration of the ramp). As the value of τ decreases, not only the amplitude of the output decreases but also the signal now is an exponential. The output falls away from the input. So, the choice of τ is dictated by the specific application.

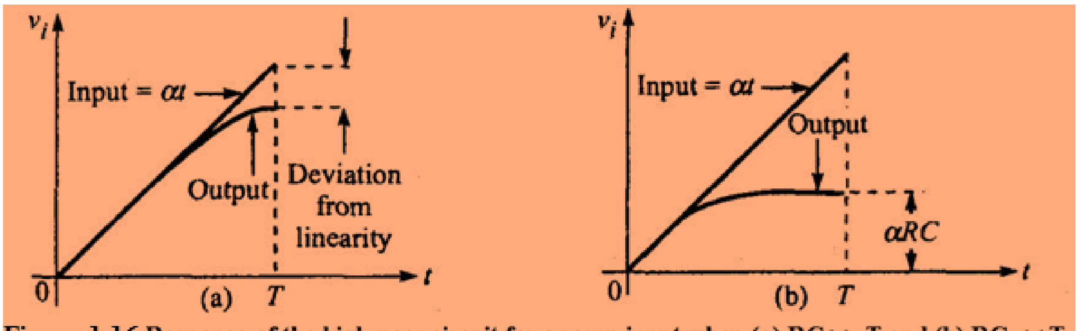


Figure 1.16 Response of the high-pass circuit for a ramp input when (a) RC >> T and (b) RC << T.

Figure 1.16 shows the response of the high-pass circuit for a ramp input when (a) $RC >> T$, and (b) $RC << T$, where T is the duration of the ramp. For small values of T (RC>>T), the output signal falls away slightly from the input as shown in the Figure 1.16 (a). When a ramp signal is transmitted through a linear network, the output departs from the input. A measure of the departure from linearity expressed as the transmission error e_t is defined as the difference between the input and the output divided by the input.

Transmission error at a time $t = T$ defines deviation from linearity and is given by

$$e_t = \frac{v_i - v_o}{v_i} \approx \frac{\alpha t - \alpha t\left(1 - \frac{t}{2RC}\right)}{\alpha t} \qquad \text{---1.32}$$

At $t = T$, $v_i = \alpha T$ and $v_o = \alpha T\,[1 - (T/2RC)]$. Therefore

$$e_t = \left.\frac{v_i - v_o}{v_i}\right|_{t=T} \approx \left.\frac{\alpha T - \alpha T\left(1 - \frac{T}{2RC}\right)}{\alpha T}\right|_{t=T} \approx \frac{T}{2RC} = \pi f_1 T \qquad \text{------------------------------------1.33}$$

Where, $f_1 = \frac{1}{2\pi RC}$ is the lower 3 dB frequency of the high-pass circuit.
For large values of t in comparison with RC, the output approaches the constant value αRC as indicated in Figure 1.16(b).

The transmission error, *et* describes how faithfully the signal is transmitted to the output. As the input is a ramp and if the output falls away from the input, *et* specifies the deviation from linearity.

Example Problem 4: A ramp is applied to an RC differentiator, [see Figure1.1]. Draw to scale the output waveform for the following cases: (i) $T = RC$, (ii) $T = 0.5RC$, (iii) $T = 10RC$.

Solution:

From Eq. (1.30)

$$v_o = \alpha\,\tau\left(1 - e^{-t/\tau}\right) \qquad v_o = V\left(\frac{\tau}{T}\right)\left(1 - e^{-t/\tau}\right) \quad \text{as} \quad \alpha = \frac{V}{T}$$

The peak of the output will occur at $t = T$.

$v_O(\text{peak}) = V\left(\frac{\tau}{T}\right)\left(1 - e^{-T/\tau}\right)$

1. When $T = \tau$, $(\tau/T) = 1$ and $(T/\tau) = 1$

$v_O(\text{peak}) = V(1)\left(1 - e^{-1}\right) = 0.632\text{ V}$

2. When $T = 0.5\tau$

$\left(\frac{T}{\tau}\right) = 0.5$ and $\left(\frac{\tau}{T}\right) = 2$

$v_O(\text{peak}) = V(2)\left(1 - e^{-0.5}\right) = 0.788\text{ V}$

3. When $T = 10\tau$

$\left(\frac{T}{\tau}\right) = 10$ $\left(\frac{\tau}{T}\right) = 0.1$

$v_o(\text{peak}) = V\,(0.1)\left(1 - e^{-10}\right) = (1 - 0.000045)$

$= 0.1\text{ V}$

The response is plotted in Figure 1.17.

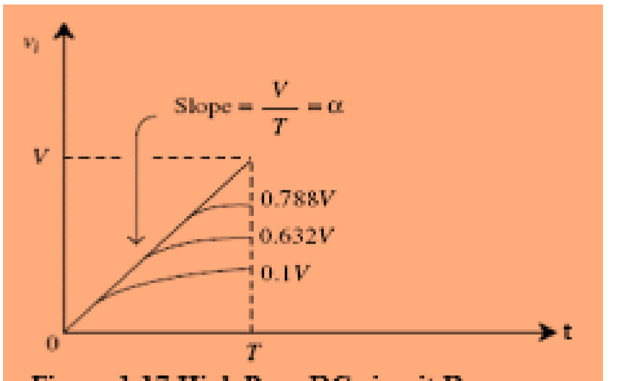


Figure 1.17 High Pass RC circuit Response to Ramp Input

1.7. High-Pass RC Response for an Exponential Input:

An exponential input as shown in Figure 1.18 is a voltage that increases or decreases exponentially with time. In such a case, voltage is zero for t < 0 and increases nonlinearly with time t called an exponential voltage.

If the input to the high-pass RC circuit in Figure 1.1 is an exponential of the form

$v_i = V\left(1 - e^{-t/\tau_1}\right)$ --1.34

Where, τ_1 is the time constant of the circuit that has generated the exponential signal as shown in Figure 1.18. From Eq. (1.18), we know

$\frac{dv_i}{dt} = \frac{v_o}{\tau} + \frac{dv_o}{dt}$, Where $\tau = RC$ is the time constant of the RC network.

Here the given input is an exponential and is

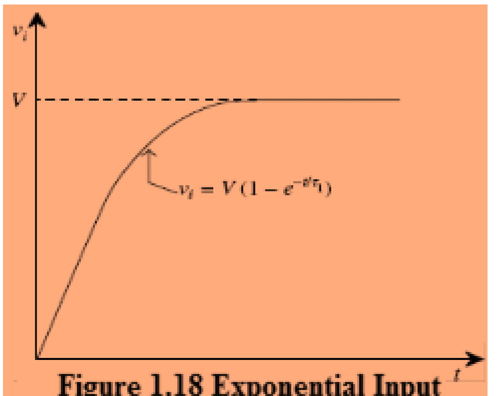


Figure 1.18 Exponential Input

$v_i = V\left(1 - e^{-t/\tau_1}\right) \Rightarrow \frac{dv_i}{dt} = \frac{V}{\tau_1}e^{-t/\tau_1}$--1.35

Substituting this value of $\frac{dv_i}{dt}$ in Eq. (1.18), we get

$\frac{V}{\tau_1}e^{-t/\tau_1} = \frac{v_o}{\tau} + \frac{dv_o}{dt}$ ---1.36

Taking Laplace transforms,

$\frac{\frac{V}{\tau_1}}{\left(s+\frac{1}{\tau_1}\right)} = \frac{v_o(s)}{\tau} + s v_o(s)$, where, τ is the time constant of the high-pass circuit.

$$\frac{\frac{V}{\tau_1}}{\left(s+\frac{1}{\tau_1}\right)} = v_o(s)\left(s+\frac{1}{\tau}\right)$$

$\therefore\ v_o(s) = \frac{\frac{V}{\tau_1}}{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau}\right)}$ --1.37

Case 1: $\tau \neq \tau 1$

Applying partial fractions, Eq. (1.37) can be written as

$v_o(s) = \frac{A}{\left(s+\frac{1}{\tau_1}\right)} + \frac{B}{\left(s+\frac{1}{\tau}\right)} = \frac{\frac{V}{\tau_1}}{\left(s+\frac{1}{\tau_1}\right)\left(s+\frac{1}{\tau}\right)}$ --1.38

Therefore,

$\frac{V}{\tau_1} = A\left(s+\frac{1}{\tau}\right) + B\left(s+\frac{1}{\tau_1}\right)$ --1.39

Put $s = -1/\tau 1$ in Eq. (1.39).

$$\frac{V}{\tau_1} = A\left(\frac{-1}{\tau_1}+\frac{1}{\tau}\right) \quad or \quad A = \frac{\frac{V}{\tau_1}}{\left(\frac{1}{\tau}-\frac{1}{\tau_1}\right)} = \frac{V}{\left(\frac{\tau_1}{\tau}-1\right)}$$

Now put $s = -1/\tau$ in Eq. (1.39), then

$$\frac{V}{\tau_1} = B\left(\frac{1}{\tau_1}-\frac{1}{\tau}\right) \quad or \quad B = \frac{\frac{V}{\tau_1}}{\left(\frac{1}{\tau_1}-\frac{1}{\tau}\right)} = \frac{-V}{\left(\frac{\tau_1}{\tau}-1\right)}$$

Substituting the values of A and B in Eq. (1.38), we get

$$v_o(s) = \frac{V}{\left(\frac{\tau_1}{\tau}-1\right)\left(s+\frac{1}{\tau_1}\right)} + \frac{-V}{\left(\frac{\tau_1}{\tau}-1\right)\left(s+\frac{1}{\tau}\right)} = \frac{V}{\left(\frac{\tau_1}{\tau}-1\right)}\left[\frac{1}{\left(s+\frac{1}{\tau_1}\right)} - \frac{1}{\left(s+\frac{1}{\tau}\right)}\right]$$

Taking inverse Laplace transform, we get

$v_o(t) = \frac{V}{\left(\frac{\tau_1}{\tau}-1\right)}\left(e^{-t/\tau_1} - e^{-t/\tau}\right)$--1.40

This is the expression for the output voltage where $\tau \neq \tau 1$.

Let $t/\tau 1 = x$ and $\tau/\tau 1 = n$. For $n \neq 1$, i.e., $\tau \neq \tau 1$, we have from Eq. (1.40)

$$v_o(t) = \frac{V}{\left(\frac{1}{n}-1\right)}\left(e^{-x} - e^{-x/n}\right) \quad \text{since } \frac{t}{\tau} = \frac{t}{\tau_1}\times\frac{\tau_1}{\tau} = \frac{x}{n}$$

Therefore,

$v_o(t) = \frac{Vn}{(1-n)}\left(e^{-x} - e^{-x/n}\right) = \frac{Vn}{(n-1)}\left(e^{-x/n} - e^{-x}\right)$ ------------------------------------1.41

If $\tau >> \tau 1$, the second term in the Eq. (1.41) is small when compared to the first. Thus,

$v_o(t) \approx \frac{Vn}{(n-1)}e^{-x/n} = \frac{Vn}{(n-1)}e^{-t/\tau}$---1.42

Case 2: $\tau = \tau 1$, that is $\tau/\tau 1 = n$, $n = 1$, then the output is given by

$$v_o(s) = \frac{\frac{V}{\tau}}{\left(s+\frac{1}{\tau}\right)\left(s+\frac{1}{\tau}\right)} = \frac{\frac{V}{\tau}}{\left(s+\frac{1}{\tau}\right)^2}$$

Taking Laplace inverse, we get

$$v_o(t) = \frac{V}{\tau} t\, e^{-t/\tau} \text{-----} 1.43$$

As $t/\tau = x = t/\tau 1$ and $\tau/\tau 1 = n = 1$

$$v_o(t) = Vx\, e^{-x} \text{-----} 1.44$$

The response of the circuit is plotted for different values of n in Figure 1.19

From the response in Figure1.19, it is seen that near the origin the output follows the input. Also, the smaller the value of n (= $\tau/\tau 1$ is small), the smaller is the output peak and the shorter is the duration of the pulse. As n increases, the peak becomes larger and the duration of the pulse becomes longer. Hence, the choice of n is based on the amplitude and duration of the pulse required for a specific application. The maximum output occurs when $(dv_o/dt) = 0$.

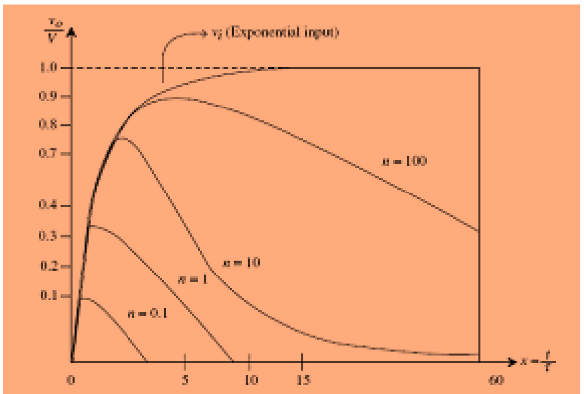


Figure 1.19 The Response of a High Pass Circuit to an Exponential Input

From Eq. (1.41), we can write

$$\frac{d}{dt}\left[\frac{Vn}{(n-1)}\left(e^{-x/n} - e^{-x}\right)\right] = 0$$

$$\frac{Vn}{(n-1)}\left[\left(\left(\frac{-1}{n}\right)\left(\frac{1}{\tau}\right)e^{-x/n} - e^{-x}\left(\frac{-1}{\tau}\right)\right)\right] = 0$$

$$\frac{Vn}{(n-1)}\left[\left(\frac{e^{-x}}{\tau} - \frac{e^{-x/n}}{n\,\tau}\right)\right] = 0$$

$$\Rightarrow \frac{e^{-x}}{\tau} = \frac{e^{-x/n}}{n\,\tau} \Rightarrow e^{-x} = \frac{e^{-x/n}}{n}$$

$$\Rightarrow n = e^{-x\left(1-\frac{1}{n}\right)} = e^{x\left(\frac{n-1}{n}\right)}$$

$$ln \quad n = \frac{x(n-1)}{n}$$

$$\therefore \; x = \frac{n}{n-1} ln \quad n \text{-----} 1.45$$

Since $x = t/\tau$, from Eq. (1.45), the time taken to rise to the peak t_p is given by

$$x = \frac{t}{\tau} = \frac{n}{n-1} ln \quad n \quad \text{and at } \mathrm{t} = \mathrm{t}_p$$

$$t_p = \tau \frac{n}{n-1} ln \quad n$$

From Eq. (1.45)

$$x = \frac{n}{n-1} ln \quad n \Rightarrow -x = -\frac{n}{n-1} ln \quad n = ln\left[n^{n/(1-n)}\right] \qquad \text{-----} 1.46$$

To obtain the maximum value of the output, substituting this value of x from Eq. (1.46) in the expression for $v_o(t)$ in Eq. (1.41), we get

$$v_o(max) = \frac{Vn}{(n-1)}\left(e^{\frac{1}{n}ln[n^{n/(1-n)}]} - e^{ln[n^{n/(1-n)}]}\right)$$
$$= V\frac{n}{(n-1)}exp\left[ln\left[n^{1/(1-n)}\right] - ln\left[n^{n/(1-n)}\right]\right]$$
$$= V\frac{n}{(n-1)}\left[\left[n^{1/(1-n)}\right] - \left[n^{n/(1-n)}\right]\right]$$
$$= V\frac{1}{(n-1)}\left[\left[n^{1+1/(1-n)}\right] - \left[n^{1+n/(1-n)}\right]\right]$$
$$= V\frac{1}{(n-1)}\left[\left[n^{(2-n)/(1-n)}\right] - \left[n^{1/(1-n)}\right]\right]$$
$$= V\frac{1}{(n-1)}\left[(n-1)n^{1/(1-n)}\right]$$
$$v_o(max) = Vn^{1/(1-n)}$$

$$\therefore \frac{v_o(max)}{V} = n^{1/(1-n)} \quad \text{for } n \neq 1 \quad \text{--}1.47$$

From the waveforms in Figure 1.19 and the subsequent mathematical relations derived, it is seen that, if an exponential signal is applied as an input to a high-pass circuit, the output is a pulse whose duration depends on $n(= \tau/\tau_1)$, where τ_1 is the time constant of the previous circuit that has generated the exponential signal and τ is the time constant of the high-pass circuit under consideration. The smaller the value of n, the smaller the duration of this output pulse and also the smaller its amplitude. As n increases, the duration as well as the amplitude of this output pulse increases. Hence, depending on our requirement, we adjust the value of n. This for the given exponential input when $\tau = \tau_1$

The input and the output waveforms are as shown in Figure 1.20.

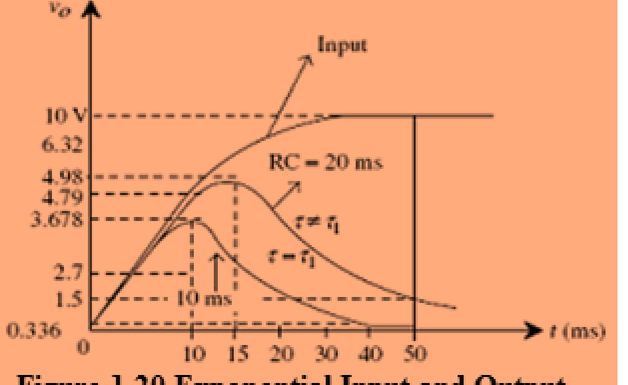


Figure 1.20 Exponential Input and Output wave forms of High-Pass Circuit

1.8. High-Pass RC Circuit as a Differentiator:

If in a circuit, the output is a differential of the input signal, then the circuit is called a differentiator. If the time constant of the high-pass *RC* circuit, shown in Figure 1.1, is much smaller than the time period of the input signal, then the circuit behaves as a differentiator. If T is to be large when compared to τ, then the frequency must be small. At low frequencies, X_c is very large when compared to R. Therefore, the voltage drop across R is very small when compared to the drop across C.

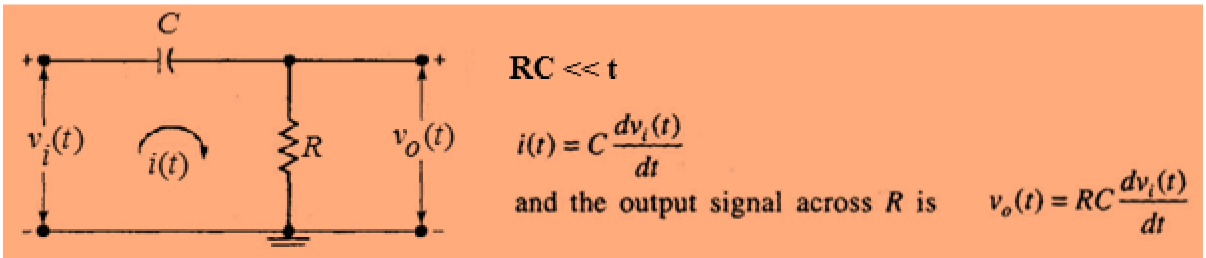


Figure 1.1 The High-Pass RC Circuit

When the time constant of the high-pass RC circuit is very small, the capacitor charges very quickly; so almost all the input *vi* (0) appears across the capacitor and the voltage across the resistor will be negligible compared to the voltage across the capacitor. Hence the current is determined entirely by the capacitance. If RC << T, then the current is given by

$i(t) = C\frac{dv_i(t)}{dt}$ and the output across the R is given by

$v_o(t) = i(t)R = R\,C\frac{dv_i(t)}{dt}$--1.48

To verify this statement, writing the KVL to the high-pass RC circuit as shown in Figure 1.1.

$v_i = \frac{q}{C} + v_o$ --1.49

Where, q is the charge on the capacitor. Differentiating with respect to t,

$\frac{dv_i}{dt} = \frac{1}{C}\frac{dq}{dt} + \frac{dv_o}{dt}$ --1.50

But, $i = \frac{dq}{dt}$

Substituting this condition in Eq. (1.50), we get

$\frac{dv_i}{dt} = \frac{i}{C} + \frac{dv_o}{dt}$

Integrating on both sides, we get

$$v_i = \frac{1}{C}\int i\,dt + v_o$$

Since $v_o = iR$, $i = v_o/R$ and $RC = \tau$.

$v_i = \frac{1}{C}\int i\,dt + i\,R$. Here $i\,R = v_o$ is negligibly small and therefore

$$v_i = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \frac{v_o}{R}\,dt = \frac{1}{RC}\int v_o dt$$ Differentiating on both sides we get

$$\frac{dv_i}{dt} = \frac{v_o}{RC} = \frac{v_o}{\tau}$$ (RC = τ is the time constant of the RC circuit)

$v_o = \tau\frac{dv_i}{dt}$---1.51

$v_o\ \alpha\ \frac{dv_i}{dt}$ --1.52

The output voltage is proportional to the derivative of the input voltage. The proportionality constant α = RC is the time constant of the RC circuit.

Thus, from Eq. (1.52), it can be seen that the output is proportional to the differential of the input signal, as shown in Figure 1.21.

Thus, we see that the output is proportional to the derivative of the input. ***The high-pass RC circuit acts as a differentiator provided the RC time, constant of the circuit is very small in comparison with the time required for the input signal to make an appreciable change.*** The derivative of a step signal is an impulse of infinite amplitude at the occurrence of the discontinuity of step. The derivative of an ideal pulse is a positive impulse followed by a delayed negative impulse, each of infinite amplitude and occurring at the points of discontinuity.

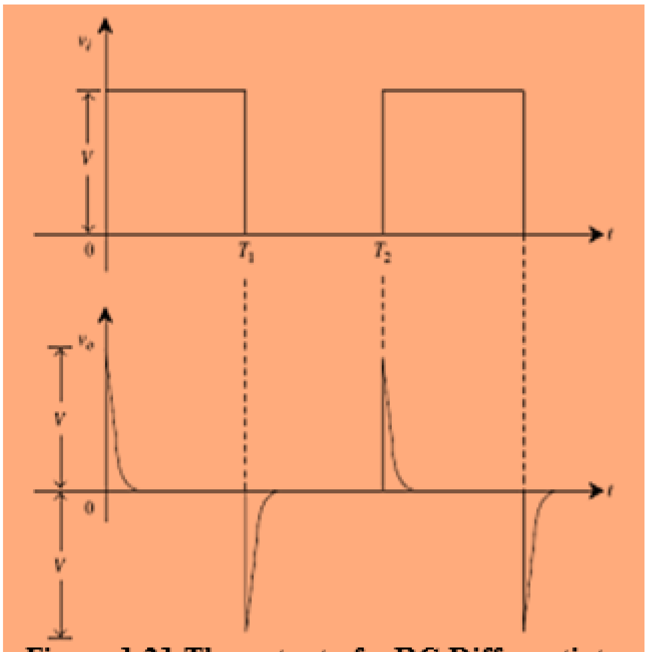


Figure 1.21 The output of a RC Differentiator

The derivative of a square wave is a waveform which is uniformly zero except, at the points of discontinuity. At these points, precise differentiation would yield impulses of infinite amplitude, zero width and alternating polarity. For a square wave input, an *RC* high-pass circuit with very small time constant will produce an output, which is zero except at the points of discontinuity. At these points of discontinuity, there will be peaks of finite amplitude *V*. This is because the voltage across *R* is not negligible compared with that across *C*. An *RC* differentiator converts a triangular wave into a square wave. For the ramp $vi = at$, the value of $RC\ (dv/dt) = a\ RC$. This is true except near the origin. The output approaches the proper derivative value only after a lapse of time corresponding to several time constants.

1.9 Low Pass RC Circuit:

Low pass circuit is one which allows low frequencies with less attenuation and high frequencies with maximum attenuation. This is because capacitance offers high reactance at low frequencies and hence there is an output. In the *RC* circuit, shown in Figure1.22, at low frequencies, the reactance of *C* is large and decreases with increasing frequency. Hence, the output is smaller for higher frequencies and vice-versa. Hence this circuit is called a low pass circuit. Let us consider the response of these low-pass circuits to different types of inputs.

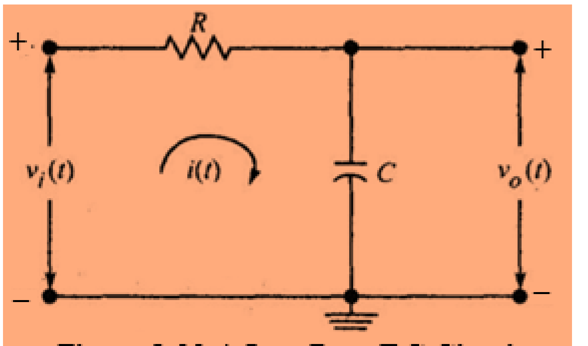


Figure 1.22 A Low Pass RC Circuit

Figure1.22 shows a low-pass *RC* circuit. A low-pass circuit is a circuit, which transmits only low-frequency signals and attenuates or stops high-frequency signals. At zero frequency, the reactance of the capacitor is infinity (i.e. the capacitor acts as an open circuit) so the entire input appears at the output, i.e. the input is transmitted to the output with zero attenuation. So, the output is the same as the input, i.e. the gain is unity. As the frequency increases the capacitive reactance decreases and so the output decreases. At very high frequencies the capacitor virtually acts as a short-circuit and the output falls to zero.

1.10 Low-Pass *RC* Response for a Sinusoidal Input:

The Laplace transformed low-pass *RC* circuit is shown in Figure 1.23 (a). The gain versus frequency curve of a low-pass circuit excited by a sinusoidal input is shown in Figure 1.23 (b). This curve is obtained by keeping the amplitude of the input sinusoidal signal constant and varying its frequency and noting the output at each frequency. At low frequencies the output is equal to the input and hence the gain is unity. As the frequency increases, the output decreases and hence the gain decreases. The frequency at which the gain is l/√2 (= 0.707) of its maximum value is called the cut-off frequency. For a low-pass circuit, there is no lower cut-off frequency. It is zero itself. The upper cut-off frequency is the frequency (in the high-frequency range) at which the gain is $\frac{1}{\sqrt{2}}$ i-e- 70.7% of its maximum value. The bandwidth of the low-pass circuit is equal to the upper cut-off frequency f_2 itself.

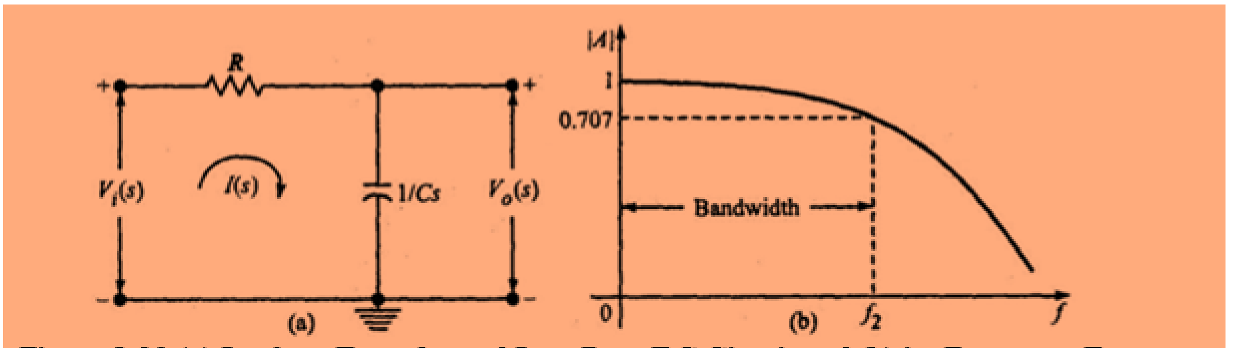


Figure 1.23 (a) Laplace Transformed Low-Pass RC Circuit and (b) its FrequencyResponse

For the circuit given in Figure 1.23 (a), if a sinusoidal signal is applied as the input, the output v_o is given by the relation

Writing KVL around the above RC Low Pass RC circuit, we get

$$v_o = v_i - iR = v_i - R\frac{v_i}{R+\frac{1}{j\omega C}} = v_i\left(1-\frac{R}{R+\frac{1}{j\omega C}}\right) = v_i\left(\frac{R+\frac{1}{j\omega C}-R}{R+\frac{1}{j\omega C}}\right) = v_i\left(\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}\right)$$

$$\therefore \frac{v_o}{v_i} = \left(\frac{1}{1+j\omega\, RC}\right) \quad \text{-----} \; 1.53$$

$$\left|\frac{v_o}{v_i}\right| = \left(\frac{1}{\sqrt{1+(\omega\, RC)^2}}\right) \quad \text{-----} \; 1.54$$

$$\therefore \; |A| \; = \; \frac{1}{\sqrt{1+(2\,\pi\, f\; R\; C)^2}}$$

$$At\;\; the\;\; upper\;\;\; cut-off\;\; frequency\;\; f_2,\;\; |A| \;\; = \; \frac{1}{\sqrt{2}}$$

$$\therefore \; \frac{1}{\sqrt{2}} \; = \; \frac{1}{\sqrt{1+(2\,\pi\, f\; R\; C)^2}}$$

Squaring both sides and equating the denominators $2 = \; 1 + \; (\, 2\,\pi\, f_2 R\; C\,)^2$

$$\therefore The\; upper\; cut-off\; frequency\;\; f_2 = \frac{1}{2\,\pi\, R\, C} \quad \text{-----} \; 1.55$$

$$So\; A \; = \frac{1}{1+j\,\frac{f}{f_2}} \quad and\; |A| = \frac{1}{\sqrt{1+\left(\frac{f}{f_2}\right)^2}}$$

$$The\; angle\;\; \theta\; by\; which\; the\; output\; leads\; the\; input\; is\; given\; by\; \theta = \; tan^{-1}\frac{f}{f_2}$$

Where, $\omega_2 = 1/RC = 1/\tau$. The phase shift θ the signal undergoes is given as $\theta = \tan-1(\omega/\omega_2) = \tan-1(\tau/T)$

Figure 1.23(c) shows a typical frequency vs. gain characteristic. Hence, f_2 is the upper half- power frequency.

At $\omega = \omega_2$,

$$\left|\frac{v_o}{v_i}\right| = |A| = \frac{1}{\sqrt{2}} = 0.707$$

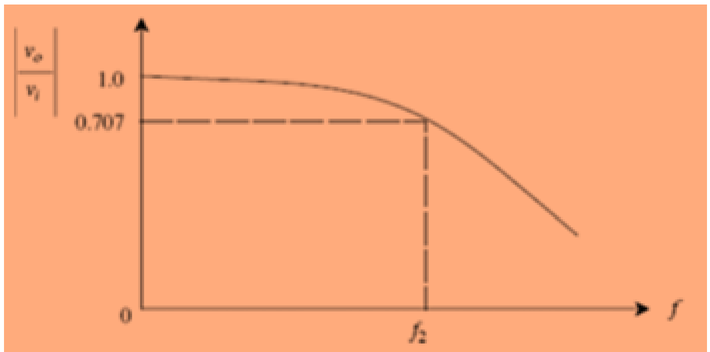


Figure 1.23 (c) Frequency Response curve of a Low-Pass Circuit for a Sinusoidal Input

The sinusoidal signal undergoes a change only in the amplitude but its shape remains preserved.

1.11. Low-Pass *RC* Response for a Step Input:

A step signal is one which maintains the value zero for all times $t < 0$, and maintains the value V for all times t > 0. The transition between the two voltage levels takes place at $t = 0$ and is accomplished in an arbitrarily small-time interval. Thus, in Figure 1.24 (a), $v_i = 0$ immediately before $t = 0$ (to be referred to as time $t = 0^-$) and $v_i = V$, immediately after t= 0 (to be referred to as time $t = 0^+$). In the low-pass RC circuit shown in Figure 1.24 (b), if the capacitor is initially uncharged, when a step input is applied, since the voltage across the capacitor cannot change instantaneously, the output will be zero at $t = 0$, and then, as the capacitor charges, the output voltage rises exponentially towards the steady-state value V with a time constant RC as shown in Figure 1.24 (c).

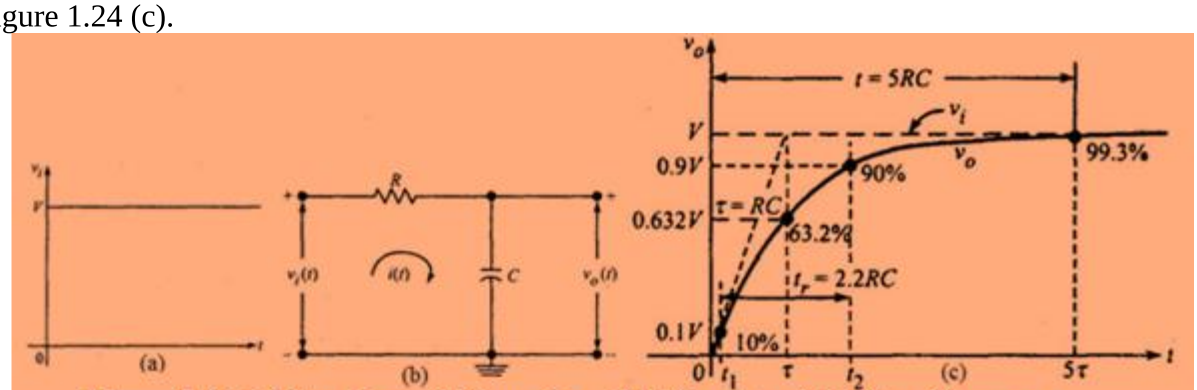


Figure 1.24 (a) Step Input,(b) Low Pass RC Circuit and (c) Step Response

Let a step voltage be applied as the input to the low-pass RC circuit shown in Figure 1.24(a). The output v_o can be obtained by writing KVL to above Figure 1.24 (b), as shown in Figure 1.24(c) we have $RC = \tau$.

Let $V^{|}$ be the initial voltage across the capacitor. Writing KVL around the loop of Low-Pass circuit as shown in the above Figure 1.24 (b), we get

$$v_i(t) = R\ i\ (t)\ + \frac{1}{C}\int i\ (t)\ dt$$

$$Differentiating\ this\ equation\ , \frac{dv_i(t)}{dt} = R\frac{di(t)}{dt} + \frac{1}{C}\ i\ (t)$$

Since

$$v_i(t) = V,\ \frac{dv_i(t)}{dt} = 0$$

$$\therefore\ 0 = R\frac{di(t)}{dt} + \frac{1}{C}\ i\ (t)$$

Taking the Laplace transform on both sides,

$$0 = RI(s)[sI(s) - I(0^+)] + \frac{1}{C} I(s)$$

$$\therefore I(0^+) = I(s)\left(s + \frac{1}{RC}\right)$$

The initial current $I(0^+)$ is given by

$$I(0^+) = \frac{V - V^{|}}{R}$$

$$\therefore I(s) = \frac{I(0^+)}{\left(s + \frac{1}{RC}\right)} = \frac{\frac{V - V^{|}}{R}}{\left(s + \frac{1}{RC}\right)} = \frac{V - V^{|}}{R\left(s + \frac{1}{RC}\right)}$$

And $\quad V_0(s) = V_i - I(s)CR = \frac{V}{s} - \frac{\frac{V-V^{|}}{R}}{\left(s + \frac{1}{RC}\right)}R = \frac{V}{s} - \frac{V-V^{|}}{\left(s + \frac{1}{RC}\right)}$

Taking the inverse Laplace transform on both sides, we get

$$V_0(t) = V - \left(V - V^{|}\right) e^{-t/RC} \quad \text{-----------------------}1.56$$

Where $V^{|}$ is the initial voltage across the capacitor $(V_{initial})$ and V is the final voltage $\left(V_{final}\right)$ to which the capacitor can charge.

So, the expression for the voltage across the capacitor of an RC circuit excited by a step voltage is given by

$$V_0(t) = V_{final} - \left(V_{final} - V_{initial}\right) e^{-t/RC} \quad \text{-----------------------}1.57$$

If the capacitor is initially uncharged, then $V_{initial}$= 0 and V_{final}=V. Therefore

$$V_0(t) = V\left(1 - e^{-t/RC}\right) \quad \text{-----------------------}1.58$$

Expression for rise time:

Voltage waveform to rise from 10% to 90% of its final value is called the rise time. It gives an indication of how fast the circuit can respond to a discontinuity in voltage. Assuming that the capacitor in Figure 1.24(b) is initially uncharged, the output voltage shown in Figure 1.24(c) at any instant of time is given by $v_0(t) = V\left(1 - e^{-t/RC}\right)$

(1)$At\ t = t_1,\ v_0(t) = 10\%\ of\ V = 0.1V$

$\therefore 0.1V = V\left(1 - e^{-t_1/RC}\right)$

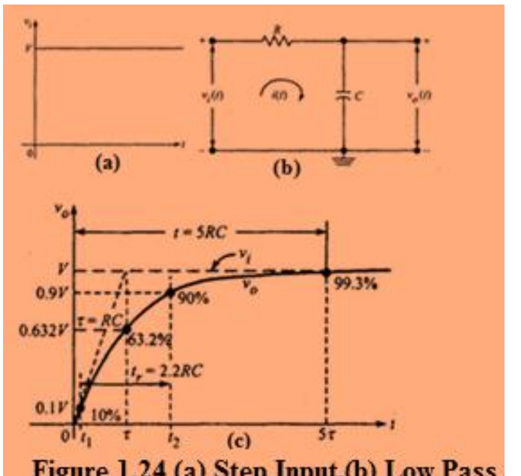


Figure 1.24 (a) Step Input,(b) Low Pass RC Circuit and (c) Step Response

$$\Rightarrow e^{-t_1/RC} = 0.9 \quad or \quad e^{t_1/RC} = \frac{1}{0.9} = 1.11$$

$$\therefore t_1 = RC\ ln(1.11) = 0.1R\,C$$

(2)$At\ \ t = t_2\ v_0(t) = 90\%\ of\ V = 0.9V$

$$\therefore 0.9V = V\left(1 - e^{-t_2/RC}\right) \Rightarrow e^{-t_2/RC} = 0.1 \quad or \quad e^{t_2/RC} = \frac{1}{0.1} = 10$$

$$\therefore t_2 = RC\ ln(10) = 2.3R\,C$$

Rise time:

The time taken for the output to reach 90% of its final value from 10% of its final value is called the rise time. From the above time instants t_2 *and* t_1, the rise time of this circuit is

Rise time $t_r = t_2 - t_1 = 2.2RC$--1.59

This indicates that the rise time *tr* is proportional to the time constant *RC* of the circuit. The larger the time constant, the slower will be the capacitor charges and the smaller the time constant the faster the capacitor charges.

Relation between rise time and upper 3-dB frequency:

We know that the upper 3-dB frequency (same as bandwidth) of a low-pass circuit is

$$f_2 = \frac{1}{2\,\pi\,R \rightleftarrows C} \quad or \; R \rightleftarrows C = \frac{1}{2\,\pi\;f_2}$$

$$\therefore\; Rise\ time,\ t_r \;=\; 2.2RC \;=\; \frac{2.2}{2\,\pi\;f_2} \;=\; \frac{0.35}{f_2} \;=\; \frac{0.35}{B.W.} \quad \text{--1.60}$$

Thus, the rise time is inversely proportional to the upper 3-dB frequency.
The *time constant* (T= *RC)* of a circuit is defined as the time taken by the output to rise to 63.2% of the amplitude of the input step. It is same as the time taken by the output to rise to 100% of the amplitude of the input step, if the initial slope of rise is maintained. See Figure 1.24 (c) the Greek letter τ is also employed as the symbol for the time constant.

Initially, as the capacitor behaves as a short circuit, the output voltage is zero. As the capacitor charges, the output reaches the steady-state value of *V* in a time interval that is dependent on the time constant τ. Let a step voltage v_i be applied to a low-pass circuit. The output does not reach the steady- state value v_i instantaneously as desired. Rather, it takes a finite time delay for the output to reach v_i, depending on the value of the time constant of the low-pass circuit employed. If this output is to drive a transistor from the OFF to the ON

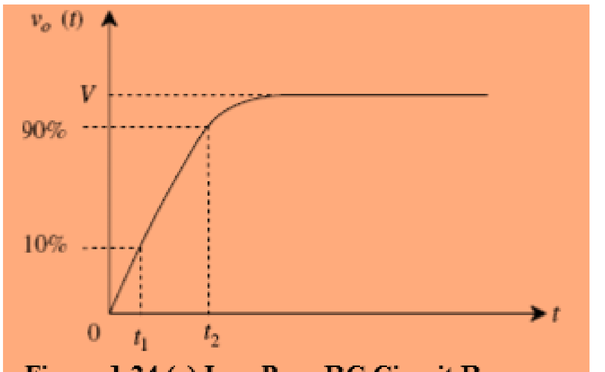


Figure 1.24 (c) Low Pass RC Circuit Response to a Step Input

state, this change of state does not occur immediately, because the output of the low-pass circuit takes some time to reach v_i. The transistor is thus said to be switched from the OFF state into the ON state only when the voltage at the output of the low-pass circuit is 90% of v_i. If this time delay is to be small, τ should be small. On the contrary, if the output is to be ramp, τ should be large.

1.12. Low-Pass *RC* Response for a Pulse Input:

Let the input of low pass RC circuit is a pulse signal which is shown in Figure 1.25(a). The pulse shown in Figure 1.25(a) is equivalent to a positive step followed by a delayed negative step as shown in Figure 1.25(b). So, the response of the low-pass *RC* circuit to a pulse for times less than the pulse width t_p is the same as that for a step input and is given by

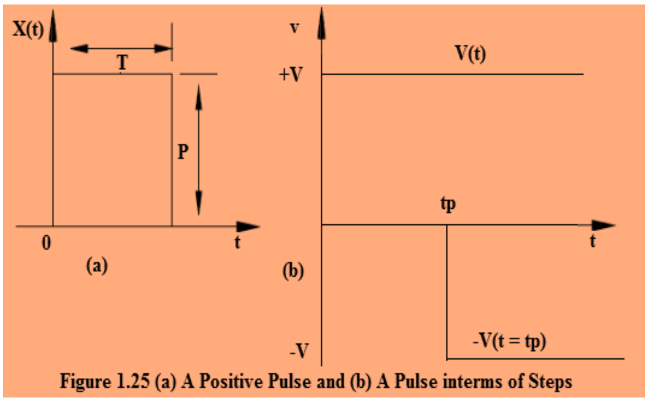


Figure 1.25 (a) A Positive Pulse and (b) A Pulse interms of Steps

$$v_o\,(t) \;=\; V\,\{1-e^{-t/RC}\} \qquad \text{-----}1.61$$

Let the input to the low-pass circuit be a positive pulse of duration t_p and amplitude V as shown in Figure 1.25(a). During the period 0 to t_p, the input is a step and the output is given by Eq. (1.61). At $t = t_p$ the input falls and the output decays exponentially as given by

$$v_o\,(t > t_p) \;=\; V_p\, e^{-(t-t_p/\tau)} \qquad \text{-----}1.62$$

For $v_i = V$, the output for different values of τ is plotted in Figure 1.26. It is seen here that the shape of the pulse at the output is preserved if the time constant of the circuit is much smaller than t_p, i.e., $\tau << t_p$. However, if a ramp is to be generated during the period of the pulse, τ is chosen such that $\tau >> t_p$. The method to compute the output is illustrated in Example 5.

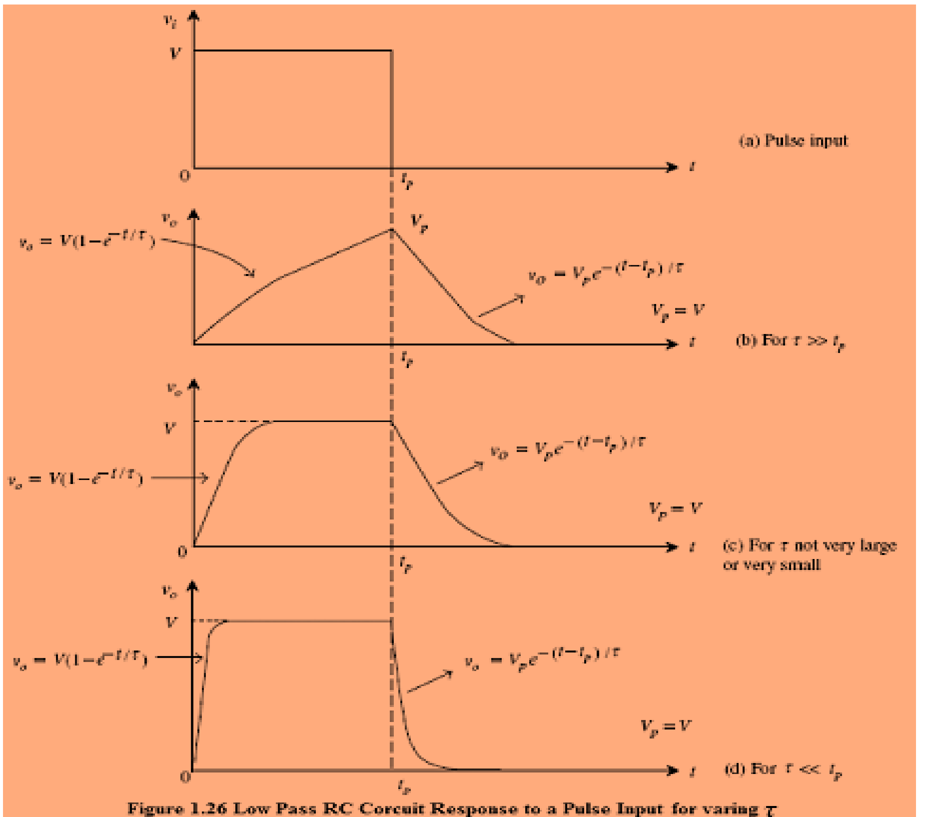


Figure 1.26 Low Pass RC Corcuit Response to a Pulse Input for varing τ

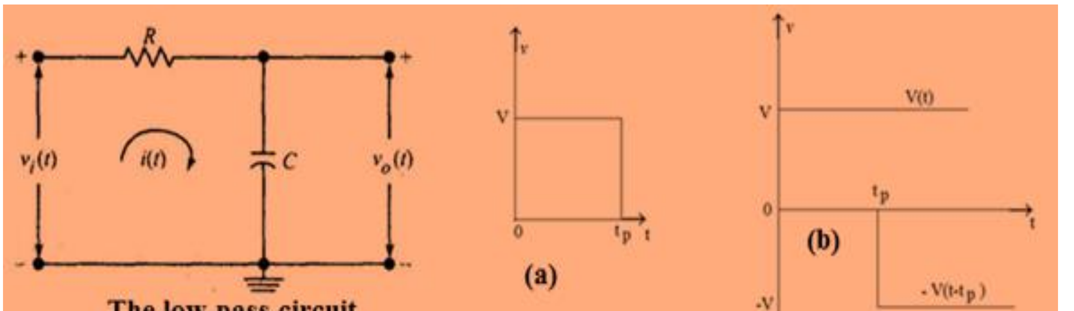
The responses of the low-pass *RC* circuit for time constant *RC* >> *tp*, *RC* < *tp* and *RC* << *tp* are shown in Figures 1.26(b), 1.26(c), and 1.26(d) respectively.

If the time constant *RC* of the circuit is very large, at the end of the pulse, the output voltage will be $V_p\,(t) \;=\; V\;(1-e^{-t/RC})$, and the output will decrease to zero from this value with a time constant *RC* as shown in Figure1.26(b). Observe that the pulse waveform is distorted when it is passed through a linear network. The output will always extend beyond the pulse width *tp*, because whatever charge has accumulated across the capacitor *C* during the pulse cannot leak off instantaneously.

If the time constant *RC* of the circuit is very small, the capacitor charges and discharges very quickly and the rise time *tr* will be small and so the distortion in the wave shape is small. For minimum distortion (i.e. for preservation of wave shape), the rise time must be small compared to the pulse width *tp.* If the upper 3-dB frequency f_2 is chosen equal to the reciprocal of the pulse width *tp*, i.e. if f_2 = 1/tp then *tr* = 0.35tp and the output is as shown in Figure 1.26(d), which for many applications is a reasonable reproduction of the input. As a rule of thumb, we can say:

A pulse shape will be preserved if the 3-dB frequency is approximately equal to the reciprocal of the pulse width. Thus to pass a 0.25 µ.s pulse reasonably well requires a circuit with an upper cut-off frequency of the order of 4 MHz.

Example Problem5: An ideal pulse of amplitude 10 V is fed to an *RC* low-pass integrator circuit. The width of the pulse is 3 μs. Draw the output waveforms for the following upper 3-dB frequencies: (a) 30 MHz, (b) 3 MHz and (c) 0.3 MHz



The low-pass circuit (a) Pulse Input (b) Pulse Interms of Steps

Solution: Consider the low-pass circuit in Figure 1.22 (a).

1. At f_2 = 30 MHz

We know that $f_2 = 1/2\pi RC$

$$\tau = RC = \frac{1}{2\pi f_2} = \frac{1}{2\pi \times 30 \times 10^6} = 5.3 \text{ ns}$$

$t_r = 2.2\tau = 2.2 \times 5.3 \times 10^{-9} = 11.67$ n

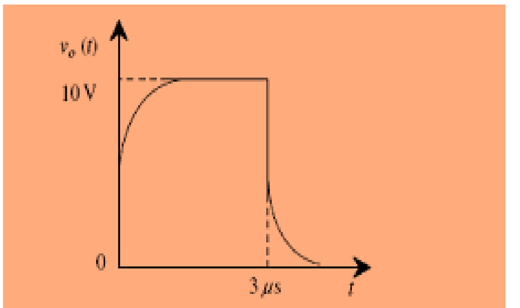


Figure 1.27 (a) Output wave form at *f*2 =30MHz

At $t = t_p$,

$V_p = V(1 - e^{-(tp/\tau)}) = 10(1 - e^{-(3\times10-6/5.3\times10-9)}) = 10$ V. The output is plotted in Figure 1.27 (a).

2. At f_2 = 3 MHz

$$\tau = RC = \frac{1}{2\pi f_2} = \frac{1}{2\pi \times 3 \times 10^6} = 53 \text{ ns}$$

$t_r = 2.2\tau = 2.2 \times 53 \times 10^{-9} = 116.6$ ns

At $t = t_p$,

$V_p = V(1 - e^{-(tp/\tau)}) = 10(1 - e^{-(3\times10-6/53\times10-9)}) = 10$ V

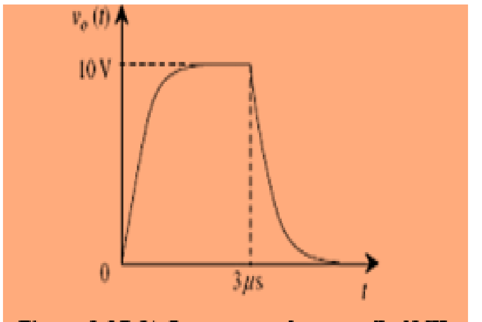


Figure 1.27 (b) Output wave forms at *f*2=3MHz

The output is plotted in Figure 1.27 (b).

3. At f_2 = 0.3 MHz

$$\tau = RC = \frac{1}{2\pi f_2} = \frac{1}{2\pi \times 0.3 \times 10^6} = 530 \text{ ns}$$

$t_r = 2.2\tau = 2.2 \times 530 \times 10^{-9} = 1.166\ \mu$s

At $t = t_p$,

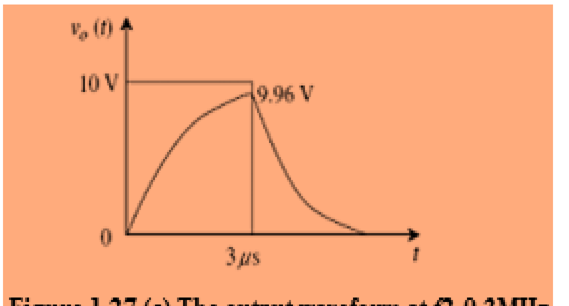


Figure 1.27 (c) The output waveform at *f*2-0.3MHz

$V_p = V(1 - e^{-(tp/\tau)}) = 10(1 - e^{-(3\times10-6/530\times10-9)}) = 9.96$ V. The output is plotted in Figure 1.27 (c).

1.13 Low-Pass *RC* Response for a Square-Wave Input:

As shown in Figure 1.28, a square wave is a periodic waveform which maintains itself at one constant level V' with respect to ground for a time T_1 and then changes abruptly to another level V'', and remains constant at that level for a time T_2, and repeats itself at regular intervals of $T = T_1 + T_2$ is applied to a RC Low-Pass. A square wave may be treated as a series of positive and negative steps. The shape of the output waveform for a square wave input depends on the time constant of the circuit. If the time constant is very small, the rise time will also be small and a reasonable reproduction of the input may be obtained.

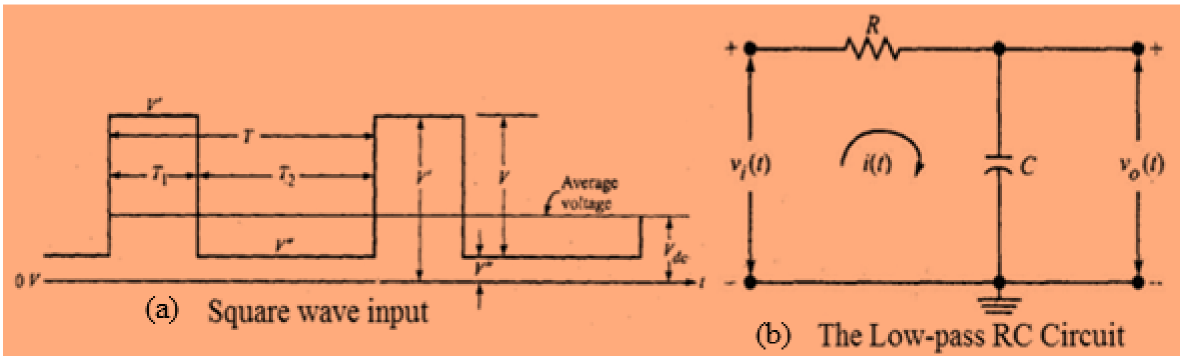


Figure 1.28 Square Wave Input to a RC Low-Pass Circuit

For the square wave shown in Figure 1.29(a), the output waveform will be as shown in Figure 1.29(b) if the time constant *RC* of the circuit is small compared to the period of the input waveform. In this case, the wave shape is preserved. If the time constant is comparable with the period of the input square wave, the output will be as shown in Figure 1.29(c). The output rises and falls exponentially. If the time constant is very large compared to the period of the input waveform, the output consists of exponential sections, which are essentially linear as indicated in Figure 1.29(d). Since the average voltage across *R* is zero, the dc voltage at the output is the same as that of the input. This average value is indicated as *Vdc* in all the waveforms of Figure 1.29.

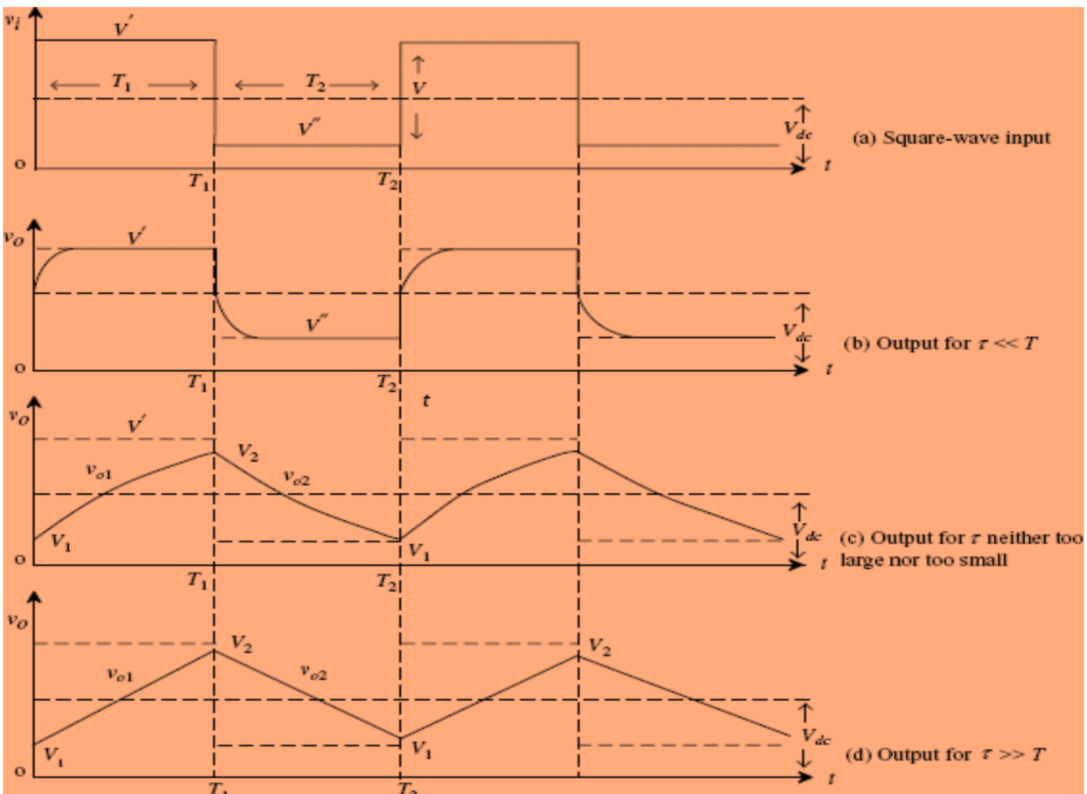


Figure 1.29 Low-Pass Circuit Response to a Square-Wave Input for Different Values of τ

In Figure 1.29(c), the equation for the rising portion is

$$V_0 = V_f - \left(V_f - V_i\right)\ e^{-t/RC}$$

From the above Figure 1.29 (c), $V_0 = v_{o1},\ V_f = V^{|}\ and\ V_i = V_1$

$v_{01} = V^{|} - (V^{|} - V_1)\ e^{-t/RC}$---1.63

Where V_1 is the voltage across the capacitor at t=0, and $V^{|}$ is the level to which the capacitor can charge.

Similarly, the equation for the falling portion is

$v_{02} = V^{||}\ -\ (V^{||} - V_2)\ e^{-(t-T_1)/RC}$ --1.64

Where V_2 is the voltage across the capacitor at t = T_1 and $V^{||}$ is the level to which capacitor can discharge.

At t = T_1 , $v_{01} = V_2$ and from Eq. (1.63) we get
$V_2 = V^{|} - (V^{|} - V_1)\ e^{-T_1/RC} = V^{|}(1 - e^{-T_1/RC}) + V_1\ e^{-T_1/RC}$ ------------------------------------1.65

At $t = T_1 + T_2$, $v_{o2} = V_1$ and from Eq. (1.64) we get

$V_1 = V^{||}\ -\ (V^{||} - V_2)\ e^{-(T_1+T_2-T_1)/RC} =\ V^{||}\,(1 - e^{-T_2/RC})\ + V_2\ e^{-T_2/RC}$ ------------------1.66

Substituting this value of V_1 in the above Eq. (1.65) we get

$$V_2 = V^{|}(1 - e^{-T_1/RC}) + \left[V^{||}\,(1 - e^{-T_2/RC})\ + V_2\ e^{-T_2/RC}\right]\ e^{-T_1/RC}$$

$i.e.\ V_2\ = \frac{V^{|}\,(1-e^{-T_1/RC})+V^{||}\,(1-e^{-T_2/RC})\,e^{-T_1/RC}}{1-e^{-(T_1+T_2)/RC}}$ ---------------------------------------1.67

Similarly substituting this value of V_2 in the above Eq. (1.65), we get

$V_1\ = \frac{V^{||}\,(1-e^{-T_2/RC})+V^{|}\,(1-e^{-T_1/RC})\,e^{-T_2/RC}}{1-e^{-(T_1+T_2)/RC}}$ ---1.68

For a symmetrical square wave form with zero average value,

$T_1 = T_2 = \frac{T}{2}$ and $V^{|} = -V^{||} = \frac{V}{2}$, hence $V_2 = -V_1$ and from Eq. (1.68), the voltage V_1 is

$$V_1\ =\ \frac{\frac{V}{2}\left(1-e^{-T/2RC}\right)\ -\frac{V}{2}\left(1-e^{-T/2RC}\right)e^{-T/2RC}}{1-e^{-T/RC}}$$
$$=\ \frac{V}{2}\left(\frac{1 - e^{-T/2RC} - e^{-T/2RC} + e^{-T/RC}}{1 - e^{-T/RC}}\right)$$
$$=\ \frac{V}{2}\frac{\left(1 - e^{-T/2RC}\right)^2}{(1 + e^{-T/2RC})(1 - e^{-T/2RC})}$$
$$=\ \frac{V}{2}\frac{\left(1 - e^{-T/2RC}\right)}{(1 + e^{-T/2RC})} = \frac{V}{2}\frac{\left(e^{T/2RC} - 1\right)}{(e^{T/2RC} + 1)} = \frac{V}{2}\frac{\left(e^{2T/4RC} - 1\right)}{(e^{2T/4RC} + 1)}$$
$$= \frac{V}{2}\frac{\left(e^{2x}-1\right)}{(e^{2x}+1)} = \frac{V}{2}\tanh x\,,\quad where\ x = \frac{T}{4RC}$$

$\therefore V_1 = \frac{V}{2}\tanh x$---1.69

Where $x = \frac{T}{4RC}$ and T is the time period of the square wave.

Similarly,

$V_2 = -V_1\ =\ -\frac{V}{2}\frac{\left(1-e^{-T/2RC}\right)}{\left(1+e^{-T/2RC}\right)} = \frac{V}{2}\frac{\left(1-e^{T/2RC}\right)}{\left(1+e^{T/2RC}\right)}$ --1.70

$$V_2 = \frac{V}{2}\frac{\left(e^{2T/4RC} - 1\right)}{(e^{2T/4RC} + 1)}$$

$V_2 = \frac{V}{2}\frac{(e^{2x}-1)}{(e^{2x}+1)} = \frac{V}{2}\tanh x$ --1.71

Using Eqs (1.70) and (1.71), it is possible to calculate V_2 and V_1 and plot the output waveforms as given in Figures 1.29 (c) and (d), respectively. If $\tau << T$, then the wave shape is maintained. And if $\tau >> T$, the wave shape is highly distorted, but the output of the low-pass circuit is now a triangular wave. So, it is possible to derive a triangular wave from a square wave by choosing τ to be very large when compared to $T/2$ of the symmetric square wave.

1.14 Low-Pass *RC* Response for a Ramp Input:

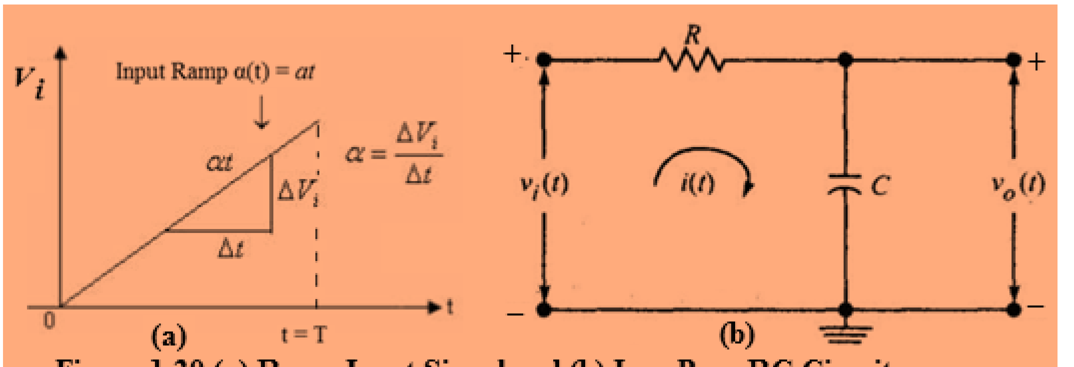


Figure 1.30 (a) Ramp Input Signal and (b) Low Pass RC Circuit

When a low-pass RC circuit shown in Figure 1.30(b) is excited by a Ramp input shown in Figure 1.30 (a) where the ramp input is given by

$$v_i = \alpha\, t,\ where\ \ \alpha \text{ is the slope of the ramp}$$

The output will be $v_o = v_i - v_R$ where $v_R$voltage is across the resistor. Assuming the elemental values of both High pass and Low Pass are identical, then the output voltage of RC high pass is the voltage drop across the resistor (v_R) of a low pass circuit. Considering the output of RC High pass from Eq. (1.30) as $V_o(t) = \ V_R(t) = \alpha\,\tau\left[1 - e^{-(t/\tau)}\right] \ = \ \alpha\,\tau\left[1 - \frac{t}{2RC}\right]$

The output from Low Pass circuit is given by

$V_o(t) = V_c(t) = V_i(t) - V_R(t)$

$V_o(t) = \alpha\, t - \alpha\,\tau\left(1 - e^{-(t/\tau)}\right)$ ---1.72

Case 1: If $\tau << T$ then $e^{-T/\tau} \approx 0$. Therefore the deviation from the input is very small

The output voltage at t = T is given by

$$V_o(t) = \alpha\, T - \alpha\,\tau\,(1 - 0) = \ \alpha\,(T - \ \tau)$$

$V_o(t) = \ \alpha\,(T - \ RC)$ --1.73

Case 2: If $\tau >> T$ then $e^{-T/\tau}$ can be expressed as a powerseries

$$V_o(t) = \alpha\, T - \alpha\,\tau\left(1 - \left(1 - \frac{T}{\tau} + \frac{T^2}{2\,\tau^2}\right)\right) = \alpha\, T - \alpha\,\tau\left(\frac{T}{\tau} - \frac{T^2}{2\,\tau^2}\right)$$

$$= \alpha\, T - \alpha\left(T - \frac{T^2}{2\,\tau}\right)$$

$V_o(t) = \alpha\, T - \alpha\,\mathrm{T} + \ \alpha\,\frac{T^2}{2\,\tau} = \alpha\,\frac{T^2}{2\,\tau}$ ---1.74

When the time constant is very small relative to the total ramp time *T*, the ramp will be transmitted with minimum distortion. The output follows the input but is delayed by one-time constant *RC* from the input (except near the origin where there is distortion) as shown in Figure 1.31(a). If the time constant is large compared with the sweep duration i.e. if RC / *T*>> 1 the output will be highly distorted as shown in Figure 1.31(b).

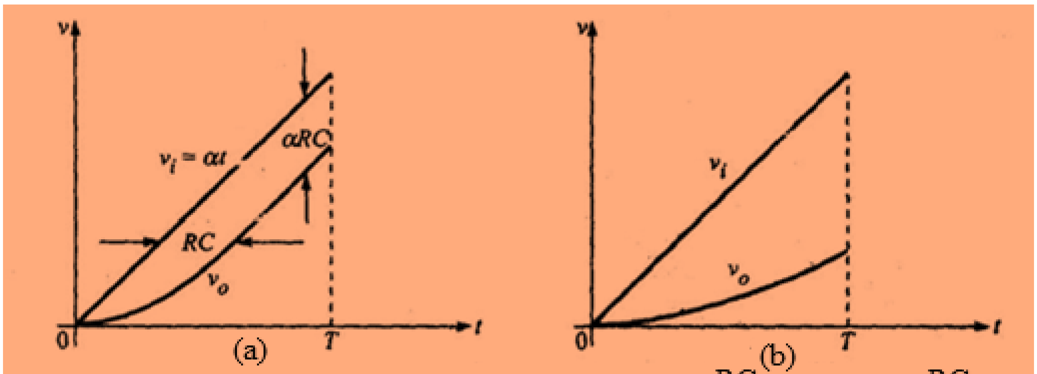


Figure 1.31Low Pass Response to a Ramp Input for (a) $\frac{RC}{T} << 1$ and (b) $\frac{RC}{T} >> 1$

This shows that a quadratic response is obtained for a linear input and hence the circuit acts as an integrator for *RC/T* >> 1.

The transmission error e_t for a ramp input is defined as the difference between the input and the output divided by the input at the end of the ramp, i.e. at *t* = *T*.

For *RC/T* << 1,

$$e_t = \frac{\alpha\, t - (\alpha t - \alpha\, R\, C)}{\alpha\, t} = \frac{\alpha\, R\, C}{\alpha\, t}$$

$$at \;\; t = T$$

$$e_t = \frac{\alpha\, RC}{\alpha\, T} = \frac{RC}{T} = \frac{1}{2\, \pi\, f_2\, T} \qquad \text{---1.75}$$

Where $f_2 = \frac{1}{2\,\pi\, R\, C}$ is the upper 3-dB frequency. For example, if we desire to pass a 2ms pulse with less than 0.1% error, the above equation yields f_2 > 80 kHz and *RC* < *2* µs.

1.15 Low-Pass *RC* Response for an Exponential Input:

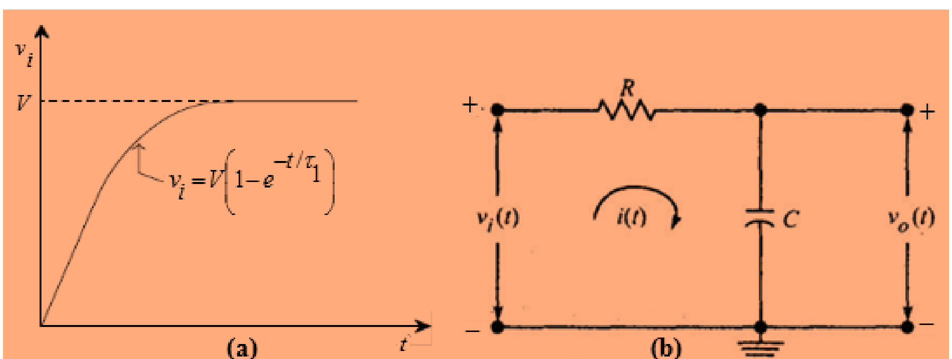


Figure 1.32 (a) Exponential Input and (b) Low Pass RC Circuit

For the low pass RC circuit which is shown in Figure 1.32(b), when an exponential input (as shown in Figure 1.32(a)) is applied, We know that from Eq. (1.34), exponential is of the form

$$v_i = V\left(1 - e^{-t/\tau_1}\right)$$

The output from Low Pass circuit is given by

$V_o(t) = V_c(t) = V_i(t) - V_R(t)$ --1.76

Case 1: Let $t/\tau_1 = x$ and $\tau/\tau_1 = n$. For $n \neq 1$, i.e., $\tau \neq \tau_1$,

Since $V_R(t)$ is the response of high-pass RC circuit for an exponential input. i.e. from Eq. (1.41)

$$v_R(t) = \frac{Vn}{(n-1)}\left(e^{-x/n} - e^{-x}\right)$$

This is the expression for the voltage across resistance R, where $\tau \neq \tau_1$. We have from Eq. (1.76)

$V_o(t) = V_c(t) = V_i(t) - V_R(t)$

$V_o(t) = V(1 - e^{-x}) - \frac{Vn}{(n-1)}\left(e^{-x/n} - e^{-x}\right)$ --1.77

Case 2: As $t/\tau = x = t/\tau_1$ and $\tau/\tau_1 = n =$, that is $\tau = \tau_1$, $n = 1$. Then from Eq. (1.44)

Since $V_R(t)$ is the response of high-pass RC circuit for an exponential input. i.e. from Eq. (1.44)

$$v_o(t) = Vx\, e^{-x}$$

So the output is given by

$$V_o(t) = V_c(t) = V_i(t) - V_R(t)$$

$$V_o(t) = V(1 - e^{-x}) - Vx\, e^{-x}$$

$v_o(t) \approx \frac{Vn}{(n-1)} e^{-x/n} = \frac{Vn}{(n-1)} e^{-t/\tau}$--1.78

1.16 The Low Pass Circuit as an Integrator:

For the low pass circuit to behave as an integrator then the voltage Variation in C is very small. From the low-pass RC circuit as shown in Figure 1.22, If the time constant of the circuit is much greater than the time period of the input signal, then the circuit behaves as an integrator. If T is to be small when compared to τ, then the frequency must be high. At high frequencies, X_c is very low when compared to R. Therefore, the voltage drop across R is very high when compared to the drop across C. If the time constant of an RC low-pass circuit is very large, the capacitor charges very slowly and so almost all the input voltage appears across the resistor for small values of time.

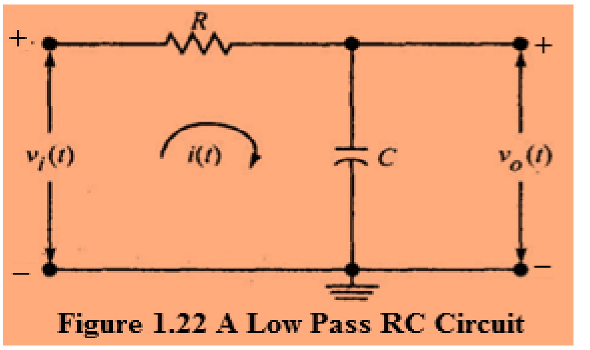


Figure 1.22 A Low Pass RC Circuit

By applying KVL to this circuit

$v_i = iR + \frac{1}{C}\int i\,dt$. And since $\frac{1}{C}\int i\,dt << iR$

$\therefore v_i = iR \quad \Rightarrow \quad i = \frac{v_i}{R}$.

Therefore, for the Low Pass RC circuit, the output is given by

$$v_o = v_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \frac{v_i}{R}\,dt = \frac{1}{RC}\int v_i dt$$

Therefore,

$$v_o \propto \int v_i dt \quad \text{---------------------} 1.79$$

So therefore, from Eq. (1.79), the output is proportional to the integral of the input signal. Hence a low pass circuit with large time constant produces an output that is proportional to the integral of the input.

As time increases, the voltage drop across *C* does not remain negligible compared with that across *R* and the output will not remain the integral of the input. The output will change from a quadratic to a linear function of time. ***If the time constant of an RC low-pass circuit is very large in comparison with the. Time required for the input signal to make an appreciable change, the circuit acts as an integrator.*** A criterion for good integration in terms of steady-state analysis is as follows: The low-pass circuit acts as an integrator provided the time constant of the circuit $RC > 15T$, where T is the period of the input sine wave. When $RC > 15T$, the input sinusoid will be shifted at least by 89.4° (instead of the ideal 90° shift required for integration) when it is transmitted through the network.

An *RC* integrator converts a square wave into a triangular wave. Integrators are almost invariably preferred over differentiators in analog computer applications for the following reasons:

1. It is easier to stabilize an integrator than a differentiator because the gain of an integrator decreases with frequency whereas the gain of a differentiator increases with frequency.

2. An integrator is less sensitive to noise voltages than a differentiator because of its limited bandwidth.

3. The amplifier of a differentiator may overload if the input waveform changes very rapidly.

4. It is more convenient to introduce initial conditions in an integrator.

1.17 Attenuators:

Attenuators are resistive networks, which are used to reduce the amplitude of the input signal. The simple resistor combination of Figure 1.33(a) would multiply the input signal by the ratio $a = R_2/(R_1 + R_2)$ independently of the frequency. If the output of the attenuator is feeding a stage of amplification, the input capacitance C_2 of the amplifier will be the stray capacitance shunting the resistor R_2 of the attenuator and the attenuator will be as shown in Figure 1.33(b), and the attenuation now is not independent of frequency. Using Thevenin's theorem, the circuit in Figure 1.33(b) may be replaced by its equivalent circuit shown in Figure 1.33(c), in which R is equal to the parallel combination of R_1 and R_2. Normally R_1 and R_2 must be large so that the nominal input impendence of the attenuator is large enough to prevent loading down the input signal. But if R_1 and R_2 are large, the rise time, $t_r = 2.2[(R_1||R_2)*C_2]$ will be large and a large rise time is normally unacceptable.

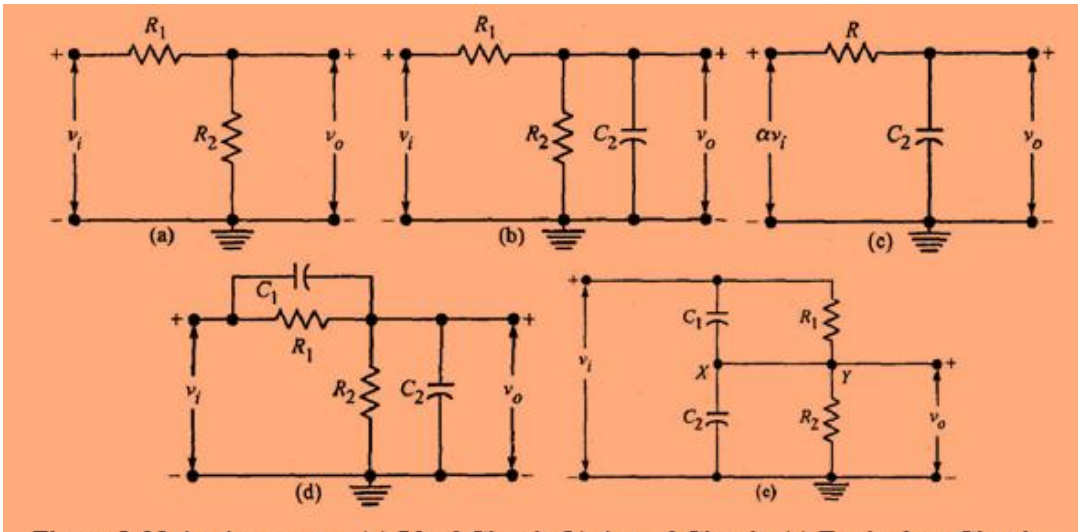


Figure 1.33 An Attenuator (a) Ideal Circuit,(b) Actual Circuit, (c) Equivalent Circuit, (d) Compensated attenuator and (e) Compensatted Attenuator redrawn as a Bridge

The attenuator may be compensated by shunting R_1 by a capacitor C_1 as shown in Figure 1.33(d), so that its attenuation is once again independent of frequency. The circuit has been drawn in Figure 1.33(e) to suggest that the two resistors and the two capacitors may be viewed as the four arms of a bridge. If $R_1 C_1 = R_2 C_2$, the bridge will be balanced and no current will flow in the branch connecting the point X to the point Y. For the purpose of computing the output, the branch X-Y may be omitted and the output will again be equal to C_M, independent of the frequency. In practice, C_1 will ordinarily have to be made adjustable. Suppose a step signal of amplitude V volts is applied to the circuit. As the input changes abruptly by V volts at $t = 0$, the voltages across C_1 and C_2 will also change abruptly. This happens because at t = 0, the capacitors act as short-circuits and a very large (ideally infinite) current flows though the capacitors for an infinitesimally small time so that a finite charge$q = \int_{0^-}^{0^+} i(t)\, dt$ is delivered to each capacitor. The initial output voltage is determined by the capacitors. Since the same current flows through the capacitors C_1 and C_2, we have the charge accumulated in the capacitor$C_1 \ = \int_{0^-}^{0^+} i(t)\, dt = q$.

$$\therefore Initial\ voltage\ across\ C_1 = \frac{q}{C_1} = V_1$$

Charge accumulated in capacitor $C_2 = \int_{0^-}^{0^+} i(t)\, dt = q$

Initial voltage across $C_2 = \frac{q}{C_2} = V_2 = v_0(0^+)$

Input signal, $V = V_1 + V_2 = \frac{q}{C_1} + \frac{q}{C_2} = q\left(\frac{C_1+C_2}{C_1 C_2}\right)$

$$\frac{v_0(0^+)}{V} = \frac{\frac{q}{C_2}}{q\left(\frac{C_1+C_2}{C_1C_2}\right)} = \frac{C_1}{C_1+C_2}$$

Or $v_0(0^+) = V\frac{C_1}{C_1+C_2} = v_i\,\frac{C_1}{C_1+C_2}$--1.80

The final output voltage is determined by the resistors R_1 and R_2, because the capacitors C_1 and C_2 act as open circuits for the applied dc voltage under steady-state conditions. Hence

$v_0(\infty) = V\frac{R_2}{R_1+R_2} = v_i\,\frac{R_2}{R_1+R_2}$ ---1.81

Looking back from the output terminals (with the input short circuited) we see a resistor $R = R_1 R_2 /(R_1 + R_2)$ in parallel with $C = C_1 + C_2$. Hence the decay or rise of the output (when the attenuator is not perfectly compensated) from the initial to the final value takes place exponentially with a time constant $\tau = RC$. The responses of an attenuator for C_1 equal to, greater than, and less than $R_2 C_2/R_1$ are indicated in Figure 1.34.

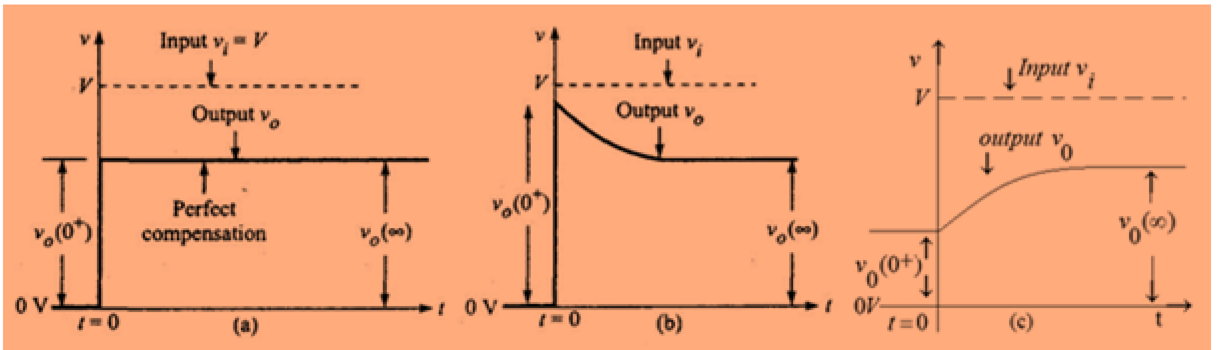


Figure 1.34 Response of Compensated Attenuator (a) Perfect compensation, (b) Over compensation and (c) Under compensation

(a) Perfect compensation means $v_0(0^+) = v_0(\infty)$: $\therefore\ C_1 = \frac{R_2C_2}{R_1}$ $i.e. R_1C_1 = R_2C_2$

(b) Over compensation means $v_0(0^+) > v_0(\infty)$: $\therefore\ C_1 > \frac{R_2C_2}{R_1}$

(c) Under compensation means $v_0(0^+) < v_0(\infty)$: $\therefore\ C_1 < \frac{R_2C_2}{R_1}$

Perfect compensation is obtained if $v_0(0^+) = v_0(\infty)$, that is, if the rise time$t_r = 0$.

$$\therefore V\frac{C_1}{C_1+C_2} = V\frac{R_2}{R_1+R_2} \qquad ---------------------------- 1.82$$
$$i.e.\ R_1C_1 + R_2C_1 = R_2C_1 + R_2C_2 \qquad ----------------------1.83$$
$$i.e.\ R_1C_1 = R_2C_2\ or\ C_1 = \frac{R_2C_2}{R_1} \qquad ----------------------1.84$$

This is the balanced bridge condition. The extreme values of $v_o(0^+)$ are 0 for $C_1 = 0$. In the above analysis we have assumed that an infinite current flow through the capacitors at $t = 0^+$ and hence the capacitors get charged instantaneously. This is valid only if the generator resistance is zero. In general, the output resistance of the generator is not zero but is of some finite value. Hence the impulse response is physically impossible. So, even though the attenuator is compensated, the ideal step response can never be obtained. Nevertheless, an improvement in rise time does result if a compensated attenuator is used. For example, if the output is one-tenth of the input, then the rise time of the output using the attenuator is one-tenth of what it would be without the attenuator. The compensated attenuator will reproduce faithfully the signal, which appears at its input terminals. However, if the output impedance of the generator driving the attenuator is not zero, the signal will be distorted right at the input to the attenuator. This situation is illustrated in Figure 1.35(a) in which a generator of step voltage V and of source resistance Rs is connected to the attenuator. Since the lead which joins the point X and point Y may be open circuited, the circuit may be redrawn as in Figure 1.35(b). Usually $R_S << R_1 + R_2$, so the input to the attenuator will be an exponential of time constant $Rs*C^{|}$, where C' is the capacitance of the series combination of C_1 and C_2 > i.e. $C^{|} = C_1 C_2/(C_1 + C_2)$. It is this exponential waveform rather than the step, which the attenuator will transmit faithfully. If the generator terminals were connected directly to the terminals to which the attenuator output is connected, the generator would see a capacitance C_2. In this case the waveform at these terminals would be an exponential with time constant $\tau = R_S*C_2$.

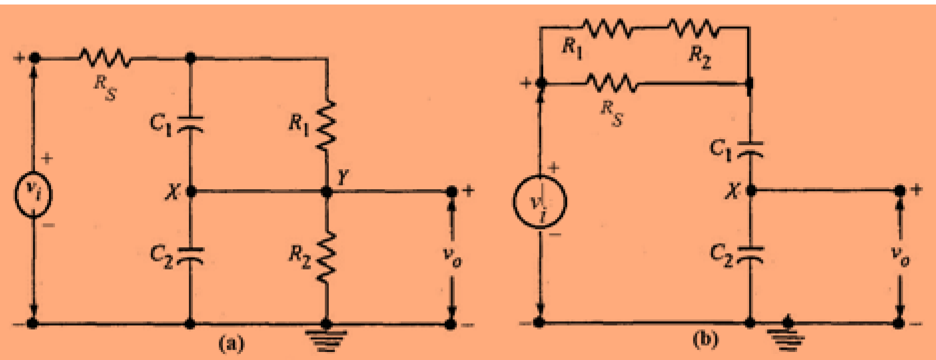


Figure 1.35 (a) Compensated Attenuator including source resistance Rs and (b) Its equivalent circuit with $v_i^{|} = V(R_1+R_2)/(R_S+R_1+R_2)$

When the attenuator is used τ > the time constant is $\tau^{|} = R_S * C^{|}$. Then, $\frac{\tau^{|}}{\tau} = \frac{C^{|}}{C_2} = \frac{C_1}{C_1+C_2} = \alpha$ is an improvement in waveform results. For example, if the attenuation is equal to 10 (a = 1/10), then the rise time of the waveform would be divided by a factor 10.

Example Problem 6: Calculate the output voltages and draw the waveforms when (a) C_1 = 75 pF, (b) C_1 = 100 pF, (c) C_1 = 50 pF for the circuit shown in Figure 1.36(a). The input step voltage is 50 V.

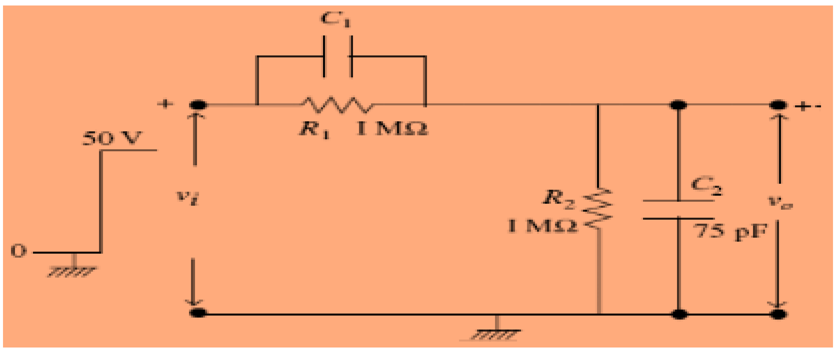


Figure 1.36(a) The given attenuator circuit

Solution: For perfect compensation, $R_1 C_1 = R_2 C_2$. Here $R_1 = R_2$.

1. When C_1 = 75 pF, then the attenuator is perfectly compensated. The rise time of the output waveform is zero. Attenuation α is,

$$\alpha = \frac{R_2}{R_1 + R_2} = \frac{1}{1+1} = 0.5$$

$v_o(0+) = v_o(\infty) = \alpha v_i = 0.5 \times 50 = 25$ V

2. When C_1 = 100 pF, then the attenuator is over-compensated, hence $v_o(0^+) > v_o(\infty)$. The output at $t = 0^+$,

$$v_o(0+) = v_i \times \frac{C_1}{C_1 + C_2} = 50 \times \frac{100}{100+75} = 28.6 \text{ V}$$

The output at $t = \infty$,

$$v_o(\infty) = v_i \times \frac{R_2}{R_1 + R_2} = 50 \times \frac{1}{1+1} = 25 \text{ V}$$

From Figure 1.36(a)

$$R = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{and} \qquad C = C_1 + C_2$$

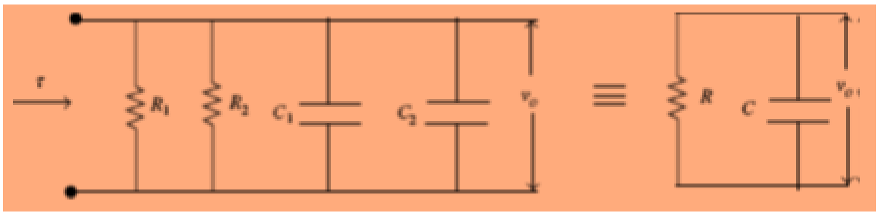


Figure 1.36(b) The equivalent circuit to get the time constant for the decay of the overshoot

From Figure 1.36(b), time constant τ_1 with which the overshoot at $t = 0^+$ decays to the steady-state value is:

$$\tau_1 = \frac{R_1 R_2}{R_1 + R_2}(C_1 + C_2) = \frac{1 \times 1}{1 + 1} \times 10^6 \times (100 + 75) \times 10^{-12} = 87.5\ \mu s$$

Fall time $t_f = 2.2\ \tau_1 = 2.2 \times 87.5 \times 10^{-6} = 192.5\ \mu s$

3. When $C_1 = 50$ pF, then the attenuator is under-compensated.

The output at $t = 0^+$:

$$v_o(0^+) = v_i \times \frac{C_1}{C_1 + C_2} = 50 \times \frac{50}{50 + 75} = 20\ \text{V}$$

The output at $t = \infty$:

$$v_o(\infty) = v_i \times \frac{R_2}{R_1 + R_2} = 50 \times \frac{1}{1 + 1} = 25\ \text{V}$$

The time constant, τ_2, with which the output rises to the steady-state value is:

$$\tau_2 = \frac{R_1 R_2}{R_1 + R_2}(C_1 + C_2) = \frac{1 \times 1}{1 + 1} \times 10^6 \times (50 + 75) \times 10^{-12} = 62.5\ \mu s$$

Rise time, $tr = 2.2\ \tau_2$

$$t_r = 2.2 \times 62.5 \times 10{-}6 = 137.5\ \mu s$$

Finally, the input and output responses for the given problem are drawn in Figure 1.36(c).

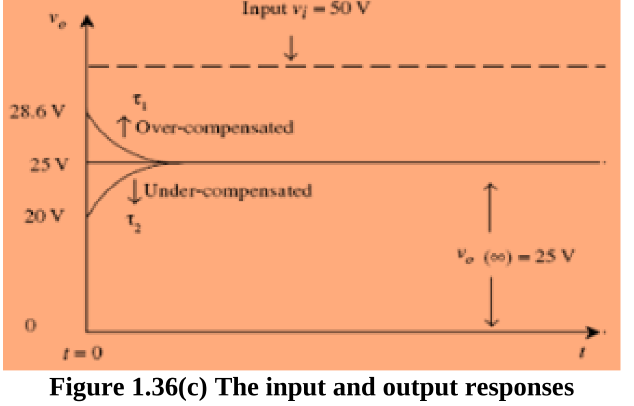


Figure 1.36(c) The input and output responses

Application of Attenuator as a CRO probe:

A perfectly compensated attenuator is sometimes used to reduce the signal amplitude when the signal is connected to a CRO to display a waveform. A typical CRO probe may be represented as in Figure 1.37. Example 6 helps to further elucidate and elaborate the functioning of the attenuator circuit.

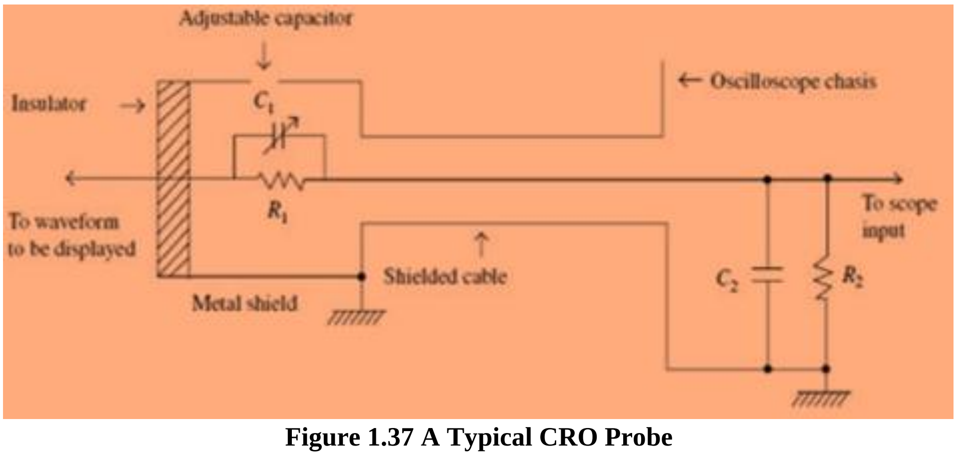
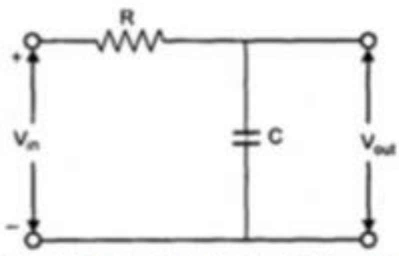
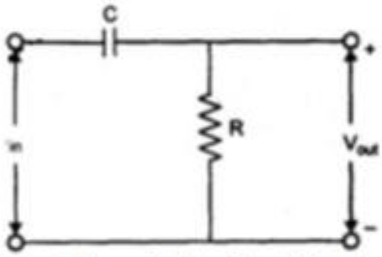
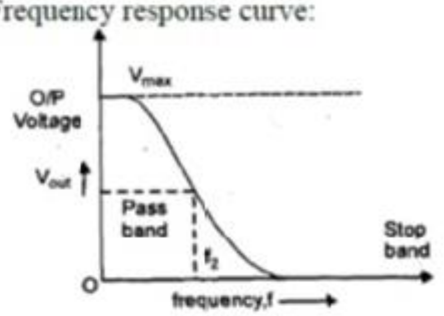
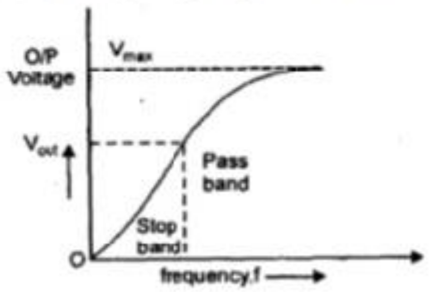


Figure 1.37 A Typical CRO Probe

1.18. Differences between Low Pass RC Circuits and High Pass RC Circuits:

Low-Pass RC Circuits	High-Pass RC Circuits
(i) In a low pass circuit v_{out} is taken across the capacitor.	(i) In high pass RC data, the O/P voltage v_{out} is taken across the resistance.
(ii) It passes low frequency signals and blocks the high frequency signals.	(ii) It blocks or attenuates low frequencies, but allows high frequency signals to pass through it.
(iii) 	(iii) 
(iv) Current through the circuit is given as: $I = \frac{V_{in}}{R - jX_C}$	(iv) Current through the circuit is given as: $I = \frac{V_{in}}{R - jX_C}$
(v) Output voltage is given as $V_{out} = \left[\frac{-jX_C}{R - jX_C}\right] . V_{in}$	(v) Output voltage is given as: $V_{out} = \left[\frac{R}{R - jX_C}\right] . V_{in}$
(vi) Magnitude of amplitude is given by: $\lvert A \rvert = \frac{1}{\sqrt{1 + \left(\frac{f}{f_2}\right)^2}}$	(vi) Magnitude amplitude is given by: $\lvert A \rvert = \frac{1}{\sqrt{1 + \left(\frac{f_1}{f}\right)^2}}$
Where f_2 = cut off frequency.	where f_1 = frequency at which $X_C = R$.
(vii) Phase angle: $\theta = \tan^{-1}\left(\frac{f}{f_2}\right)$	(vii) Phase angle: $\theta = \tan^{-1}\left(\frac{f_1}{f}\right)$
(viii) Frequency response curve: 	(viii) Frequency response curve: 
(ix) At very high frequencies the capacitive reactance become very small so 0/p becomes equal to i/p.	(ix) With the increase in frequency the reactance of the capacitor decreases and therefore, the output will be zero and gain increase.
(x) R-C circuits with time constant larger than time period of the input signal are used as by-pass capacitors.	(x) R-C circuits with RC >> T is employed in R-C coupling of amplifiers where distortion and differentiation of waveform is to be avoided.
(xi) It is used is generation of triangular and ramp waveforms.	(xi) R-C circuits with RC << T is employed generate pipes for triggering electronic circuit such as flip-flop multivibrators.