→ Optimal of Coherent Reception : PSK, FSK, QPSK

(or)

Comparison of digital modulation techniques with Pe

(or)

Pe for various Data Txion techniques :-

→ BPSK system :- It provides minimum Pe because it satisfies optimum condition $s_1(t) = -s_2(t)$

For BPSK system, $\left.\begin{aligned} s_1(t) &= A\cos\omega_0 t \\ s_2(t) &= -A\cos\omega_0 t \end{aligned}\right\} \quad 0 \le t \le T$

$\therefore\ s_1(t) = -s_2(t)$ ✓

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_s}{\eta}\right)^{1/2}$$

$$E_s = \int_0^T s_1^2(t)\,dt = \int_0^T s_2^2(t)\,dt = \int_0^T (A\cos\omega_0 t)^2\,dt$$

$$= A^2\int_0^T \frac{(1+\cos 2\omega_0 t)}{2}\,dt$$

$$= \frac{A^2}{2}\left[t + \frac{\sin 2\omega_0 t}{2\omega_0}\right]_0^T$$

$$\Rightarrow E_s = \frac{A^2}{2}T.$$

$$\boxed{\therefore\ P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{A^2 T}{2\eta}\right]^{1/2}}$$

→ BPSK in a correlator (imperfect phase synchronization)

$$s_o(T) = \frac{1}{\gamma}\int_0^T s_i(t)\left[s_1(t) - s_2(t)\right]dt$$

let $s_1(t)$ is Rxed by correlator, for BPSK system, the sampled o/p is

$$S_{01}(T) = \frac{1}{\tau}\int_0^{T} A\cos\omega_0 t\left[A\cos\omega_0 t - (-A\cos\omega_0 t)\right]dt$$

$$= \frac{A^2}{\tau}\int_0^T 2\cos^2\omega_0 t \, dt$$

$$= \frac{2A^2}{\tau}\int_0^T \left[\frac{1+\cos 2\omega_0 t}{2}\right]dt$$

$$\Rightarrow S_{01}(T) = \frac{A^2}{\tau}\left[t + \frac{\sin 2\omega_0 t}{2\omega_0}\right]_0^T = \frac{A^2}{\tau}T = CA^2T$$

$$\left\{\because C = \frac{1}{\tau}\right\}$$

similarly when $S_2(t)$ is Rxed, then

$$S_{02}(T) = \frac{1}{\tau}\int_0^T (-A\cos\omega_0 t)\left[A\cos\omega_0 t - (-A\cos\omega_0 t)\right]dt$$

$$= \frac{1}{\tau}\int_0^T -2A^2\cos^2\omega_0 t \, dt$$

$$\Rightarrow S_{02}(T) = -\frac{2A^2}{\tau}\int_0^T \cos^2\omega_0 t \, dt = -CA^2T.$$

$$\therefore P_0(T) = S_{01}(T) - S_{02}(T) = CA^2T - (-CA^2T) = 2CA^2T$$

Suppose now local s/g used at correlator is $2A\cos(\omega_0 t + \phi)$ instead of $2A\cos\omega_0 t$, where $\phi$ is some fixed phase offset then correlator o/p $S_0(T)$ is either $CA^2T\cos\phi$ or $-CA^2T\cos\phi$.

$$\Rightarrow S_0(T) = \pm CA^2T\cos\phi.$$

$$S_0(T) = \frac{1}{\tau}\int_0^T A\cos\omega_0 t\,(2A\cos(\omega_0 t + \phi))\,dt$$

$$= \frac{2A^2}{\tau}\int_0^T \cos\omega_0 t\cos(\omega_0 t + \phi)\,dt$$

$$= \frac{2A^2}{2\tau}\int_0^T\left[\cos(2\omega_0 t + \phi) + \cos\phi\right]dt$$

$$= \frac{A^2}{\tau}\left[\frac{\sin(2\omega_0 t + \phi)}{2\omega_0} + t\cos\phi\right]_0^T$$

$$\Rightarrow S_{01}(T) = \frac{A^2}{\tau}T\cos\phi = CA^2T\cos\phi$$

$$\text{llly } S_{02}(T) = -CA^2T\cos\phi \qquad \therefore S_0(T) = \pm CA^2T\cos\phi$$

$$P_0(T) = 2CA^2T\cos\phi$$

& energy becomes $E_s \cos^2\phi$

$\therefore P_e = \frac{1}{2}\,\text{erfc}\left(\frac{E_s}{\eta}\right)^{1/2}$ can be replaced as

$$P_e = \frac{1}{2}\,\text{erfc}\left(\frac{E_s \cos^2\phi}{\eta}\right)^{1/2}$$

$$\Rightarrow \boxed{P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_s \cos^2\phi}{\eta}}}$$

$\{\because \cos\phi$ is decreasing fn, $P_e$ is low, if $\phi$ is low.

When there is a phase offset $\phi$ in the BPSK Rxer using a correlator Rxer the phase offset $\phi$ increases $P_e$ which inturn deteriorates the performance of BPSK system. In comm. system with $P_e$ in the range $10^{-4}$ to $10^{-7}$, if $\phi = 25°$, $P_e$ is increased by a factor 10 than that of a system with $\phi = 0$.

→ BFSK system :-

$S_1(t) = A\cos(\omega_0+\Omega)t$ ; $P_e = \frac{1}{2}\,\text{erfc}\left(\frac{\nu^2}{8}\right)^{1/2}$

$S_2(t) = A\cos(\omega_0-\Omega)t$

where $\nu^2_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)}\,df = \int_0^T \frac{P^2(t)\,dt}{\eta/2}$

$\left\{\because \text{Using Parseval's theorem, } \int_{-\infty}^{\infty}|P(f)|^2\,df = \int_0^T P^2(t)\,dt\right]$

$$\therefore \nu^2_{max} = \frac{2}{\eta}\int_0^T \left(A\cos(\omega_0+\Omega)t - A\cos(\omega_0-\Omega)t\right)^2 dt$$

$$= \frac{2}{\eta}A^2\left[\int_0^T (-2\sin\omega_0 t\,\sin\Omega t)^2\,dt\right.$$

$$= \frac{2}{\eta}A^2\, 4\int_0^T \sin^2\omega_0 t\,\sin^2\Omega t\,\Big]\,dt$$

$$= \frac{8A^2}{\eta}\int_0^T \left(\frac{1-\cos 2\omega_0 t}{2}\right)\left(\frac{1-\cos 2\Omega t}{2}\right)dt$$

$$\frac{8A^2}{4\eta}\int_0^T \left[1-\cos 2\omega_0 t - \cos 2\Omega t + \cos 2\omega_0 t\,\cos 2\Omega t\right]dt$$

$$\Rightarrow v_{max}^2 = \frac{8A^2}{4\eta}\int_0^1 1 - \cos 2\omega_0 t - \cos 2\Omega t + \frac{1}{2}\left[\cos 2(\omega_0+\Omega)t + \cos 2(\omega_0-\Omega)t\right] dt$$

$$\Rightarrow v_{max}^2 = \frac{2A^2}{\eta}\left[t - \frac{\sin 2\omega_0 t}{2\omega_0} - \frac{\sin 2\Omega t}{2\Omega} + \frac{1}{2}\left[\frac{\sin 2(\omega_0+\Omega)t}{2(\omega_0+\Omega)} + \frac{\sin 2(\omega_0-\Omega)t}{2(\omega_0-\Omega)}\right]\right]_0^T$$

$$\Rightarrow v_{max}^2 = \frac{2A^2}{\eta}\left[T - \frac{\sin 2\omega_0 T}{2\omega_0} - \frac{\sin 2\Omega T}{2\Omega} + \frac{1}{4}\frac{\sin 2(\omega_0+\Omega)T}{(\omega_0+\Omega)} + \frac{1}{4}\frac{\sin 2(\omega_0-\Omega)T}{(\omega_0-\Omega)}\right]$$

2nd method

$$v_{max}^2 = \frac{2}{\eta}\int_0^T \left(A\cos(\omega_0+\Omega)t - A\cos(\omega_0-\Omega)t\right)^2 dt$$

$$= \frac{2A^2}{\eta}\int_0^T \cos^2(\omega_0+\Omega)t + \cos^2(\omega_0-\Omega)t - 2\cos(\omega_0+\Omega)t\cos(\omega_0-\Omega)t \, dt$$

$$= \frac{2A^2}{\eta}\int_0^T \left[\frac{1+\cos 2(\omega_0+\Omega)t}{2}\right] + \left[\frac{1+\cos 2(\omega_0-\Omega)t}{2}\right] - \left[\cos 2\omega_0 t + \cos 2\Omega t\right] dt$$

$$= \frac{2A^2}{\eta}\left\{\frac{1}{2}\left(t + \frac{\sin 2(\omega_0+\Omega)t}{2(\omega_0+\Omega)}\right] + \frac{1}{2}\left[t + \frac{\sin 2(\omega_0-\Omega)t}{2(\omega_0-\Omega)}\right] - \frac{\sin 2\omega_0 t}{2\omega_0} - \frac{\sin 2\Omega t}{2\Omega}\right]_0^T$$

$$= \frac{2A^2}{\eta}\left[\frac{T}{2} + \frac{\sin 2(\omega_0+\Omega)T}{4(\omega_0+\Omega)} + \frac{1}{2}T + \frac{\sin 2(\omega_0-\Omega)T}{4(\omega_0-\Omega)} - \frac{\sin 2\omega_0 T}{2\omega_0} - \frac{\sin 2\Omega T}{2\Omega}\right]$$

$$\Rightarrow v_{max}^2 = \frac{2A^2}{\eta}\left[T - \frac{\sin 2\omega_0 T}{2\omega_0} - \frac{\sin 2\Omega T}{2\Omega} + \frac{1}{4}\frac{\sin 2(\omega_0+\Omega)T}{(\omega_0+\Omega)} + \frac{1}{4}\frac{\sin 2(\omega_0-\Omega)T}{\omega_0-\Omega}\right]$$

For $\omega_0 >> \Omega$

then $$v_{max}^2 = \frac{2A^2}{\eta}\left[T - \cancel{\frac{\sin 2\omega_0 T}{2\omega_0}} - \frac{\sin 2\Omega T}{2\Omega} + \frac{1}{4}\cancel{\frac{\sin 2\omega_0 T}{\omega_0}} + \frac{1}{4}\cancel{\frac{\sin 2\omega_0 T}{\omega_0}}\right]$$

$$\boxed{v_{max}^2 = \frac{2A^2}{\eta}\left[T - \frac{\sin 2\Omega T}{2\Omega}\right]}$$

$$\Rightarrow \gamma^2_{max} = \frac{2A^2T}{\eta}\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right]$$

$\therefore \gamma^2$ is max when $\boxed{\sin 2\Omega T = -1} \Rightarrow 2\Omega T = 3\pi/2$

$$\therefore \gamma^2_{max} = \frac{2A^2T}{\eta}\left[1 - \frac{(-1)}{3\pi/2}\right]$$

$$= \frac{2A^2T}{\eta}\left[1 + \frac{2}{3\pi}\right] = 2.42\,\frac{A^2T}{\eta}$$

$$= 4.84\,\frac{A^2T}{2\eta}$$

$$= \frac{4.84}{\eta}\left(\frac{A^2T}{2}\right)$$

$$= \frac{4.84}{\eta}\,E_s \qquad \left\{\because E_s = \frac{A^2T}{2}\right\}$$

$$\therefore P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\gamma^2}{8}\right)^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{4.84}{\eta\, 8}E_s\right)^{1/2}$$

$$\Rightarrow \boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(0.6\,\frac{E_s}{\eta}\right)^{1/2}}$$

Comparing $P_e|_{BPSK}$ & $P_e|_{BFSK}$ systems, the same probability of error can be achieved if the s/g energy in BPSK is 0.6 time as large as that of FSK. Hence a two decibels increase in the Txed s/g power is required for BFSK system to posess the same probability of error, the reason for this is in BPSK, $\boxed{S_1(t) = -S_2(t)}$ where this condition has been failed in BFSK system. $\therefore \boxed{P_e|_{BPSK} < P_e|_{BFSK}}$

→ Non-coherent detection of FSK :- When the phase of incoming s/g is not used in receiving a s/g at Rxer, it is called non-coherent detection. $P_e|_{\text{Non-coherent detection of FSK}}$, $\boxed{P_e = \frac{1}{2}e^{-E_s/2\eta}}$

→ Differential Phase shift Keying (DPSK)

In DPSK, always a previous bit should be taken as reference. Decision boundary is considered at an angle $\frac{\pi}{2}$. Therefore when the phase difference b/w two consecutive bits differ by $< \frac{\pi}{2}$, we decide it as bit '1' was sent & if the phase difference b/w two consecutive bits differ by $> \pi/2$, we decide it as '0' bit was sent.

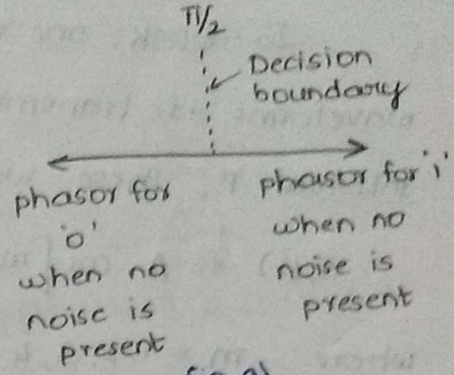


fig a)

From fig(b), consider an example of 111 as Rxed bits. DPSK Rxer compares 1st bit with 2nd bit & reads an angle $\theta_1$, $\theta_1 < \pi/2$. Therefore it decides as '1' bit is received as 2nd bit.

Again DPSK Rxer compares 2nd bit with 3rd bit & reads an angle $\theta_2$, $\theta_2 > \pi/2$. Therefore it decides that '0' bit was transmitted which is an error.

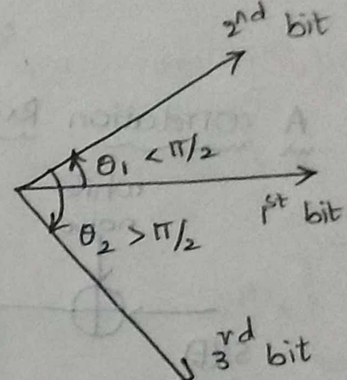


fig b): phasors when noise is present.

$$\boxed{P_e|_{DPSK} = \frac{1}{2} e^{-E_s/\eta}}$$

→ Comparison of $P_e$ for different systems :-

$$P_e|_{BPSK} = \frac{1}{2}\,\mathrm{erfc}\sqrt{E_s/\eta}$$

$$P_e|_{DPSK} = \frac{1}{2} e^{-E_s/\eta}$$

$$P_e|_{\text{Non-coherent FSK}} = \frac{1}{2} e^{-E_s/2\eta}$$

$$P_e|_{BFSK} = \frac{1}{2}\,\mathrm{erfc}\sqrt{0.6\frac{E_s}{\eta}}$$