5. Noise in PCM & DM systems

→ S/N ratio in PCM system :-

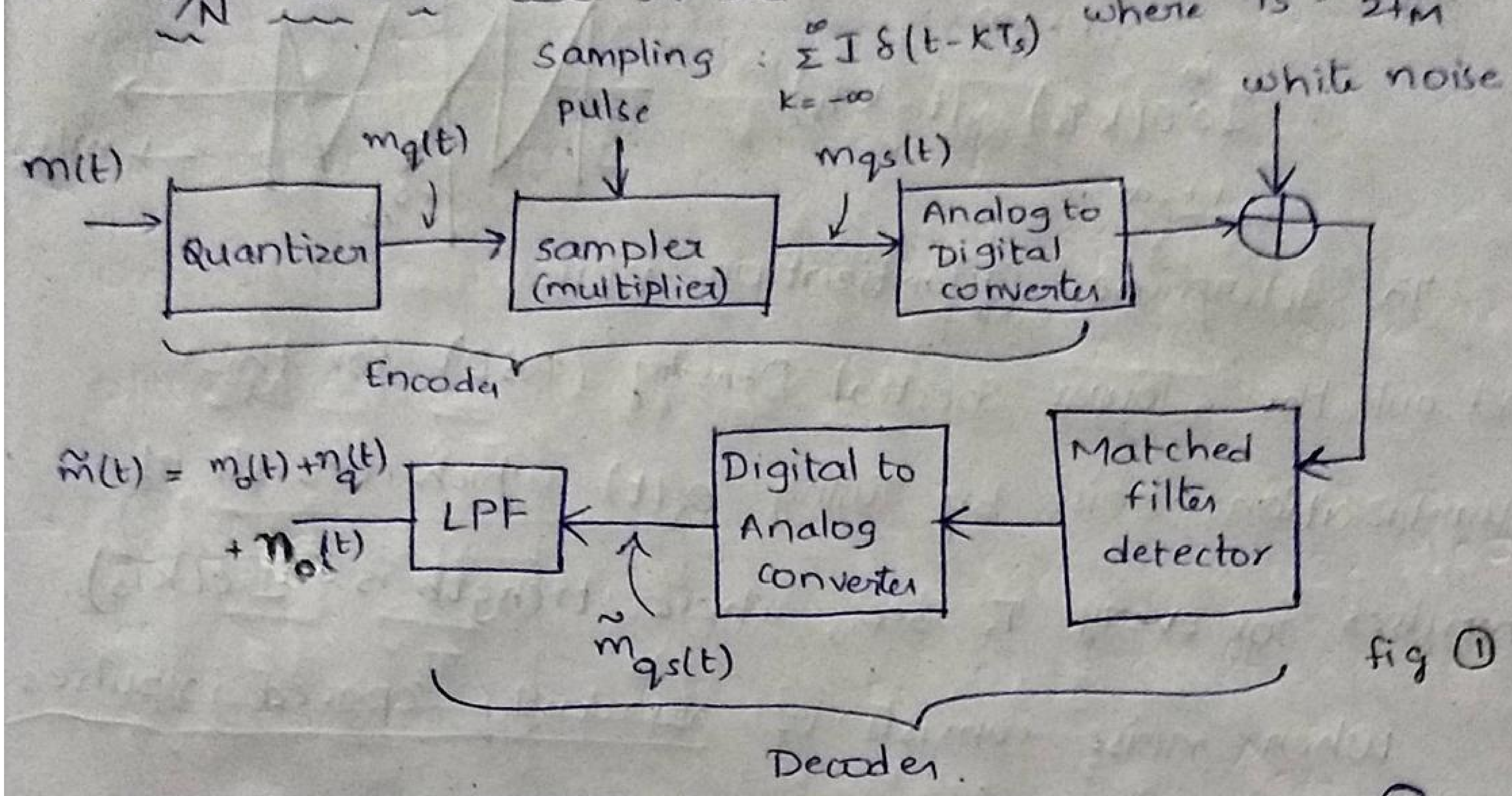


fig ①

$$\boxed{m_q(t) = m(t) + e(t)} \longrightarrow ①$$

$$\& \quad m_{qs}(t) = m_q(t)\, s(t) = m_s(t) + e_s(t) \quad ②$$

$$= m(t)\, I \sum_{k=-\infty}^{\infty} \delta(t-kT_s) + e(t)\, I \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$$

where $e(t)$ is, the error s/g which results from the process of quantization

The base band s/g is quantized, giving rise to quantized s/g $m_q(t)$ where $m_q(t) = m(t) + e(t)$

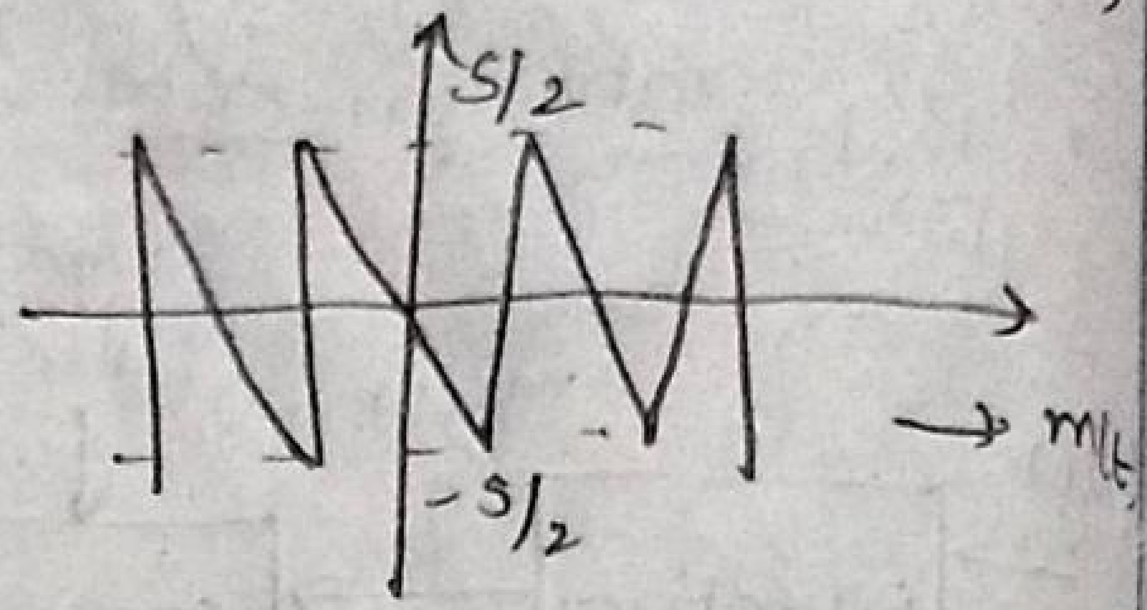
The o/p of the sampled ckt is $m_{qs}(t)$ which is consisting of instantaneous samples corresponding to base band s/g $m(t)$ & the samples corresponding to quantization error $e(t)$.

By maintaining unity gain from i/p of ADC to the o/p of DAC, the same sample values can be recovered at the Rxing end. By considering the predominant noises, quantization noise N_q & thermal noise N_{th},

$$\frac{S_o}{N_o} = SNR\Big|_{PCM} = \frac{S_o}{N_q + N_{th}}$$

Measurement of Quantization noise (N_q) :-

$e(t) = m_q(t) - m(t)$



$$e_s(t) = e(t)\, I \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= I \sum_{k=-\infty}^{\infty} e(kT_s)\, \delta(t - kT_s)$$

To determine quantization noise, find out the Power Spectral Density (PSD) for this quantization noise by using $e_s(t)$ which consists of impulses for every T_s sec, whose strength is $I\, e(kT_s)$

When noise consists of <u>randomly spaced impulses</u>

PSD for this noise is

$$\boxed{G_n(f) = \frac{1}{T_s} |P(f)|^2}$$ where $P(f) = F[p(t)]$

$\therefore$ PSD of sampled quantization error is

$$G_{es}(f) = \frac{1}{T_s} |P(f)|^2$$

$$= \frac{1}{T_s} I^2 \overline{e^2(kT_s)}$$

$$= \frac{I^2}{T_s} \frac{S^2}{12}$$

{ Here $P(f) = I\, e(kT_s)$ & $\overline{e^2(t)} = \frac{S^2}{12}$ for uniform quantizer $= \overline{e^2(kT_s)}$

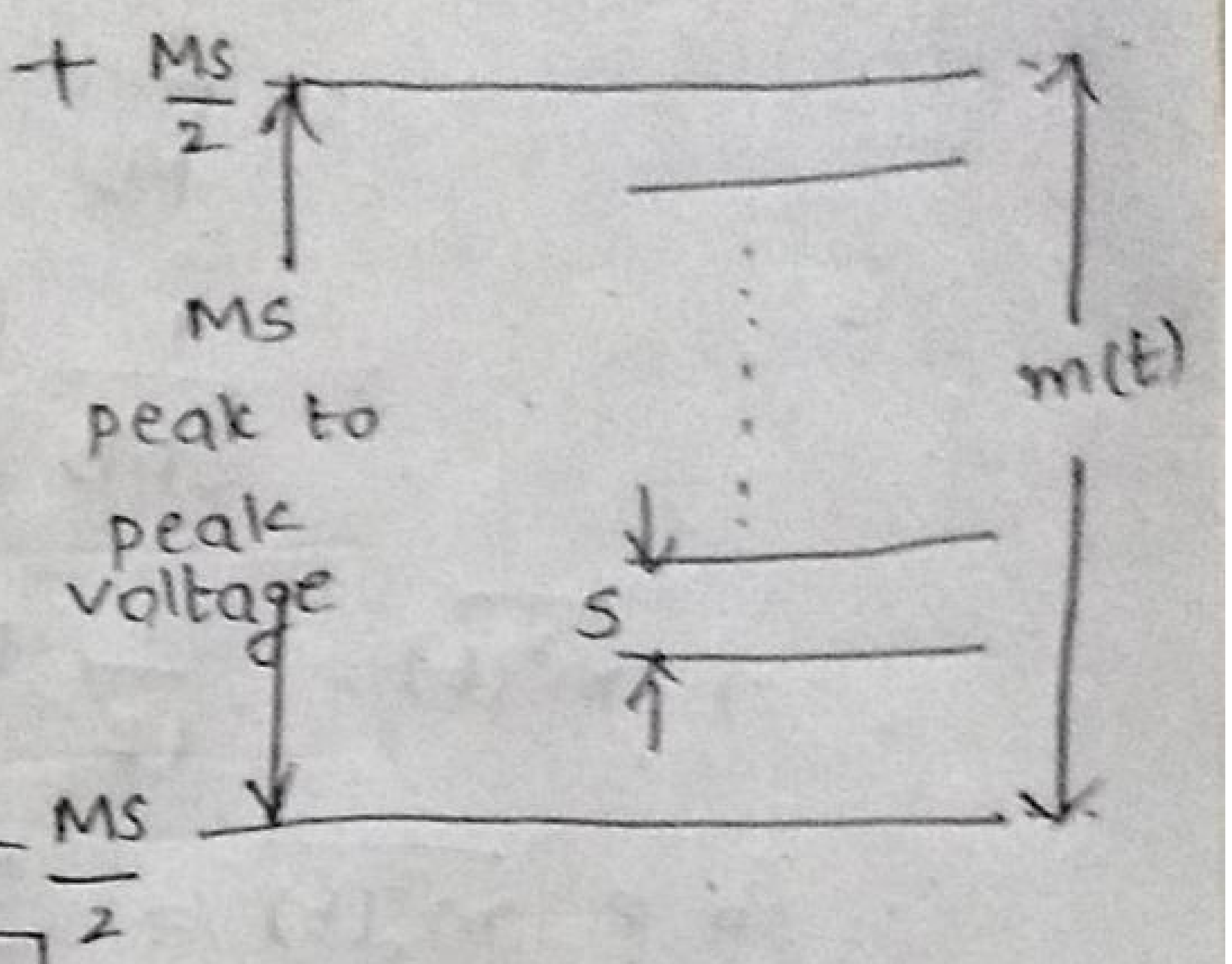
$\therefore$ Quantization noise $N_q = \int_{-f_M}^{f_M} G_{es}(f)\, df = \int_{-f_M}^{f_M} \frac{I^2 S^2}{12 T_s} df$

$$= \frac{I^2 S^2}{12 T_s} 2f_M$$

$$\Rightarrow \boxed{N_q = \frac{I^2 S^2}{12 T_s^2}} \quad \left\{ \because T_s = \frac{1}{2f_M} \right.$$

O/p s/g power :-

$$m_s(t) = m(t)\, I \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$



pdf of instantaneous value of m is

$$\boxed{f(m) = \frac{1}{MS} \quad \text{for} \quad -\frac{MS}{2} < m < \frac{MS}{2}}$$

The sampled o/p consists of various impulses of strength I & are separated by T_s sec.

$$m_s(t) = m(t)\, I \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$= m(t)\left[\frac{I}{T_s} + \sum_{k=1}^{\infty} \frac{2I}{T_s} \cos\frac{2\pi kT}{T_s}\right]$$

$$= m(t)\frac{I}{T_s} + \frac{I}{T_s}\left[2m(t)\cos\frac{2\pi T}{T_s} + 2m(t)\cos\frac{4\pi t}{T_s} + \cdots\right]$$

The 1st term in the fourier series is a dc component and all other terms lies outside the base band filter with cut-off freq f_M.

∴ The o/p of base band filter is $m_o(t) = m(t)\frac{I}{T_s}$

$$\therefore \boxed{m_o(t) = m(t)\frac{I}{T_s}}$$

∴ o/p s/g power, $S_o = \overline{m_o^2(t)} = \frac{I^2}{T_s^2}\overline{m^2(t)}$

where $\overline{m^2(t)} = \int_{-MS/2}^{MS/2} m^2 f(m)\, dm$

$$= \int_{-MS/2}^{MS/2} m^2 \frac{1}{MS}\, dm$$

$$= \frac{1}{Ms}\left[\frac{m^3}{3}\right]_{-Ms/2}^{Ms/2}$$

$$= \frac{1}{3Ms}\left[2\,\frac{M^3S^3}{8}\right]$$

$$\Rightarrow \overline{m^2(t)} = \frac{1}{12} M^2S^2$$

$$\therefore S_o = \overline{m_o^2(t)} = \frac{I^2}{T_s^2}\,\overline{m^2(t)} = \frac{I^2}{T_s^2}\,\frac{1}{12} M^2S^2$$

$$\Rightarrow \boxed{S_o = \frac{I^2}{T_s^2}\,\frac{M^2S^2}{12}}$$

$$\therefore \frac{S_o}{N_q} = \frac{\frac{I^2}{T_s^2}\,\frac{M^2S^2}{12}}{\frac{I^2S^2}{12\,T_s^2}} = M^2 = (2^N)^2 = 2^{2N}$$

$\{\because M = 2^N\}$ where N is word length

$\therefore \frac{S_o}{N_q}$ increases as N increases

$\therefore$ As word length (N) increases, the efficiency of PCM system increases.

Effect of Thermal Noise :-

eg ①:- consider an eq of $P_e = 10^{-5} = \frac{1}{100,000}$ i.e, 1 bit is in error for 1,00,000 bits

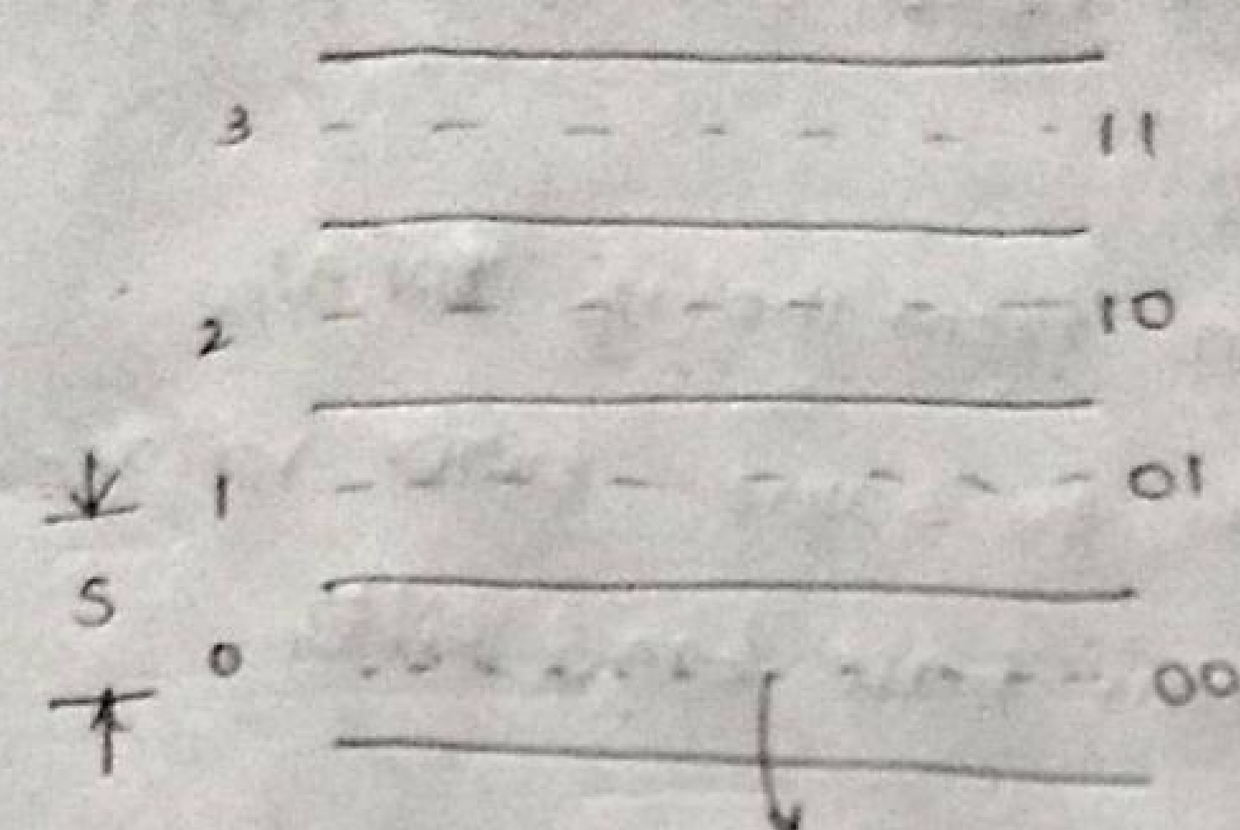
where P_e is bit error probability

if 1 word has 8 bits i.e, for N = 8

then no. of words = $\frac{1,00,000}{8} = 12,500 \Rightarrow \left\{\frac{1}{NP_e}\right\}$

i.e, 1 word is in error for 12,500 words Txed.

eg ⑥: Consider a quantizer with 4 levels



① let 00 be the Txed code word

Txed code word 0 0
Rxed code word 0 1
$\rightarrow$ error mag $\}$ S

② Txed code word 0 0
Rxed code word 1 0
$\}$ 2S
$\hookrightarrow$ error

s → step size   quantization level

∴ Let Δm_s is the error. & $\overline{\Delta m_s^2}$ is the variance.

let us consider a quantizer consisting of 'M' levels & codeword length is 'N' bits. then the least significant level is assigned as 00..... 000, the next level as 0,0..... 001 & so on. & the most significant level as 11..... 111

most significant level	11 111
	⋮
	00 001
least significant level	00 000

If there is an error in LSB of code word, then error magnitude is 'S' v. If the error occurs in the next significant bit. then error magnitude is '2S'. then error magnitude is $\overline{\Delta m_s^2}$

let the error be Δm_s. Hence the variance is $\overline{\Delta m_s^2}$

$$\therefore \overline{\Delta m_s^2} = \frac{1}{N}\left[S^2 + (2S)^2 + (4S)^2 + \dots (2^{N-1}S)^2\right]$$

$$= \frac{S^2}{N}\left[1 + 2^2 + 4^2 + \dots (2^{N-1})^2\right]$$

$$= \frac{S^2}{N}\left[\frac{4^N - 1}{4-1}\right]$$

$$= \frac{S^2}{N}\left[\frac{(2^2)^N - 1}{3}\right] = \frac{S^2}{3N}\left[2^{2N} - 1\right]$$

{ in the form of G.P where $r = \frac{4^2}{2^2} = 4$

$\because \frac{r^n - 1}{r - 1}$ for $r > 1$ }

$$\Rightarrow \boxed{\overline{\Delta m_s^2} = \frac{S^2}{3N}\left[2^{2N} - 1\right] \simeq \frac{2^{2N} S^2}{3N}} \text{ for } N \geq 2$$

The effect of thermal noise errors may be taken into

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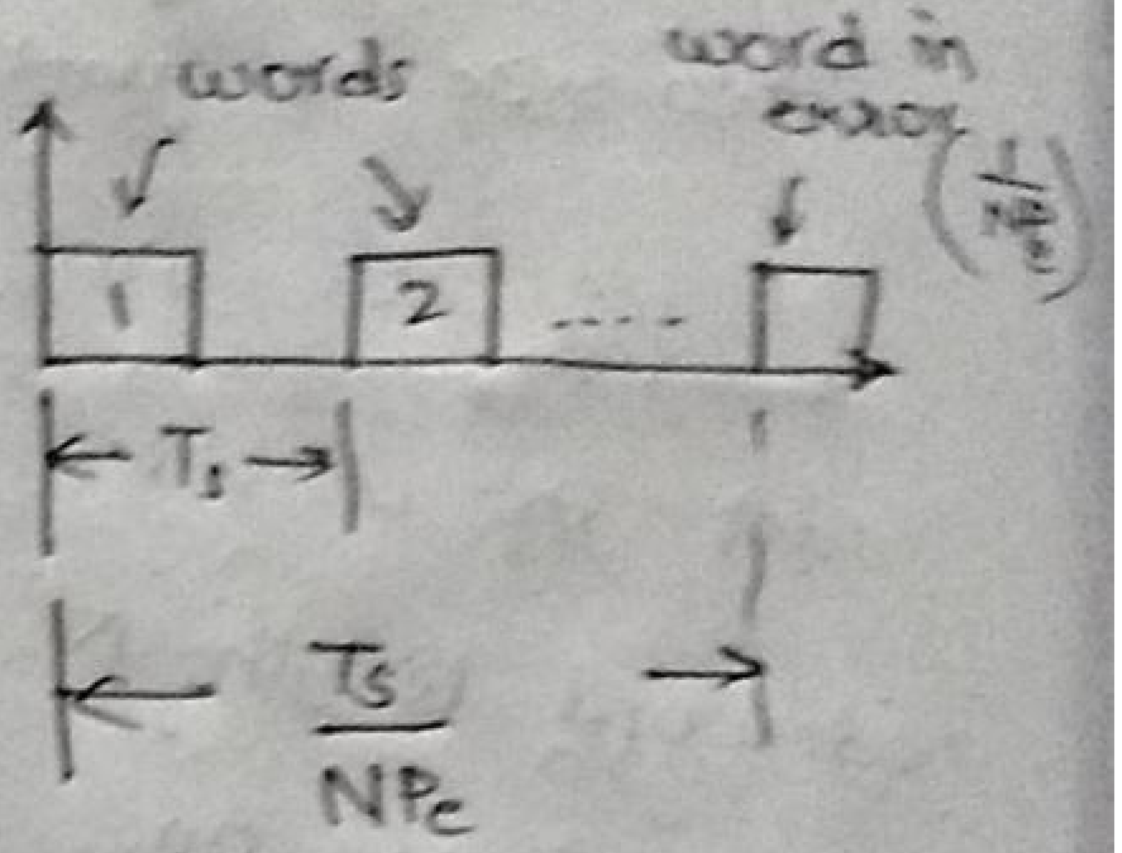
account by adding an error voltage Δm_s at the i/p of A/D converter & by deleting white noise source & matched filter in fig ① {∵ ie, Block diagram}

By maintaining unity gain from the i/p of A/D converter to the o/p of D/A converter, the same error voltage appears at the i/p of the low-pass base band filter.

The result of succession of errors results a train of impulses of random amplitude & random time of occurence each of strength $\underline{I\Delta m_s}$. If words are sampled in time by T_s with N bits/word



The mean time separation b/w words which are in error is $\boxed{T = \frac{T_s}{NP_e}}$

∴ PSD of thermal noise impulse train is

$$G_{th}(f) = \frac{1}{T_s}|P(f)|^2 = \frac{I^2\,\overline{\Delta m_s^2}}{T} \qquad \left\{ \begin{array}{l} \text{Here } P(f) = I\Delta m_s \\ T_s = T \end{array} \right.$$

$$= \frac{I^2\, 2^{2N} S^2}{3N\, T_s} NP_e = \frac{I^2 S^2\, 2^{2N} P_e}{3T_s}$$

∴ O/p power due to thermal-noise error is

$$N_{th} = \int_{-f_M}^{f_M} G_{th}(f)\, df = \int_{-f_M}^{f_M} \frac{I^2 S^2\, 2^{2N} P_e}{3T_s}\, df$$

$$= \frac{I^2 S^2\, 2^{2N} P_e}{3T_s}[2f_M]$$

$$\Rightarrow \boxed{N_{th} = \frac{I^2 S^2\, 2^{2N} P_e}{3T_s^2}} \qquad \left\{ \because T_s = \frac{1}{2f_M} \right\}$$

O/P s/g to Noise ratio for PCM :-

$$S_o = \frac{I^2}{T_s^2}\frac{M^2 S^2}{12} \; ; \; N_q = \frac{I^2 S^2}{12 T_s^2} \; ; \; N_{th} = \frac{I^2 S^2 2^{2N} P_e}{3 T_s^2}$$

$$\therefore \frac{S_o}{N_o} = \frac{S_o}{N_q + N_{th}} = \frac{I^2 M^2 S^2 / T_s^2 \, 12}{\frac{I^2 S^2}{12 T_s^2} + \frac{I^2 S^2 2^{2N} P_e}{3 T_s^2}} = \frac{M^2}{1 + 4 \, 2^{2N} P_e}$$

$$\boxed{SNR\Big|_{PCM} \Rightarrow \frac{S_o}{N_o}\Big|_{PCM} = \frac{(2^N)^2}{1+4(2^{2N}P_e)} = \frac{2^{2N}}{1+4(2^{2N}P_e)}} \quad \left\{ \because M = 2^N \right\}$$

$$P_e\Big|_{BPSK} = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_s}{\eta}}$$

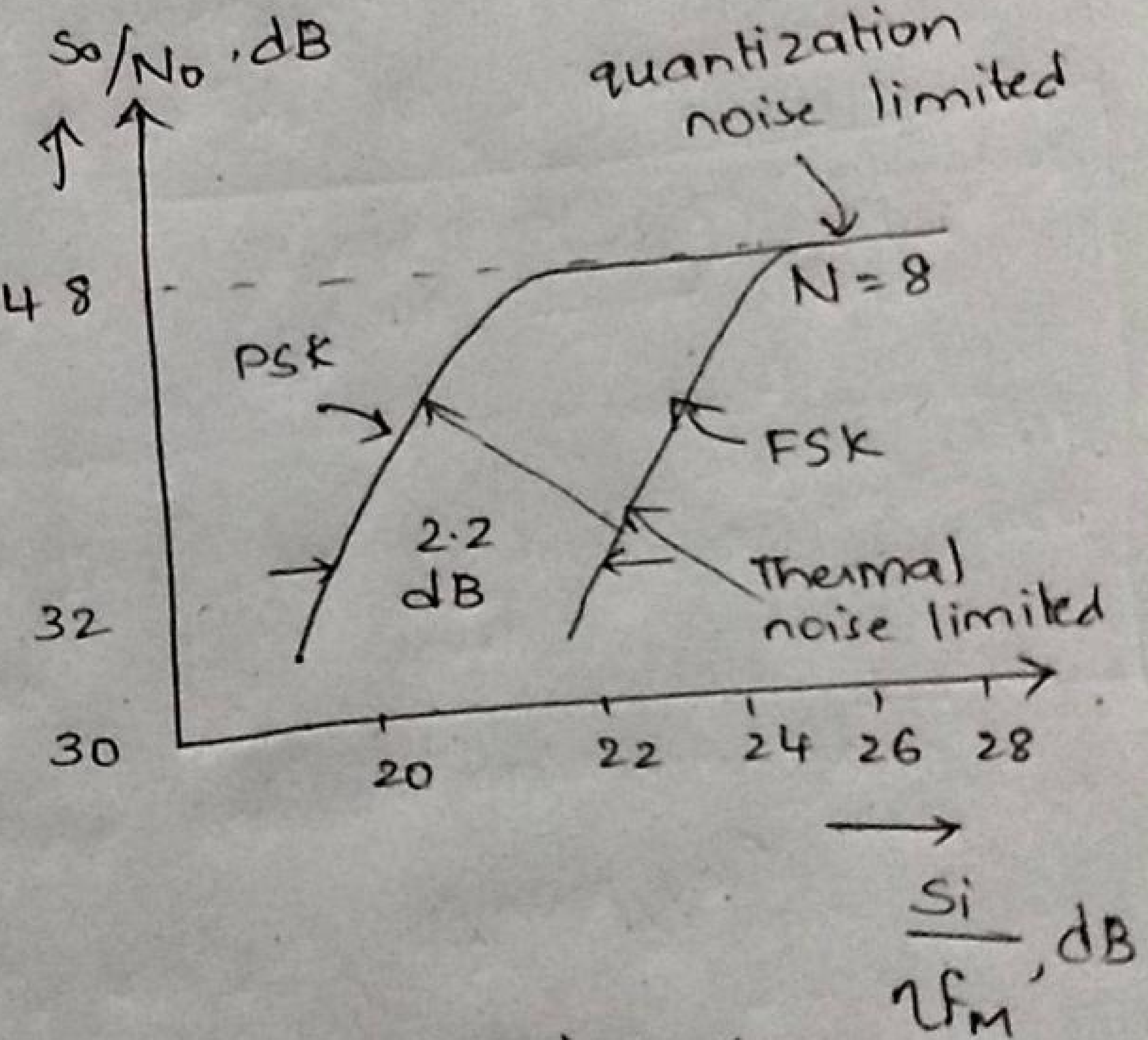
$$P_e\Big|_{BFSK} = \frac{1}{2}\,\mathrm{erfc}\sqrt{0.6\frac{E_s}{\eta}}$$

where E_s is bit energy

S_i is Rxed s/g power

N is word length

$$E_s = S_i \frac{T_s}{N}$$



$\therefore$ SNR can be increased either by increasing N value or by decreasing P_e. The FSK threshold occurs at a value 2.2 dB greater than that of BPSK system. For lower bit s/g energy thermal noise is limited, & for higher bit s/g energy quantization noise is limited.

———— x ———— x ————

→ For lower s/g energy $\Rightarrow E_s \downarrow \Rightarrow P_e \uparrow$
$\therefore$ as $P_e \uparrow \Rightarrow$ SNR $\downarrow$ es } $\therefore N_{th}$ is dependent of $P_e \therefore N_{th}$ should be limited

For higher s/g energy $\Rightarrow E_s \uparrow \Rightarrow P_e \downarrow$
as $P_e \downarrow \Rightarrow$ SNR $\uparrow$ es. } $\therefore N_{th}$ is small & N_q should be limited to ↑ s/