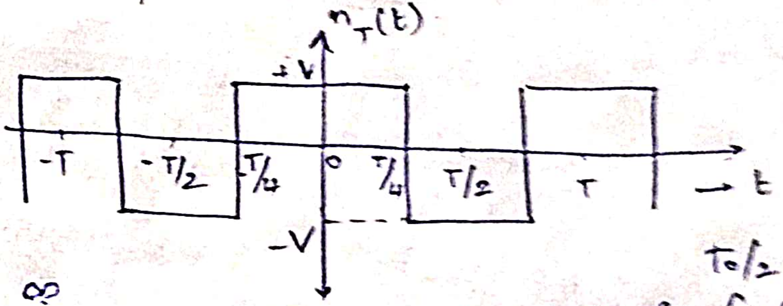
Problem :-

→ A symmetrical square wave shown in fig. makes excursions b/w +V & −V volts where V is a random variable, uniformly distributed b/w 1 V and 2 V. It has a fundamental freq of 1 KHz. Make a normalized plot of two sided power spectral density waveform.



Ans)

$$n_T(t) = \sum_{k=1}^{\infty} C_k \cos\frac{2\pi k t}{T_0} \quad \text{where} \quad C_k = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} v(t)\cos\frac{2\pi k t}{T_0}\,dt$$

$$\therefore C_k = \frac{2}{T}\left[\int_{-T/2}^{-T/4}(-V)\cos\frac{2\pi k t}{T}\,dt + \int_{-T/4}^{0}(+V)\cos\frac{2\pi k t}{T}\,dt + \int_{0}^{T/4} V\cdot\cos\frac{2\pi k t}{T}\,dt + \int_{T/4}^{T/2}(-V)\cos\frac{2\pi k t}{T}\,dt\right]$$

$$\Rightarrow C_k = \frac{2}{T}\left[\left[\frac{-V\sin\frac{2\pi k t}{T}}{\frac{2\pi k}{T}}\right]_{-T/2}^{-T/4} + \left[\frac{V\sin 2\pi k t/T}{\frac{2\pi k}{T}}\right]_{-T/4}^{0} + \left[\frac{V\sin\frac{2\pi k t}{T}}{\frac{2\pi k}{T}}\right] + \left(\frac{-V\sin\frac{2\pi k t}{T}}{\frac{2\pi k}{T}}\right)_{T/4}^{T/2}\right]$$

$$\Rightarrow C_k = \frac{2}{T}\left[\frac{VT}{2\pi k}\left\{-\sin\left(\frac{-2\pi k}{4}\right) + \sin\left(\frac{-2\pi k}{2}\right) + 0 - \sin\left(\frac{-2\pi k T}{4T}\right) + \sin\frac{2\pi k}{4} - 0 + \left[-\sin\frac{2\pi k}{2} + \sin\frac{2\pi k}{4}\right.\right.\right.$$

$$= \frac{V}{\pi k}\left\{+\sin\frac{\pi}{2}k - \cancel{\sin \pi k}^{\,0} + \sin\frac{\pi k}{2} + \sin\frac{\pi k}{2} - \cancel{\sin \pi k}^{\,0} + \sin\frac{\pi k}{2}\right]$$

$\Rightarrow C_k = \frac{V}{\pi k}\left[4 \sin\frac{\pi}{2}k\right] = \frac{4\overline{V}}{\pi k} \sin\frac{\pi}{2}k$

Power spectral density $G_n(k\Delta f) = \frac{C_k^2}{4\Delta f}$

$\therefore G_n(k\Delta f) = \frac{C_k^2}{4\Delta f} = \frac{16\overline{V^2}}{4\pi^2 k^2 (\Delta f)^2} \sin^2\frac{\pi}{2}k$

where V is a random variable uniformly distributed b/w 1 V & 2 V

$\therefore f(v) = 1 \longrightarrow$ pdf of V uniformly distributed b/w 1 V & 2 V

then pdf $f(v) = \frac{1}{b-a} = \frac{1}{2-1} = 1$

$\therefore \overline{V^2} = \int v^2 f(v)\,dv$

$= \int_1^2 v^2 (1)\,dv = \left[\frac{v^3}{3}\right]_1^2 = \frac{1}{3}[8-1] = 7/3$

$\therefore G_n(k\Delta f) = \frac{16}{4\pi^2 k^2 (\Delta f)^2} \times \frac{7}{3} \underbrace{\sin^2\left(\frac{\pi}{2}k\right)}$ value is '1' for k is odd, '0' for k is even

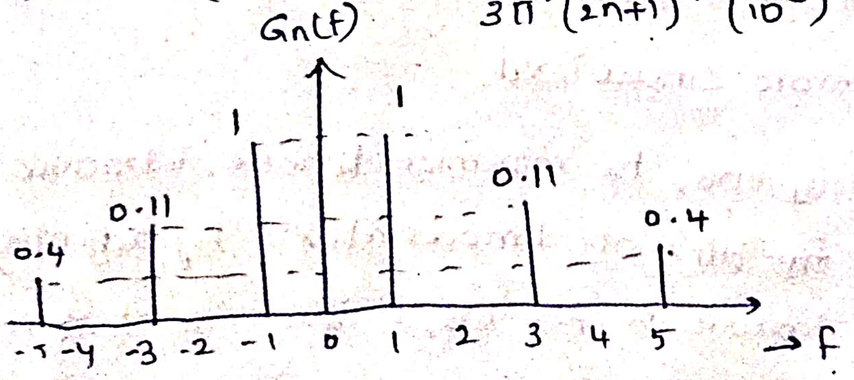
$= \frac{28}{3\pi^2 k^2 (\Delta f)^2}$

Given $\Delta f = 1$ KHz $\Rightarrow G_n(k\Delta f) = \frac{28}{3\pi^2 k^2 (10)^3} \underbrace{\sin^2\left(\frac{\pi}{2}k\right)}$

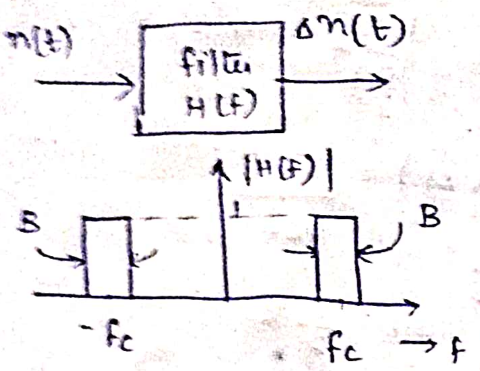
value is '1' for k is odd, '0' for k is even

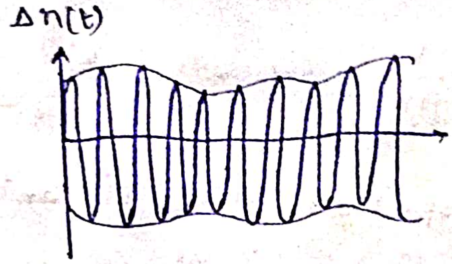
for k is odd, then let $k = (2n+1)$

then $G_n((2n+1)\Delta f) = \frac{28}{3\pi^2 (2n+1)^2 (10^3)}$



→ Response of a Narrow Band filter to Noise :-





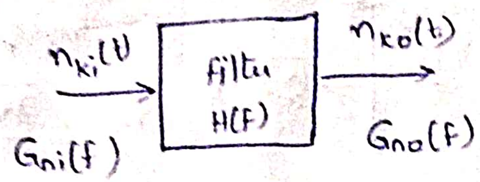
When noise is passed through a narrow band filter the o/p of the filter looks like a sinusoid except that the amplitude varies randomly.

The spectral range of the envelope of the filter o/p encompasses from $-B/2$ to $B/2$ where B is the filter bandwidth. The average freq of the waveform is the centre freq f_c of the filter. If $B << f_c$, the envelope changes very slowly & makes an appreciable change only over many cycles. Thus the spacings of zero crossings of the waveform are not precisely constant, the change from cycle to cycle is small & when averaged over many cycles is quite constant at the value $\frac{1}{2f_c}$.

Finally, as 'B' becomes progressively smaller, so also does the average amplitude & the waveform becomes more and more sinusoidal.

→ Note :- At the receiving end, to minimize noise, introduce a filter with transfer fn H(f). at demodulator & by adjusting BW of the filter as narrow as possible, finite amount of noise is passed through the filter so that S/N ratio can be improved.

→ Effect of filter on Power Spectral density (PSD) of noise :



let spectral component of noise at $k\Delta f$ freq $n_{ki}(t)$ is applied to a filter whose transfer fn at freq $k\Delta f$ is

$$H(k\Delta f) = |H(k\Delta f)| \, e^{j\phi_k}$$

$$= |H(k\Delta f)| \angle \phi_k \longrightarrow ①$$

∴ i/p noise at $k\Delta f$ freq is

$$n_{ki}(t) = a_k \cos 2\pi k\Delta f t + b_k \sin 2\pi k\Delta f t$$

Power associated with $n_{ki}(t)$ is

$$P_{ki} = \frac{\overline{a_k^2}}{2} + \frac{\overline{b_k^2}}{2} = \frac{\overline{a_k^2} + \overline{b_k^2}}{2} \longrightarrow ②$$

The corresponding o/p spectral component of noise will be

$$n_{ko}(t) = a_k |H(k\Delta f)| \cos(2\pi k\Delta f t + \phi_k) + b_k |H(k\Delta f)| \sin(2\pi k\Delta f t + \phi_k)$$

Power associated with $n_{ko}(t)$ is

$$P_{ko} = \frac{\overline{[a_k |H(k\Delta f)|]^2}}{2} + \frac{\overline{[b_k |H(k\Delta f)|]^2}}{2} \longrightarrow ③$$

Since $|H(k\Delta f)|$ is deterministic fn,

$$\left.\begin{aligned} \overline{[|H(k\Delta f)| \, a_k]^2} &= |H(k\Delta f)|^2 \, \overline{a_k^2} \\ \& \quad \overline{[|H(k\Delta f)| \, b_k]^2} &= |H(k\Delta f)|^2 \, \overline{b_k^2} \end{aligned}\right\} \rightarrow ④$$

then eq ③ implies

$$P_{ko} = \frac{\overline{a_k^2}}{2} |H(k\Delta f)|^2 + \frac{\overline{b_k^2}}{2} |H(k\Delta f)|^2$$

$$\Rightarrow P_{ko} = \left[\frac{\overline{a_k^2} + \overline{b_k^2}}{2}\right] |H(k\Delta f)|^2$$

$$\Rightarrow \boxed{P_{ko} = P_{ki} \; |H(k\Delta f)|^2} \rightarrow (5) \quad \left\{ \because \text{from eq } (3) \quad P_{ki} = \frac{\overline{a_k^2} + \overline{b_k^2}}{2} \right.$$

we know that power $P_k = 2 G_n(k\Delta f)\,\Delta f$, so that we can write

$$\boxed{\begin{array}{l} P_{ko} = 2\, G_{no}(k\Delta f)\,\Delta f \;\; \& \\ P_{ki} = 2\, G_{ni}(k\Delta f)\,\Delta f \end{array}} \rightarrow (6)$$

{ where $G_{ni}(k\Delta f)$ & $G_{no}(k\Delta f)$ are power spectral densities of noise at i/p & o/p respectively at $k\Delta f$ freq.

Substituting eqs (6) in eq (5), we get

$$P_{ko} = P_{ki} \; |H(k\Delta f)|^2$$

$$\Rightarrow 2\, G_{no}(k\Delta f)\,\Delta f = 2\, G_{ni}(k\Delta f)\,\Delta f \; |H(k\Delta f)|^2$$

$$\Rightarrow G_{no}(k\Delta f) = |H(k\Delta f)|^2 \, G_{ni}(k\Delta f)$$

$$\Rightarrow \boxed{G_{no}(k\Delta f) = |H(k\Delta f)|^2 \, G_{ni}(k\Delta f)} \rightarrow (7)$$

In the limit as $\Delta f \to 0$, & $k\Delta f$ is replaced by a continuous variable f, then eq (7) becomes

$$G_{no}(f) = |H(f)|^2 \, G_{ni}(f)$$

$$\Rightarrow \boxed{G_{no}(f) = |H(f)|^2 \, G_{ni}(f)}$$

Note :-

→ o/p Noise power, $N_o = \int_{-\infty}^{\infty} G_{no}(f)\, df$

$$= \int_{-\infty}^{\infty} |H(f)|^2 \, G_{ni}(f)\, df.$$

where $G_{no}(f)$ is o/p noise PSD ; $G_{ni}(f)$ is i/p noise PSD