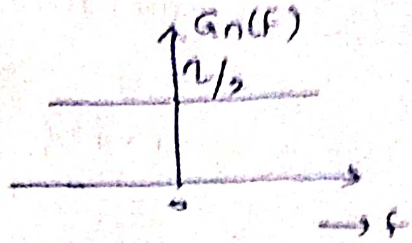
→ Linear filtering of noise :-

① The PSD of white noise is uniform over entire range of frequencies



$G_n(f) = \eta/2 \quad -\infty < f < \infty$

② PSD of thermal noise is uniform upto frequencies of the order of 10^{13} Hz.

③ PSD of shot noise is constant upto frequencies of the order of

$$\frac{1}{\text{Transit time of charge carriers across the junction.}}$$

Let us consider the effect of certain types of filters on the noise. In each case we consider the i/p noise applied to the filter. as white noise, whose PSD is $\boxed{G_{ni}(f) = \eta/2}$ over the entire range of frequencies.

① RC Low pass filter (RC LPF) :-

An RC-LPF with a 3 dB Freq f_c, has a transfer fn

$$H(f) = \frac{1}{1+j\,f/f_c}.$$

We know that

$$\boxed{\begin{aligned} G_{no}(f) &= |H(f)|^2\, G_{ni}(f) \\ \& \quad N_o &= \int_{-\infty}^{\infty} G_{no}(f)\, df. \end{aligned}}$$

where N_o is o/p noise power ; $G_{no}(f)$ is o/p noise PSD.

$G_{ni}(f)$ is i/p noise PSD , white noise is applied as i/p $\therefore G_{ni}(f) = \eta/2$ for all frequencies

& $|H(f)|^2 = H(f)\, H^*(f)$

$$\therefore G_{no}(f) = |H(f)|^2 G_{ni}(f)$$

$$= H(f)\, H^*(f)\, G_{ni}(f)$$

$$= \left[\frac{1}{1+jf/f_c}\right]\left[\frac{1}{1-jf/f_c}\right] \eta/2 .$$

$$= \frac{1}{1+(f/f_c)^2}\, \eta/2$$

$$\therefore N_o = \int_{-\infty}^{\infty} G_{no}(f)\,df = \int_{-\infty}^{\infty} \frac{1}{1+(f/f_c)^2}\, \eta/2 \, df = \frac{\eta f_c}{2}\left(\tan^{-1}(f/f_c)\right)_{-\infty}^{\infty}$$

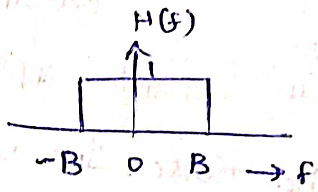
$$= \eta/2 \left[\pi/2 - (-\pi/2)\right] f_c$$

$$= \eta/2 \, f_c \, \pi .$$

$$\Rightarrow \boxed{N_o : \pi/2\, \eta f_c} \quad \text{for RC LPF}$$

② Rectangular (Ideal) LPF :-

$$H(f) = \begin{cases} 1 & |f| \le B \\ 0 & \text{elsewhere} \end{cases}$$



$$\therefore N_o = \int_{-\infty}^{\infty} G_{no}(f)\, df$$

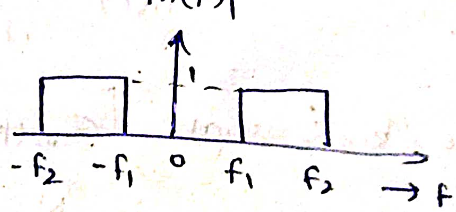
$$= \int_{-\infty}^{\infty} |H(f)|^2 G_{ni}(f)\, df = \int_{-B}^{B} (1)\, \eta/2 + 0 = \eta/2\,[B+B]$$

$$= \frac{\eta}{2}(2B)$$

$$\Rightarrow \boxed{N_o = \eta B}$$

③ Rectangular Bandpass Filter :

$$H(f) = \begin{cases} 1 & |f_1| \le f \le |f_2| \\ 0 & \text{elsewhere} \end{cases}$$



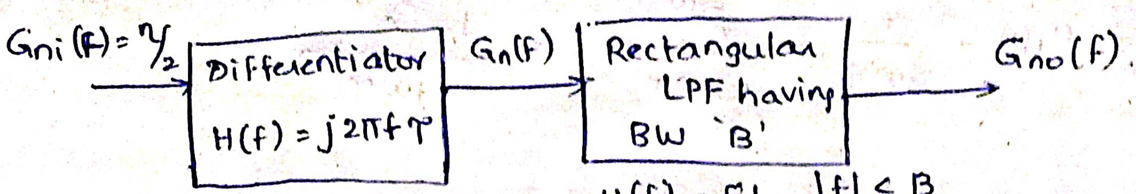
$$N_o = \int_{-\infty}^{\infty} G_{no}(f)\, df .$$

$$= \int_{-f_2}^{-f_1} (1)\, \eta/2 \, df + \int_{f_1}^{f_2} (1)\, \eta/2 \, df = \eta/2\left[-f_1 + f_2\right] + \left[f_2 - f_1\right]$$

$$= \eta/2 \left[2(f_2 - f_1)\right]$$

$$\Rightarrow \boxed{N_o = \eta(f_2 - f_1)}$$

④ Differentiating filter :-



$$H_1(f) = \begin{cases} 1 & |f| \le B \\ 0 & , \text{elsewhere} \end{cases}$$

A differentiating filter is a n/w which yields at its o/p a waveform which is proportional to the time derivative of the i/p waveform. This filter has a transfer fn $\boxed{H(f) = j2\pi f\tau}$ where τ is a constant factor of proportionality.

This filter is followed by a rectangular LPF having bandwidth `B'. Let the i/p noise is white, then the o/p noise-power spectral density is calculated as follows :-

$$G_{no}(f) = |H_1(f)|^2 \, G_n(f)$$

where

$$G_n(f) = |H(f)|^2 \, G_{ni}(f)$$

$$= (j2\pi f\tau)(-j2\pi f\tau)\,\eta/2$$

$$= 4\pi^2 f^2 \tau^2 \,\eta/2 = 2\pi^2 f^2 \tau^2 \eta.$$

$$\therefore G_{no}(f) = |H_1(f)|^2 \, G_n(f)$$

$$= 1 \cdot (2\pi^2 f^2 \tau^2 \eta) = 2\pi^2 f^2 \tau^2 \eta.$$

$$\therefore N_o = \int_{-\infty}^{\infty} G_{no}(f)\,df = \int_{-B}^{B} 2\pi^2 f^2 \tau^2 \eta \, df$$

$$= 2\pi^2 \tau^2 \eta \left(\frac{f^3}{3}\right)_{-B}^{B}$$

$$= \frac{2\pi^2 \tau^2 \eta}{3} \left[B^3 + B^3 \right]$$

$$\Rightarrow \boxed{N_o = \frac{4\pi^2 \tau^2 \eta B^3}{3}}$$

⑤ An Integrator :-

$$H(f) = \frac{1}{j\omega\tau} - \frac{e^{-j\omega T}}{j\omega\tau} = \frac{1-e^{-j\omega T}}{j\omega\tau}$$

$$G_{ni}(f) = \eta/2 \longrightarrow \boxed{\frac{1-e^{-j\omega T}}{j\omega\tau}} \longrightarrow G_{no}(f)$$

N_o

Let the noise $n(t)$ be applied to the i/p of an integrator at time $t=0$, and we calculate the noise power at o/p of integrator at time $t=T$.

A n/w which performs the operation of integration has a transfer fn $\frac{1}{j\omega\tau}$. A delay by an interval T is represented by a factor $e^{-j\omega T}$. Hence a n/w which performs an integration over an interval T may be represented by a n/w whose transfer fn is

$$H(f) = \frac{1}{j\omega\tau} - \frac{e^{-j\omega T}}{j\omega\tau} = \boxed{\frac{1-e^{-j\omega T}}{j\omega\tau}}$$

where τ is a constant

$$\therefore G_{no}(f) = |H(f)|^2 G_{ni}(f)$$

$$= \left[\frac{1-e^{-j\omega T}}{j\omega\tau}\right]\left[\frac{1-e^{j\omega T}}{-j\omega\tau}\right]\eta/2$$

$$= \left[\frac{1-e^{-j\omega T}-e^{j\omega T}+1}{-j^2\omega^2\tau^2}\right]\eta/2$$

$$= \left[\frac{2-(e^{j\omega T}+e^{-j\omega T})}{\omega^2\tau^2}\right]\eta/2$$

$$\Rightarrow G_{no}(f) = \frac{2 - 2\cos \omega T}{\omega^2 \tau^2} \, \eta/2 \qquad \left\{ \because \cos 2\theta = 1 - 2\sin^2\theta \right\}$$

$$= \frac{2\,(2\sin^2 \omega T/2)}{\omega^2 \tau^2} \, \eta/2$$

$$= 2\eta \, \frac{\sin^2 \omega T/2}{\omega^2 \tau^2}$$

$$= \frac{2\eta}{\tau^2} (T/2)^2 \left(\frac{\sin \omega T/2}{\omega T/2} \right)^2$$

$$\Rightarrow G_{no}(f) = \frac{\eta T^2}{2\tau^2} \left(\frac{\sin \omega T/2}{\omega T/2} \right)^2 \qquad \{ \omega = 2\pi f$$

$$= \frac{\eta T^2}{2\tau^2} \left(\frac{\sin \pi f T}{\pi f T} \right)^2$$

$\therefore$ O/p Noise Power, $N_o = \int_{-\infty}^{\infty} G_{no}(f)\, df = \int_{-\infty}^{\infty} \frac{\eta T^2}{2\tau^2} \left(\frac{\sin \pi f T}{\pi f T} \right)^2 df$

$$= \frac{\eta T^2}{2\tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin \pi f T}{\pi f T} \right)^2 df$$

$$= \frac{\eta T^2}{2\tau^2} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 \frac{dx}{\pi T} \qquad \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 = \pi \right.$$

let $x = \pi f T$

$dx = \pi T \, df$

$$= \frac{\eta T^2}{2\tau^2} \, \frac{\cancel{\pi}}{\cancel{\pi} T}$$

$$\Rightarrow \boxed{N_o = \frac{\eta T}{2\tau^2}}$$

→ Problem :-

① Given a white noise of amplitude $\eta = 0.001\ \mu W/Hz$ is fed to the following filters

a) An RC low pass filter of $R = 1\ k\Omega$ & $C = 0.1\ \mu F$

b) An ideal LPF of $B = 1\ kHz$

c) A differentiator followed by an ideal LPF defined as in ⓑ. Consider constant of proportionality $\gamma = 0.1$

Find the o/p noise power in each case.

d) How does the result change in each case if cut-off freq is doubled in each case.

Ans) a) RC low pass filter :-

for RC LPF ; $N_o = \frac{\pi}{2} \eta f_c$

Given $R = 1\ k\Omega$

$C = 0.1\ \mu F$

$\eta = 0.001\ \mu W/Hz$

where $\boxed{f_c = \frac{1}{2\pi RC}}$

$= \frac{1}{2\pi \times 10^3 \times 0.1 \times 10^{-6}}$

$\therefore N_o = \frac{\pi}{2} \eta f_c$

$= \frac{\pi}{2} \times 0.001 \times 10^{-6} \times \frac{1}{2\pi \times 10^3 \times 0.1 \times 10^{-6}}$

$\Rightarrow \boxed{N_o = 2.5\ \mu W}$

if cut-off freq is doubled then $N_o \propto f_c$

$\therefore N_{o1} = 2N_o$

$= 2 \times 2.5\ \mu W$

$\Rightarrow \boxed{N_{o1} = 5\ \mu W}$

b) ideal LPF :-

$N_o = \eta B$

$= 0.001 \times 10^{-6} \times (1 \times 10^3)$

$\Rightarrow N_o = 1\ \mu W$

If cut-off freq is doubled then $N_{o1} \propto B$

$\therefore N_{o1} = 2N_o$

$= 2\ \mu W.$

c) $N_o = \frac{4}{3}\pi^2 \eta B^3 \gamma^2$ Given $\gamma = 0.1$

$= \frac{4}{3}\pi^2 (0.001 \times 10^{-6})(10^3)^3 (0.1)^2$

$\simeq 0.1315\ W = 131.5\ mW$

If cut-off freq is doubled, $N_{o1} \propto B^3$

$\Rightarrow N_{o1} \propto (2B)^3$

$\Rightarrow N_{o1} = 8 \times N_{o1}$

$= 8 \times 131.5$

$= 1052\ mW.$

② → The Gaussian noise in a very narrow spectrum range is represented by $n_k(t) = a_k \cos 2\pi k\Delta f t + b_k \sin 2\pi k \Delta f t$. The normalized power of noise is 0.01 μW. Write an expression for probability density function of coefficients a_k, b_k when the noise sample function is stationary.

Ans] The probability density fn for Gaussian noise is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$$

where m is mean, σ^2 is variance (or) normalized power.

For noise is stationary random process,

① mean $= 0 \Rightarrow E[a_k] = E[b_k] = 0$

② equal variance $\Rightarrow E[a_k^2] = E[b_k^2]$

③ uncorrelated to each other $\Rightarrow E[a_k b_k] = 0.$

∴ The probability density fn of coefficients a_k, b_k are

$$f(a_k) = \frac{1}{\sqrt{2\pi \times (0.01 \times 10^{-6})}} e^{-a_k^2/2 \times 0.01 \times 10^{-6}}$$

& $$f(b_k) = \frac{1}{\sqrt{2\pi (0.01 \times 10^{-6})}} e^{-b_k^2/2 \times 0.01 \times 10^{-6}}$$