Abstract

Gravitational waves are 'ripples' in space-time caused by some of the most violent and energetic processes in the Universe. Albert Einstein first theorized the presence of gravitational waves in 1916 within his General Theory of Relativity. Einstein's mathematics revealed that the movement of massive accelerating objects, such as neutron stars or black holes orbiting each other, would create a disturbance in space-time, causing ripples of undulating space-time to spread in all directions from the source.

The momentous finding of gravitational waves took place in 2015, when the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors, located in Washington and Louisiana, successfully detected them using ground-based facilities. When a gravitational wave is detected, the arms of LIGO also experience fluctuations in length. As a result, the lasers within the arms travel varying distances before coming together again. The technical term for this oscillation is "Differential Arm" motion, since the arms simultaneously change lengths in opposing ways. Now, as the beams recombine, they are out of sync, or "out of phase", and no longer totally destructively interfere. Instead, the merging beams shift in and out of phase, and a flicker of light emerges from the interferometer.

Traveling at the speed of light, these cosmic ripples provide valuable insights into the true nature of gravity and reveal information about their origins, making them a fascinating area of study for scientists.

Theoretical Construct

Let,

- Mass of first body = m_1
- Mass of second body = m_2
- Distance between the bodies = r
- Angular frequency of revolution = ω
- Distance of m_1 from the centre of mass of the system, $r_1 = \frac{m_2 r}{m_1+m_2}$
- Distance of m_2 from the centre of mass of the system, $r_2 = \frac{m_1 r}{m_1+m_2}$

$\Rightarrow$ Instantaneous velocity of first body, $v_1 = \omega r_1 = \frac{m_2 r\omega}{m_1+m_2}$

$\Rightarrow$ Instantaneous velocity of second body, $v_2 = \omega r_2 = \frac{m_1 r\omega}{m_1+m_2}$

Now, the Centripetal force on each mass is equal to the Gravitational force experienced by them respectively

$\therefore F_{gravitation} = \frac{Gm_1m_2}{r^2} = m_1 r_1 \omega^2 = F_{centripetal}$ ----- (here G = Gravitational constant)

$$\Rightarrow \frac{Gm_1m_2}{r^2} = \frac{m_1m_2r\omega^2}{m_1+m_2}$$

$$\Rightarrow \frac{G(m_1+m_2)}{r^3} = \omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}$$

Assuming that relativistic effects are negligible,

Total energy, $E = Kinetic\ enrgy + Potential\ Energy$

$$\Rightarrow E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{r}$$

$$\Rightarrow E = \frac{1}{2}m_1\left(\frac{G(m_1+m_2)}{r^3}\right)\left(\frac{m_2r}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{G(m_1+m_2)}{r^3}\right)\left(\frac{m_1r}{m_1+m_2}\right)^2 - \frac{Gm_1m_2}{r}$$

$$\therefore E = -\frac{Gm_1m_2}{2r}$$

Substituting r from $\omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}$

$$\Rightarrow E = \left(\frac{Gm_1m_2}{2}\right)\left(\frac{\omega^2}{G(m_1+m_2)}\right)^{1/3}$$

$$\Rightarrow E = -\frac{(G\omega)^{\frac{2}{3}} m_1 m_2}{2(m_1 + m_2)^{\frac{1}{3}}}$$

$$\Rightarrow E = -\frac{\left(G\omega(m_1 + m_2)\right)^{\frac{2}{3}} m_1 m_2}{2(m_1 + m_2)}$$

$$\Rightarrow E = -\frac{\left(G\omega(m_1 + m_2)\right)^{\frac{2}{3}} \mu}{2}$$

Here, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ which is the reduced mass of the system

Power radiated through electromagnetic waves by a time-varying dipole is given by Larmor's formula:

$$P_{EW} = \frac{\omega^4 |\mathrm{p}_\omega|^2}{12\pi\varepsilon c v^2}$$

$$P_{EW} = \frac{\mathrm{k}\omega^4 |\mathrm{p}_\omega|^2}{3 c v^2}$$

Where,

P_{EW}: Power radiated through electromagnetic waves

p_ω: Dipole moment

ε: Permittivity of the medium

c: Speed of light

v: Wave speed (=c for both electromagnetic and gravitational waves)

k: Coulomb's constant

Comparing with Larmor's formula, the power radiated by gravitational waves (P_{GW}) are expected to rely on:

G: The gravitational constant is equivalent to k

ω: The angular frequency of the system (as Gravitational waves are produced by the revolution/oscillation of the masses)

c: Speed of light=Speed of the gravitational waves

Q: The quadrupole moment of the binary system≡Moment of inertia of the system

Q is equivalent to p_ω in Larmor's formula

Moment of inertia, $|Q| = m_1 r_1^2 + m_2 r_2^2$

$$\Rightarrow Q = m_1 \left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2 \left(\frac{m_1 r}{m_1 + m_2}\right)^2$$

$$\Rightarrow Q = \frac{m_1 m_2 r^2}{m_1 + m_2}$$

$$\Rightarrow |Q| = \mu r^2$$

Where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$: Reduced mass of the system

Power radiated by electromagnetic radiation is proportional to the square of dipole moment.

$$P_{EW} \propto |p_\omega|^2$$

Comparing the given formula with gravitational waves,

$$P_{GW} \propto |Q|^2$$

$\therefore$ Power radiated, $P_{GW} = \alpha G^a \omega^b (\mu r^2)^2 c^d$ ------ (α is the proportionality constant)

Comparing the dimensions of the quantities,

$$\Rightarrow [ML^2T^{-3}] = [M^{-1}L^3T^{-2}]^a . [T^{-1}]^b . [ML^2]^2 . [LT^{-1}]^d$$

$$\Rightarrow 1 = -a + 2$$

$$2 = 3a + 4 + d$$

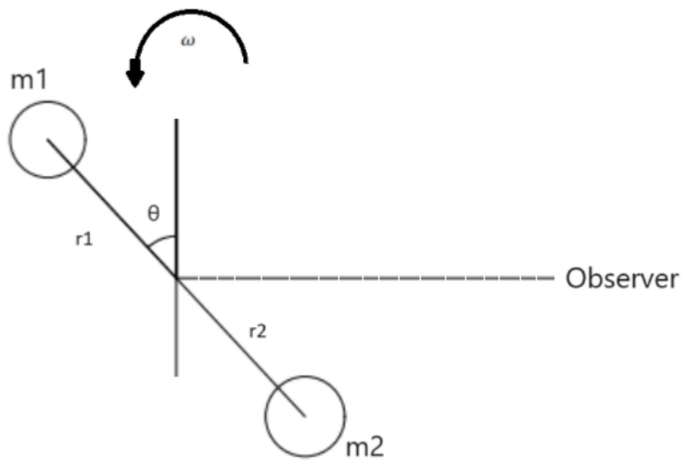
$$-3 = -2a - b - d$$

$$\therefore a = 1, d = -5, b = 6$$

Given: Constant of proportionality, $\alpha = 32/5$

$$\therefore \textit{Power emitted by gravitational waves}, P_{GW} = \frac{32}{5} \frac{G\omega^6 \mu^2 r^4}{c^5}$$

For a Binary System



Now, $\theta = \omega t$

Moment of Inertia of the system along one direction (One component of quadrupole moment tensor)

$$= \mathrm{m_1}(\mathrm{r_1}\cos\theta)^2 + \mathrm{m_2}(\mathrm{r_2}\cos\theta)^2$$

$$= (\mathrm{m_1 r_1^2} + \mathrm{m_2 r_2^2})\cos^2\theta$$

$$= \left(m_1\left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\frac{m_1 r}{m_1 + m_2}\right)^2\right) cos^2(\omega t)$$

$$= \frac{m_1 m_2}{m_1 + m_2} r^2 \left(\frac{1 + \cos(2\omega t)}{2}\right)$$

$$= \frac{\mu r^2}{2}\left(1 + \cos(2\omega t)\right)$$

$$= \frac{\mu r^2}{2}\left(1 + \cos(\omega_e t)\right)$$

Where, ω_e is the angular frequency of the system.

[Similarly, the moment of inertia in other directions (Other components of the quadrupole moment tensor) are trigonometric functions of $2\omega t$.]

$$\therefore \omega_e = 2\omega$$

ω_e is the angular frequency with which the quadrupole moment (moment of inertia here) changes, which is the same as the frequency of emission of gravitational waves.

Therefore, the frequency of emission of gravitational waves is twice the orbital angular frequency.

Let us express the Power radiated by gravitational wave in terms of only angular frequency ω and the masses in the binary system

$$\Rightarrow P_{GW} = \frac{32}{5}\frac{G\mu^2 r^4 \omega^6}{c^5}$$

$$putting\ r = \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{1/3}$$

$$\Rightarrow P_{GW} = \frac{G\mu^2\omega^6}{c^5}\left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{4/3}$$

$$\Rightarrow P_{GW} = \frac{G^{7/3}\mu^2\omega^{10/3}(m_1 + m_2)^{4/3}}{c^5}$$

$$\Rightarrow P_{GW} = \frac{32}{5}\left(\frac{G^7\mu^6\omega^{10}(m_1 + m_2)^4}{c^{15}}\right)^{1/3}$$

Now, we have already derived the relation between the energy of the system and the orbital angular frequency ω

$$\Rightarrow\ E = -\frac{1}{2}(Gm_1m_2\omega)^{2/3}.\mu^{1/3}$$

Differentiating both sides with respect to time

$$\Rightarrow \frac{dE}{dt} = -\frac{1}{3}(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}\frac{d\omega}{dt}$$

$$\Rightarrow -\frac{dE}{dt} = \frac{1}{3}(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}\frac{d\omega}{dt}$$

The Power of the gravitational waves emitted is the first derivative of

$$P_{GW} = -\frac{dE}{dt}$$

$$\Rightarrow P_{GW} = \frac{1}{3}(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}\frac{d\omega}{dt}$$

$$\Rightarrow P_{GW} = \frac{32}{5}\frac{G\mu^2r^4\omega^6}{c^5} = \frac{1}{3}(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}\frac{d\omega}{dt}$$

$$\Rightarrow \frac{32}{5}\frac{G\mu^2r^4\omega^6}{c^5} = \frac{1}{3}(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}\frac{d\omega}{dt}$$

$$\Rightarrow \frac{96}{5}\frac{G\mu^2r^4\omega^6}{(Gm_1m_2)^{2/3}.\left(\frac{\mu}{\omega}\right)^{1/3}c^5} = \frac{d\omega}{dt}$$

$$\Rightarrow \frac{96}{5}\frac{G^{1/3}\mu^{5/3}r^4\omega^{17/3}}{(m_1m_2)^{2/3}c^5} = \frac{d\omega}{dt}$$

$$putting\ r = \left(\frac{G(m_1+m_2)}{\omega^2}\right)^{1/3}$$

$$\Rightarrow \left(\frac{96}{5}\frac{G^{1/3}\mu^{5/3}\omega^{17/3}}{(m_1m_2)^{2/3}c^5}\right).\left(\frac{G(m_1+m_2)}{\omega^2}\right)^{4/3} = \frac{d\omega}{dt}$$

$$\Rightarrow \left(\frac{96}{5}\frac{G^{5/3}\mu^{5/3}\omega^{11/3}}{c^5}\right).\left(\frac{(m_1+m_2)^{4/3}}{(m_1m_2)^{2/3}}\right) = \frac{d\omega}{dt}$$

$$\Rightarrow \left(\frac{96}{5}\frac{G^{5/3}\mu^{5/3}\omega^{11/3}}{c^5}\right).\left(\frac{1}{\mu}\right)^{5/3}.\left(\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right) = \frac{d\omega}{dt}$$

$$\Rightarrow \left(\frac{96}{5}\frac{G^{5/3}\omega^{11/3}}{c^5}\right).\left(\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right) = \frac{d\omega}{dt}$$

$$\Rightarrow \frac{m_1m_2}{(m_1+m_2)^{1/3}} = \left(\frac{5}{96}\frac{c^5}{G^{5/3}\omega^{11/3}}\right)\frac{d\omega}{dt}$$

$$\Rightarrow \left(\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right)^{3/5} = \left(\frac{5}{96}\right)^{3/5}\left(\frac{c^3}{G\omega^{11/5}}\right)\left(\frac{d\omega}{dt}\right)^{3/5}$$

$$\Rightarrow M_{chirp} = \left(\frac{m_1m_2}{(m_1+m_2)^{1/3}}\right)^{3/5}$$

$$\Rightarrow M_{chirp} = \left(\frac{5}{96}\right)^{3/5}\left(\frac{c^3}{G}\right)\omega^{-11/5}\dot{\omega}^{3/5}$$

Now, we need to express Chirp Mass in terms of the frequency of emitted gravitational radiation and its temporal derivative

The frequency of radiation emitted is twice of the orbital angular frequency ω

$$\therefore \omega = 2\pi f_{orbital} = \pi f_{emitted}$$

$$\Rightarrow \frac{d\omega}{dt} = \pi\frac{df_{emitted}}{dt}$$

$$\Rightarrow M_{chirp} = \left(\frac{5}{96}\right)^{3/5}\left(\frac{c^3}{G}\right)(\pi f_e)^{-11/5}(\pi\dot{f}_e)^{3/5}$$

$$\Rightarrow M_{chirp} = \left(\frac{5}{96}\right)^{3/5}\left(\frac{c^3}{G}\right)\pi^{8/5}\, f_e{}^{-11/5}\, \dot{f}_e{}^{3/5}$$

$$\Rightarrow M_{chirp} = \left(\frac{5}{96}\pi^{8/3}\right)^{3/5}\left(\frac{c^3}{G}\right)\, f_e{}^{-11/5}\, \dot{f}_e{}^{3/5}$$

Estimating the Distance

$$Power\ emitted\ by\ gravitational\ waves, P_{GW} = \frac{32}{5}\frac{G\omega^6\mu^2r^4}{c^5}$$

Intensity can be related to power as:

$$I_{GW} = \frac{\gamma P_{GW}}{d^2}$$

Where, γ is a constant and d is the distance of Gravitational wave detector from the source

$$\therefore I_{GW} \propto \frac{G\omega^6\mu^2r^4}{c^5d^2}$$

$$\Rightarrow I_{GW} = \frac{\pi}{2}\frac{G\omega^6\mu^2r^4}{c^5d^2}$$

Now, rearranging the above equation

$$\Rightarrow d^2 = \frac{\pi}{2}\frac{G\omega^6\mu^2r^4}{c^5I_{GW}}$$

$$\Rightarrow d = \left(\frac{\pi}{2}\frac{G\omega^6\mu^2r^4}{c^5I_{GW}}\right)^{1/2}$$

Using the waveform we have drawn and Figure 1 the distance to our source is

$$\Rightarrow d > 1\ billion\ Light\ Years$$