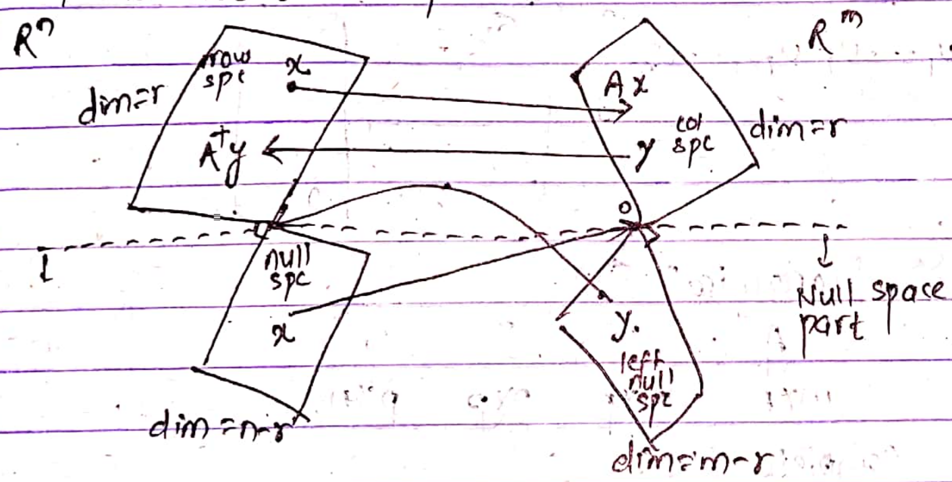
Pseudoinverse and Least Squares.

I. PSEUDOINVERSE

A : $m \times n$. A^{+} is a $n \times m$ matrix.

if A^{-1} exists $\Leftrightarrow$ $A^{+} = A^{-1}$ [square and full rank]

Pseudoinverse = best possible 'inverse'



→ in the upper part, A and A^{+} are inverses of each other, i.e.

$$A^{+}Ax = x \quad \text{if } x \text{ is in rowspace}$$

$$AA^{+}x = x \quad \text{if } x \text{ is in column space}$$

Formula for A^{+}.

Remember:

→ A transforms vector from row to column space

→ A also moves any vector from null space to 0

So, A^{+} should do the opposite

→ In 'non-nullspace' part, A^{+} should move a vector from colspace to rowspace

→ A^{+} should move a vector from left nullspace to zero. [see figure]

For A:

$A.v_1 = \sigma_1 u_1$

$\vdots$

$A.v_r = \sigma_r u_r$

$Av_{r+1} = 0$

$\vdots$

$Av_n = 0.$

$R^n \begin{cases} \{v_1 \cdots v_r\} \Rightarrow \text{forms basis of } \cancel{\text{col}}\text{ row space} \\ \{v_{r+1} \cdots v_n\} \Rightarrow \text{forms basis of nullspace.} \end{cases}$

$R^m \begin{cases} \{u_1 \cdots u_r\} \Rightarrow \text{forms basis of rowspace} \\ \{u_{r+1} \cdots u_m\} \Rightarrow \text{forms basis of left nullspace} \end{cases}$

$$A \begin{bmatrix} v_1 \cdots v_r \; v_{r+1} \cdots v_n \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_r \; u_{r+1} \cdots u_m \end{bmatrix} \cdot \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

$AV = U\Sigma$

$\Rightarrow A = U\Sigma V^T$

Economical SVD

$$\underset{m\times n}{A} = \underset{m\times p}{U} \; \underset{p\times p}{\Sigma} \; \underset{p\times n}{V^T}.$$

Complete SVD:

$$\underset{m\times n}{A} = \underset{m\times m}{U} \; \underset{m\times n}{\Sigma} \; \underset{n\times n}{V^T}$$

out of which first $p\times p$ is diagonal.

Notes:

A^TA

$= (V\Sigma^T U^T)(U\Sigma V^T)$

$= V(\Sigma^T\Sigma)V^T$

$A^TAV = V(\Sigma^T\Sigma)$

$\Rightarrow V$ is eigenvector matrix of A^TA

similarly,

U is eigenvector matrix of AA^T ..

For A^{+}

$$\left.\begin{aligned} A^{+}u_1 &= \frac{1}{\sigma_1} v_1 \\ &\vdots \\ A^{+}u_r &= \frac{1}{\sigma_r} v_r \end{aligned}\right\}$$ Transform the vectors from column space to row space back again.

$$\left.\begin{aligned} A^{+}u_{r+1} &= 0 \\ &\vdots \\ A^{+}u_m &= 0 \end{aligned}\right\}$$ Transform vectors from left nullspace to zero.

$$A^{+}\left[u_1 \cdots u_r\, u_{r+1} \cdots u_m\right] = \left[v_1 \cdots v_r\, v_{r+1} \cdots v_n\right] \left[\begin{array}{ccc|c} \frac{1}{\sigma_1} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_r} & \\ \hline & 0 & & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} A^{+}U &= V\Sigma^{+} \\ \therefore A^{+} &= V\Sigma^{+}U^{T} \end{aligned}}$$

Economical:

$$\underset{n\times m}{A^{+}} = \underset{n\times r}{V}\ \underset{r\times r}{\Sigma^{+}}\ \underset{r\times m}{U^{T}}$$

Complete:

$$\underset{n\times m}{A^{+}} = \underset{n\times n}{V}\ \underset{n\times m}{\Sigma^{+}}\ \underset{m\times m}{U^{+}}$$

Out of which first $r\times r$ is diagonal.

$[p=r]$ from left page is the rank of A.

Notes:

$\Sigma\Sigma^{+}$ is closest to I as it can be

$\Sigma\Sigma^{+} = m\times m$

$\Sigma^{+}\Sigma = n\times n$.

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II. LEAST SQUARES.

$Ax = b$ $\boxed{A}$

If A is invertible,

$m = n = r$

$\therefore$ The system is solvable.

Normally, there is some noise in the measurement which does not allow the system to be solvable.

We project b onto the column space of A.

$$Ax = \begin{bmatrix} 1 & x_1 \\ 1 & \vdots \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

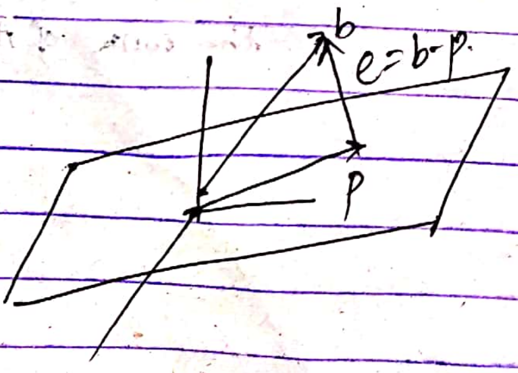
Minimize the error:

$\min. \|Ax - b\|^2$ [l^2 norm]

$L = (Ax-b)^T(Ax-b)$

$$\frac{dL}{dx} = 2A^TAx - 2A^Tb = 0$$

$$\therefore \hat{x} = (A^TA)^{-1}A^Tb$$



$\boxed{B}$

The calculations on the left are true if ~~exists~~ A has independent columns

because

if A has independent columns [most likely in tall and thin matrices]

The big picture is:

This part shrinks to zero

$N(A) = 0$.

$\rightarrow$ A^TA is invertible if A has independent columns

Proof:

$A^TAx = 0$

$A^T(Ax) = 0.$

$A^Tb = 0$

b is in column space of A, which is row space of A^T, so, for product to be zero, $b = 0$

$\Rightarrow Ax = 0.$

but A has independent columns

$\Rightarrow x = 0.$

$\Rightarrow (A^TA)$ is invertible.

→ If columns of A are independent

$N(A) = 0 \quad , r = n$

Claim:

$A^{+}b = (A^TA)^{-1}A^Tb$

But, AA^T is not invertible because $N(A^T)$ exists

$A^{+} = V\Sigma^{+}U^T = (A^TA)^{-1}A^T$

Multiply by A on left & right

$(A^TA)^{-1}(A^TA) = I$

$A(A^TA)^{-1}A^T$ = closest to identity

$\boxed{C}$

What if the columns of A are not independent?

→ There will be multiple solutions