Weekly presentation SCM research

5th presentation: Special distributions (part 2)

Summary

I/Gamma distribution

1) Definition

2)Exponential distribution

3) Life test

4) Relation to poisson process

II/Bivariate Normal distribution

1) Generalities

2) Definition

3) Properties

I/ Gamma distribution
1) Definition

- Popular model for random variables that are known to be positive

- Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Interesting mathematical properties

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

$$\Gamma(n) = (n - 1)!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

I/ Gamma distribution
1) Definition

- Gamma distribution

a random variable X has a gamma distribution with parameters α and β if X has a continuous distribution for which the p.d.f. is :

$$f(x/\alpha,\beta) = \begin{cases} \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & for\ x > 0 \\ 0 & for\ x \leq 0 \end{cases}$$

I/ Gamma distribution
1) Definition

- Mean and Variance

$$E(X^k) = \int_0^\infty x^k f(x/\alpha,\beta)dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} e^{-\beta x}\, dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}}$$

$$E(X) = \frac{\alpha}{\beta}$$

$$Var(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$Var(X) = \frac{\alpha}{\beta^2}$$

I/ Gamma distribution
1) Definition

- Moment generating function

$$\psi(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$$

- Properties

*The sum of independent random variables that have gamma distrubtions with **a common value of the parameter β** will also have a gamma distribution*

I/ Gamma distribution
2) Exponential distribution

•Definition

A random variable X following ***a gamma distribution with parameters*** $\boldsymbol{\alpha}$***=1 and*** $\boldsymbol{\beta}$ *is said to have an exponential distribution*

$$f(x/\beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$E(X) = \frac{1}{\beta} \qquad Var(X) = \frac{1}{\beta^2} \qquad \psi(t) = \frac{\beta}{\beta - t}$$

I/ Gamma distribution
3) Life test

- Theorem
  - *$X_1 \ldots X_n$ are som random samples from an exponential distribution with parameter β, then the distribution $Y = \min[X_1 \ldots X_n]$ will be an exponential distribution with parameters $n\beta$*
- Example (light bulbs)
  - N light bulbs are burning simultaneously and independently in a test to determine the lengths of their lives
  - Each one of them X_i has an exponential distribution with parameter β
  - <u>What is the distribution of time Y_1 until one of the n bulbs fails?</u>

  - $Y = \min[X_1 \ldots X_n]$
  - $Pr(Y_1 > t) = Pr(X_1 > t, \ldots, X_n > t) = Pr(X_1 > t) \ldots Pr(X_n > t) = e^{-n\beta t}$

  - *Using the memoryless property, Y_k will have an exponential distribution with parameter $(n-k-1)\beta$*

I/ Gamma distribution
4) Relation to Poisson process

•An exponential distribution is often used in a practical problem to represent the distribution of the time that elapses before the occurrence os some event

•If the events being considered occur with a Poisson process then both the waiting time until an event occurs and the period of time between two successive events will hqve exponentiql distributions

•More generally the waiting time until the kth occurrence in a Poisson process with rate λ has a gamma distribution with parameters k and λ

I/ Bivariate Normal distribution

1) Generalities

• Often we are interested in the intersection of two or more events, or the outcome of two or more random variables at the same time.

•Joint distribution

$$F(x, y) = P(X \leq x, Y \leq y).$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(y)$$

$$\int_x \int_y f_{X,Y}(x,y)\, dy\, dx = 1.$$

I/ Bivariate Normal distribution
1) Generalities

- Marginal probability

•Given two random variables X and Y whose joint distribution is known, the marginal distribution of X is simply the probability distribution of X averaging over information about Y. This is typically calculated by summing or integrating the joint probability distribution over Y.

$$p_X(x) = \int_y p_{X,Y}(x,y)\, \mathrm{d}y = \int_y p_{X|Y}(x|y)\, p_Y(y)\, \mathrm{d}y,$$

•Intuitively, the marginal probability of X is computed by examining the conditional probability of X given a particular value of Y, and then averaging this conditional probability over the distribution of all values of Y.

I/ Bivariate Normal distribution
1) Generalities

- Conditional probability

•Given two jointly distributed random variables X and Y, the conditional probability distribution of Y given X is the probability distribution of Y when X is known to be a particular value

$$f_Y(y \mid X = x) = \frac{f_{X,Y}(x, y)}{f_X(x)},$$

$$f_Y(y \mid X = x) f_X(x) = f_{X,Y}(x, y) = f_X(x \mid Y = y) f_Y(y).$$

•Relation to independence

•Random variables X, Y are independent if and only if the conditional distribution of Y given X is equal to the unconditional distribution of Y

I/ Bivariate Normal distribution

1) Generalities

- Covariance and correlation

•covariance is a measure of how much two random variables change together(statistical relationship between two random variables or two sets of data)

$$\sigma(x,y) = \mathrm{E}\left[(x - \mathrm{E}[x])(y - \mathrm{E}[y])\right],$$

- Correlation refers to any of a broad class of statistical relationships involving dependence

$$\rho_{X,Y} = \mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y},$$

I/ Bivariate Normal distribution
2) Definition

•Two random variables X1 and X2 are said to have a bivariate normal distribution of their joint p.d.f. is:

$$f(x_1,x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}\sigma_1\sigma_2} e^{-\frac{1}{2(1-\rho^2)}[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2]}$$

$$E(Xi) = \mu_i \qquad Var(Xi) = \sigma^2{}_i$$

$$\rho(X_1,X_2) = \rho$$

I/ Bivariate Normal distribution
2) Definition

- Origin
  - Z1 and Z2 are two independent variables, with standard normal distribution. Their joint p.d.f. is therefore

$$g(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}({z^2}_1 + {z^2}_2)}$$

  - We define

$$X_1 = \sigma_1 Z_1 + \mu_1$$

$$X_2 = \sigma_2 \left[\rho Z_1 + \sqrt{1 - \rho^2} Z_2 \right] + \mu_2$$

I/ Bivariate Normal distribution
2) Definition

•Origin

$$J = \frac{1}{\Delta} = \frac{1}{\sqrt{1 - \rho^2 \sigma_1 \sigma_2}}$$

•But the bivariate normal distribution arises directly and naturally in many practical problems (at least approximatively)

I/ Bivariate Normal distribution
3) Properties

- Marginal distributions

  - Since X_1 and X_2 are defined with Z_1 and Z_2 we can say that the marginal distributions of X_i is a normal distribution with mean μ_i and variance σ_i^2

- Independence and Correlation

  - Two random variables X_1 and X_2 that have a bivariate normal distribution are independent if and only if they are uncorrelated

I/ Bivariate Normal distribution
3) Properties

•Conditional distributions

•The conditional distribution of X2 given that X1=x1 is a normal distribution, with

- •Mean

$$E(X_1/X_2) = \mu_2 + \rho\sigma_2\left(\frac{x_1 - \mu_1}{\sigma_1}\right)$$

- •Variance

$$Var(X_1/X_2) = (1-\rho)^2{\sigma^2}_2$$

Joint p.d.f. f(x1,x2) is symmetric in the two variables (x1-μ1)/σ1 and (x2-μ2)/σ2, so the other conditional distribution can be easily derived by just interchanging the parameters

I/ Bivariate Normal distribution
3) Properties

- Linear combinations
  - If two random variables X_1 and X_2 have a bivariate normal distribution, then each linear combination $Y=a_1X_1+a_2X_2+b$ will have a normal distribution

- Mean
  - $E(Y)= a_1\mu_1+a_2\mu_2+b$

- Variance
  - $Var(Y)= a^2_1\sigma^2_1+a^2_2\sigma^2_2+2a_1a_2\rho\sigma_1\sigma_2$

Thank you for your attention!