Weekly presentation SCM research

6th presentation: Introduction to Markov Chains

Summary

I/Introdction

1) Definition

2) Examples

II/ One-step transition theory

1) Theory

2) Examples

III/ Inventory control

I/ Introduction
1) Definition

• dynamic aspect of probability

•System that is in a particular state at each moment in time, as a result of various random influences, the state changes from one moment to the next

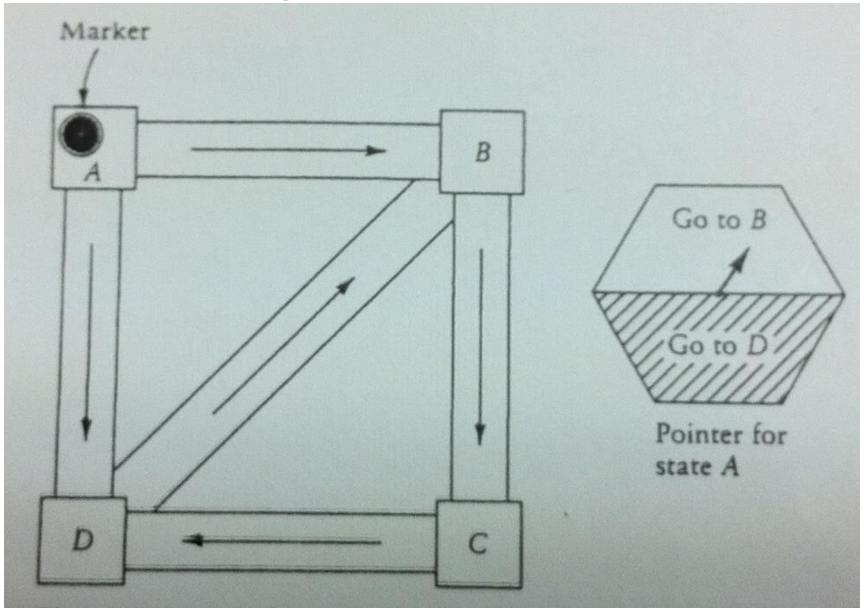
•Stochastic system represented as a transition diagram made up of nodes and one-way paths

•To be called a Markov chain, two rules need to apply:

- ***No-memory rule***
- ***Time-homogeneous rule***

I/ Introduction
1) Definition

- Markov Chain as a board game



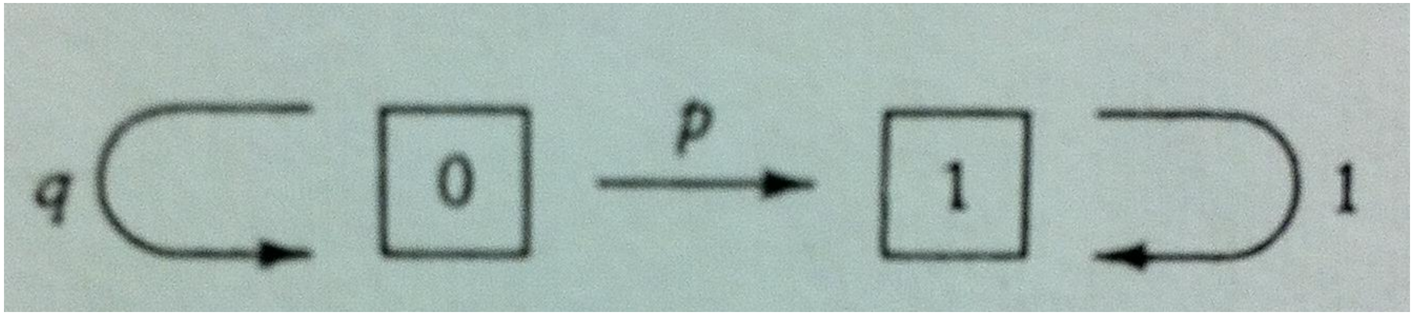
- To be called a Markov chain, two rules need to apply:
    - ***No-memory rule***
    - ***Time-homogeneous rule***

I/ Introduction
2) Examples

Waiting game

•2 states, 0 and 1

•Represents several independents rounds of flips of a coin (probability of head is p, otherwise q = 1 – p)



I/ Introduction
2) Examples

Bold play

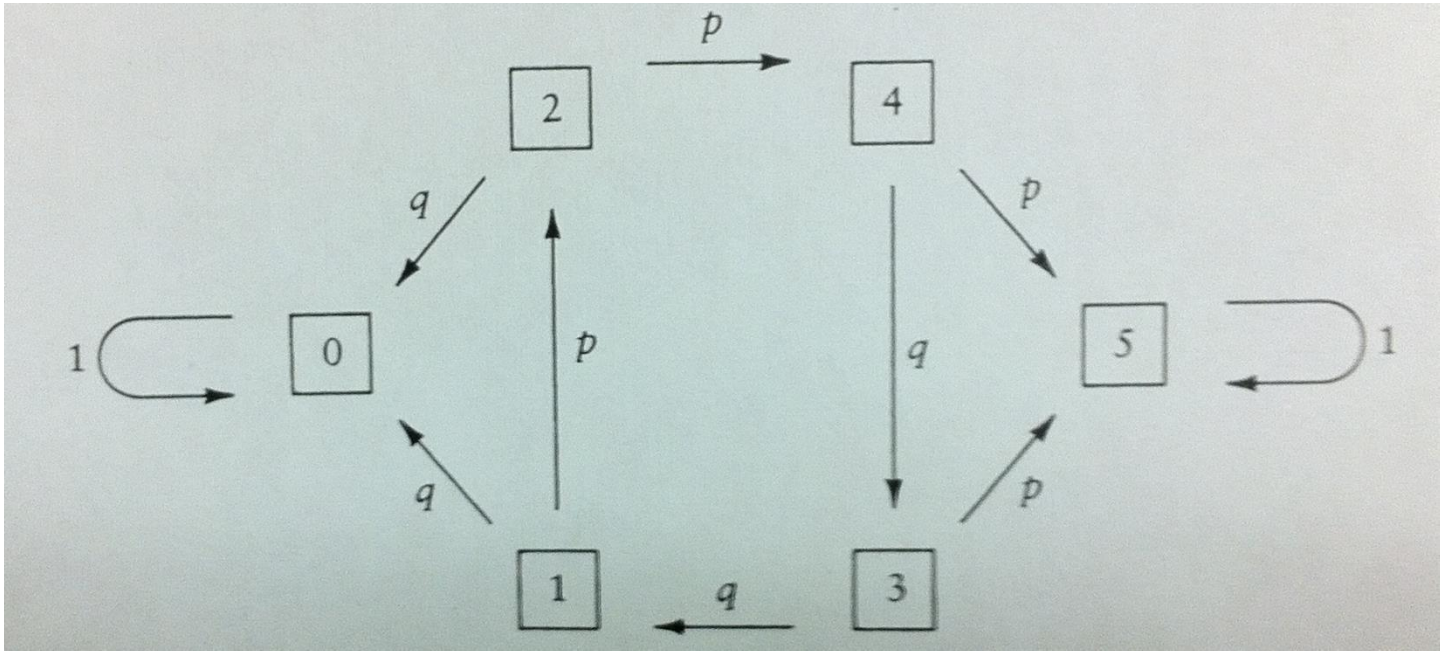
•Suppose we have $1, but desperatly need $5

•We play a game with win probability p at each round, and a double or nothing payoff

•Bold play strategy -> bet everything we have if wining the round would give us the goal of $5 or less; if not we bet only the necessary amount to go up to $5 in case of wining

I/ Introduction
2) Examples

Bold play



I/ Introduction
2) Examples

Bold play

Table 6.3 Simulation of 1000 Games of Bold Play

p	0.2	0.3	0.4	0.5	0.6	0.7
WIN FREQUENCY	0.010	0.042	0.105	0.202	0.342	0.483
AVERAGE LENGTH	1.30	1.52	1.78	2.05	2.25	2.47

Table 6.1 Simulation of 1000 Games of Gambler's Ruin ($N = 5$, Initial State $= 1$)

p	0.2	0.3	0.4	0.5	0.6	0.7
WIN FREQUENCY	0.004	0.017	0.087	0.208	0.394	0.582
AVERAGE LENGTH	1.58	2.21	3.06	3.97	4.38	4.81

II/ One-step transtion
1) Theory

• finite state system, goes from one state to another in discrete time steps by the random mechanism

• one-step transition probability : Pij = probability of moving from state i to state j in one step

$$0 \le p_{ij} \le 1 \qquad \sum_{j} p_{ij} = 1$$

•Answer two basic questions

- *What is the chance of winning (to move the marker from a designated starting node to a finishing node) ?*
- *what is the expected length of time needed to win?*

II/ One-step transition
1) Theory

Joint probabilities

•Random choice of the starting node, can be any of them

->πi is the probability of picking node I as the starting position

• we define a discrete random variable X(n) by setting X(n) = i if the marker is at node i at the nth move

-> the sequence {X(n)} describes the possible random paths that can be taken in playing the game

•By definition {πi} is the probability mass function of X(0)

II/ One-step transition

1) Theory

Joint probability mass function

- $A_k = \{X(k) = i_k\}$
- $B_k = \{X(0) = i_0, \ldots, X(k) = i_k\}$
- $P\{A_{n+1}/B_n\} = P\{A_{n+1}/A_n\}$

- Theorem : Multiplication rule for path probabilities

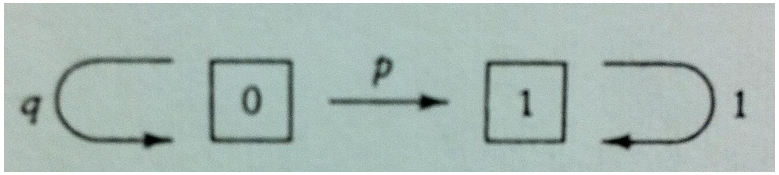
  For any transition diagram with one-step transition probabilities p_{ij} and any initial distribution π for $X(0)$, the probability of B_k is the product of the probability π_{i_0} of starting at node i_0 multiplied by the probabilities $P_{i_s,i_{s+1}}$ of each step $i_s \to i_{s+1}$

$$P(B_k) = P(A_k/A_{k-1})\, P(A_{k-1}/A_{k-2}) \ldots P(A_1/A_0) P(A_0)$$

II/ One-step transition
2) *Examples*

Waiting game

•Considering that the marker is at 0, what is the probability that after k flips it will be at 1?



•If the first outcome of heads occurs on the mth flip then the corresponding path is (0) -> … -> (0) -> (1) -> … -> (1) with probability $q^{m-1}p$

•Then we sum for all possible values of m

$$P\{X(k) = 1 / X(0) = 0\} = 1 - q^k$$

II/ One-step transition
2) *Examples*

Bold play VS Timid Play

•Bold play $\dfrac{p^3(1+q)}{1-p^2q^2}$ Timid play $p^4(1+3pq)$

•Choice of black or red in roulette, p = 18/38 = 0.4737

•Unlimited number of trial, goal N = 5

Bold play 0.1730 *Timid play* 0.1602

•Limited of 6 or fewer trials

Bold play 0.1622 *Timid play* 0.0880

III/ Inventory control

- Method
  - *Calculate the one-step transition probabilities Pij*
  - *Verify that the no-memory and time-homogeneity properties are valid*

Inventory model

•*Wholesaler distributor with randonmly fluctuating inventory of a certain item. The orders are shipped out at the end of the day, with possible back orders for the next day*

•*S=2, full inventory size at the end of the day (- back orders)*

•*Conitnious review policy, with critical level s=0 < S*

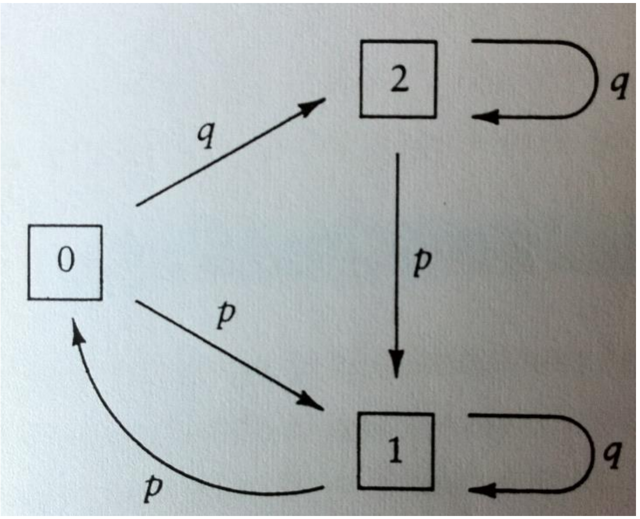
•*D(n) =1 or 0 is the total items order on the nth day*

•*X(n) state of the system at the end of the day n*

$$X(n+1) = \begin{cases} X(n) - D(n+1) & if\ X(n) > s \\ S - D(n+1) & if\ X(n) \leq s \end{cases}$$

III/ Inventory control

- Diagram



- Simulation

- Compute random daily demands D(n)

$$g(u) = \begin{cases} 0 & for\ 0 < u \leq 0.1 \\ 1 & for\ 0.1 < u \leq 0.3 \\ 2 & for\ 0.3 < u \leq 0.6 \\ 3 & for\ 0.6 < u \leq 0.85 \\ 4 & for\ 0.85 < u < 1 \end{cases}$$

III/ Inventory control

- $X(0) = 3$
- $S = 3$
- $s = 0$

$$x_2 = \begin{cases} x_1 - d_2 \ \ if\ x_1 \geq 1 \\ 3 - d_2 \ \ if\ x_1 \leq 0 \end{cases}$$

- Numerical example

•u1= 0.405	d1=2	x1 = 3 - d1 = 1
•u2= 0.829	d2=3	x2 = 1 – d2 = -2 (5 items restocked)
•u3= 0.152	d3=1	x3 = 3 – d3 = 2
•u4=0.927	d4=4	x4 = 2 – d4 = -2 (again 5 items)
•u5=0.556	d5=2	x5 = 3 – d5 = 1

- *Average unfilled demand is 4/5 order per day*
- *We can compare effectiveness of different inventory policies by changing s and S*

Thank you for your attention!