

5th presentation: Special distributions (part 2)

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#### I/Gamma distribution

- 1) Definition
- 2)Exponential distribution
- *3) Life test*
- 4) Relation to poisson process

#### II/Bivariate Normal distribution

- 1) Generalities
- 2) Definition
- 3) Properties





- Popular model for random variables that are known to be positive
- •Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} dx$$

*Interesting mathematical properties* 

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$



Gamma distribution

a random variable X has a gamma distribution with parameters  $\alpha$  and  $\beta$  if X has a continuous distribution for which the p.d.f. is :

$$f(x/\alpha,\beta) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & for \ x > 0\\ 0 & for \ x \le 0 \end{cases}$$



#### Mean and Variance

$$E(X^k) = \int_0^\infty x^k f(x/\alpha, \beta) dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} e^{-\beta x} dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}}$$

$$E(X) = \frac{\alpha}{\beta}$$

$$Var(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$Var(X) = \frac{\alpha}{\beta^2}$$



Moment generating function

$$\psi(t) = (\frac{\beta}{\beta - t})^{\alpha}$$

Properties

The sum of independent random variables that have gamma distributions with a common value of the parameter  $\beta$  will also have a gamma distribution

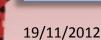




A random variable X following **a gamma distribution with parameters**  $\alpha$ =1 **and**  $\beta$  is said to have an exponential distribution

$$f(x/\beta) = \begin{cases} \beta e^{-\beta x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

$$E(X) = \frac{1}{\beta}$$
  $Var(X) = \frac{1}{\beta^2}$   $\psi(t) = \frac{\beta}{\beta - t}$ 



# I/ Gamma distribution 3) Life test

#### •Theorem

•X1...Xn are som random samples from an exponential distribution with parameter  $\beta$ , then the distribution  $Y=\min [X1 ... Xn]$  will be an exponential distribution with parameters  $n\beta$ 

#### •Example (light bulbs)

- •N light bulbs are burning simultaneously and independently in a test to determine the lengths of their lives
- •Each one of them Xi has an exponential distribution with parameter β
- •What is the distribution of time Y1 until one of the n bulbs fails?
- •Y = min [X1 ... Xn]

•
$$Pr(Y_1 > t) = Pr(X_1 > t, ..., X_n > t) = Pr(X_1 > t) ... Pr(X_n > t) = e^{-n\beta t}$$

•Using the memoryless property, Yk will have an exponential distribution with parameter  $(n-k-1)\beta$ 

#### I/ Gamma distribution 4) Relation to Poisson process

•An exponential distribution is often used in a practical problem to represent the distribution of the time that elapses before the occurrence os some event

•If the events being considered occur with a Poisson process then both the waiting time until an event occurs and the period of time between two successive events will have exponential distributions

•More generally the waiting time until the kth occurrence in a Poisson process with rate  $\lambda$  has a gamma distribution with parameters k and  $\lambda$ 



- Often we are interested in the intersection of two or more events, or the outcome of two or more random variables at the same time.
- Joint distribution

$$F(x,y) = P(X \le x, Y \le y).$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y)$$

$$\int_{x} \int_{y} f_{X,Y}(x,y) \ dy \ dx = 1.$$

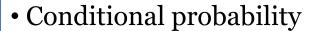


•Given two random variables X and Y whose joint distribution is known, the marginal distribution of X is simply the probability distribution of X averaging over information about Y. This is typically calculated by summing or integrating the joint probability distribution over Y.

$$p_X(x) = \int_y p_{X,Y}(x,y) \, dy = \int_y p_{X|Y}(x|y) \, p_Y(y) \, dy,$$

•Intuitively, the marginal probability of X is computed by examining the conditional probability of X given a particular value of Y, and then averaging this conditional probability over the distribution of all values of Y.





•Given two jointly distributed random variables X and Y, the conditional probability distribution of Y given X is the probability distribution of Y when X is known to be a particular value

$$f_Y(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)},$$

$$f_Y(y \mid X = x) f_X(x) = f_{X,Y}(x, y) = f_X(x \mid Y = y) f_Y(y).$$

•Relation to independence

•Random variables X, Y are independent if and only if the conditional distribution of Y given X is equal to the unconditional distribution of Y





•covariance is a measure of how much two random variables change together( statistical relationship between two random variables or two sets of data)

$$\sigma(x, y) = \mathrm{E}\left[(x - \mathrm{E}[x])(y - \mathrm{E}[y])\right],$$

• Correlation refers to any of a broad class of statistical relationships involving dependence

$$\rho_{X,Y} = \operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y},$$

### I/ Bivariate Normal distribution 2) Definition



$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{1 - \rho^2}\sigma_1\sigma_2} e^{-\frac{1}{2(1 - \rho^2)} \left[ \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1}{\sigma_1} \right) \left( \frac{x_2 - \mu_2}{\sigma_2} \right) + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2 \right]}$$

$$E(Xi) = \mu_i$$

$$Var(Xi) = \sigma_i^2$$

$$\rho(X_1, X_2) = \rho$$

### I/ Bivariate Normal distribution 2) Definition



Origin

•Z1 and Z2 are two independent variables, with standard normal distribution. Their joint p.d.f. is therefore

$$g(z_1, z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z^2_1 + z^2_2)}$$

•We define

$$X_1 = \sigma_1 Z_1 + \mu_1$$

$$X_2 = \sigma_2 \left[ \rho Z_1 + \sqrt{1 - \rho^2} Z_2 \right] + \mu_2$$

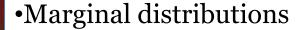
### I/ Bivariate Normal distribution 2) Definition



$$J = \frac{1}{\Delta} = \frac{1}{\sqrt{1 - \rho^2 \sigma_1 \sigma_2}}$$

•But the bivariate normal distribution arises directly and naturally in many practical problems (at least approximatively)

## I/ Bivariate Normal distribution 3) Properties

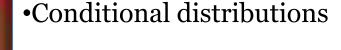


•Since X1 and X2 are defined with Z1 and Z2 we can say that the marginal distributions of Xi is a normal distribution with mean  $\mu i$  and variance  $\sigma i^2$ 

•Independence and Correlation

•Two random variables X1 and X2 that have a bivariate normal distribution are independent if and only if they are uncorrelated

## I/ Bivariate Normal distribution 3) Properties



- •The conditional distribution of X2 given that X1=x1 is a normal distribution, with
  - •Mean

$$E(X_1/X_2) = \mu_2 + \rho \sigma_2(\frac{x_1 - \mu_1}{\sigma_1})$$

Variance

$$Var(X_1/X_2) = (1-\rho)^2 \sigma_2^2$$

Joint p.d.f.  $f(x_1,x_2)$  is symmetric in the two variables  $(x_1-\mu_1)/\sigma_1$  and  $(x_2-\mu_2)/\sigma_2$ , so the other conditional distribution can be easily derived by just interchanging the parameters

## I/ Bivariate Normal distribution 3) Properties



#### Linear combinations

•If two random variables X1 and X2 have a bivariate normal distribution, then each linear combination Y=a1X1+a2X2+b will have a normal distribution

•Mean

•
$$E(Y) = a1\mu 1 + a2\mu 2 + b$$

Variance

•
$$Var(Y) = a^21\sigma^21 + a^22\sigma^22 + 2a1a2\rho\sigma1\sigma2$$



#### Thank you for your attention!