Weekly presentation SCM research

7th presentation: Theory of Markov Chains (part 1)

Summary

I/N-step transition matrix

1) Definitions

2) Matrix multiplication theory

II/ Classification of states

1) Definitions

2) Standard form for transition matrix

III/ Classes periodicity

I/n-step transition probabilities

1) Definitions

- {X(n)} Markov chain
- ***N-step transition probabilities***

$$p_{ij}(n) = P\{X(n) = j/X(0) = i\}$$

- *Probability of moving the marker from node I to node j in n trials*

- Chapman-kolmogorov equation

$$p_{ij}(m+n) = \sum\nolimits_k p_{ik}(m)p_{kj}(n)$$

- Finding all the paths of length n in the transition diagram that start at state I and end at state j

I/n-step transition probabilities
2) Matrix multiplication theory

- other method : Matrix multiplication
  - *Avoid the combinatories of path-enumeration*
  - *Suited to computer calculations*

- P, one-step transition matrix for the Markov Chain (N states)

P(1) = P

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NN} \end{bmatrix}$$

- P(n), is the n-step transition matrix, with pij(n) in row I nd column j

$$0 \leq p_{ij}(n) \leq 1 \qquad \sum_j p_{ij}(n) = 1$$

- -> stochastic matrix

I/n-step transition probabilities
2) *Matrix multiplication theory*

- Theorem

$$P(n) = P^n$$

- Initial probabitilities

$$P\{X(n) = j\} = \sum_i P\{X(0) = i\}.P\{X(n) = j/X(0) = i\}$$

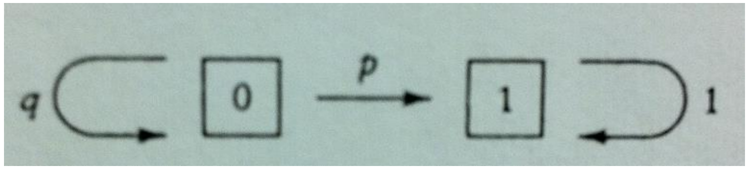
$$= \sum_i \pi_i (P^n)_{ij}$$

$$= (\pi . P^n)_j$$

I/n-step transition probabilities

2) Matrix multiplication theory

- Example : Waiting game



$$P = \begin{array}{ccc} State & 0 & 1 \\ 0 & q & p \\ 1 & 0 & 1 \end{array}$$

- P is upper triangular

$$P^n = \begin{bmatrix} q^n & 1 - q^n \\ 0 & 1 \end{bmatrix}$$

II/ Classification of states

1) Definitions

- Now we are interested in knowing which states can the marker possibily visit during the course of the Markov chain seen as a game, and starting from a node i

- i -> j means j can be reached from i with positive probability (at least one path exists)

- For any chain with N states, i -> j can be determined by checking all paths starting at i and having length less than N

- Using matrix, this means that i -> j if and only if there is an integer $0 \leq n \leq N$ so that $pij(n) \geq 0$

- Or with the matrix sum : $G = I + P + P^2 + \ldots + P^{N-1}$

  i->j if and only if Gij > 0

II/ Classification of states

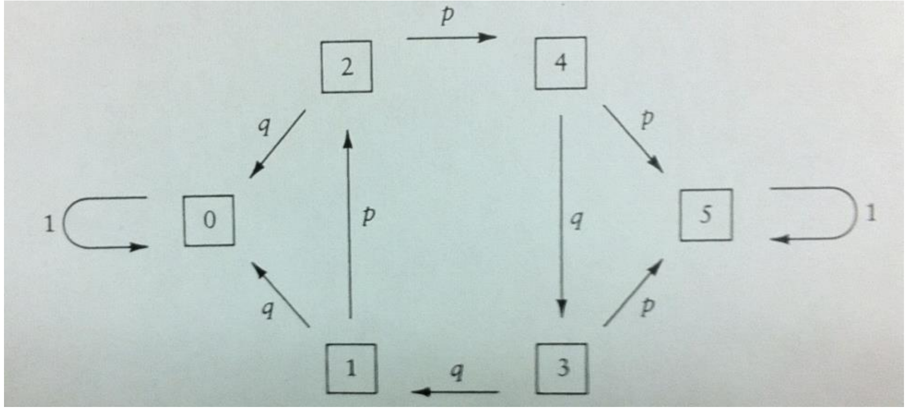
1) Definitions

- States i and j **communicate** if i-> j and also j -> i (i<->j)
- A Markov chain is **irreductible** if every state communicates with every other states
- C(i), class of the state i

$$C(i) = \{ \textit{All states } j \textit{ such that } i \leftrightarrow j\}$$

- Example : bold play

C(0) = {0}
C(5) = {5}
C(1) = {1,2,3,4}



II/ Classification of states

1) Definitions

- Properties
  - *i € C(i) for every state i*
  - *If j € C(i), then i € C(j)*
  - *For any two states i and j either C(i) = C(j) or C(i) is disjoint from C(j)*

- This creates a partition of the state space S (union of classes)

- It is possible to leave on class and enter another one, but it is impossible to come back to the first one

- The state i is **absorbing** if pii = 1

II/ Classification of states

1) Definitions

- A class C is called **ergodic** if every path that starts in C remains in C

$$\sum_{j \in C} p_{ij} = 1$$

- A class C is transient if there is a path out of C

$$\sum_{j \in C} p_{ij} < 1$$

II/ Classification of states

1)Standard form for transition matrix

- For the gambler’s ruin, N = 4, the states grouped into classes are {0}, {4}, {1,2,3}

- And the matrix P is

$$\begin{array}{cccccc}
State & 0 & 4 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 & 0 & 0 \\
1 & q & 0 & 0 & p & 0 \\
2 & 0 & 0 & q & 0 & p \\
3 & 0 & p & 0 & q & 0
\end{array}$$

II/ Classification of states

1)Standard form for transition matrix

- Standard form for transition matrix
- A stochastic matrix P can be written

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{R} & \boldsymbol{Q} \end{bmatrix}$$

- With

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = \begin{matrix} q & 0 \\ 0 & 0 \\ 0 & p \end{matrix} \qquad Q = \begin{matrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{matrix}$$

- *Q describes the transient -> transient mouvements in the chain*
- *R describes transient -> ergodic movements in the chain*
- *S describes the movements within each ergodic class in the chain*

II/ Classification of states

1)Standard form for transition matrix

- S describes the movements within each ergodic class in the chain

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- If we suppose the chain has d erodic classes, S has the shape (where Si is the transition matrix within th ith ergodic class)

$$S = \begin{bmatrix} S_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_d \end{bmatrix}$$

III/ Classes periodicity

- Periodic and aperiodic classes

- A state is **periodic**, with period d > 1, if every path that starts and ends at this state has length nd (n integer)

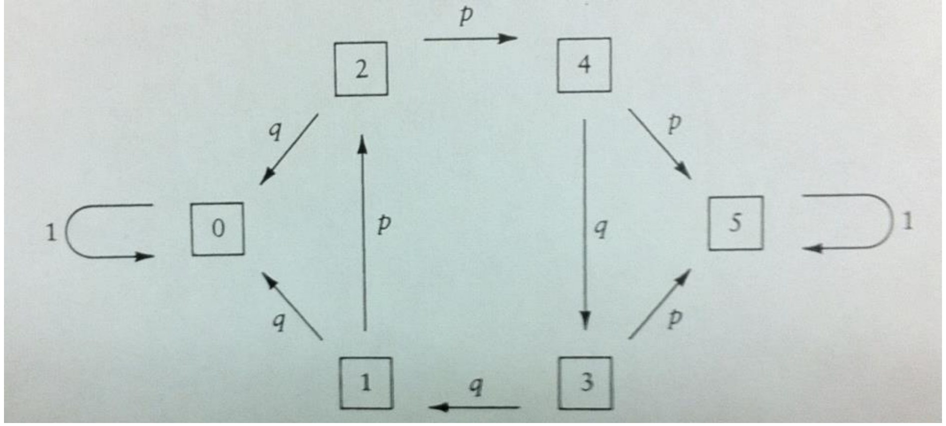
- A state is **aperiodic** if not periodic

- Example : gambler's ruin
  - P11(k) = 0 when k is odd

- If i is periodic, and i<->j, then j is periodic witht the same period as i
  - ➢demonstration

III/ Classes periodicity

- Example : Bold play



- *States 1,2,3 and are all periodic of period 4*
- *Loop 1->2->4->3->1*

Thank you for your attention!