Weekly presentation SCM research

8th presentation: Theory of Markov Chains (part 2)

Summary

I/Absorbing Chains

1) Definition

2) Behavior of the transition matrix

3) Length of the game

4) Application

II/ Regular Chains

1) Definition

2) Limit theorem

I/ Absorbing Chains
1) Definition

- A Markov Chain is said to be **absorbing** if every state in it is either absorbing or transient

- Each ergodic class consists of a single absorbing state

- Waiting game, gambler's ruin, bold play …

- Property

If we start in a transient state, we are certain of reaching an absorbing state after a finite number of trials

I/ Absorbing Chains

2) Behavior of the transition matrix

- Reminder

$$P = \begin{bmatrix} S & 0 \\ R & Q \end{bmatrix}$$

- *Q describes the transient -> transient mouvements in the chain*
- *R describes transient -> ergodic movements in the chain*
- *S describes the movements within each ergodic class in the chain*

- Absorbing Chain

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

I/ Absorbing Chains
2) Behavior of the transition matrix

- Matrix multiplication

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

$$P^2 = \begin{bmatrix} I & 0 \\ R + QR & Q^2 \end{bmatrix} \qquad P^3 = \begin{bmatrix} I & 0 \\ R + QR + Q^2R & Q^3 \end{bmatrix}$$

$$P^n = \begin{bmatrix} I & 0 \\ N_nR & Q^n \end{bmatrix}$$

I/ Absorbing Chains
2) Behavior of the transition matrix

- Matrix multiplication

$$P^n = \begin{bmatrix} I & 0 \\ N_n R & Q^n \end{bmatrix}$$

- With

$$N_{n=} I + Q + Q^2 + \cdots + Q^{n-1}$$

- Theorem

When n-> ∞, then Q^n*-> 0 and* $N_n \to (1 - Q)^{-1}$

I/ Absorbing Chains

2) Behavior of the transition matrix

•Demonstration

•*Define a norm*

$$\|A\| = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

•*Show that there exist constants b>0 and r<1 such that*

$$\|Q^n\| \le br^n$$

•*Show that I – Q is invertible using geometric series*

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2) Behavior of the transition matrix

•Limiting form of the transition matrix

$$\lim_{n\to\infty} P^n = \begin{bmatrix} I & 0 \\ NR & 0 \end{bmatrix}$$

Where $N_n \to (1-Q)^{-1}$

The matrices N and NR contain the important information about what happens in the chain in the long run

I/ Absorbing Chains

3) Length of the game

Pi(A) = P{A / X(0) = i}

Ei[Y] = E[Y / X(0) = i]

T, set of transient states

Vj, total number of visits of state j during the entire game (Vj < ∞ and has finite expectations)

The matrix N gives the expected number of visits to each transient state

I/ Absorbing Chains
3) Length of the game

•Theorem

For every pair of transient states i and j

$$E_i[V_j] = n_{ij}$$

•Demontration

$$E_i[V_j] = \delta_{ij} + \sum_{k \in T} p_{ik} E_k[V_j]$$

M = I + QM

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3) Length of the game

- W, number of steps taken until an absorbing state is reached

- We have then

  - $W < \infty$

  - $E_i[W] = \sum_{j \in T} n_{ij}$

This gives the expected length of the game when starting at a transient state i

I/ Absorbing Chains

3) Length of the game

- Probability of reaching each absorbing state

$$P_i\{Game\ ends\ at\ j\} = (NR)_{ij}$$

- Demonstration

  - *The marker moves to j at first step, ending the game*
  - *The marker moves to an absorbing state other than j at first step, ending the game*
  - *The marker moves to a transient state k at first step*

  - *Rule of Total Causes* $P_i[A_j] = p_{ij} + \sum_{k \in T} p_{ik} P_k[A_j]$

I/ Absorbing Chains
4) Application

- To use the theorems, the matrix I – Q needs to be inverted *(Cramer's rule, Gauss-Jordan method …)*

- Example : the waiting game

$$P = \begin{array}{ccc} State & 0 & 1 \\ 0 & q & p \\ 1 & 0 & 1 \end{array}$$

$$N = \left[\frac{1}{1-q}\right] = \left[\frac{1}{p}\right] \qquad NR = [1]$$

I/ Absorbing Chains

4) Application

•Example : the gambler's ruin, N = 3

$$Q = \begin{array}{ccc} \text{State} & 1 & 2 \\ 1 & 0 & p \\ 2 & q & 0 \end{array} \qquad R = \begin{array}{ccc} \text{State} & 0 & 3 \\ 1 & q & 0 \\ 2 & 0 & p \end{array}$$

•We can invert I - Q

$$N = \begin{array}{ccc} \text{State} & 1 & 2 \\ 1 & \dfrac{1}{1-pq} & \dfrac{p}{1-pq} \\ 2 & \dfrac{q}{1-pq} & \dfrac{1}{1-pq} \end{array} \qquad NR = \begin{array}{ccc} \text{State} & 0 & 3 \\ 1 & \dfrac{q}{1-pq} & \dfrac{p^2}{1-pq} \\ 2 & \dfrac{q^2}{1-pq} & \dfrac{p}{1-pq} \end{array}$$

I/ Absorbing Chains

4) Application

- Example : the gambler's ruin, N = 3

- The matrix theory matches the earlier analysis

$$(NR)_{state\ 1, state\ 0} = \frac{q}{1 - pq} \qquad \frac{1 - r^2}{1 - r^3}$$

- The row sums of the matrix NR are 1 (one of the players is certain to be ruined)

- Duration of the game $E_1[W] = \dfrac{1 + p}{1 - pq}$

II/ Regular Chains
1) Definition

- A finite Markov Chain is called regular it it is irreductible and aperiodic
  - *i<->j for every state*
  - *There are paths of any length*

- Property
  - *A Markov chain is regular if and only if there exists an interger n>0 such that every entry of Pn is positive (and for every m>n as well)*

- Examples of regular matrix
  - *A stochastic matrix in which all entries are positive*
  - *Stochastic matrix irreductible and that have at least one positive diagonal entry*
  - *Machine maintenance, inventory models are of this type*

II/ Regular Chains
2) Limit Theorem

•LIMIT THEOREM

If P is a regular transition matrix, then

- *The powers of P converge to a stochastic matrix A*
- *Each row of A (* ***limiting matrix*** *) is the same vector α (* ***limiting vector*** *) which every component is positive*

Thank you for your attention!