Weekly presentation SCM research

4th presentation: Special distributions

Summary

I/ Introduction

- An incredible variety of special distributions have been studied over the years, and new ones are constantly being added to the literature
- two general classes of distributions: location-scale families and exponential families
- Discrete/ Continious families
- Special properties

II/ Poisson distribution
1) Definition

• Discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time and/or space if these events occur with a known average rate and independently of the time since the last even

•Can also be used for the number of events in other specified intervals such as distance, area or volume

•Useful approximation to binomial distributions with very small success probabilities

I/ Poisson distribution
1) Definition

- Probability function

Random variable with discrete distribution, X is said to have a Poisson distribution with mean λ (λ>0) if

$$f(x/\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & for\ x = 0,1,2\ldots \\ 0 & otherwise \end{cases}$$

II/ Poisson distribution
1) Definition

•Mean and Variance

λ is both the mean and the variance of the Poisson distribution

•Proof

$$E(X) = \sum_{x=1}^{\infty} x f(x/\lambda) = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} = \lambda \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^{x-1}}{x-1!}$$

$$= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!} = \lambda$$

II/ Poisson distribution

1) Definition

$$E(X(X-1)) = \sum_{x=2}^{\infty} x(x-1)f(x/\lambda) = \lambda^2$$

$$E(X^2) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - E(X)^2 = \lambda^2$$

• Moment Generating function

$$\psi(t) = E(e^{tX}) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!} = e^{\lambda(e^{t}-1)}$$

II/ Poisson distribution
2) Properties

- If the random variables X1 … Xk are independent and if Hi has a Poisson distribution with mean λi then the sum X1+…+Xk has a Poisson distribution with mean λ1+…+ λk

  Proof using the moment generating function

- **Example (customer arrivals)**

  - The number of customer who arrive in disjoint hours can be represented by independent random variable having a poisson distribution with mean 4.5
  - *Probability that at least 12 customers arrive in a two-hour period?*

  - Using the property the total number X=X1+X2 has a Poisson distribution with mean 9
  - **P(X≥12) = 0;1970**

II/ Poisson distribution

3) Binomial approximation

- If the value of n is large and the value of p is close to 0, the binomial distribution with parameters n and p can be approximated by a Poisson distribution with mean np

- Proof

$$f(x/n,p) = \frac{n(n-1)\dots(n-x+1)}{x!}p^x(1-p)^{n-x}$$

$$f(x/n,p) = \frac{\lambda^x}{x!}\frac{n}{n}\frac{n-1}{n}\dots\frac{n-x+1}{n}(1-\frac{\lambda}{n})^{-x}(1-\frac{\lambda}{n})^n$$

$$f(x/n,p) \to \frac{e^{-\lambda}\lambda^x}{x!}$$

II/ Poisson distribution

3) Binomial approximation

- **Example**

  - In a large population the proportion of peaople that have a certain disease is 0.01
  - *Probability that in a group of 200 at least 4 of them have the disease?*

  - Binomial distribution with parameters n=200 and p = 0.01
  - Approximated by a Poisson distribution with mean 2

  - Reading on the Poisson tables, **$P(X \geq 4)=0.1428$**
  - The actual value is 0.1420

II/ Poisson distribution

4) Poisson process

•A poisson process with rate λ per unit time is a process that satisfies:

- ➢*The number of arrivals in every fixed interval of time of length t has a Poisson distribution for which the mean is λt*
- ➢*The numbers of arrivals in every two disjoint time intervals are independent*

•Poisson processes are more general (particles emitted from a radioactive source, defects on the surface of a manufactured product…)

•Popular process

- ➢ *computationnaly convenient*
- ➢ *Mathematical justification*

III/ Normal distribution

1) Definition

- Most widely used model for random variables with continuous distribution and single most important distribution in statistics
  - *Mathematical convenience*
  - *Various physical experiments often have distributions that are approximately normal*
  - *For a large random sample from any distribution that has a finite variance, the distribution of the sample will be approximatively normal (consequence of the central limit theorem)*

- It is said that a random variable X has a normal distribution with mean μ and variance σ^2 if:

$$f(x/\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

- The p.d.f. cannot be integrated in closed form (hence d.f. tables are used in computer programs)

III/ Normal distribution

1) Definition

- Moment generating function

$$\psi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\left[tx - \frac{(x-\mu)^2}{2\sigma^2}\right]} dx$$

$$\psi(t) = e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}$$

- Mean and Variance

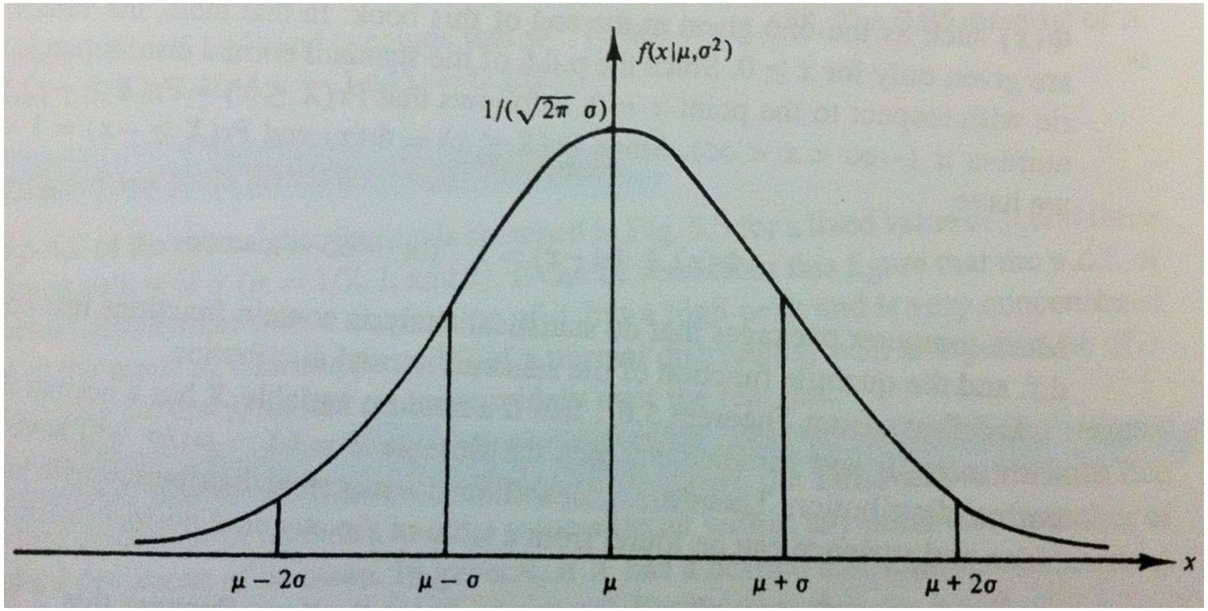
$$E(X) = \psi'(0) = \mu$$

$$Var(X) = \psi''(0) - [\psi'(0)]^2 = \sigma^2$$

III/ Normal distribution
2) Properties

- Shape

  - μ is the mode of the distribution



III/ Normal distribution

2) *Properties*

- Linear transformation

If X has a normal distribution with mean μ and variance σ² then Y=aX+b (with a≠0) has a normal distribution with mean aμ+b and vacriance a²σ² + generalization

- The normal distriution with mean 0 and variance 1 is called the standard normal distribution

$$\phi(x) = f\left(\frac{x}{0,1}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)du$$

III/ Normal distribution

2) *Properties*

Example (heights of men and women)

- women heights follow a normal distribution with $\mu=65$ and $\sigma=1$
- men heights follow a normal distribution with $\mu=68$ and $\sigma=2$

Probability that a randomly selected woman will be taller than a randomly selected man?

W, height of the woman

M, height of the man

The difference W-M follows a normal distribution with mean 65-68=-3 and variance $1^2+2^2=5$

We define

$$Z = \frac{W - M + 3}{\sqrt{5}}$$

It follows from the properties that Z has a standard normal distribution

$P(W>M)=P(W-M>0)=P(Z>1.342)=1-\Phi(1.342)=0.090$

III/ Normal distribution

3) Central Limit Theorem

•**Theorem**

Whenever a random sample of size n is taken from ANY distribution with mean μ and variance σ², the sample mean Xn will have a distribution that is appromaxitively normal with mean μ and variance σ²/n

If the random variables X1,…,Xn from a random sample of size n from a given distribution with mean μ and variance σ², then for each fixed number x :

$$\lim_{n\to\infty} P\left[\frac{\sqrt{n}(\overline{X_n}-\mu)}{\sigma} \leq x\right] = \Phi(x)$$

$\dfrac{\sqrt{n}(\overline{X_n}-\mu)}{\sigma}$ will be approximatively a standard normal distribution

III/ Normal distribution
3) Central Limit Theorem

- Theorem effects

  - *Provides a plausible explanation for the fact that the distributuions of many random variables studied in physical experiments are approximatively normal*

  - *The distribution of the sum of many random variables can be approximatively normal even though the distribution of each random variable in the sum differs from the normal*

Thank your for your attention!