Variational Inference

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P70 Variational Inference — Background (Review)

Frequecy Perspctive — Optimization Issues $\begin{cases} \text{analytical solution: } \frac{\partial Loss}{\partial w} = 0 \\ \text{numerical solution: } GD \end{cases}$

Bayesian Perespctive — Itegral Issues

X Samples $\hat{x}$ new samples → Find $P(\hat{x}|X)$

$$P(\hat{x}|X) = \int_\theta P(\hat{x},\theta|X)\,d\theta = \int_\theta P(\hat{x}|\theta,X)P(\theta|X)\,d\theta$$
$$= \int_\theta P(\hat{x}|\theta)P(\theta|X)\,d\theta = E_{\theta|X}[P(\hat{x}|\theta)]$$

P71,72 Formula Deduction (Mean Field VI → Classical VI)

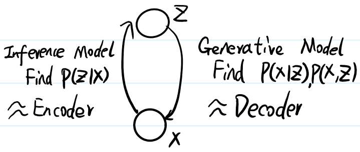
X: observed data Z: latent variable + parameter (X,Z): complete data

$$\log p(x) = \log p(x,z) - \log p(z|x) = \log\frac{p(x,z)}{q(z)} - \log\frac{p(z|x)}{q(z)}$$

左边 $= \int_z \log p(x)\,q(z)\,dz = \log p(x)$ 右边 $= \underbrace{\int_z q(z)\log\frac{p(x,z)}{q(z)}\,dz}_{\text{evidence lower bound (ELBO)}} - \underbrace{\int_z q(z)\log\frac{p(z|x)}{q(z)}\,dz}_{KL(q\|p)} = \underbrace{\mathcal{L}(q)}_{\text{variation}} + \underbrace{KL(q\|p)}_{\geq 0}$

To find $p(z|x) \Rightarrow q(z) \approx p(z|x) \Rightarrow \hat{q}(z) = \arg\min_{q(z)} KL(q\|p) = \arg\max_{q(z)} \mathcal{L}(q)$

Mean Field Theory: Divide $q(z)$ into M independent groups $\Rightarrow q(z) = \prod_{i=1}^{M} q_i(z_i)$



$$\mathcal{L}(q) = \underbrace{\int_z q(z)\log p(x,z)\,dz}_{①} - \underbrace{\int_z q(z)\log q(z)\,dz}_{②}$$

Focus on item j

$$① = \int_z \prod_{i=1}^{M} q_i(z_i)\log p(x,z)\,dz = \int_{z_j} q_j(z_j)\Big[\underbrace{\int_{z_1\cdots z_{j-1}z_{j+1}\cdots z_M} \prod_{i=1,i\neq j}^{M} q_i(z_i)\log p(x,z)\,dz_1\cdots dz_{j-1}dz_{j+1}\cdots dz_M}_{\int_{z_i}\prod_i q_i(z_i)\log p(x,z)\,dz_i\ (i=1\cdots j-1,j+1\cdots M)}\Big]dz_j$$

$$= \int_{z_j} q_j(z_j)\, E_{\prod_i q_i(z_i)}[\log p(x,z)]\,dz_j \quad (i=1,\cdots j-1,j+1\cdots M) \qquad \text{Let } E_{\prod_{i\neq j} q_i(z_i)}[\log p(x,z)] = \log\hat{p}(x,z_j)$$

$$\therefore ① = \int_{z_j} q_j(z_j)\log\hat{p}(x,z_j)\,dz_j$$

$$② = \int_z \prod_{i=1}^{M} q_i(z_i)\sum_{i=1}^{M}\log q_i(z_i)\,dz = \int_z \prod_{i=1}^{M} q_i(z_i)[\log q_1(z_1) + \cdots + \log q_M(z_M)]\,dz$$

Take item 1 as a example: $\int_z \prod_{i=1}^{M} q_i(z_i)\log q_1(z_1)\,dz = \int_{z_1} q_1(z_1)\log q_1(z_1)\,dz_1 \cdot \underbrace{\int_{z_2} q_2(z_2)\,dz_2}_{1} \cdot \cdots \cdot \underbrace{\int_{z_M} q_M(z_M)\,dz_M}_{1}$

$$= \int_{z_1} q_1(z_1)\log q_1(z_1)\,dz_1$$

$\therefore ② = \sum_{i=1}^{M}\int_{z_i} q_i(z_i)\log q_i(z_i)\,dz_i$ $\because$ Focus on item j and treat the rest items as constant

$$\therefore ② = \int_{z_j} q_j(z_j)\log q_j(z_j)\,dz_j + C$$

$$\therefore ①-② = \int_{z_j} q_j(z_j)\log\frac{\hat{p}(x,z_j)}{q_j(z_j)}\,dz_j + C = -KL\big(q_j(z_j)\|\hat{p}(x,z_j)\big) + C$$

$-KL \leq 0$ The equation holds if and only if $q_j(z_j) = \hat{p}(x,z_j)$

$\log q_j(z_j) = E_{\prod_{i\neq j} q_i(z_i)}[\log p(x,z)] = \int_{q_1}\cdots\int_{q_{j-1}}\int_{q_{j+1}}\cdots\int_{q_M} q_1\cdots q_{j-1}q_{j+1}\cdots q_M \log p(x,z)\,dq_1\cdots dq_{j-1}dq_{j+1}\cdots dq_M$ Change to integrate over q

$$\log\hat{q}_1(z_1) = \int_{q_2}\cdots\int_{q_M} q_2\cdots q_M\log p(x,z)\,dq_2\cdots dq_M$$
$$\log\hat{q}_2(z_2) = \int_{\hat{q}_1}\int_{q_3}\cdots\int_{q_M}\hat{q}_1 q_3\cdots q_M\log p(x,z)\,d\hat{q}_1\,dq_3\cdots dq_M$$
$$\cdots$$
$$\log\hat{q}_M(z_M) = \int_{\hat{q}_1}\cdots\int_{\hat{q}_{M-1}}\hat{q}_1\cdots\hat{q}_{M-1}\log p(x,z)\,d\hat{q}_1\cdots d\hat{q}_{M-1}$$

Repeat just like Coordinate Ascend

P73 SGVI

Default $q(z|x)$ is an exponential family distribution

So we find parameter ϕ of $q(z)$ instead of finding $q(z)$ directly

$$\therefore \mathcal{L}(\phi) = E_{q_\phi(z)}[\log P_\theta(x,z) - \log q_\phi(z)] \qquad \hat{\phi} = \arg\max_\phi \mathcal{L}(\phi)$$

$$\nabla_\phi\mathcal{L}(\phi) = \nabla_\phi E_{q_\phi(z)}[\log P_\theta(x,z) - \log q_\phi(z)] = \nabla_\phi\int q_\phi(z)[\log P_\theta(x,z) - \log q_\phi(z)]\,dz$$

$\because$ Integration and derivation are interchangeable $\therefore = \underbrace{\int[\log P_\theta(x,z) - \log q_\phi(z)]\nabla_\phi q_\phi(z)\,dz}_{①} + \underbrace{\int q_\phi(z)\nabla_\phi\underbrace{[\log P_\theta(x,z)}_{与\phi无关} - \log q_\phi(z)]\,dz}_{②}$

$$② = -\int q_\phi(z)\nabla_\phi\log q_\phi(z)\,dz = -\int q_\phi(z)\frac{1}{q_\phi(z)}\nabla_\phi q_\phi(z)\,dz = -\int\nabla_\phi q_\phi(z)\,dz = -\nabla_\phi\int q_\phi(z)\,dz = -\nabla_\phi 1 = 0$$

$\because \nabla_\phi q_\phi(z) = q_\phi(z)\nabla_\phi\log q_\phi(z)$ $\therefore ① = \int q_\phi(z)[\log P_\theta(x,z) - \log q_\phi(z)]\nabla_\phi\log q_\phi(z) = E_{q_\phi(z)}[(\log P_\theta(x,z) - \log q_\phi(z))\nabla_\phi q_\phi(z)]$

$$\therefore \nabla_\phi\mathcal{L}(\phi) = E_{q_\phi(z)}[(\log P_\theta(x,z) - \log q_\phi(z))\nabla_\phi q_\phi(z)] \Leftarrow \text{MCMC}\cdots$$