机器学习第四次作业

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题目：

对以下数据点，求出其最优的超平面。

$$x_1 = (1,2,3), y_1 = +1$$

$$x_2 = (4,1,2), y_2 = +1$$

$$x_3 = (-1,2,-1), y_3 = -1$$

解：

设超平面方程为：

$$w_1 x + w_2 y + w_3 z + b = 0$$

则上述问题等价于下列约束最优化问题：

$$\min_{w_i, b, i=1,2,3} \frac{1}{2}(w_1^2 + w_2^2 + w_3^2)$$

$$s.t.$$

$$\begin{cases} w_1 + 2w_2 + 3w_3 + b \geq 1 \\ 4w_1 + w_2 + 2w_3 + b \geq 1 \\ w_1 - 2w_2 + w_3 - b \geq 1 \end{cases}$$

构建拉格朗日函数：

$$\begin{aligned} L(w_1, w_2, w_3, b, \mu_1, \mu_2, \mu_3) \\ &= \frac{1}{2}(w_1^2 + w_2^2 + w_3^2) + \mu_1(1 - w_1 - 2w_2 - 3w_3 - b) \\ &+ \mu_2(1 - 4w_1 - w_2 - 2w_3 - b) + \mu_3(-w_1 + 2w_2 - w_3 + b + 1) \end{aligned}$$

分别求L对w_1、w_2、w_3和b的偏微分：

$$\frac{\partial L}{\partial w_1} = w_1 - \mu_1 - 4\mu_2 - \mu_3 = 0$$

$$\frac{\partial L}{\partial w_2} = w_2 - 2\mu_1 - \mu_2 + 2\mu_3 = 0$$

$$\frac{\partial L}{\partial w_3} = w_3 - 3\mu_1 - 2\mu_2 - \mu_3 = 0$$

$$\frac{\partial L}{\partial b} = -\mu_1 - \mu_2 + \mu_3 = 0$$

用 KKT 条件对其进行约束：

$$
\begin{cases}
w_1 + 2w_2 + 3w_3 + b - 1 \geq 0 \\
4w_1 + w_2 + 2w_3 + b - 1 \geq 0 \\
w_1 - 2w_2 + w_3 - b - 1 \geq 0 \\
\mu_i \geq 0,\ i = 1,2,3 \\
\mu_1(w_1 + 2w_2 + 3w_3 + b - 1) = 0 \\
\mu_2(4w_1 + w_2 + 2w_3 + b - 1) = 0 \\
\mu_3(w_1 - 2w_2 + w_3 - b - 1) = 0
\end{cases}
$$

分情况进行讨论：

1) 假设$\mu_1 = 0$，可以计算得：$w_1 = 5\mu_2$，$w_2 = -\mu_2$，$w_3 = 3\mu_2$，代入 3 个不等式得：$b \geq 1 - 12\mu_2$，$b \geq 1 - 25\mu_2$，$b \leq 10\mu_2 - 1$。此时，$\mu_2 = \mu_3 \geq \frac{1}{11}$，则$4w_1 + w_2 + 2w_3 + b - 1 = 0$，$w_1 - 2w_2 + w_3 - b - 1 = 0$，即：$25\mu_2 + b - 1 = 0$，$10\mu_2 - b - 1 = 0$，解得：$\mu_2 = \frac{2}{35}$，矛盾，所以$\mu_1 \geq 0$；

2) 假设$\mu_2 = 0$，可以计算得：$w_1 = 2\mu_1$，$w_2 = 0$，$w_3 = 4\mu_1$，代入 3 个不等式得：$b \geq 1 - 14\mu_1$，$b \geq 1 - 16\mu_1$，$b \leq 6\mu_1 - 1$。此时，$\mu_1 = \mu_3 \geq \frac{1}{10}$，则$w_1 + 2w_2 + 3w_3 + b - 1 = 0$，$w_1 - 2w_2 + w_3 - b - 1 = 0$，即：$25\mu_2 + b - 1 = 0$，$10\mu_2 - b - 1 = 0$，解得：$\mu_1 = \frac{1}{10}$。此时，$\mu_2 = 0$，$\mu_1 = \mu_3 = \frac{1}{10}$，$w_1 = \frac{1}{5}$，$w_2 = 0$，$w_3 = \frac{2}{5}$，$b = -\frac{2}{5}$；

3) 假设$\mu_3 = 0$，可以计算得：$w_1 = -3\mu_1$，$w_2 = \mu_1$，$w_3 = \mu_1$，代入 3 个不等式得：$b \geq 1 - 2\mu_1$，$b \geq 1 + 9\mu_1$，$b \leq -4\mu_1 - 1$。此时，$\mu_1 = -\mu_2 \leq -1$，矛盾，所以$\mu_3 \geq 0$。

综上所述，$\mu_2 = 0$，$\mu_1 = \mu_3 \neq 0$，此时，$\mu_2 = 0$，$\mu_1 = \mu_3 = \frac{1}{10}$，$w_1 = \frac{1}{5}$，$w_2 = 0$，$w_3 = \frac{2}{5}$，$b = -\frac{2}{5}$。所以，超平面方程为：$\frac{1}{5}x + \frac{2}{5}z - \frac{2}{5} = 0$，即：$x + 2z = 2$。