Matrix Analysis Assignment 3

October 24, 2020

Question 1.

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1. 由线性变换的定义我们可以得到:

$$
\begin{aligned}
\mathbf{A}\left(\mathbf{e}_1\right) &= \mathbf{A}(1,0,0) = (1,0,1) = (1)\mathbf{e}_1 + (0)\mathbf{e}_2 + (1)\mathbf{e}_3 \\
\mathbf{A}\left(\mathbf{e}_2\right) &= \mathbf{A}(0,1,0) = (2,-1,0) = (2)\mathbf{e}_1 + (-1)\mathbf{e}_2 + (0)\mathbf{e}_3 \\
\mathbf{A}\left(\mathbf{e}_3\right) &= \mathbf{A}(0,0,1) = (-1,0,7) = (-1)\mathbf{e}_1 + (0)\mathbf{e}_2 + (7)\mathbf{e}_3
\end{aligned}
$$

所以,

$$
[\mathbf{A}]_{\mathcal{S}} = \left(\left[\mathbf{A}\left(\mathbf{e}_1\right)\right]_{\mathcal{S}} \mid \left[\mathbf{A}\left(\mathbf{e}_2\right)\right]_{\mathcal{S}} \mid \left[\mathbf{A}\left(\mathbf{e}_3\right)\right]_{\mathcal{S}}\right) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix}
$$

2.

$$
\mathbf{Q} = \left([\mathbf{y}_1]_{\mathcal{S}} \mid [\mathbf{y}_2]_{\mathcal{S}} \mid [\mathbf{y}_3]_{\mathcal{S}}\right) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
[\mathbf{A}]_{\mathcal{S}'} = \mathbf{Q}^{-1}[\mathbf{A}]_{\mathcal{S}}\mathbf{Q} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 2 & -9 \\ 1 & 1 & 8 \end{pmatrix}
$$

Question 2.

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1. 由线性变换的定义我们可以得到:

$$
\begin{aligned}
T\left(e_1\right) &= T(1,0,0,0) = (1,0,0,0)^T = e_1 \in \mathcal{X} \\
T\left(e_2\right) &= T(0,1,0,0) = (1,1,0,0)^T = e_1 + e_2 \in \mathcal{X}
\end{aligned}
$$

对于空间 $\mathcal{X}$ 的所有向量 $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 \in \mathcal{X}$,

$$
T(\mathbf{x}) = (\alpha+\beta, \beta, 0, 0)^T = (\alpha+\beta)\,\mathbf{e}_1 + \beta\mathbf{e}_2 \in \mathcal{X}
$$

所以空间 $\mathcal{X}$ 在 T 下是不变子空间

2. 因为 $\mathbf{T}(e_1)=e_1, \mathbf{T}(e_2)=e_1+e_2$

$$\left[\mathbf{T}_{/\mathcal{X}}\right]_{e_1,e_2}=\left(\left[\mathbf{T}_{/\mathcal{X}}\left(\mathbf{x}_1\right)\right]_{e_1,e_2} \mid \left[\mathbf{T}_{/\mathcal{X}}\left(\mathbf{x}_2\right)\right]_{e_1,e_2}\right)=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

3. 假设 $\mathcal{B}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{u}_1,\mathbf{u}_2,\}$ 是 $\mathcal{R}^4$ 空间的一组基,因此 $[\mathbf{T}]_{\mathcal{B}}=\left(\left[\mathbf{T}\left(\mathbf{e}_1\right)\right]_{\mathcal{B}} \mid \left[\mathbf{T}\left(\mathbf{e}_2\right)\right]_{\mathcal{B}} \left|\left[\mathbf{T}\left(\mathbf{u}_1\right)\right]_{\mathcal{B}}\right| \left[\mathbf{T}\left(\mathbf{u}_2\right)\right]_{\mathcal{B}}\right)$ 于是，我们有

$$[\mathbf{T}]_{\mathcal{B}}=\begin{pmatrix} \left[\mathbf{T}_{/\mathcal{X}}\right]_{\mathcal{B}_{\mathcal{X}}} & \mathbf{B}_{2\times 2} \\ 0 & \mathbf{C}_{2\times 2} \end{pmatrix}$$

将 $\left[\mathbf{T}_{/\mathcal{X}}\right]_{e_1,e_2}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 代入得:

$$[\mathbf{T}]_{\mathcal{B}}=\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \mathbf{B}_{2\times 2} \\ 0 & \mathbf{C}_{2\times 2} \end{pmatrix}$$