Matrix Analysis Assignment 4

November 8, 2020

1. Exercise 3

1. 对于矩阵 A, 先计算矩阵 A^TA 的特征值:

$$A^TA-\lambda I=\begin{pmatrix}1&-1\\-2&2\end{pmatrix}\begin{pmatrix}1&-2\\-1&2\end{pmatrix}-\lambda I=\begin{pmatrix}2-\lambda&-4\\-4&8-\lambda\end{pmatrix}$$

由 $(\lambda-2)(\lambda-8)-16=0$, 解得 $\lambda_1=10,\lambda_2=0$, 因此可得

$$\begin{aligned}&\|A\|_2=\sqrt{\lambda_{\max}}=\sqrt{10}\\&\|A\|_F=\left(\sum_{i,j}|a_{ij}^2|\right)^{\frac{1}{2}}=\sqrt{10}\\&\|A\|_1=\max_j\sum_i|a_{ij}|=4\\&\|A\|_\infty=\max_i\sum_j|a_{ij}|=3\end{aligned}$$

2. 对于矩阵 B, 先计算矩阵 B^TB 的特征值:

$$B^TB-\lambda I=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}-\lambda I=\begin{pmatrix}1-\lambda&0&0\\0&1-\lambda&0\\0&0&1-\lambda\end{pmatrix}$$

由 $(1-\lambda)^3=0$, 解得 $\lambda_1=1$, 因此可得:

$$\begin{aligned}&\|B\|_2=\sqrt{\lambda_{\max}}=1\\&\|B\|_F=\left(\sum_{i,j}|a_{ij}^2|\right)^{\frac{1}{2}}=\sqrt{3}\\&\|B\|_1=\max_j\sum_i|a_{ij}|=1\\&\|B\|_\infty=\max_i\sum_j|a_{ij}|=1\end{aligned}$$

3. 对于矩阵 C, 先计算矩阵 C^TC 的特征值:

$$C^TC-\lambda I=\begin{pmatrix}36-\lambda&-18&36\\-18&9-\lambda&-18\\36&-18&36-\lambda\end{pmatrix}$$

由 $\lambda((36-\lambda)(45-\lambda)-18\times 90)=0$, 解得 $\lambda_1=81,\lambda_2=0$, 因此可得:

$$\begin{aligned}&\|C\|_2=\sqrt{\lambda_{\max}}=9\\&\|C\|_F=\left(\sum_{i,j}|a_{ij}^2|\right)^{\frac{1}{2}}=9\\&\|C\|_1=\max_j\sum_i|a_{ij}|=10\\&\|C\|_\infty=\max_i\sum_j|a_{ij}|=10\end{aligned}$$

2. Exercise 12

1. k=1: $r_{11} \longleftarrow \|a_1\| = \sqrt{3}$ 并且 $q_1 = \longleftarrow \frac{a_1}{r_{11}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

   k=2: $r_{12} \longleftarrow q_1^T a_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \sqrt{3}$

   $q_2 \longleftarrow a_2 - r_{12}q_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad r_{22} \longleftarrow \|q_2\| = \sqrt{3} \quad q_2 \longleftarrow \frac{q_2}{\|r_{22}\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

   k=3:$r_{13} \longleftarrow q_1^T a_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \\ 1 \end{pmatrix} = -\sqrt{3}$

   $r_{23} \longleftarrow q_2^T a_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \\ 1 \end{pmatrix} = \sqrt{3}$

   $q_3 \longleftarrow a_3 - r_{13}q_1 - r_{23}q_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \quad r_{33} \longleftarrow \|q_3\| = \sqrt{6} \quad q_3 \longleftarrow \frac{q_3}{\|r_{33}\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$

   所以，我们可以得到:

$$Q = \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{pmatrix}$$

2. 将 $A = QR$ 带入 $Ax = b$ 得到:

$$QRx = b \Longleftrightarrow Q^TQRx = Q^Tb \Longleftrightarrow Rx = Q^Tb$$

即

$$\begin{pmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ \frac{\sqrt{3}}{3} \\ 0 \end{pmatrix}$$

回代解得

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}$$

3. Exercise 16

1.

$$P_v = \frac{vv^*}{v^*v} = \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix}$$

则 u 在 v 张成的空间投影为:

$$P_v u = \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix}$$

2.

$$P_u = \frac{uu^T}{u^Tu} = \frac{1}{15} \begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix}$$

则 v 在 u 张成的空间投影为:

$$P_u v = \frac{1}{15} \begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}$$

3.

$$P_{v^\perp} = I - \frac{vv^T}{v^Tv} = I - \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}$$

则 u 在 $v^\perp$ 张成的空间投影为:

$$P_{v\perp}u = \frac{1}{18}\begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{13}{6} \\ \frac{1}{3} \\ 3 \\ -\frac{5}{6} \end{pmatrix}$$

4.

$$P_{u^\perp} = I - \frac{uu^T}{u^Tu} = \frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}$$

则 v 在 $u^\perp$ 张成的空间投影为:

$$P_{u^\perp}v = \frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ 19 \\ -3 \\ -4 \end{pmatrix}$$