

# Three views of filtering

- Image filters in spatial domain (空域)
  - Filter is a mathematical operation of a grid of numbers
  - Smoothing, sharpening, measuring texture
- Image filters in the frequency domain (频域)
  - Filtering is a way to modify the frequencies of images
  - Denoising, sampling, image compression
- Image pyramids (图像金字塔)
  - Scale-space representation allows coarse-to-fine operations

# Image filtering

- Image filtering:
  - Compute function of local neighborhood at each position

h=output

f=filter

I=image

$$h[m,n] = \sum_{k,l} f[k,l] I[m+k,n+l]$$

2d coords=k,l

2d coords=m,n

[ ]

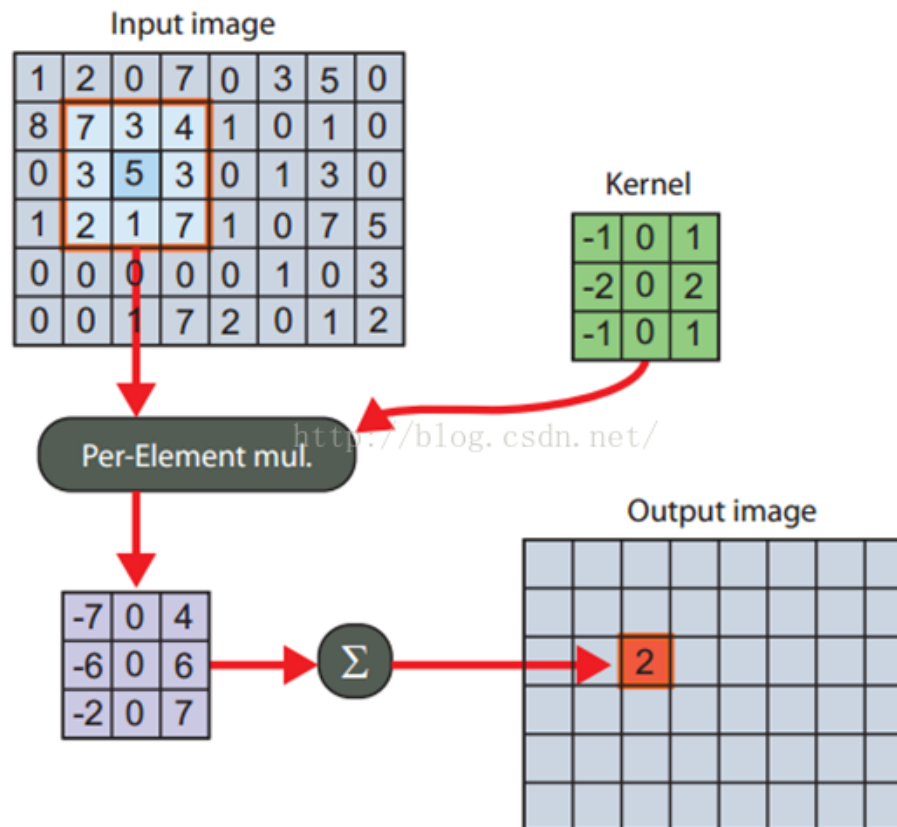
[ ]

[ ]

# Image filtering

- Image filtering:
  - Compute function of local neighborhood at each position

$$h[m,n] = \sum_{k,l} f[k,l] I[m+k, n+l]$$



# Example: box filter

$f[\cdot, \cdot]$

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

$h[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0


$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10							

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10	20						

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$



# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10	20	30					

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$



# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10	20	30	30				

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$I[\cdot, \cdot]$$
$$h[\cdot, \cdot]$$
[illegible][illegible]

$$h[m,n] = \sum_{k,l} f[k,l] I[m+k,n+l]$$

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9}$$

1	1	1
1	1	1
1	1	1

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10	20	30	30				
							?		
				50					

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

# Image filtering

$$f[\cdot, \cdot] \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$I[\cdot, \cdot]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$h[\cdot, \cdot]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

# Box Filter

What does it do?

- Replaces each pixel with an average of its neighborhood
- Achieve smoothing effect (remove sharp features)

$$\frac{1}{9} f[\cdot, \cdot]$$

1	1	1
1	1	1
1	1	1

# Box Filter

What does it do?

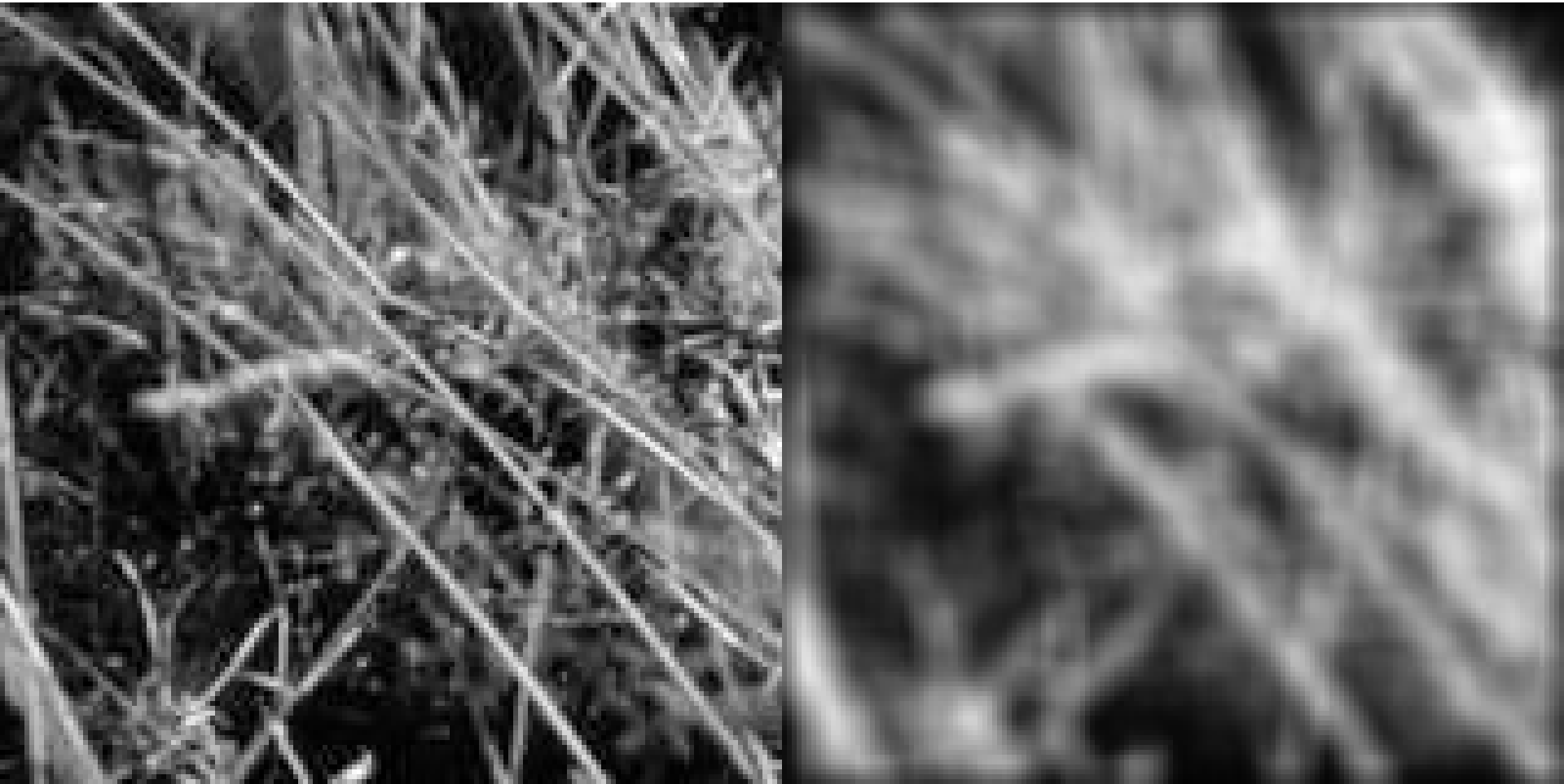
- Replaces each pixel with an average of its neighborhood
- Achieve smoothing effect (remove sharp features)
- Why does it sum to one?

$$\frac{1}{9} f[\cdot, \cdot]$$

1	1	1
1	1	1
1	1	1

# Smoothing with box filter

James Hays



# Image filtering

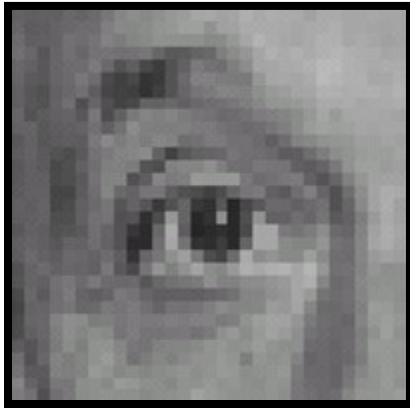
- Image filtering:
  - Compute function of local neighborhood at each position

$$h[m, n] = \sum_{k, l} f[k, l] I[m + k, n + l]$$

- Really important!
  - Enhance images
    - Denoise, resize, increase contrast, etc.
  - Extract information from images
    - Texture, edges, distinctive points, etc.
  - Detect patterns
    - Template matching



# Think-Pair-Share time



1.

0	0	0
0	1	0
0	0	0

2.

0	0	0
0	0	1
0	0	0

3.

1	0	-1
2	0	-2
1	0	-1

4.

0	0	0
0	2	0
0	0	0

—

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

# 1. Practice with linear filters



Original

0	0	0
0	1	0
0	0	0

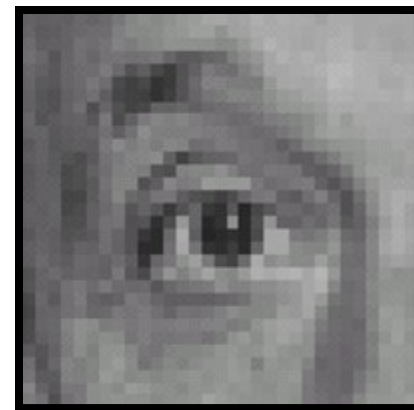
?

# 1. Practice with linear filters



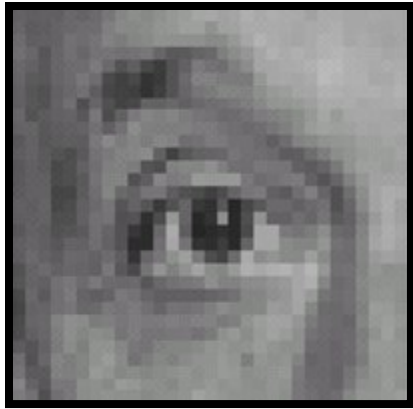
Original

0	0	0
0	1	0
0	0	0



Filtered  
(no change)

## 2. Practice with linear filters



Original

0	0	0
0	0	1
0	0	0

?

## 2. Practice with linear filters



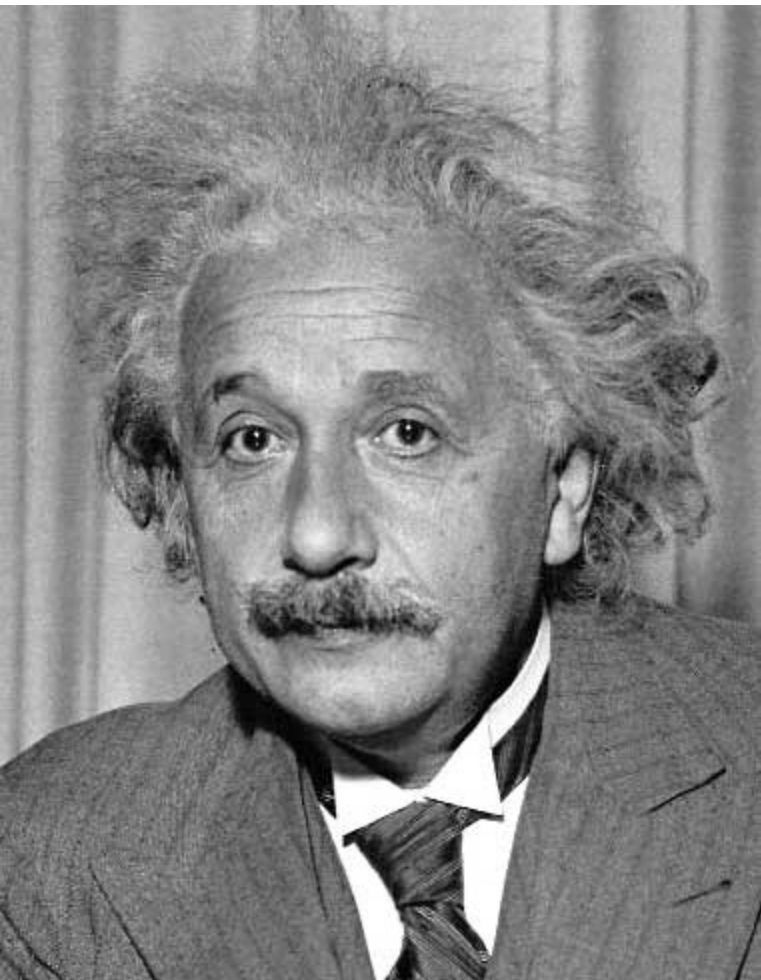
Original

0	0	0
0	0	1
0	0	0



Shifted left  
By 1 pixel

# 3. Practice with linear filters



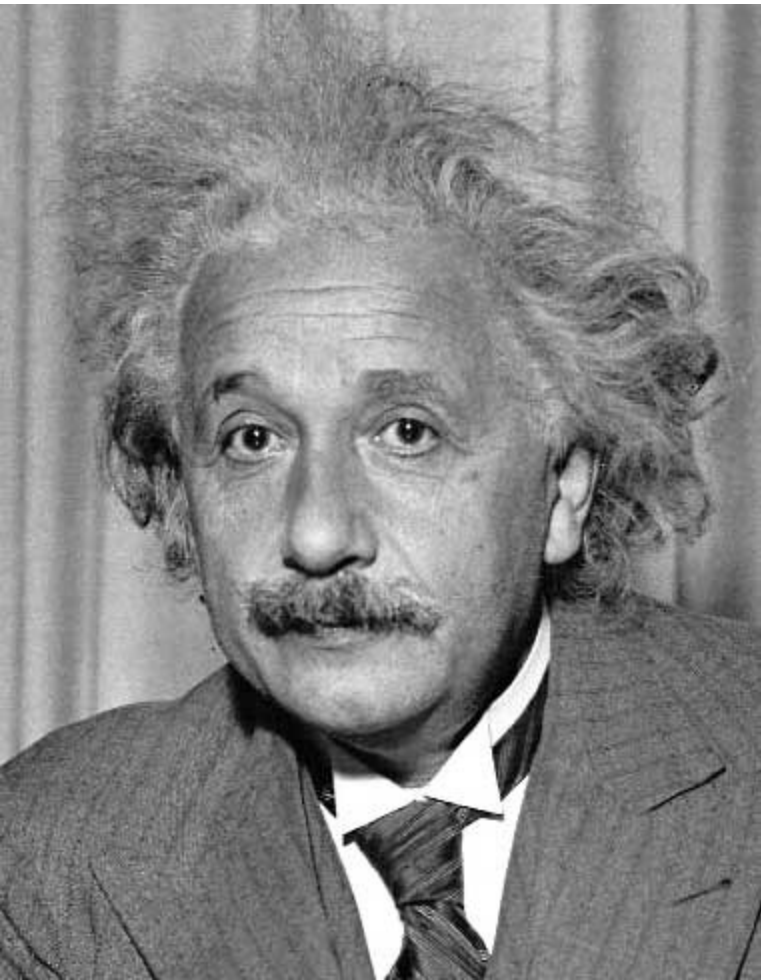
1	0	-1
2	0	-2
1	0	-1

Sobel



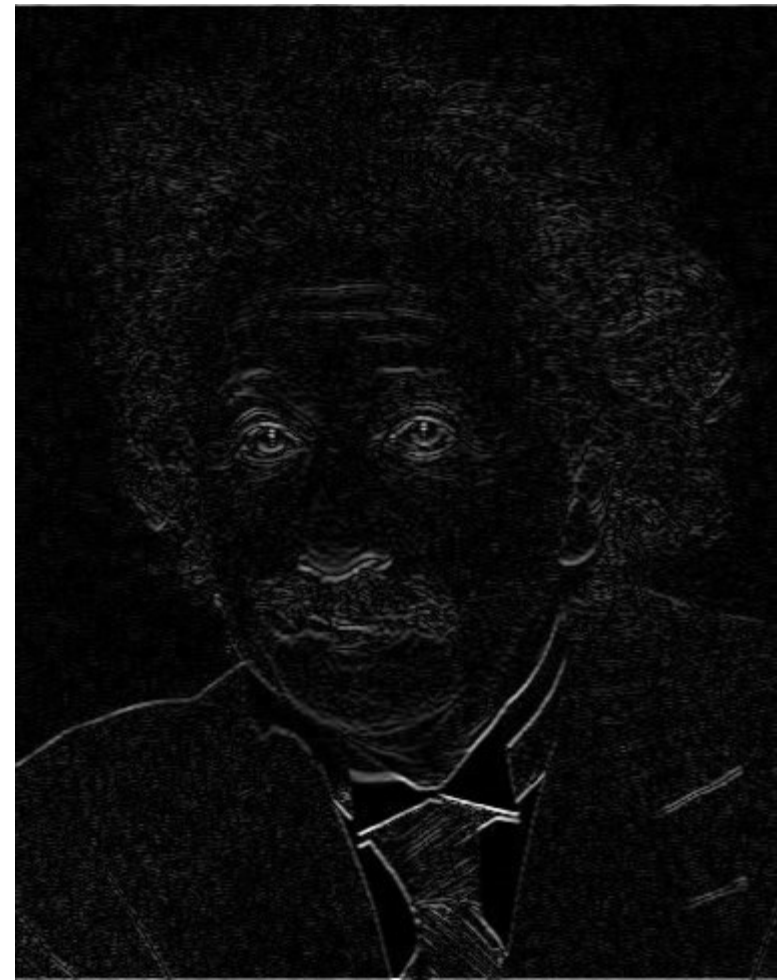
Vertical Edge  
(absolute value)

# 3. Practice with linear filters



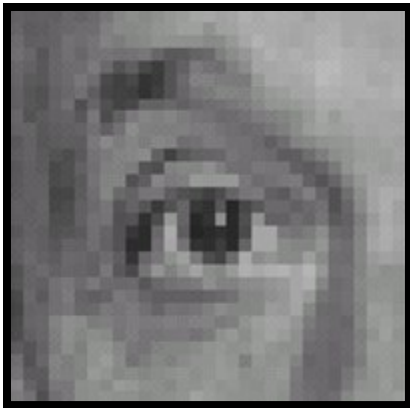
1	2	1
0	0	0
-1	-2	-1

Sobel



Horizontal Edge  
(absolute value)

# 4. Practice with linear filters



Original

0	0	0
0	2	0
0	0	0

—

$\frac{1}{9}$

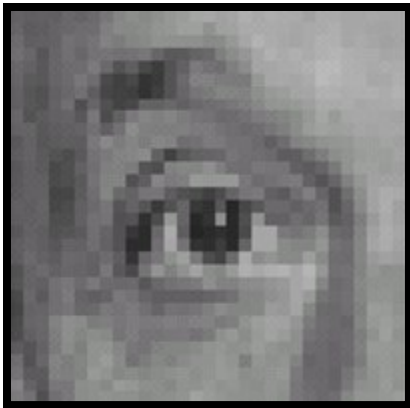
1	1	1
1	1	1
1	1	1

?

(Note that filter sums to 1)



# 4. Practice with linear filters



Original

0	0	0
0	2	0
0	0	0

—

$\frac{1}{9}$

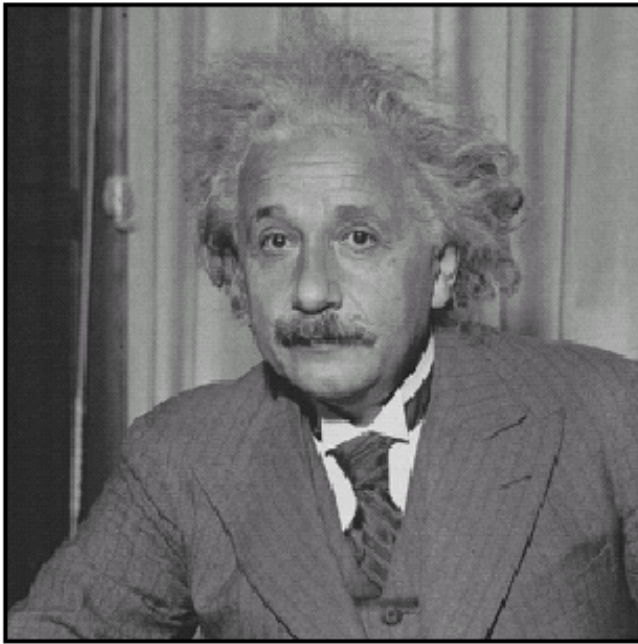
1	1	1
1	1	1
1	1	1



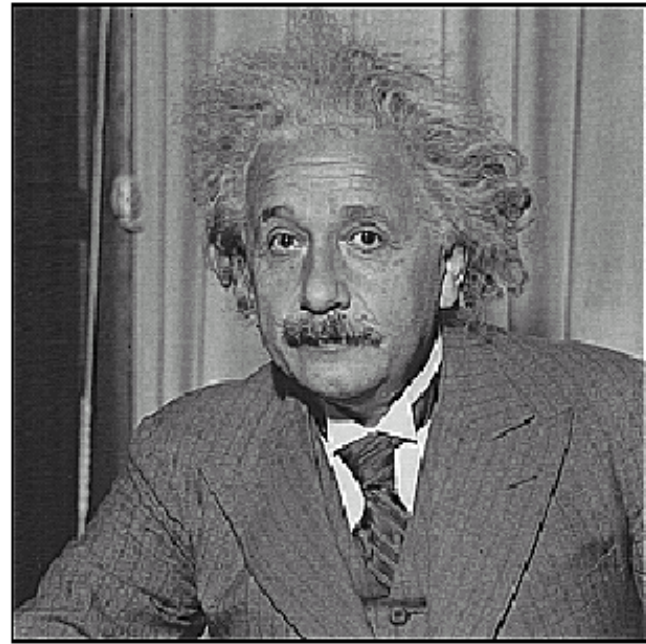
## Sharpening filter

- Accentuates differences with local average

## 4. Practice with linear filters



before



after

# Correlation and Convolution

- 2d correlation

$$h[m,n] = \sum_{k,l} f[k,l] I[m+k,n+l]$$

`h=filter2(f,I);` or `h=imfilter(I,f);`

# Correlation and Convolution

- 2d correlation

$$h[m,n] = \sum_{k,l} f[k,l] I[m+k,n+l]$$

`h=filter2(f,I);` or `h=imfilter(I,f);`

- 2d convolution

$$h[m,n] = \sum_{k,l} f[k,l] I[m-k,n-l]$$

`h=conv2(f,I);` or `h=imfilter(I,f,'conv');`

`conv2(I,f)` is the same as `filter2(rot90(f,2),I)`

Correlation and convolution are identical when the filter is symmetric.

# Key properties of linear filters

## Linearity:

$$\text{imfilter}(I, f_1 + f_2) = \text{imfilter}(I, f_1) + \text{imfilter}(I, f_2)$$

## Shift invariance:

Same behavior regardless of pixel location

$$\text{imfilter}(I, \text{shift}(f)) = \text{shift}(\text{imfilter}(I, f))$$

Any linear, shift-invariant operator can be represented as a convolution.

# Convolution properties

Commutative:  $a * b = b * a$

- Conceptually no difference between filter and signal
- But particular filtering implementations might break this equality, e.g., image edges

Associative:  $a * (b * c) = (a * b) * c$

- Often apply several filters one after another:  $((a * b_1) * b_2) * b_3$
- This is equivalent to applying one filter:  $a * (b_1 * b_2 * b_3)$

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- Correlation is not associative (rotation effect)

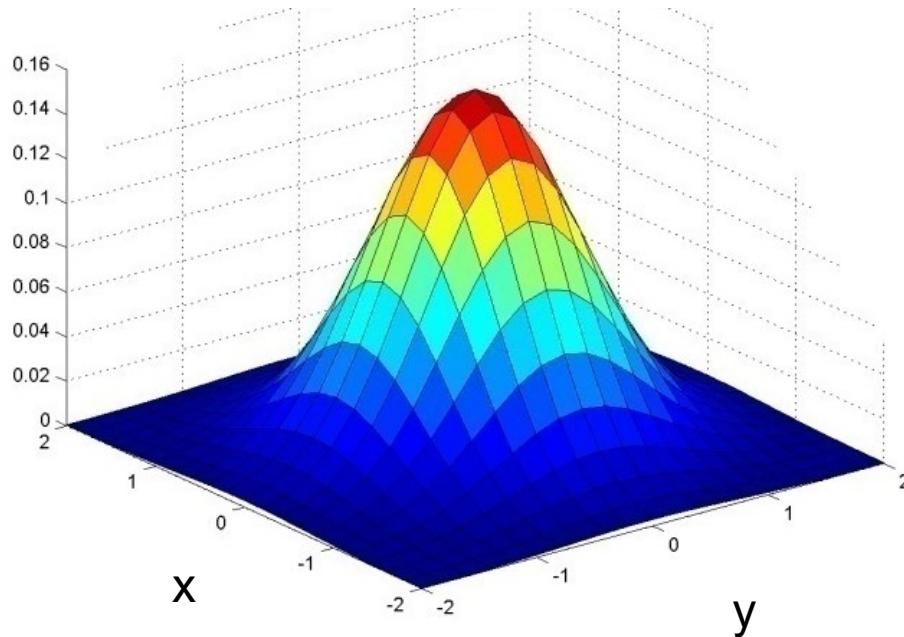
# Convolution properties

- Commutative:  $a * b = b * a$ 
  - Conceptually no difference between filter and signal
  - But particular filtering implementations might break this equality, e.g., image edges
- Associative:  $a * (b * c) = (a * b) * c$ 
  - Often apply several filters one after another:  $((a * b_1) * b_2) * b_3$
  - This is equivalent to applying one filter:  $a * (b_1 * b_2 * b_3)$
  - Correlation is not associative (rotation effect)
- Distributes over addition:  $a * (b + c) = (a * b) + (a * c)$
- Scalars factor out:  $ka * b = a * kb = k(a * b)$
- Identity: unit impulse  $e = [0, 0, 1, 0, 0]$ ,  $a * e = a$



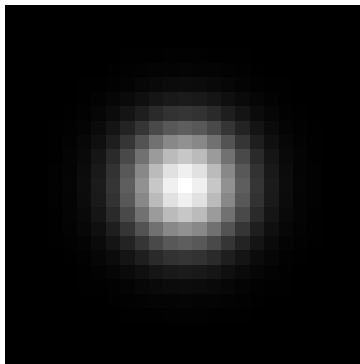
# Important filter: Gaussian

- Weight contributions of neighboring pixels by nearness



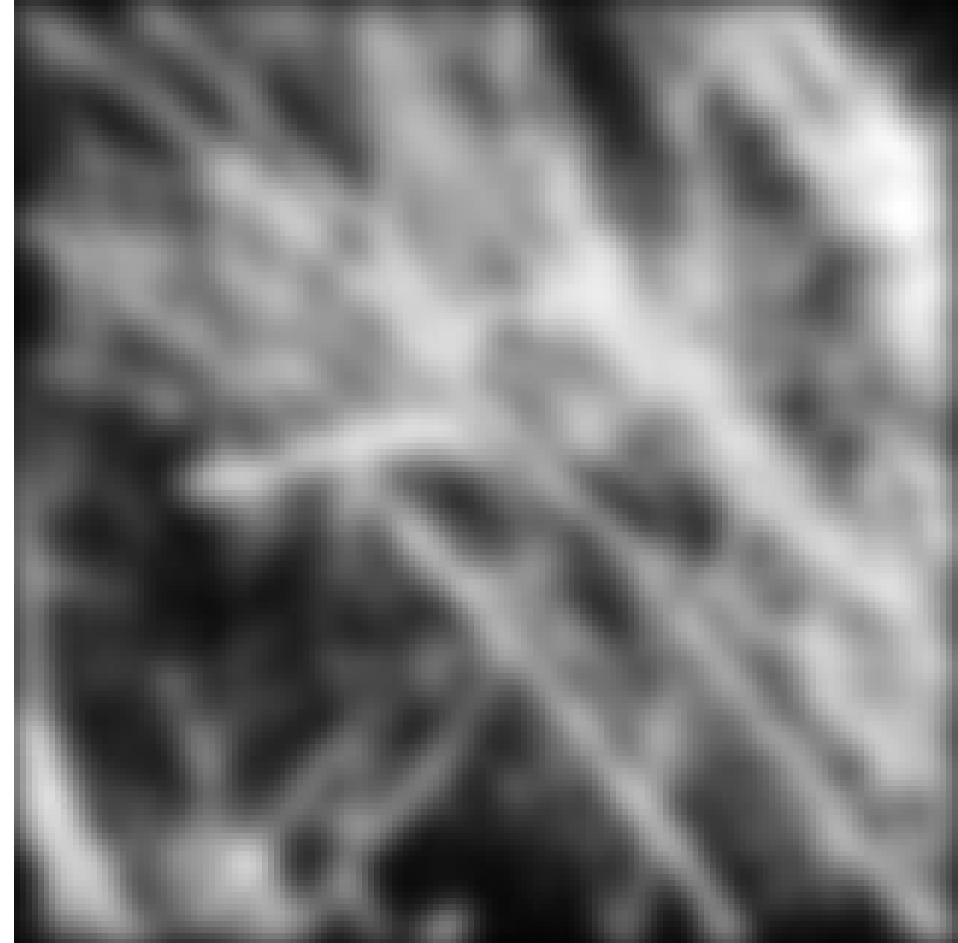
x				
0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

5 x 5,  $\sigma = 1$

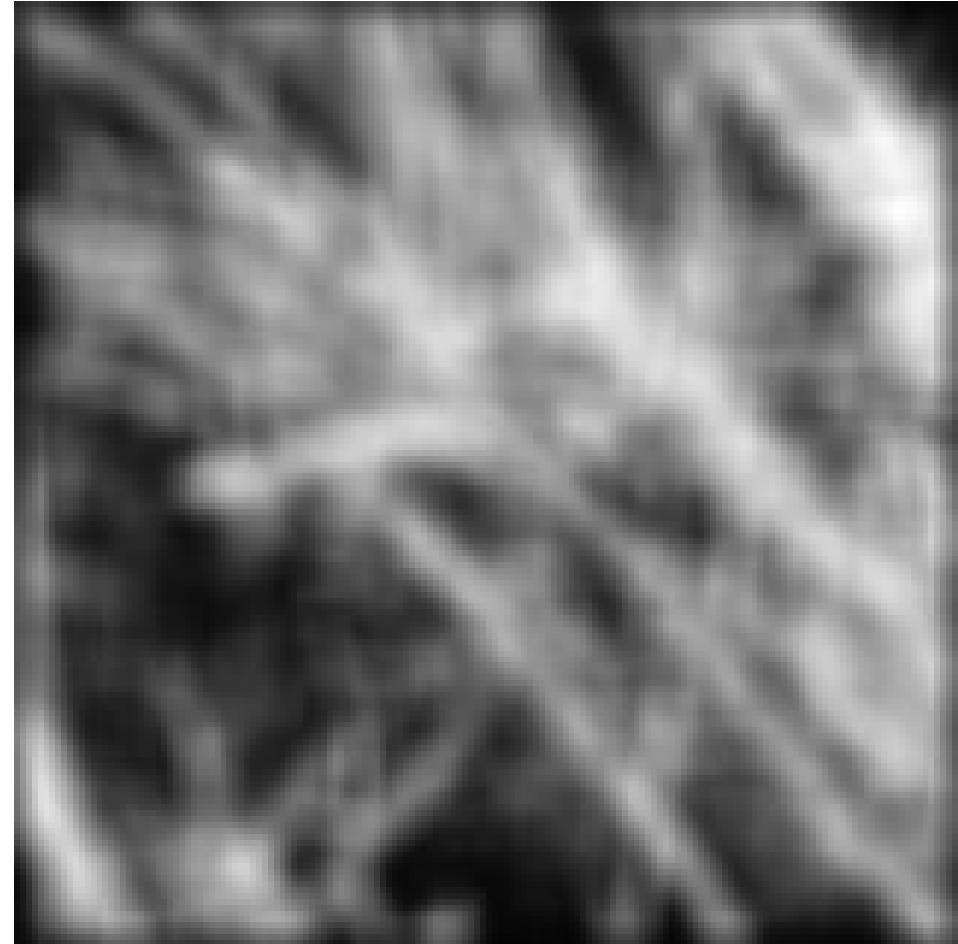


$$G_{\sigma} = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

# Smoothing with Gaussian filter



# Smoothing with box filter



# Gaussian filters

- Remove “high-frequency” components from the image (low-pass filter)
  - Images become more smooth
- Gaussian convolved with Gaussian...
  - ...is another Gaussian
    - So can smooth with small-width kernel, repeat, and get same result as larger-width kernel would have
- *Separable* kernel
  - Factors into product of two 1D Gaussians

# Separability of the Gaussian filter

$$\begin{aligned} G_{\sigma}(x, y) &= \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2 + y^2}{2\sigma^2}} \\ &= \left( \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{x^2}{2\sigma^2}} \right) \left( \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{y^2}{2\sigma^2}} \right) \end{aligned}$$

The 2D Gaussian can be expressed as the product of two functions, one a function of  $x$  and the other a function of  $y$

In this case, the two functions are the (identical) 1D Gaussian



# Separability example

2D convolution  
(center location only)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix}$$

The filter factors  
into a product of 1D  
filters:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Perform convolution  
along rows:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix} = \begin{bmatrix} & 11 & \\ & 18 & \\ & 18 & \end{bmatrix}$$

Followed by convolution  
along the remaining column:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} * \begin{bmatrix} & 11 & \\ & 18 & \\ & 18 & \end{bmatrix} = \begin{bmatrix} & & \\ & 65 & \\ & & \end{bmatrix}$$

# Separability

Why is separability useful in practice?

# Separability

Why is separability useful in practice?

MxN image, PxQ filter

- 2D convolution:  $\sim MN PQ$  multiply-adds
- Separable 2D:  $\sim MN(P+Q)$  multiply-adds

Speed up =  $PQ/(P+Q)$

9x9 filter =  $\sim 4.5x$  faster



# Summary of Typical Image Filter

- Box Filter (Averaging)


$$\begin{array}{|c|c|c|} \hline 0 & 0.2 & 0 \\ \hline 0.2 & 0 & 0.2 \\ \hline 0 & 0.2 & 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$


- Sharpness Filter



$$\begin{array}{|c|c|c|} \hline -1 & -1 & -1 \\ \hline -1 & 9 & -1 \\ \hline -1 & -1 & -1 \\ \hline \end{array}$$



$$\begin{array}{|c|c|c|c|c|} \hline -1 & -1 & -1 & -1 & -1 \\ \hline -1 & 2 & 2 & 2 & -1 \\ \hline -1 & 2 & 8 & 2 & -1 \\ \hline -1 & 2 & 2 & 2 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 \\ \hline \end{array}$$





- Sharpness Filter



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}; \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}; \begin{bmatrix} -k & -k & -k \\ -k & 8k+1 & -k \\ -k & -k & -k \end{bmatrix}$$

– The summation of all elements in the filter is 1

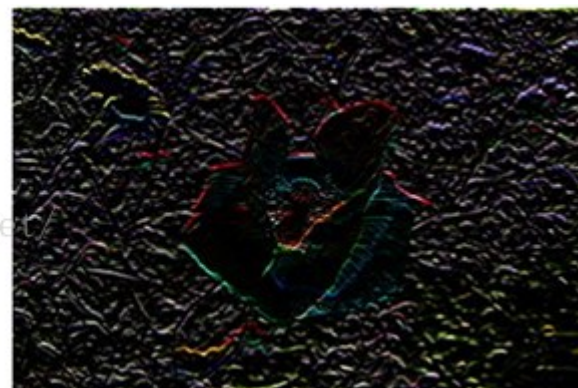
- Edge detection Filter



\*

0	0	0	0	0
0	0	0	0	0
-1	-1	2	0	0
0	0	0	0	0
0	0	0	0	0

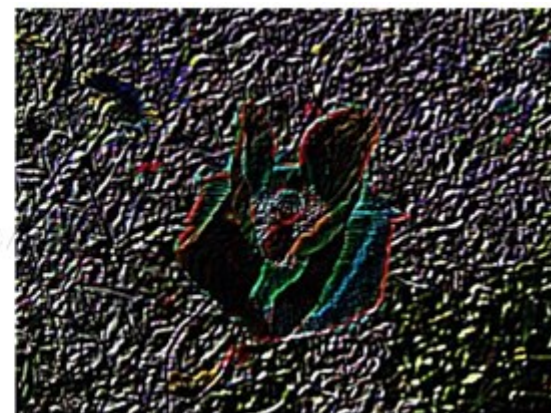
=



\*

-1	0	0	0	0
0	-2	0	0	0
0	0	6	0	0
0	0	0	-2	0
0	0	0	0	-1

=

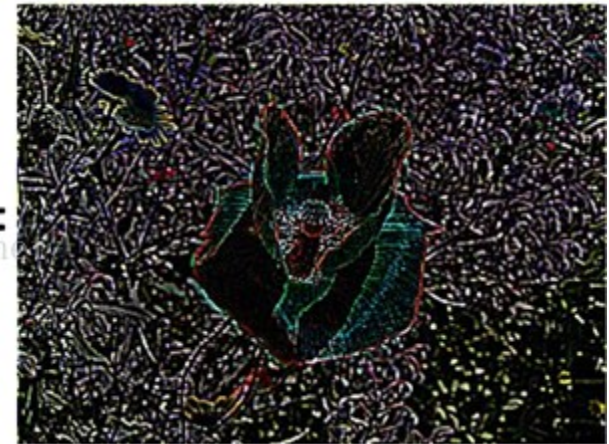




- Edge detection Filter



$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$



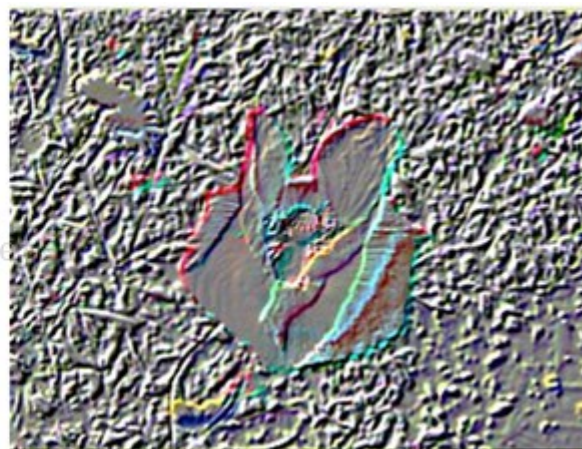
$$\begin{bmatrix} -1/8 & -1/8 & -1/8 \\ -1/8 & 1 & -1/8 \\ -1/8 & -1/8 & -1/8 \end{bmatrix} ; \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} ;$$

– The summation of all elements in the filter is 0

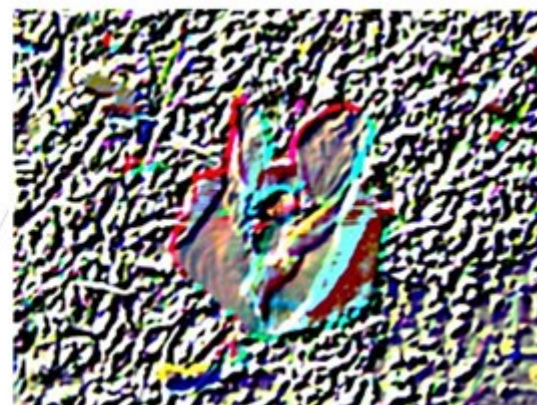
- Embossing Filter



$$\begin{array}{|c|c|c|} \hline -1 & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}$$



$$\begin{array}{|c|c|c|c|c|} \hline -1 & -1 & -1 & -1 & 0 \\ \hline -1 & -1 & -1 & 0 & 1 \\ \hline -1 & -1 & 0 & 1 & 1 \\ \hline -1 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

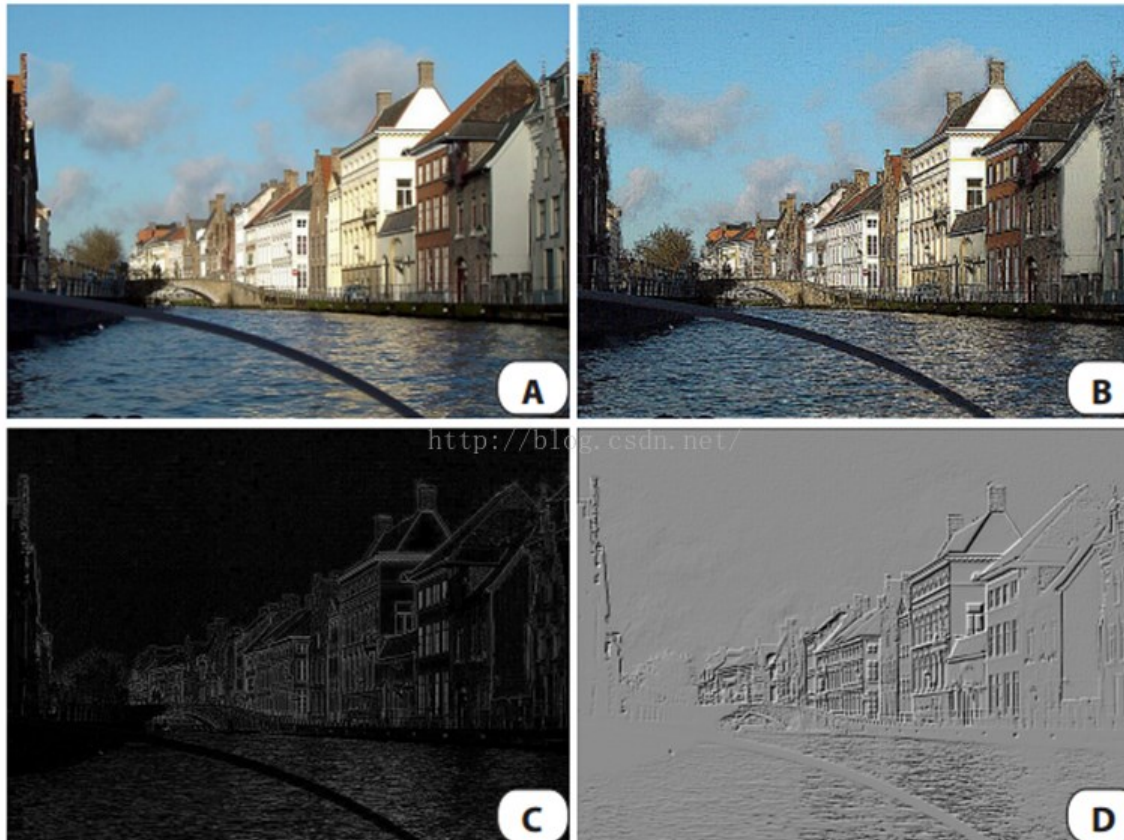




- Embossing Filter

$$\begin{bmatrix} 2 & -0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

<http://blog.csdn.net/>



– The filter is unsymmetrical (directional)

- Motion Blur Filter

```
1, 0, 0, 0, 0, 0, 0, 0, 0
0, 1, 0, 0, 0, 0, 0, 0, 0
0, 0, 1, 0, 0, 0, 0, 0, 0
0, 0, 0, 1, 0, 0, 0, 0, 0
0, 0, 0, 0, 1, 0, 0, 0, 0
0, 0, 0, 0, 0, 1, 0, 0, 0
0, 0, 0, 0, 0, 0, 1, 0, 0
0, 0, 0, 0, 0, 0, 0, 1, 0
0, 0, 0, 0, 0, 0, 0, 0, 1
```



– Directional



- Gaussian Blur Filter

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{x^2}{2\sigma^2}}; G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

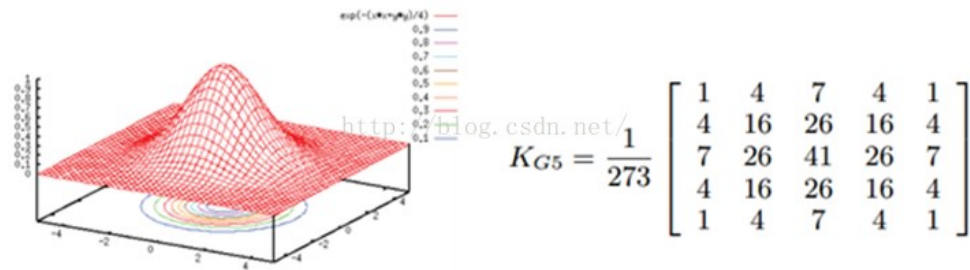


Figure 4: The 2D Gaussian function.



# Take home message for Image Filter

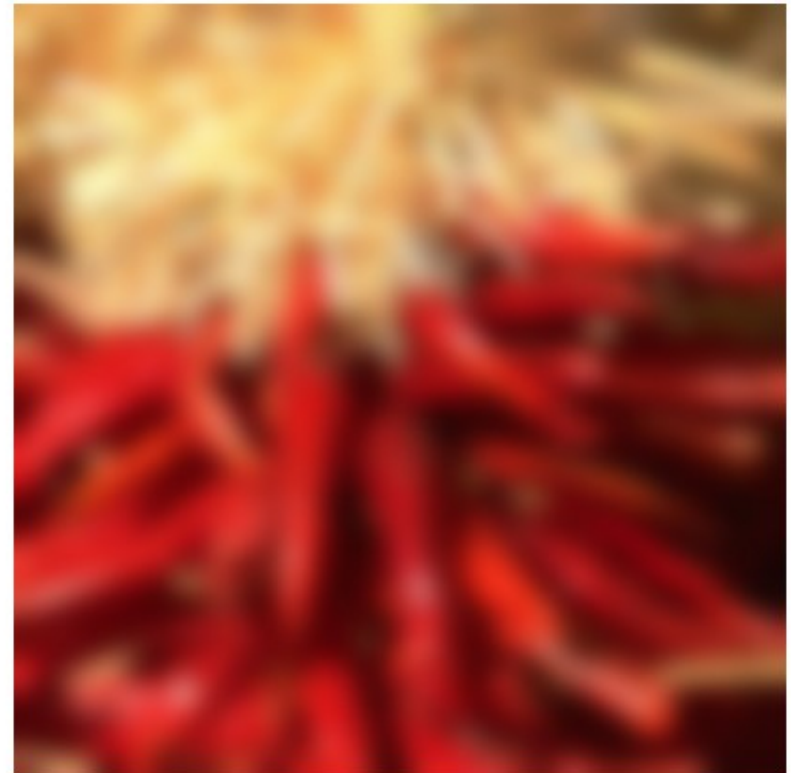
- 滤波器的大小应该是奇数，这样它才有一个中心，例如3x3，5x5或者7x7。有中心了，也有了半径的称呼，例如5x5大小的核的半径就是2。
- 滤波器矩阵所有的元素之和应该要等于1，这是为了保证滤波前后图像的亮度保持不变。当然了，这不是硬性要求。
- 如果滤波器矩阵所有元素之和大于1，那么滤波后的图像就会比原图像更亮，反之，如果小于1，那么得到的图像就会变暗。如果和为0，图像不会变黑，但也会非常暗。
- 对于滤波后的结构，可能会出现负数或者大于255的数值。对这种情况，我们将他们直接截断到0和255之间即可。对于负数，也可以取绝对值。

# Take home message for Image Filter

- Correlation（协相关）和 Convolution（卷积）是图像处理最基本的操作，但却非常有用。这两个操作有两个非常关键的特点：它们是线性的，而且具有平移不变性。
- 积和协相关的差别是，卷积需要先对滤波矩阵进行180的翻转，但如果矩阵是对称的，那么两者就没有什么差别。
- 平移不变性指我们在图像的每个位置都执行相同的操作。线性指这个操作是线性的，也就是我们用每个像素的邻域的线性组合来代替这个像素。
- 2D卷积需要4个嵌套循环4-double loop，所以它并不快。如果卷积核可分离为1D，则可加快卷积计算速度。

# Practical matters

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - reflect across edge



# Convolution in Convolutional Neural Networks

- Convolution is the basic operation in CNNs
- Learning convolution kernels allows us to learn which `features' provide useful information in images.

# Next class: Thinking in Frequency

