2 广义逆矩阵 A^{-}

定义1 设 $A\in C^{m\times n}$,如果存在矩阵$G\in C^{n\times m}$,使得

$$AGb=b \quad (\forall\, b\in R(A))$$

则称G为A的广义逆矩阵,记为$G=A^{-}$.

定理1 设 $A\in C^{m\times n}$,则A存在广义逆矩阵的充要条件是存在$G\in C^{n\times m}$,使其满足

$$AGA=A$$

proof 必要性：$\forall u \in C^n \Rightarrow b = Au \in R(A)$

$$\xrightarrow{AGb=b} AGAu = AGb = b = Au \Rightarrow$$

$$AGA = A$$

充分性：$Ax = b \Rightarrow AGb = AGAx$

$= Ax = b \Rightarrow$ G为A的一个广义逆矩阵

推论1 设 $A\in C^{m\times n}$, 且$A^{-}\in C^{n\times m}$是A的一个广义逆矩阵, 则

$$rank(A^{-})\geq rank(A)$$

proof $rank(A)=rank(AA^{-}A)\leq rank(AA^{-})\leq rank(A^{-})$

定义 $A\{1\}=\{G\mid AGA=A,\quad \forall\, G\in C^{n\times m}\}$

定理2 设 $A\in C^{m\times n}$, 且A^{-}是A的任一广义逆矩阵, 则有

$$A\{1\}=\{G\mid G=A^{-}+U-A^{-}AUAA^{-},\ \forall\, U\in C^{n\times m}\}$$

$$=\{G\mid G=A^{-}+(E_n-A^{-}A)V+W(E_m-AA^{-}),\ \forall\ V,W\in C^{n\times m}\}$$

分析:只需证明这三个集合依次相互包含

proof (1) $\forall\ G \in A\{1\} \Rightarrow AGA = A \Rightarrow$

$$G = A^{-} + G - A^{-} - A^{-}A(G - A^{-})AA^{-} \xrightarrow{U = G - A^{-}}$$

$$G = A^{-} + U - A^{-}AUAA^{-} \Rightarrow$$

$$A\{1\} \subset \{G \mid G = A^{-} + U - A^{-}AUAA^{-},\ \forall\ U \in C^{n\times m}\}$$

(2) $\forall\ U \in C^{n\times m} \Rightarrow G = A^{-} + U - A^{-}AUAA^{-}$

$$\Rightarrow G = A^{-} + U - UAA^{-} + \underline{UAA^{-} - A^{-}AUAA^{-}}$$

$$\Rightarrow \quad G = A^{-} + (E_n - A^{-}A)UAA^{-} + U(E_m - AA^{-})$$

$$W = U, \quad V = UAA^{-}$$

$$\xrightarrow{} \quad G = A^{-} + (E_n - A^{-}A)V - W(E_m - AA^{-})$$

$$\Downarrow$$

$$\{G \mid G = A^{-} + U - A^{-}UAA^{-}, \forall\, U \in C^{n\times m}\} \subset$$

$$\{G \mid G = A^{-} + (E_n - A^{-}A)V + W(E_m - AA^{-}),$$

$$\forall\, V, W \in C^{n\times m}\}$$

$$(3)\forall M \in \{G \mid G = A^{-} + (E_n - A^{-}A)V + W(E_m - AA^{-}),\ \forall V, W \in C^{n\times m}\}$$

$$\xrightarrow{\forall\ V,W \in C^{n\times m}} M = A^{-} + (E_n - A^{-}A)V + W(E_m - AA^{-})$$

$$\Downarrow$$

$$AMA = A[A^{-} + (E_n - A^{-}A)V + W(E_m - AA^{-})]A$$

$$\Downarrow$$

$$AMA = AA^{-}A + (A - AA^{-}A)VA + AW(A - AA^{-}A)$$

$$\Downarrow$$

$$AMA = A + (A - A)VA + AW(A - A) = A$$

$$\{G \mid G = A^{-} + (E_n - A^{-}A)V + W(E_m - AA^{-}),$$

$$\forall V, W \in C^{n\times m}\} \subset A\{1\}$$

定理3 设 $A \in C^{m\times n}, \lambda \in C$, 则

(i) $(A^T)^{-} = (A^{-})^T$, $(A^H)^{-} = (A^{-})^H$

(ii) AA^{-}与$A^{-}A$都是幂等矩阵且

$$rank(A) = rank(AA^{-}) = rank(A^{-}A)$$

(*iii*) $\lambda^{-}A^{-}$为λA的广义逆矩阵其中$\lambda^{-}=\begin{cases}0 & \lambda=0\\ \lambda^{-1} & \lambda\neq 0\end{cases}$

(*iv*) 设S是m阶可逆矩阵，T是n阶可逆矩阵，且

$B=SAT$，则$T^{-1}A^{-}S^{-1}$是B的广义逆矩阵

(*v*) $R(AA^{-})=R(A),\quad N(A^{-}A)=N(A);$

proof (*i*) $AA^{-}A=A \rightarrow A^{T}=A^{T}(A^{-})^{T}A^{T} \rightarrow$

$(A^{-})^{T}=(A^{T})^{-}$ 同理可证 $(A^{-})^{H}=(A^{H})^{-}$

其他由广义逆的充分要条件AGA = A，容易验证。

推论2 设$A \in C^{m \times n}$，则

(1) $rank(A) = n$的充要条件是$A^{-}A = E_n$;

(2) $rank(A) = m$的充要条件是$AA^{-} = E_m$.

proof 充分性:由定理3的（2）可知

$$rank(A) = rank(A^{-}A) = n.$$得证.

必要性： $rank(A^{-}A)=rank(A)=n$ $\Rightarrow$

$A^{-}A$是n阶可逆矩阵 $\Rightarrow$ $E_n=(A^{-}A)(A^{-}A)^{-1}$

$=A^{-}(AA^{-}A)(A^{-}A)^{-1}=A^{-}A(A^{-}A)(A^{-}A)^{-1}$

$=A^{-}A$

引理1 设 $A\in C^{m\times n},P\in C^{m\times m},Q\in C^{n\times n}$都是可逆矩阵,则

$$Q(PAQ)^{-}P\in A\{1\}$$

证： $PAQ(PAQ)^{-}PAQ=PAQ$ $\Rightarrow$

$$A[Q(PAQ)^{-}P]A = A \longrightarrow Q(PAQ)^{-}P \in A\{1\}$$

引理 2 设 $A = \begin{pmatrix} A_{11} & \\ & A_{22} \end{pmatrix}$，则存在$X_{12}, X_{21}$满足

$A_{11}X_{12}A_{22} = 0,\ A_{22}X_{21}A_{11} = 0$, 使得

$$\begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix} \in A\{1\}$$

证： $A\begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix}A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$

$$=\begin{pmatrix} A_{11}A_{11}^{-} & A_{11}X_{12} \\ A_{22}X_{21} & A_{22}A_{22}^{-} \end{pmatrix}\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

$$=\begin{pmatrix} A_{11}A_{11}^{-}A_{11} & A_{11}X_{12}A_{22} \\ A_{22}X_{21}A_{11} & A_{22}A_{22}^{-}A_{22} \end{pmatrix}$$

$$=\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

$$=A$$

定理4 设 $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$，则

(*i*) 如果A_{11}^{-1}存在,则存在X_{12}, X_{21}满足

$$X_{12}(A_{22}-A_{21}A_{11}^{-1}A_{12})=0, (A_{22}-A_{21}A_{11}^{-1}A_{12})X_{21}=0,$$

使得

$$\begin{bmatrix} E_r & -A_{11}^{-1} \\ 0 & E_{n-r} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & X_{12} \\ X_{21} & (A_{22}-A_{21}A_{11}^{-1}A_{12})^{-} \end{bmatrix}\begin{bmatrix} E_r & 0 \\ -A_{21}A_{11}^{-1} & E_{m-r} \end{bmatrix}$$

$$\in A\{1\}$$

(ii) 如果A_{22}^{-1}存在,则存在Y_{12}, Y_{21}满足

$$Y_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21}) = 0, (A_{11} - A_{12}A_{22}^{-1}A_{21})Y_{12} = 0,$$

使得

$$\begin{bmatrix} E_r & 0 \\ -A_{22}^{-1}A_{21} & E_{n-r} \end{bmatrix} \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-} & Y_{12} \\ Y_{21} & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} E_r & -A_{11}^{-1}A_{12} \\ 0 & E_{m-r} \end{bmatrix}$$

$$\in A\{1\}$$

proof

$$\begin{bmatrix} E_r & 0 \\ -A_{21}A_{11}^{-1} & E_{m-r} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} E_r & -A_{11}^{-1} \\ 0 & E_{n-r} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$