

2 广义逆矩阵 A^-

定义1 设 $A \in C^{m \times n}$, 如果存在矩阵 $G \in C^{n \times m}$, 使得

$$AGb = b \quad (\forall b \in R(A))$$

则称 G 为 A 的广义逆矩阵记为 $G = A^-$.

定理1 设 $A \in C^{m \times n}$, 则 A 存在广义逆矩阵的充要条件是存在 $G \in C^{n \times m}$, 使其满足

$$AGA = A$$

proof 必要性: $\forall u \in C^n \longrightarrow b = Au \in R(A)$

$$\underline{AGb = b} \longrightarrow AGAu = AGb = b = Au \longrightarrow$$

$$AGA = A$$

充分性: $Ax = b \longrightarrow AGb = AGAx$

$= Ax = b \longrightarrow G$ 为 A 的一个广义逆矩阵

推论 1 设 $A \in C^{m \times n}$, 且 $A^- \in C^{n \times m}$ 是 A 的一个广义逆矩阵, 则

$$\text{rank}(A^-) \geq \text{rank}(A)$$

proof $\text{rank}(A) = \text{rank}(AA^-A) \leq \text{rank}(AA^-) \leq \text{rank}(A^-)$

定义 $A\{1\} = \{G \mid AGA = A, \quad \forall G \in C^{n \times m}\}$

定理 2 设 $A \in C^{m \times n}$, 且 A^- 是 A 的任一广义逆矩阵, 则有

$$A\{1\} = \{G \mid G = A^- + U - A^-AUA A^-, \quad \forall U \in C^{n \times m}\}$$

$$= \{G \mid G = A^- + (E_n - A^-A)V + W(E_m - AA^-), \quad \forall V, W \in C^{n \times m}\}$$

分析: 只需证明这三个集合依次相互包含

proof (1) $\forall G \in A\{1\} \rightarrow AGA = A \rightarrow$

$$G = A^- + G - A^- - A^-A(G - A^-)AA^- \quad \underline{U = G - A^-}$$

$$G = A^- + U - A^-AUAA^- \rightarrow$$

$$A\{1\} \subset \{G \mid G = A^- + U - A^-AUAA^-, \forall U \in C^{n \times m}\}$$

$$(2) \forall U \in C^{n \times m} \rightarrow G = A^- + U - A^-AUAA^-$$

$$\rightarrow G = A^- + U - UAA^- + \underline{UAA^-} - A^-AUAA^-$$

$$\longrightarrow G = A^- + (E_n - A^- A)UAA^- + U(E_m - AA^-)$$

$$\begin{array}{l} W = U, \\ \underline{V = UAA^-} \end{array} \quad G = A^- + (E_n - A^- A)V - W(E_m - AA^-)$$



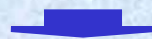
$$\{G \mid G = A^- + U - A^- UAA^-, \forall U \in C^{n \times m}\} \subsetneq$$

$$\{G \mid G = A^- + (E_n - A^- A)V + W(E_m - AA^-),$$

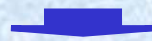
$$\forall V, W \in C^{n \times m}\}$$

$$(3) \forall M \in \{G \mid G = A^- + (E_n - A^- A)V + W(E_m - AA^-), \\ \forall V, W \in C^{n \times m}\}$$

$$\xrightarrow{\forall V, W \in C^{n \times m}} M = A^- + (E_n - A^- A)V + W(E_m - AA^-)$$



$$AMA = A[A^- + (E_n - A^- A)V + W(E_m - AA^-)]A$$



$$AMA = AA^-A + (A - AA^-A)VA + AW(A - AA^-A)$$



$$AMA = A + (A - A)VA + AW(A - A) = A$$



$$\{G \mid G = A^- + (E_n - A^-A)V + W(E_m - AA^-),$$

$$\forall V, W \in C^{n \times m} \} \subset A\{1\}$$

定理 3 设 $A \in C^{m \times n}$, $\lambda \in C$, 则

$$(i) (A^T)^- = (A^-)^T, (A^H)^- = (A^-)^H$$

(ii) AA^- 与 A^-A 都是幂等矩阵, 且

$$\text{rank}(A) = \text{rank}(AA^-) = \text{rank}(A^-A)$$

(iii) $\lambda^- A^-$ 为 λA 的广义逆矩阵, 其中 $\lambda^- = \begin{cases} 0 & \lambda = 0 \\ \lambda^{-1} & \lambda \neq 0 \end{cases}$

(iv) 设 S 是 m 阶可逆矩阵, T 是 n 阶可逆矩阵, 且

$B = SAT$, 则 $T^{-1} A^- S^{-1}$ 是 B 的广义逆矩阵

(v) $R(AA^-) = R(A)$, $N(A^-A) = N(A)$;

proof (i) $AA^-A = A \xrightarrow{\quad} A^T = A^T (A^-)^T A^T \xrightarrow{\quad}$

$(A^-)^T = (A^T)^- \quad \text{同理可证} \quad (A^-)^H = (A^H)^-$

其他由广义逆的充分要条件 $AGA = A$ ，容易验证。

推论 2 设 $A \in C^{m \times n}$ ，则

(1) $rank(A) = n$ 的充要条件是 $A^-A = E_n$;

(2) $rank(A) = m$ 的充要条件是 $AA^- = E_m$.

proof 充分性: 由定理3的 (2) 可知

$$rank(A) = rank(A^-A) = n. \text{ 得证}$$

必要性: $\text{rank}(A^-A) = \text{rank}(A) = n \rightarrow$

$$\begin{aligned} A^-A \text{ 是 } n \text{ 阶可逆矩阵} &\rightarrow E_n = (A^-A)(A^-A)^{-1} \\ &= A^- \underline{(AA^-A)} (A^-A)^{-1} = A^-A(A^-A)(A^-A)^{-1} \\ &= A^-A \end{aligned}$$

引理1 设 $A \in C^{m \times n}$, $P \in C^{m \times m}$, $Q \in C^{n \times n}$ 都是可逆矩阵, 则

$$Q(PAQ)^-P \in A\{1\}$$

证: $\underline{PAQ}(PAQ)^- \underline{PAQ} = \underline{PAQ} \rightarrow$

$$A[Q(PAQ)^{-}P]A = A \longrightarrow Q(PAQ)^{-}P \in A\{1\}$$

引理2 设 $A = \begin{pmatrix} A_{11} & \\ & A_{22} \end{pmatrix}$, 则存在 X_{12}, X_{21} 满足

$$A_{11}X_{12}A_{22} = 0, A_{22}X_{21}A_{11} = 0, \text{使得}$$

$$\begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix} \in A\{1\}$$

证: $A \begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix} A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-} & X_{12} \\ X_{21} & A_{22}^{-} \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$

$$= \begin{pmatrix} A_{11}A_{11}^{-} & A_{11}X_{12} \\ A_{22}X_{21} & A_{22}A_{22}^{-} \end{pmatrix} \begin{pmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}A_{11}^{-}A_{11} & \underline{A_{11}X_{12}A_{22}} \\ \underline{A_{22}X_{21}A_{11}} & A_{22}A_{22}^{-}A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{pmatrix}$$

$$= A$$

定理4 设 $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, 则

(i) 如果 A_{11}^{-1} 存在, 则存在 X_{12}, X_{21} 满足

$$X_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12}) = 0, (A_{22} - A_{21}A_{11}^{-1}A_{12})X_{21} = 0,$$

使得

$$\begin{bmatrix} E_r & -A_{11}^{-1} \\ 0 & E_{n-r} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & X_{12} \\ X_{21} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^- \end{bmatrix} \begin{bmatrix} E_r & 0 \\ -A_{21}A_{11}^{-1} & E_{m-r} \end{bmatrix}$$

$$\in A\{1\}$$

(ii) 如果 A_{22}^{-1} 存在, 则存在 Y_{12}, Y_{21} 满足

$$Y_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21}) = 0, (A_{11} - A_{12}A_{22}^{-1}A_{21})Y_{12} = 0,$$

使得

$$\begin{bmatrix} E_r & 0 \\ -A_{22}^{-1}A_{21} & E_{n-r} \end{bmatrix} \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-} & Y_{12} \\ Y_{21} & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} E_r & -A_{11}^{-1}A_{12} \\ 0 & E_{m-r} \end{bmatrix}$$

$$\in A\{1\}$$

proof

$$\begin{aligned}
 & \begin{bmatrix} E_r & \mathbf{0} \\ -A_{21}A_{11}^{-1} & E_{m-r} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} E_r & -A_{11}^{-1} \\ \mathbf{0} & E_{n-r} \end{bmatrix} \\
 &= \begin{bmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}
 \end{aligned}$$