10. Partial Differential Equations. Finite Difference Method.

NUMERICAL ANALYSIS. Prof. Y. Nishidate (323-B, nisidate@u-aizu.ac.jp)
http://web-int.u-aizu.ac.jp/~nisidate/na/

Partial Differential Equation Classification

Many physical problems lead to the second order partial differential equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

where A, B, C, D, E, F and G are given functions of x and y, which are continuous in some region R of the xy-plane.

The partial differential equations are classified in one of three types:
If $B^2 - 4AC < 0$ then the equation is *elliptic*.
If $B^2 - 4AC = 0$ then the equation is *parabolic*.
If $B^2 - 4AC > 0$ then the equation is *hyperbolic*.

Finite Difference Method

The *finite difference method* is based on the approximation of partial derivatives by finite differences. Typically finite difference equations are established on rectangular grid of points where the unknown function values are sought.

Finite difference formulas for first and second derivatives. By Taylor series,

$$u(x+h) = u(x) + hu'(x) + \tfrac{h^2}{2}u''(x) + \tfrac{h^3}{3!}u^{(3)}(x) + \dots$$
$$u(x-h) = u(x) - hu'(x) + \tfrac{h^2}{2}u''(x) - \tfrac{h^3}{3!}u^{(3)}(x) + \dots$$

The first derivative is approximated by the central difference as

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

The central difference approximation for the second derivative is

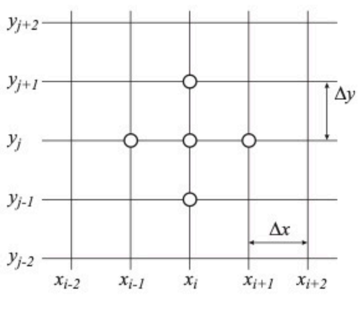
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

The first derivative could also be approximated by the forward or backward difference, but this approximation would have an error of $O(h)$. When u is a function of both x and y, we get the second partial derivative with respect to x, $\partial^2 u/\partial x^2$, by holding y constant and evaluating the function at three points x, $x+h$ and $x-h$. The partial derivative $\partial^2 u/\partial y^2$ is similarly computed, holding x constant.

Laplace equation. To solve the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on a region in the xy-plane, we subdivide the region with equally spaced lines parallel to the x- and y-axes.



Replacing the derivatives by difference approximations at the point (x_i, y_j), we get

$$\nabla^2 u(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{(\Delta y)^2} = 0$$

It is convenient to let double subscripts on u indicate the x- and y-values:

$$\nabla^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

It is common to take $\Delta x = \Delta y = h$, resulting in considerable simplification, so that

$$\nabla^2 u_{i,j} = \frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) = 0$$

It is convenient to represent the relationship pictorially, where the linear combinations of us is represented symbolically. The above equation for Laplace equation becomes:

$$\nabla^2 u_{i,j} = \frac{1}{h^2}\left\{ \begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} \right\} u_{i,j} = 0$$

The approximation has $O(h^2)$ error. This formula is called *five-point* or *five-point star* formula.

Laplace Equation on a Rectangular Region

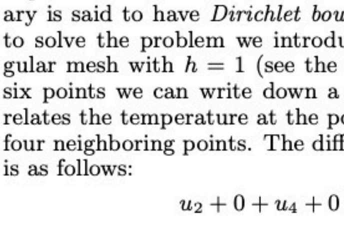
A typical steady-state heat flow problem is as follows: A thin metallic plate is 4 by 3 rectangle. One of short edges is held at 100°C and the other three edges are held at 0°C. What are steady-state temperatures at interior points?

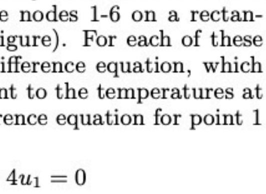
Assuming that heat flows only in x- and y-directions we can state the problem mathematically in the following way:

Find $u(x, y)$ such that

$$\nabla^2 u = 0$$

with $u(x,0) = u(x,3) = u(0,y) = 0$, $u(4,y) = 100$





A problem with function values known on the whole boundary is said to have *Dirichlet boundary conditions*. In order to solve the problem we introduce nodes 1-6 on a rectangular mesh with $h = 1$ (see the Figure). For each of these six points we can write down a difference equation, which relates the temperature at the point to the temperatures at four neighboring points. The difference equation for point 1 is as follows:

$$u_2 + 0 + u_4 + 0 - 4u_1 = 0$$

Two zeros on the left-hand side present since at two neighboring nodes the temperature is known and equal to 0. The difference equations for node 2 has the appearance:

$$u_3 + u_1 + u_5 + 0 - 4u_2 = 0$$

The equation for node 3 is:

$$100 + u_2 + u_6 + 0 - 4u_3 = 0$$

Combining equations for all inner nodes we arrive at the following linear equation system:

$$\begin{bmatrix} 4 & -1 & & -1 & & \\ -1 & 4 & -1 & & -1 & \\ & -1 & 4 & & & -1 \\ -1 & & & 4 & -1 & \\ & -1 & & -1 & 4 & -1 \\ & & -1 & & -1 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 100 \end{Bmatrix}$$

Symmetry of the system is apparent by inspection. This can be taken into account for memory and operation count reduction. Of course, with such a course grid as was used, we cannot expect reasonable accuracy. In practice, much smaller grid parameter h should be used; this leads to equation systems of thousands equations or more.

Liebman's method for Laplace equation Liebman's method is one of the simplest solution methods for difference equations, which does not require assembly of the equation system. According to this method the solution is obtained through the following steps:

1) Select initial approximation $u_{i,j}^{(0)}$ for interior nodes. Initial approximation can be calculated using linear interpolation of the boundary conditions.

2) Calculate successive approximations $u_{i,j}^{(k+1)}$ by

$$u_{i,j}^{(k+1)} = \frac{1}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)})$$

Here we suppose that nodes are numbered in order defined by growth of indices i and j. It can be shown that the process converges. The number of iterations necessary for an iterative method to converge is not known beforehand. Therefore, the iterative method is continued until a convergence test is satisfied. For example, iteration process is terminated if

$$\left|(u_{i,j}^{(k+1)} - u_{i,j}^{(k)})/u_{i,j}^{(k+1)}\right| < \varepsilon$$

is satisfied at all grid points , where ε is a prescribed error tolerance.

SOR method The chief drawback in Liebman's method is its slow convergence. The use of *successive overrelaxation* (SOR) can give significantly faster convergence. In the SOR method the increment of unknown function value is multiplied by an *overrelaxation factor* ω

$$u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + \frac{\omega}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - 4u_{i,j}^{(k)})$$

Maximum acceleration is obtained for some optimum value of ω. This optimum value will always lie between 1.0 and 2.0 for Laplace equation.

Poisson Equation

The methods of the previous sections are readily applied to Poisson's equation. We illustrate it with an analysis of torsion in a rectangular bar subject to twisting. The torsion function u satisfies the Poisson equation:

$$\nabla^2 u + 2 = 0, \qquad u = 0 \text{ on boundary}$$

The tangential stresses are proportional to the partial derivatives of u for a twisted prismatic bar of constant cross section.

If we subdivide cross section into a square grid with stepsize h then the difference equation for an inner point is

$$\frac{1}{h^2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}) + 2 = 0$$

Iteration formula for the Liebman's process looks like

$$u_{i,j}^{(k+1)} = \frac{1}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} + 2h^2)$$

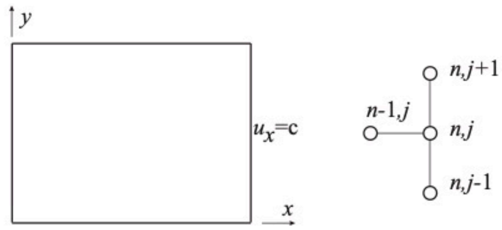
The SOR method can be applied to the Poisson difference equation using the relation

$$u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + \frac{\omega}{4}(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - 4u_{i,j}^{(k)} + 2h^2)$$

Derivative boundary conditions

The *Neumann boundary conditions* specify the directional derivative of $u(x, y)$ normal to an edge. For temperature problems this means that the heat flux through the edge is specified.

Let us illustrate implementation of a derivative boundary condition for a rectangular area. Suppose that the derivative boundary condition $\partial u/\partial x = u_x = c$ is prescribed at the right boundary where $x = x_n$.



The Laplace equation for the point (x_n, y_j) is

$$u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

The value $u_{n+1,j}$ does not exist because the point located outside the region. However, for this fictitious point we can write the numerical differentiation formula

$$u_x(x_n, y_j) = \frac{u_{n+1,j} - u_{n-1,j}}{2h} = c$$

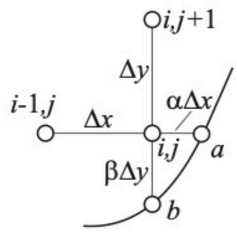
If we express $u_{n+1,j}$ from the above formula and put into Laplace difference equation we obtain the relation specifying derivative boundary condition on the right boundary:

$$2u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} + 2hc = 0$$

Relations for derivative boundary conditions at other edges of the region can be easily derived in a similar manner.

Irregular Regions

We assumed that the domains for partial differential equations are rectangular. In practice, however, the geometries often have curved boundaries. In order to deal with curved boundaries it is possible to use the rectangular grid with adjustment to the difference equations for the grid points near the boundary.



Let us introduce special grid points at intersections of regular grid lines and the curved boundary a and b. The Laplace difference equation at point (x_i, y_j) may be written as

$$\left(\frac{u_a - u_{i,j}}{\alpha\Delta x} - \frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right) \Big/ \left(\frac{\Delta x}{2}(1+\alpha)\right) + \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y} - \frac{u_{i,j} - u_b}{\beta\Delta y}\right) \Big/ \left(\frac{\Delta y}{2}(1+\beta)\right) = 0$$