DTU Compute

02425
December 4, 2024
UHT/uht

Exercise 12: Stability

Consider the stochastic Pella-Tomlinson model

$$dX_t = rX_t(1 - (X_t/K)^p)\ dt + \sigma X_t\ dB_t, \quad X_0 = x \quad .$$

where $r > 0$, $K > 0$, $\sigma > 0$, $p > 0$ and $x \geq 0$. This is a generalization of the stochastic logistic growth model, which describes growth of biological populations.

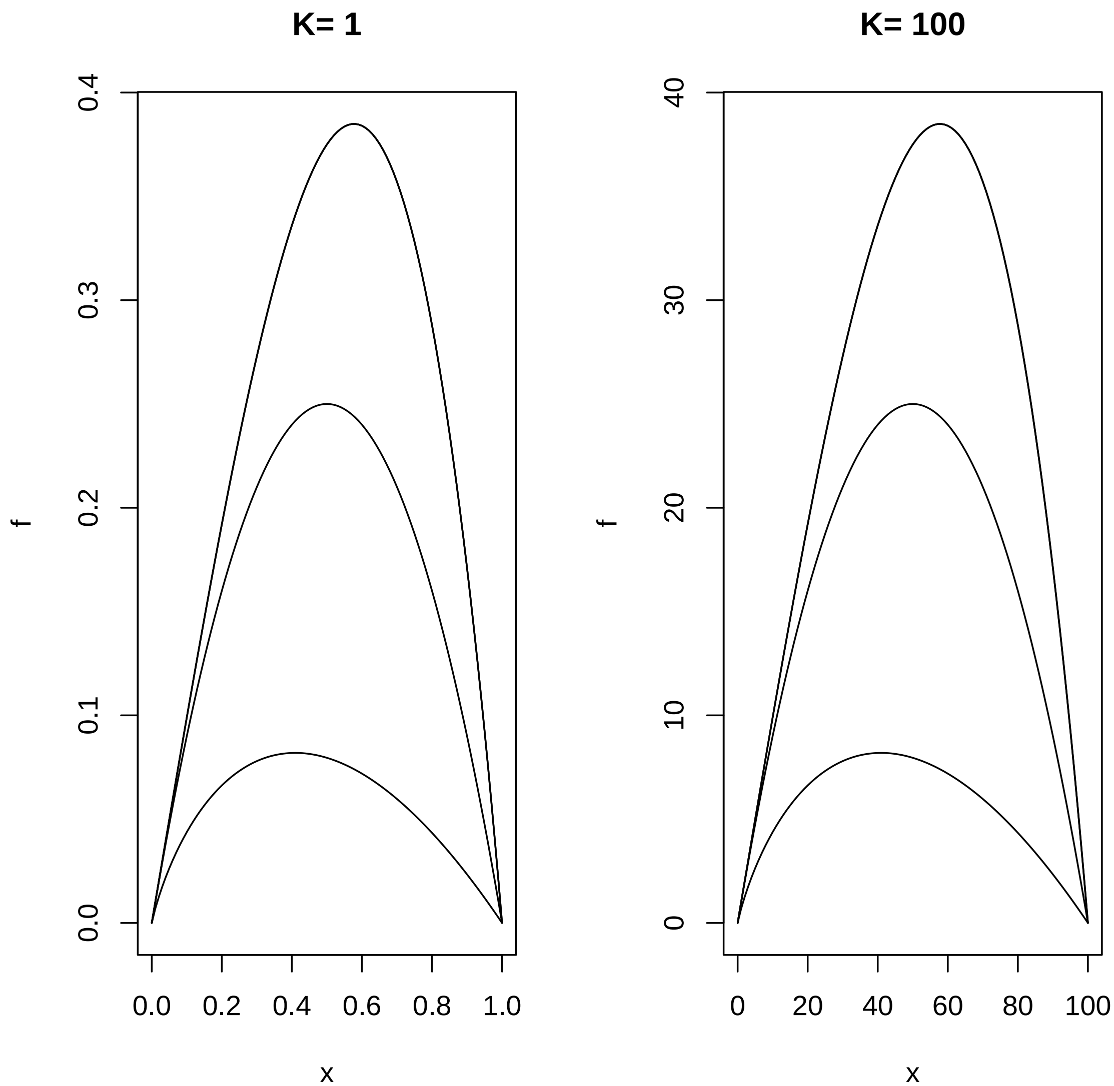
Question 1 **Warm-up:** Sketch the drift function. Convince yourself that p is a "shape" parameter in the drift, while r and K are "scale" parameters. Verify that the process $\{X_t\}$ with $X_t = 0$ satisfies the equation when $x = 0$.

Solution: r is clearly a scale parameter which scales time while K is a scale parameter which scales abundance. I.e., if we define $Y_s = X_{s/r}/K$, then $dt = ds/r$ and $W_s = \sqrt{r}\ B_{s/r}$ is standard Brownian motion, so the dimensionless version is:

$$dY_s = Y_s(1 - Y_s^p)\ ds + \sqrt{\sigma/r}Y_s\ dW_s$$

The following code shows the drift for different value of the scale parameter K and shape parameter p.

```
## Q1


f <- function(x) r*x*(1-(x/K)^p)
g <- function(x) sigma * x

par(mfrow=c(1,2))

## arbitrary parameters as numerical examples
r <- 1
sigma <- 1

for(K in c(1,100))
    {
        ps <- c(2,1,0.25)

        p <- ps[1]
        plot(f,from=0,to=K,main=paste("K=",K))
        for(p in ps)
            plot(f,from=0,to=K,add=TRUE)
    }
```



$X_t \equiv 0$ satisfies the equation since $f(0) = g(0) = 0$:

```
## Verify the zero solution
print(f(0))

## [1] 0

print(g(0))

## [1] 0
```

Question 2 **Sensitivity of the zero solution:** Pose the sensitivity equation of the zero solution and state its solution. State the Lyapunov exponent of the zero solution and state the condition on the parameters r, K, σ and p for the Lyapunov exponent to be negative.

Solution: The sensitivity S_t satisifes

$$dS_t = f'(X_t)S_t\ dt + g'(X_t)S_t\ dB_t, \quad S_0 = 1.$$

Here,

$$f'(x) = r(1 - (p+1)(x/K)^p), \quad g'(x) = \sigma$$

With the zero solution we have

$$f'(0) = r, \quad g'(0) = \sigma.$$

Thus, the sensitivity equation becomes

$$dS_t = rS_t\ dt + \sigma S_t\ dB_t$$

i.e., the same as for geometric Brownian motion. This reflects that the linearization of the system at the origin is geometric Brownian motion. We know from GBM that the Lyapunov exponent is

$$\lambda = r - \frac{1}{2}\sigma^2$$

so that the origin is stable when

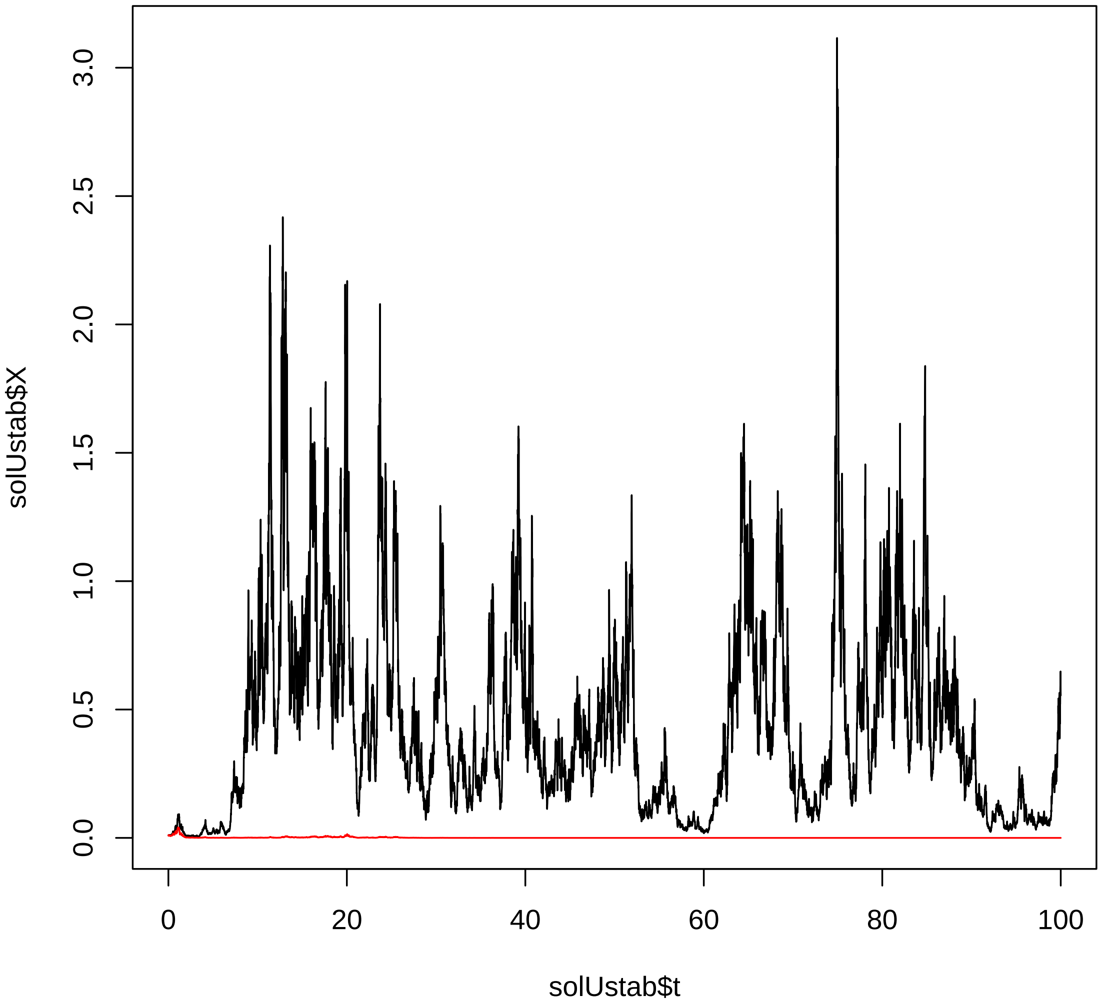
$$\lambda < 0 \Leftrightarrow r < \frac{1}{2}\sigma^2.$$

Question 3 **Simulation of stable and unstable scenarios:** Choose parameters such that the Lyapounov exponent is negative. Simulate the system - does the simulation support the conclusion that the zero solution is stable? Repeat for a parameter combination where the Lyapunov exponent is positive.

Solution: For stable parameters, we choose $r = 0.25$ and $\sigma = 1$ so that $\lambda = -0.25$. For unstable parameters, we choose $r = 1$ and the same $\sigma = 1$, so that $\lambda = 0.5$.

When we simulate the system with the Euler scheme, we sometimes see negative values. To avoid this, we reflect the solution at 0. We implement this by using `abs` as a "projection" argument to the Euler method; at each time step, this replaces X with $|X|$.

```r
r <- 0.25
sigma <- 1
K <- 1
p <- 1.5

require(SDEtools)


## Loading required package: SDEtools


tv <- seq(0,100,0.01)
B <- rBM(tv)

solStab <- euler(f,g,tv,0.01,B,abs)
r <- 1
solUstab <- euler(f,g,tv,0.01,B,abs)

plot(solUstab$t,solUstab$X,type="l")
lines(solStab$t,solStab$X,col="red")
```

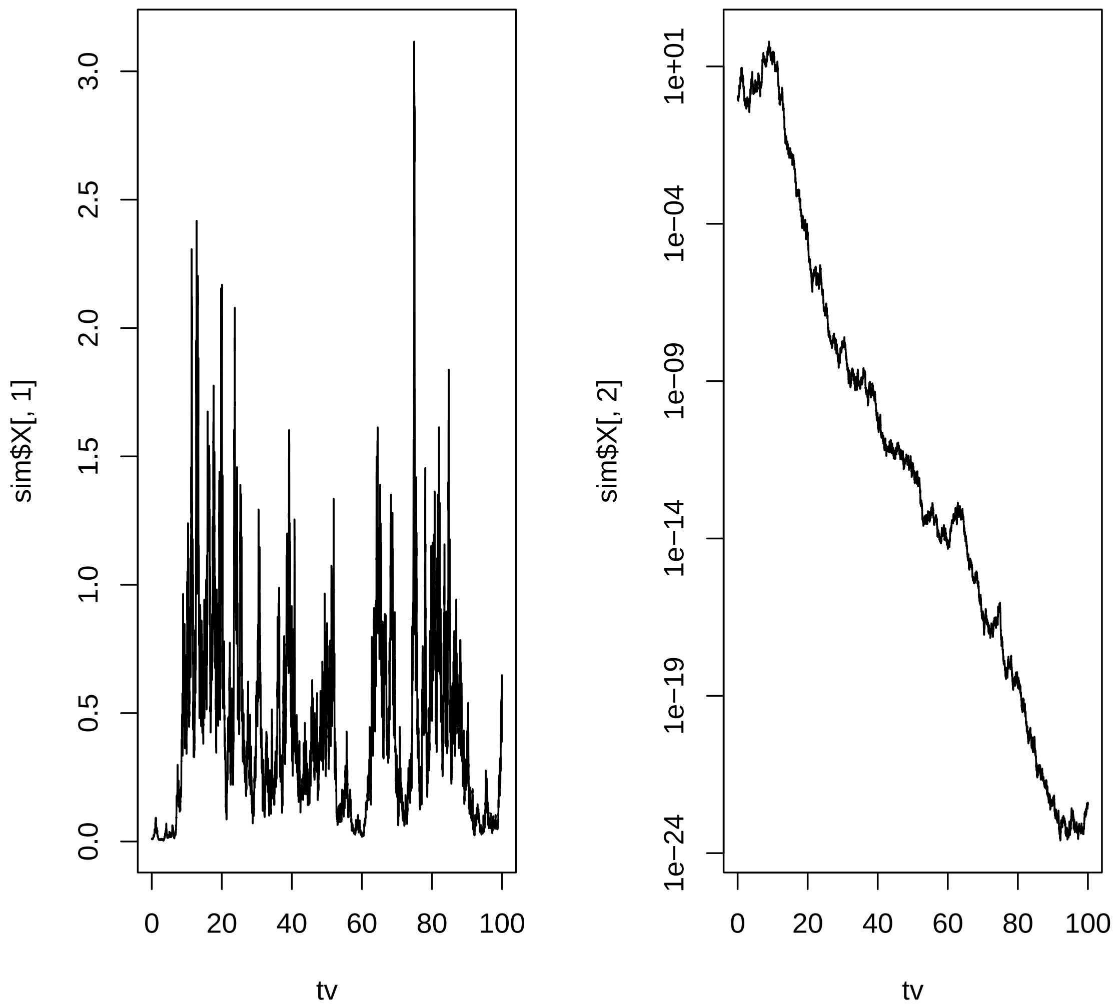


We see that in the stable case, the population goes extinct. In the unstable case, it does not - even if there are still large fluctuations which brings the state close to 0.

Question 4 **Stochastic Lyapunov exponent:** For the unstable parameter from the previous, simulate the system and the sensitivity equation simultaneously. Take $X_0 = 0.01K$. Give a (rough) estimate of the stochastic Lyapunov exponent.

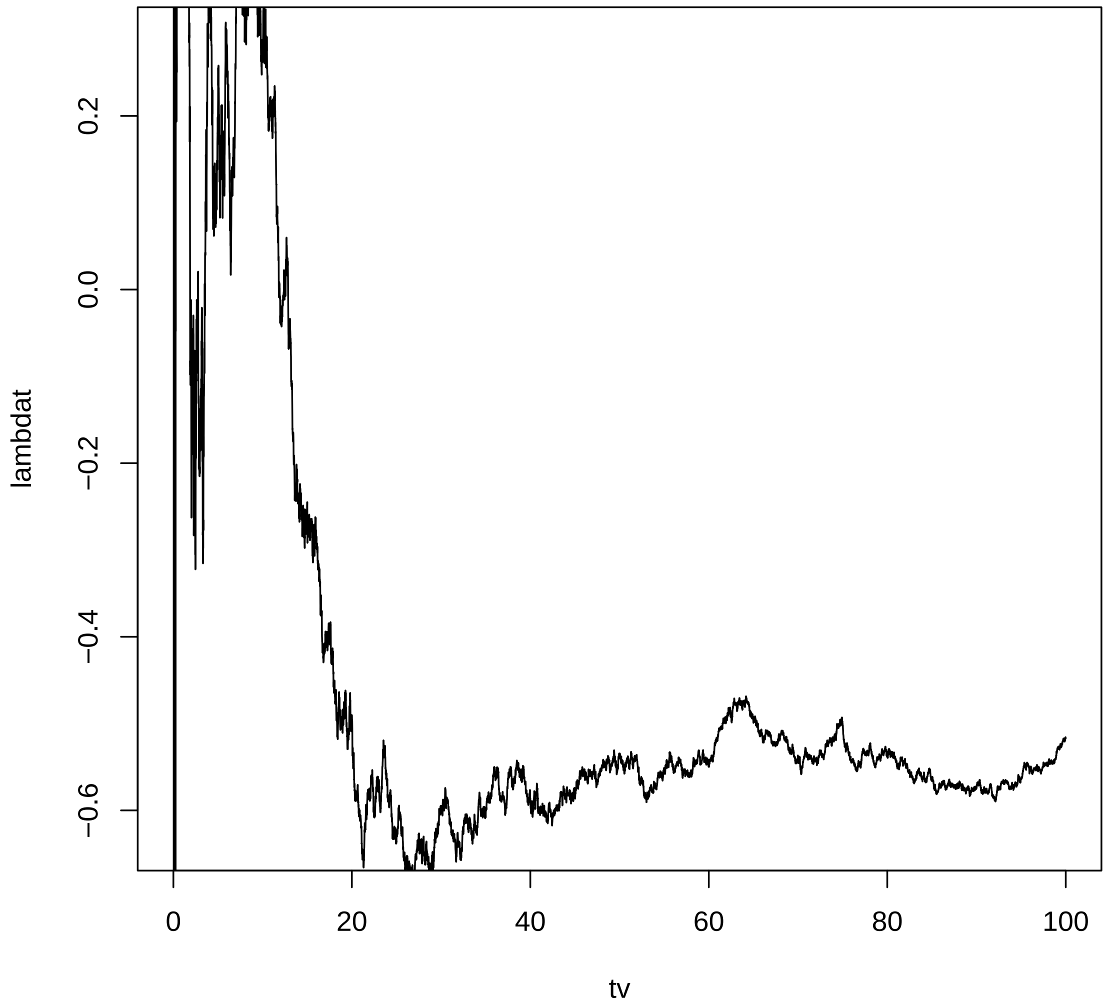
Solution: The following code simulates the joint system

```
fp <-  function(x) r*(1-(p+1)*(x/K)^p)
gp <-  function(x) sigma
fXS <- function(xs) c(f(xs[1]),fp(xs[1])*xs[2])
gXS <- function(xs) c(g(xs[1]),gp(xs[1])*xs[2])

xs0 <- c(0.01*K,1)

sim <- euler(fXS,gXS,tv,xs0,B,abs)

par(mfrow=c(1,2))
plot(tv,sim$X[,1],type="l")
plot(tv,sim$X[,2],type="l",log="y")
```



The following plot shows that the finite-time Lyapunov exponent appears to be about to converge, even if longer simulations are needed for an accurate estimate:

```
lambdat <-  log(sim$X[,2])/tv
plot(tv,lambdat,type="l",ylim=quantile(lambdat,c(0.05,0.95),na.rm=TRUE))
```



Question 5 **Stochastic Lyapunov functions:** Follow example 12.7.1 (p. 232) and use theorem 12.7.1 to show that $V(x) = x^q$ is a stochastic Lyapunov function which verifies stochastic stability of the zero solution, provided $0 < q < 1 - 2r/\sigma^2$. Compare the stability condition with what you found in question 2.

Solution: First, note that $V(x) = x^q$ (for $x \geq 0$) when $q > 0$. is proper. Next, we get

$$LV(x) = qx^{q-1}f(x) + \frac{1}{2}q(q-1)x^{q-2}g^2(x) = x^q(qr(1-(x/K)^p) + \frac{1}{2}q(q-1)\sigma^2)$$

This paranthesis is negative for all $x > 0$, if

$$qr + \frac{1}{2}q(q-1)\sigma^2 < 0 \Leftrightarrow q < 1 - 2r/\sigma^2$$

so that such a $q > 0$ can be found provided $1 - 2r/\sigma^2 > 0$. This condition is equivalent to $r - \sigma^2 < 0$, which is the condition we found previously. Notice that the reasoning here is exactly the same as for geometric Brownian motion. In this case the condition $LV \leq 0$ holds globally, because the superlinearity $-rx^{p+1}/K^p$ is stabilizing.

Question 6 Stability in the mean square: Try to show stability in the mean square with a candidate Lyapunov function $V(x) = x^2$, and show that it is never possible. Simulate the system for parameters where the zero solution is stochastically stable, but not stable in the mean square, and explain how these contradicting properties are visible in the simulation. *Note:* The stability theorem, which we have covered in class, only gives a *necessary* condition for stability. There exist *converse stability theorems* which can be used to shows that the zero solution is not stable.

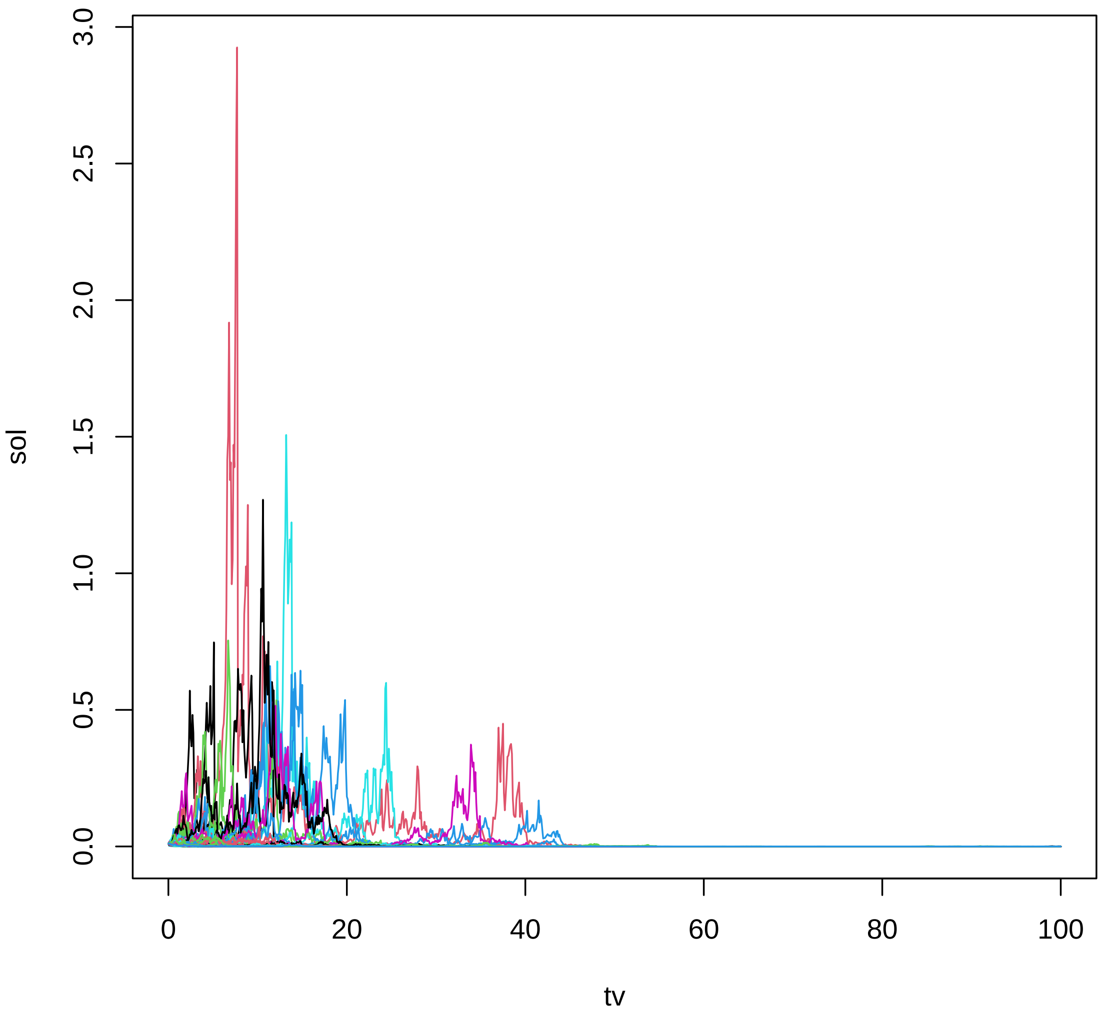
Solution: With $V(x) = x^2$ we get

$$LV(x) = x^2(2r(1-(x/K)^p) + \sigma^2) \ .$$

The second order terms in a Taylor expansion of this around $x = 0$ is $x^2(2r + \sigma^2)$, which can never be bounded above by a negative quadratic $-kx^2$. Therefore we cannot show stability in the mean square using this Lyapunov function.

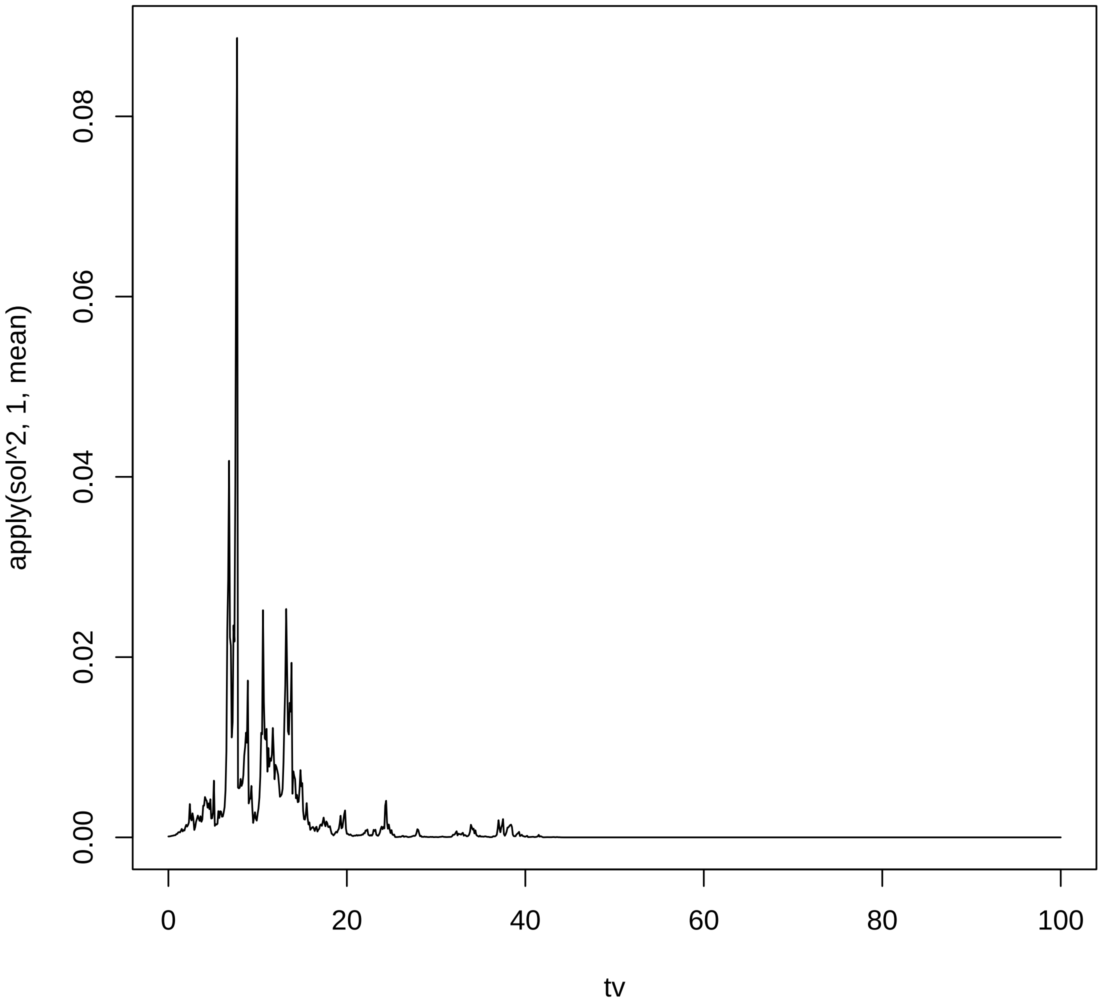
We can use our previous “stochastically stable” parameters for simulation:

```r
r <- 0.25
sigma <- 1
K <- 1
p <- 1.5

tv <- seq(0,100,0.1)

sol <- sapply(1:100,function(i)euler(f,g,tv,0.01,B=NULL,abs)$X)
matplot(tv,sol,type="l",lty="solid")
```



We see that all the solutions die out, as expected, but that they display large transients before doing so. Initially, the mean square grows:

```
plot(tv,apply(sol^2,1,mean),type="l")
```



Notice that that mean square fluctuates wildly. This reflects that it is dominated by single realizations, which are in the upper tail of the their distribution.

Question 7 **Boundedness:** (Compare exercise 12.7.) Consider the candidate Lyapunov function $V(x) = x - \log x$. Show that V is positive and proper and that $LV(x)$ is bounded above and negative when x is sufficiently large. (Re)state the condition for $LV(x)$ to be negative for sufficiently small $x > 0$. Conclude that, under this condition, there exists an interval $[a, b]$ with $0 < a < b$ which is positively recurrent.

Question 8 **The stationary distribution:** Solve the stationary forward Kolmogorov equation, at first neglecting normalization. *Hint: Remember detailed balance. Still,* `maple` *or similar is probably a good idea.* Then show that the solution can be normalized to a probability density function, if and only if $2r/\sigma^2 > 1$. *(Hint: The function* x^q *is integrable on any interval* $(0, a]$ *with* $a > 0$*, if and only if* $q > -1$*.)* Finally, compare the condition for existence of a stationary distribution with the stability conditions obtained in the previous.

Solution: For simplicity we set $K = 1$ (this can always be done by rescaling variables). We write the stationary forward Kolmogorov equation in advection-diffusion form:

$$D(x) = \frac{\sigma^2}{2}x^2, \quad u(x) = rx(1-x^p) - \sigma^2 x.$$

Then, the stationary equation is

$$u\pi = D\pi' \Leftrightarrow \pi = e^{\int u/D \ dx}$$

For the exponent, we get

$$\int u/D \ dx = \int 2(r/\sigma^2 - 1)x^{-1} - 2r/\sigma^2 x^{p-1} \ dx = 2(r/\sigma^2 - 1)\log x - 2r/(\sigma^2 p)x^p$$

and therefore an unnormalized density

$$\pi(x) = x^{2(r/\sigma^2-1)}e^{-2r/(\sigma^2 p)x^p}$$

With $p = 1$, this is a gamma distribution. With $p \neq 1$, this is not a "standard" distribution. This density can be normalized if it is integrable on the real axis. Since the density is continuous for $x \in (0, \infty)$, integrability mail fail due to a singularity at 0, or at $+\infty$.

First, focusing on the interval $[0, 1]$, the exponential term is bounded and therefore π is integrable if and only if $2(r/\sigma^2 - 1) > -1$. We can rewrite this

$$r/\sigma^2 > 1/2 \text{ or } r - \frac{1}{2}\sigma^2 > 0.$$

Next, regarding the interval $[1, +\infty)$, the exponential decays faster than any polynomial may grow, so π is always integrable on this interval. Therefore π can be normalized to a probability density function if and only if it is integrable on $(0, 1]$, i.e. for $r > \sigma^2/2$.

If you are not convinced by the previous reasoning, because the exponent has x^p and not just x, then do the variable substitution

$$y = x^p, \quad dx = y^{1/p-1}p^{-1} \ dy$$

to find

$$\pi(x) \ dx = y^{2(r/\sigma^2-1)/p}e^{-2r/(\sigma^2 p)y} \ y^{1/p-1}p^{-1} \ dy$$

and notice that now the density w.r.t. y is in the form "monomial times a negative exponential", hence integrable on $[1, \infty)$. Incidentally, we have now shown that $Y_t = X_t^p$ has a stationary distribution which is in the family of gamma distributions, with rate parameter $2r/(\sigma^2 p)$ and scale parameter $(2r/\sigma^2 - 1)/p$. Since the gamma distribution is a probability distribution (i.e., normalized to 1) whenever the scale parameter is positive, we see that a stationary probability distribution exists when $2r/\sigma^2 - 1 > 0$, or $r > \sigma^2/2$, as before.

In conclusion, a stationary probability density exists if and only if the origin is unstable. This sounds reasonable: If the origin is stable, then nearby trajectories will be absorbed at the origin, and eventually the interior $(0, \infty)$ will be evacuated.