DTU Compute

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UHT/uht

Exercise 13: Optimal control and fisheries management

Following the notes (section 13.8), we consider a population $\{X_t : t \geq 0\}$ which is governed by the Itō sde

$$dX_t = X_t(1 - X_t)\ dt - U_t\ dt + \sigma X_t\ dB_t$$

Here, $\{U_t : t \geq 0\}$ is a *catch rate*. The total profit arising from the harvesting is the random variable:

$$J = \int_0^T \sqrt{U_t}\ dt$$

Here, $\sqrt{U_t}$ is the instantaneous income, which grows slower than the harvest U_t because the price will decrease with the supply. We allow only catches $\{U_t\}$ such that $X_t \geq 0$.

Question 1 Analytical steady-state solution: Verify the solution in the notes. I.e., Show that a steady-state solution of the HJB equation is $V(x,t) = V_0(x) - \gamma t$ with $V_0(x) = \frac{1}{2}\log x + b$, $\gamma = \frac{1}{2}(1 - \frac{1}{2}\sigma^2)$, with the optimal control $\mu^*(x) = x^2$.

Question 2 Simulation of the closed-loop system: Simulate the system on the interval $t \in [0,T]$. Take $\sigma = 1$, $T = 10$. Simulate also the system with constant harvest rate $U_t = \frac{1}{2}X_t$. Compare the two policies in terms of total payoff J; how much better is the optimal strategy than the strategy with constant harvest rate? Why is it better?

Question 3 Numerical solution of the HJB equation.: Use the supplied code `HJB.R` to compute the value function and the optimal control from question 1; compare the numerical results with the analytical result. Take the domain to be $[0,5]$ and use reflection at both end points. Take $\sigma = 0.5$. *Note:* The formula (13.12) for the optimal control leads to numerical problems when V_0' is 0 or very small. A quick hack is to replace $V_0'(x)$ in this formula with

$$\max\{V_0'(x), \bar{v}(x)\}$$

where $\bar{v}(x)$ in ∞ in the first grid cell and 0.01 in the remaining grid cells.

Question 4 Extension to the Pella-Tomlinson model.: Repeat the numerical analysis for the Pella-Tomlinson model

$$dX_t = X_t(1 - X_t^p)\ dt - U_t\ dt + \sigma X_t\ dB_t$$

for $p = 0.5$ and $p = 2$. Show how the optimal policy depends on p.