DTU Compute

02425 Diffusions and SDEs
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UHT/uht

Exercise for session 3: Stochastic processes

Question 1 Simulation of Brownian motion: (Compare exercise 4.2 in the book; p. 67). Implement a function which simulates Brownian motion on an interval $[0, T]$; for example as given in the book: The function should take as input a partition $0 \leq t_1 < \cdots < t_n = T$, and should compute and return $B_{t_1}, B_{t_2}, \ldots, B_{t_{n-1}}, B_T$. Test the function by simulating sufficiently many replicates of $(B_0, B_{1/2}, B_{3/2}, B_2)$ to verify the covariance of this vector, and the distribution of B_2. Save the function for future use.

Question 2 Extrema of Brownian motion and hitting times:

1. Generate $N = 1000$ sample paths of Brownian motion on the time interval [0,1] using a time step of 0.001.
2. For each sample path $\{B_t : 0 \leq t \leq 1\}$, compute the maximum $S_1 = \max\{B_t : 0 \leq t \leq 1\}$. Plot the histogram of S_1 (or the empirical c.d.f.) and compare with the theoretical distribution of S_1.
3. For each sample path $\{B_t : 0 \leq t \leq 1\}$, compute the hitting time $\tau = \min\{t : B_t \geq b\}$ with $b = 0.5$. *Note:* If the sample path does not hit b in the time interval $[0, 1]$, then define $\tau = 1$. Plot the histogram of τ (or the emprirical c.d.f.) and compare with the theoretical distribution.

Question 3 Total and quadratic variation of Brownian motion: Reproduce figure 4.3 (page 70) in the book:

1. Generate one sample path of standard Brownian motion on the time interval $[0, 1]$ using a time step of 2^{-20}.
2. Compute the discretized total variation $V_\Delta = \sum |\Delta B|$ and quadratic variation $[B]_1 = \sum |\Delta B|^2$.
3. Subsample the Brownian motion at every other time step.
4. Repeat the previous steps until you have reached a time step of 2^{-10}.
5. Plot the discretized total variation and quadratic variation as function of the time step in double logarithmic plot.

Include the analytical predictions in the graphs, using exercise 4.34 (page 63) for the total variation. *Optional:* Solve exercise 4.34 so that you know where the analytical prediction comes from.

Basic Martingales

Question 4: Show that $\{B_t^2 - t\}$ is a martingale (exercise 4.8 in the book; p. 79).

Question 5: Solve exercise 4.19 in the book (p. 87) concerning *Doob's martingale.*

Question 6: Solve exercise 4.10 (p. 82) concerning the increasing variance of a martingale. *Extra:* When we apply this result to Doob's martingale (exercise 4.19), we conclude that the variance of the estimator increases as we accumulate information. Does this sound obvious or counter-intuitive to you? In the latter case, think carefully about how it should be understood; for example by decomposing the variance of X according to the information $\mathcal{F}_t$.