DTU Compute

02425 Diffusions and SDEs
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UHT/uht

Exercise 2: Probability measures

Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with $\Omega = [0,1]^2$, $\mathcal{F}$ the usual Borel-algebra on Ω, and $\mathbf{P}$ the Lebesgue measure, i.e. area. For $\omega = (x,y) \in \Omega$, define $X(\omega) = x$, $Y(\omega) = y$, and $Z(\omega) = x + y$.

Question 1 Independence and marginal distributions of X and Y: Verify that X and Y are independent, and that each are uniformly distributed on $[0,1]$.

Question 2 Level sets and σ-algebras: Sketch level sets (contour lines) for X, Y, and Z. *Note:* A level set for a function is a subset of the domain where the function attains one specific value, such as $X^{-1}(1/2) = \{(x,y) : x = 1/2\}$. Show typical elements in the σ-algebras $\sigma(X)$, $\sigma(Y)$ and $\sigma(Z)$.

Question 3 Monte Carlo simulation and conditional expectations of Z given X: Simulate a larger number, for example $N = 10,000$, of realizations of X, Y and Z. Binning the realized values of X into bins $[0, 1/n, 2/n, \ldots, 1]$, estimate empirically the conditional expectations of Z given X

$$\mathbf{E}\left\{Z | X \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\}$$

for $n = 10$ and $i = 1, \ldots, n$. Plot the result against i/n.

Question 4 Conditional expectations of Z given X, analytically: Find a function $g : \mathbf{R} \mapsto \mathbf{R}$ such that $\mathbf{E}\{Z|X\} = g(X)$. Use inspiration (!) from the Monte Carlo experiments and elementary arguments. Sketch contour lines of $\mathbf{E}\{Z|X\}$. Verify that this “candidate” conditional expectation satisfies the defining property

$$\int_H g(X(\omega))\ d\mathbf{P}(\omega) = \int_H Z(\omega)\ d\mathbf{P}(\omega)$$

for any $H \in \sigma(X)$.

Question 5 Conditional expectations of X given Z: Repeat the two previous questions for $\mathbf{E}\{X|Z\}$: First estimate the function $h : \mathbf{R} \mapsto \mathbf{R}$ such that $h(Z) = \mathbf{E}\{X|Z\}$ from the Monte Carlo experiments. Then find that function and verify that it satisfies its defining properties.

Question 6 The law of total expectation (The simple Tower property): Verify from the Monte Carlo experiments that

$$\mathbf{E}\{\mathbf{E}\{X|Z\}\} = \mathbf{E}\{X\}$$

Question 7 Variance decomposition/Law of total variance: Find an analytical expression for $\mathbf{V}\{X|Z\}$ and verify it against the Monte Carlo simulations. Then verify from the Monte Carlo simulations the variance decomposition formula

$$\mathbf{V}\{X\} = \mathbf{E}\mathbf{V}\{X|Z\} + \mathbf{V}\mathbf{E}\{X|Z\}$$