DTU Compute

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UHT/uht

# Exercise 13: Optimal control and fisheries management

Following the notes (section 13.8), we consider a population $\{X_t : t \geq 0\}$ which is governed by the Itō sde

$$dX_t = X_t(1 - X_t)\ dt - U_t\ dt + \sigma X_t\ dB_t$$

Here, $\{U_t : t \geq 0\}$ is a *catch rate*. The total profit arising from the harvesting is the random variable:

$$J = \int_0^T \sqrt{U_t}\ dt$$

Here, $\sqrt{U_t}$ is the instantaneous income, which grows slower than the harvest $U_t$ because the price will decrease with the supply. We allow only catches $\{U_t\}$ such that $X_t \geq 0$.

**Question 1** **Analytical steady-state solution:** Verify the solution in the notes. I.e., Show that a steady-state solution of the HJB equation is $V(x,t) = V_0(x) - \gamma t$ with $V_0(x) = \frac{1}{2}\log x + b$, $\gamma = \frac{1}{2}(1 - \frac{1}{2}\sigma^2)$, with the optimal control $\mu^*(x) = x^2$.

**Solution:** The HJB equation is

$$\dot{V} + \sup_u \left[ V'\ x(1-x) - V'u + \frac{1}{2}\sigma^2 V'' + \sqrt{u} \right] = 0 \quad .$$

We first identify the optimal control. Note that the terms in the bracket make a concave function of $u$ which has vertical tangent at $u = 0$, and that it will be decreasing for $u$ large enough whenever $V' > 0$. Therefore we can find the optimizing $u$ by identifying the unique stationary point:

$$V' = \frac{1}{2\sqrt{u}} \Leftrightarrow u = \frac{1}{4(V')^2}.$$

assuming $V' > 0$. With this, we get the HJB equation

$$\dot{V} + V'\ x(1-x) + \frac{1}{4V'} + \frac{1}{2}\sigma^2 x^2 V'' = 0 \quad .$$

We prepare to insert the candidate solution, and first write down its derivatives:

$$\dot{V} = -\gamma, \quad V' = \frac{1}{2x}, \quad V'' = -\frac{1}{2x^2}.$$

Inserting these in the HJB equation, we find

$$-\gamma + \frac{1}{2}(1-x) + \frac{1}{2}x - \frac{1}{4}\sigma^2 = 0$$

We see that the terms involving $x$ cancel out, as claimed, and that the equation is satisfied everywhere provided

$$\gamma = \frac{1}{2} - \frac{1}{4}\sigma^2$$

as claimed.

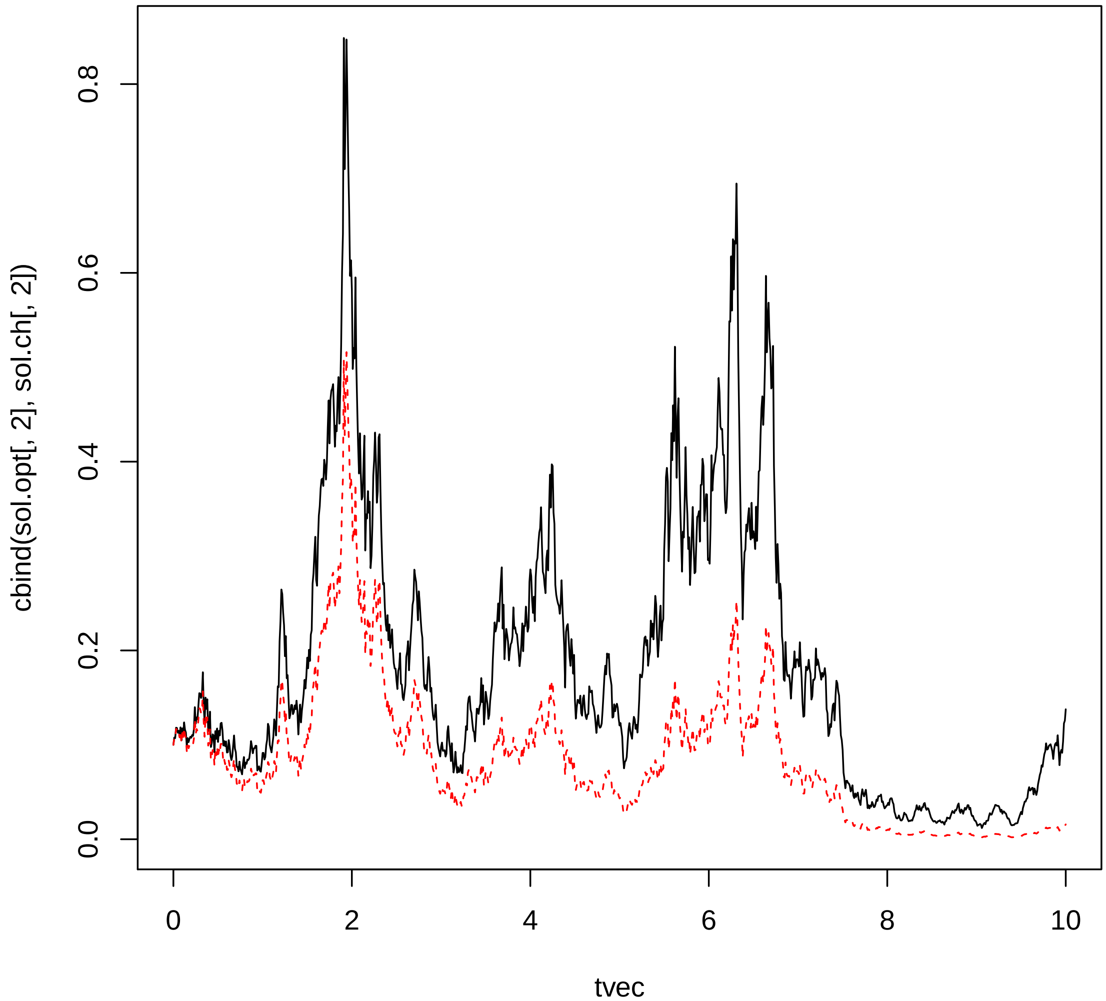
**Question 2** **Simulation of the closed-loop system:** Simulate the system on the interval $t \in [0, T]$. Take $\sigma = 1$, $T = 10$. Simulate also the system with constant harvest rate $U_t = \frac{1}{2} X_t$. Compare the two policies in terms of total payoff $J$; how much better is the optimal strategy than the strategy with constant harvest rate? Why is it better?

**Solution:**

```
require(SDEtools)

## Loading required package:  SDEtools

require(Matrix)

## Loading required package:  Matrix

set.seed(123456)  ## Fix the seed so that I know what the plot looks like :)

sigma <- 1
T <- 10

dt <- 0.01
tvec <- seq(0,T,dt)
x0 <- 0.1

fopt <- function(x) x*(1-x) - x^2
fch <- function(x) x*(1-x) - 0.5*x

g <- function(x) sigma*x

## Number of realizations
M <- 100

## Generate noise for all realizations
B <- rvBM(tvec,n=M)

sol.opt <- sapply(1:M,function(i)euler(fopt,g,tvec,x0,B=B[,i],p=abs)$X)
sol.ch <-  sapply(1:M, function(i)euler(fch,g,tvec,x0,B=B[,i],p=abs)$X)

J.opt <- mean(sqrt(sol.opt^2))
J.ch <- mean(sqrt(0.5*sol.opt))
```

We plot one particular realization with the two strategies:

```
matplot(tvec,cbind(sol.opt[,2],sol.ch[,2]),type="l")
```




We see that the optimal strategy is better at letting the population recover, once a random fluctuation has pushed it down. This is because the optimal strategy $u = x^2$ is below the heuristic ($u = x/2$) when the biomass $x$ is low.

We can compare the average performance of the two systems:

```
print(c(J.opt,J.ch))

## [1] 0.2422412 0.3134881
```

and we see that there is some difference. This depends on our choice of initial condition. Here, I used an initial condition which was low, so it was important that the optimal strategy is able to rebuild the population faster. If you choose an initial condition which is close to the steady state ($x = 1/2$), then the relative difference will be smaller.

It is possible to give analytical results for the expected harvest in the limit of long horizons. For the optimal strategy, this is $\gamma$. For the heuristic $u = x/2$, we could use that the stationary distribution is in the family of gamma distributions. We do not pursue this further here.

**Question 3 Numerical solution of the HJB equation.:** Use the supplied code `HJB.R` to compute the value function and the optimal control from question 1; compare the numerical results with the analytical result. Take the domain to be $[0, 5]$ and use reflection at both end points. Take $\sigma = 0.5$. *Note:* The formula (13.12) for the optimal control leads to numerical problems when $V_0'$ is 0 or very small. A quick hack is to replace $V_0'(x)$ in this formula with

$$\max\{V_0'(x), \bar{v}(x)\}$$

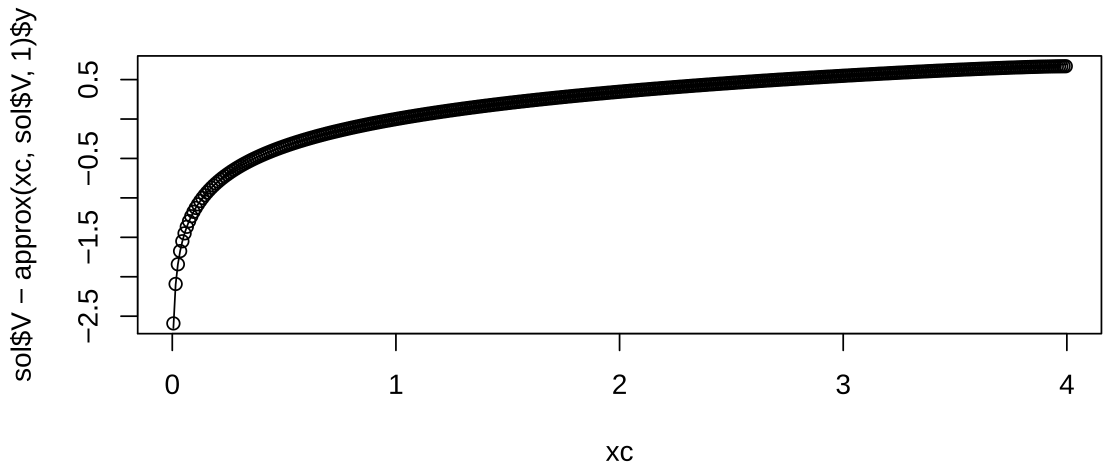
where $\bar{v}(x)$ in $\infty$ in the first grid cell and 0.01 in the remaining grid cells.

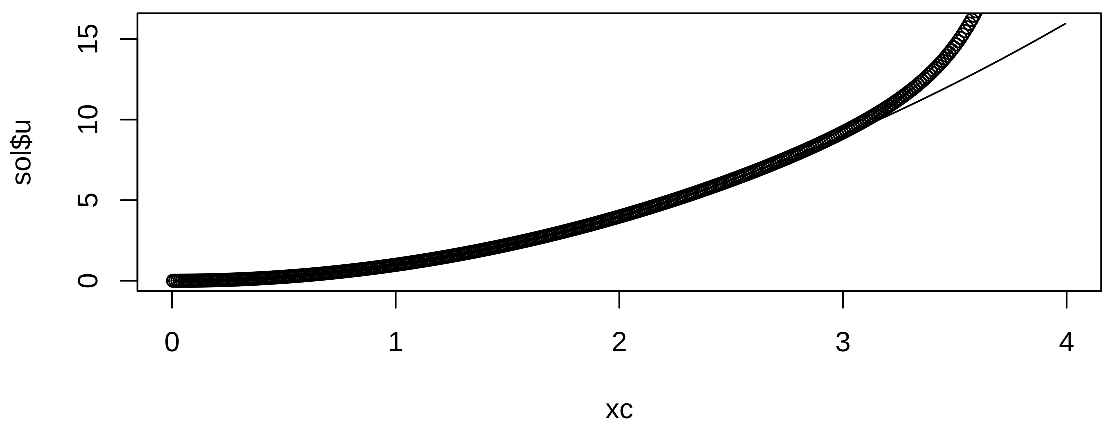
**Solution:**

```
require(SDEtools)

### Discretization of state space
Xmax <- 4
dx <- 0.01
xi <- seq(0,Xmax,dx)
xc <- xi[-1] - 0.5*diff(xi)

sigma <- 1

### Functions entering into the model

### Uncontrolled system:
D <- function(x) 1/2*sigma^2*x^2
dD <- function(x) sigma^2*x

f <- function(x) x*(1-x)
advection <- function(x) f(x) - dD(x)

G0 <- fvade(advection,D,xi,'r')

### Effect of the fishing: The "generator" d/dx
G1 <- fvade(function(x)-1,function(x)0,xi,'r')

ubound <- c(0,rep(100,nrow(G1)-1))

k <- function(u) sqrt(u)

vbar <- c(Inf,rep(0.01,nrow(G0)-1))
hack <- function(dV) pmax(-dV,vbar)
uopt <- function(dV) 1/4/hack(dV)^2

sol <- PolicyIterationSingular(G0,G1,k,uopt,do.minimize = FALSE)
```

We plot the solution along the analytical result $V_0(x) = 0.5 \log x$, $u = x^2$. For the value function, we have to remember that it is only determined up to an additive constant, so we shift it to match at $x = 1$ where we have $V_0(1) = 0$.

```
par(mfrow=c(2,1))
plot(xc,sol$V - approx(xc,sol$V,1)$y)
lines(xc,0.5*log(xc))

plot(xc,sol$u,ylim=c(0,max(xc)^2))
lines(xc,xc^2)
```




We see a good agreement for the value function. For the control, we see that there is some disagreement at the upper boundary. This is because the numerical approximation enforces reflection at the upper boundary. This corresponds to biomass "going to waste", so the discretized systems increases the harvest near this boundary in order to reduce the probability of hitting the boundary. For reliable results, we must do computations in a domain which is quite large.

**Question 4** **Extension to the Pella-Tomlinson model.:** Repeat the numerical analysis for the Pella-Tomlinson model

$$dX_t = X_t(1 - X_t^p)\ dt - U_t\ dt + \sigma X_t\ dB_t$$

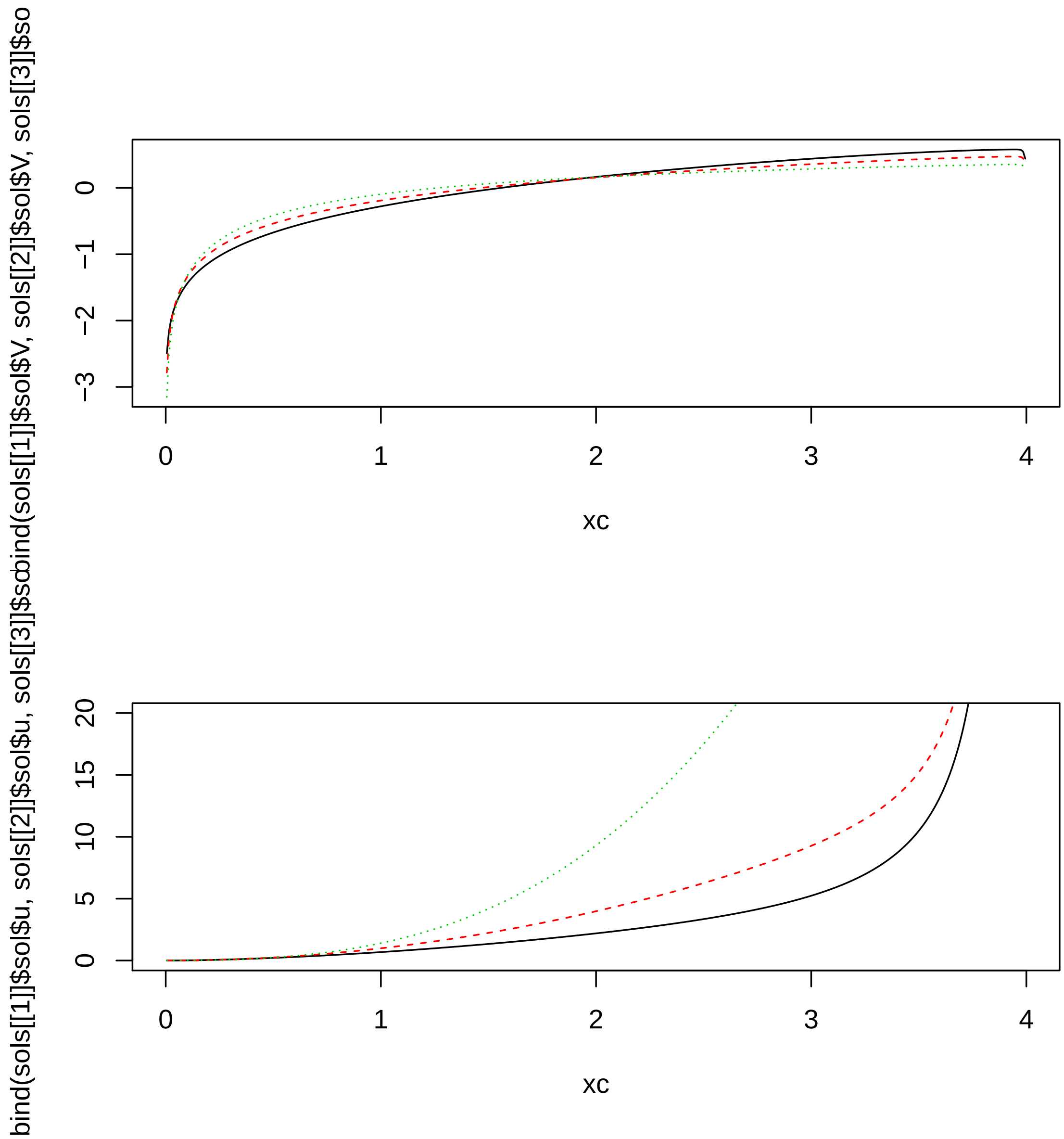
for $p = 0.5$ and $p = 2$. Show how the optimal policy depends on $p$.

**Solution:**

```
pv <- c(0.5,1,2)

sols <- list()


for(p in pv)
{
    f <- function(x) x*(1-x^p)
    advection <- function(x) f(x) - dD(x)
    G <- fvade(advection,D,xi,'rc')

    sol <- PolicyIterationSingular(G,G1,k,uopt,do.minimize = FALSE)

    sols[[length(sols)+1]] <- list(p=p,sol=sol)
}
```

We can now plot the value function and the policy.

```
par(mfrow=c(2,1))
matplot(xc,cbind(sols[[1]]$sol$V,sols[[2]]$sol$V,sols[[3]]$sol$V),type="l")
matplot(xc,cbind(sols[[1]]$sol$u,sols[[2]]$sol$u,sols[[3]]$sol$u),type="l",ylim=c(0,20))
```




We see that the low value of $p$ leads to a lower harvest, for all values of $x$.