DTU Compute

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Exercises for session 6: Itô calculus and SDE's

Using Itō's lemma to verify of solution

Question 1: Verify that $Y_t = \sinh B_t$ satisfies the Itô SDE

$$dY_t = \frac{1}{2} Y_t \; dt + \sqrt{1 + Y_t^2} \; dB_t$$

Stochastic logistic growth

Consider the Itō stochastic differential equation governing $\{X_t\}$, the abundance of bacteria in a population:

$$dX_t = X_t(1 - X_t) \; dt + \sigma X_t \; dB_t$$

Question 2 Using Itō's lemma to perform a coordinate transformation: Identify a Lamperti transform h, i.e. find a transformed coordinate $Y_t = h(X_t)$ such that the Itō equation for $\{Y_t\}$ has additive noise. Write up this Itō equation.

Numerical analysis with the Euler-Maruyama method

Question 3: Simulate a sample path of $\{X_t\}$ with the Euler-Maruyama discretization method on the time interval $[0, 10]$. Take $\sigma = 0.2$ and $X_0 = 0.1$. Choose a sufficiently (but not excessively) small time step. Plot the solution.

Question 4: Extend the Euler-Maruyama simulation so that it solves the Itō equation for $\{X_t\}$ *and* the Itō equation for $\{Y_t\}$ in the same loop. Finally, determine numerically the solution to

$$dZ_t = h'(X_t) \; dX_t + \frac{1}{2} h''(X_t) \; d[X]_t$$

Plot in the same window X_t, $h^{-1}(Y_t)$, and $h^{-1}(Z_t)$ versus time.

Question 5: Extend the simulation of $\{X_t\}$ (or $\{Y_t\}$) for sufficiently long that the process appears to reach a steady state. Plot a histogram of the estimated stationary distribution of $\{X_t\}$, and estimate the mean and variance in steady state. Repeat with stronger noise, e.g. $\sigma = 0.5$, $\sigma = 1$ and $\sigma = 2$. What is the effect of increasing the noise?

Mean and variance in a the noisy harmonic oscillator

Consider the SDE (compare also exercise **??**)

$$dX_t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X_t \; dt + \sigma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; dB_t$$

with the initial condition $X_0 = x$. Here $X_t \in \mathbf{R}^2$ and $\{B_t : t \geq 0\}$ is two-dimensional Brownian motion.

Question 6: Verify that

$$\mathbf{E}X_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x$$

for all t.

Question 7: Write $S_t = \|X_t\|^2$ as an Itô process and find $\mathbf{E}S_t$ as a function of t.

Question 8: Pose and solve the differential Lyapunov equation governing the variance-covariance matrix of X_t. *Hint:* To solve the equation, first guess that the variance-covariance matrix is diagonal.