DTU Compute

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UHT/uht

Exercise 10: Filtering in a non-linear equation

This exercise reproduces the running example in chapter 10, with a few minor deviations. A bacterial population evolves according to the Cox-Ingersoll-Ross model

$$dX_t = \lambda \cdot (\xi - X_t)\ dt + \gamma\sqrt{X_t}\ dB_t \qquad \text{with } \lambda = \xi = \gamma = 1.$$

Here X_t is a measure of the concentration of the bacteria (cells per volume) at time t. We do not specify the volume unit.

Question 1 The forward Kolmogorov equation: Write up the forward Kolmogorov equation for the transition probabilities, in advection-diffusion form, and discretize the generator. Use a uniform grid on $[0, 5]$. Compute the transition probabilities over a time step of 0.1 and plot them.

An observer measures the population abundance, i.e. the state X_t, in the following way: A small volume $v = 0.5$ is sampled and all bacteria in the volume are counted. The count data is Poisson distributed with (conditional) mean $\mathbf{E}\{Y_t|X_t\} = v \cdot X_t$, i.e.

$$Y_t|X_t \sim \text{Poisson}(\text{mean} = v \cdot X_t)$$

Question 2 Measurements: Import the measurements from the file `hmm-obs.txt` and plot them.

Question 3 The state likelihood: Write up the "state likelihood function" of the state $X_t = x$, for a given measurement $y = Y_{t_i}(\omega)$, i.e. identify

$$\mathbb{P}\{Y_{t_i} = y|X_{t_i} = x\}$$

as a function of x and y. Plot the state likelihood function as a function of x, for $y = 0, \ldots, 5$.

Question 4: Then construct a table which, for each sampling time $t = t_i$ and each possible value of the state $x = X_t$, holds the state likelihood function $l_i(x)$. Plot this table as a pseudocolor image.

Question 5 Implementing the filter: Implement the HMM filter to estimate the state, based on iterating time update and data update.

Question 6: Make a color plot of the estimated state distribution. Add to the plot time series of the mean (alternatively, median or mode) in the estimated distribution.

Download the file `hmm-states.txt`, which holds the true states, and add this to the plot

Question 7 **Maximum likelihood estimation:** We now consider the parameter ξ unknown. Run the filter in the previous with values of ξ ranging from 0 to 2, in steps of 0.1. For each value of ξ, compute the likelihood. Plot the likelihood function of ξ and commment - the data was simulated with a value of $\xi = 1$.