DTU Aqua/Informatics

02425
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UHT/uht

Solutions for session 12: Optimal control

Question 1: Veriying the analytical solution

The steady-state HJB equation is, from the notes

$$\sup_u \left[V_0'(x)(x(1-x) - ux) + \frac{1}{2}\sigma^2 x^2 V_0''(x) + \sqrt{xu} \right] = \nu$$

The term in the bracket, which is to be maximized w.r.t. $u \geq 0$, is concave in u and initially increasing, so there exists a unique maximum point which is found as the unique stationary point. To find this, we differentiate w.r.t. u to find the optimality condition

$$-V_0'(x)x + \sqrt{\frac{x}{2u}} = 0 \Leftrightarrow u = \mu^*(x) = \frac{1}{4x(V_0'(x))^2}$$

Insert the guess $V_0(x) = \frac{1}{2}\log x + b$, we get

$$V_0'(x) = \frac{1}{2x}, \quad V_0''(x) = -\frac{1}{2x^2}, \quad \mu^*(x) = x$$

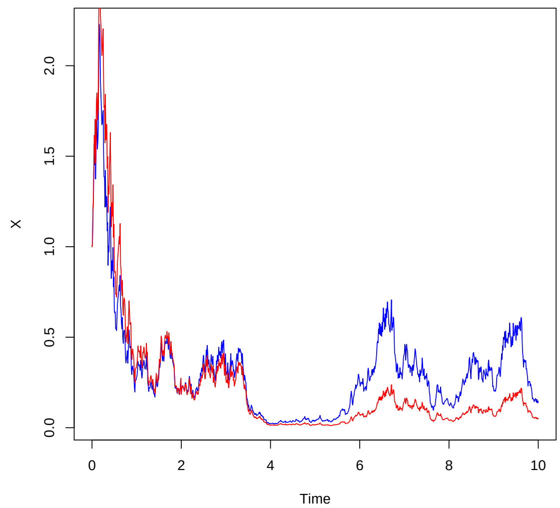
and inserting this in the HJB equation we find

$$\frac{1}{2x}(x(1-x) - x^2) + \frac{1}{2}\sigma^2 x^2 \frac{-1}{2x^2} + x = \frac{1}{2} - x - \frac{1}{4}\sigma^2 + x = \frac{1}{2}(1 - \frac{1}{2}\sigma^2)$$

This is independent of x, which verifies that our guess on V_0 is correct. The constant ν is the found as $\frac{1}{2}(1 - \frac{1}{2}\sigma^2)$.

Question 2: Simulation of the two systems

See `R` code. Here are the results:



Note that the optimal strategy initially results in a faster decay of the fish population. The constant harvest rate underexploits the resource at this point. Later, when a random even brings the population down to low levels, the constant harvest rate overexploits the resource, so that the population is unable to recover. The optimal strategy, in turn, reduces the harvest rate, so that the resource may recover.

This run is made with a very high noise level σ, so the results are rather extreme. With the constant harvest rate $u = 1/2$, the noise-free equilibrium is still $x = 1/2$ which matches Maximum Sustainable Yield, but under the closed-loop dynamics, the equilibrium point $x = 0$ is at the boundary of stability.

Question 3: The non-stationary HJB equation

$$\dot{V} + \sup\left[V'(x,t)(x(1-x) - ux) + \frac{1}{2}\sigma^2 x^2 V''(x,t) + \sqrt{xu}\right] = 0$$

leading, as before, to the optimal control

$$u = \mu^*(x,t) = \frac{1}{4x(V'(x,t))^2)}$$

Question 4: Numerical solution of the HJB equation

See `R` code.

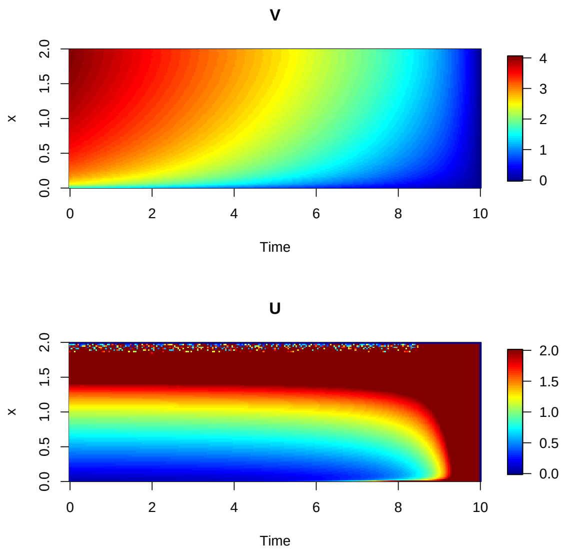
The Euler method for the HJB equation requires very small time steps. To avoid this, I have replaced the Euler step of the uncontrolled system with an exponential matrix. I.e., the terms

```
V[,i+1] + L0 %*% V[,i+1] * dt
```

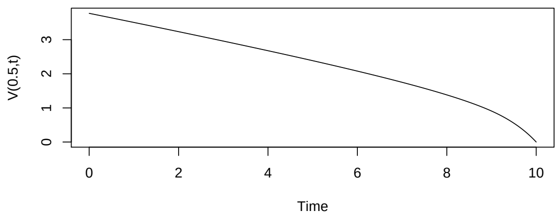
have been replaced with

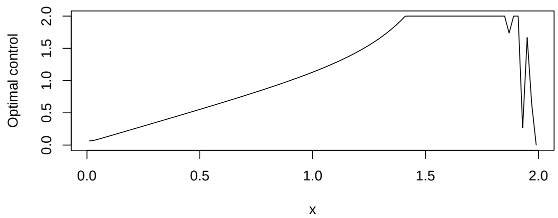
```
P %*% V[,i+1]
```

where `P=expm(L0*dt)`. It is possible to improve further on this, but we will not pursue it now, since it isn't necessary. The result are displayed in this graph:



The following plot checks that the value function has reached its linear asymptote, and displays the stationary control strategy





Note the wiggles near the boundaries of the computational domain. They are effects of the boundary condition employed.