DTU Compute

02425 Diffusions and SDE's
September 20, 2022
UHT/uht

Exercise 4: Linear systems

Steady-state variance structure for a mass-spring-damper system

Consider the mass-spring-damper system in the notes 5.1,5.2, p. 104, with the force $\{F_t : t \geq 0\}$ being white noise with a given intensity $S_{FF}(\omega) = \sigma^2$.

Question 1: Write the system in the standard form $dX_t = AX_t\ dt + G\ dB_t$, i.e. specify A and G.

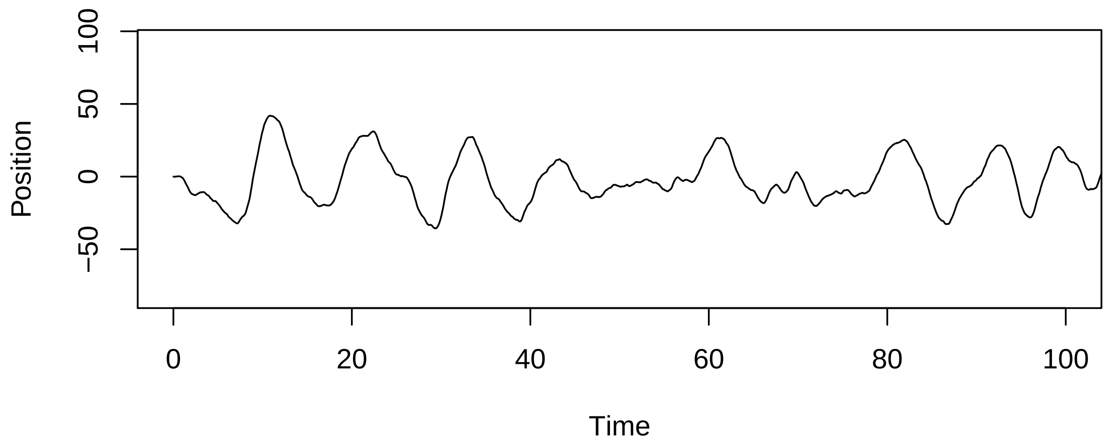
Solution: The system is written as $dX_t = AX_t\ dt + GdB_t$ where

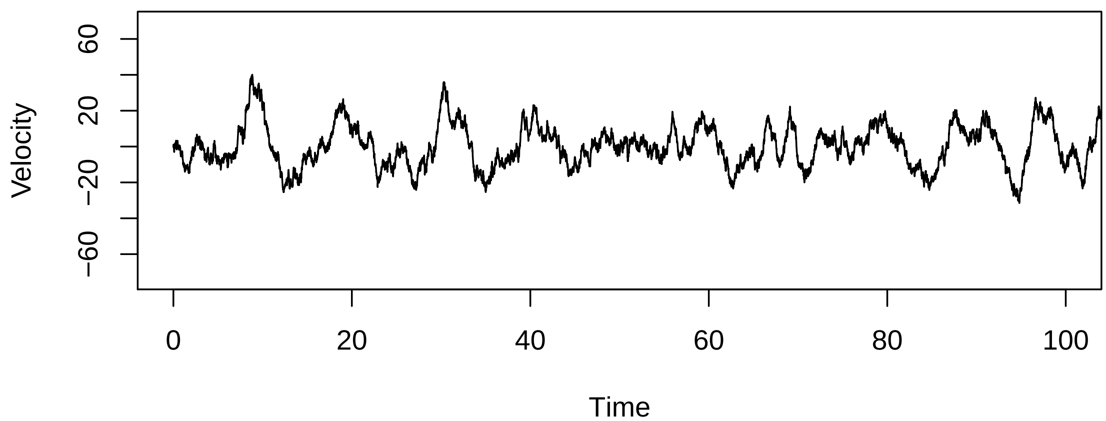
$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, G = \begin{bmatrix} 0 \\ \sigma/m \end{bmatrix},$$

Question 2: Using the general form, simulate the system on the time interval $t \in [0, 1000]$ using the Euler method. Take system parameters $m = 1$ kg, $k = 0.5$ N/m, $c = 0.2$ Ns/m, $\sigma^2 = 100$ $\mathrm{N}^2 s$. Let the system start at rest at $t = 0$. Use a time step of $\Delta t = 0.01$ s. Plot the sample path.

Solution: The following code simulates the system using the Euler method. We time step also the Lyapunov differential equation governing the variance.

```r
## Helper function for simulation of Brownian motion, as per week 3
rBM <- function(tvec)
{
    return(cumsum(rnorm(length(tvec),mean=0,sd=sqrt(diff(c(0,tvec))))))
}

## System parameters
m <- 1    # [kg]
k <- 0.5  # [N/m]
c <- 0.2  # [N*s/m]
sigma <- 10 # [N sqrt(s)]

## Q 1.1
A <- array(c(0,-k/m,1,-c/m),c(2,2))
G <- array(c(0,sigma/m),c(2,1))


## Simulation parameters
T <- 1000
dt <- 0.01

## Setup arrays
tvec <- seq(0,T,dt)
P <- array(0,c(2,2,length(tvec)))  # This also computes the solution of the Lyapunov equation
X <- array(0,c(2,length(tvec)))

## Simulate sample path of Brownian motion
B <- rBM(tvec)
dB <- diff(B)

## Main time loop, Euler stepping the SDE and the Lyapunov equation
for(i in 1:(length(tvec)-1))
{
    X[,i+1] <- X[,i] + A %*% X[,i] * dt + G * dB[i]
    P[,,i+1] <- P[,,i] + (A %*% P[,,i] + P[,,i] %*% t(A) + G %*% t(G)) * dt
}

## Q 2.2: Plot the sample path; first part only for clarity
par(mfrow=c(2,1))
plot(tvec,X[1,],type="l",xlim=c(0,100),xlab="Time",ylab="Position")
plot(tvec,X[2,],type="l",xlim=c(0,100),xlab="Time",ylab="Velocity")
```





Question 3: Estimate from your simulation the steady-state variance of position Q_t, of velocity V_t, and the covariance between the two. Compare with the solution of the algebraic Lyapunov equation governing the variance. *Note:* In `Matlab`, use built-in `lyap.m`. In `R`, use the function `lyap.R` on FileSharing.

Solution: We use the supplied solver for the Lyapunov equation:

```
lyap <- function(A,Q)
{
    A <- as.matrix(A)
    I <- diag(rep(1,nrow(A)))
    P <- kronecker(I,A)+kronecker(A,I)
    X <- -solve(P,as.numeric(Q))
    return(matrix(X,nrow=nrow(A)))
}

Pinf <- P[,,length(tvec)]
Pinf2 <- lyap(A,G%*%t(G))

## Print the empirical covariance and compare with the solution of the Lyapunov equation
print(cov(t(X)))
```

```
##             [,1]       [,2]
## [1,] 544.643379  -1.253909
## [2,]  -1.253909 272.142162
```

```
print(Pinf)
```

```
##                [,1]          [,2]
## [1,]  5.000000e+02 -1.315445e-12
## [2,] -1.315445e-12  2.500000e+02
```

```
print(Pinf2)
```

```
##      [,1] [,2]
## [1,]  500    0
## [2,]    0  250
```

Question 4: The kinetic energy is $\frac{1}{2}mV_t^2$ while the potential energy is $\frac{1}{2}kQ_t^2$. In steady-state, what is the expected kinetic energy and the expected potential energy? *Note:* The result is an example of equipartitioning of energy, a general principle in statistical mechanics, both quantum and classical.

Solution:

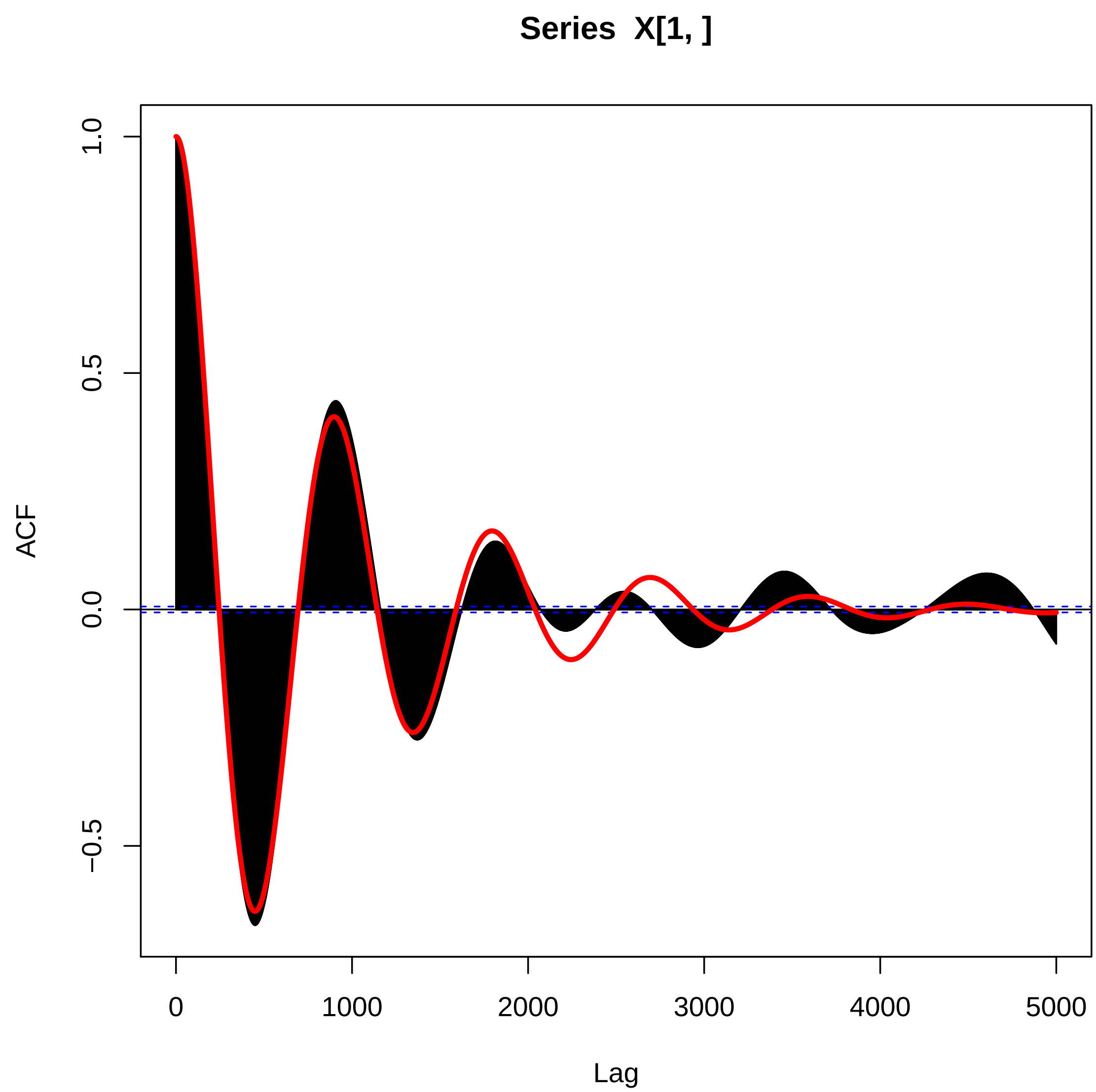
```
print(Ekin <- 0.5*m*var(X[2,]))

## [1] 136.0711

print(Epot <- 0.5*k*var(X[1,]))

## [1] 136.1608

## ... add the analytical predictions:
print(0.5*m*Pinf2[2,2])

## [1] 125

print(0.5*k*Pinf2[1,1])

## [1] 125
```

Note that analytically, the expected kinetic and potential energies are equal, and that they are also quite close, emperically. The empirical result agree reasonably with the analytical result, considering that the simulation is somewhat short compared to the period of oscillations.

Question 5: For the simulation, compute and plot the empirical a.c.f. of $\{Q_t\}$ up to lag 50 s. *Hint:* In `Matlab` and `R`, use `acf`. Add to the plot the theoretical prediction.

Solution: We plot empirical a.c.f. of the solution. Note: Lags up to 50 seconds, not 50 time steps!

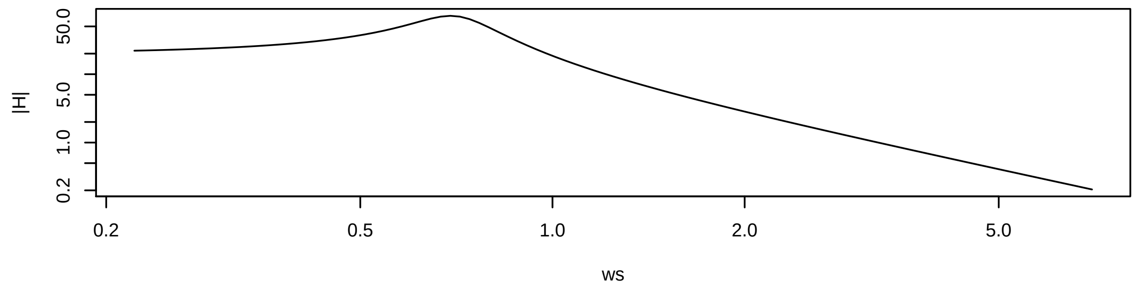
```
acf(X[1,],lag.max=50/dt)

ivec <- seq(0,50/dt,1)
tvec <- ivec * dt

require(expm)


## Loading required package:  expm

## Loading required package:  Matrix

##
## Attaching package:  'expm'

## The following object is masked from 'package:Matrix':
##
##     expm


rhovec <- sapply(tvec,function(t) (Pinf2 %*% expm(t(A)*t) )[1,1])
lines(ivec,rhovec/rhovec[1],col="red",lwd=3)
```

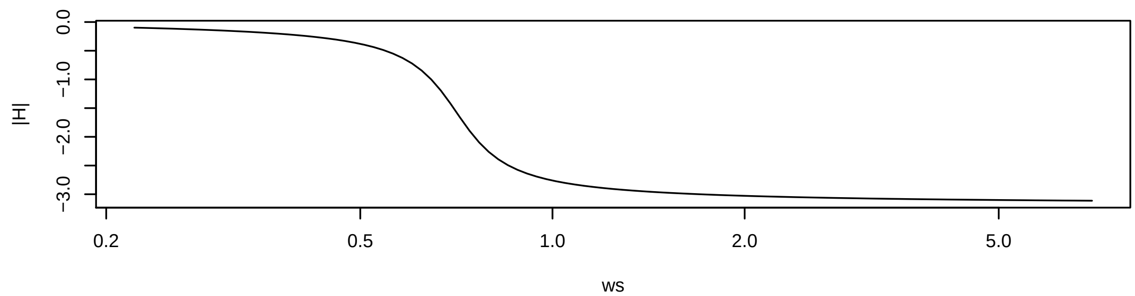


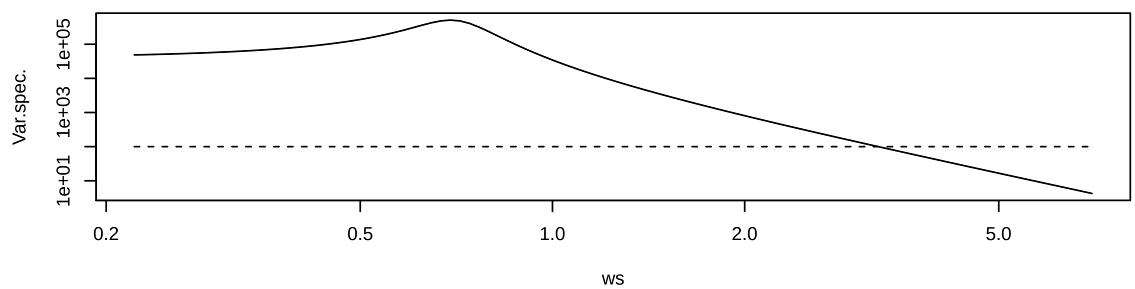
Question 6: Plot, as a function of the frequency ω, the amplitude and phase of the frequency response from the noise to the position. Plot also the theoretical variance spectrum of the position.

Solution:

```
I <- diag(c(1,1))
H <- function(w) (solve(1i*w*I-A) %*% G)[1]
omegaR <- abs(Im(eigen(A)$values[1]))
ws <- omegaR*10^(seq(-0.5,1,length=101))
Hs <- sapply(ws,H)
par(mfrow=c(3,1))
plot(ws,abs(Hs),log="xy",type="l",ylab="|H|")
plot(ws,Arg(Hs),log="x",type="l",ylab="|H|")
plot(ws,abs(Hs)^2*sigma^2,type="l",log="xy",ylab="Var.spec.")
lines(ws,rep(sigma^2,length(ws)),lty="dashed")
```







Variance in a scalar linear system

Consider the scalar linear system

$$\dot{X}_t = aX_t + gU_t, \quad X_0 = x$$

where $\{U_t : t \geq 0\}$ is Gaussian "white noise", i.e. the formal derivative of standard Brownian motion.

Question 7: Write up the mean $\mathbf{E}X_t$ as a function of time.

Solution: The analytical mean is $\mathbb{E}X_t = xe^{at}$.

Question 8: Write up the differential Lyapunov equation governing the variance $\mathbf{V}X_t$, and solve it.

Solution: The differential Lyapunov equation is

$$\dot{\Sigma} = 2a\Sigma + g^2$$

and its solution (with $\Sigma(0) = 0$ is the initial condition is deterministic) is

$$\Sigma(t) = \frac{g^2}{2a}(e^{2at} - 1)$$

Question 9: Assume that the system is stable. What is the steady-state variance, $\lim_{t\to\infty} \mathbf{V} X_t$?

Solution: If $a < 0$, then $\Sigma(\infty) = -g^2/(2a)$.

Question 10: Verify that the steady-state variance is an equilibrium point of the Lyapunov equation.

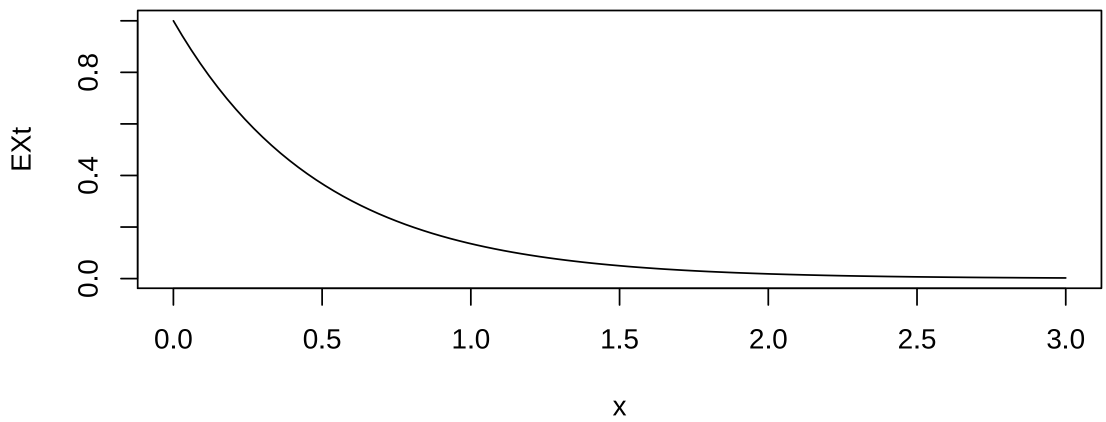
Solution: If $\Sigma(t) = -g^2/(2a)$, then we get $\dot{\Sigma}(t) = -2ag^2/(2a) + g^2 = 0$.

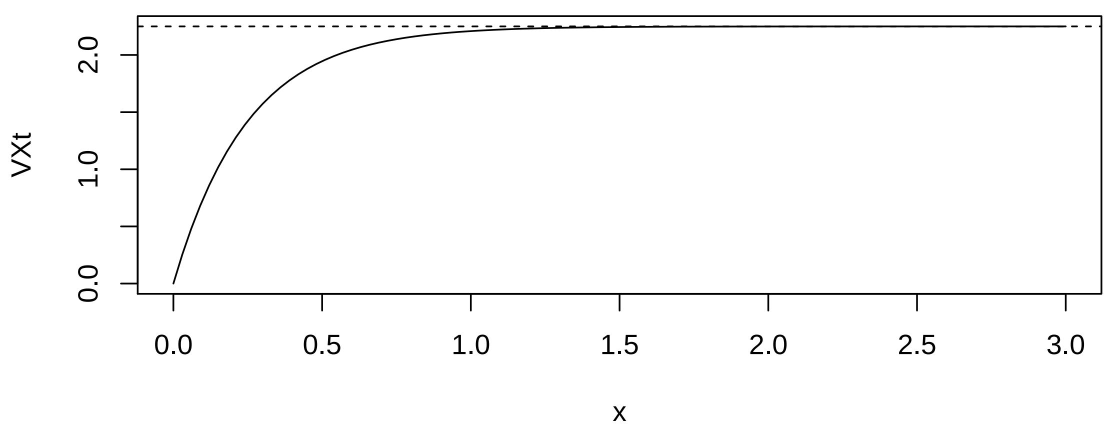
The following is a numerical example (which is not asked for in the exercise)

```
## Specific examples of system parameters
a <- -2
g <- 3
x <- 1

EXt <- function(t) x*exp(a*t)

## Differential lyapunov equation
Lyap <- function(V) 2*a*V + g^2

## Analytical solution of the equation
VXt <- function(t) g^2/2/a*(exp(2*a*t)-1)

## Limit as time goes to infinity, assuming a<0
VXinf <- -g^2/2/a

## Test that it is an equilibrium point for the Lyapunov equation
print(Lyap(VXinf))
```

```
## [1] 0
```

```
par(mfrow=c(2,1))
plot(EXt,from=0,to=3)
plot(VXt,from=0,to=3)
abline(h=VXinf,lty="dashed")
```


EXt
0.0
0.4
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x


VXt
0.0
1.0
2.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x