DTU Compute

02425 Diffusion and SDE's
September 27, 2022
UHT/uht

Exercise 5: The Euler method and the Ito integral

In the following, we use the following implementation of the Itō integral:

```
itointegral <- function(G,B) c(0,cumsum(head(G,-1)*diff(B)))
```

Question 1: Verify the implementation by computing and plotting

$$\int_0^T \cos t \; d\sin t$$

for $T \in [0, 2\pi]$ on a sufficiently fine grid (at least 100 grid points).

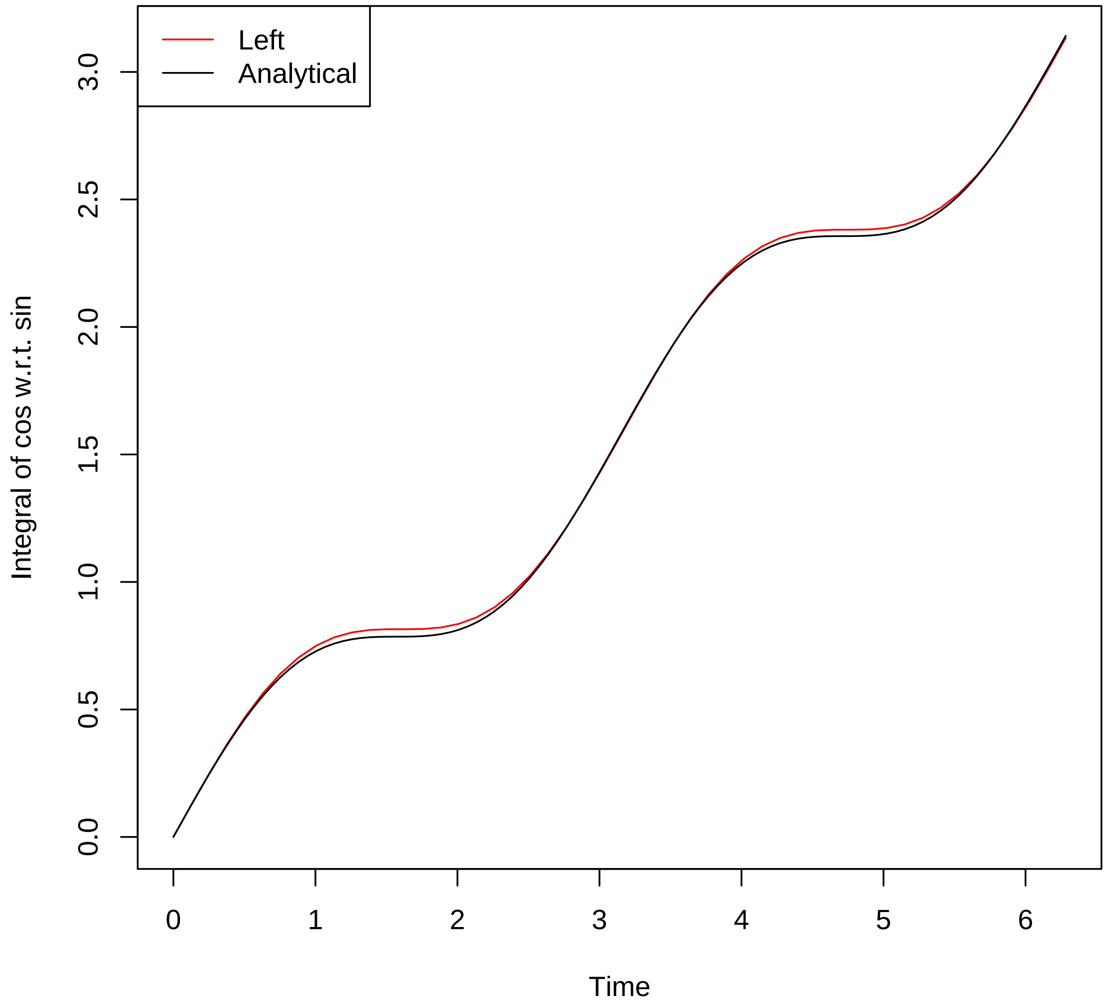
Establish the analytical result

$$\int_0^T \cos t \; d\sin t = \frac{1}{2}T + \frac{1}{4}\sin 2T$$

and compare with the numerical result. *Hint:* Use $d\sin t = \cos t \; dt$ and $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$.

Solution: The following code tests the integral.

```
## Time grid
tvec <- seq(0,2*pi,length=51)

## Compute integral
I <- itointegral(cos(tvec),sin(tvec))

## Plots
plot(tvec,I,type="l",col="red",
     xlab="Time",ylab="Integral of cos w.r.t. sin")
plot(function(t)t/2+sin(2*t)/4,col="black",from=0,to=2*pi,add=TRUE)

legend(x="topleft",leg=c("Left","Analytical"),
       lty="solid",col=c("red","black"))
```



To establish the analytical result, use

$$\begin{aligned}\int_0^T \cos t \ d\sin t &= \int_0^T \cos t \ \cos t \ dt \\ &= \int_0^T \frac{1}{2}(1+\cos 2T) \ dt \\ &= \frac{1}{2}[t]_0^T + \frac{1}{4}[\sin 2t]_0^T \\ &= \frac{1}{2}T + \frac{1}{4}\sin 2T\end{aligned}$$

Integrating Brownian motion w.r.t. itself

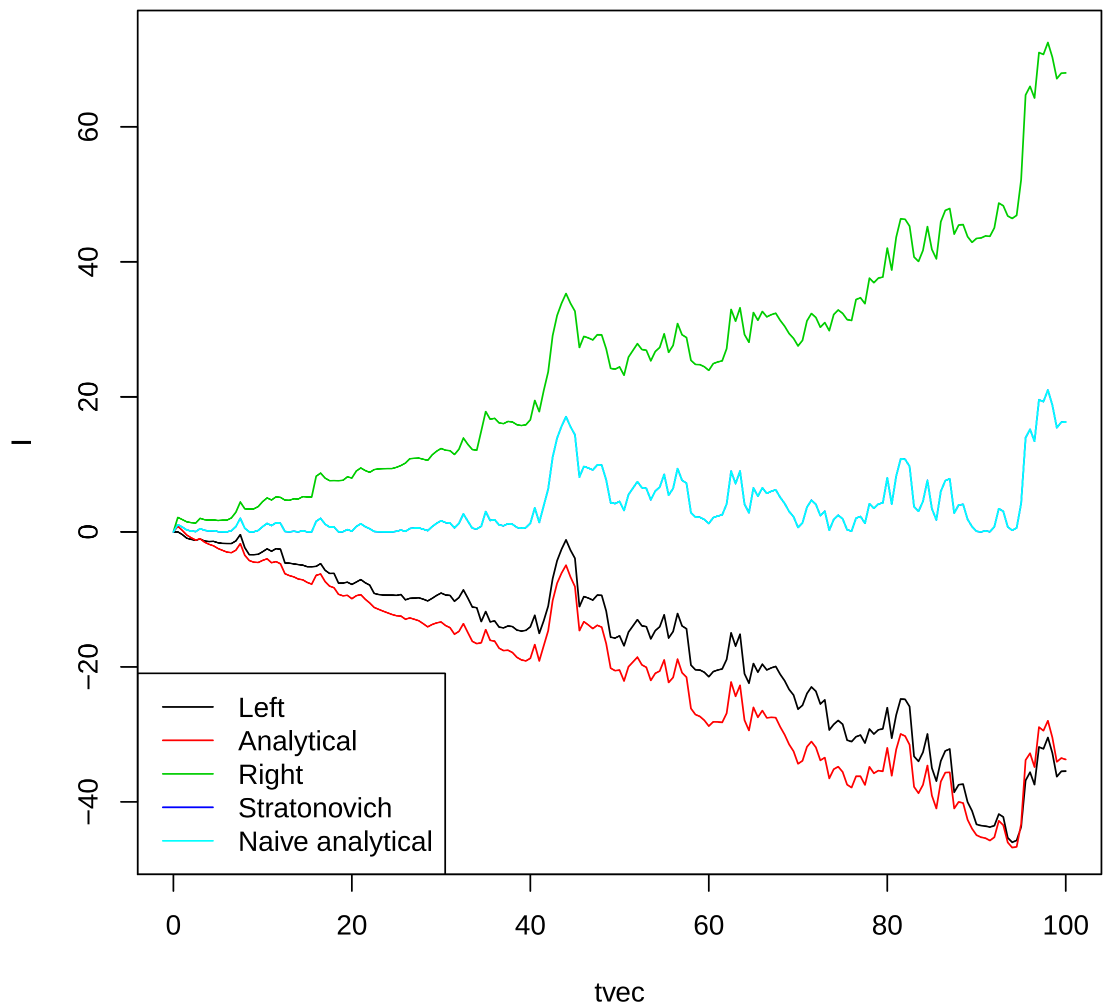
Question 2: Re-create figure 6.5 in the notes (p. 131). Specifically, apply your integrating function to compute one realization of the Itô integral

$$I_t = \int_0^t B_s \, dB_s$$

for $t \in \{0, 0.5, 1.0, \ldots, 100\}$, where $\{B_s : 0 \le s \le 100\}$ is Brownian motion. Plot the result as function of t. Compare with the analytical result $I_t = \frac{1}{2}(B_t^2 - t)$.

Solution: The following code simulates a sample path of Brownian motion (as we did in past weeks), computes the integral, and plots the result along with the analytical result. The code also includes the answers to the subsequent question, i.e. the right hand approximation and the Stratonovich results.

```
## Time grid
tvec <- seq(0,100,0.5) ## Use a finer time step for increased accuracy

## Simulate Brownian motion
rBM <- function (times) cumsum(rnorm(length(times),sd=sqrt(diff(c(0,times)))))
Bvec <- rBM(tvec)

## Compute integral
I <- itointegral(Bvec,Bvec)

## Compute also "right hand rule"
rightintegral <- function(G,B) c(0,cumsum(tail(G,-1)*diff(B)))
Ir <- rightintegral(Bvec,Bvec)

## ... and Stratonovich approximation
Is <- (I+Ir)/2

## Plots
plot(tvec,I,type="l",col=1,ylim=range(c(I,Ir)))
lines(tvec,0.5*(Bvec^2-tvec),type="l",col=2)
lines(tvec,Ir,type="l",col=3)
lines(tvec,Is,type="l",col=4)
lines(tvec,0.5*Bvec^2,type="l",col=5)

legend(x="bottomleft",leg=c("Left","Analytical","Right","Stratonovich","Naive analytical"),
       lty="solid",col=1:5)
```



Note the reasonable agreement between the numerical Itō integral and the analytical result. There is some discretization error, which can be traced back to the difference between the discretized total variation of Brownian motion

$$\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2$$

and the limit, $t_n = 100$. The magnitude of the discretization error therefore also varies quite a bit between realizations.

Question 3: Write a modified integrator which computes the "right hand approximation" given by (for general integrand f and integrator g)

$$I_m^R = \sum_{i=1}^{m} f(t_i) \cdot [g(t_i) - g(t_{i-1})]$$

Add this integral to the plot. Furthermore, add the Stratonovich approximation

$$I_m^S = \sum_{i=1}^{m} \frac{1}{2}(f(t_{i-1}) + f(t_i)) \cdot [g(t_i) - g(t_{i-1})]$$

Finally, add the result that one could expect from deterministic calculus, i.e. $\frac{1}{2}B_t^2$.

Solution: The answers to this question have already been included in the previous answers. Note that the “right hand” integral lies above the “left hand”; the difference between the two is exactly t. Note that the “naive analytical results” $\frac{1}{2}B_t^2$ lies centered between the “left hand” and the “right hand” integrals and agrees with the Stratonovich interpretation.

Simulating an SDE with the Euler-Maruyama method

Consider the Cox-Ingersoll-Ross process (also termed the Square Root Process)

$$dX_t = \lambda(\xi - X_t)\ dt + \gamma\sqrt{|X_t|}\ dB_t$$

Question 4: Simulate the process using the Euler-Maruyama method. Take parameters $\lambda = 1/2$, $\xi = 2$, $\gamma = 1$. Use an initial condition of $X_0 = \xi$ and simulate the process on the time interval $[0, 100]$ with a time step of $h = 0.01$. Plot the solution $\{X_t\}$ versus time t.

Solution: The following piece of code sets up the system, the parameters, the vector of time points, and simulates a sample path of the driving Brownian motion

```
## Parameters and model equations
lambda <- 0.5
xi <- 2
gamma <- 1

f <- function(x) lambda*(xi-x)
g <- function(x) gamma*sqrt(abs(x))

## Initial condition
x0 <- xi

## Time points
tvec <- seq(0,100,0.01)
dt <- diff(tvec)

## Brownian motion
B <- rBM(tvec)
dB <- diff(B)

## Setup array for solution
X <- numeric(length(tvec))
X[1] <- x0
```

The core in the simulation is the following Euler stepping:

```
## Simulation using the Euler stepping
for(i in 1:(length(tvec)-1))
    X[i+1] <- X[i] + f(X[i])*dt[i] + g(X[i])*dB[i]
```

See the plot in the following.

Effect of the time step

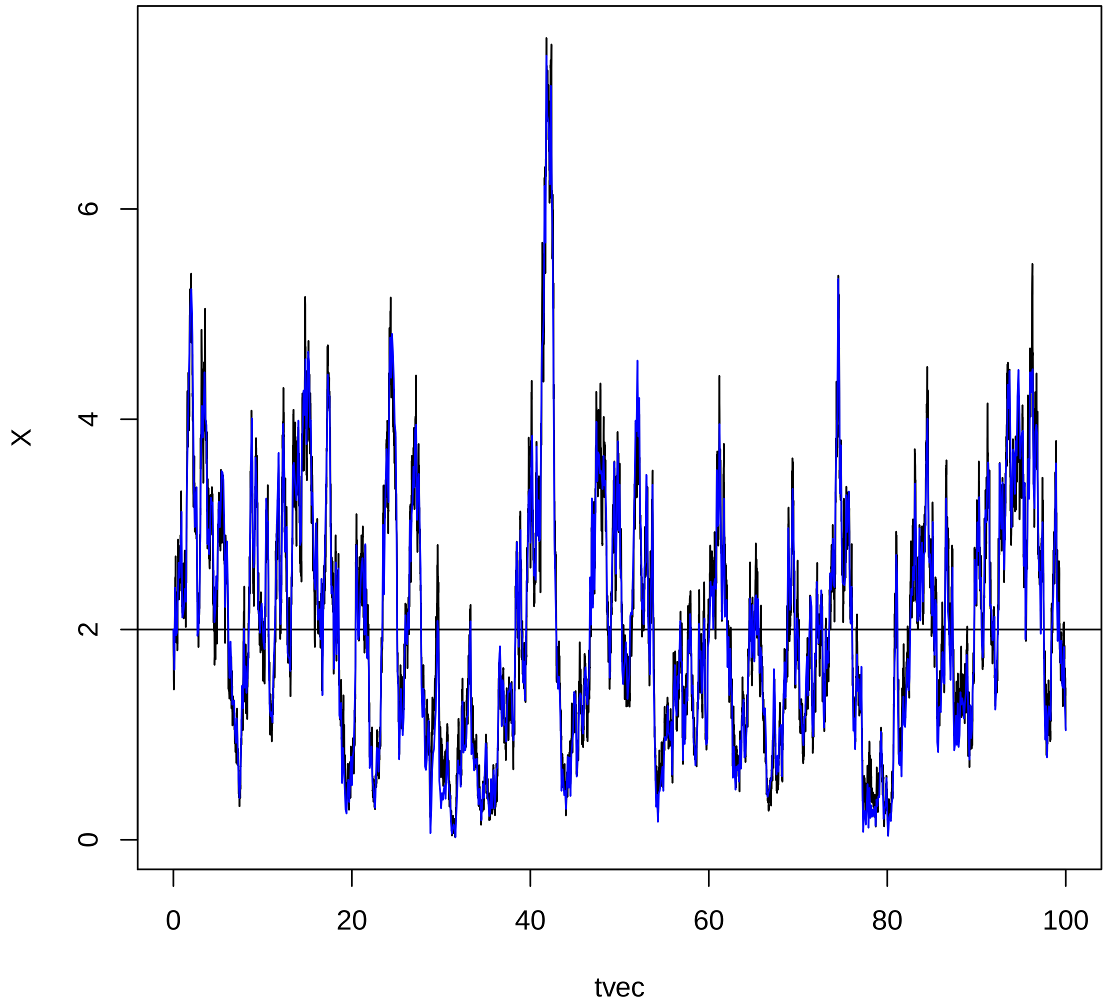
Question 5: Repeat the simulation with the same sample path of Brownian motion, but now using a time step of $10h$. Verify that both the long term and the short term behavior is roughly the same.

Solution: To repeat with coarser time steps, we first sub-sample both the time vector, and the Brownian motion. We then repeat the simulation.

```
## Repeat with coarser time steps
t2 <- tvec[seq(1,length(tvec),10)]
dt2 <- diff(t2)

B2 <- B[seq(1,length(B),10)]
dB2 <- diff(B2)

X2 <- numeric(length(t2))
X2[1] <- x0

for(i in 1:(length(t2)-1))
    X2[i+1] <- X2[i] + f(X2[i])*dt2[i] + g(X2[i])*dB2[i]
```

We now plot the resulting trajectories; both with coarse and fine time steps. We include also the stationary expectation of X_t. A close inspection reveals that there is some difference between the coarse and the fine time steps.

```
## Plot the sample path
plot(tvec,X,type="l")

## Add line for the expectation
abline(h=xi)

lines(t2,X2,type="l",col="blue")
```



Verification of the integral version

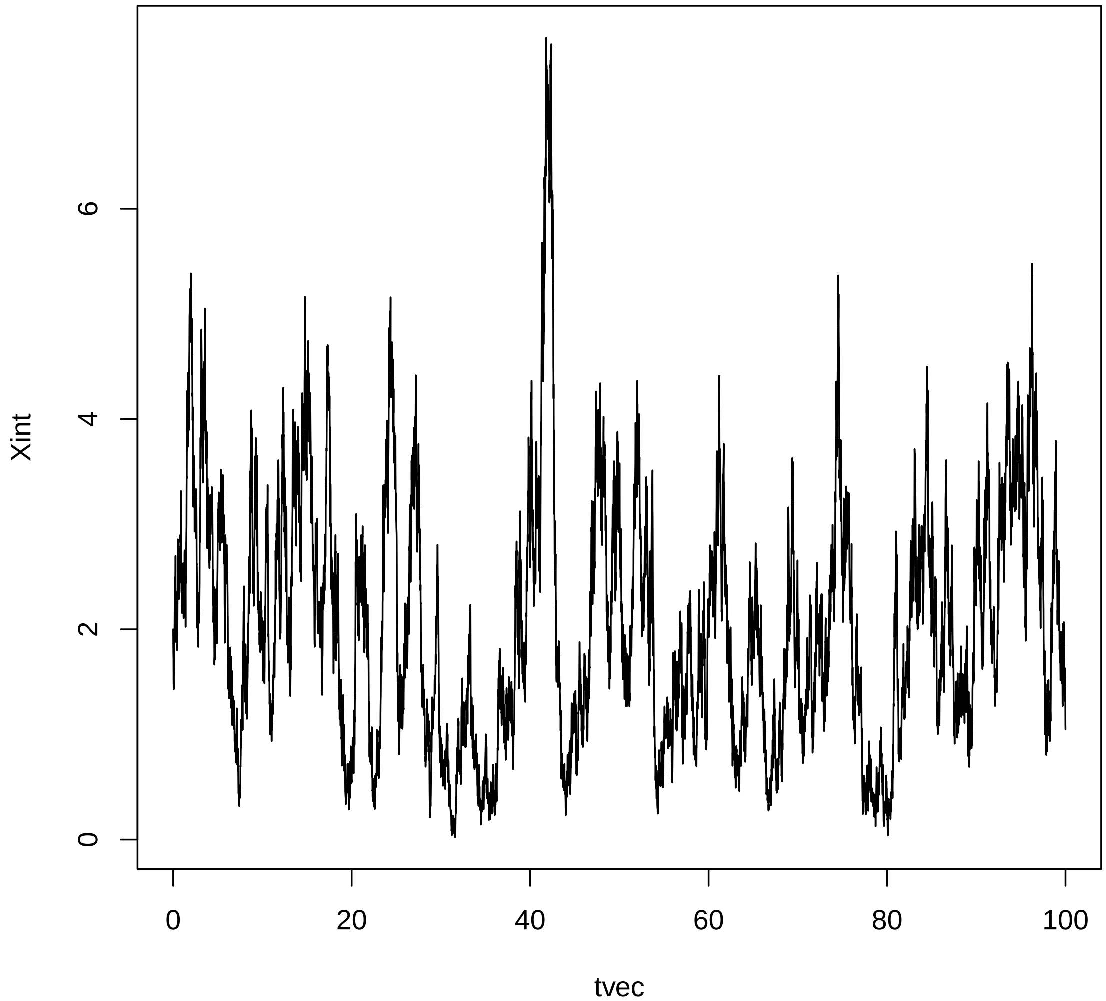
Question 6: Verify numerically that the simulated sample paths of $\{B_t\}$ and $\{X_t\}$ satisfy the integral version

$$X_t = X_0 + \int_0^t \lambda(\xi - X_s)\ ds + \int_0^t \gamma\sqrt{|X_s|}\ dB_s$$

for each $t = 0,\ 0.01,\ 0.02, \ldots, 100$. Here, you use the numerical integration routine that you made in question 1, and the verification should be by plotting the sum of the integrals on top of the Euler solution.

Solution: The following code computes the integrals and adds the sum to the plot:

```
## Compare with the integral version
Xint <- x0+itointegral(f(X),tvec) + itointegral(g(X),B)
plot(tvec,Xint,type="l")
```



It appears to be quite similar. We can see just how similar:

```
print(max(abs(Xint-X)))

## [1] 3.952394e-14
```

The difference is not due to discretization error, but due to roundoff (the finite precision arithmetic). Both the Euler-Maruyama solution, and the numerical Itō integration, are subject to time discretization error, but these discretization errors are *exactly* the same.

Mean and variance of $\int_0^t B_s \, dB_s$

(Exercise 6.9 in the notes)

Question 7: Consider the Itō integral

$$I_t = \int_0^t B_s \, dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Use the properties of Gaussian variables (exercise 3.13) to determine the mean and variance of I_t, and verify that this agrees with the properties of Itō integrals.

Solution: See the solution in the notes.