DTU Compute

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UHT/uht

Exercise 8: The Kolmogorov equations

Consider the Cox-Ingersoll-Ross process

$$dX_t = \lambda(\xi - X_t)\ dt + \gamma\sqrt{|X_t|}\ dB_t$$

where λ, ξ and γ are positive parameters.

The stationary distribution

Question 1: (Compare exercise 9.2)

Write the advection-diffusion form of the forward Kolmogorov equation, i.e. determine the advection $u(x)$ and the diffusivity $D(x)$. Then, show that in stationarity, X_t is Gamma distributed with rate parameter $\omega = 2\lambda/\gamma^2$ and shape parameter $\nu = 2\lambda\xi/\gamma^2$, i.e. density

$$\phi(x) = \frac{\omega^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\omega x}\ .$$

Solution: We have (when $x > 0$)

$$D(x) = \frac{1}{2}g^2(x) = \frac{1}{2}\gamma^2 x$$

$$u(x) = f(x) - D'(x) = \lambda(\xi - x) - \frac{1}{2}\gamma^2$$

To find the stationary distribution using (9.6) on p. 209), we first identify the antiderivative

$$\int_{x_0}^{x} \frac{u(y)}{D(y)}\ dy = \int_{x_0}^{x} \left[2\frac{\lambda(\xi-y)}{\gamma^2}\frac{1}{y} - \frac{1}{y}\right]\ dy = \left(\frac{2\lambda\xi}{\gamma^2} - 1\right)\log(x/x_0) - \frac{2\lambda}{\gamma^2}(x - x_0)$$

so that the un-normalized density is

$$\phi(x) = \frac{1}{Z} x^{2\lambda\xi/\gamma^2 - 1} e^{-2\lambda/\gamma^2\ x}$$

We see that this has the form of a Gamma distribution with the rate and shape parameter as claimed. The normalization constant follows from the properties of the Gamma distribution. *Note:* This could also be found from the drift and intensity (f, g) using (9.7); the calculations are essentially the same.

The mean and mode in the stationary distribution

Question 2: Using the properties of the Gamma distribution, find the stationary expectation $\mathbf{E}X_t$ and check that it agrees with what we already know about the Cox-Ingersoll-Ross process.

Solution: For a Gamma distributed random variable with rate parameter ω and shape parameter ν, the expectation is ν/ω (c.f. e.g. Wikipedia). Here, we find a stationary expectation $\mathbf{E}X_t = \xi$. We knew this already (exercise 6.12).

Question 3: For a general stationary scalar advection-diffusion equation $(u\pi - D\pi')' = 0$, show that stationary points of π are those where $u = 0$. Use this to find the mode of the stationary distribution in the Cox-Ingersoll-Ross process from the previous question. Compare with what Wikipedia (or your favorite reference) has to say about Gamma distributions.

Solution: The stationary distribution, in general, satisfies

$$u\pi - D\pi' = 0$$

Therefore, at a stationary point x we have $\pi'(x) = 0$ and therefore also $u(x) = 0$ (assuming that $\pi(x) \neq 0$). So the mode of the stationary distribution is characterized by the advective flow vanishing. For the Cox-Ingersoll-Ross process, this criterion becomes

$$u(x^*) = \lambda(\xi - x^*) - \frac{1}{2}\gamma^2 = 0$$

or

$$x^* = \xi - \frac{\gamma^2}{2\lambda}.$$

Note that the mode is lower than the equilibrium of the drift, i.e. ξ; the noise shifts the mode to the left.

According to the properties of the Gamma distribution, the mode is found at

$$(\nu - 1)/\omega = \frac{2\lambda\xi/\gamma^2 - 1}{2\lambda/\gamma^2} = \xi - \frac{\gamma^2}{2\lambda}.$$

We see that the two results agree (as they should).

Numerical analysis

In the following we consider the Cox-Ingersoll-Ross model with $\lambda = 1/2$, $\xi = 2$, $\gamma = 1$. We truncate the real axis, so that we only consider the process on the interval $[0, H]$. Here, H is for you to choose, so that the process rarely exceeds H. Use trial-and-error or and informed guess based on the previous. We then use reflecting boundary boundary conditions at 0 and H.

Question 4: Use the supplied code `fvade.R` or `fvade.m` to discretize the forward Kolmogorov equation to a system of ordinary differential equations. Use a uniform spatial grid. First, try it with 5 or 6 grid points, so that you can inspect the entire generator. Then, for actual computations, use at least 100 grid cells. Compute the stationary probability density and plot it. Include in the plot the analytical solution from the previous.

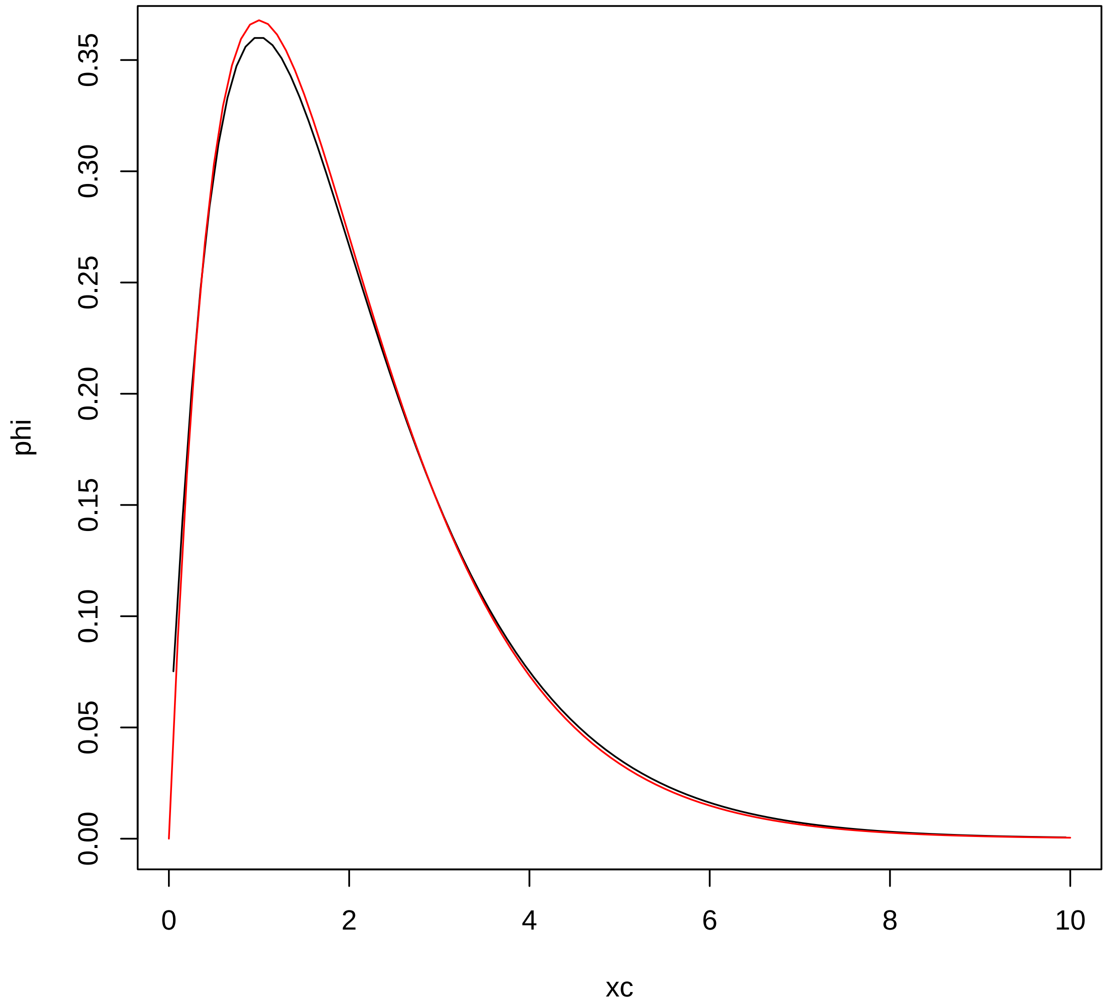
Solution:

```r
require(SDEtools)
require(fields)
```

```r
## Define dynamics

## Drift and intensity
f <- function(x) lambda * (xi - x)
g <- function(x) gamma*sqrt(abs(x))

## Diffusivity and its spatial derivative
D <- function(x) 0.5*gamma^2*x
Dp <- function(x) 0.5*gamma^2

## Advective flow field
u <- function(x) f(x) - Dp(x)

## Parameters
xi <- 2
gamma <- 1
lambda <- 1/2

## Spatial grid
xmax <- 10
xv <- seq(0,xmax,length=101)
dx <- diff(xv)
xc <- xv[-1] - 0.5*dx

## Discretize the generator
G <- fvade(u,D,xv,'r')
```

```
## Loading required package: Matrix

##
## Attaching package: 'Matrix'

## The following object is masked from 'package:spam':
##
##     det
```

```r
## Compute the stationary density
pi <- StationaryDistribution(G)
phi <- pi / dx
plot(xc,phi,type="l")
plot(function(x)dgamma(x,rate=2*lambda/gamma^2,shape=2*lambda*xi/gamma^2),
     from=0,to=xmax,add=TRUE,col="red")
```



We now pursue transient solutions to the forward Kolmogorov equation. We use the initial condition $X_0 = \xi/4$ for the SDE, i.e. the initial condition for the Kolmogorov equation is a Dirac delta at X_0. Solve the forward Kolmogorov equation using the matrix exponentiel:

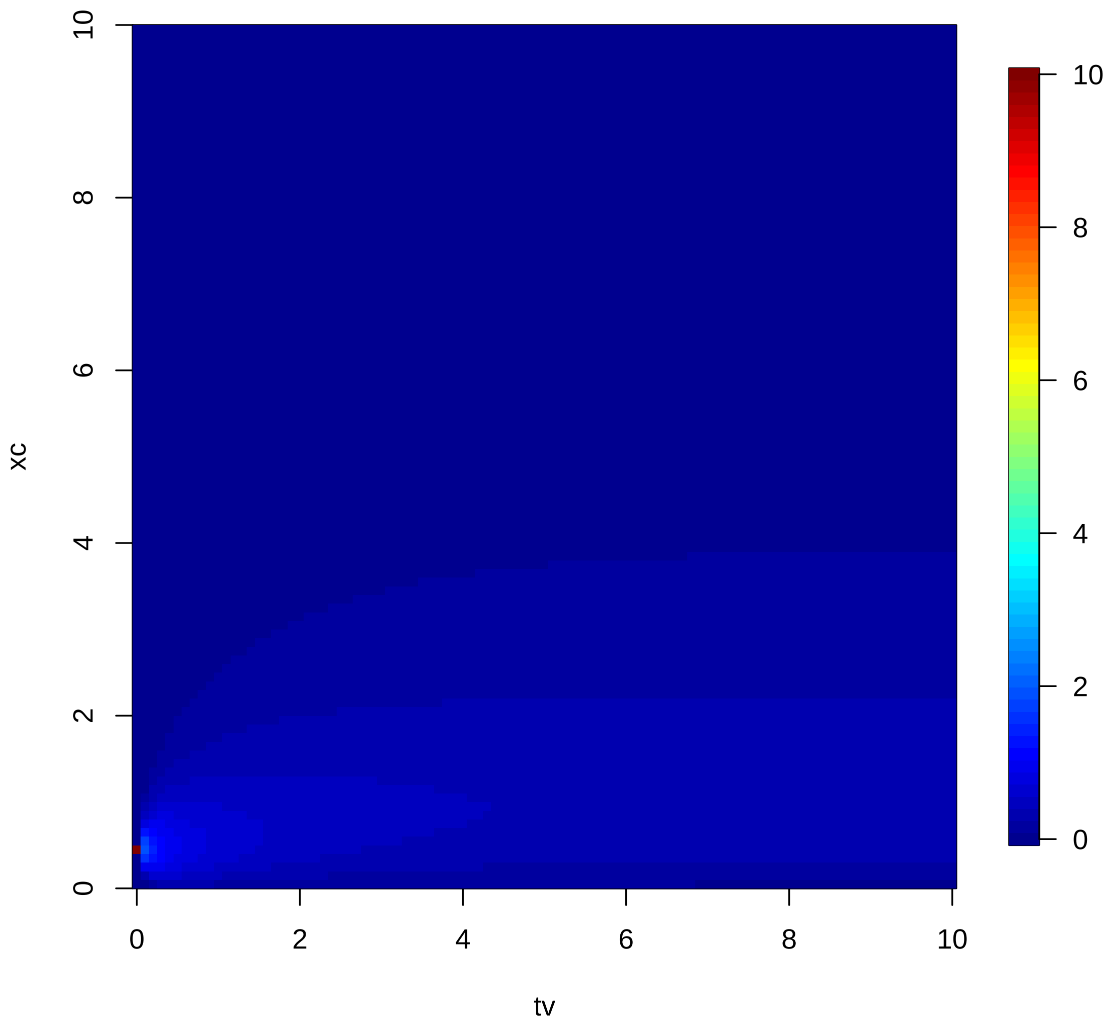
$$\phi(t) = \phi(0)\exp(Gt)$$

(or, if you prefer, another method).

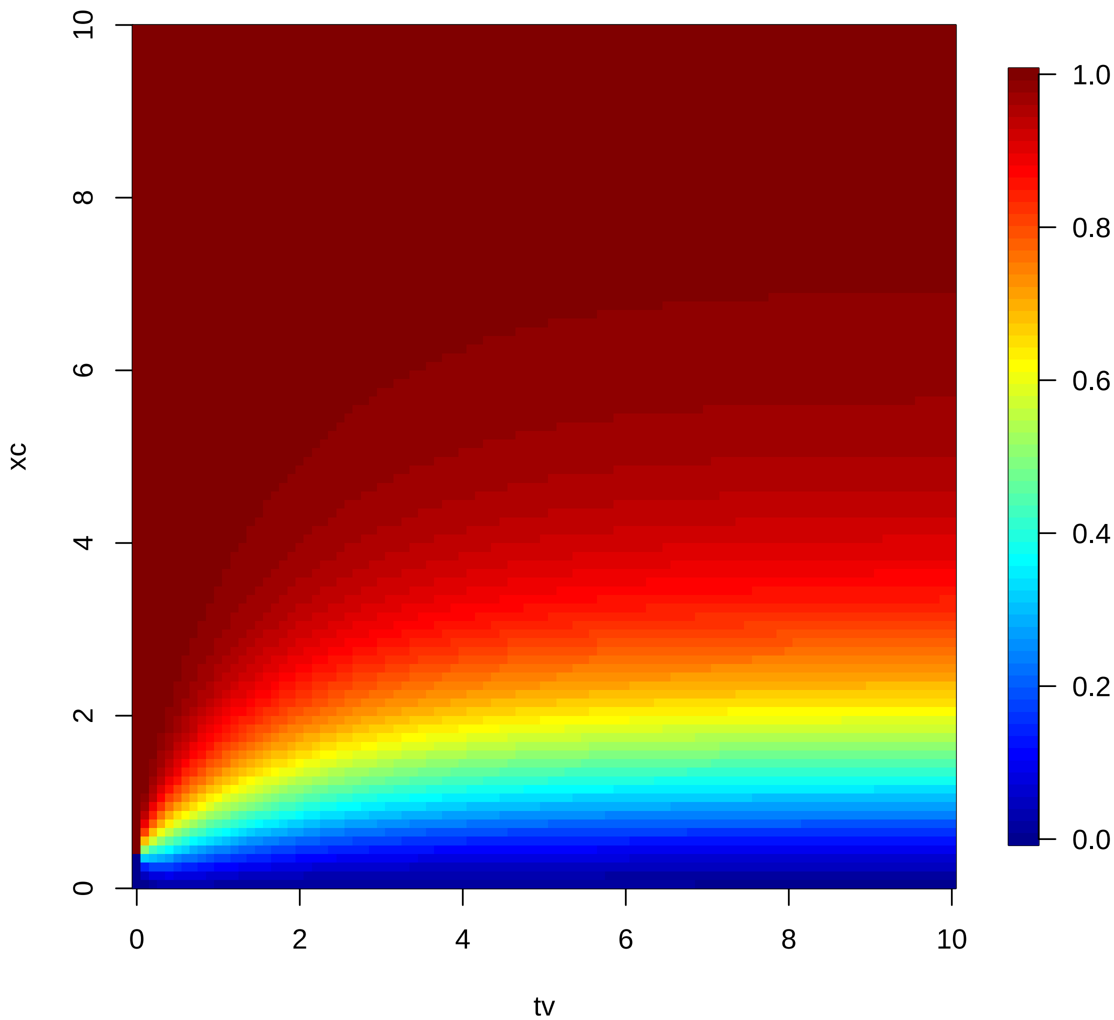
Question 5: Solve the equation on a sufficiently large time interval $\{0, \Delta t, 2\Delta t, \ldots, T\}$ so the convergence to the steady-state solution is visible. Plot the solution as a pseudocolor image. Since p.d.f.'s are badly scaled, plot also the c.d.f.

Solution:

```r
## Transients

## Initial condition for the SDE
x0 <- xi/4

## Initial condition for the FKE
phi0 <- numeric(length(xc))
phi0[sum(xc<x0)] <- 1

## Time grid
tv <- seq(0,10,0.1)

## Solve the FKE
PHI <- sapply(tv,function(t) as.numeric(phi0 %*% expm(G*t)))/dx

image.plot(tv,xc,t(PHI))
```



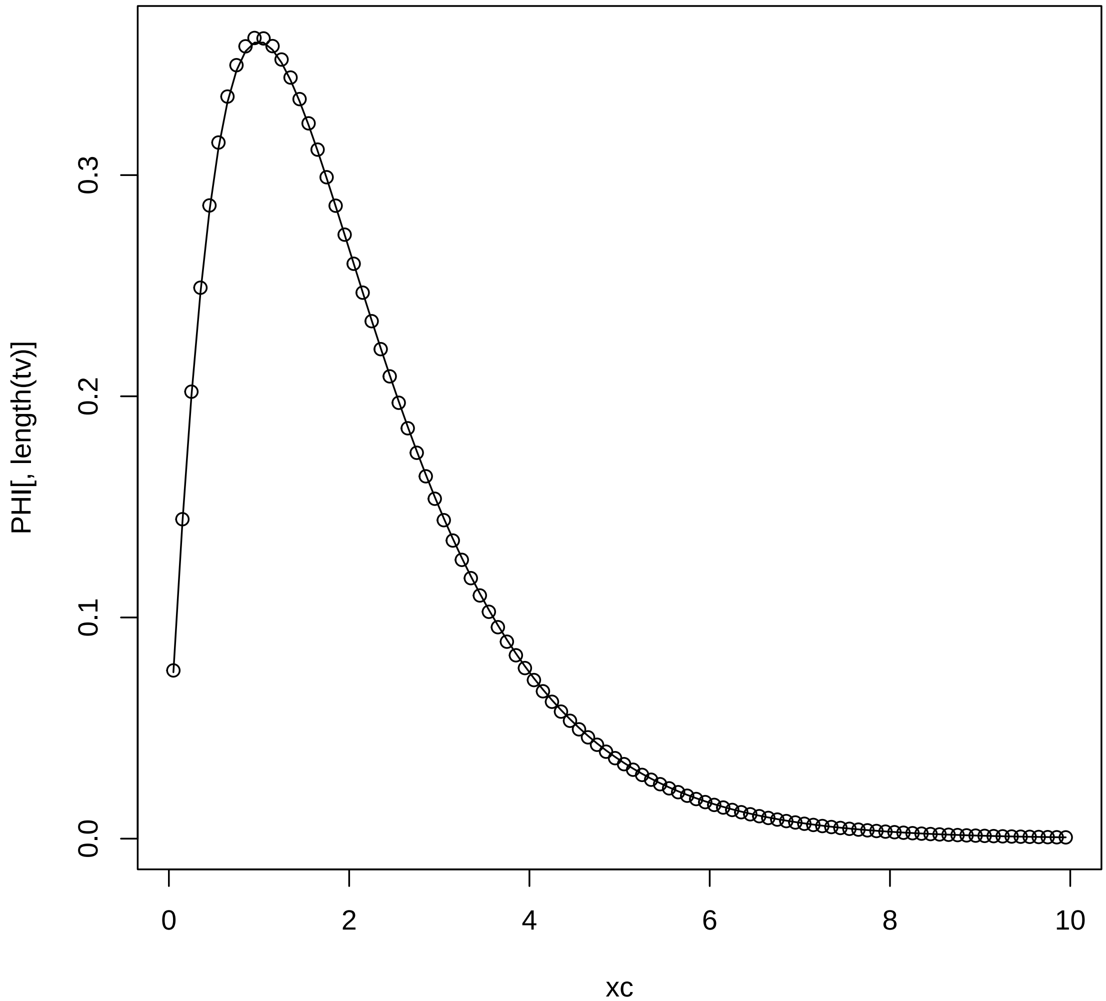

```
CDF <- apply(PHI*dx,2,cumsum)

image.plot(tv,xc,t(CDF))
```



Question 6: Make a line plot of the p.d.f. at time T and compare with the stationary p.d.f. found in the previous.

Solution:

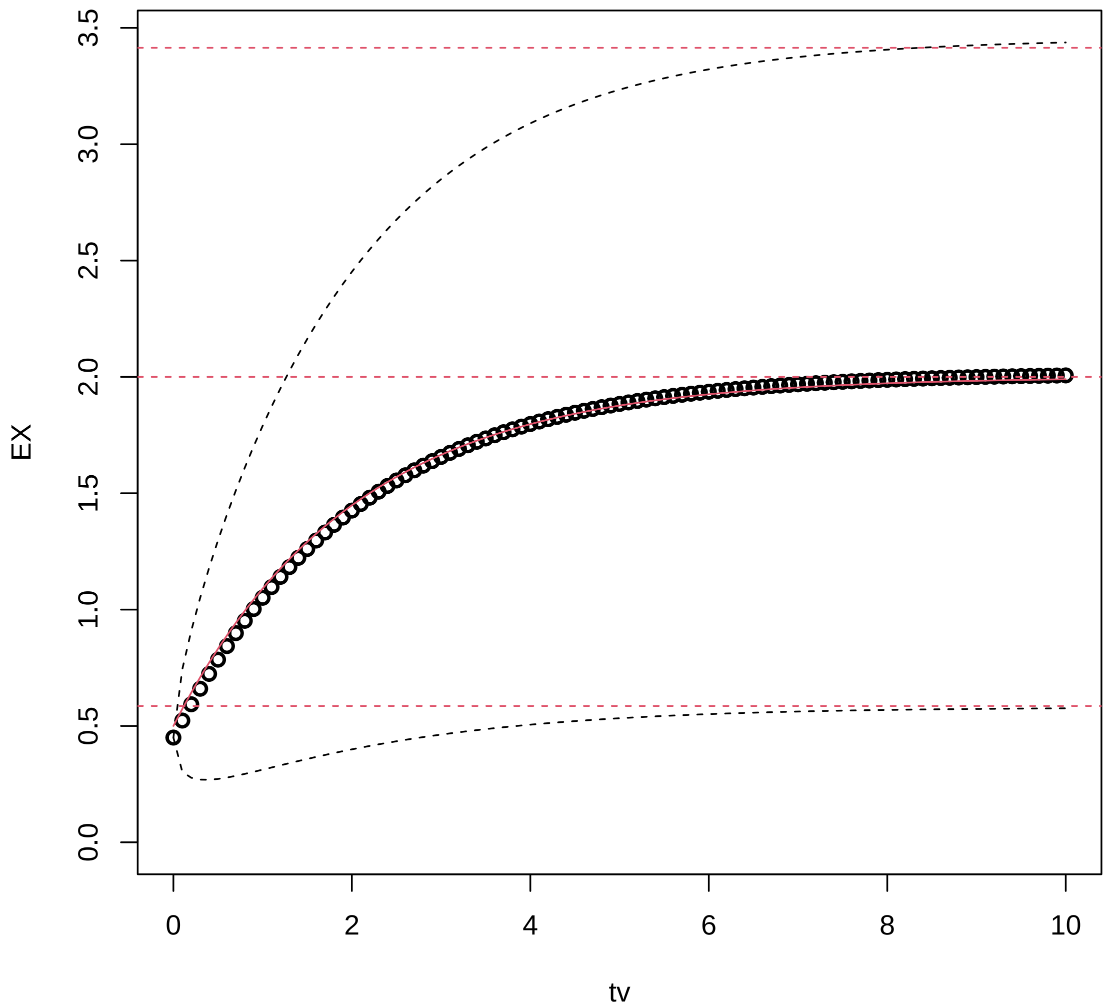
```
plot(xc,PHI[,length(tv)])
lines(xc,phi)
```



Question 7: From the p.d.f., compute and plot the expectation as a function of time, and the mean $\pm$ the standard deviation. Include the mean computed using the result from exercise 8.5 (p. 192). Include also the stationary mean and variance from the same exercise.

Solution:

```
EX <- apply(PHI*dx*xc,2,sum)

EX2 <- apply(PHI*dx*xc^2,2,sum)
VX <- EX2 - EX^2
sX <- sqrt(VX)

plot(tv,EX,lwd=2,ylim=c(0,max(EX+sX)))
lines(tv,EX+sX,lty="dashed")
lines(tv,EX-sX,lty="dashed")

EXa <- xi+(x0-xi)*exp(-lambda*tv)
lines(tv,EXa,col=2)

sXa <- sqrt(gamma^2*xi/2/lambda)

abline(h=xi,col=2,lty=2)
abline(h=xi+sXa,col=2,lty=2)
abline(h=xi-sXa,col=2,lty=2)
```



The backward Kolmogorov equation

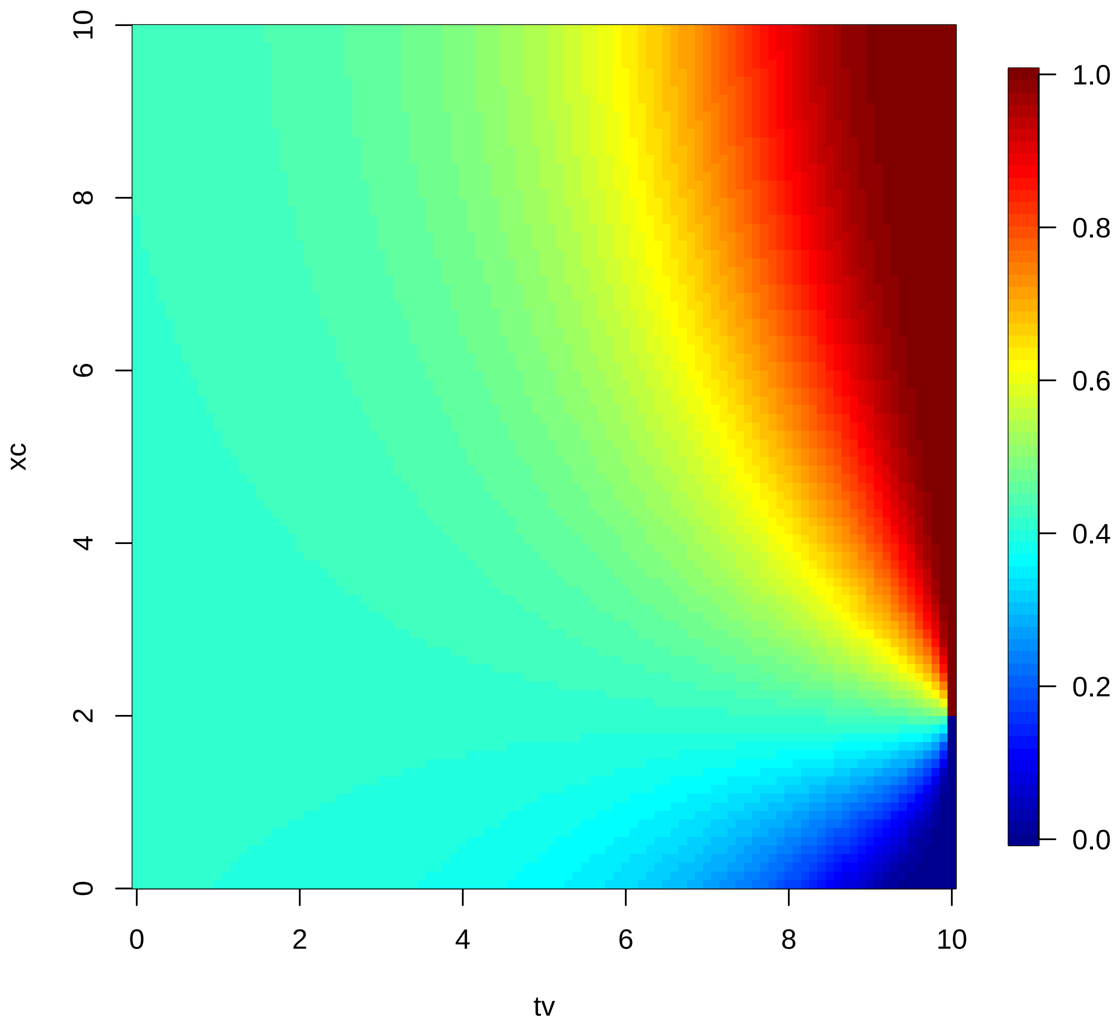
Question 8: Use the backward Kolmogorov equation to find

$$\mathbb{P}^{X_t=x}(X_T \geq 2)$$

for the same values of x and $t < T$ as in the previous. *Hint:* Define $h(x) = \mathbf{1}(()x \geq 2)$; then we aim to find $k(x, s) = \mathbf{E}^{X_s=x} h(X_T)$. This function k satisfies $\dot{k} + Lk = 0$.

Solution:

```
h <- (xc>=2)
T <- tail(tv,1)
k <- sapply(tv,function(t) as.numeric(expm(G*(T-t)) %*% h))
image.plot(tv,xc,t(k))
```



Comparison with the Euler method

Question 9: Simulate the Cox-Ingersoll-Ross process with the Euler method. Take the same model parameters as in the previous. Choose sufficiently small time steps for accuracy, and sufficiently long time series that the process appears stationary. Plot the histogram of the process and compare with the stationary solution from the previous.

Solution:

```
tsim <- seq(0,1000,0.1)

sim <- euler(f,g,tsim,x0,p=abs)
hist(sim$X,freq=FALSE,breaks=50)
lines(xc,phi)
```