DTU Compute

02425
November 24, 2022
UHT/uht

Exercise 12: Stability

Consider the stochastic Pella-Tomlinson model

$$dX_t = rX_t(1 - (X_t/K)^p)\ dt + \sigma X_t\ dB_t, \quad X_0 = x \quad .$$

where $r > 0$, $K > 0$, $\sigma > 0$, $p > 0$ and $x \geq 0$. This is a generalization of the stochastic logistic growth model, which describes growth of biological populations.

Question 1 Warm-up: Sketch the drift function. Convince yourself that p is a "shape" parameter in the drift, while r and K are "scale" parameters. Verify that the process $\{X_t\}$ with $X_t = 0$ satisfies the equation when $x = 0$.

Question 2 Sensitivity of the zero solution: Pose the sensitivity equation of the zero solution and state its solution. State the Lyapunov exponent of the zero solution and state the condition on the parameters r, K, σ and p for the Lyapunov exponent to be negative.

Question 3 Simulation of stable and unstable scenarios: Choose parameters such that the Lyapounov exponent is negative. Simulate the system - does the simulation support the conclusion that the zero solution is stable? Repeat for a parameter combination where the Lyapunov exponent is positive.

Question 4 Stochastic Lyapunov exponent: For the unstable parameter from the previous, simulate the system and the sensitivity equation simultaneously. Take $X_0 = 0.01K$. Give a (rough) estimate of the stochastic Lyapunov exponent.

Question 5 Stochastic Lyapunov functions: Follow example 12.7.1 (p. 232) and use theorem 12.7.1 to show that $V(x) = x^q$ is a stochastic Lyapunov function which verifies stochastic stability of the zero solution, provided $0 < q < 1 - 2r/\sigma^2$. Compare the stability condition with what you found in question 2.

Question 6 Stability in the mean square: Try to show stability in the mean square with a candidate Lyapunov function $V(x) = x^2$, and show that it is never possible. Simulate the system for parameters where the zero solution is stochastically stable, but not stable in the mean square, and explain how these contradicting properties are visible in the simulation. *Note:* The stability theorem, which we have covered in class, only gives a *necessary* condition for stability. There exist *converse stability theorems* which can be used to shows that the zero solution is not stable.

Question 7 Boundedness: (Compare exercise 12.7.) Consider the candidate Lyapunov function $V(x) = x - \log x$. Show that V is positive and proper and that $LV(x)$ is bounded above and

negative when x is sufficiently large. (Re)state the condition for $LV(x)$ to be negative for sufficiently small $x > 0$. Conclude that, under this condition, there exists an interval $[a, b]$ with $0 < a < b$ which is positively recurrent.

Question 8 **The stationary distribution:** Solve the stationary forward Kolmogorov equation, at first neglecting normalization. *Hint: Remember detailed balance. Still,* `maple` *or similar is probably a good idea.* Then show that the solution can be normalized to a probability density function, if and only if $2r/\sigma^2 > 1$. *(Hint: The function x^q is integrable on any interval $(0, a]$ with $a > 0$, if and only if $q > -1$.)* Finally, compare the condition for existence of a stationary distribution with the stability conditions obtained in the previous.