DTU Compute

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November 4, 2022
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# Exercise 9: The Kolmogorov equations II

**Question 1 Random directions and the von Mises distribution:** (Compare exercise 9.7 in the notes) Consider the Itō stochastic differential equation

$$dX_t = -\sin X_t \; dt + \sigma \; dB_t,$$

which can be viewed as a random walk on the circle which is biased towards $X_t = 2n\pi$ for $n \in \mathbb{N}$. This is a popular model for random reorientations, when there is a preferred direction.

1. Write up the forward Kolmogorov equation and show that a stationary solution is the so-called *von Mises* (or Tikhonov) distribution $\rho(x) = Z^{-1} \exp(\kappa \cos x)$, where $\kappa = 2/\sigma^2$. *Note:* This $\rho$ integrates to 1 over an interval of length $2\pi$, so we consider the state $X_t$ an angle which is only well defined up to adding a multiple of $2\pi$.

2. Take $\sigma = 1$. Simulate the process starting at $x = 0$ over the time interval $t \in [0, 100]$. Plot the trajectory and the histogram of the state. Compare the histogram with the stationary distribution.

3. Discretize the generator on $x \in [-\pi, \pi]$ using periodic boundary conditions. Determine the stationary distribution numerically from the generator and compare it, graphically, with the results from the previous question. *Note:* Unless the spatial grid is very fine, some numerical diffusion stemming from the discretization will affect the numerical solution.

4. Estimate the autocovariance function of $\{\sin X_t\}$ from the time series (using a built-in routine in your favorite software environment) and plot it. Use a sufficient large number of lags until you can see how long it takes for the process to decorrelate.

5. Compute the aucovariance function numerically from the following formula:

$$\mathbf{E}[(h(X_0) - \mu)h(X_t)] = \int_{\mathbb{X}} \rho(x)(h(x) - \mu)[e^{Lt}h](x) \; dx.$$

   Here, we take $h = \sin$. We have $\mu = \mathbf{E}h(X_0) = \int_{\mathbb{X}} \rho(x)h(x) \; dx$. Add this autocovariance to the empirical plot from the previous.

6. Compute the slowest 3 eigenmodes of $L$ from the numerical discretization. Add to the plot of the autocovariance an exponentially decaying function $e^{-\lambda t}\mathbf{V} \sin X_0$ where $\lambda$ is the largest non-zero eigenvalue of $L$. Comment on the agreement. Then, in a different plot, plot the slowest 3 eigenfunctions of $L$ as well as of $L^*$ and describe their role.