Exercise 3: Solution

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Q1: Simulation

```
rBM <- function(t) cumsum(rnorm(length(t),mean=0,sd=sqrt(diff(c(0,t)))))

t <- c(0,0.5,1.5,2)
N <- 1000

B <- sapply(1:N,function(i)rBM(t))
print(apply(B,1,mean))
```

```
## [1] 0.00000000 0.01249073 0.03003401 0.02325187
```

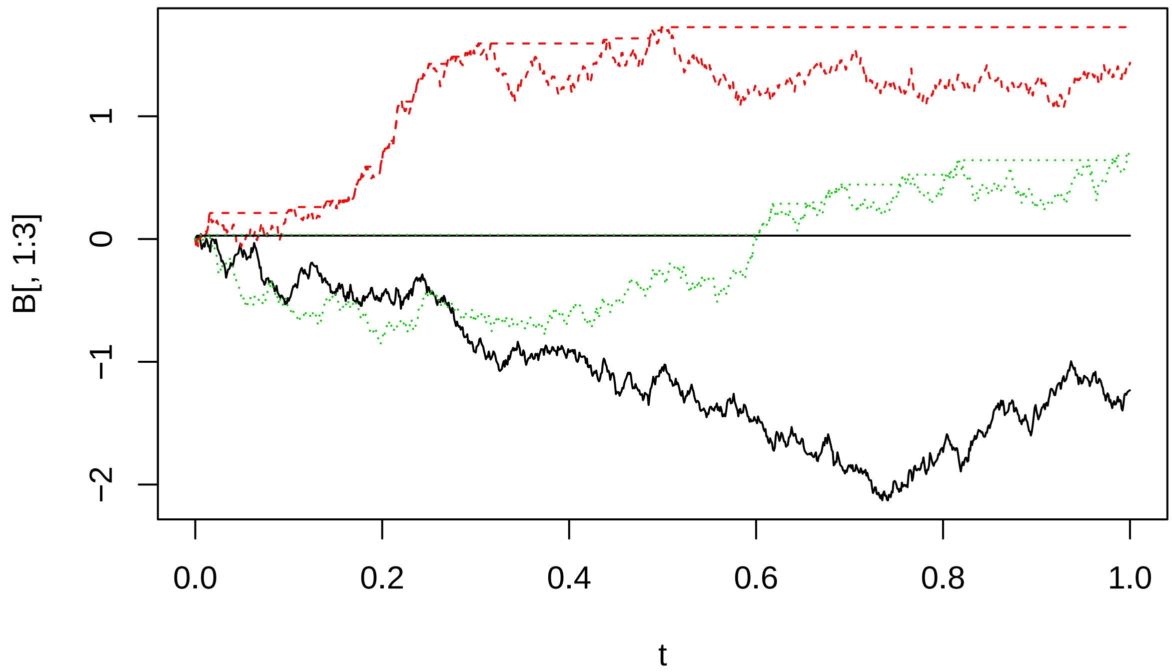
```
print(var(t(B)))
```

```
##      [,1]      [,2]      [,3]      [,4]
## [1,]    0 0.0000000 0.0000000 0.0000000
## [2,]    0 0.5017251 0.5146145 0.5246508
## [3,]    0 0.5146145 1.5604180 1.5517133
## [4,]    0 0.5246508 1.5517133 2.0573251
```

Q2: Extrema

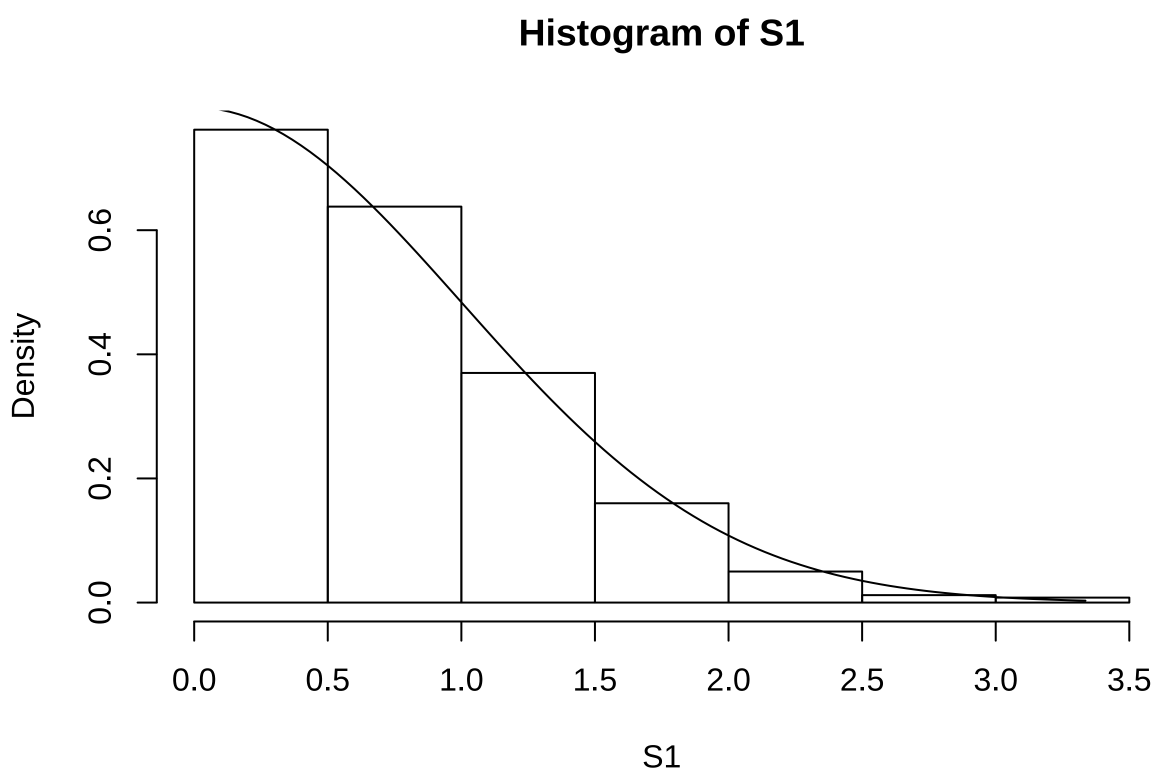
We first generate the sample paths and show the running max (even if it is not asked for).

```
N <- 1000
t <- seq(0,1,0.001)
B <- sapply(1:N,function(i)rBM(t))
S <- (apply(B,2,cummax))
S1 <- apply(B,2,max)
matplot(t,B[,1:3],type="l")
matplot(t,S[,1:3],type="l",add=TRUE)
```



We plot the histogram and compare with the pdf:

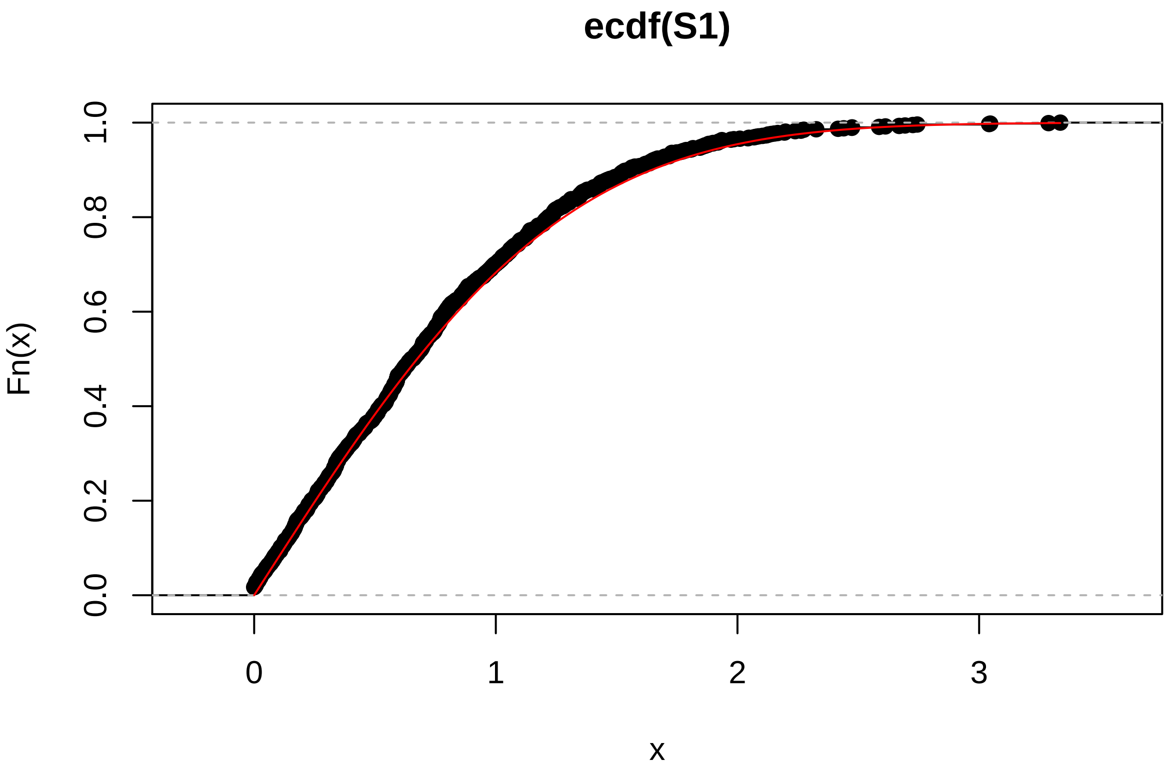
```
hist(S1,freq=FALSE)
Spdf <- function(x) 2*dnorm(x)
plot(Spdf,add=TRUE,from=0,to=max(S1))
```



We also do this for the empirical and theoretical cdf:

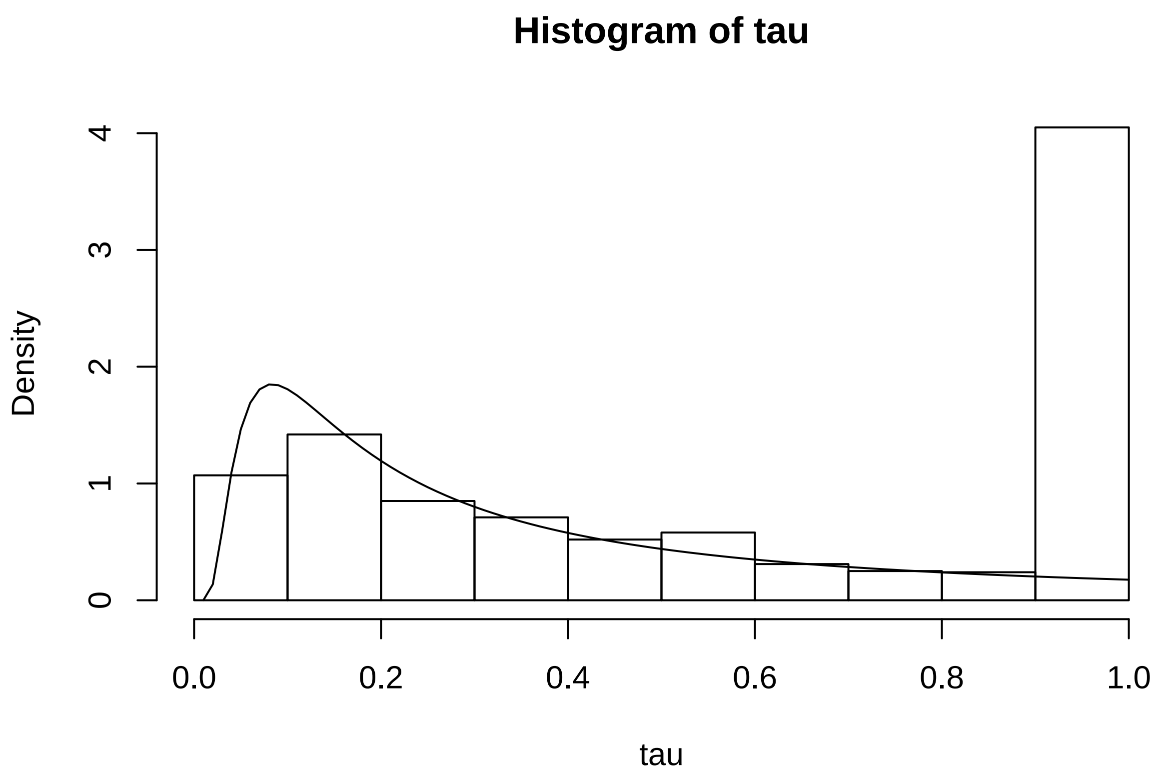
```
plot(ecdf(S1))
Scdf <- function(x) 2*pnorm(x)-1
```

```
plot(Scdf,add=TRUE,from=0,to=max(S1),col="red")
```



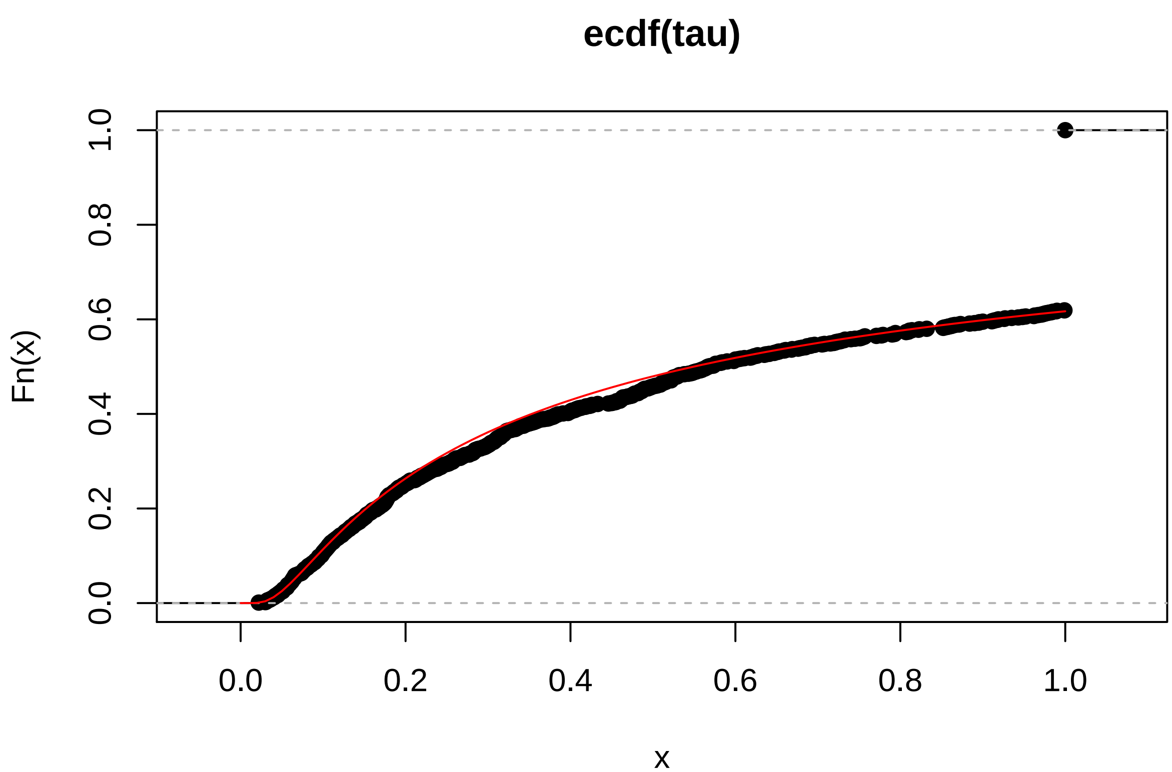
We compute *tau* for each sample path (a little coarsely) and plot the empirical and theoretical p.d.f.:

```
s <- 0.5
tau <- apply(S,2,function(x)t[sum(x<s)])
hist(tau,freq=FALSE)
taucdf <- function(t)2-2*pnorm(s/sqrt(t))
taupdf <- function(t)dnorm(s/sqrt(t))*s*t^(-3/2)
curve(taupdf,add=TRUE,from=0,to=max(tau))
```



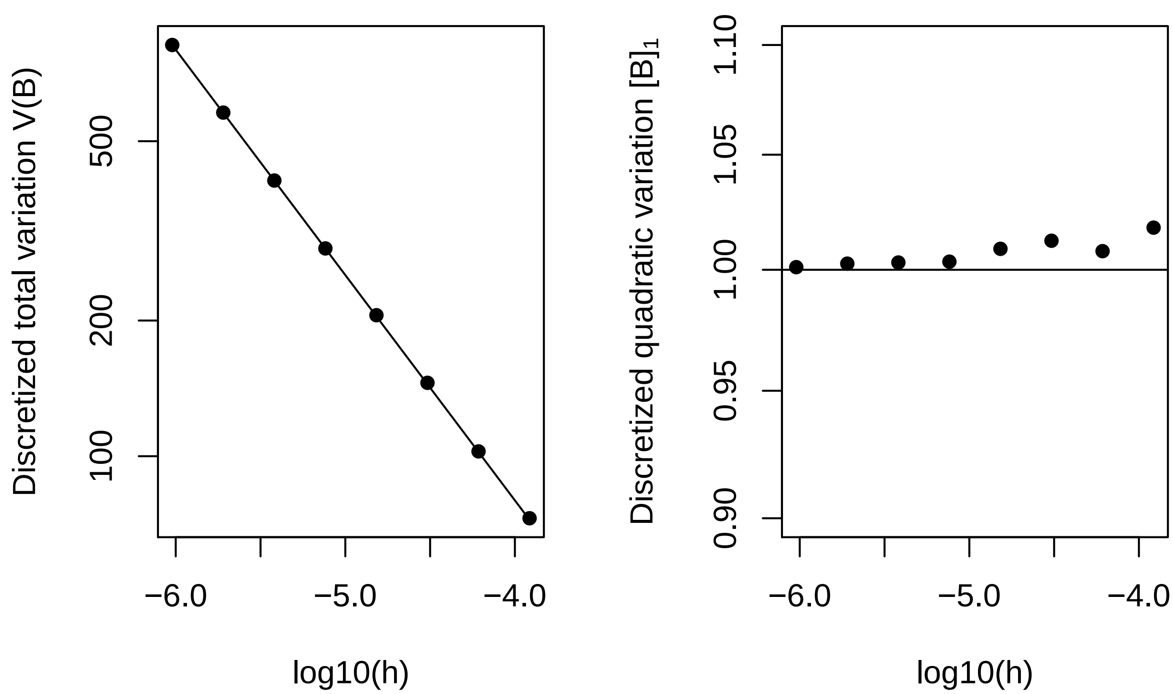
... repeat for empirical and theoretical c.d.f.:

```
plot(ecdf(tau))
curve(taucdf,from=0,to=max(tau),add=TRUE,col="red")
```



Total and quadratic variation

```
T <- 1

N <- 2^20
h <- T/N

Ndouble <- 8

B <- rBM(seq(0,T,h))
dB <- diff(B)

var <- array(0,c(Ndouble,2))

for(i in 1:Ndouble)
    {
        var[i,1] <- sum(abs(dB))
        var[i,2] <- sum(dB^2)
        dB <- apply(array(dB,c(2,length(dB)/2)),2,sum)
    }

hs <- h*2^(1:Ndouble)/2

par(mfrow=c(1,2))
plot(log10(hs),var[,1],xlab='log10(h)',ylab="Discretized total variation V(B)",pch=16,log="y")
lines(log10(hs),sqrt(2/pi/hs))
plot(log10(hs),var[,2],xlab='log10(h)',ylab=expression("Discretized quadratic variation [B]"[1]),ylim=c
abline(h=1)
```



For the total variation, analytically, we first use that

$$X \sim N(0,1) \Rightarrow \mathbb{E}|X| = \sqrt{2/\pi} \approx 0.797$$

To see this:

$$\mathbb{E}|X| = \int_0^\infty x\sqrt{2/\pi}\ e^{-\frac{1}{2}x^2}\ dx = \sqrt{2/\pi}\int_0^\infty e^{-u}\ du = \sqrt{2/\pi}$$

We therefore get $\mathbb{E}|\Delta B| = \sqrt{2\Delta t/\pi}$. Assume a regular grid with n subintervals each of length $\Delta t = 1/n$, then

$$\mathbb{E}\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| = \sqrt{2n\pi}$$

Q4: Basic martingales

See solutions in the notes.