

## Exercise 5: The Euler method and the Ito integral

In the following, we use the following implementation of the Itô integral:

```
itointegral <- function(G,B) c(0,cumsum(head(G,-1)*diff(B)))
```

**Question 1:** Verify the implementation by computing and plotting

$$\int_0^T \cos t \, d \sin t$$

for  $T \in [0, 2\pi]$  on a sufficiently fine grid (at least 100 grid points).

Establish the analytical result

$$\int_0^T \cos t \, d \sin t = \frac{1}{2}T + \frac{1}{4} \sin 2T$$

and compare with the numerical result. *Hint:* Use  $d \sin t = \cos t \, dt$  and  $\cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t$ .

## Integrating Brownian motion w.r.t. itself

**Question 2:** Re-create figure 6.5 in the notes (p. 131). Specifically, apply your integrating function to compute one realization of the Itô integral

$$I_t = \int_0^t B_s \, dB_s$$

for  $t \in \{0, 0.5, 1.0, \dots, 100\}$ , where  $\{B_s : 0 \leq s \leq 100\}$  is Brownian motion. Plot the result as function of  $t$ . Compare with the analytical result  $I_t = \frac{1}{2}(B_t^2 - t)$ .

**Question 3:** Write a modified integrator which computes the “right hand approximation” given by (for general integrand  $f$  and integrator  $g$ )

$$I_m^R = \sum_{i=1}^m f(t_i) \cdot [g(t_i) - g(t_{i-1})]$$

Add this integral to the plot. Furthermore, add the Stratonovich approximation

$$I_m^S = \sum_{i=1}^m \frac{1}{2} (f(t_{i-1}) + f(t_i)) \cdot [g(t_i) - g(t_{i-1})]$$

Finally, add the result that one could expect from deterministic calculus, i.e.  $\frac{1}{2}B_t^2$ .

## Simulating an SDE with the Euler-Maruyama method

Consider the Cox-Ingersoll-Ross process (also termed the Square Root Process)

$$dX_t = \lambda(\xi - X_t) dt + \gamma\sqrt{X_t} dB_t$$

**Question 4:** Simulate the process using the Euler-Maruyama method. Take parameters  $\lambda = 1/2$ ,  $\xi = 2$ ,  $\gamma = 1$ . Use an initial condition of  $X_0 = \xi$  and simulate the process on the time interval  $[0, 100]$  with a time step of  $h = 0.01$ . Plot the solution  $\{X_t\}$  versus time  $t$ .

### Effect of the time step

**Question 5:** Repeat the simulation with the same sample path of Brownian motion, but now using a time step of  $10h$ . Verify that both the long term and the short term behavior is roughly the same.

### Verification of the integral version

**Question 6:** Verify numerically that the simulated sample paths of  $\{B_t\}$  and  $\{X_t\}$  satisfy the integral version

$$X_t = X_0 + \int_0^t \lambda(\xi - X_s) ds + \int_0^t \gamma\sqrt{X_s} dB_s$$

for each  $t = 0, 0.01, 0.02, \dots, 100$ . Here, you use the numerical integration routine that you made in question 1, and the verification should be by plotting the sum of the integrals on top of the Euler solution.

### Mean and variance of $\int_0^t B_s dB_s$

(Exercise 6.9 in the notes)

**Question 7:** Consider the Itô integral

$$I_t = \int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Use the properties of Gaussian variables (exercise 3.13) to determine the mean and variance of  $I_t$ , and verify that this agrees with the properties of Itô integrals.