DTU Compute

02425 Diffusion and SDEs
December 4, 2024
UHT/uht

Exercises for session 7:
Existence and uniqueness, numerics, and Stratonovich SDEs

Conditions for existence and uniqueness

Question 1: Determine which of the following Initial Value Problems satisfy the sufficient condition for existence and uniqueness of a solution.

1. $dX_t = \sin X_t \; dt + \cos X_t \; dB_t$, $X_0 = 0$
2. $dX_t = e^{X_t} \; dt + dB_t$, $X_0 = 0$
3. $dX_t = -X_t^3 \; dt + dB_t$, $X_0 = 0$
4. $dX_t = \lambda(\mu - X_t) \; dt + \sigma\sqrt{|X_t|} \; dB_t$, $X_0 = \mu$, where $\mu > 0$ and $\lambda > 0$.

Optional: For each case, if the conditions are not met, can we expect solutions to fork or explode?

The physical pendulum

Consider the two coupled SDE's which model a physical pendulum with friction and noise:

$$dX_t = V_t \; dt, \qquad dV_t = \sin X_t \; dt - \lambda V_t \; dt + \sigma \; dB_t \quad .$$

Here, X_t denotes the angle with $x = 0$ pointing upwards. V_t is the agnular velocity.

Question 2: Show that the system satisfies the conditions for existence and uniqueness of solutions, and that the Itō interpretation and the Stratonovich interpretation of the equation are identical.

Question 3: Simulate the system with the Euler method for $t \in [0, 1000]$, taking $\lambda = 0.1$ and $\sigma = 0.01$. Use $dt = 0.01$ and $X_0 = 0$, $V_0 = 0$. Plot the solution as a function of time, and a histogram of the X. Does it appear that the solution reflects the long term behavior of the process? Repeat for larger values of σ, up to $\sqrt{2}$, say.

Conversion between Itō and Stratonovich equations

Consider (again) the Itō equation governing the *Cox-Ingersoll-Ross process* (a.k.a. the *square root process*):

$$dX_t = \lambda(\mu - X_t)\; dt + \sigma\sqrt{X_t}\; dB_t \quad .$$

Question 4: Identify the equivalent Stratonovich equation; i.e., the Stratonovich equation that has the same solution.

Question 5: Simulate the solution using the Euler-Maruyama method for the Itō equation: Take $\mu = \lambda = 1$. Start with small values of σ (say, 0.1) and then increase σ gradually, using the same realization of Brownian motion for all values of σ. Plot the resulting trajectory as well as the histogram of the solution. *Note:* Theoretically, the solution $\{X_t\}$ stays non-negative, but the discrete-time approximation may take on negative values which can cause havoc. Take appropriate action to address this if necessary.

Question 6 Euler-Maruyama vs. Heun; the effect of time step: Take $\mu = \lambda = 1$ and $\sigma = 1.25$. Generate a single realization of Brownian motion for $t \in [0, 100]$ with a time step of 0.01. Simulate the process with both the Euler-Maruyama method (for the Itō formulation) and the Heun method (for the Stratonovich formulation). Check that the two solutions are reasonably close to eachother. Then subsample the Brownian motion to a time step of e.g. 0.1 and repeat the two simulations. Assess the sensitivity of the two schemes to the time step.