DTU Compute

02425 Diffusion and SDE's
December 4, 2024
UHT/uht

Exercise 5: The Euler method and the Ito integral

In the following, we use the following implementation of the Itō integral:

```
itointegral <- function(G,B) c(0,cumsum(head(G,-1)*diff(B)))
```

Question 1: Verify the implementation by computing and plotting

$$\int_0^T \cos t \; d\sin t$$

for $T \in [0, 2\pi]$ on a sufficiently fine grid (at least 100 grid points).

Establish the analytical result

$$\int_0^T \cos t \; d\sin t = \frac{1}{2}T + \frac{1}{4}\sin 2T$$

and compare with the numerical result. *Hint:* Use $d\sin t = \cos t \; dt$ and $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$.

Integrating Brownian motion w.r.t. itself

Question 2: Re-create figure 6.6 in the notes (p. 130). Specifically, apply your integrating function to compute one realization of the Itô integral

$$I_t = \int_0^t B_s \; dB_s$$

for $t \in \{0, 0.5, 1.0, \dots, 100\}$, where $\{B_s : 0 \le s \le 100\}$ is Brownian motion. Plot the result as function of t. Compare with the analytical result $I_t = \frac{1}{2}(B_t^2 - t)$.

Question 3: Write a modified integrator which computes the "right hand approximation" given by (for general integrand f and integrator g)

$$I_m^R = \sum_{i=1}^{m} f(t_i) \cdot [g(t_i) - g(t_{i-1})]$$

Add this integral to the plot. Furthermore, add the Stratonovich approximation

$$I_m^S = \sum_{i=1}^{m} \frac{1}{2}(f(t_{i-1}) + f(t_i)) \cdot [g(t_i) - g(t_{i-1})]$$

Finally, add the result that one could expect from deterministic calculus, i.e. $\frac{1}{2}B_t^2$.

Question 4: (Compare exercise 6.9 in the book) Consider the Itō integral

$$I_t = \int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t$$

Use the properties of Gaussian variables (exercise 3.13) to determine the mean and variance of I_t, and verify that this agrees with the properties of Itō integrals, viz., the martingale property and the Itō isometry.

Simulating an SDE with the Euler-Maruyama method

Consider the Cox-Ingersoll-Ross process (also termed the Square Root Process)

$$dX_t = \lambda(\xi - X_t) \, dt + \gamma \sqrt{|X_t|} \, dB_t$$

Question 5: Simulate the process using the Euler-Maruyama method. Take parameters $\lambda = 1/2$, $\xi = 2$, $\gamma = 1$. Use an initial condition of $X_0 = \xi$ and simulate the process on the time interval $[0, 100]$ with a time step of $h = 0.01$. Plot the solution $\{X_t\}$ versus time t.

Effect of the time step

Question 6: Repeat the simulation with the same sample path of Brownian motion, but now using a time step of $10h$. Verify that both the long term and the short term behavior is roughly the same.

Verification of the integral version

Question 7: Verify numerically that the simulated sample paths of $\{B_t\}$ and $\{X_t\}$ satisfy the integral version

$$X_t = X_0 + \int_0^t \lambda(\xi - X_s) \, ds + \int_0^t \gamma \sqrt{|X_s|} \, dB_s$$

for each $t = 0, \ 0.01, \ 0.02, \ldots, 100$. Here, you use the numerical integration routine that you made in question 1, and the verification should be by plotting the sum of the integrals on top of the Euler solution.

Mean, variance, and autocovariance function

Question 8: Solve exercise 6.12 in the book, where we use the Euler-Maruyama method to reach ODE's for the mean, variance, and autocovariance function of the Cox-Ingersoll-Ross process.