DTU Compute

02425 Diffusions and SDE's
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UHT/uht

Exercise 4: Linear systems

Steady-state variance structure for a mass-spring-damper system

Consider the mass-spring-damper system in the notes (5.1), (5.2), p. 90, with the force $\{F_t : t \geq 0\}$ being white noise with a given intensity $S_{FF}(\omega) = \sigma^2$.

Question 1: Write the system in the standard form $dX_t = AX_t\ dt + G\ dB_t$, i.e. specify A and G. *Note:* Yes, the answer is *almost* given in the textbook.

Question 2: Using the general form, simulate the system on the time interval $t \in [0, 1000]$ using the Euler-Maruyama method. Take system parameters $m = 1$ kg, $k = 0.5$ N/m, $c = 0.2$ Ns/m, $\sigma^2 = 100$ N$^2 s$. Let the system start at rest at $t = 0$. Use a time step of $\Delta t = 0.01$ s. Plot the first part of the sample path, $t \in [0, 100]$.

Question 3: Estimate from your simulation the steady-state variance of position Q_t, of velocity V_t, and the covariance between the two. Compare with the solution of the algebraic Lyapunov equation governing the variance. *Note:* In `Matlab`, use built-in `lyap.m`. In `R`, use the function `lyap.R` in `SDEtools`. In `python`, use `scipy.linalg.solve_continuous_lyapunov`.

Question 4: The kinetic energy is $\frac{1}{2}mV_t^2$ while the potential energy is $\frac{1}{2}kQ_t^2$. In steady-state, what is the expected kinetic energy and the expected potential energy? *Note:* The result is an example of equipartitioning of energy, a general principle in statistical mechanics, both quantum and classical.

Question 5: For the simulation, compute and plot the empirical a.c.f. of $\{Q_t\}$ up to lag 50 s. *Hint:* In `Matlab` and `R`, use `acf`. In `python`, use e.g. `statsmodels.tsa.stattools.acf`. Add to the plot the theoretical prediction.

Question 6: Plot, as a function of the frequency ω, the amplitude and phase of the frequency response from the noise to the position. Plot also the theoretical variance spectrum of the position.

Variance in a scalar linear system

The objective of this question is to reproduce figure 5.7 in the book (p. 108). So, consider the scalar linear system

$$\dot{X}_t = -\lambda X_t + \sigma U_t, \quad X_0 = x$$

where $\{U_t : t \geq 0\}$ is Gaussian "white noise", i.e. the formal derivative of standard Brownian motion. For numerical work, we take parameters $x = 1$, $\lambda = 1$, $\sigma = 1$.

Question 7: Plot the mean $\mathbf{E}X_t$ as a function of time.

Question 8: Write up the differential Lyapunov equation governing the variance $\mathbf{V}X_t$, and its solution. Add to the plot the mean plus/minus the standard deviation.

Question 9: Assume that $\lambda > 0$, write up the steady-state variance, $\lim_{t\to\infty} \mathbf{V}X_t$. Compute its numerical value and compare with the value of $\Sigma(t)$ for t large.

Question 10: Simulate a sample path of $\{X_t\}$ and add it to the graph.