DTU Compute

02425
December 4, 2024
UHT/uht

Exercise 11: The generator

Following section 11.2.3, consider the linear stochastic differential equation

$$dX_t = -X_t\ dt + \sqrt{2}\ dB_t, \quad X_0 = x$$

evolving on the interval $X_t \in (-l, l)$ Let $\tau = \inf\{t : |X_t| \geq l\}$ denote the first time of exit from the interval. We aim to find $\mathbf{E}^{X_0=x}\tau$. Moreover, we want to find the probability of exiting to the right, i.e. $\mathbb{P}^{X_0=x}(X_\tau = 1)$

Question 1 Monte Carlo simulation: Estimate $\mathbb{P}^{X_0=x}(X_\tau = l)$ and $\mathbf{E}^{X_0=x}\tau$, for $x = l/2$, by simulating $N = 1,000$ realizations of X_t until τ. Take $l = 1$ and use a sufficiently fine time step. Repeat for $l = 2$ and even larger values of l.

Question 2 Finding $\mathbb{P}^{X_0=x}(X_\tau = l)$ using a backward equation: Write a boundary value problem which governs $h(x) := \mathbb{P}^{X_0=x}(X_\tau = l)$. Solve the boundary value problem using whichever method you prefer. Plot the solution for the values of l you used in the previous and compare the results. *Hint:* Applicable methods are: Analytical solution, e.g. in terms of the scale function. Numerical solution using a built-in solver of boundary value problems (e.g. `bvp4c` in `Matlab`, `pracma::bvp` in `R`, `scipy.solve_bvp` in `python`). Numerical solution using the generator obtained from `fvade` - in that case, use boundary condition `'e'` to "extend" the generator and include the absorbing boundary points.

Question 3 Finding $\mathbf{E}\tau$ using a backward equation: Write a boundary value problem which governs $k(x) := \mathbf{E}^{X_0=x}\tau$. Solve the problem, plot the solution, and compare with the Monte Carlo solution as well as with the analytical result. *Hint:* The same methods apply. If using `fvade`, it is slightly easier to use absorbing boundary conditions (`'a'`).

Question 4 Finding the expected total pay-off: Find, as a function of x, the expected total pay-off

$$\mathbf{E}^{X_0=x} \int_0^\tau X_t^2\ dt$$

Extend the Monte Carlo simulation to compute also this payoff and compare with the numerical solution.

Is the origin attainable under the CIR and GBM processes?

Question 5 The CIR process: Example 11.7.1 in the notes establishes a criterion under which the Cox-Ingersoll-Ross process may reach the origin. Fill in the details in the argument, thus verifying the criterion.

Question 6 **Geometric Brownian motion:** Example 11.7.2 establishes a criterion for which geometric Brownian motion may converge to the origin. Fill in the details in the argument, thus verifying the criterion.